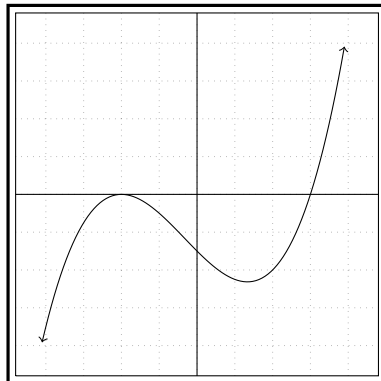


Chapter 1

Polynomials



Introduction and Terminology (L43)

Objective: Identify key features of and classify a polynomial by degree and number of nonzero terms.

A *polynomial* in terms of a variable x is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0,$$

where each *coefficient*, a_i , is a real number ($a_n \neq 0$) and the exponent, or *degree* of the polynomial, n , is a nonnegative integer.

Examples of polynomials include: $f(x) = x^2 + 5$, $f(x) = x$ and $f(x) = -3x^7 + 4x^3 - 5x$. Before classifying polynomials, we will take a moment to establish some key terminology. For our general polynomial above, the:

<i>degree</i>	is	n
<i>coefficients</i>	are	$a_n, a_{n-1}, \dots, a_1, a_0$
<i>leading coefficient</i>	is	a_n
<i>leading term</i>	is	$a_n x^n$
<i>constant term</i>	is	$a_0 x^0 = a_0$.

A concrete example will help to clarify each of these terms.

Example 1. Identify the degree, leading coefficient, leading term and constant term for the polynomial

$$f(x) = -19x^5 + 4x^4 - 6x + 21.$$

The degree of this polynomial is $n = 5$, since five is the greatest exponent.

The leading term, which is the term that contains the greatest exponent (degree), is $a_n x^n = -19x^5$.

The leading coefficient is the real number being multiplied by x^n in the leading term, namely $a_n = -19$.

The constant term is $a_0 = 21$, which also represents the y -intercept for the graph of the given polynomial, just as it did in the chapter on quadratics.

The complete set of coefficients for the given polynomial is

$$\{a_5 = -19, a_4 = 4, a_3 = 0, a_2 = 0, a_1 = -6, a_0 = 21\}.$$

It is important to point out the fact that the previous example contains no *cubic* or *quadratic* terms, since the respective coefficients are both zero. This example demonstrates that not every polynomial will contain a nonzero coefficient for every term. As another example, the *power function* $f(x) = x^{10}$ is also characterized as a polynomial having degree $n = 10$, leading coefficient $a_{10} = 1$, and trailing coefficients $a_i = 0$ for $i = 9, 8, \dots, 1, 0$.

Before we can identify and begin to classify a polynomial, we may need to simplify the given expression for x , by distributing and combining all like terms. The general form of a polynomial should be reminiscent of the standard form of a quadratic, with possibly more terms. Hence the name “polynomial”, meaning “many terms”.

The following example shows how to identify a polynomial after the necessary simplification has taken place.

Example 2. Identify the degree, leading coefficient, leading term and constant term for the given polynomial function.

$$\begin{aligned} f(x) &= 3(x+1)(x-1) + 4x^3 + 2x + 3 \\ &= 3(x^2 - 1) + 4x^3 + 2x + 3 \\ &= 3x^2 - 3 + 4x^3 + 2x + 3 \\ &= 4x^3 + 3x^2 + 2x \end{aligned}$$

Upon simplifying, we see that f has degree $n = 3$, since three is the greatest exponent.

The leading term is $4x^3$ with a leading coefficient of $a_n = 4$.

Since no constant term is shown, $a_0 = 0$ is our constant term.

Now that we can identify the essential components of a polynomial, we will categorize polynomials based upon their degree, as well as the number of terms, after all necessary simplification.

Types of Polynomials

Degree	Type	Example
0	Constant	-1
1	Linear	$2x + \sqrt{5}$
2	Quadratic	$5x^2 - 32x + 2$
3	Cubic	$(-1/2)x^3$
4	Quartic	$-3x^4 + 2x^2 + 3x + 1$
5	Quintic	$-2x^5$
6 or more	n^{th} Degree	$-2x^7 + 52x^6 + 12$

One point of note in the table above is the appearance of both rational and irrational coefficients ($-1/2$ and $\sqrt{5}$). The appearance of such coefficients is permissible in polynomials, since our coefficients a_i are only required to be real numbers. A coefficient containing the imaginary number $i = \sqrt{-1}$, on the other hand, is not permitted.

Polynomial Characterizations by Number of Nonzero Terms

Number of Terms	Name	Example
1	Monomial	$4x^5$
2	Binomial	$2x^3 + 1$
3	Trinomial	$-23x^{18} + 4x^2 + 3x$
4	Tetranomial	$-23x^{18} + 4x^2 + 3x + 1$
5 or more	Polynomial	$-2x^4 + x^3 + 15x^2 - 41x + 12$

Example 3. Describe the type and characterization (number of terms) of the polynomial function shown below.

$$f(x) = -19x^5 + 4x^4 - 6x + 21$$

Polynomials are typically named by their degree first and then their number of terms. The polynomial above is a *quintic tetranomial*; quintic because it is degree five and tetranomial because it contains four terms.

Example 4. Describe the type and characterization (number of terms) of the polynomial function shown below.

$$f(x) = x^3 + x^2$$

The polynomial above is a *cubic binomial*, since it has degree three and contains two terms.

Example 5. Describe the type and characterization (number of terms) of the polynomial function shown below.

$$f(x) = 21x^4 + 12x^2 - 3x^2 - 9x^2 - 22x^4$$

Upon simplifying, we see that the given polynomial reduces to $f(x) = -x^4$. As a result, our polynomial is a quartic (degree four) monomial (one term).

This section “sets the table” for the basic terminology that will be used throughout the chapter. In the next section, we will review some additional prerequisite factoring techniques which will be necessary for working with certain polynomials, and provide a brief summary of all factoring methods that have been discussed up to this point. Once we have finished our review of factoring, we will be ready to begin the natural (albeit lengthy) method of analyzing and graphing a polynomial function.

Sign Diagrams (L44)

Objective: Construct a sign diagram for a given polynomial expression.

If a polynomial function or expression is completely factored, it will be beneficial to us to construct a sign diagram for the polynomial, in order to answer questions about its graph and confirm any other findings. Therefore, we devote this section to the construction of a sign diagram for a factored polynomial. Note that expanded polynomials first require us to find a complete factorization prior to constructing a sign diagram. This will require us to first employ factoring techniques and possibly polynomial division, which we reserve for a later section.

Recall that the roots of a quadratic expression represent the dividers in its corresponding sign diagram. This carries over directly to a polynomial expression.

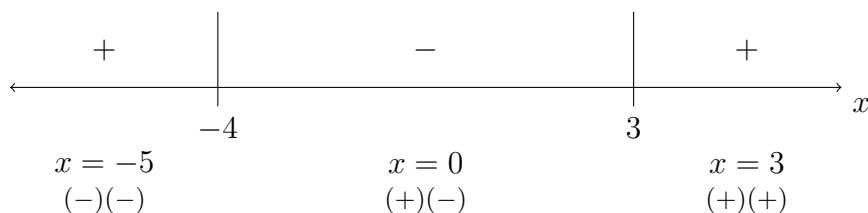
We begin with an example for quadratics.

Example 6. Construct a sign diagram for the polynomial function $f(x) = 2x^2 + 3x - 20$.

Although our first example is not factored, we can apply the ac -method to quickly factor our function.

$$\begin{aligned} f(x) &= 2x^2 + 3x - 20 \\ &= 2x^2 + 8x - 5x - 20 \\ &= 2x(x + 4) - 5(x + 4) \\ &= (x + 4)(2x - 5) \end{aligned}$$

This gives us two roots, $x = -4$ and $x = \frac{5}{2}$, which serve as the dividers in our accompanying diagram. For our three test values, we will use $x = -5, 0$, and 3 .



