Lesson 55: Sign Diagrams for Rational Functions

Objective: Solve rational inequalities by constructing a sign diagram.

Students will be able to:

- Create a sign diagram for a rational expression.
- Identify and express the solution to a rational inequality using interval notation.

Prerequisite Knowledge:

- Factoring.
- The Rational Root Theorem.
- Polynomial and /or synthetic division.
- Definition and associated terminology of a rational function.
- Domain of a rational function.
- Interval notation.

Lesson:
I - Motivating Example(s):

Whenever we are asked to find when a rational expression or function f is positive, negative, ≥ 0, or ≤ 0, we can always apply the following steps.

1. Identify a complete factorization of the expression.
2. Construct a sign diagram.
3. Find all intervals that correspond to the desired inequality.
4. In the case of $\geq$ or $\leq$, make sure to include any $x-$intercepts.

Solve the inequality $f(x) > 0$ for the following function.

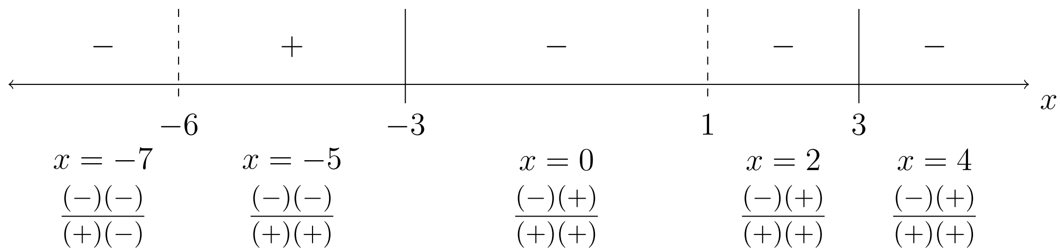
$$f(x) = \frac{-2x^3 + 6x^2 + 18x - 54}{3x^3 + 12x^2 - 33x + 18}$$

Using Desmos to graph our function, we can see that our answer should be $(-6, -3)$. To reach this result algebraically, we will need to find a factored form for f in order to construct a sign diagram. Using our prerequisite knowledge from factoring polynomials, specifically the Rational Root Theorem, one can obtain the following factorization.

$$f(x) = \frac{-2(x+3)(x-3)^2}{3(x+6)(x-1)^2}$$

To find the sign diagram for f, we need to identify our $x-$intercepts, as well as those x not in the domain. From our factorization, we see that this is the set $x = \{-6, -3, 1, 3\}$, with $x \pm 3$ being our intercepts.

Our diagram is shown below.



One important observation in our diagram is in the calculation of each sign. For each test value, we have excluded the *squared* factors in the numerator and denominator, since both $(x-3)^2$ and $(x-1)^2$ will always contribute a positive sign and not affect the end result. For example, when $x = 0$, we get

$$\frac{(-)(+)(-)^2}{(+)(+)(-)^2},$$

which reduces to the result that we see above. Similarly, we could have excluded the $(+)$ that appears in the denominator of each test value's sign calculation, since the constant multiplier of 3 will have no impact on sign.

At this point we are essentially done, since the factorization and construction of our diagram has done the bulk of the work for us. Since we are asked to find all x such that $f(x) > 0$, we see that this equals to the union of all intervals that correspond to a $+$ sign. This gives us our anticipated answer of $(-6, -3)$.

Recalling our discussion of the last example, if we wished to answer the follow-up question of when $f(x) \geq 0$, we would just need to include all boundary values in our diagram that correspond to $x-$intercepts (when $f(x) = 0$). From our diagram, this would be any value of x that has a *solid* divider, remembering that dashed dividers correspond to values not in our domain. In this case, the function $f(x) \geq 0$ for all x in the set $(-6, -3] \cup \{3\}$.

II - Demo/Discussion Problems:

Solve each of the following inequalities. Use Desmos to confirm your answers.

1. $\dfrac{4x^2 - 4x + 1}{x^3 - x^2 - 17x - 15} \leq 0$

2. $\dfrac{x-6}{x} \geq \dfrac{-2}{x-1}$

III - Practice Problems:

Solve each of the following inequalities. Use Desmos to confirm your answers.

1. $\dfrac{(x-5)(x+4)}{x-1} \leq 0$
2. $\dfrac{x+1}{x-1} \geq 0$
3. $\dfrac{(x+1)(x-3)}{x+2} \geq 0$
4. $\dfrac{x^2-25}{x^2-1} \leq 0$
5. $\dfrac{x^2+x-12}{x^3-25x} \leq 0$