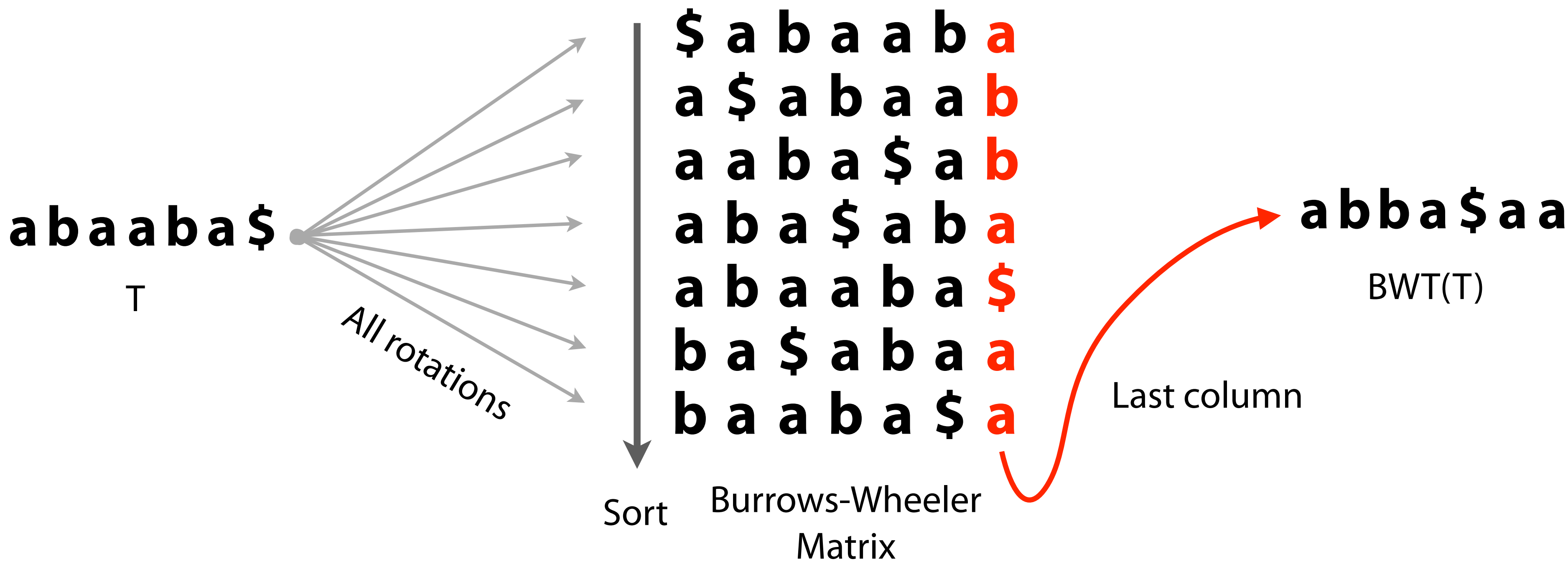
Burrows-Wheeler Transform

Reversible permutation of the characters of a string, used originally for compression



How is it useful for compression? How is it reversible? How is it an index?

Burrows M, Wheeler DJ: A block sorting lossless data compression algorithm. *Digital Equipment Corporation, Palo Alto, CA* 1994, Technical Report 124; 1994

