Linear Algebra Notes

Linear Transformations

Linearity Properties for Linear Transformations

1. $T(u+v) = Tu + Tv$
2. $T(cu) = cTu$

Briefly show: $T(cu+v) = cTu + Tv$

Kernel and Range

$$Ker(T) = Nullsp(T) = v \in V : Tv = 0$$

$$Range(T) = R(T) = \{w \in W : w = Tv, v \in V\}$$

Note: Every vector in $Ker(T)$ is mapped to the zero vector in W.

$$dim(Ker(T)) + dim(Range(T)) = dimV$$

$$Nullity(T) + Rank(T) = dimV$$

So, $dim(Ker(T)) = Nullity(T), dim(Range(T)) = Rank(T)$.

Existance and Uniqueness of Linear Transformation

If V and W are finite dimensional vector spaces over same field, there is **unique** linear tranformation that

Let B is a bases with vectors $v_i, i = \overline{1,n}$

$$Tv_i = w_i$$

The set of all linear transformations from V to W is a vector space, denoted by $L(V,W)$

Product (Composition) of Linear Transformations

$T_1 \in L(U,V), T_2 \in L(V.W)$, then

$$(T_2T_1)u = T_2(T_1u)$$

Properties:

1- Associativity:

$$(T_3T_2)T_1 = T_3(T_2T_1)$$

2- Distributivity:

$$(S_1+S_2)T = S_1T + S_2T$$

$$S(T_1+T_2) = ST_1 + ST_2$$

3- Identity Transformation:

$$I_v : V \to V$$

$$I_w : W \to W$$

Then,

$$TI_v = T, TI_w = T$$

One-to-one and onto

T is **one-to-one** $\iff$ T maps **every linearly independent set of vectors** in V to W.

T is **one-to-one** $\iff$ $Ker(T) = 0$

T is **onto** $\iff$ $Range(T) = W$.

1. If T is **one-to-one** $\Rightarrow dimV \leq dimW$
2. If T is **onto** $\Rightarrow dimV \geq dimW$
3. If T is **one-to-one and onto (bijective)** $\Rightarrow dimV = dimW$

Invertibility

T is invertible if $\exists S : W \to V$ s.t.

$$ST = I_v \quad \text{and} \quad TS = I_w$$

S is called inverse of T.

T is **invertibe** $\iff$ T is **one-to-one and onto**.

If $dimV = dimW = n$, then followings are equal:

1. T is **invertible**
2. T is **one-to-one**
3. T is **onto**

Coordinate Representation of Vectors

Let $\beta = \{v_1, \ldots, v_n\}$ is a ordered bases of V. Then $\forall v \in V$ has a unique representation:

$$v = c_1 v_1 + \cdots + c_n v_n$$

coordinate vector of v related to the base β is

$$[v]_\beta = [c_1 \ldots c_n]^t = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Transformation $L_0 : V \to M_{nx1}$ is: $L_0(v) = [v]_\beta$

1. is one-to-one and onto (invertible)
2. is linear

Thus, L_0 is an isomorphism.

Transition matrix P

Let $dimV = n$, $\beta = \{v_1, \dots, v_n\}$ and $\Upsilon = \{u_1 \dots, u_n\}$ be two **ordered** bases.

Then, there exists a unique invertible (nxn) matrix P s.t.

$[v]_\beta P[v]_\Upsilon, \forall v \in V$

and since P is invertible.

$[v]_\Upsilon = P^{-1}[v]_\beta$

where

$$P = \begin{bmatrix} [u_1]_\beta & \dots & [u_n]_\beta \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = [p_{ij}]$$

and P is called **transition matrix**.

Matrix Representation of Operator T

Every linear transformation defined on a finite dimensional space can be represented by a matrix.

Matrix A is called the matrix representation of operator T relative to the bases β, Υ for V and β', Υ' for W. Then, for each linear transformation $T : V \to W$ there is a **unique** matrix A_{mxn}.