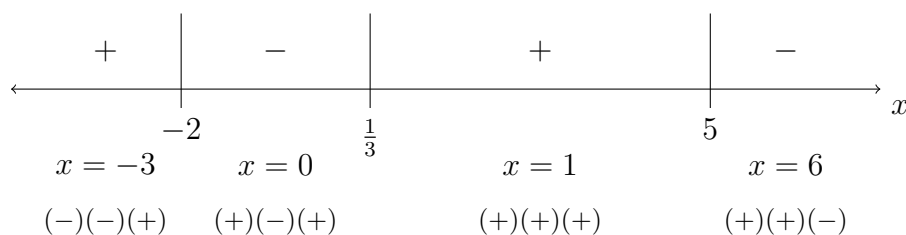
The previous example should be a familiar one, and one that we are comfortable with, since it ties in directly with the chapter on quadratics (degree-2 polynomials). For polynomials

with a degree of $n \geq 3$, our diagram should look similar. The primary exceptions will be number of factors in our expression, and the number of dividers in our diagram. Again, we will focus primarily on polynomials which are already factored for our examples.

Example 7. Construct a sign diagram for the factored polynomial function

$$g(x) = (x + 2)(3x - 1)(5 - x).$$

Our roots are $x = -2, \frac{1}{3}$, and 5. Consequently, the following diagram shows three dividers.

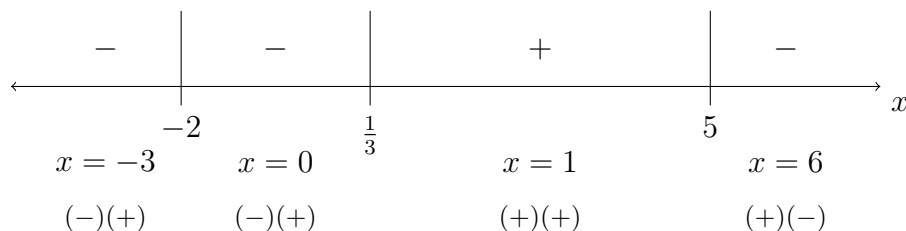


For our next example, we will make a slight change to the function g from the previous example, by including an extra factor of $x + 2$.

Example 8. Construct a sign diagram for the factored polynomial function

$$h(x) = (x + 2)^2(3x - 1)(5 - x).$$

Since the roots of h equal those from g , our diagram will have the same dividers and test values.



In the previous diagram, we see that each of our sign calculations have excluded the $(x + 2)^2$ factor, since it will always contribute a positive sign and therefore has no impact on the end result. For example, for the test value $x = -3$, we get

$$(-)^2(-)(+) = \cancel{(-)^2}(-)(+),$$

which reduces to a negative sign. This simplification in our sign calculation can be employed for any factor that appears in our function with an *even* exponent.

Additionally, our last two diagrams look almost identical, with the lone exception being the sign associated with our first interval, $(-\infty, -2)$. This should make some sense, however,

since we only changed the factor associated with the root $x = -2$ from one example to the next. The reason behind the change in diagram will become more clear to us as we explore polynomials further.

For our last example, we will present both the sign diagram and the accompanying graph for the given polynomial. Although the techniques to graphing a polynomial have not yet been discussed, for any function it is often helpful to utilize a graphing utility such as [Desmos](#), in order to better understand the makeup of the function and how its graph is related.

Example 9. Construct a sign diagram for the factored polynomial function

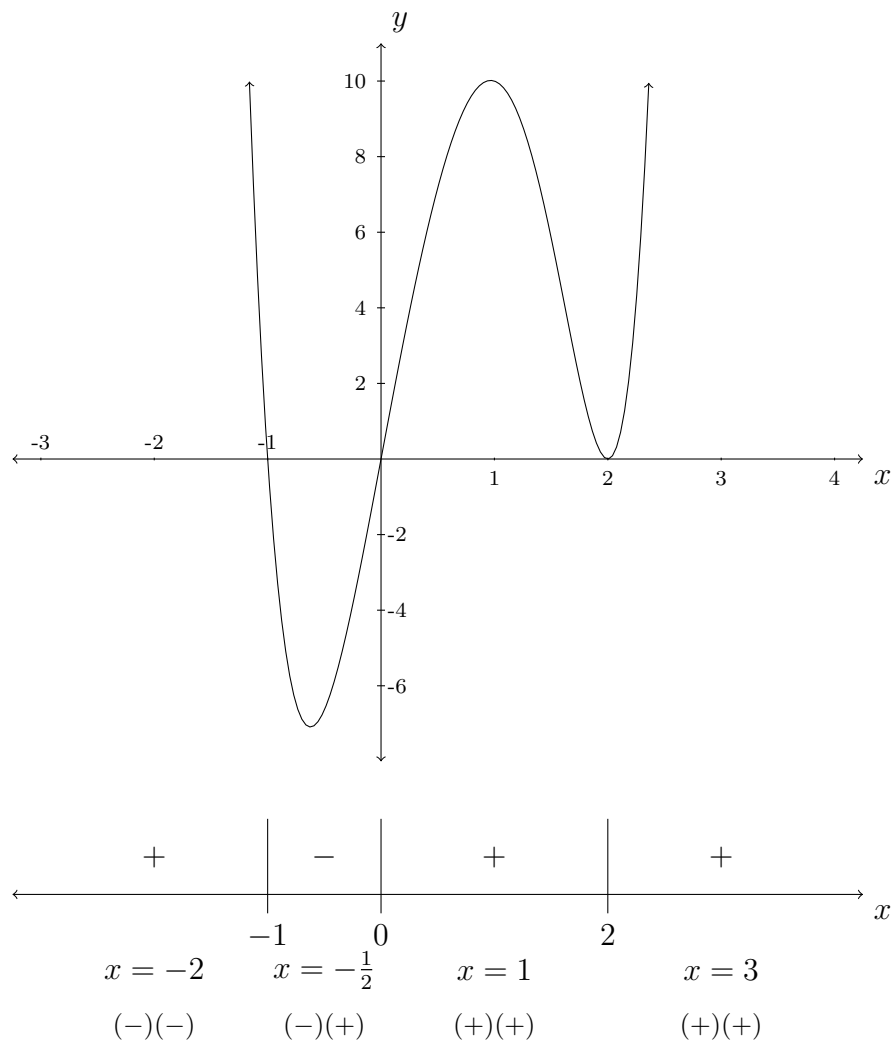
$$f(x) = x(x + 1)(x - 2)^2(x^2 + 4).$$

Use [Desmos](#) or a similar graphing utility to construct a graph of f .

Before we get started, it is important to spend some time discussing the factorization of f . Specifically, the factor of x will contribute a root of $x = 0$. This is the only instance in which our diagram requires a divider at $x = 0$.

Additionally, the factor of $x^2 + 4$ is often misinterpreted. By setting the expression equal to zero and solving for x , we see that the factor contributes two *imaginary* roots at $x = \pm 2i$. Furthermore, if we look more closely at this factor, we see that for any real number x , $x^2 + 4$ will always be positive. Hence, this factor will have no impact on our sign diagram calculations, and will be omitted. One should caution, however, that this factor does have an impact on the graph of f .

We can now conclude that the set of roots for f are $x = -1, 0$, and 2 . The accompanying diagram and graph are shown below. As before, we have also omitted the factor of $(x - 2)^2$, since the squared factor will not impact our signs.



By looking at the graph of our last example, one should begin to notice the relationship that the graph of a polynomial has with its precise makeup and, consequently, its sign diagram. In particular, close attention should be paid to the nature of the graph of f near its real roots. In the case of $x = -1$ and $x = 0$ in our last example, we see that the graph *crosses over* the x -axis. Alternatively, our graph *turns around* or “bounces off” at the root $x = 2$. This difference in the local behavior of the graph of f at its roots is not just a coincidence, but rather a consequence of the makeup of the function f , as we will see when we discuss the *multiplicity* of the root of a polynomial in a later section.

Factoring

Some Special Cases (L45)

Objective: Factor a general polynomial expression using one or more factorization methods.

When factoring polynomials there are a few special products that, if we can recognize them, can be easily broken down. The first is one we have seen before, when factoring some quadratics in which there is no linear term.

