Lesson 42: Absolute-Value as a Piecewise Function

Objective: Interpret a function containing an absolute value as a piecewise-defined function.

Students will be able to:

- Express an absolute value function as a piecewise function.

Prerequisite Knowledge:

- Evaluate and solve absolute value expressions.
- Graphing piecewise functions.
- Finding $x-$ and $y-$intercepts.

Lesson:

By definition, we know that

$$|x| = \begin{cases} -x & \text{if } \; x < 0 \\ x & \text{if } \; x \geq 0 \end{cases}.$$

If $m \neq 0$ and b is a real number, we may generalize the definition above as follows.

$$\begin{aligned} |mx+b| &= \begin{cases} -(mx+b) & \text{if } \; mx+b < 0 \\ mx+b & \text{if } \; mx+b \geq 0 \end{cases} \\ &= \begin{cases} -mx-b & \text{if } \; mx+b < 0 \\ mx+b & \text{if } \; mx+b \geq 0 \end{cases}. \end{aligned}$$

Notice that since we have never specified whether m is positive or negative above, it would not be wise to attempt to simplify either inequality in our new definition. Once we are given a value for m, as in our next example, we will be able to simplify our piecewise representation completely.

I - Motivating Example(s):

Example: Express $g(x) = |x-3|$ as a piecewise-defined function.

$$g(x) = |x-3| = \begin{cases} -(x-3) & \text{if } \; x-3 < 0 \\ x-3 & \text{if } \; x-3 \geq 0 \end{cases}$$

Simplifying, we get

$$g(x) = \begin{cases} -x+3 & \text{if } \ x < 3 \\ x-3 & \text{if } \ x \geq 3 \end{cases}.$$

Our piecewise answer above should begin to make sense, when one considers the graph of g as a horizontal shift of $y = |x|$ to the right by 3 units.

Example: Express $h(x) = |x| - 3$ as a piecewise-defined function.

Since the variable within the absolute value remains unchanged, the domains for each piece in our resulting function will not change. Instead, we need only subtract 3 from each piece of our answer. Thus, we get the following representation.

$$h(x) = \begin{cases} -x-3 & \text{if } \ x < 0 \\ x-3 & \text{if } \ x \geq 0 \end{cases}.$$

Similarly, this answer again seems reasonable, as the graph of $h(x) = |x| - 3$ represents a vertical shift of $y = |x|$ down by 3 units.

II - Demo/Discussion Problems:

Express each function as a piecewise-defined function and identify any $x-$ and $y-$intercepts on its graph. Determine the domain and range of of the function from its graph. Use Desmos to confirm your answers.

1. $r(x) = |x - 10|$
2. $s(x) = \dfrac{1}{2}|x - 10| + 5$
3. $g(x) = |3x + 1|$
4. $h(x) = -2|3x + 1|$
5. $k(x) = 4 - 2|3x + 1|$

III - Practice Problems:

Express each function below as a piecewise-defined function. Graph the function and use Desmos to confirm your answers.

1. $f(x) = -|x| - 7$
2. $g(x) = 4|x| + 2$
3. $h(x) = 5|x - 3| + 6$
4. $k(x) = -2|x + 1| + 10$
5. $m(x) = |x - 3|$
6. $n(x) = -4|x - 1| + 1$