Burrows-Wheeler Transform

Characters of the BWT are sorted by their *right-context*

This lends additional structure to BWT(T), tending to make it more compressible

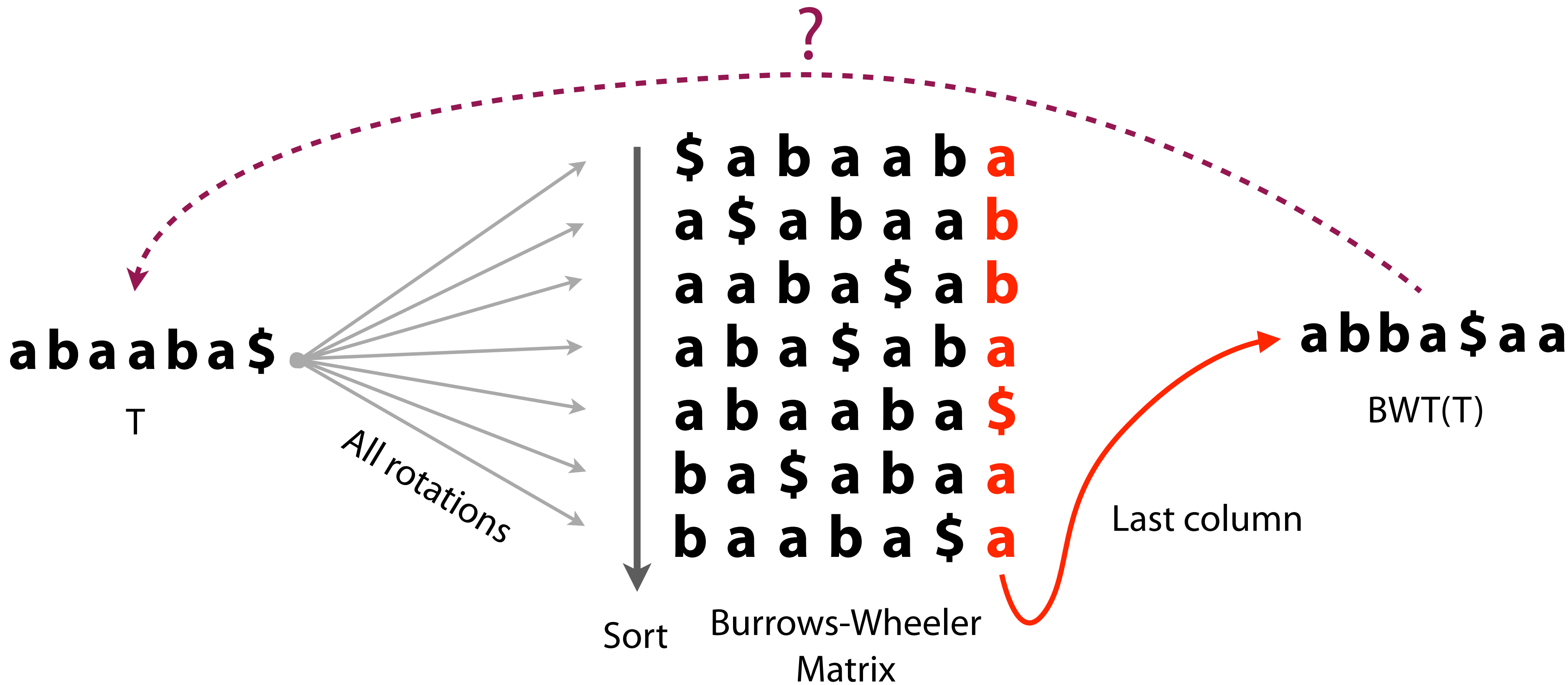
final char (*L*)	sorted rotations
a	n to decompress. It achieves compression
o	n to perform only comparisons to a depth
o	n transformation} This section describes
o	n transformation} We use the example and
o	n treats the right-hand side as the most
a	n tree for each 16 kbyte input block, enc
a	n tree in the output stream, then encodes
i	n turn, set $L[i]$ to be the
i	n turn, set $R[i]$ to the
o	n unusual data. Like the algorithm of Man
a	n use a single set of probabilities table
e	n using the positions of the suffixes in
i	n value at a given point in the vector $R
e	n we present modifications that improve t
e	n when the block size is quite large. Ho
i	n which codes that have not been seen in
i	n with ch appear in the {\em same order
i	n with ch. In our exam
o	n with Huffman or arithmetic coding. Bri
o	n with figures given by Bell~\cite{bell}.

Figure 1: Example of sorted rotations. Twenty consecutive rotations from the sorted list of rotations of a version of this paper are shown, together with the final character of each rotation.

Burrows M, Wheeler DJ: A block sorting lossless data compression algorithm. *Digital Equipment Corporation, Palo Alto, CA* 1994, Technical Report 124; 1994

Burrows-Wheeler Transform

How to reverse the BWT?



BWM has a key property called the *LF Mapping*...

Burrows-Wheeler Transform: T-ranking

Give each character in *T* a rank, equal to # times the character occurred previously in *T*. Call this the *T-ranking*.

$$\mathbf{a}_0\ \mathbf{b}_0\ \mathbf{a}_1\ \mathbf{a}_2\ \mathbf{b}_1\ \mathbf{a}_3\ \$$$

Now let's re-write the BWM including ranks...

Burrows-Wheeler Transform

BWM with T-ranking:

F						*L*
\$	a_0	b_0	a_1	a_2	b_1	$\mathbf{a_3}$
$\mathbf{a_3}$	\$	a_0	b_0	a_1	a_2	b_1
$\mathbf{a_1}$	a_2	b_1	a_3	\$	a_0	b_0
$\mathbf{a_2}$	b_1	a_3	\$	a_0	b_0	$\mathbf{a_1}$
$\mathbf{a_0}$	b_0	a_1	a_2	b_1	a_3	\$
b_1	a_3	\$	a_0	b_0	a_1	$\mathbf{a_2}$
b_0	a_1	a_2	b_1	a_3	\$	$\mathbf{a_0}$

Look at first and last columns, called *F* and *L*

And look at just the **a**s

as occur in the same order in *F* and *L*. As we look down columns, in both cases we see: $\mathbf{a_3}$, $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_0}$