When expanding, we know that the product of a sum and difference of the same two terms results in a difference of two squares.

Difference of Two Squares: $a^2 - b^2 = (a + b)(a - b)$

Consequently, if faced with the difference of two squares, one can conclude that such an expression will always factor as a product of the sum and difference of their square roots. Our first four examples demonstrate this fact.

Example 10. Factor each of the given binomial expressions completely over the real numbers.

1. $x^2 - 16$
2. $9x^2 - 25y^2$
3. $x^2 - 24$
4. $2x^2 - 5$

1. In this first example, we see that $a = x$ and $b = 4$ for our difference of two squares.

$$\begin{aligned} x^2 - 16 &= (x)^2 - (4)^2 \\ &= (x + 4)(x - 4) \end{aligned}$$

2. Taking the square roots of $9x^2$ and $25y^2$ gives us $a = 3x$ and $b = 5y$ for our second expression.

$$\begin{aligned} 9x^2 - 25y^2 &= (3x)^2 - (5y)^2 \\ &= (3x + 5y)(3x - 5y) \end{aligned}$$

3. Our third expression poses a bit of a challenge, since it is the first which does not present us with the difference of two *perfect* squares. In this case, $a = x$, but $b = \sqrt{24} = \sqrt{4 \cdot 6} = 2\sqrt{6}$.

$$\begin{aligned} x^2 - 24 &= (x)^2 - (2\sqrt{6})^2 \\ &= (x + 2\sqrt{6})(x - 2\sqrt{6}) \end{aligned}$$

4. Similarly, our final expression presents us with two terms, neither of which are perfect squares. In this case, $a = \sqrt{2x^2} = \sqrt{2}x$ and $b = \sqrt{5}$.

$$\begin{aligned} 2x^2 - 5 &= \left(\sqrt{2}x\right)^2 - \left(\sqrt{5}\right)^2 \\ &= \left(\sqrt{2}x + \sqrt{5}\right)\left(\sqrt{2}x - \sqrt{5}\right) \end{aligned}$$

Note that in this last case, we have $\sqrt{2}x$ (or $x\sqrt{2}$) in our factorization, and not $\sqrt{2x}$.

It is important to note that, unlike differences, a *sum* of squares will never factor over the real numbers. Such expressions only factor over the complex numbers. Hence, we say that they are *irreducible* over the reals. This can be seen in our next example, where we will attempt to employ the *ac*-method to factor.

Example 11. Factor the expression $x^2 + 36$ completely over the real numbers and over the complex numbers.

For the expression $x^2 + 36$, $ac = 36$ and $b = 0$, as we have no linear term. So we need to identify two integers, m and n , such that $m + n = 0$ and $m \cdot n = 36$. Our choices for $m \cdot n$ are $1 \cdot 36$, $2 \cdot 18$, $3 \cdot 12$, $4 \cdot 9$ and $6 \cdot 6$. But, since there are no combinations from these that will both multiply to 36 *and* add to 0, we conclude that the given expression is irreducible over the reals.

Notice that $x^2 + 36$ does, however, factor over the complex numbers.

$$\begin{aligned} x^2 + 36 &= x^2 - (-36) \\ &= x^2 - \left(\sqrt{-36}\right)^2 \\ &= x^2 - \left(\sqrt{36}\sqrt{-1}\right)^2 \\ &= x^2 - (6i)^2 \\ &= (x - 6i)(x + 6i) \end{aligned}$$

We can further make sense of this result by recalling the methods from the chapter on quadratics. Since the discriminant of $x^2 + 36$ is

$$\begin{aligned} b^2 - 4ac &= 0^2 - 4(1)(36) \\ &= -144 \\ &< 0, \end{aligned}$$

we know that the given expression has no real roots. Hence, any factorization must contain imaginary numbers. By setting the expression equal to zero and extracting square roots, we get $x = \pm 6i$, which further supports our factorization.

We present the general factorization for the sum of two squares over the complex numbers below.

Sum of Two Squares: $a^2 + b^2 = (a + bi)(a - bi)$
--

For graphing purposes, we will primarily be concerned with factorization over the real numbers.

In many cases, we can also recognize an entire expression as a perfect square (or a squared binomial).

Perfect Square: $a^2 + 2ab + b^2 = (a + b)^2$

While it might seem difficult to recognize a perfect square at first glance, by employing the ac -method, we can see that in the case where $m = n$, the resulting factorization will be a perfect square. In this case, we can factor by identifying the square roots of the first and last terms and using the sign from the middle term. This is demonstrated in the following example.

Example 12. Factor each of the given trinomial expressions completely over the real numbers.

1. $x^2 - 6x + 9$

2. $4x^2 + 20xy + 25y^2$

1. For our first expression, $a = 1$, $b = -6$, and $c = 9$. So we must find two integers m and n such that $m + n = -6$ and $mn = ac = 9$. In this case, the numbers we need are -3 and -3 . Consequently, we will have a perfect square.

Using the square roots of $a = 1$ and $c = 9$ and the negative sign from the linear term, our factorization is

$$x^2 - 6x + 9 = (x - 3)^2.$$

2. For our second expression, $a = 4$, $b = 20$, and $c = 25$. So we are looking for an m and n such that $m + n = 20$ and $mn = ac = 100$. Quickly, we see that $m = n = 10$, and again, we have a perfect square.

In this case, our factorization is

$$4x^2 + 20xy + 25y^2 = (2x + 5y)^2.$$

Another factoring shortcut involves sums and differences of cubes. Both sums and differences of cubes have very similar factorizations.

Sum of Cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ Difference of Cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

As with all of the formulas in this section, we can verify those for a sum and difference of cubes by expanding the right-hand side. For example, for the difference of cubes,

$$\begin{aligned} (a - b)(a^2 + ab + b^2) &= a(a^2 + ab + b^2) - b(a^2 + ab + b^2) \\ &= a^3 + \cancel{a^2b} + \cancel{ab^2} - \cancel{a^2b} - \cancel{ab^2} - b^3 \\ &= a^3 - b^3 \end{aligned}$$

Comparing the formulas one may notice that the only difference resides in the signs between the terms. One way to remember these two formulas is to think of “**SOAP**”:

- S** The first sign in our factorization is the **Same** sign as the given expression.
- O** The second sign in our factorization is the **Opposite** sign as the given expression.
- AP** The last sign in our factorization is **Always Positive**.

Example 13. Factor each of the given binomial expressions completely over the real numbers.

1. $m^3 - 27$

2. $125p^3 + 8r^3$

1. In our first expression, our desired cube roots for each term are $a = m$ and $b = 3$. Using the “Same, Opposite, Always Positive” acronym, we have the following factorization.

$$\begin{aligned} m^3 - 27 &= (m - 3) ((m)^2 + 3m + (3)^2) \\ &= (m - 3) (m^2 + 3m + 9) \end{aligned}$$

2. In second expression, our desired cube roots for each term are

$$\begin{aligned} a &= \sqrt[3]{125p^3} & b &= \sqrt[3]{8r^3} \\ &= \sqrt[3]{125} \sqrt[3]{p^3} & &= \sqrt[3]{8} \sqrt[3]{r^3} \\ &= 5p & &= 2r \end{aligned}$$

Using the “Same, Opposite, Always Positive” acronym, we have the following factorization.

$$\begin{aligned} 125p^3 + 8r^3 &= (5p + 2r) ((5p)^2 - (5p)(2r) + (2r)^2) \\ &= (5p + 2r) (25p^2 - 10pr + 4r^2) \end{aligned}$$

The second expression in our last example illustrates an important point. When we identify the first and last terms of the trinomial in our factorization, we must square each cube root in its entirety. In this case, both the coefficients and variables are squared, so that $(5p)^2$ becomes $25p^2$, and $(2r)^2$ becomes $4r^2$.

After factoring a sum or difference of cubes, it should be natural to attempt to factor the resulting trinomial expression (our second factor). As a general rule, however, this factor should always be irreducible over the reals, with the main exception being that of a GCF in the given expression that might have been initially overlooked.

Our last special case comes up frequently enough that we will devote the next subsection to it.

