Subjects and Algorithms Summaries

Limitss

Existance

Limit exists $\iff$ left-hand and right-hand limits exists and $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.

Functions

Monotonic Functions

If function f is increasing ($f'(x) > 0$) or decreasing ($f'(x) < 0$) for an interval $\implies$ f is Monotonic function.

Solving Inverse of a Functions

1)Solve $y = f(x)$ for x

$$x = f^{-1}(y)$$

2)Interchange x and y to obtain

$$y = f^{-1}(x)$$

Continuity

$f(x)$ is continous $\iff$ followings holds:

$$1)\exists f(c) \quad (c \in D(f))$$

$$2)\exists \lim_{x \to c} f(x)$$

$$3) \lim_{x \to 0} f(x) = f(c)$$

(Function is continous $\iff$ there exists $f(c)$ and limit exists at c and limit of f(x) at when x $\rightarrow$ c is equalt to $f(c)$).

One-variable Calculus

Vertical Asymptote

If

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \vee \quad \lim_{x\to a^-} f(x) = \pm\infty$$

a is vertical asymptote.

Horizontal Asymptote

If

$$\lim_{x\to\infty} f(x) = b \quad \vee \quad \lim_{x\to-\infty} f(x) = b$$

horizontal asymptote.

Oblique or Slant Asymptote

An oblique or a slant asymptote is an asymptote that is neither vertical or horizontal.

If the degree of the numerator is one more than the degree of the denominator, then the graph of the rational function will have a slant asymptote. Slant asymptote find by long division of the rational function. When denumerator's degree is equal to one, numerator is the asymptote function.

!* A graph can have both a vertical and a slant asymptote, but it CANNOT have both a horizontal and slant asymptote.

L'Hospital's Rule

!This rule is can be used in only one-variable calculus

If $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0} \vee \frac{\infty}{\infty}$, take derivates of numerator and denumerator seperately ($\frac{f'(x)}{g'(x)}$) until it is $\neq \frac{0}{0}$ or $\frac{\infty}{\infty}$

Sandwich Theorem

Suppose $g(x) \leq f(x) \leq h(x)$ and $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$. Then,

$$\lim_{x\to c} f(x) = L$$

Tangent Line to the Curve

Let $y = f(x)$ at (x_0, y_0)

1. Calculate slope m: $m = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = f'(x_0)$
2. If limit exists, find tangent line: $y = y_0 + m(x - x_0)$

Instantaneous Speed

$$\lim_{\triangle t\to 0} \frac{\triangle s}{\triangle t} = f'(x_0)$$

Graphing steps for y=f(x)

1. Identify domain of f and symmetries (even/odd)(horizontal/vertical/slant)
2. Find y', y''
3. Find critical points(bounds, $f'(x) = 0, \nexists f'(x)$) as (x,y).
4. Find intervals of f that it is increasing ($f'(x) > 0$) or decreasing ($f'(x) < 0$)
5. Find local max and minimums (critical points that change signs $f'(x)$).
6. Calculate all critical points and find absolute min and abosulute max.
7. Find points of inflection ($f''(x) = 0$) and concavity ($f'' < 0$ and $f'' > 0$)
8. Plot key points, intersections then concavities.

Shifting and Strecthing Graph

Let $y = af(b(x + c)) + d$.

a: (+) $\rightarrow$ horizontal stretch or compression, (-) $\rightarrow$ reflection about y-axis

b: (+) $\rightarrow$ vertical strecth or compression, (-) $\rightarrow$ reflection about x-axis

c: horizontal Shifting

d: vertical shift

Rolle’s Theorem

Let $f(x) = y$ is continuous at every point of [a,b] and differentiable at every point of (a,b). Then, if $f(a) = f(b)$ then $\exists c$ that $f'(c) = 0$.

Mean-Value Theorem

Mean-Value theorem use Rolle’s Theorem to get this equation:

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Integral

Integration by Parts

$$\int (f(x)g'(x))dx = f(x)g(x) - \int (f'(x)g(x))dx$$

Let $u = f(x)$ and $v = g(x)$. Then the statement will be

$$\int (u)dv = uv - \int (v)du$$

*Priority of picking u is shortened by LIATE (L-Logarithms, I-Inverse Trigonometric, A-Algebraic, T-Trigonometric, E-Exponential)

Taking Integral of Powers of Cosinus and Sinus

Let $\int(\sin^m(x).\cos^n(x))dx$.

