

## Lesson 60: Rational Functions Graphing Summary

CC attribute: *College Algebra* by C. Stitz and J. Zeager.



**Objective:** Graph a rational function in its entirety.

**Students will be able to:**

- Identify all important aspects of the graph of a rational function: intercepts, holes, and any vertical, horizontal, slant, or curvilinear asymptotes.
- Sketch a complete graph of a rational function using a sign diagram.

**Prerequisite Knowledge:**

- Evaluating a function.
- Factoring.
- The Rational Root Theorem.
- Polynomial and /or Synthetic Division.
- Multiplicative identity /inverse.
- Sign Diagrams.
- Rational function features.

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**Lesson:**

At this point, we have addressed all key features of rational functions individually. This section pulls each of these aspects together, for a detailed analysis of a rational function, culminating in a complete sketch of its graph. Along the way, we will need to address each of the following aspects for our rational function  $f(x) = \frac{p(x)}{q(x)}$ . It is important to note that there is no universally accepted order to this checklist.

- Find the  $y$ -intercept of the graph of  $f$ ,  $(0, f(0))$ , if it exists.
- Use the degrees and leading coefficients of  $p$  and  $q$  to determine whether the graph of  $f$  has a horizontal asymptote. If the graph of  $f$  has a slant asymptote, use polynomial division to find where it is located.
- Identify a complete factorization of  $f$ , and use it to find the domain of the function. This is the set of all  $x$ , such that  $q(x) \neq 0$ .
- Find any  $x$ -intercepts of the graph of  $f$ . This is the set of all  $x$  in the domain of  $f$ , such that  $p(x) = 0$ . Using multiplicities, classify each  $x$ -intercept as a crossover or turnaround (“bounce”) point.

- Find the simplified expression  $g$  for the given function  $f$ , and use it to identify any vertical asymptotes or holes in the graph of  $f$ . Use multiplicities to help visualize the nature of the graph of  $f$  near its vertical asymptotes. If  $f$  has a hole at  $x = c$ , use  $g$  to help plot the hole's precise location at  $(c, g(c))$ .
- Using both the  $x$ -intercepts and the discontinuities (those  $x$  not in the domain), construct a sign diagram for  $f$ .

In each rational function we encounter, we will carefully examine the function, making sure not to omit any of the checklist items above and to compare each item to those that precede it along the way for accuracy. Although the process will take some time, if we are thorough, our end result should be a complete, accurate sketch of the given rational function.

### I - Motivating Example(s):

**Example:** Sketch a complete graph of the rational function below, making sure to have a clearly defined scale and label all key features of your graph (intercepts, asymptotes, and holes).

$$f(x) = \frac{4x^2}{x^3 + 3x^2 - 4x}$$

In this example, we see that the graph of  $f$  will not have a  $y$ -intercept, since  $f(0) = \frac{0}{0}$ , which is undefined.

Since the degree of the numerator is less than the degree of the denominator, we conclude that the graph of  $f$  has a horizontal asymptote along the  $x$ -axis,  $y = 0$ .

Our graph also has no  $x$ -intercepts, since our numerator only equals zero when  $x = 0$ , which we know is not in our domain of  $f$ .

Furthermore, using the knowledge from our previous lessons, we can find the following factorization of  $f$ .

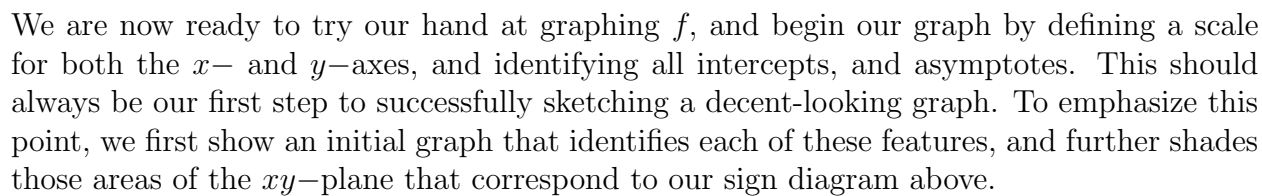
$$f(x) = \frac{4x^2}{x(x+4)(x-1)}$$

Consequently,  $f$  has corresponding domain  $x \neq -4, 0, 1$ , and related simplified expression

$$g(x) = \frac{4x}{(x+4)(x-1)}.$$

The graph of  $f$  has vertical asymptotes at  $x = -4$  and  $x = 1$  and a hole at  $(0, g(0)) = (0, 0)$ .

Since the multiplicities of both  $x = -4$  and  $x = 1$  in the denominator of  $f$  are both one (odd), we know that the graph of  $f$  will approach each vertical asymptote from opposite sides of the  $x$ -axis. The following sign diagram confirms this observation.



## II - Demo/Discussion Problems:

Sketch a complete graph of each rational function below, making sure to have a clearly defined scale and label all key features of your graph (intercepts, asymptotes, and holes). Use [Desmos](#) to compare your results.

$$1. f(x) = \frac{x^3 - 16x}{3x^2 - 6x - 24}$$

$$2. g(x) = \frac{4x^2}{x^3 + 3x^2 - 4x}$$

## III - Practice Problems:

Sketch a complete graph of each rational function below, making sure to have a clearly defined scale and label all key features of your graph (intercepts, asymptotes, and holes). Use [Desmos](#) to compare your results.

$$1. a(x) = \frac{2x + 6}{x + 3}$$

$$2. g(x) = \frac{x^3 - 16x}{x^2 - 4x}$$

$$3. h(x) = \frac{x^2 - 4x - 9}{x + 2}$$

$$4. j(x) = \frac{18x^3 - 4x - 1}{x^2 - 4}$$

$$5. k(x) = \frac{x^2 - 17x + 72}{x - 9}$$

$$6. p(x) = \frac{x^2 + 6x + 8}{2x + 4}$$

$$7. q(x) = \frac{x^5}{x(x - 5)}$$

$$8. r(x) = \frac{x^2 - 11x + 30}{x^2 - 36}$$

$$9. t(x) = \frac{3x^2 - 12x + 12}{x^2 + 2x - 8}$$

$$10. v(x) = \frac{x^2 + 2x - 8}{3x^2 - 12x + 12}$$