Quadratic Type (L46)

Objective: Recognize and factor a polynomial expression of quadratic type.

Recall that a quadratic expression in terms of a variable x is an expression of the form

$$ax^2 + bx + c.$$

If y is any algebraic expression, we say that the expression

$$ay^2 + by + c$$

is an expression of *quadratic type*.

In just about every case we will see, we will consider y as a power of x , $y = x^n$, so that our expression of quadratic type will appear as follows.

<p>Quadratic Type:</p> $ax^{2n} + bx^n + c = a[x^n]^2 + b[x^n] + c$

If $y = x^3$, then the expression

$$ay^2 + by + c = ax^6 + bx^3 + c$$

would be an expression of quadratic type.

Similarly, if $y = x^4$, then the expression

$$ay^2 + by + c = ax^8 + bx^4 + c$$

would be an expression of quadratic type.

In each of these last two examples, notice the exponential pattern, where the middle term has an exponent that is half that of the leading term's. This will always be apparent, as long as the middle coefficient b is nonzero.

By viewing certain expressions as quadratic type, we can often apply more familiar methods, such as the ac -method, to obtain a complete factorization.

For example, if we let $y = x^2$, then the difference of fourth powers $x^4 - 16$ can be rewritten as a difference of squares, $y^2 - 4^2$, leading us to the complete factorization over the real numbers shown below.

$$\begin{aligned}
 x^4 - 16 &= (x^2)^2 - 4^2 \\
 &= y^2 - 4^2, \quad y = x^2 \\
 &= (y + 4)(y - 4) \\
 &= (x^2 + 4)(x^2 - 4) \\
 &= (x^2 + 4)(x + 2)(x - 2)
 \end{aligned}$$

Example 14. Factor the trinomial expression $x^4 + 2x^2 - 24$ completely over the real numbers.

Notice that the given trinomial exhibits quadratic type characteristics, since the degree of four is double the exponent appearing in the middle term. Consequently, we will let $y = x^2$ and rewrite the expression in terms of y .

$$y^2 + 2y - 24$$

Applying the ac -method, we see the following.

$$\begin{aligned} y^2 + 2y - 24 &= y^2 + 6y - 4y - 24 \\ &= y(y + 6) - 4(y + 6) \\ &= (y + 6)(y - 4) \end{aligned}$$

Substituting back for x , we have $(x^2 + 6)(x^2 - 4)$. The first factor is a sum of squares, which is irreducible over the reals. The second factor of $x^2 - 4$ is a difference of perfect squares, which we know is factorable as $(x + 2)(x - 2)$.

Our final factorization is

$$x^4 + 2x^2 - 24 = (x^2 + 6)(x + 2)(x - 2).$$

Example 15. Find all real roots of the polynomial expression $x^4 - 12x^2 + 27$.

In this example, we are not asked to factor the given expression, but instead to solve for when the expression equals zero. Still, we can start by finding a complete factorization, again substituting $y = x^2$ and employing the ac -method.

$$\begin{aligned} x^4 - 12x^2 + 27 &= y^2 - 12y + 27 \\ &= y^2 - 3y - 9y + 27 \\ &= y(y - 3) - 9(y - 3) \\ &= (y - 3)(y - 9) \\ &= (x^2 - 3)(x^2 - 9) \\ &= (x + \sqrt{3})(x - \sqrt{3})(x - 3)(x + 3) \end{aligned}$$

Here, we see that after using the ac -method and substituting back for x , we end up with two quadratic factors which can *both* be factored as a difference of squares. In the case of the first factor, $x^2 - 3$, our factorization requires a square root, since 3 is not a perfect square.

Setting each of the four factors equal to zero gives us our set of real roots, $\{\pm\sqrt{3}, \pm 3\}$.

In each of our last two examples, we have seen a degree-four polynomial having two and four real roots, respectively. We can also easily identify degree-four polynomials having no, one, or three real roots. The expression $x^4 + x^2 + 1$, for example, factors as $(x^2 + 1)^2$, which has only complex roots at $\pm i$. In general, a degree- n polynomial can have as few as zero and as many as n unique real roots. This is a fact which we will more formally state in a later section, once we have discussed the *multiplicity* of a root.

The following example should look familiar.

Example 16. Factor each of the following polynomial expressions completely over the real numbers.

1. $x^8 + 2x^4 - 24$

2. $x^6 + 2x^3 - 24$

Before we begin, notice that the coefficients for each of the given expressions match those in Example 14, with the only difference being the exponents appearing in each expression.

1. Despite the fact that the first expression has a higher degree, its factorization will be simpler than the second expression's. In this case, we will let $y = x^4$, and apply the *ac*-method as before.

$$\begin{aligned} x^8 + 2x^4 - 24 &= (x^4)^2 + 2(x^4) - 24 \\ &= y^2 + 2y - 24 \\ &= (y + 6)(y - 4) \\ &= (x^4 + 6)(x^4 - 4) \end{aligned}$$

Though it might not be obvious, our first factor $x^4 + 6$ is in fact irreducible over the real numbers. One way we can realize this is to think of $x^4 + 6$ as a vertical shift of the graph of x^4 up six units. The resulting graph will lie entirely in the upper-half of the xy -plane, and therefore will not intersect the x -axis. Hence, the factor of $x^4 + 6$ will have no real roots, and consequently any factorization will involve the introduction of imaginary numbers. Alternatively, one might also notice that raising any real number to the fourth power and adding six will never produce an output of zero, leading us to again conclude that the expression has no real roots. Lastly, we could also recognize $x^4 + 6$ as a sum of squares, namely $(x^2)^2 + (\sqrt{6})^2$, which we have already discussed as one expression type that is irreducible over the real numbers.

On the other hand, we can view our second factor $x^4 - 4$ as a difference of two squares, and factor it as follows.

$$\begin{aligned} x^4 - 4 &= (x^2)^2 - 2^2 \\ &= (x^2 + 2)(x^2 - 2) \\ &= (x^2 + 2)(x + \sqrt{2})(x - \sqrt{2}) \end{aligned}$$

Our complete factorization over the reals is then

$$x^8 + 2x^4 - 24 = (x^4 + 6)(x^2 + 2)(x + \sqrt{2})(x - \sqrt{2}).$$

2. In the case of the second expression, if we let $y = x^3$, we start out with the same two factors for y , which we rewrite as

$$(x^3 + 6)(x^3 - 4).$$

Although neither 6 nor 4 are perfect cubes, we can still break down each of the factors above by using the formulas for the sum and difference of cubes from earlier in the section. For our first factor, letting $a = x$ and $b = \sqrt[3]{6}$, we can write $x^3 + 6$ as

$$(a + b)(a^2 - ab + b^2) = (x + \sqrt[3]{6}) \left(x^2 - \sqrt[3]{6}x + (\sqrt[3]{6})^2 \right).$$

Similarly, for the second factor, if $a = x$ and $b = \sqrt[3]{4}$, we can write $x^3 - 4$ as

$$(a - b)(a^2 + ab + b^2) = (x - \sqrt[3]{4}) \left(x^2 + \sqrt[3]{4}x + (\sqrt[3]{4})^2 \right).$$

Our complete factorization over the reals is then

$$\begin{aligned} x^6 + 2x^3 - 24 &= (x^3 + 6)(x^3 - 4) \\ &= (x + \sqrt[3]{6}) \left(x^2 - \sqrt[3]{6}x + (\sqrt[3]{6})^2 \right) (x - \sqrt[3]{4}) \left(x^2 + \sqrt[3]{4}x + (\sqrt[3]{4})^2 \right). \end{aligned}$$

We end the subsection on quadratic type with one final example.

Example 17. Factor each polynomial expression completely over the reals and find its set of real roots.

1. $x^4 - 49$

2. $x^6 - 4x^3 - 5$

- Setting $y = x^2$, we can quickly factor our first expression as a difference of squares, breaking down one of its factors in a similar manner.

$$\begin{aligned} x^4 - 49 &= (x^2)^2 - 49 \\ &= y^2 - (7)^2 \\ &= (y + 7)(y - 7) \\ &= (x^2 + 7)(x^2 - 7) \\ &= (x^2 + 7)(x + \sqrt{7})(x - \sqrt{7}) \end{aligned}$$

From our two linear factors, we obtain $x = \pm\sqrt{7}$ as our two real roots.