1. If m-odd $\rightarrow$ replace $\sin^2(x)$ with $\sin^2(x) = 1 - \cos^2(x)$
2. If m-even and n-odd $\rightarrow$ replace $\cos^2(x)$ with $\cos^2(x) = 1 - sin^2(x)$
3. If m-even and n-even $\rightarrow$ replace $\sin^2(x)$ with $\sin^2(x) = \frac{1-\cos(2x)}{2}$ and $\cos^2(x) = \frac{1+\cos(2x)}{2}$

Vector Calculus

Area of Parallelogram

$$||u \times v|| = |u||v|\sin(\theta)$$

Volume of Parallelpiped (Triple Scalar Product)

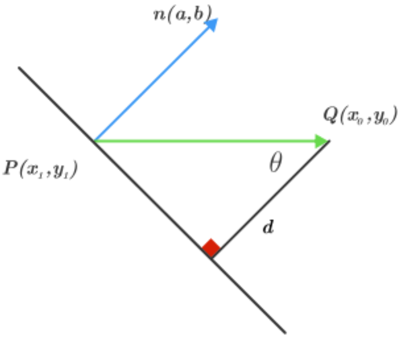
$$(u \times v).w = \begin{vmatrix} u_1 & y_3 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Vector Equation for line

r_0 is initial point, t is parameter (scalar), v is direction vector (usually unit vector)

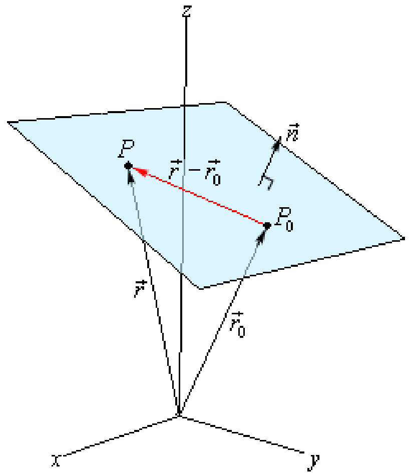
$$r(t) = r_0 + tv$$

Distance from a Point to a line



$$\frac{|\overrightarrow{PQ} \times \overrightarrow{n}|}{||\overrightarrow{n}||} = distance$$

Plane Equation



Normal Vector to a plane: $n = Ai + Bj + Ck$

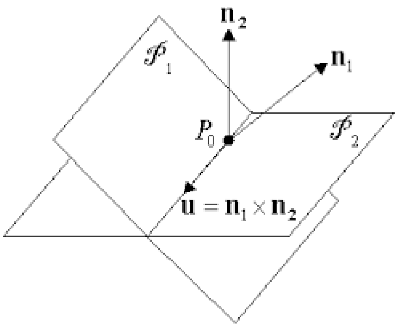
Plane = $n.\overrightarrow{P_0P}$

= $A(x - x_0) + B(y - y_0) + C(z - z_0) = D$

So, the final equation for plane is:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) - D = 0$$

Line of Intersection of Two Planes



Let planes P_1 and P_2 are $P_1 = A_1x + B_1y + C_1z = D_1$ and $P_2 = A_2x + B_2y + C_2z = D_2$

1. Find an intersection point (P_0) of two planes: $P_1 = P_2$

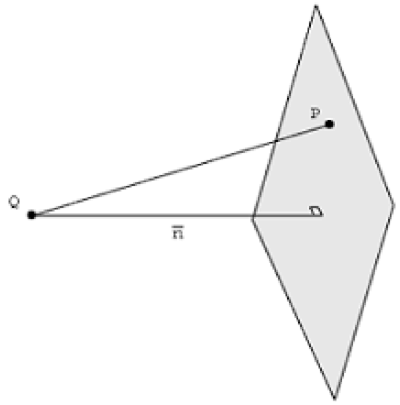
2. Find direction vector ($\overrightarrow{u}$) of intersection line

$$\overrightarrow{u} = \overrightarrow{n_1} \times \overrightarrow{n_2} = \begin{vmatrix} i & j & k \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix}$$

3. Write new plane equation as follows:

$$A_3(x - x_0) + B_3(y - y_0) + C_3(z - z_0) - D = 0$$

Distance from a Point to a Plane



$$\frac{|\overrightarrow{PQ}.\ \overrightarrow{n}\,|}{||\overrightarrow{n}\,||} = distance$$

Angle between two vectors

$$\cos(\theta) = \frac{\overrightarrow{u}\ .\ \overrightarrow{v}}{||\overrightarrow{u}\,||.\,||\overrightarrow{v}\,||} \implies \theta = \cos^{-1}\left(\frac{\overrightarrow{u}\ .\ \overrightarrow{v}}{||\overrightarrow{u}\,||.\,||\overrightarrow{v}\,||}\right)$$