Burrows-Wheeler Transform

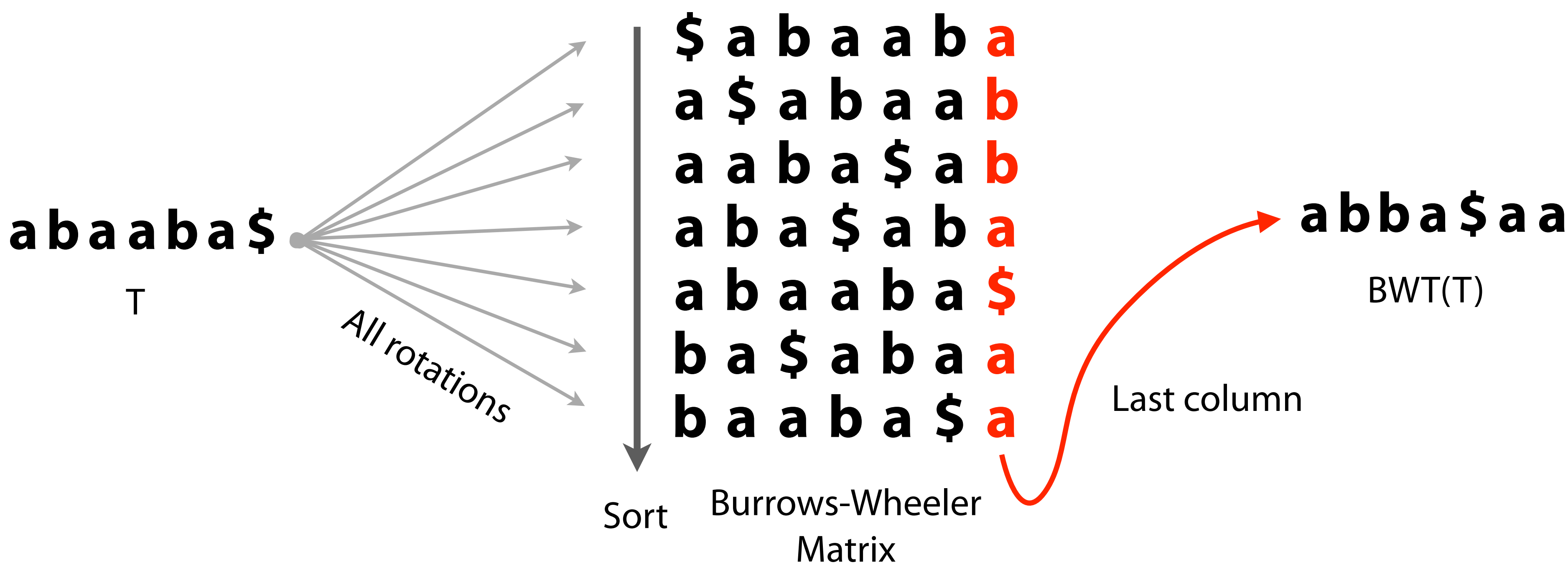
BWM with T-ranking:

F						L
\$	a_0	b_0	a_1	a_2	b_1	a_3
a_3	\$	a_0	b_0	a_1	a_2	$\mathbf{b}_1$
a_1	a_2	b_1	a_3	\$	a_0	$\mathbf{b}_0$
a_2	b_1	a_3	\$	a_0	b_0	a_1
a_0	b_0	a_1	a_2	b_1	a_3	\$
$\mathbf{b}_1$	a_3	\$	a_0	b_0	a_1	a_2
$\mathbf{b}_0$	a_1	a_2	b_1	a_3	\$	a_0

Same with **b**s: $\mathbf{b}_1$, $\mathbf{b}_0$

Burrows-Wheeler Transform

Reversible permutation of the characters of a string, used originally for compression



How is it useful for compression? How is it reversible? How is it an index?

Burrows M, Wheeler DJ: A block sorting lossless data compression algorithm. *Digital Equipment Corporation, Palo Alto, CA* 1994, Technical Report 124; 1994

Burrows-Wheeler Transform: LF Mapping

BWM with T-ranking:

F						*L*
\$	a_0	b_0	a_1	a_2	b_1	a_3
a_3	\$	a_0	b_0	a_1	a_2	b_1
a_1	a_2	b_1	a_3	\$	a_0	b_0
a_2	b_1	a_3	\$	a_0	b_0	a_1
a_0	b_0	a_1	a_2	b_1	a_3	\$
b_1	a_3	\$	a_0	b_0	a_1	a_2
b_0	a_1	a_2	b_1	a_3	\$	a_0

LF Mapping: The i^{th} occurrence of a character c in L and the i^{th} occurrence of c in F correspond to the *same* occurrence in T

However we rank occurrences of c, ranks appear in the same order in F and L

Burrows-Wheeler Transform: LF Mapping

Why does the LF Mapping hold?

Why are these **a**s in this order relative to each other?

\$	a	b	a	a	b	a_3
a_3	\$	a	b	a	a	b_1
a_1	a	b	a	\$	a	b_0
a_2	b	a	\$	a	b	a_1
a_0	b	a	a	b	a	\$
b_1	a	\$	a	b	a	a_2
b_0	a	a	b	a	\$	a_0

They're sorted by right-context

\$	a	b	a	a	b	a_3
a_3	\$	a	b	a	a	b_1
a_1	a	b	a	\$	a	b_0
a_2	b	a	\$	a	b	a_1
a_0	b	a	a	b	a	\$
b_1	a	\$	a	b	a	a_2
b_0	a	a	b	a	\$	a_0

Why are these **a**s in this order relative to each other?

They're sorted by right-context

Occurrences of *c* in *F* are sorted by right-context. Same for *L*!