- For our second expression, setting $y = x^3$, we apply the ac -method.

$$\begin{aligned} x^6 - 4x^3 - 5 &= (x^3)^2 - 4(x^3) - 5 \\ &= y^2 - 4y - 5 \\ &= (y + 1)(y - 5) \\ &= (x^3 + 1)(x^3 - 5) \\ &= (x^3 + 1)(x + \sqrt[3]{5})(x - \sqrt[3]{5}) \end{aligned}$$

Our two linear factors give us $x = \pm\sqrt[3]{5}$ as the real roots for our given expression.

Division

Polynomial (Long) Division (L47)

Objective: Apply polynomial division to a rational expression.

Up until this point, every polynomial expression that we have encountered has either already been provided in its factored form or is easily factorable using one or more of the many techniques that we have learned. We must be careful, however, to consider the very likely possibility that a given polynomial is not factorable using elementary methods (GCF, grouping, *ac*-method, etc.). In many cases, obtaining a complete factorization can prove almost impossible without the aid of mathematical computing software. Although, there is still one powerful tool that can help us to dissect certain factorable, yet formidable, polynomials. This tool is known as the *Rational Root Theorem*, and we will see its use in a later section.

In order to successfully employ the Rational Root Theorem, we first must understand polynomial division. As we will see, dividing polynomials is a process very similar to long division of whole numbers.

Before we begin with our first example, let's recall the terminology and format associated with division.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

Alternatively, multiplying both sides of the above equation by the divisor, we have the following.

$$\begin{aligned} \frac{\text{dividend}}{\text{divisor}} \cdot \cancel{\text{divisor}} &= \text{quotient} \cdot \text{divisor} + \frac{\text{remainder}}{\text{divisor}} \cdot \cancel{\text{divisor}} \\ \text{dividend} &= \text{quotient} \cdot \text{divisor} + \text{remainder} \end{aligned}$$

We begin with dividing a polynomial by a monomial, which simply utilizes the distributive property. In the following example, we can think of the stated division as a distribution of the divisor (denominator) to each term in the dividend (numerator).

Example 18. Divide and simplify the following expressions.

1. $\frac{9x^5 + 6x^4 - 18x^3 - 24x^2}{3x^2}$

2. $\frac{8x^3 + 4x^2 - 2x + 6}{4x^2}$

1. By distributing the expression $\frac{1}{3x^2}$ to each of the four terms in the numerator, our first expression becomes

$$\frac{9x^5 + 6x^4 - 18x^3 - 24x^2}{3x^2} = \frac{9x^5}{3x^2} + \frac{6x^4}{3x^2} - \frac{18x^3}{3x^2} - \frac{24x^2}{3x^2}.$$

We can then reduce each individual quotient to produce the following expression.

$$3x^3 + 2x^2 - 6x - 8$$

In this case, our answer is $3x^3 + 2x^2 - 6x - 8$, and we can summarize our results as follows.

$$\frac{9x^5 + 6x^4 - 18x^3 - 24x^2}{3x^2} = 3x^3 + 2x^2 - 6x - 8$$

In this first example, our expression reduced completely, producing our quotient polynomial expression and a remainder of zero.

2. Again, we will begin with the second expression by splitting it up, or distributing the denominator to each of the three terms in the numerator. Reducing each individual quotient gives us our answer.

$$\begin{aligned}\frac{8x^3 + 4x^2 - 2x + 6}{4x^2} &= \frac{8x^3}{4x^2} + \frac{4x^2}{4x^2} - \frac{2x}{4x^2} + \frac{6}{4x^2} \\ &= 2x + 1 - \frac{1}{2x} + \frac{3}{2x^2}\end{aligned}$$

Unlike our first expression, here our expression does not reduce completely, i.e., it contains a nonzero remainder. Because of this, since our answer includes two rational (or fractional) expressions, one could also combine them to form a single rational expression.

$$\begin{aligned}\frac{8x^3 + 4x^2 - 2x + 6}{4x^2} &= 2x + 1 - \frac{1}{2x} \cdot \frac{x}{x} + \frac{3}{2x^2} \\ &= 2x + 1 - \frac{x}{2x^2} + \frac{3}{2x^2} \\ &= 2x + 1 + \frac{3 - x}{2x^2}\end{aligned}$$

Furthermore, if we wanted to identify the remainder in this example, we could multiply both sides of the equation by the divisor, $4x^2$.

$$\begin{aligned}\frac{8x^3 + 4x^2 - 2x + 6}{\cancel{4x^2}} \cdot \cancel{4x^2} &= (2x + 1) \cdot 4x^2 + \left(\frac{3 - x}{\cancel{2x^2}}\right) \cdot \cancel{4x^2} \\ \underbrace{8x^3 + 4x^2 - 2x + 6}_{\text{dividend}} &= \underbrace{(2x + 1)}_{\text{quotient}} \underbrace{4x^2}_{\text{divisor}} + \underbrace{6 - 2x}_{\text{remainder}}\end{aligned}$$

Lastly, one final observation to point out from this particular answer is the initial reduction of the second term, $\frac{4x^2}{4x^2}$, which equals one (not zero), and therefore does not simply disappear from the expression altogether.

For division by polynomial expressions that contain more than a single term, long division is usually required. To illustrate the relationship between polynomial division and standard numerical long division, an example with whole numbers is provided in order to review the (general) steps that will also be used for polynomial long division.

Example 19. Divide 631 by 4.

$$\begin{array}{r} 157 \\ 4 \overline{) 631} \\ \underline{400} \\ 231 \\ \underline{200} \\ 31 \\ \underline{28} \\ 3 \end{array}$$

Recalling the process associated with long division, we begin to construct our quotient (157) by comparing the divisor (4) with the largest placeholder of our dividend (631), then subtracting, and repeating this process until we have worked our way down to the ones digit. We know we have finished, once we are left with a remainder (3) that is less than our divisor.

Expressing our answer in the form

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}},$$

we have $\frac{631}{4} = 157 + \frac{3}{4}$. Or, in the alternate form

$$\text{dividend} = \text{quotient} \cdot \text{divisor} + \text{remainder},$$

we have $631 = 157 \cdot 4 + 3$.

The general process for division of polynomials follows closely with that for dividing integers. The only real difference is in the terminology that we use: *term* in place of *number/digit*, and *degree* instead of *value*.

General Steps for Polynomial (Long) Division

Let $D(x)$ and $d(x)$ represent two nonzero polynomial functions. The steps for simplifying the rational expression $\frac{D(x)}{d(x)}$ are as follows.

1. Divide the leading term of the dividend D by the leading term of the divisor d . Label the resulting term $a_n x^n$, and write it above the dividend. This will be the leading term of the quotient, $q(x)$.
2. Multiply $a_n x^n$ by the divisor, distribute, and simplify. Label this as $d_1(x)$ and write it directly below the dividend, D , making sure to align terms according to exponents.
3. Subtract the resulting terms from the dividend. Label the new expression D_1 .
4. Repeat steps (1)-(3) for the divisor d and the new expression D_i until the degree of D_i is *less than* the degree of the divisor. Relabel the final new dividend as the remainder, $r(x)$. The entire polynomial expression appearing above the original dividend is the quotient, $q(x)$.

$$\begin{array}{r}
 q(x) \\
 d(x) \overline{) D(x)} \\
 \underline{- d_1(x)} \\
 D_1(x) \\
 \underline{- d_2(x)} \\
 \dots \\
 \underline{- d_i(x)} \\
 D_i(x) = r(x)
 \end{array}$$

Step (3) above often tends to pose the greatest challenge for students. It is important to keep in mind that we are always subtracting the top term from the bottom term, which is why we must change the signs of the term(s) on the bottom. In most cases, we will need to utilize the distributive property.

A basic example should clear up any confusion, and we begin by revisiting Example 18.

