Lesson 57: Slant and Curvilinear Asymptotes

Objective: Identify a slant or curvilinear asymptote in the graph of a rational function.

Students will be able to:

- Determine the existence of a slant or curvilinear asymptote in the graph of a rational function.
- Identify the equation of a slant or curvilinear asymptote for the graph of a rational function.

Prerequisite Knowledge:

- Rational function end behavior criteria.
- Polynomial and /or synthetic division.

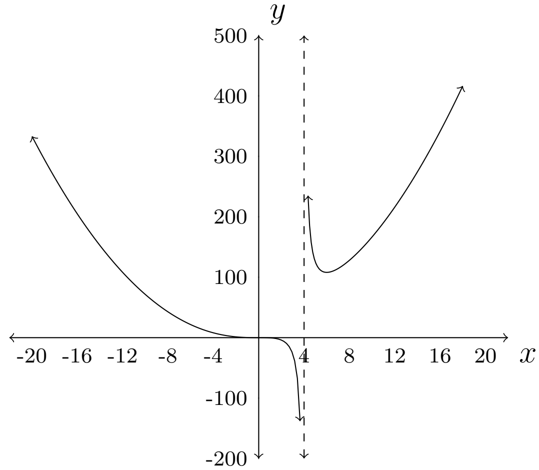
Lesson:

For a rational function

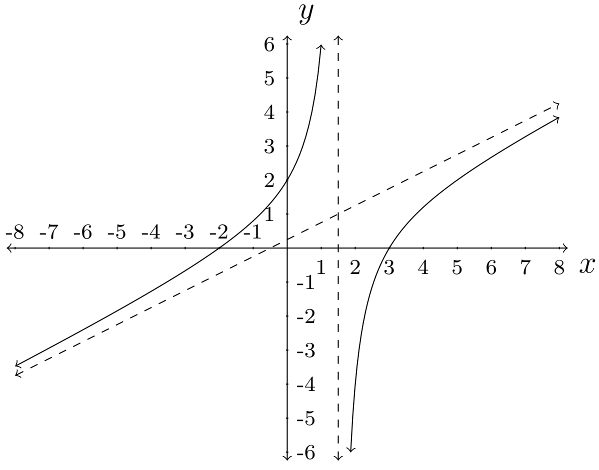
$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0},$$

when $n > m$, we know that the graph of f will have no horizontal asymptotes. Depending upon the difference between n and m, however, there is more to discover about the nature of the graph of f, as $x \to \pm\infty$.

I - Motivating Example(s):



$$f(x) = \frac{x^3}{x-4}$$



$$g(x) = \frac{x^2 - x - 6}{2x - 3}$$

In the case of $g(x) = \dfrac{x^2 - x - 6}{2x - 3}$, we see that as $x \to \pm\infty$, the graph of g actually approaches a linear asymptote. Whereas horizontal asymptotes are horizontal lines, having a slope of zero, this new type of linear asymptote has a non-zero slope and is consequently *slanted*. Hence, we say that the graph of g contains a *slant* or *oblique asymptote*. This occurs when the degree of the numerator is *one* more than the denominator, resulting in a *slanted* or *linear* asymptote.

On the other hand, the graph of $f(x) = \dfrac{x^3}{x - 4}$ does not appear to contain a slant asymptote. In fact, as $x \to \pm\infty$, the graph of f resembles a parabola. In cases such as these, we could say that the graph of f contains a *curvilinear asymptote*. In other words, the graph of f approaches some identifiable non-linear curve, as x approaches $\pm\infty$. This happens because the difference in degree between the numerator and the denominator is 2, hence the asymptote is quadratic.

II - Demo/Discussion Problems:

Use division to identify the equation of the slant or curvilinear asymptote for the graph of each of the following rational functions. Use Desmos to verify your answers.

1. $g(x) = \dfrac{x^2 - x - 6}{2x - 3}$
2. $f(x) = \dfrac{-2x^3 + x^2 - 2x + 3}{x^2 + 1}$
3. $h(x) = \dfrac{x^2 - 5x + 7}{x - 2}$

III - Practice Problems:

Use division to identify the equation of the slant or curvilinear asymptote for the graph of each of the following rational functions. Use Desmos to verify your answers.

1. $a(x) = \dfrac{5x^2 - 1}{x + 3}$
2. $g(x) = \dfrac{x^2 - 1}{x - 4}$
3. $h(x) = \dfrac{x^2 - 4x - 9}{x + 2}$
4. $j(x) = \dfrac{18x^3 - 4x - 1}{3x - 8}$
5. $k(x) = \dfrac{-x^2 + 4}{x - 9}$
6. $p(x) = \dfrac{-x^2 + 4}{x - 9}$
7. $q(x) = \dfrac{x^5}{x^4 - 4x^3 - 2x + 1}$
8. $r(x) = \dfrac{x^4 - 4}{x^3}$
9. $t(x) = \dfrac{15x^2 - 10}{5x - 19}$