Whatever ranking we give to characters in *T*, rank orders in *F* and *L* will match

Burrows-Wheeler Transform: LF Mapping

BWM with T-ranking:

F						*L*
$	a_0	b_0	a_1	a_2	b_1	a_3
a_3	$	a_0	b_0	a_1	a_2	b_1
a_1	a_2	b_1	a_3	$	a_0	b_0
a_2	b_1	a_3	$	a_0	b_0	a_1
a_0	b_0	a_1	a_2	b_1	a_3	$
b_1	a_3	$	a_0	b_0	a_1	a_2
b_0	a_1	a_2	b_1	a_3	$	a_0

We'd like a different ranking so that for a given character, ranks are in ascending order as we look down the F / L columns...

Burrows-Wheeler Transform: LF Mapping

BWM with B-ranking:

F						*L*
\$	a_3	b_1	a_1	a_2	b_0	$\mathbf{a_0}$
$\mathbf{a_0}$	\$	a_3	b_1	a_1	a_2	$\mathbf{b_0}$
$\mathbf{a_1}$	a_2	b_0	a_3	\$	a_3	$\mathbf{b_1}$
$\mathbf{a_2}$	b_0	a_0	\$	a_3	b_1	$\mathbf{a_1}$
$\mathbf{a_3}$	b_1	a_1	a_2	b_0	a_0	**\$**
$\mathbf{b_0}$	a_0	\$	a_3	b_1	a_1	$\mathbf{a_2}$
$\mathbf{b_1}$	a_1	a_2	b_0	a_0	\$	$\mathbf{a_3}$

Ascending rank

F now has very simple structure: a **$**, a block of **a**s *with ascending ranks*, a block of **b**s *with ascending ranks*

Burrows-Wheeler Transform

	F	L
	$\$$	a_0
	a_0	b_0
	a_1	b_1 ← Which BWM row *begins* with b_1?
	a_2	a_1
	a_3	$\$$
	b_0	a_2
row 6 →	b_1	a_3

Skip row starting with **$** (1 row)

Skip rows starting with **a** (4 rows)

Skip row starting with b_0 (1 row)

Answer: row 6

Burrows-Wheeler Transform

Say T has 300 **A**s, 400 **C**s, 250 **G**s and 700 **T**s and **$** < **A** < **C** < **G** < **T**

Which BWM row (0-based) begins with $\mathbf{G}_{100}$? (Ranks are B-ranks.)

Skip row starting with **$** (1 row)

Skip rows starting with **A** (300 rows)

Skip rows starting with **C** (400 rows)

Skip first 100 rows starting with **G** (100 rows)

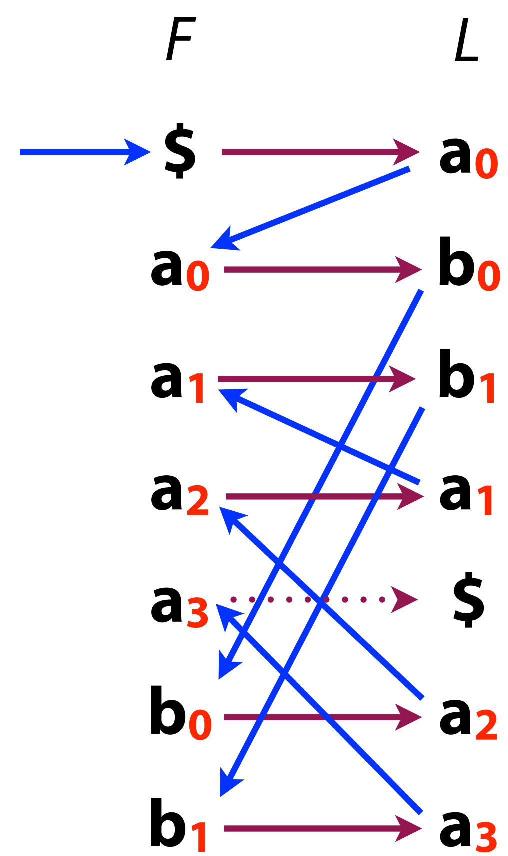
Answer: row 1 + 300 + 400 + 100 = **row 801**

Burrows-Wheeler Transform: reversing

Reverse BWT(T) starting at right-hand-side of *T* and moving left

Start in first row. *F* must have **$**. *L* contains character just prior to **$**: $\mathbf{a}_0$

$\mathbf{a}_0$: LF Mapping says this is same occurrence of **a** as first **a** in *F*. Jump to row *beginning* with $\mathbf{a}_0$. *L* contains character just prior to $\mathbf{a}_0$: $\mathbf{b}_0$.