Angle between two Planes

$$\theta = \cos^{-1}\left(\frac{\overrightarrow{n_1}.\overrightarrow{n_2}}{||\overrightarrow{n_1}||.\,||\overrightarrow{n_2}||}\right)$$

Velocity, Speed and Acceleration

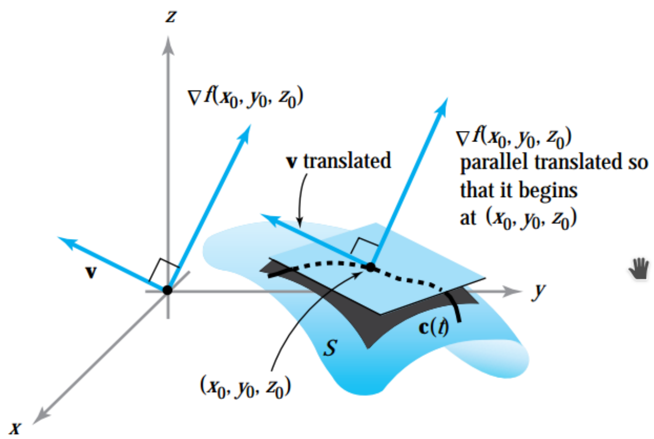
Let $c(t) = (x(t), y(t), z(t))$ is path.

$$c'(t) = v \to \text{velocity}$$

$$|c'(t)| = |v| \to \text{speed}$$

$$a = c''(t) = v' \to \text{acceleration}$$

Gradient of f

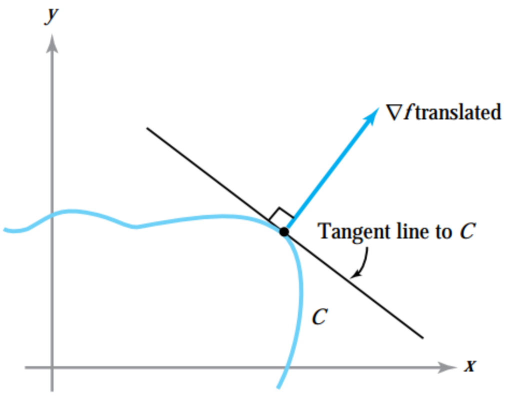


$$\triangledown f=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)$$

*$\triangledown f \neq 0 \implies \triangledown f$ gives direction of f is increasing fastest.

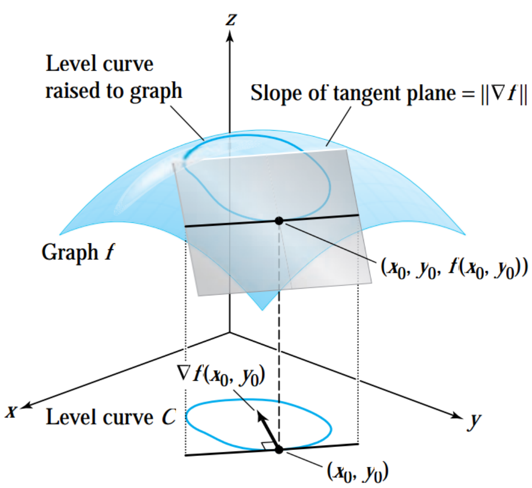
*$\triangledown f$ is normal to Level Surfaces.

Tangent Line



$$l(t)=c(t0)+(t-t0).\,c'(t0)$$

Tangent Planes to Level Surfaces



$$\text{Tangent Plane} = \triangledown f(x_0, y_0, z_0).\,(x - x_0, y - y_0, z - z_0) = 0$$

$$\text{Slope of tangent Plane} = || \triangledown f||$$

Taylor Theorem

The main point of the single-variable Taylor theorem is to find approximations of a function near a given point that are accurate to a higher order than the linear approximation.