$A = [T]_{\beta}^{\beta^\prime} = \begin{bmatrix} [Tu_1]_{\beta^\prime} & \dots & [Tu_n]_{\beta^\prime} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$

and satisfying

$$A[v]_\beta = [Tv]_{\beta'} \quad ,\forall v \in V$$

This means that, for given β and β', the correspondance $T \xmapsto{L_0} [T]_\beta^{\beta'}$ defines a mapping $L_0 : L(V, W) \to M_{mxn}$

1. This mapping is **linear**
2. This mapping is **one-to-one and onto (invertible)**

Notations for 2 case:

1. If $\beta \neq \beta' \implies A = [T]_\beta^{\beta'}$ then,

$$[T]_\beta^{\beta'}[v]_\beta = [Tv]_{\beta'} \quad ,\forall v \in V$$

2. If $\beta = \beta' \implies A = [T]_\beta$, then

$$[T]_\beta[v]_\beta = [Tv]_\beta$$

With two different bases, let P be be the transition matrix from $\beta \to \Upsilon$ and Q from $\beta' \to \Upsilon'$.

$$[T]_\Upsilon^{\Upsilon'} = Q^{-1}[T]_\beta^{\beta'}P$$

If $T : V \to V$ is a linear operator and β, Υ two bases for V, and P is transition matrix s.t.

$$[v]_\beta = P[v]_\Upsilon$$

then relation between representations $[T]_\beta$ and $[T]_\Upsilon$ is:

$$[T]_\Upsilon = P^{-1}[T]_\beta P$$

Two vector spaces V and W over the same field that is **isomorphic** notated as $V \cong W$.

Isomorphism Properties

Isomorphism is an equivalence relation on the set of all n-dimensional vector spaces over the same field. That's, if U, V, W are any n-dimensional vector spaces over $\mathbb{F}$, then

1. $V \cong V$ (reflexive)
2. $V \cong U \implies U \cong V$ (symmetric)
3. $V \cong U and U \cong W \implies V \cong W$ (transitive)

Then,

$$V_\mathbb{F} \cong M_{nx1}(\mathbb{F}) \cong \mathbb{F}^n$$

$$v \longleftrightarrow \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \longleftrightarrow (x_1, \ldots, x_n)$$

Let $dimV = n$ and $dimW = m$

Then,

$$L(V, W) \cong M_{mxn}(\mathbb{F})$$

and

$$dimL(V, W) = dimM_{mxn} = m \cdot n$$

Sum and Direct Sum

Let $U_1, \ldots, U_m$ be subspaces of V. Then the **sum of $U_1, \ldots, U_m$** is defined as

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m : u_j \in U_j, j = \overline{1, m}\}$$

if each element $v \in V$ **has a unique** representation as a sum $\implies$ V is the **direct sum** its subspaces $U_!, \ldots, U_m$ written as

$$V = U_1 \oplus \cdots \oplus U_m$$

$$v = u_1 + \cdots + u)m$$

where $u_j \in U_j, j = \overline{1, m}$

If V is a finite dimensional vector space and U is subspace of V, $\exists W$ subspace of V s.t.

$$V = U \oplus W$$

$$V = U_1 \oplus \cdots \oplus U_m$$

$$\iff$$

1. $v = U_1 + \cdots + U_m$
2. $u_1 + \cdots + u_m = 0 \implies u_j = 0 \quad , \forall j = \overline{1, m}$

Let U, W be subspaces of V, then

$$V = U \oplus W$$

$$\iff$$

$$V = U + W \quad \textbf{and} \quad U \cap W = \{0\}$$

A mapping $T : V \to V$ that is linear, called **linear operator**. In that case $T \in L(V)$.

For U_j that is subspace of V, restriction of T to smaller domain U_j notated as $T|_{U_j}$

The subspace U is called to be **invariant under operator T** if $u \in U \implies Tu \in U$.

(In other words, U is **invariant under T** if $T|_U$ is an operator on U.

Linear Operator Properties

1. $\{0\}$ is **invariant** under T.
2. V is **invariant** under T.
3. $Ker(T)$ is **invariant** under T.
4. $Range(T)$ is **invariant** under T.

