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THE $M/M/c/N$ QUEUE WITH BALKING AND RENEGING

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Scope and Purpose—The truncated queueing models, especially those with both reneging and balking, fit many real-life problems, such as those arising in communication systems, machine manning, banks and military forces. This paper introduces an analytical solution in terms of the hypergeometric function. For specific values of the parameters, the solution can easily be found using a computer.

Abstract—In this paper, a truncated multi-channel queue with reneging and general balk function is considered. The steady-state distribution of the number of units in the system is derived. The expected number of units in both the system and the queue are also obtained. Some previously published results are shown to be special cases of the present results.

INTRODUCTION

The *multi-channel queue* $M/M/c$ with *balking*, *reneging* or both has been discussed by many researchers. Reynolds [1] studied the multi-channel Markovian queue $M/M/c$ with the *discouragement* (a kind of balking: an arriving unit ceases to enter the queue) concept only. He derived the probability generating function of the number of units in the system in terms of the incomplete gamma function and from it he derived expressions for the probability of finding all servers busy on arrival. Reynolds [1] also obtained some measures of effectiveness, such as the expected number of units in the system and the queue and the *waiting-time* distribution. Haghighi *et al.* [2] discussed the multi-channel Markovian queue $M/M/c$ (without truncation) with both balking ($b_n = \beta$, $0 \leqslant \beta < 1$) and reneging (a unit leaves the queue before being served) concepts and derived the steady-state probabilities. While neglecting the reneging concept, Haghighi *et al.* [2] calculated the expected number of units in the system and gave an expression for the average loss of units during a fixed duration of time.

In this paper, the truncated multi-channel queue $M/M/c/N$ is treated with both balking and reneging concepts. The chosen balk function generalizes most previously used ones. This choice of balk function should enable many previously published works to be deduced as special cases of the present model. The steady-state probabilities of the model herein are derived together with some measures of effectiveness where these measures are analytically deduced in terms of the hypergeometric function. Finally, some previously published works are shown to be special cases of the model herein.

DERIVATION OF THE MODEL

Consider the situation where the units arrive according to a Poisson process, with arrival rate

$$\lambda_n = \begin{cases} \lambda, & 0 \leqslant n \leqslant c-1, \\ b_n\lambda, & c \leqslant n \leqslant N, \end{cases}$$

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and the units are served on a FIFO basis according to an exponential distribution with service rate μ_n given by:

$$\mu_n = \begin{cases} n\mu, & 1 \leqslant n < c; \\ c\mu, & c \leqslant n \leqslant N. \end{cases}$$

When there are n units in the system, it is assumed that each unit joins the queue with probability b_n, where

$$b_n = \begin{cases} 1, & 0 \leqslant n < c, \\ \dfrac{\beta(1-(n-c+1)/N)}{(n-c+2)^m}, & c \leqslant n \leqslant N, \end{cases}$$

where β is a measure of a unit's willingness to join the queue and the parameter m is a non-negative integer. It is clear that $(1 - b_n)$ is the probability a unit may balk, and that $0 \leqslant b_{n+1} \leqslant b_n \leqslant 1$.

It is also assumed that the units may renege according to an exponential distribution, $f(t) = \alpha e - ^{\alpha t}$, $t > 0$, with parameter α. The probability of reneging in a short period of time Δt is given by $r_n = (n - c)\alpha\, \Delta t$, for $c \leqslant n \leqslant N$ and $r_n = 0$, for $1 \leqslant n < c$.

