Lesson 33: Composite Functions

Objective: Construct, evaluate, and interpret composite functions.

Students will be able to:

- Evaluate composite functions for at a specified value, $x = c$.
- Find and simplify composite functions in terms of a variable.

Prerequisite Knowledge:

- Order of operations.
- Evaluating Functions.
- Function Arithmetic.

Lesson:

In addition to the four basic arithmetic operations $(+, -, \cdot, \div)$, we will now discuss a fifth operation, known as a *composition* and denoted by $\circ$ (not to be confused with a product, $\cdot$). The result of a composition is called a *composite function* and is defined as follows.

$$(f \circ g)(x) = f(g(x))$$

The notation $(f \circ g)(x)$ above should always be interpreted as "f of g of x". In this situation, we consider g to be the *inner* function, since it is being substituted into f for x. Consequently, we refer to f as the *outer* function.

Similarly, if we reversed the order of the two functions f and g, then the resulting composite function $(g \circ f)(x) = g(f(x))$ will have inner function f and outer function g, and should be interpreted as "g of f of x". As we will see, one should never assume that the two composite functions $f \circ g$ and $g \circ f$ will be equal.

We will begin by evaluating a composite function at a single value. This is accomplished by first evaluating the inner function at the specified value, and then substituting ("plugging in") the corresponding *output* into the outer function.

We can also identify a composite function in terms of the variable. In our second example, we will substitute the inner function into the outer function for every instance of the variable and then simplify. This approach is sometimes referred to as the "inside-out" approach.

It is important to note that $(f \circ g)(x)$ usually will *not* equal $(g \circ f)(x)$, as our first Demo/Discussion problem will show.

I - Motivating Example(s):

Example: Find $(f \circ g)(3)$, where $f(x) = x^2 - 2x + 1$ and $g(x) = x - 5$.

$$
\begin{aligned}
(f \circ g)(3) &= f(g(3)) && \text{Rewrite } f \circ g \text{ as inner and outer functions} \\
g(3) &= (3) - 5 = -2 && \text{Evaluate inner function at } x = 3 \\
& && \text{Use output of } -2 \text{ as input for } f \\
f(-2) &= (-2)^2 - 2(-2) + 1 && \text{Evaluate outer function at } x = -2 \\
&= 4 + 4 + 1 && \text{Simplify} \\
(f \circ g)(3) &= 9 && \text{Our solution}
\end{aligned}
$$

Example: Find $(f \circ g)(x)$, where $f(x) = x^2 - x$ and $g(x) = x + 3$.

$$
\begin{aligned}
(f \circ g)(x) &= f(g(x)) && \text{Rewrite } f \circ g \text{ as inner and outer functions} \\
& && \text{Our inner function is } g(x) = x + 3 \\
&f(x + 3) && \text{Replace each } x \text{ in } f \text{ with } (x + 3) \\
& && \text{Make sure to include parentheses!} \\
&(x + 3)^2 - (x + 3) && \text{Simplify; expand binomial} \\
&(x^2 + 6x + 9) - (x + 3) && \text{Distribute negative} \\
&x^2 + 6x + 9 - x - 3 && \text{Combine like terms} \\
(f \circ g)(x) &= x^2 + 5x + 6 && \text{Our solution} \\
&= (x + 3)(x + 2) && \text{in factored form}
\end{aligned}
$$

II - Demo/Discussion Problems:

Find each of the following.

1. $(g \circ f)(x)$, where $f(x) = x^2 - x$ and $g(x) = x + 3$
2. $(g \circ g)(x)$, where $g(x) = x^2 - 2x$

In each problem below, use the given pair of functions to find the following six composite values if they exist.

- $(g \circ f)(0)$
- $(g \circ f)(-3)$
- $(f \circ g)(-1)$
- $(f \circ g)\left(\frac{1}{2}\right)$
- $(f \circ f)(2)$
- $(f \circ f)(-2)$

3. $f(x) = 4 - x, \quad g(x) = 1 - x^2$
4. $f(x) = |x - 1|, \quad g(x) = x^2 - 5$
5. $f(x) = 4x + 5, \quad g(x) = \sqrt{x}$
6. $f(x) = \dfrac{3}{1 - x}, \quad g(x) = \dfrac{4x}{x^2 + 1}$

Use the given pair of functions to find and simplify expressions for the following three composite functions. Then state the domain of each using interval notation.