Projection Operators in Vector Space

Let $V = U \oplus W$, then $P : V \to V$ is called **projection operator** that maps V onto W along U, if $\forall v \in V$ having a **unique** representation

$$v = u + w$$

we define

$$Pv = w$$

Moreover, ($I - P$) is a projection of V onto U, where I identity operator on V.

Also, $U = Ker(P), \quad W = Range(P)$.

Properties

1. P is a **linear operator**.
2. $P^2 = P$ (P is ideupatent operator).
3. $Pw = w \iff w \in Range(P)$.

Eigenvalue and Eigenvector of Linear Transformations

Let $T \in L(V)$ is a **linear operator**.

1. An **eigenvalue** of T is a scalar λ s.t. $\exists v \in V \quad v \neq 0$ satisfying $Tv = \lambda v$. v is **eigenvector**.
2. The subspace W_λ that $W_\lambda = \{\forall v \in V : Tv = \lambda v, \quad \lambda \in \mathbb{F}\}$ is called **eigenspace**.

The set of all **eigenvalues** of T is called **spectrum** of T, denoted by $spec(T)$.

$$spec(T) = \{\lambda \in \mathbb{F} : Tv = \lambda v \quad , \exists v \in V\}$$

- **existance of eigenvalues depends on the field $\mathbb{F}$ whether is real or complex.**

Every linear operator $T : V \to V$ where V is a finite dimensional vector space over the field of **complex numbers has at least one eigenvalue**.

T is a linear operator, if $\lambda_1, \ldots, \lambda_m$ are **distinct eigenvalues** of T, and $u_1, \ldots, u_m$ are corresponding **eigenvectors**, then set of these eigenvectors $\{u_1, \ldots, u_m\}$ is a **linearly independent set**.

If $dimV = n \implies T$ **has at most** n distinct eigenvalues.

Fan of Subspaces of V and Triangular Matrix Representations

The set $\{W_1, \dots, W_n\}$ of subspaces of V is called a **fan of T** in V, if the followings hold:

1. $W_1 \subset \dots \subset W_n = V$
2. $dim(W_k) = k$ for $\forall k = \overline{1, n}$
3. W_k is **invariant** under T, that's $T(W_k) \subseteq W_k \quad, \forall k = \overline{1, n}$

If $\{W_1, \dots, W_n\}$ is a fan of T, then $\beta = \{v_1, \dots, v_n\}$ is called a fan basis for V, if

$\{v_1\}$ is a basis for W_1

$\vdots$

$\{v_1, \dots, v_n\}$ is a basis for $W_n = V$

If $T \in L(V)$ and β is a fan basis for V, then the matrix representation of T w.r.t. β is **upper trianguler**.

Let $T \in L(V)$ and $\beta = \{v_1, \dots, v_n\}$ is a basis of V. Then the followings are equivalent.

1. The matrix of T w.r.t. basis β is **upper triangular**.
2. $Tv_k \in W_k = span\{v_1, \dots, v_k\}$ for each $k = \overline{1, n}$
3. $W_k = span\{v_1, \dots, v_k\}$ is **inavriant** under T for each $k = \overline{1, n}$

If $T \in L(V)$ and V is a finite dimensional vector space over the field of **complex numbers**, then ∃ a fan of T in V.

Every lin. op. T defined on a **complex** vector space V can be represented by a **triangular matrix$$.

Diagonalizable Matrices

A square matrix A_{nxn} is **diagonalizable** if $\exists P$ that is invertible s.t.

$$P^{-1}AP = D$$

A lin. op. T defined on a finite dim. v.sp. V is **diagonalizable** if basis $\exists \beta$ for V s.t. the **matrix representation of T w.r.t. this basis is diagonal**, that

$$[T]_\beta = D$$

not every operator diagonalizable, even in complex vector space.

$T \in L(V)$ on a finite dim. v.sp. V is diagonalizable $\iff$ there exists a basis β for V consisting entirely of **eigenvectors** of T.

