Lesson 24: Vertex Form and Graphing

Objective: Graph quadratic equations in both standard and vertex forms.

Students will be able to:

- Find the vertex, axis of symmetry, and $x-$intercept(s) of a quadratic equation in vertex or standard form.

Prerequisite Knowledge:

- State the concavity of the graph of a quadratic equation.
- Find the $y-$intercept of the graph of an equation.
- Evaluate an equation and create a table of values.
- Plot points on the Cartesian $xy-$coordinate plane.

Lesson:

Recall the two forms of a quadratic equation, shown below. In both forms, we assume $a \neq 0$.

Standard Form: $y = ax^2 + bx + c$, where a, b, and c are real numbers

Vertex Form: $y = a(x - h)^2 + k$, where a, h, and k are real numbers

Unlike the standard form, a quadratic equation written in vertex form allows for immediate recognition of the vertex (h, k), which will always coincide with either a maximum (if $a < 0$) or a minimum (if $a > 0$) on the accompanying graph, called a parabola. Additionally, using the vertex form, we can easily identify the *axis of symmetry* for the parabola, which is a vertical line $x = h$ that passes through the x-coordinate of the vertex and "splits" the graph into two identical halves.

If $y = ax^2 + bx + c$ $(a \neq 0)$, we can identify the x-coordinate for the vertex (and consequently the equation for the axis of symmetry) using the following formula.

$$h = -\frac{b}{2a}$$

Consequently, the y-coordinate for our vertex can be found by plugging $x = h$ back into the given equation for our quadratic, and simplifying to find the y-coordinate, which we will relabel as k.

Once we have h and k, we can use them, along with a, to write the vertex form for our quadratic,

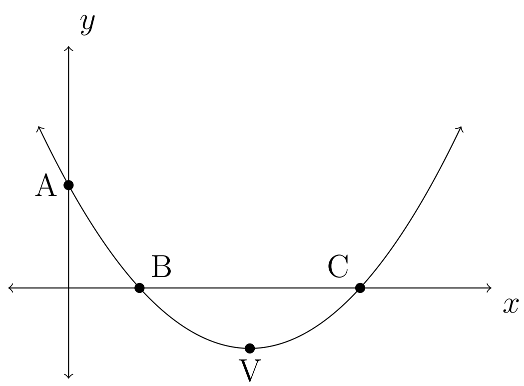
$$y = a(x-h)^2 + k.$$

On the other hand, if we are given a quadratic equation in vertex form, we can expand it to obtain the corresponding vertex form, as shown below.

$$\begin{aligned} y &= a(x-h)^2 + k \\ &= a(x-h)(x-h) + k \\ &= a(x^2 - 2hx + h^2) + k \\ &= ax^2 - 2ahx + ah^2 + k \end{aligned}$$

If we set the linear coefficient $-2ah$ above equal to b and solve for h, we can see the connection with the formula for the vertex.

It can be difficult to find a sufficient collection of points to determine the overall shape of our parabola. For this reason, we will now formally identify several key points on a parabola, which will enable us to always determine a complete graph. These points are the $y-$intercept, $x-$intercepts, and the vertex.



Point A: y-intercept; where the graph crosses the vertical y-axis (when $x = 0$).

Points B and C: x-intercepts; where the graph crosses the horizontal x-axis (when $y = 0$)

Point V: vertex (h, k); The point of the minimum (or maximum) value, where the graph changes direction.

We will use the following method to find each of the key points on our parabola.

Steps for graphing a quadratic in standard form, $y = ax^2 + bx + c$.

1. Identify and plot the vertex: $h = -\dfrac{b}{2a}$. Plug h into the equation to find k. Resulting point is (h, k).
2. Identify and plot the y-intercept: Set $x = 0$ and solve. The y-intercept will correspond to the constant term c. Resulting point is $(0, c)$.
3. Identify and plot the x-intercept(s): Set $y = 0$ and solve for x. Depending on the expression, we will end up with zero, one or two $x-$intercepts.
4. After plotting these points we can connect them with a smooth curve.

Important: Up until now, we have only discussed how to solve a quadratic equation for x by factoring. If an expression is not easily factorable, we may not be able to identify the x-intercepts. In our next few lessons, we will learn of two additional methods for finding x-intercepts, which will prove especially useful, when an equation is not easily factorable.

I - Motivating Example(s):

Example: Consider $y = -2(x+1)^2 + 3$.

