Lesson 12: Functions and Relations

Objective: Define a relation and a function; determine if a relation is a function.

Students will be able to:

- Apply the Vertical Line Test to a graph to determine whether a relation represents a function.
- Determine whether a relation given as either a set, an equation, or a table of points represents a function.

Prerequisite Knowledge:

- Graph a set or table of points on the coordinate plane.
- Isolate a variable in an equation.

Lesson:

A **relation** R is a set of points in the xy-plane. A relation in which each x-coordinate is paired with exactly one y-coordinate is said to describe y as a **function** of x. Relations which represent functions of x will often be denoted by f, or $f(x)$, rather than R. The set of all x-coordinates of the points in a function f is called the **domain** of f, and the set of all y-coordinates of the points in f is called the **range** of f.

One major test that is used to determine whether or not a graph of a relation represents y as a function of x is known as the Vertical Line Test. We will now state the Vertical Line Test as a mathematical theorem and then demonstrate its use.

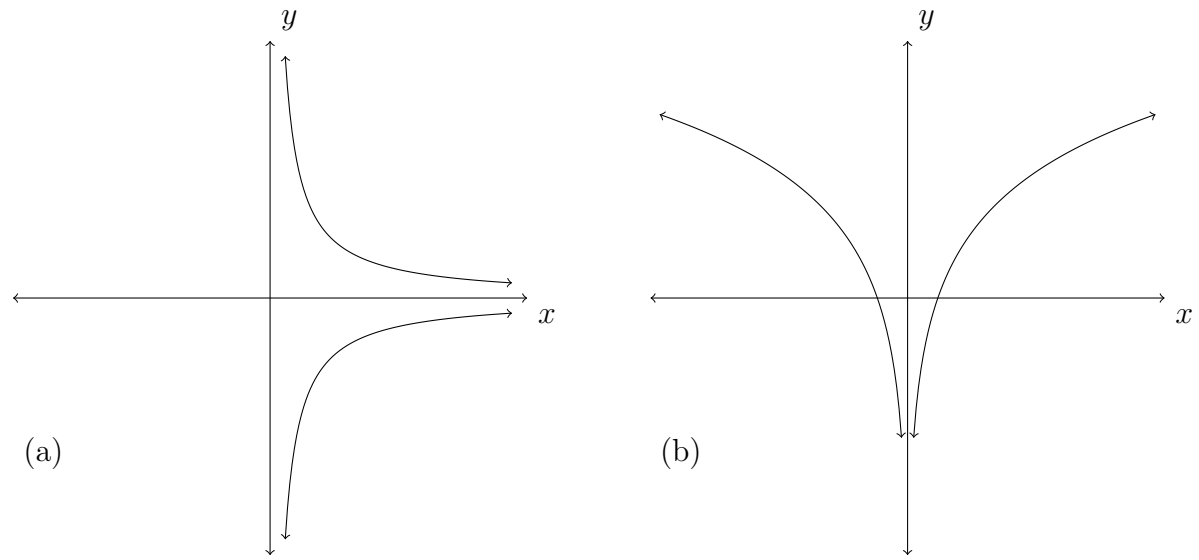
Vertical Line Test: A set of points in the xy-plane represents y as a function of x if and only if no two points lie on the same vertical line.

Alternatively stated, if a graph is known to represent y as a function of x, then there can be no vertical line that intersects the graph in more than one point. Conversely, if a known graph has the property that no vertical line intersects it in more than one point, then the given graph represents y as a function of x.

When we are presented with an equation, instead of a graph, we can still determine whether or not the equation for our relation represents y as a function of x by solving the equation for y and carefully considering the result. When solving for y, the existence of the $\pm$ in our solution will cause our corresponding graph to fail the VLT.

I - Motivating Example(s):

Example: Use the Vertical Line Test to determine whether each of the following graphs represent y as a function of x.



Graph (a) above fails the VLT, since any vertical line drawn in the right half-plane (where $x > 0$) intersects the relation at two points. Graph (b) passes the VLT, since no vertical line intersects the graph at more than one point.