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n$$

Forms of the Remainder

$$R_n(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}$$

$\exists c$ between a and x such that

Arc Length

$$L = \int_a^b |\vec{v}|dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Arc Length

$$s(t) = \int_a^b |v(u)|du$$

Arc Length Parameter

$$s'(t) = |v(t)|$$

Speed on Smooth Curve

Tangent Vector

Tangent Vector $\rightarrow \vec{v} = r'(t)$

$$\vec{T}(t) = \frac{v}{|v|} = \frac{r'(t)}{|r'(t)|}$$

Unit Tangent Vector

$$K = \left|\frac{dT}{ds}\right| = \frac{1}{|v|}.\left|\frac{dT}{dt}\right| = \frac{|v \times a|}{|v|^3}$$

Curvature

$$P = \frac{1}{K}$$

Radius of Curvature

$$N(t) = \frac{1}{K} \cdot \frac{dT}{ds} = \frac{T'(t)}{|T'(t)|}$$

$$B = T \times N$$

Unit Binormal Vector

Linear Algebra

System and Matrix Relation

$$\text{System} \rightarrow \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_n \end{cases}$$

$$\text{Matrix form } AX = b \rightarrow \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$\text{Augmented Matrix form } [A|b] \rightarrow \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & | & b_1 \\ \vdots & & & & | & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & | & b_n \end{bmatrix}$$

Rank

Rank of matrix is equal to nonzero rows of echelon form of a matrix.

$*$ So this means that linear independent ‘row’ vector count of a matrix.

$Rank(A) \neq Rank[A|b] \implies$ no solution

$Rank(A) = Rank[A|b] = n \implies$ unique solution

$Rank(A) = Rank[A|b] < n \implies$ infinitely many solution

Determinant

$$det(A) = \sum_{k=1}^{n} a_{ik}C_{ik}$$

where $C_{ij} = (-1)^{i+j}M_{ij}$ and M_{ij} is the minor of A

$*det(A) = det(A^t)$

$det(A) \neq 0 \implies$ A is invertible and $Rank(A) = n$

Algebraic Structures

Group

- is a set G
- one binary operation defined with properties:
  - Closure: a $\diamond$ b $\in$ G
  - Identity Element: $a \diamond e = e \diamond a = a$
  - Associavity: $a \diamond (b \diamond c) = (a \diamond b) \diamond c$
- is Abelain Group $\implies a \diamond b = b \diamond a$

Field

- is a set F
- two operation (+), $(\times)$ defined:
  - F-$\{0\}$ is also Abelian Group under $(\times)$ operation

Vector Spaces

- non-empty set
- two operations defined;
  - addition (+)
  - scalar multiplication $(\times)$
- following properties holds;
  - Closed under $+$ and $\times$
  - Commutative under $(+)$
  - Associative under $+$ and $\times$ $(u + v = v + u)$
  - There exists Additive Identity: $\exists \overrightarrow{0} \in V(u + (-u) = (-u) + u = \overrightarrow{0})$

    and Multiplicative Identity ($\overrightarrow{u} \times e = e \times \overrightarrow{u} = \overrightarrow{u}$)$
  - Distrubutive $(k \times (\overrightarrow{u} + \overrightarrow{v}) = k\overrightarrow{u} + k\overrightarrow{l})$ and $((k + l)\overrightarrow{u} = k\overrightarrow{u} + l\overrightarrow{u})$

$*$ To determine whether a set V with addition and scalar multiplication defined on V is a vector space requires verification of the 10 vector space axioms.

$*$ In all vector spaces, additive inverses are unique.

Subspace

A subspace W of a vector space V is a nonempty subset that is itself a vector space with respect to the inherited operations of vector addition and scalar multiplication on V .

$*$ The intersection of any collection of subspaces of a vector space is a subspace of the vector space.

Proving Subspace

$*$ If W is nonempty subset of the vector space V, then W is a subspace of V $\iff$ W is **closed under addition and scalar multiplication** (and $\overrightarrow{0}$ is in W).

To prove that, we need to show $\overrightarrow{u} + (c \cdot \overrightarrow{v}) \in W$ with 3 steps.

Step 1 - Closed Under Addition: Suppose that $\overrightarrow{u}, \overrightarrow{v} \in W$; then $u + (1 \cdot v) = u + v \in W$

Step 2 - W is Nonempty (has $\overrightarrow{0}$ vector): $\overrightarrow{0} = u + ((-1) \cdot u)$

Step 3 - Closed Under Scalar Multiplication: $c \times u = 0 + (c \cdot u) \in W$

Step 4 - Converse: if W is a subspace with u and v in W, and c a scalar, then since W is closed under addition and scalar multiplication, we know that $u + (c \cdot v) \in W$ $\square$.

Polynomial Vector Spaces

P_n the set of all polynomials of degree n or less.

$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n$

$*$Note that $V = P_n \cup \{0\}$ is a real vector space, where 0 is the zero polynomial.

