Numerical methods
Project : Resolution of the heat equation using the finite difference method

This project covers the numerical resolution of the time-independent heat equation using the finite difference method (FDM).

1 The time-independent heat equation

The distribution of the temperature in a solid body Ω is governed by the following time-dependent heat equation :

$$\rho c_p \frac{\partial T}{\partial t} - \nabla \cdot (k^{\mathrm{The}} \nabla T) = s(t,x,y) \quad \text{in } [0,T] \times \Omega, \tag{1a}$$

$$T(t=0,x,y) = T_0(x,y) \quad \text{in } \Omega, \tag{1b}$$

$$T(t,x,y) = T_{\mathrm{bnd}}(x,y) \quad \text{on } \Gamma_{\mathrm{Dir}} \times [0,T], \tag{1c}$$

$$\boldsymbol{n} \cdot (k^{\mathrm{The}} \nabla T) = q_{\mathrm{conv}}(x,y) \quad \Gamma_{\mathrm{Neu}} \times [0,T], \tag{1d}$$

$$\sigma \cdot (T^4 - T_{\mathrm{bnd}}^4) = q_{\mathrm{rad}}(x,y) \quad \Gamma \times [0,T]. \tag{1e}$$

In equation (1), Γ is the boundary of Ω with $\Gamma = \Gamma_{\mathrm{Dir}} \cup \Gamma_{\mathrm{Neu}}$ where Γ_{Dir} and Γ_{Neu} are part of Γ where the boundary condition on the temperature and on the convection flux are imposed. $T : [0,T] \times \Omega \to \mathbb{R} : (t,x,y) \mapsto T(t,x,y)$ is the temperature [K], $\dfrac{\partial T}{\partial t}$ is the time derivative of the temperature and

$$\nabla \cdot (k^{\mathrm{The}} \nabla T) = \frac{\partial}{\partial x}\left(k^{\mathrm{The}} \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k^{\mathrm{The}} \frac{\partial T}{\partial y}\right).$$

The variables ρ, c_p, k^{The}, s and σ are respectively the mass density [kg m^{-3}], the specific heat capacity [J kg^{-1} K^{-1}], the thermal conductivity [W m^{-1} K^{-1}], the volumetric heat source [W m^{-3}] and the Stefan-Boltzmann constant with $\sigma = 5.670373 \times 10^{-8}$ W m^{-2} K^{-4}. The (known) fields $T_0(x,y)$, $T_{\mathrm{bnd}}(x,y)$, $q_{\mathrm{conv}}(x,y)$, $q_{\mathrm{rad}}(x,y)$ are respectively the initial temperature, the temperature at the boundary, the heat flux exchanged by convection and the heat flux exchanged by radiation. The characteristic thermal diffusion time is defined by :

$$T_c = \frac{\rho c_p L_c^2}{k^{\mathrm{The}}},$$

with L_c which is the characteristic length of the device (L_c is typically the dimension of the device).

Electronic components containing chips produce periodic heat sources at a rate that is very large compared to the characteristic thermal diffusion time. As an example, for a piece of copper with the dimensions 10cm $\times$ 10cm, the characteristic thermal diffusion time $T_c \approx 87$s which is too large compared to the period of the network voltage in France which is 20ms. For heat problems with such a periodic heat source or with a constant heat source, the temperature field settles to a steady-state temperature after a transient phase. This temperature is governed by the following steady-state heat equation :

$$-\nabla \cdot (k^{\mathrm{The}} \nabla T) = s(x,y) \quad \text{in } \Omega, \tag{2a}$$

$$T(x,y) = T_{\mathrm{bnd}}(x,y) \quad \text{on } \Gamma_{\mathrm{Dir}}, \tag{2b}$$

$$\boldsymbol{n} \cdot (k^{\mathrm{The}} \nabla T) = q_{\mathrm{conv}}(x,y) \quad \Gamma_{\mathrm{Neu}}, \tag{2c}$$

$$\sigma \cdot (T^4 - T_{\mathrm{bnd}}^4) = q_{\mathrm{rad}}(x,y) \quad \Gamma. \tag{2d}$$

In the rest of this project, we only consider the Dirichlet boundary condition of type (2b) and we neglect convection and radiation fluxes of type (2c)-(2d). We also consider the following heat source for the cheap :

$$s(x,y) = P_{\mathrm{ave}} = \frac{1}{T}\int_0^T p(t)\mathrm{d}t = \frac{1}{T}\int_0^T v(t)i(t)\mathrm{d}t$$

where P_{ave} is the average power dissipated over a period T, $p(t)$ is the instantaneous power and $v(t)$ and $i(t)$ are the voltage and the current, respectively.

2 The finite difference method for the heat equation

2.1 Description

Problem (2) is discretized using the finite difference method. To this end, we consider a square domain defined by $\Omega = [-0.5L, 0.5L] \times [-0.5L, 0.5L]$ and we define a 2D spatial grid with the node $(x_i, y_j) = (-0.5L + i\,\Delta x, -0.5L + j\,\Delta y)$ where $\Delta x = L/N_x$ and $\Delta y = L/N_y$ are the steps along the x and y directions, $i = 0, 1, 2, \ldots, N_x$ and $j = 0, 1, 2, \ldots, N_y$.

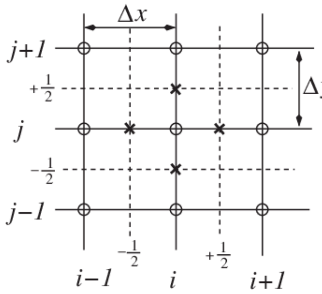

FIGURE 1 – The spatial grid in the neighborhood of the node (x_i, y_j).