$T \in L(V)$ on v.sp. V with $dimV = n$. If T has n distinct **eigenvalues**, then T is diagonalizable.

$T \in L(V)$ and $dimV = n$. Let $\lambda_1, \dots, \lambda_m$ be the **distinct eigenvalues of T** and $W_i = Ket(T - \lambda T) \quad , i = \overline{1, m}$ denote the correspondant eigenspaces. Then, the followings are equivalent

1. T is **diagonalizable** operator.
2. V has a **basis consisting of eigenvectors** of op. T.
3. There exists one-dimensional subspaces $U_1, \dots, U_n$ of V, each of which is **invariant** under T, and $V = U_1 \oplus \dots \oplus U_n$.
4. $V = W_1 \oplus \dots W_m \quad , W_i = Ker(T - \lambda_i I)$.
5. $dimV = dimW_1 + \dots + dimW_m$

Remember for Matrices

- If A_{mxn} and $m \neq n$, then we **can not define powers** of A.
- If A_{nxn} is a square matrix, then we **can define the powers of A**
  $A, A^2, \dots, A^k, \quad k = 1, 2, \dots$

Similarly

- If $T : V \to W$ is a linear mapping and $V \neq W$, then we **can't define the powers of T**.But,
- if $T \in L(V)$ then powers of T are defined as:

$$T^2 = TT$$

$$T^3 = TTT$$

$$\vdots$$

$$T^m = \underbrace{T \dots T}_{\text{m times}}$$

and $T^0 = T$

Invariant Subspaces on Real Vector Spaces

- On a **Complex Vector Space V every linear operator T** has an **eigenvalue** and therefore has an **eigenvector**. It means, in complex spaces a linear opreator always has an **invarian** subspace of dimension 1.

- On a **Real Vector Space V** a linear operator **may not have** an **eigenvalue**, and therefore **may not have an invariant subspace** of dimension 1. However, an invariont subspace of dimension 1 or 2 **always exist**.

Remember, T has a 1-dimensional invariant subspace $\iff$ T has an eigenvalue(eigenvector).

Every lin. op. T on a finite-dimensional, nonzero, **real vector space V** has an **invariant subspace of dimension 1 or 2**.

Every operator T on a **odd-dimensional real v.sp. V** has an eigenvalue.

Inner Product Spaces

Introduction

1. Vector Space operations $\rightarrow \begin{cases} u+v & (\text{addition}) \\ \alpha u & (\text{scalar multiplication}) \end{cases}$

2. Inner Product Space $\rightarrow$ Vector Space $+$ inner product $< u, v >$

- In every inner product space one has Norm: $||u|| = \sqrt{< u, v >}$, Distance: $||u-v|| = \sqrt{< u-v, u-v >}$

- Most important notion in inner product spaces: **Orthogonality**: $< u, v >= 0$

- An **inner product** on a vector space is a **function** that takes **each ordered pair** $(u, v) \in X \times X$ to a number $< u, v > \in \mathbb{F}$ and satisfies the following properties $\forall u, v, w \in X, \quad \alpha \in \mathbb{F}$:

1. $< u, v > \geq 0$ for $\forall v \in X$ **(positivity)**

2. $< v, v >= 0 \iff v = 0$ **(definiteness)**

3. $< u+v,w >=< u,w > + < v,w >$ **(additivity in first component)**

4. $< \alpha u,v >= \alpha < u,v >$ **(homogenity in first component)**

5. $< u,v >= \overline{< v,u >}$ **(conjugate symmetry)**

- **Inner product space X is a vector space** on which an **inner product (function) defined**: $\{X,<.,.>\}$
- If $\mathbb{F} = \mathbb{R}$ then complex conjugation is not needed, else $(\mathbb{F} = \mathbb{C})$ complex conjugation needed.