Repeat for $\mathbf{b}_0$, get $\mathbf{a}_2$

Repeat for $\mathbf{a}_2$, get $\mathbf{a}_1$

Repeat for $\mathbf{a}_1$, get $\mathbf{b}_1$

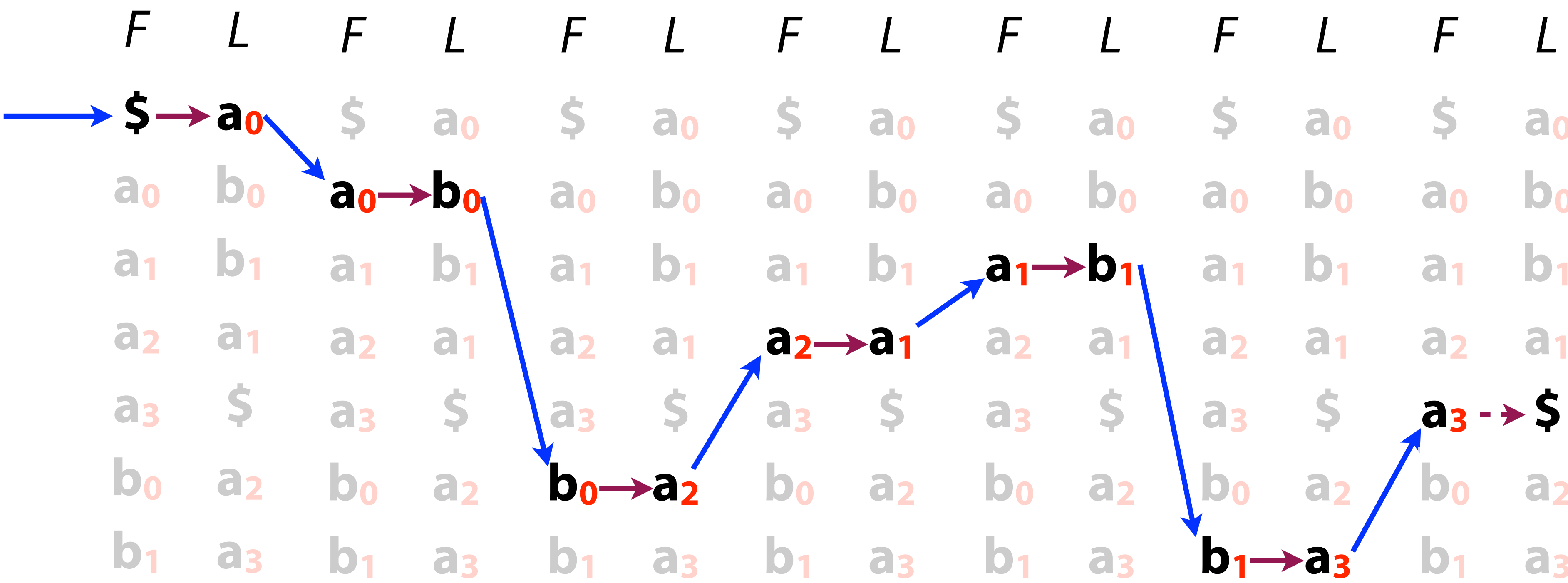
Repeat for $\mathbf{b}_1$, get $\mathbf{a}_3$

Repeat for $\mathbf{a}_3$, get **$**, done

Reverse of chars we visited = $\mathbf{a}_3\ \mathbf{b}_1\ \mathbf{a}_1\ \mathbf{a}_2\ \mathbf{b}_0\ \mathbf{a}_0\ \mathbf{\$} = T$

Burrows-Wheeler Transform: reversing

Another way to visualize reversing BWT(T):



T: $a_3\ b_1\ a_1\ a_2\ b_0\ a_0\ \$$

Burrows-Wheeler Transform

- We've seen how BWT is useful for compression:
  - Sorts characters by right-context, making a more compressible string
- And how it's reversible:
  - Repeated applications of LF Mapping, recreating *T* from right to left
- How is it used as an index?

FM Index

FM Index: an index combining the BWT *with a few small auxilliary data structures*

“FM” supposedly stands for “Full-text Minute-space.” (But inventors are named Ferragina and Manzini)

Core of index consists of *F* and *L* from BWM:

F can be represented very simply (1 integer per alphabet character)

And *L* is compressible

Potentially very space-economical!

```
F             L
$ a b a a b a
a $ a b a a b
a a b a $ a b
a b a $ a b a
a b a a b a $
b a $ a b a a
b a a b a $ a
  |---------|
Not stored in index
```

Paolo Ferragina, and Giovanni Manzini. "Opportunistic data structures with applications." *Foundations of Computer Science, 2000. Proceedings. 41st Annual Symposium on*. IEEE, 2000.

FM Index: querying

Though BWM is related to suffix array, we can't query it the same way

$	a	b	a	a	b	a
a	$	a	b	a	a	b
a	a	b	a	$	a	b
a	b	a	$	a	b	a
a	b	a	a	b	a	$
b	a	$	a	b	a	a
b	a	a	b	a	$	a

6	$
5	a $
2	a a b a $
3	a b a $
0	a b a a b a $
4	b a $
1	b a a b a $

We don't have these columns; binary search isn't possible

FM Index: querying

Look for range of rows of BWM(T) with *P* as prefix

Do this for *P*'s shortest suffix, then extend to successively longer suffixes until range becomes empty or we've exhausted *P*

P = **aba**

Easy to find all the rows beginning with **a**, thanks to *F*'s simple structure

F						*L*
$\$$	a	b	a	a	b	**a_3**
a_0	$\$$	a	b	a	a	**b_1**
a_1	a	b	a	$\$$	a	**b_0**
a_2	b	a	$\$$	a	b	**a_1**
a_3	b	a	a	b	a	**$\$$**
b_0	a	$\$$	a	b	a	**a_2**
b_1	a	a	b	a	$\$$	**a_0**

FM Index: querying

We have rows beginning with **a**, now we seek rows beginning with **ba**

P = **aba**

F						L
$	a	b	a	a	b	a_0
a_0	$	a	b	a	a	b_0
a_1	a	b	a	$	a	b_1
a_2	b	a	$	a	b	a_1
a_3	b	a	a	b	a	$
b_0	a	$	a	b	a	a_2
b_1	a	a	b	a	$	a_3

Look at those rows in *L*.

b_0, b_1 are **b**s occuring just to left.

