Lesson 52: Polynomials Graphing Summary

Objective: Graph a polynomial function in its entirety.

Students will be able to:

- Identify all important aspects of the graph of a polynomial function: $y-$intercept, $x-$intercepts (including their nature), and end behavior.
- Sketch a complete graph of a polynomial function.

Prerequisite Knowledge:

- Evaluating a function.
- Polynomial terminology and end behavior.
- Factoring.
- The Rational Root Theorem.
- Polynomial and /or Synthetic Division.
- Sign Diagrams.

Lesson:

At this point, we have addressed all key features of polynomials individually. This lesson pulls each of these aspects together, for a detailed analysis of a polynomial, culminating in a complete sketch of its graph. Along the way, we will need to address each of the following aspects for our polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$. It is important to note that there is no universally accepted order to this checklist.

- Find the $y-$intercept of the graph of f, $(0, f(0)) = (0, a_0)$.
- Use the degree n and leading coefficient a_n to determine the end behavior of the graph of f.
- Identify a complete factorization of f, and use it to find any $x-$intercepts of the graph of f. Using multiplicities, classify each $x-$intercept as a crossover or turnaround ("bounce") point.
- Using the $x-$intercepts, construct a sign diagram for f.

In each polynomial we encounter, we will carefully examine the function, making sure not to omit any of the checklist items above and to compare each item to those that precede it along the way for accuracy. Although the process will take some time, if we are thorough, our end result should be a complete, accurate sketch of the given polynomial.

I - Motivating Example(s):

Example: Sketch a complete graph of $f(x) = 14x^4 - 17x^3 - 6x^2 + 7x + 2$.

We will start with the $y-$intercept, which is $(0, 2)$.

Next, we see that f has even degree and positive leading coefficient. So, the tails of the graph of f both point upwards. In other words, as $x \to \pm\infty, f(x) \to \infty$.

Since f is degree-4, contains more than four terms, and is not of quadratic type, we will apply the Rational Root Theorem. In this case, our set of possible rational roots is

$$\left\{\pm 1, \pm 2, \pm\frac{1}{2}, \pm\frac{1}{7}, \pm\frac{1}{14}, \pm\frac{2}{7}\right\}$$

Fortunately, we see that $f(1) = 14 - 17 - 6 + 7 + 2 = 0$. So, $x - 1$ is a factor of f. Dividing, we get:

$$\begin{array}{r|l}
 & 14x^3 - 3x^2 - 9x - 2 \\
\hline
x-1 & 14x^4 - 17x^3 - 6x^2 + 7x + 2 \\
 & -14x^4 + 14x^3 \\
\hline
 & \quad -3x^3 - 6x^2 \\
 & \quad 3x^3 - 3x^2 \\
\hline
 & \qquad -9x^2 + 7x \\
 & \qquad 9x^2 - 9x \\
\hline
 & \qquad\quad -2x + 2 \\
 & \qquad\quad 2x - 2 \\
\hline
 & \qquad\qquad 0
\end{array}
\qquad
\begin{array}{r|rrrrr}
1 & 14 & -17 & -6 & 7 & 2 \\
 & & 14 & -3 & -9 & -2 \\
\hline
 & 14 & -3 & -9 & -2 & \mathbf{0}
\end{array}$$

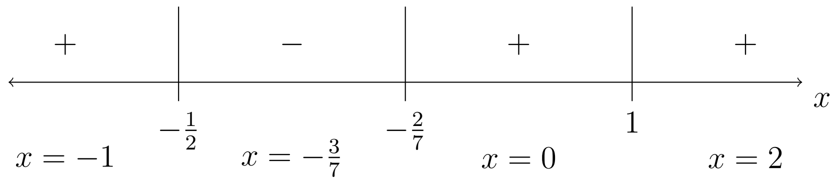
So, $f(x) = (x - 1)(14x^3 - 3x^2 - 9x - 2)$. Applying the Rational Root Theorem a second time, we can see that $x = 1$ is also a root of the cubic factor of f, since $14 - 3 - 9 - 2 = 0$. Again, we can divide to factor f further.

$$\begin{array}{r|l}
 & 14x^2 + 11x + 2 \\
\hline
x-1 & 14x^3 - 3x^2 - 9x - 2 \\
 & -14x^3 + 14x^2 \\
\hline
 & \quad 11x^2 - 9x \\
 & \quad -11x^2 + 11x \\
\hline
 & \qquad 2x - 2 \\
 & \qquad -2x + 2 \\
\hline
 & \qquad\quad 0
\end{array}
\qquad
\begin{array}{r|rrrr}
1 & 14 & -3 & -9 & -2 \\
 & & 14 & 11 & 2 \\
\hline
 & 14 & 11 & 2 & \mathbf{0}
\end{array}$$