If X is an inner product space with an inner product $< u,v >$ defined on it, then $\forall u,v,w \in X$ and $\forall \alpha \in \mathbb{F}$

1. $< 0,u >= 0$
2. $< u,0 >= 0$
3. $< u,v+w >=< u,v > + < u,w >$
4. $< u,\alpha v >= \overline{\alpha} < u,v >$
5. $ <\alpha u, \beta_1 v_1 + \beta_2 v_2> = \alpha \overline{\beta_1} <u,v_1> + \alpha \overline{\beta_2} <u,v_2>$
6. $< u, \sum_{i=1}^{n} >= \sum_{i=1}^{n} \overline{c_i} < u,v_i >$

Norm of a Vector

Let X be an inner product space. Then, the orml of $v \in X$ is $||v|| = \sqrt{< v,v >}$.

Then,

1. $||v|| \geq 0$ and $||v|| = 0 \iff v = 0$
2. $||\alpha v|| = ||\alpha|| ||v||$
3. $||u+v|| \leq ||u|| + ||v||$ (Triangle Inequality)

Hence every inner product space is also a normed space.

Orthogonality

Let X be an inner product space. Two vectors $u,v \in X$ are orthogonal if $< u,v >= 0$.

Pythogorean Theorem

If u and v orthogonal vectors in X then $||u+v||^2 = ||u||^2 + ||v||^2$

Cauchy-Schwartz Inequality

If $u,v \in X$ then $| < u,v > | \leq ||u||.||v||$

Parallelogram Equality

If $u, v \in X$ then $||u+v||^2 + ||u-v||^2 = 2(||u||^2 + ||v||^2)$

Distance

Distance between two vector is: $||u-v|| = \sqrt{< u-v, u-v >}$

Orthonormal Set Of Vectors

A set of vectors in an inner product space X $S = \{u_1, \ldots, u_m\}$ is said to be orthonormal if

$$< u_i, u_j >= \delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$$

If the set S is **orthonormal**, then it is **linearly independent**.

If the set S is **orthonormal**,

$$||c_1 u_1 + \cdots + c_m u_m||^2 = |c_1|^2 + \cdots + |c_m|^2 = ||\sum_{i=1}^{m} c_i u_i||^2 = \sum_{i=1}^{m} |c_i|^2$$

Orthonormal Basis

Let X be an inner pr. sp. over $\mathbb{F}$. A set of vectors in X, say $\beta_0 = \{u_1, \ldots, u_n\}$ is an **orthonormal basis** for X if

$$dim X = n \quad \textbf{and} \quad < u_i, u_j >= \delta_{ij}, \quad \forall i, j = \overline{1, n}$$

If X is an inner product space and β_0 is an orthonormal basis for X then every $v \in X$ has the unique representation

1. $v = \sum_{i=1}^{n} < v, u_i > u_i$,
2. $||v||^2 = \sum_{i=1}^{n} | < v, u_i > |^2$

TODO: Gram-Schmidt Procedure (Page 101)

Every finite-dimensional inner product space has an orthonormal basis.

Every orthonormal set of vectors in X can be extended to an orthonormal basis for X.

Orthogonal Projection Operator

If U is a subset of an inner product space X then the **orthogonal compliment** of U, denoted by $U^{\perp}$, is the set of all vectors in X that are orthogonal to **every** vector in U:

$$U^{\perp} = \{w \in X :< w, u >= 0 \quad \forall u \in U\}$$

$U^{\perp}$ is always a **subspace of X**. Take $w_1, w_2 \in U^{\perp}$, then by defn.

$$< w_1, u >= 0, < w_2, u >= 0 \quad \forall u \in U$$. It follows that $$< \alpha w_1 + w_2, u >= \alpha < w_1, u > + < w_2, u >= 0 \quad \forall u \in U$$

$$\implies \alpha w_1 + w_2 \in U^{\perp}$$

$\implies$ **$U^{\perp}$ is a subspace**.

- **Note that**, $X^{\perp} = \{0\}, \quad \{0\}^{\perp} = X$

If U is a subspace of an inner product space X then

$$X = U \oplus U^{\perp}$$

Let $X = U \oplus U^{\perp}$, which means any $v \in X$ has a unique representation.