Let $p_n(t)$ be the probability that there are n units in the system at time t. There are five cases, namely, $n = 0$, $1 \leqslant n < c$, $n = c$, $c < n < N$ and $n = N$. Proceeding as usual, the steady-state probability difference equations are:

$$-\lambda p_0 + \mu p_1 = 0, \qquad n = 0, \tag{1}$$

$$-(\lambda + n\mu)p_n + \lambda p_{n-1} + (n+1)\mu p_{n+1} = 0, \qquad 1 \leqslant n < c, \tag{2}$$

$$-(b_c\lambda + c\mu)p_c + \lambda p_{c-1} + (c\mu + \alpha)p_{c+1} = 0, \qquad n = c, \tag{3}$$

$$-[b_n\lambda + c\mu + (n-c)\alpha]p_n + b_{n-1}\lambda p_{n-1} + [c\mu + (n-c+1)\alpha]p_{n+1} = 0, \qquad c < n < N \tag{4}$$

and

$$-[c\mu + (N-c)\alpha]p_N + b_{N-1}\lambda p_{N-1} = 0, \qquad n = N. \tag{5}$$

These $(N + 1)$ linear equations in the unknown probabilities $p_0, \ldots, p_N$ are solved as follows. Solving equations (1) and (2) obtain

$$p_n = p_0 \frac{\rho^n}{n!}, \quad 0 \leqslant n \leqslant c. \tag{6}$$

Let $\gamma = \lambda/\alpha$ and $\delta = (c\mu)/\alpha$, then from equations (3)–(5) and relation (6)

$$p_n = p_0 \frac{\rho^c}{c!} \frac{(1-N)_{n-c}}{[(2)_{n-c}]^m(1+\delta)_{n-c}} a^{n-c}, \quad c < n \leqslant N, \tag{7}$$

is obtained, where $(d)_n = d(d+1)\ldots(d+n-1)$ for $n > 0$, $(d)_0 = 1$ and $a = (-\beta\gamma)/N = (-\beta\lambda)/(N\alpha)$. Thus, p_n can be written as follows:

$$p_n = \begin{cases} p_0 \dfrac{\rho^n}{n!}, & 0 \leqslant n \leqslant c; \\ p_0 \dfrac{\rho^c}{c!} \dfrac{(1-N)_{n-c}}{[(2)_{n-c}]^m(1+\delta)_{n-c}} a^{n-c}, & c < n \leqslant N. \end{cases} \tag{8}$$

Note that relation (8) could be obtained using the rate coefficients λ_n and μ_n defined above and $p_n = p_0 \prod_{i=1}^{n} (\lambda_i - 1)/\mu_i$. To find p_0, the boundary condition $\sum_{n=0}^{N} p_n = 1$ is used, to give

$$\begin{aligned} p_0^{-1} &= \sum_{n=0}^{c} \frac{\rho^n}{n!} + \frac{\rho^c}{c!} \sum_{n=c+1}^{N} \frac{(1-N)_{n-c}}{[(2)_{n-c}]^m(1+\delta)_{n-c}} a^{n-c} \\ &= \sum_{n=0}^{c} \frac{\rho^n}{n!} + \frac{\rho^c}{c!} \frac{(1-N)a}{2^m(1+\delta)} \sum_{n=0}^{N-c-1} \frac{(1)_n(2-N)_n}{n![(3)_n]^m(2+\delta)_n} a^n \end{aligned}$$

$$= \sum_{n=0}^{c} \frac{\rho^n}{n!} + \frac{\rho^c}{c!} \frac{(1-N)a}{2^m(1+\delta)} {}_2F_{m+1}\left(\begin{matrix} 1, 2-N \\ 3, \ldots, 3, 2+\delta; a \end{matrix}\right), \tag{9}$$

where

$${}_pF_q\left(\begin{matrix} d_1 0, \ldots, d_p \\ e_1, \ldots, e_q; x \end{matrix}\right) = \sum_{n=0}^{\infty} \frac{(d_1)_n \ldots (d_p)_n}{n!(e_1)_n \ldots (e_q)_n} x^n$$

is the hypergeometric function. Note that when $n \geqslant N - c$, then either $(2-N)_n = 0$ or $(1)_n = 0$, which terminates the infinite hypergeometric series at $n = N - c - 1$.