P = **aba**

F						L
$	a	b	a	a	b	a_0
a_0	$	a	b	a	a	b_0
a_1	a	b	a	$	a	b_1
a_2	b	a	$	a	b	a_1
a_3	b	a	a	b	a	$
b_0	a	$	a	b	a	a_2
b_1	a	a	b	a	$	a_3

Use LF Mapping. Let new range delimit those **b**s

Now we have the rows with prefix **ba**

FM Index: querying

We have rows beginning with **ba**, now we seek rows beginning with **aba**

$P =$ **aba**

F						L
$\$$	a	b	a	a	b	a_0
a_0	$\$$	a	b	a	a	b_0
a_1	a	b	a	$\$$	a	b_1
a_2	b	a	$\$$	a	b	a_1
a_3	b	a	a	b	a	$\$$
b_0	a	$\$$	a	b	a	a_2
b_1	a	a	b	a	$\$$	a_3

a_2, a_3 occur just to left.

Use LF Mapping

$P =$ **aba**

F						L
$\$$	a	b	a	a	b	a_0
a_0	$\$$	a	b	a	a	b_0
a_1	a	b	a	$\$$	a	b_1
a_2	b	a	$\$$	a	b	a_1
a_3	b	a	a	b	a	$\$$
b_0	a	$\$$	a	b	a	a_2
b_1	a	a	b	a	$\$$	a_3

Now we have the rows with prefix **aba**

FM Index: querying

$P =$ **aba**

Now we have the same range, [3, 5), we would have got from querying suffix array

	F						*L*
	\$	a	b	a	a	b	a_0
	a_0	\$	a	b	a	a	b_0
	a_1	a	b	a	\$	a	b_1
[3, 5)	a_2	b	a	\$	a	b	a_1
	a_3	b	a	a	b	a	\$
	b_0	a	\$	a	b	a	a_2
	b_1	a	a	b	a	\$	a_3

Where are these?

	6	\$
	5	a \$
	2	a a b a \$
[3, 5)	3	a b a \$
	0	a b a a b a \$
	4	b a \$
	1	b a a b a \$

Unlike suffix array, we don't immediately know *where* the matches are in T...

FM Index: querying

When *P* does not occur in *T*, we will eventually fail to find the next character in *L*:

$P =$ **b**<u>**ba**</u>

	F						*L*	
	$	a	b	a	a	b	a_0	
	a_0	$	a	b	a	a	b_0	
	a_1	a	b	a	$	a	b_1	
	a_2	b	a	$	a	b	a_1	
	a_3	b	a	a	b	a	$	
Rows with **ba** prefix	b_0	a	$	a	b	a	a_2	No **b**s!
	b_1	a	a	b	a	$	a_3	

FM Index: querying

If we *scan* characters in the last column, that can be very slow, $O(m)$

$P =$ **aba**

F					L
$	a	b	a	a	b	a_3
a_0	$	a	b	a	a	b_1
a_1	a	b	a	$	a	b_0
a_2	b	a	$	a	b	a_1
a_3	b	a	a	b	a	$
b_0	a	$	a	b	a	a_2
b_1	a	a	b	a	$	a_0

Scan, looking for **b**s

FM Index: lingering issues

(1) Scanning for preceding character is slow

$	a	b	a	a	b	a_0
a_0	$	a	b	a	a	b_0
a_1	a	b	a	$	a	b_1
a_2	b	a	$	a	b	a_1
a_3	b	a	a	b	a	$
b_0	a	$	a	b	a	a_2
b_1	a	a	b	a	$	a_3

$O(m)$ scan

(2) Storing ranks takes too much space

m integers

```
def reverseBwt(bw):
    """ Make T from BWT(T) """
    ranks, tots = rankBwt(bw)
    first = firstCol(tots)
    rowi = 0
    t = "$"
    while bw[rowi] != '$':
        c = bw[rowi]
        t = c + t
        rowi = first[c][0] + ranks[rowi]
    return t
```

(3) Need way to find where matches occur in *T*:

Where?

$	a	b	a	a	b	a_0
a_0	$	a	b	a	a	b_0
a_1	a	b	a	$	a	b_1
a_2	b	a	$	a	b	a_1
a_3	b	a	a	b	a	$
b_0	a	$	a	b	a	a_2
b_1	a	a	b	a	$	a_3

FM Index: fast rank calculations

Is there an O(1) way to determine which **b**s precede the **a**s in our range?