1. $v = u + w, \quad u \in U, \quad w \in U^{\perp}$

Then the operator $P : X \to X$ defined as

2. $Pv = u, \quad \forall v \in X$

is called **orthogonal projection** of X onto U.

clearly, $\begin{cases} Ker(P) = U^{\perp} \\ Range(P) = U \end{cases}$

- Also, $(I - P)v = w \in U^{\perp}$ is an orthogonal projection of X onto $U^{\perp}$
- Moreover, one can show that

1. $P^2 = P$
2. $||Pv|| \leq ||v||, \quad \forall v$

TODO: The Best Approximation (Minimizin The Error) (Page 106)

Linear Functions and Adjoint Operators

Linear Functionals on Vector Spaces

A **linear mapping** f from v.sp. V to the field $\mathbb{F}$ is called a **linear functional** and we write $f : V \to \mathbb{F}$ or $f \in L(V, \mathbb{F})$.

So, a linear functional $f : V \to \mathbb{F}$ is a special linear transformation, where the **output space** is $\mathbb{F}$($\mathbb{R}$ or $\mathbb{C}$)

Notes

Since linear functionals special linear maps then we can use our previous knowledge to say that:

1. If $f : V \to \mathbb{F}$ is a nonzero linear functional, then $Range(f) = \mathbb{F}$ (one-dimensional)
2. **By dimension theorem**, $dim(Ker(f)) + dim(Range(f)) = dimV$. If $dimV = n \implies dim(Ker(f)) = n - 1$, since $dimRange(f) = 1$
3. **By Existance and Uniqueness Theorem** for linear transformations, a linear function $f : V \to \mathbb{F}$ is uniquely determined by its action on basis vectors $\beta = \{v_1, \ldots, v_n\}$ of V.

- The collection of all linear functionals on a v. sp. V over $\mathbb{F}$ forms a vector space denoted by $V^* = L(V, \mathbb{F})$.

Linear Functionals on Inner Product Spaces

Linear functionals on inner product spaces have some special properties, which we will describe now.

First, let X be an inner product space over $\mathbb{F}$. Then, for given any $w \in X$, the inner product function

$$f(v) =< v, w >, \quad \forall u \in V$$

defines a linear functional.

This is because inner product is linear in the first variable v i.e.

$$f(\alpha v_1 + v_2) =< \alpha v_1 + v_2, w >= \alpha < v_1, w > + < v_2, w > \$\$ \$\$\alpha f(v_1) + f(v_2)$$

Conversely, next theorem shows that every linear functional $f : X \to \mathbb{F}$ can be represented **as an inner product**.

Riesz Representation Theorem (RRT) (for finite-dimensional inner product spaces)

Let X be a finite-dimensional inner product space over $\mathbb{F}$.

If $f : X \to \mathbb{F}$ is a linear functional on X, then there exists a unique vector $w \in X$ s.t.

$$f(v) =< v, w >, \quad \forall v \in X$$

The Adjoint Operator

Let V and W be finite dimensional inner product spaces.

If $T : V \to W$ is a linear transformation, there exists a unique linear transformation $T^* : W \to V$ s.t.

$$< Tv, w >_W = < v, T^*w >_W. \quad \forall v \in V, \quad \forall w \in W$$

If $T : V \to W$ is a linear transformation, then $T^* : W \to V$ is called the adjoint of T if

$$< Tv, w > = < v, T^*w >, \quad \forall v \in V, \quad \forall w \in W$$

Properties of Adjoint

If $T, S \in L(V, W), \quad \alpha \in \mathbb{F}$ then

1. $(T + S)^* = T^* + S^*$ **(additivity)**
2. $(\alpha T)^* = \overline{\alpha} T^*$ **(conjugate homogenity)**
3. $(T^*)^* = T$
4. $(ST)^* = T^*S^*$ for $T \in L(V, W)$ and $S \in L(W, U)$
5. $(T^*)^{-1} = (T^{-1})^*$