MEASURES OF EFFECTIVENESS

To derive the expected number of units in the system, a result due to Abou-El-Ata [3] is used, which states that for a simple birth–death process

$$L = -\lambda \frac{\partial \ln p_0}{\partial \lambda} = -\rho \frac{\mathrm{d} \ln p_0}{\mathrm{d}\rho}.$$

Thus,

$$L = -\lambda \frac{\partial \ln p_0}{\partial \lambda} = p_0 \left[\sum_{n=1}^{c} \frac{\rho^n}{(n-1)!} + \frac{(c+1)\rho^c}{c!} \frac{(1-N)a}{2^m(1+\delta)} {}_2F_{m+1}\left(\begin{matrix} 1, 2-N \\ 3, \ldots, 3; 2+\delta; a \end{matrix}\right) \right.$$
$$\left. + \frac{\rho^c}{c!} \frac{(1-N)(2-N)a^2}{(2.3)^m(1+\delta)(2+\delta)} {}_2F_{m+1}\left(\begin{matrix} 2, 3-N \\ 4, \ldots, 4; 3+\delta; a \end{matrix}\right) \right]. \tag{10}$$

The expected number of units in the queue is

$$L_q = \sum_{n=c}^{N} (n-c)p_n = L - (c - \bar{c}) = L - \frac{\lambda(1-p_N)}{\mu} \tag{11}$$

and the expected waiting time in both the system and the queue are given by $w = L/\lambda_{\text{eff}}$ and $w_q = L_q/\lambda_{\text{eff}}$, where $\bar{c} = \sum_{n=0}^{c} (c-n)p_n$ is the expected number of idle servers and $\lambda_{\text{eff}} = \lambda(1-p_N) = \mu(c-\bar{c})$, which is known as the effective arrival rate.

Some special cases

With regard to relations (8)–(10), there are the following special cases:

(i) Putting $\alpha = 0, m = c = \beta = \lambda = 1$ ($M/M/1/N, b_n = (N-n)/(N(n+1)), n \geqslant 0$), these relations reduce to Haight's [4].

(ii) Unlike Reynolds's [1] approximate relations for p_0 and L when $c \geqslant 2$, the following exact results (putting $\alpha = 0$, $m = \beta = 1$ and as $N \to \infty$: i.e. $M/M/c$ with balking only, $b_n = 1/(n-c+2), n \geqslant c$) were obtained for these measures:

$$p_0^{-1} = \sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^{c-1}}{c!}(e^{\rho/c} - 1)$$

and

$$L = p_0 \left\{ \sum_{n=1}^{c} \frac{\rho^n}{(n-1)!} + \frac{\rho^{c-1}}{c!} \left[\frac{\rho e^{\rho/c}}{c} + (c-1)(e^{\rho/c} - 1) \right] \right\};$$

while, for $c = 1$, the present results coincide with those of Reynolds [1].

(iii) Putting $\alpha = m = 0$, $c = 2$ and as $N \to \infty$ ($M/M/2$ with balk function $b_n = \beta$), the present relations reduce to Singh's [5] results in the case when $\mu_1 = \mu_2$.

(iv) Finally, the results obtained by Haghighi *et al.* [2], could be achieved by putting $\alpha = m = 0$ and as $N \to \infty$ ($M/M/c$, with balk function $b_n = \beta$). Unfortunately, when $\alpha \neq 0$ (i.e. considering the reneging concept also), they calculated p_n and

p_0 only. However, in this case, it is easily shown that

$$L = p_0\left[\sum_{n=1}^{c} \frac{\rho^n}{(n-1)!} + \frac{\rho^c}{c!}\sum_{n=c+1}^{\infty} \frac{n(\beta\lambda)^{n-c}}{\prod_{r=1}^{n-c}(c\mu + r\alpha)}\right].$$

CONCLUSION

In this paper, the truncated multi-channel queue $M/M/c/N$ is studied with both balking and reneging. The chosen balk function generalizes all previously used ones and contains a parameter m which gives flexibility to the rate of balking of units. This choice enabled many previously published works to be deduced as special cases of the present model.

Both p_0 and L were derived analytically in an exact form in terms of the hypergeometric function. Hence, the present results improve Reynolds's [1] approximate relations for the non-truncated $M/M/c$ queue with balking only and the results of Haghighi *et al.* [2] who neglected the reneging concept when calculating L for the non-truncated $M/M/c$ queue with both balking and reneging concepts.

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