The points of the grid can be classified into two sets : (1) the boundary nodes with $T_{i,j} = T_{\text{bnd}}$ and the interior nodes (see 2.1) where the terms $\left(\nabla \cdot (k^{\text{The}} \nabla T)\right)_{i,j}$ and $s_{i,j}$ can be discretized as :

$$
\begin{aligned}
\left(\nabla \cdot (k^{\text{The}} \nabla T)\right)_{i,j} &= \frac{1}{\Delta x^2} \left(\nabla_x (k^{\text{The}}_{i+\frac{1}{2},j} \Delta_x)\right) T_{i,j} + \frac{1}{\Delta_y^2} \left(\nabla_y (k^{\text{The}}_{i,j+\frac{1}{2}} \Delta_y)\right) T_{i,j} + \mathcal{O}(\Delta x^2, \Delta y^2) \\
&= \frac{1}{\Delta x^2} \left(k^{\text{The}}_{i+\frac{1}{2},j} T_{i+1,j} - (k^{\text{The}}_{i+\frac{1}{2},j} + k^{\text{The}}_{i-\frac{1}{2},j}) T_{i,j} + k^{\text{The}}_{i-\frac{1}{2},j} T_{i-1,j}\right) \\
&+ \frac{1}{\Delta y^2} \left(k^{\text{The}}_{i,j+\frac{1}{2}} T_{i,j+1} - (k^{\text{The}}_{i,j+\frac{1}{2}} + k^{\text{The}}_{i,j-\frac{1}{2}}) T_{i,j} + k^{\text{The}}_{i,j-\frac{1}{2}} T_{i,j-1}\right) \\
&= (m_{i+1,j}\, T_{i+1,j} + m_{i-1,j}\, T_{i-1,j} + m_{i,j+1}\, T_{i,j+1} + m_{i,j-1}\, T_{i,j-1} + m_{i,j}\, T_{i,j})
\end{aligned}
\tag{3}
$$

with the terms $m_{u,v}$ that can be determined from (3), and

$$
s_{i,j} = \begin{cases} s_{\text{chip}}(t) & \text{if } (x_i, y_j) \in \Omega_{\text{chip}}, \\ 0 & \text{if } (x_i, y_j) \in \Omega_{\text{board}}. \end{cases}
\tag{4}
$$

The unknowns in (3) are the temperatures $T_{i+1,j}$, $T_{i-1,j}$, $T_{i,j+1}$, $T_{i,j-1}$ and $T_{i,j}$ (assuming that the respective nodes do not belong to the boundary Γ). The 2D arrays of nodal temperatures $T_{i,j}$ and nodal sources $s_{i,j}$ can be converted into 1D arrays x_k and b_k using the following indicial mappings :

$$
k = N_x(j-1) + i \qquad \text{and} \qquad i = (k \bmod N_x) \text{ and } j = \text{floor}(k, N_x),
$$

where i and j are numbered from 1 and not 0 as it is the case for Python !

From now on, we assume $N_x = N_y = N$ and $\Delta x = \Delta y$. Using this notation, we can derive a following linear system :

$$
\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},
$$

where the rows of the matrix $\boldsymbol{A}$ are given by :

$$
\boldsymbol{A}[k,:] = \begin{cases} [0, 0, \ldots, \underbrace{1}_{\text{k}^{\text{th}}\text{position}}, \ldots, 0, 0] & \text{if } (x_i, y_j) \in \Gamma, \\ [0, 0, \ldots, m_{i,j-1}, 0, \ldots, 0, m_{i-1,j}, \underbrace{m_{i,j}}_{\text{k}^{\text{th}}\text{position}}, m_{i+1,j}, 0, \ldots, m_{i,j+1}, 0, 0] & \text{if } (x_i, y_j) \text{ is an interior node.} \end{cases}
$$

The vector of the temperature $\boldsymbol{x} = [T_0, T_1, T_2, \ldots, T_{N^2}]$ and the right-hand side $\boldsymbol{b} = [s_0, s_1, s_2, \ldots, s_{N^2}]$ with $b_k = T_{\text{bnd}}$ if k corresponds to a boundary node and $b_k = s_{i,j}$ if k corresponds to an interior node.

2.2 Pseudocode

A possible algorithm for the FDM is provided in the following pseudocode :

Algorithm 1 Pseudocode for the FD method for the heat equation

INPUT: Thermal conductivities $k^{\mathrm{The}}_{\mathrm{chip}}$, $k^{\mathrm{The}}_{\mathrm{board}}$, the source s_{chip} and the boundary conditions T_{bnd}.
OUTPUT: The array 2D of the temperature T.

```
procedure FINITE__DIFFERENCE__METHOD
    • Build the matrix A and the right-hand side b,
    • Solve equation (2.1) for the 1D array of temperatures x.
    • Build and output the 2D array of the temperature T.
end procedure
```

You may find it easier to use a function that allows to get the right indices to-and-fro $k \leftrightarrow (i, j)$ and a functions that output the thermal conductivity k^{The} and the source term s given the indices (i, j).

3 Applications

Use the developed algorithm to solve the heat transfer for the following two cases :

1. The case of a chip on the board.
2. The case of a chip on the heat sink.

You will use the following parameters :

a. Geometric dimensions : length of the board/heat sink $L = 80$cm and centered around $(0, 0)$, the length of the squared chip $l = 20$cm and centered around $(c_x, c_y) = (-20\mathrm{cm}, 20\mathrm{cm})$.

b. Material properties : $k^{\mathrm{The}}_{\mathrm{chip}} = 0.023$W m^{-1}K^{-1}, $k^{\mathrm{The}}_{\mathrm{board}} = 0.15$W m^{-1}K^{-1} and $k^{\mathrm{The}}_{\mathrm{heatsink}} = 237$W m^{-1}K^{-1}.

You will also consider time-independent densities of heat source with increasing values.