Let $T \in L(V, W)$. Then,

1. $Ker(T^*) = (Range(T))^\perp$
2. $Range(T^*) = (Ker(T))^\perp$
3. $Ker(T) = (Range(T^*))^\perp$
4. $Range(T) = (Ker(T)^*)^\perp$

according to that, these are always true

$$V = Ker(T) \oplus (Ker(T))^\perp = R(T^*) \oplus (R(T^*))^\perp$$

$$W = R(T) \oplus (R(T))^\perp = Ker(T^*) \oplus (Ker(T^*))^\perp$$

This proposition says that: $\begin{cases} V = Ker(T) \oplus R(T^*) \\ W = Ker(T^*) \oplus R(T) \end{cases}$

Matrix Representation using Orthonormal Bases

Let V be an inner pr. sp. with orthonormal basis $\beta = \{v_1, \ldots, v_n\}, \quad dimV = n$

and W be an inner pr. sp. with orthonormal basis $\Upsilon = \{w_1, \ldots, w_m\}, \quad dimW = m$

If $T : V \to W$ is a linear transformation, then

$$[T]_\beta^\Upsilon = \begin{bmatrix} < Tv_1, w_1 > & \dots & < Tv_n, w_1 > \\ < Tv_1, w_2 > & \dots & < Tv_n, w_2 > \\ \vdots & \vdots & \vdots \\ < Tv_1, w_m > & \dots & < Tv_n, w_m > \end{bmatrix}$$

or briefly

$$[T]_\beta^\Upsilon = A = (< Tv_j, w_i >) = (a_{ij}), \quad j = \overline{1, n}, \quad i = \overline{1, m}$$

If $T^* : W \to V$ is the adjoint of $T : V \to W$, then

$$[T^*]_\Upsilon^\beta = \begin{bmatrix} \overline{< Tv_1, w_1 >} & \dots & \overline{< Tv_n, w_m >} \\ \overline{< Tv_2, w_1 >} & \dots & \overline{< Tv_n, w_m >} \\ \vdots & \vdots & \vdots \\ \overline{< Tv_n, w_1 >} & \dots & \overline{< Tv_n, w_m >} \end{bmatrix}$$

If $A = [T]_\beta^\Upsilon$ is the matrix representation of T w.r.t. orthonormal bases β and Υ, and $B = [T^*]_\Upsilon^\beta$ is the matrix representation of T^*, then

$$B = \overline{A}^t, \quad \text{i. e.} [T^*]_\Upsilon^\beta = (\overline{[T]_\beta^\Upsilon})^t$$

- **Note**: With respect to nonorthonormal bases, the matrix of T^* does not necessarily equal the conjugate transpose of the matrix of T.

Special Operators in Inner Product Spaces and The Sepctral Theorem

$V \to$ finite dimensional inner product space

$T : V \to V$ linear operator

$T^* : V \to V$ adjoint operator of T defined as $< Tu, v > = < u, T^*v >, \quad \forall u, v \in V$

In this chapter we will study:

1. Self-Adjoint ops. $(T = T^*)$
2. Normal ops. $(TT^* = T^*T)$
3. Complex Spectral Theorem
4. Real Spectral Theorem

Inclussion of some classes of operators:

Orthogonal Adjoint $\subset \overbrace{Self - Adjoint}^{T=T^*} \subset \overbrace{Normal Ops.}^{TT^*=T^*T} \supset \overbrace{Unitary Ops.}^{TT^*=T^*T=I}$

Analogy with Complex Numbers : $z \in \mathbb{C}$

1. $z = \overline{z} \implies z$ is real
2. $z = -\overline{z} \implies z$ is pure imaginary
3. $z\overline{z} = |z|^2 = 1 \implies z$ on the unit circle
4. $z\overline{z} = \overline{z}z \implies$ any z

Self-Adjoint Operators

T is called self-adjoint if

$$T = T^* \quad i.e. \quad < Tu, u >=< u, Tv >, \quad \forall u, v \in V$$

- If $T = T^*$, then its matrix representation with respect to an orthonormal basis β for V, say $A = [T]_\beta$ is **Hermitian**, that's $A = A^t$, when $\mathbb{F} = \mathbb{C}$.
- If $T = T^*$, then $A = [T]_\beta$ is **symmetric**, that's $A = A^t$, when $\mathbb{F} = \mathbb{R}$.
- Every complex number can be written using two real numbers. Similarly, any operator can be written in terms of two self-adj. ops.