In this example we can see that the vertex is at $(-1, 3)$. It is important that we not overlook the negative value for h. The axis of symmetry, passes through the x-coordinate for the vertex, $x = -1$. Expanding our vertex form gives us the corresponding standard form, shown below.

$$\begin{aligned} y &= -2(x+1)(x+1) + 3 \\ &= -2(x^2 + 2x + 1) + 3 \\ &= -2x^2 - 4x + 1 \end{aligned}$$

The value of c from our standard form gives us or $y-$intercept of $(0, 1)$, which we also confirm using the vertex form.

$$y = -2(0+1)^2 + 3 = -2(1) + 3 = 1$$

Example: Identify the vertex and axis of symmetry for the parabola represented by the given quadratic equation.

$$\begin{array}{rl} y = x^2 + 8x - 12 & \text{Given an equation in standard form} \\ a = 1, \quad b = 8, \quad c = -12 & \text{Identify } a, b, \text{ and } c \\ h = -\dfrac{b}{2a} = -\dfrac{8}{2(1)} = -4 & \text{Identify } h \\ x = -4 & \text{Use } h \text{ for axis of symmetry, a vertical line} \\ k = (-4)^2 + 8(-4) - 12 & \text{Plug in } h \text{ to find } k \\ k = 16 - 32 - 12 = -28 & \text{Simplify} \\ (-4, -28) & \text{Write the vertex as an ordered pair } (h, k) \end{array}$$

II - Demo/Discussion Problems:

Identify the vertex and axis of symmetry for each quadratic equation below. Write the vertex form for each equation. Sketch a complete graph of the corresponding parabola, labeling any $x-$ and $y-$intercepts.

1. $y = x^2 - 4x + 3$
2. $y = x^2 + 4x + 3$
3. $y = -3x^2 + 12x - 9$
4. $y = x^2 - 6x + 9$
5. $y = -x^2 + 16$
6. $y = -x^2 - 25$
7. $y = x^2 + 1$
8. $y = -3x^2 + 6x - 1$

III - Practice Problems:

Expand each equation below to find the corresponding standard form. Sketch a graph of the corresponding parabola, identifying the vertex, $y-$intercept, and axis of symmetry. If possible, identify any $x-$intercepts by factoring.

1. $y = (x-3)^2 + 4$
2. $y = (x-2)^2 + 5$
3. $y = 6(x+3)^2 + 4$
4. $y = -2(x-3)^2 + 4$
5. $y = -2(x-1)^2 - 7$
6. $y = -(x+1)^2$
7. $y = -\frac{1}{5}(x+1)^2$

Identify whether the quadratic is in vertex form, standard form, or both. If it is in vertex form, then identify the vertex (h, k).

8. $y = (x-12)^2 + 5$
9. $y = -3(x-3)^2 + 5$
10. $y = x^2 + 8$
11. $y = 2(x-4)^2$
12. $y = -4(x-1)^2 + 2$
13. $y = -5(x-7)^2$
14. $y = x^2 + 3x + 4$
15. $y = x^2 - 1$
16. $y = x^2 - 3$
17. $y = (x-1)^2 - 3$
18. $y = (x-1)^2$
19. $y = x^2$

Each quadratic equation below has been given in standard form. Rewrite each equation in vertex form.

20. $y = x^2 + 2x - 1$
21. $y = -3x^2 - 12x - 5$
22. $y = 3x^2 + 12x - 1$
23. $y = x^2 + 2x$
24. $y = x^2 + 6$
25. $y = -5x^2 - 40x$
26. $y = x^2 + 8x$
27. $y = x^2$
28. $y = x^2 + 4x - 2$
29. $y = x^2 + 16x - 2$
30. $y = 4x^2 + 10x$

Find the vertex and any intercepts ($x-$ and $y-$) of the following quadratics. Use this information to graph the resulting parabola. Identify the axis of symmetry on your graph, and write the corresponding vertex form.

31. $y = x^2 - 2x - 8$
32. $y = x^2 - 2x - 3$
33. $y = 2x^2 - 12x + 10$
34. $y = 2x^2 - 12x + 16$
35. $y = -2x^2 + 12x - 18$
36. $y = -2x^2 + 12x - 10$
37. $y = -3x^2 + 24x - 45$
38. $y = -3x^2 + 12x - 9$
39. $y = -x^2 + 4x + 5$
40. $y = -x^2 + 4x - 3$
41. $y = -x^2 + 6x - 5$
42. $y = -2x^2 + 16x - 30$
43. $y = -2x^2 + 16x - 24$
44. $y = 2x^2 + 4x - 6$
45. $y = 3x^2 + 12x + 9$
46. $y = 5x^2 + 30x + 45$
47. $y = 5x^2 - 40x + 75$
48. $y = 5x^2 + 20x + 15$
49. $y = -5x^2 - 60x - 175$
50. $y = -5x^2 + 20x - 15$