- $(g \circ f)(x)$
- $(f \circ g)(x)$
- $(f \circ f)(x)$

7. $f(x) = 3x - 5, \quad g(x) = \sqrt{x}$
8. $f(x) = 3 - x^2, \quad g(x) = \sqrt{x+1}$
9. $f(x) = 3x - 1, \quad g(x) = \dfrac{1}{x+3}$
10. $f(x) = \dfrac{2x}{x^2 - 4}, \quad g(x) = \sqrt{1-x}$

Write each of the given functions as a composition of two or more non-identity functions. (There are several correct answers, so check your answer using function composition.)

11. $p(x) = (2x+3)^3$
12. $H(x) = |7 - 3x|$
13. $Q(x) = \dfrac{2x^3+1}{x^3-1}$
14. $w(x) = \dfrac{x^2}{x^4+1}$

III - Practice Problems:

In each problem below, use the given pair of functions to find the following six composite values if they exist.

- $(g \circ f)(0)$
- $(f \circ g)(-1)$
- $(f \circ f)(2)$
- $(g \circ f)(-3)$
- $(f \circ g)\left(\frac{1}{2}\right)$
- $(f \circ f)(-2)$

1. $f(x) = x^2, \quad g(x) = 2x + 1$
2. $f(x) = 4 - 3x, \quad g(x) = |x|$
3. $f(x) = \sqrt{3-x}, \quad g(x) = x^2 + 1$
4. $f(x) = \dfrac{x}{x+5}, \quad g(x) = \dfrac{2}{7-x^2}$

Use the given pair of functions to find and simplify expressions for the following three composite functions. Then state the domain of each using interval notation.

- $(g \circ f)(x)$
- $(f \circ g)(x)$
- $(f \circ f)(x)$

5. $f(x) = 2x + 3, \quad g(x) = x^2 - 9$
6. $f(x) = x^2 - x + 1, \quad g(x) = 3x - 5$
7. $f(x) = x^2 - 4, \quad g(x) = |x|$
8. $f(x) = |x+1|, \quad g(x) = \sqrt{x}$
9. $f(x) = |x|, \quad g(x) = \sqrt{4-x}$
10. $f(x) = x^2 - x - 1, \quad g(x) = \sqrt{x-5}$

11. $f(x) = \dfrac{3x}{x-1}, \quad g(x) = \dfrac{x}{x-3}$

12. $f(x) = \dfrac{x}{2x+1}, \quad g(x) = \dfrac{2x+1}{x}$

Write each of the given functions as a composition of two or more non-identity functions. (There are several correct answers, so check your answer using function composition.)

13. $P(x) = \left(x^2 - x + 1\right)^5$

14. $h(x) = \sqrt{2x-1}$

15. $r(x) = \dfrac{2}{5x+1}$

16. $R(x) = \dfrac{7}{x^2-1}$

17. $q(x) = \dfrac{|x|+1}{|x|-1}$

18. $v(x) = \dfrac{2x+1}{3-4x}$

19. Let $g(x) = -x$, $h(x) = x+2$, $j(x) = 3x$ and $k(x) = x-4$. In what order must these functions be composed with $f(x) = \sqrt{x}$ to create $F(x) = 3\sqrt{-x+2} - 4$?

20. What linear functions could be used to transform $f(x) = x^3$ into $F(x) = -\frac{1}{2}(2x-7)^3 + 1$? What is the proper order of composition?

Let f be the function defined by

$$f = \{(-3,4), (-2,2), (-1,0), (0,1), (1,3), (2,4), (3,-1)\}$$

and let g be the function defined

$$g = \{(-3,-2), (-2,0), (-1,-4), (0,0), (1,-3), (2,1), (3,2)\}$$

. Find each composite value if it exists.

21. $(f \circ g)(3)$

22. $f(g(-1))$

23. $(f \circ f)(0)$

24. $(f \circ g)(-3)$

25. $(g \circ f)(3)$

26. $g(f(-3))$

27. $(g \circ g)(-2)$

28. $(g \circ f)(-2)$

29. $g(f(g(0)))$