F					*L*
\$	a	b	a	a	b	$\mathbf{a}_0$
$\mathbf{a}_0$	\$	a	b	a	a	$\mathbf{b}_0$
$\mathbf{a}_1$	a	b	a	\$	a	$\mathbf{b}_1$
$\mathbf{a}_2$	b	a	\$	a	b	$\mathbf{a}_1$
$\mathbf{a}_3$	b	a	a	b	a	**\$**
$\mathbf{b}_0$	a	\$	a	b	a	$\mathbf{a}_2$
$\mathbf{b}_1$	a	a	b	a	\$	$\mathbf{a}_3$

Idea: pre-calculate # **a**s, **b**s in *L* up to every row:

Tally

F	*L*	**a**	**b**
\$	**a**	1	0
a	**b**	1	1
a	**b**	1	2
a	**a**	2	2
a	**\$**	2	2
b	**a**	3	2
b	**a**	4	2

We infer $\mathbf{b}_0$ and $\mathbf{b}_1$ appear in *L* in this range

O(1) time, but requires $m \times |\Sigma|$ integers

FM Index: fast rank calculations

Another idea: pre-calculate # **a**s, **b**s in *L* up to *some* rows, e.g. every 5th row. Call pre-calculated rows *checkpoints*.

		Tally		
F	*L*	**a**	**b**	
$	**a**	1	0	Lookup here succeeds as usual
a	**b**			
a	**b**			
a	**a**			
a	**$**			Oops: not a checkpoint
b	**a**	3	2	But there's one nearby
b	**a**			

To resolve a lookup for character *c* in non-checkpoint row, scan along *L* until we get to nearest checkpoint. Use tally at the checkpoint, *adjusted for # of cs we saw along the way*.

FM Index: fast rank calculations

What's my rank?

482 + 2 - 1 = 483

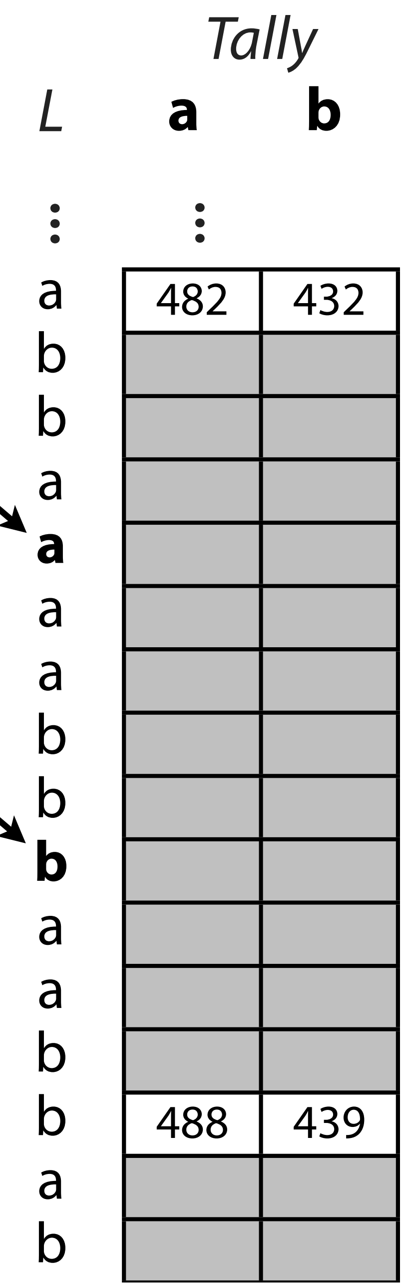
Checkpoint

as along the way

tally -> rank

What's my rank?

439 - 2 - 1 = 436



Assuming checkpoints are spaced $O(1)$ distance apart, lookups are $O(1)$

FM Index: a few problems

Solved! At the expense of adding checkpoints ($O(m)$ integers) to index.

(1)

F						L
\$	a	b	a	a	b	a_0
a_0	\$	a	b	a	a	b_0
a_1	a	b	a	\$	a	b_1
a_2	b	a	\$	a	b	a_1
a_3	b	a	a	b	a	\$
b_0	a	\$	a	b	a	a_2
b_1	a	a	b	a	\$	a_3

This scan is $O(m)$ work

With checkpoints it's $O(1)$

(2) Ranking takes too much space

m integers

```
def reverseBwt(bw):
    """ Make T from BWT(T) """
    ranks, tots = rankBwt(bw)
    first = firstCol(tots)
    rowi = 0
    t = "$"
    while bw[rowi] != '$':
        c = bw[rowi]
        t = c + t
        rowi = first[c][0] + ranks[rowi]
    return t
```

With checkpoints, we greatly reduce # integers needed for ranks

But it's still O(*m*) space - there's literature on how to improve this space bound

FM Index: a few problems

Not yet solved: **(3)** Need a way to find where these occurrences are in *T*:

\$	a	b	a	a	b	**a_0**
a_0	\$	a	b	a	a	**b_0**
a_1	a	b	a	\$	a	**b_1**
a_2	b	a	\$	a	b	**a_1**
a_3	b	a	a	b	a	**\$**
b_0	a	\$	a	b	a	**a_2**
b_1	a	a	b	a	\$	**a_3**