$*$Note that n degree Polynomial Vector Space's dimension is $dim(P_n) = n + 1$

Vector Space Properties

- $\overrightarrow{0}$ vector is unique
- $0 \cdot \overrightarrow{v} = \overrightarrow{0}$ and $k \cdot \overrightarrow{0} = \overrightarrow{0}$
- Additive inverse of a vector $\overrightarrow{v}$ is unique
- If $k \cdot \overrightarrow{v} = \overrightarrow{0} \implies k = 0 \vee \overrightarrow{v} = \overrightarrow{0}$

Linear Combination

Linear combination of vectors defined as the form:

$$(c_1 \cdot \overrightarrow{v_1}) + (c_2 \cdot \overrightarrow{v_2}) + \cdots + (c_k \cdot \overrightarrow{v_k})$$

$*$ Let u and v are vectors in same vector space. For any scalar c, $u = cv \implies$ then u is linear combination of v.

Linear Dependence

$S = \{v_1, v_2, \ldots, v_m\}$ in a vector space V is **linearly independent**
$\implies c_1v_1 + \cdots + c_mv_m = 0$ is **only have trivial solution** $c_1 = c_2 = \cdots = c_m = 0$.

If the equation has a nontrivial solution the set S is **linearly dependent**.

$*$ Linear dependance is a property of vector sets.

$*A_{n \times n}$ and column vectors of A **linearly independent** $\iff det(A) \neq 0$

$*A_{n \times n}$ and column vectors of A **linearly independent** $\iff$ A is invertible.

$*$ If $\overrightarrow{0}$ vector contained by S, then S is **linearly dependent**.

$*$ Let $S = \{v_1, v_2, \ldots, v_n\}$ is set of nonzero vectors that in R^m. $n > m \implies$ S is **linearly dependent**.

$*$ Set of nonzero vectors is **linearly dependent** $\iff$ at least one of the vectors is linear combination of other vectors in the set.

$*$ S is **linearly independent** set of vectors $\implies$ subset of S is linearly independent.

$*$ T is **linearly dependent** set of vectors and S contains T $\implies$ S is linearly dependent.

$*$ Let $Ax = b$ consistent. The solution is unique $\iff$ column vectors of A is **linearly independent**.

$*$ Let $Ax = 0$. The only solution is trivial $\iff$ column vectors of A is **linearly independent**.

Determining set of vectors linearly independent

Let column vectors

$$v_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} v_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} \ldots v_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

1. Write column vectors as a linear combination $\rightarrow c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$
2. obtain a system based on scalars $\rightarrow \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

3. Solve the system.
4. If only solution is the trivial solution $c_1 = c_2 = \cdots = c_n = 0 \implies$ vectors are linearly independent.

Span of Set of Vectors

Let $S = \{v_1, v_2, \ldots, v_k\}$ be a vector set.

$$span(S) = \{(c_1 \cdot \overrightarrow{v_1}) + (c_2 \cdot \overrightarrow{v_2}) + \cdots + (c_k \cdot \overrightarrow{v_k})\}$$

Proposition - $span(S)$ is a Subspace

$*$ If S is a set of vectors in vector space V $\implies$ $span(S)$ is a **subspace**.

- **Proof**

  $\overrightarrow{u} + c\overrightarrow{w} = (c_1v_1 + \cdots + c_nv_n) + c(d_1v_1 + \cdots + d_nv_n) = (c_1 + cd_1)v_1 + \cdots + (c_n + cd_n)v_n$

  Since $c_n + cd_n$ is a scalar, $u + cw \in span(S)$, and hence span is a subspace $\square$.

Null Space

Null space of A is the set $N(A) = \{\overrightarrow{x} : A\overrightarrow{x} = \overrightarrow{0}, x \in R^n\}$.

Column Space

Column space of A is denoted by $col(A)$ is the set of all linear combinations of column vectors of A.

Theorem - $Ax = b$ consistence

Let A be an m × n matrix. The linear system $Ax = b$ is consistent $\iff$ $b \in col(A)$.

Theorem - $N(A)$ and $col(A)$ are Subspace

If A is $m \times n$ matrix, $col(A)$ is subspace of R^m and $N(A)$ is subspace of R^n.

Basis and Dimension

Span

Let $S = \{v_1, v_2, \ldots, v_m\}$ and $\forall v \in V$ can be written as $v = c_1v_1 + c_2v_2 + \cdots + c_mv_m \implies span(S) = V$ (S span V).

Let $dim(V) = n$.

$*$ If $k < n \implies span(S) \neq V$.