Example: Determine whether the following equation represents y as a function of x.

$$x^2 + y^2 = 9$$

Solve the equation for y.

$$\begin{aligned}
x^2 + y^2 &= 9 && \text{Solve for } y \\
\underline{-x^2} \quad & \underline{\quad -x^2} && \text{Subtract } x^2 \\
y^2 &= 9 - x^2 && \\
\sqrt{y^2} &= \pm\sqrt{9 - x^2} && \text{Introduce a square root} \\
& && \quad \text{include a } \pm \text{ on right side} \\
y &= \pm\sqrt{9 - x^2} && y \text{ is not a function of } x
\end{aligned}$$

Due to the $\pm$, we can conclude that the equation does *not* represent y as a function of x.

II - Demo/Discussion Problems:

Determine whether each of the following relations represent y as a function of x. Use Desmos to sketch a graph of each relation.

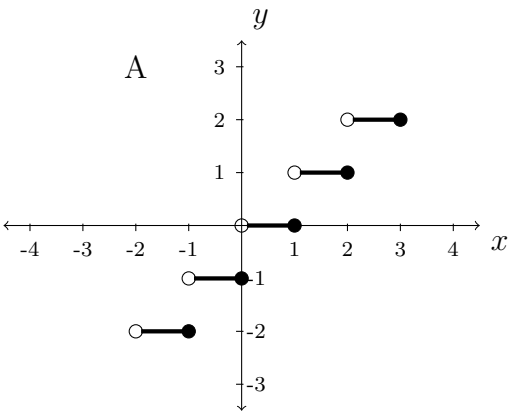
1. $\{(1,1),(2,-3),(2,0),(0,3),(-2,1/2)\}$
2. $\{(x,y) \mid x > 3 \text{ and } y \leq 2\}$
3. $x^2 = 1 - y^2$
4. $x = y^2$
5. $y = x^2$
6. $y = 3 - 2x$

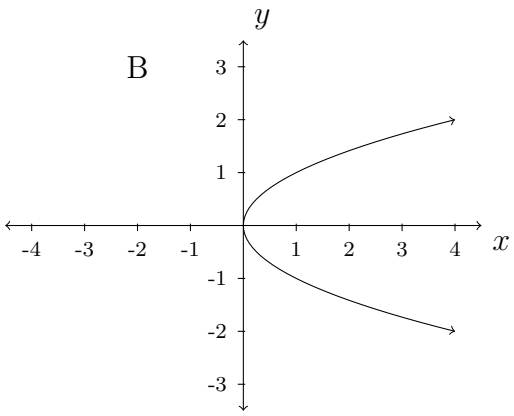
III - Practice Problems:

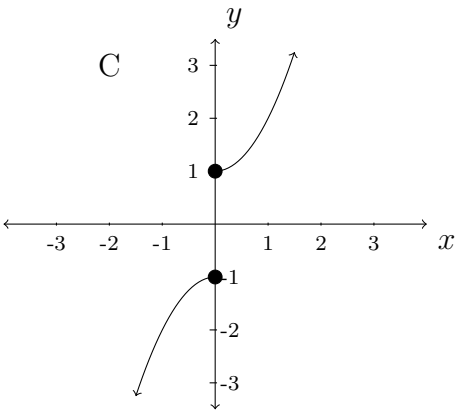
Determine if the following relations represent y as a function of x by making a table of values and graphing. Explain your reasoning. Use Desmos to confirm your results.

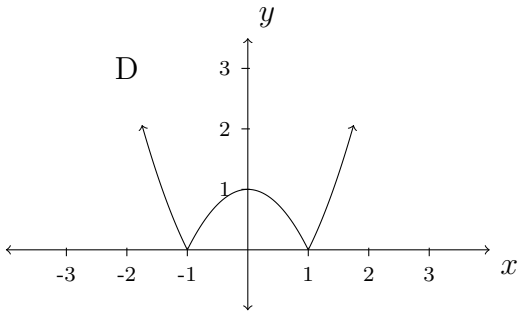
1. $x = y^3$
2. $y = x$
3. $xy = 1$
4. $y = (x-3)^2$
5. $x = (y-3)^2$
6. $y < 2x - 5$

Circle the letter of each graph/table below that represents y as a function of x.









E

x	y
3	-3
2	-2
1	-1
0	0
1	1
2	2
3	3

F

x	y
-3	3
-2	2
-1	1
0	0
1	1
2	2
3	3

G

x	y
-3	0
-2	0
-1	0
0	0
1	0
2	0
3	0

H

x	y
3	8
2	4
1	2
0	1
-1	$1/2$
-2	$1/4$
-3	$1/8$