Example 20. Divide $9x^5 + 6x^4 - 18x^3 - 24x^2$ by $3x^2$. Simplify and express your answer in the form

$$\frac{D(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

$$3x^2 \overline{) 9x^5 + 6x^4 - 18x^3 - 24x^2}$$

We set up our division process by first writing the dividend and the divisor in the appropriate locations.

$$\begin{array}{r}
 3x^3 \\
 3x^2 \overline{) 9x^5 + 6x^4 - 18x^3 - 24x^2} \\
 \underline{9x^5} \\
 6x^4
 \end{array}$$

Notice that in this example we need only carry down the remaining terms from the dividend when our new expression contains terms which are alike to them. In this case, for example, while it is perfectly fine to carry down $-18x^3 - 24x^2$, it is not necessary until these terms play a role in the subtraction step.

$$\begin{array}{r}
 3x^3 + 2x^2 \\
 3x^2 \overline{) 9x^5 + 6x^4 - 18x^3 - 24x^2} \\
 \underline{9x^5} \\
 6x^4 \\
 \underline{6x^4} \\
 -18x^3 \\
 3x^3 + 2x^2 - 6x \\
 3x^2 \overline{) 9x^5 + 6x^4 - 18x^3 - 24x^2} \\
 \underline{9x^5} \\
 6x^4 \\
 \underline{6x^4} \\
 -18x^3 \\
 18x^3 \\
 \underline{18x^3} \\
 -24x^2 \\
 3x^3 + 2x^2 - 6x - 8 \\
 3x^2 \overline{) 9x^5 + 6x^4 - 18x^3 - 24x^2} \\
 \underline{9x^5} \\
 6x^4 \\
 \underline{6x^4} \\
 -18x^3 \\
 18x^3 \\
 \underline{18x^3} \\
 -24x^2 \\
 24x^2 \\
 \underline{24x^2} \\
 0
 \end{array}$$

$$\frac{9x^5 + 6x^4 - 18x^3 - 24x^2}{3x^2} = 3x^3 + 2x^2 - 6x - 8 + \frac{0}{3x^2}$$

Next, we identify the leading term for our quotient, $3x^3$.

Multiplying and subtracting produces our new expression, $D_1(x) = 6x^4$.

Repeating our division steps gives us the second term in our quotient, $2x^2$.

Multiplying this term by the divisor, subtracting, and carrying down the next term in the dividend finishes up the second round of our steps for division. Our new expression is $D_2(x) = -18x^3$.

Again, we repeat our division steps to produce the third term in our quotient, $-6x$.

Multiplying and subtracting produces another new expression, $D_3(x) = -24x^2$. Since the degree of D_3 equals that of our divisor, $d(x) = 3x^2$, we will need to apply our steps for division one final time.

After our fourth and final round of steps, our new expression produces a remainder of $r(x) = 0$. This should come as no real surprise, based upon our earlier calculations from Example 18. Many examples that we will encounter after this first one will not work out as nicely.

We express the results of our division in the required form as follows.

Before we begin, we wish to point out the important prerequisite for polynomial division that all expressions be written in *descending power order*. In this case, we will start out by rewriting our divisor as $2x + 4$.

$$\begin{array}{r}
 3x^2 - 10x + 25 \\
 2x + 4 \overline{) 6x^3 - 8x^2 + 10x + 103} \\
 \underline{- 6x^3 - 12x^2} \\
 -20x^2 + 10x \\
 \underline{20x^2 + 40x} \\
 50x + 103 \\
 \underline{- 50x - 100} \\
 3
 \end{array}$$

This example is similar to the previous one. Specifically, the divisor is a linear binomial, and the dividend has a degree of 3.

Consequently, the steps for polynomial division are employed three times, yielding a constant remainder.

Our answer should be expressed as follows.

$$\frac{6x^3 - 8x^2 + 10x + 103}{2x + 4} = 3x^2 - 10x + 25 + \frac{3}{2x + 4}$$

In each of the previous two examples the dividend contained only nonzero coefficients. In other words, no term was “skipped over” in the expression for $D(x)$. Our last example will address the importance of keeping track of polynomial *place holders*, in the event that a specific term carries with it a zero coefficient, and is therefore omitted from the original expression for the dividend, $D(x)$.

Our last example demonstrates the importance of these preliminary steps.

Example 23. Divide and simplify the given expression.

$$\frac{2x^4 + 42x - 4x^2}{x^2 + 3x} \quad \text{Reorder dividend; need } x^3 \text{ term, add } 0x^3$$

$$x^2 + 3x \overline{) 2x^4 + 0x^3 - 4x^2 + 42x} \quad \text{Divide the leading terms : } \frac{2x^4}{x^2} = 2x^2$$

$$\begin{array}{r}
 2x^2 \\
 x^2 + 3x \overline{) 2x^4 + 0x^3 - 4x^2 + 42x} \\
 \underline{- 2x^4 - 6x^3} \\
 -6x^3 - 4x^2
 \end{array}$$

Multiply this term by divisor : $2x^2(x^2 + 3x) = 2x^4 + 6x^3$
 Subtract, changing terms
 Bring down the next term, $-4x^2$

$$\begin{array}{r}
 2x^2 - 6x \\
 x^2 + 3x \overline{) 2x^4 + 0x^3 - 4x^2 + 42x} \\
 \underline{- 2x^4 - 6x^3} \\
 -6x^3 - 4x^2 \\
 \underline{+ 6x^3 + 18x^2} \\
 14x^2 + 42x
 \end{array}$$

Repeat, divide new leading term by x^2 : $\frac{-6x^3}{x^2} = -6x$
 Multiply this term by divisor : $-6x(x^2 + 3x) = -6x^3 - 18x^2$
 Subtract, changing signs
 Bring down the next term, $42x$

$$\begin{array}{r}
 2x^2 - 6x + 14 \\
 x^2 + 3x \overline{) 2x^4 + 0x^3 - 4x^2 + 42x} \\
 \underline{-2x^4 - 6x^3} \\
 -6x^3 - 4x^2 \\
 \underline{+6x^3 + 18x^2} \\
 14x^2 + 42x \\
 \underline{-14x^2 - 42x} \\
 0
 \end{array}$$

Repeat, divide new leading term by $x^2 : \frac{14x^2}{x^2} = 14$

Multiply this term by the divisor : $14(x^2 + 3x) = 14x^2 + 42x$

Subtract, changing signs

Zero remainder

$2x^2 - 6x + 14$ Our solution

So we have,

$$\frac{2x^4 - 4x^2 + 42x}{x^2 + 3x} = 2x^2 - 6x + 14$$

It is important to take a moment to check each problem, to verify that the exponents decrease incrementally and that none are skipped.

This final example also illustrates that, just as with classic numerical long division, sometimes our remainder will be zero.

Synthetic Division (L48)

Objective: Apply synthetic division to a rational expression.

Next, we will introduce a method of division that can be used to streamline the polynomial division process and is often preferred over the more traditional long division method. This method, known as *synthetic division*, although usually quicker than traditional polynomial division, this alternative method can only be implemented when the given divisor is *linear*. Specifically, we will require $d(x)$ to be of the form $x - c$.

For our first example, we will divide $x^3 + 4x^2 - 5x - 14$ by $x - 2$, which one can check will produce a quotient of $x^2 + 6x + 7$ and a remainder of zero using polynomial long division.

$$\frac{x^3 + 4x^2 - 5x - 14}{x - 2} = x^2 + 6x + 7$$

The method of synthetic division focuses primarily on the coefficients of both the divisor and dividend. We must still pay careful attention, however, to the powers of our exponents, which will serve as placeholders throughout the process. To start the process, we will write our coefficients in what we will refer to as a *synthetic division tableau* prior to dividing.

To divide $x^3 + 4x^2 - 5x - 14$ by $x - 2$, we first write 2 in the place of the divisor since 2 is zero of the factor $x - 2$ and we write the coefficients of $x^3 + 4x^2 - 5x - 14$ in for the dividend.