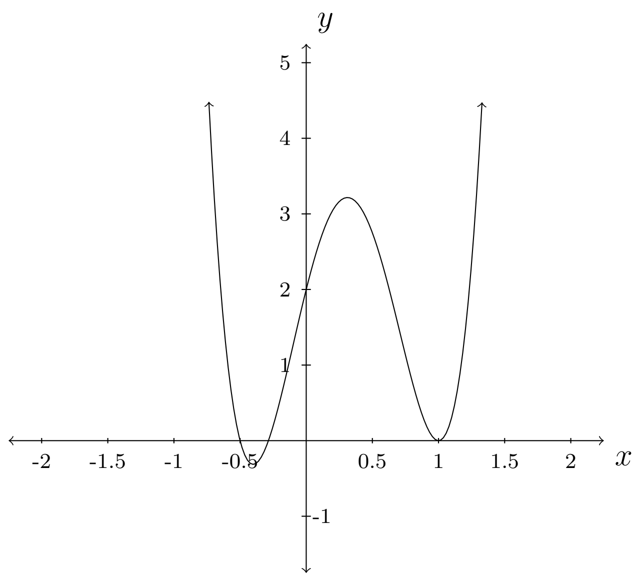
So, $f(x) = (x - 1)^2(14x^2 + 11x + 2)$. Factoring the remaining quadratic, we have $f(x) = (x - 1)^2(7x + 2)(2x + 1)$, with set of roots $\{1, -\frac{1}{2}, -\frac{2}{7}\}$.

Using multiplicities, we conclude that the $x-$intercept $(1,0)$ is a turnaround point, and the intercepts $\left(-\frac{1}{2},0\right)$ and $\left(-\frac{2}{7},0\right)$ are crossover points.

Though not necessary for graphing, a sign diagram confirms our end and local behavior findings.



Putting all of this information together results in the following graph.



II - Demo/Discussion Problems:

Factor each polynomial below, and sketch a complete graph of the function, making sure to have a clearly defined scale and label any intercepts. Use Desmos to compare your results.

1. $f(x)=-x^3+7x^2-x+7$
2. $f(x)=3x^4-5x^3-12x^2$
3. $f(x)=2x^3-5x^2-x$
4. $f(x)=-x^4-2x^2+15$

Use the Rational Root Theorem and polynomial division to get a complete factorization of each polynomial function below. Then sketch a graph of the function, making sure to have a clearly defined scale and label any intercepts. Use Desmos to compare your results.

5. $f(x) = x^4 + 4x^3 - x - 4$
6. $f(x) = 2x^3 - 5x^2 - 52x + 60$
7. $f(x) = -x^3 - x^2 + 39x + 45$
8. $f(x) = -2x^4 + 7x^3 + 17x^2 - 28x - 36$
9. $f(x) = x^7 - 5x^6 - 24x^5 + 120x^4 - 25x^3 + 125x^2$

III - Practice Problems:

Factor each polynomial below, and sketch a complete graph of the function, making sure to have a clearly defined scale and label any intercepts. Use Desmos to compare your results.

1. $f(x) = -17x^3 + 5x^2 + 34x - 10$
2. $f(x) = x^4 - 9x^2 + 14$
3. $f(x) = 3x^4 - 14x^2 - 5$
4. $f(x) = 2x^4 - 7x^2 + 6$
5. $f(x) = x^5 - 2x^4 - x + 2$
6. $f(x) = 2x^5 + 3x^4 - 32x - 48$
7. $f(x) = x^6 - 6x^3 - 16$
8. $f(x) = 2x^6 - 7x^3 + 5$

Get a complete factorization of each polynomial below by first dividing the function by $x-1$. Then sketch a graph of the function, making sure to have a clearly defined scale and label any intercepts. Use Desmos to compare your results.

9. $f(x) = x^3 - 2x^2 - 5x + 6$
10. $f(x) = x^3 + 4x^2 - 11x + 6$
11. $f(x) = x^5 - x^4 - 37x^3 + 37x^2 + 36x - 36$
12. $f(x) = x^4 - 2x^3 + 5x^2 - 8x + 4$

Use the Rational Root Theorem and polynomial division to get a complete factorization of each polynomial function below. Then sketch a graph of the function, making sure to have a clearly defined scale and label any intercepts. Use Desmos to compare your results.

13. $f(x) = x^4 - 9x^2 - 4x + 12$
14. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
15. $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$
16. $f(x) = 3x^3 + 3x^2 - 11x - 10$
17. $f(x) = 6x^3 + 19x^2 - 34x - 40$
18. $f(x) = -2x^3 + 19x^2 - 49x + 20$
19. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$