any $z \in \mathbb{C} \implies z = \underbrace{\frac{1}{2}(z + \overline{z})}_{\text{real}} + i. \underbrace{\frac{1}{2i}(z - \overline{z})}_{\text{real}}$

any T linaer $\implies T = \underbrace{\frac{1}{2}(T + T^*)}_{\text{self-adj}} + i. \underbrace{\frac{1}{2}(T - T^*)}_{\text{self-adj}}$

Eigenvalues and Eigenvectors of Self-Adj Ops

1. Every eigenvalue of a self-adj operator is **real**.
2. Eigenvectors of a self-adj op. corresponding to **distinct** eigenvalues are orthogonal.

If V is a **complex** inner product space and $T : V \to V$ is any linear operator s.t. $< Tv, v >= 0$ for $\forall v \in V$, then $T = 0$.

- **Note**: This is false for **real** inner product spaces.

Let V be a **complex** inner product space and $T \in L(V)$. Then,

T is self-adj $\iff < Tv, v > \in \mathbb{R}$ for $\forall v \in V$

If T is **self-adj.** op. on V s.t. $< Tv, v >= 0$ for $\forall v \in V$, then $T = 0$.

Existance of Eigenvalue(Eigenvector) of a Self-Adjoint Operator

Let T be a self-adjoint operator on V. If $\alpha, \beta \in \mathbb{R}$ satisfy $\alpha^2 < \mu\beta$, then $T^2 + \alpha T + \beta I$ is **invertible**.

A **self-adj.** op. $T : V \to V$ on a finite dim. Complex or Real inner product space **has an eigenvalue(eigenvector)**.

Normal Operators

We expect natural generalizations of complex numbers ($z\overline{z}=\overline{z}z) only when operators commute. Thus we look at normal operators.

A linear op. T is **normal** if $TT^* = T^*T$. (T and T^* commute).

A linear op. T is **normal** $\iff ||Tv|| = ||T^*v||, \quad \forall v \in V.$

Let T be a normal op. A vector $v \neq 0$ is an eigenvector of T with an eigenvalue $\lambda \iff v$ is an eigenvector of T^* with eigenvalue $\overline{\lambda}$. That's for normal op.

$$Tv = \lambda v \iff T^*v = \overline{\lambda}v$$

If T is **normal**, then eigenvectors of T corresponding to **distinct** eigenvalues are **orthogonal**.

The Spectral Theorem

- Our goal was to learn more about the structure of linear operators.
- In applications, one of the most important operators are the diagonalizable ones.

- If V is a **vector space**, $dimV = n$, then we have seen that $T : V \to V$ is diagonalizable $\iff$ it has n-linearly independent eigenvectors.
- If V is an **inner product space**, then **the spectral theorem** gives the necessary and sufficient conditions, under which a linear $T : V \to V$ is unitarily diagonalizable. We shall see that these are precisely the operators $T \in L(V)$, s.t. V has an **orthonormal basis** consisting of eigenvectors of T.
- The structure of these operators depend on the field $\mathbb{F}$. Namely, we have

1. Complex Spectral Theorem (Normal Ops.)
2. Real Spectral Theorem (Self-Adj. Ops.)

Let $V = \oplus W^{\perp}$ be a finite dimensional inner product space, and $T : V \to V$ be any linear operator. If W is invariant under T, then $W^{\perp}$ is invariant under T^*.

Complex Spectral Theorem

Suppose that V is a **complex** inner product space and $T \in L(V)$. Then, V has an orthonormal basis consisting of eigenvectors of $T \iff Tisnormal$.

Real Spectral Theorem

Suppose that V is **real** inner product space and $T \in L(V)$. Then,

V has an orthonormal basis consisting of eigenvectors of $T \iff T$ is self-adjoint.