As our next step, we ‘bring down’ the first coefficient of the dividend. We will then multiply and add repeatedly.

$$\begin{array}{r|rrrrr} 2 & 1 & 4 & -5 & -14 & \\ & & & & & \\ \hline & & & & & \end{array} \qquad \begin{array}{r|rrrrr} 2 & 1 & 4 & -5 & -14 & \\ & & & & & \\ \hline & & & & & 1 \end{array}$$

Next, take the 2 from the divisor and multiply by the 1 that was brought down to get 2. Write this underneath the 4, then add to get 6.

$$\begin{array}{r|rrrrr} 2 & 1 & 4 & -5 & -14 & \\ & & 2 & & & \\ \hline & & & & & \\ 1 & & 6 & & & \end{array} \qquad \begin{array}{r|rrrrr} 2 & 1 & 4 & -5 & -14 & \\ & & 2 & & & \\ \hline & & & & & \\ 1 & & 6 & & & \end{array}$$

Now multiply the 2 from the divisor by the 6 to get 12, and add it to the -5 to get 7.

$$\begin{array}{r|rrrrr} 2 & 1 & 4 & -5 & -14 & \\ & & 2 & 12 & & \\ \hline & & & & & \\ 1 & & 6 & & & \end{array} \qquad \begin{array}{r|rrrrr} 2 & 1 & 4 & -5 & -14 & \\ & & 2 & 12 & & \\ \hline & & & & & \\ 1 & & 6 & 7 & & \end{array}$$

Finally, multiply the 2 in the divisor by the 7 to get 14, and add it to the -14 to get 0.

$$\begin{array}{r|rrrrr} 2 & 1 & 4 & -5 & -14 & \\ & & 2 & 12 & 14 & \\ \hline & & & & & \\ 1 & & 6 & 7 & & \end{array} \qquad \begin{array}{r|rrrrr} 2 & 1 & 4 & -5 & -14 & \\ & & 2 & 12 & 14 & \\ \hline & & & & & \\ 1 & & 6 & 7 & \mathbf{0} & \end{array}$$

The first three numbers in the last row of our tableau will be the coefficients of the desired quotient polynomial. Remember, we started with a third degree polynomial and divided by a first degree polynomial, so the quotient will be a second degree polynomial. Hence the quotient is $x^2 + 6x + 7$. The number in bold represents the remainder, which is zero in this case.

Due in large part to its speed, synthetic division is often a ‘tool of choice’ for dividing polynomials by divisors of the form $x - c$. It is important to reiterate that synthetic division will *only* work for these kinds of divisors (linear divisors with leading coefficient 1), and we will need to use polynomial long division for divisors having degree larger than 1.

Another observation worth mentioning is that when a polynomial (of degree at least 1) is divided by $x - c$, the result will be a quotient polynomial of exactly one less degree than the original polynomial. This is a direct result of the divisor being a linear expression.

For a more complete understanding of the relationship between long and synthetic division, students are encouraged to trace each step in synthetic division back to its corresponding step in long division.

We conclude this section with three examples using synthetic division. We will summarize each example using the form below.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

Example 24. Use synthetic division to perform the following polynomial division. Find the quotient and the remainder polynomials.

$$\frac{5x^3 - 2x^2 + 1}{x - 3}$$

When setting up the synthetic division tableau, we need to enter 0 for the coefficient of x in the dividend as a placeholder, just like in polynomial division.

Setting up and working through the tableau gives us the following result.

$$\begin{array}{r|rrrr} 3 & 5 & -2 & 0 & 1 \\ & & 15 & 39 & 117 \\ \hline & 5 & 13 & 39 & \mathbf{118} \end{array}$$

Since the dividend was a third degree polynomial, the quotient is a quadratic polynomial with coefficients 5, 13 and 39. Our quotient is then $q(x) = 5x^2 + 13x + 39$ and the remainder is $r(x) = 118$.

Putting this all together, we have the following equation.

$$\frac{5x^3 - 2x^2 + 1}{x - 3} = 5x^2 + 13x + 39 + \frac{118}{x - 3}$$

Example 25. Use synthetic division to perform the following polynomial division. Find the quotient and the remainder polynomials.

$$\frac{x^3 + 8}{x + 2}$$

For this division, since we have a factor of $x + 2$, we must use the zero of $x = -2$ to begin. Here, we will once again stress that it is critical to take the time in order to ensure we have set the synthetic division tableau up correctly at the onset of the problem. Failure to do so will result in an incorrect answer, as well as a considerable amount time spent re-doing the problem.

$$\begin{array}{r|rrrr} -2 & 1 & 0 & 0 & 8 \\ & & -2 & 4 & -8 \\ \hline & 1 & -2 & 4 & \mathbf{0} \end{array}$$

We then obtain a quotient of $q(x) = x^2 - 2x + 4$ and remainder of $r(x) = 0$. This gives us the following equation.

$$\frac{x^3 + 8}{x + 2} = x^2 - 2x + 4$$

This answer is a great reminder of the factoring rules for cubic polynomials that we outlined earlier in the chapter.

Example 26. Use synthetic division to perform the following polynomial division. Find the quotient and the remainder polynomials.

$$\frac{4 - 8x - 12x^2}{2x - 3}$$

To divide $4 - 8x - 12x^2$ by $2x - 3$, two things must be done. First, we write the dividend in descending powers of x as $-12x^2 - 8x + 4$. Second, since synthetic division works only for factors of the form $x - c$, we factor $2x - 3$ as $2(x - \frac{3}{2})$. Our strategy is to first divide $-12x^2 - 8x + 4$ by 2, to get $-6x^2 - 4x + 2$. Next, we divide by $x - \frac{3}{2}$. The tableau becomes

$$\begin{array}{r|rrr} \frac{3}{2} & -6 & -4 & 2 \\ & & -9 & -\frac{39}{2} \\ \hline & -6 & -13 & -\frac{35}{2} \end{array}$$

From this, we get a quotient of $q(x) = -6x - 13$ and a remainder of $r(x) = -\frac{35}{2}$. This gives us the following equation.

$$\frac{-6x^2 - 4x + 2}{x - \frac{3}{2}} = -6x - 13 + \frac{-\frac{35}{2}}{x - \frac{3}{2}}$$

Multiplying both sides by of our equation by $\frac{2}{2}$ and distributing gives us our desired answer.

$$\frac{-12x^2 - 8x + 4}{2x - 3} = -6x - 13 + \frac{-35}{2x - 3}$$

Note that we could also multiply both sides of our last equation by $2x - 3$ to obtain the following equation.

$$-12x^2 - 8x + 4 = (2x - 3)(-6x - 13) - 35$$

While both of the forms above are certainly equivalent, the previous one may remind us of the familiar classic division algorithm for integers, shown below.

$$\text{dividend} = (\text{divisor}) \cdot (\text{quotient}) + \text{remainder}$$

The first form, however, will be particularly useful when we graph more complicated rational functions in the next chapter.

End Behavior (L49)

Objective: Identify and describe the end behavior of the graph of a polynomial function.

The *end behavior* of any function refers to what happens near the extreme ends of its graph. We also often refer to these as the “tails” of the graph. The ends of the graph of a function correspond to points having large positive or negative x -coordinates. Because of this, we can associate the expressions

$$x \rightarrow \infty \quad \text{and} \quad x \rightarrow -\infty$$

to the end behavior of a function. For example, the sentence

$$\text{As } x \rightarrow \infty, f(x) \rightarrow \infty.$$

describes a function for which the right-hand side of its graph, i.e. when $x \rightarrow \infty$, points upward. Alternatively, the sentence

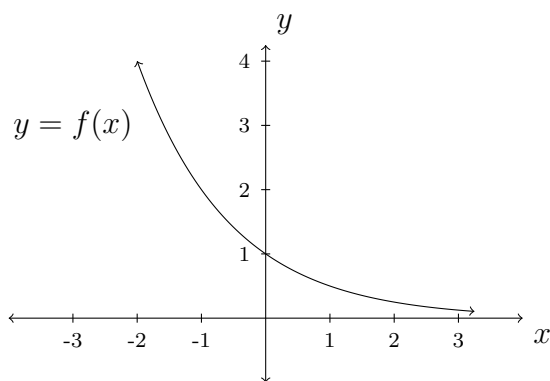
$$\text{As } x \rightarrow \infty, f(x) \rightarrow -\infty.$$

describes a function for which the right-hand side of its graph points downward.