If suffix array were part of index, we could simply look up the offsets

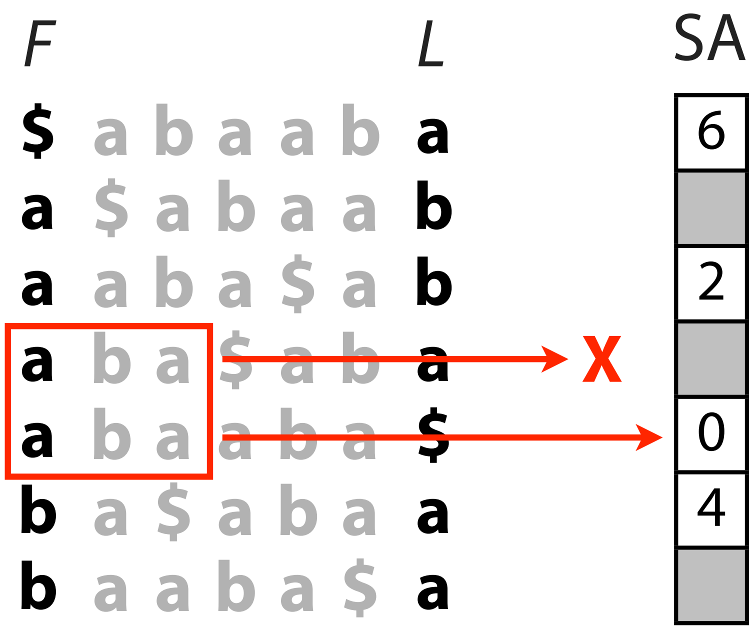
F						*L*	SA	
\$	a	b	a	a	b	**a**	6	**\$**
a	\$	a	b	a	a	**b**	5	**a \$**
a	a	b	a	\$	a	**b**	2	**a a b a \$**
a	b	a	\$	a	b	**a**	3	**a b a \$**
a	b	a	a	b	a	**\$**	0	**a b a a b a \$**
b	a	\$	a	b	a	**a**	4	**b a \$**
b	a	a	b	a	\$	**a**	1	**b a a b a \$**

Offsets: 0, 3

But SA requires *m* integers

FM Index: resolving offsets

Idea: store some, but not all, entries of the suffix array



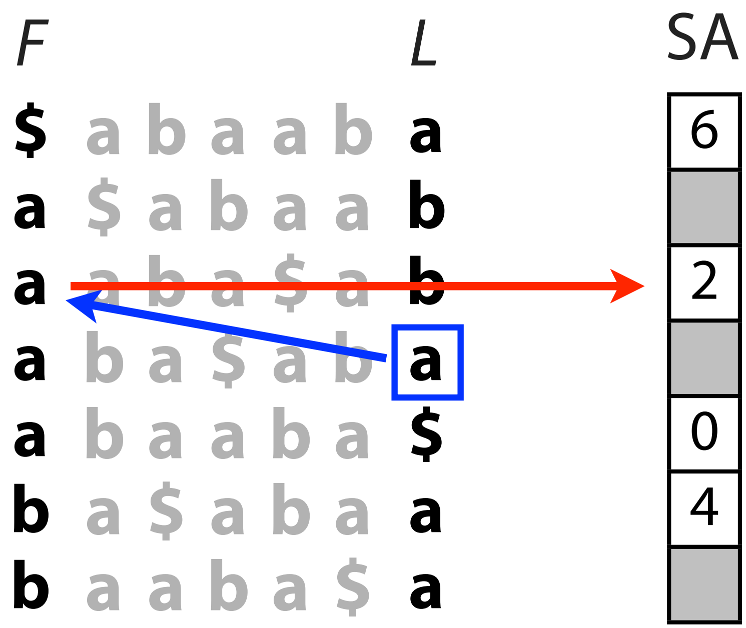
Lookup for row 4 succeeds - we kept that entry of SA

Lookup for row 3 fails - we discarded that entry of SA

FM Index: resolving offsets

But LF Mapping tells us that the **a** at the end of row 3 corresponds to...

...the **a** at the begining of row 2



And row 2 has a suffix array value = 2

So row 3 has suffix array value = 3 = 2 (row 2's SA val) + 1 (# steps to row 2)

If saved SA values are O(1) positions apart in *T*, resolving offset is O(1) time

FM Index: problems solved

Solved!

At the expense of adding some SA values ($O(m)$ integers) to index

Call this the "SA sample"

(3) Need a way to find where these occurrences are in *T*:

\$	a	b	a	a	b	a_0
a_0	\$	a	b	a	a	b_0
a_1	a	b	a	\$	a	b_1
a_2	b	a	\$	a	b	a_1
a_3	b	a	a	b	a	**\$**
b_0	a	\$	a	b	a	a_2
b_1	a	a	b	a	\$	a_3

With SA sample we can do this in *O*(1) time per occurrence

FM Index: small memory footprint

Components of the FM Index:

First column (F):	$\sim |\Sigma|$ integers
Last column (L):	m characters
SA sample:	$m \cdot a$ integers, where a is fraction of rows kept
Checkpoints:	$m \times |\Sigma| \cdot b$ integers, where b is fraction of rows checkpointed

Example: DNA alphabet (2 bits per nucleotide), T = human genome, $a = 1/32$, $b = 1/128$

First column (F):	16 bytes
Last column (L):	2 bits * 3 billion chars = 750 MB
SA sample:	3 billion chars * 4 bytes/char / 32 = ~ 400 MB
Checkpoints:	*3 billion* * 4 bytes/char / *128* = ~ 100 MB
	Total < 1.5 GB