How to show S spans R^n

1. Take $v = (a_1, a_2, \ldots, a_n) \in R^n$

2. Let $v = (a_1, a_2, \dots, a_n) = c_1 v_1 + \dots + c_n v_n$ s.t. c_i are constants.

3. Show that $\begin{bmatrix} v_1 & v_2 & \dots & v_k & | & a_1 \\ \downarrow & \downarrow & \dots & \downarrow & | & \vdots \\ & & & & | & a_n \end{bmatrix}$ is consistent (by echelon form).

Basis for a Vector Space - Definition

If following properties hold:

1. B is linearly independent set ($c_1 v_1 + \dots + c_m v_m = 0$ is **only have trivial solution**).

2. B spans V (Every item in V should be written as linear combination of vectors in B).

B is a **basis** for vector space V.

$*\forall v \in V$ has a unique representation.

Corollary Let $S = \{v_1, v_2, \dots, v_k\}$ st. $v_1, v_2, \dots, v_k \in V$ and $A = [v_1|v_2|\dots|v_k]$. For Rank(V) = n.

- if $k > n$, then S is **linearly dependent**
- if $k = n$ and $det(A) = 0$ (also means that $Rank(A) \neq n$) then S is **linearly dependent**
- if $k = nand\$det(A) \neq 0$ (also means that $Rank(A) = n$) then S is **linearly independent**

Wronskian of Function set

Function set $S = f_1, f_2, \dots, f_k$ where $f_1, f_2, \dots, f_k \in C^{k-1}(I)$ (means that those functions differentiable k-1 times)

$$W[f_1, f_2, \dots, f_k] = \begin{vmatrix} f_1(x) & f_2(x) & \dots & f_k(x) \\ \vdots & & & \\ f_1^{(k-1)}(x) & f_2^{(k-1)} & \dots & f_k^{(k-1)}(x) \end{vmatrix}$$

$*$ Wronskian is scalar since it is a determinant.

$*$ If $W[f_1, f_2, \dots, f_k] \neq 0$ for any $x_0 \in I$, then S is linearly independent.

Dimension

The number of vectors in any basis for V and denoted by $dim(V)$.

$*$ Every basis of V has n vectors.

$*$ If $V = \{\overrightarrow{0}\} \implies dim(V) = 0$.

$* \, dim(M_{m\times n}(IR)) = m \cdot n$, $dim(P_n(R)) = n + 1$

$*$ Let $dim(V) = n$ and $B = \{v_1, v_2, \ldots, v_k\}$ s.t. $v_1, v_2, \ldots, v_k \in V$

1. If B linearly independent $\implies$ B is basis for V (B spans V also).
2. If B spans V $\implies$ B is basis for V (B is linearly independent also).

Ordered Basis

Ordered basis are basis that ordered set of vectors.

Transition Matrix of Ordered Basis from B to B'

Let $B = \{v_1, v_2, \ldots, v_n\}$ and $B' = \{v'_1, v'_2, \ldots, v'_n\}$

Transition matrix $[I]_B^{B'} = [\,[v_1]_{B'} \quad [v_2]_{B'} \quad \ldots \quad [v_n]_{B'}\,]$

Finding Transition Matrix

1. Let $c_1, c_2, \ldots, c_n$ are coordinate vectors. Than solve equation for $c_1 v'_1 = v_1, c_2 v'_2 = v_2, \ldots, c_n v'_n = v_n$
2. Then put them in form of $[I]_B^{B'} = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \\ \downarrow & \downarrow & \ldots & \downarrow \end{bmatrix}$

Eigenvalues and Eigenvectors

A is matrix, $\overrightarrow{v}$ is eigenvector and λ is eigenvalue.

$$A\overrightarrow{v} = \lambda\overrightarrow{v}$$

λ is eigenvalue of matrix $A \iff det(A - \lambda I) = 0$

$*A\overrightarrow{v} = \lambda\overrightarrow{v}$ says that eigenvectors keep same direction when multiplied by A.

$*$ The eigenvalues of $A^2 \to \lambda^2$ and $A^{-1} \to \lambda^{-1}$ with the same eigenvectors.

$*$ The sum of λ's = sum of main diagonal of A.

$*$ The product of λ's is equal to determinant of A.

Finding Eigenvalues and Eigenvectors

1. Compute $det(A - \lambda I) = 0$
2. Find the roots of this polynomial (λ's)
3. Solve $(a - \lambda I)\overrightarrow{v} = 0$ to find eigenvectors $\overrightarrow{v}$