In each of the above mathematical statements, we are identifying both a horizontal direction and a vertical direction:

1. the independent variable x getting large (either positively or negatively),
2. and the effect this has on the values of $f(x)$.

Example 27. Describe the end behavior of the function f whose graph is shown below.



Although this graph is one that we typically see in a precalculus setting (known as an exponential function), we can still discuss its end behavior. In this case, as the values of x increase, we see that the points on the graph approach the x -axis. This translates to the following statement.

$$\text{As } x \rightarrow \infty, f(x) \rightarrow 0.$$

On the other hand, as the values of x tend towards $-\infty$, we see that the y -coordinates for their respective points continue to increase. Hence, we can say the following.

$$\text{As } x \rightarrow -\infty, f(x) \rightarrow \infty.$$

Prior to this chapter, we have not had much need to discuss end behavior at great length, since most of the functions which we have been exposed to have been relatively easily diagnosed and graphed. A quadratic function, $f(x) = ax^2 + bx + c$, for example, will either open up or down, depending on the sign of the leading coefficient, a . As we begin to graph polynomials, however, we will see our graphs take more than a few turns, which will require us to have a better understanding about the nature of their tails.

For each algebraic function, the corresponding graph will describe two such statements: one for the left-hand side of the graph ($x \rightarrow -\infty$) and one for the right-hand side of the graph ($x \rightarrow \infty$). In the case of polynomials, there are only four cases for these two statements, summarized as follows.

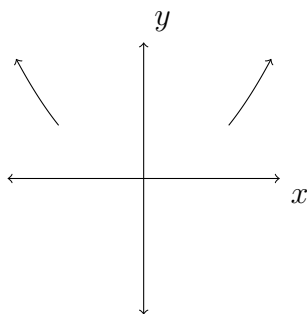
Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

be a polynomial function with degree n and nonzero leading coefficient a_n .

The end behavior of f is described by one of the following four cases.

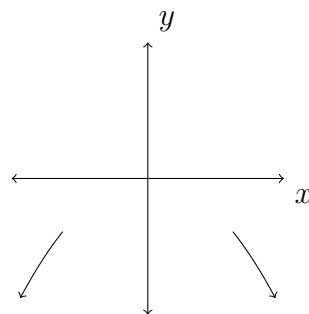
I. n even, $a_n > 0$



As $x \rightarrow \infty$, $f(x) \rightarrow \infty$

As $x \rightarrow -\infty$, $f(x) \rightarrow \infty$

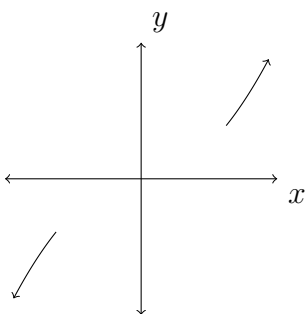
II. n even, $a_n < 0$



As $x \rightarrow \infty$, $f(x) \rightarrow -\infty$

As $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$

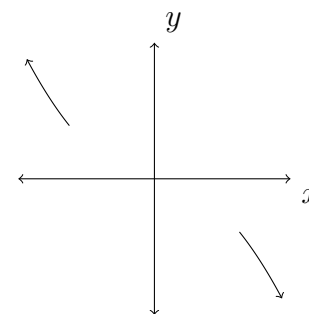
III. n odd, $a_n > 0$



As $x \rightarrow \infty$, $f(x) \rightarrow \infty$

As $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$

IV. n odd, $a_n < 0$



As $x \rightarrow \infty$, $f(x) \rightarrow -\infty$

As $x \rightarrow -\infty$, $f(x) \rightarrow \infty$

An important initial observation of the previous figure is that the cases for the end behavior of a polynomial only depend on its leading term, $a_n x^n$. More specifically, the end behavior of a polynomial depends only on the parity of its degree n (even or odd) and the sign of its leading coefficient a_n (positive or negative). Additionally, we can see that cases I and II also include all quadratic functions (when $n = 2$).

Identifying the end behavior for an expanded polynomial, is much more straightforward than for a factored polynomial, as we will see in our next example.

Example 28. Determine the end behavior of each of the following functions.

1. $f(x) = 1 - 3x^4$
2. $g(x) = -2x^3 + 10000x^2 + 1000$
3. $h(x) = x(2x - 1)(x - 5)^2$
4. $k(x) = -2(1 - 3x)^2(x + 1)(x - 1)(x^2 + 1)$

1. The polynomial $f(x) = 1 - 3x^4$ is in expanded form, though not written in descending-power order. We can easily re-write f as $f(x) = -3x^4 + 1$. In this case, the degree $n = 4$ is even, and the leading coefficient $a_n = -3$ is negative. Hence, we are in case II:

$$\text{As } x \rightarrow -\infty, f(x) \rightarrow -\infty.$$

$$\text{As } x \rightarrow \infty, f(x) \rightarrow -\infty.$$

2. The polynomial g is also in expanded form, with odd degree $n = 3$ and negative leading coefficient, $a_n = -2$. The “large” quadratic and constant terms will not affect the end behavior of g , and so we are in case IV:

$$\text{As } x \rightarrow -\infty, g(x) \rightarrow \infty.$$

$$\text{As } x \rightarrow \infty, g(x) \rightarrow -\infty.$$

3. The polynomial h is written in factored form, which is helpful for identifying roots/ x -intercepts, but not necessarily for describing the tails of the graph of h . Although we could, with enough time, expand h completely to describe the function’s end behavior, this will quickly prove to be an inefficient strategy. Recall, however, that for end behavior we need only focus on finding the leading term, $a_n x^n$. We do this by identifying any parts of h that will contribute to the leading term. In this case, we identify in boldface font all contributing components to the leading term of h below.

$$h(x) = \mathbf{x(2x - 1)(x - 5)^2}$$

So, when we expand, the leading term of h will be

$$a_n x^n = x(2x)(x)^2 = 2x^4.$$

We can now see that h has even degree $n = 4$, and positive leading coefficient, $a_n = 2$. Hence, we are in case I:

$$\text{As } x \rightarrow -\infty, h(x) \rightarrow \infty.$$

$$\text{As } x \rightarrow \infty, h(x) \rightarrow \infty.$$

4. Similarly, the polynomial k is written in factored form, and will require us to find the leading term, $a_n x^n$. Again, we identify all contributing components to the leading term of k in boldface font below.

$$k(x) = -\mathbf{2}(1-\mathbf{3x})^2(\mathbf{x}+1)(\mathbf{x}-1)(\mathbf{x}^2+1)$$

So, when we expand, the leading term of k will be

$$\begin{aligned} a_n x^n &= -2(-3x)^2(x)(x)(x^2) \\ &= -2(9x^2)(x^4) \\ &= -18x^6. \end{aligned}$$

Through careful analysis, we see that k has an even degree, $n = 6$, and a negative leading coefficient, $a_n = -18$. Hence, we are in case II:

$$\text{As } x \rightarrow -\infty, k(x) \rightarrow -\infty.$$

$$\text{As } x \rightarrow \infty, k(x) \rightarrow -\infty.$$

In the previous example, we witnessed a new technique to quickly identify the end behavior of a polynomial that is given in factored form. The idea behind this technique is to only focus on the contributing components to a polynomial's leading term, $a_n x^n$, ignoring all others. This essentially boils down to focusing on three things:

- any constant multiplier,
- the leading term of each factor,
- and the power associated with each factor.

In general, if we suppose that a polynomial f has the factorization

$$f(x) = c \cdot (\text{Factor 1})^{k_1} \cdot (\text{Factor 2})^{k_2} \cdot \dots \cdot (\text{Factor m})^{k_m},$$

then the leading term for f will equal

$$a_n x^n = c \cdot (\text{Leading Term 1})^{k_1} \cdot (\text{Leading Term 2})^{k_2} \cdot \dots \cdot (\text{Leading Term m})^{k_m}.$$

Note that “Leading Term 1” refers to the leading term of Factor 1, and so on for the other factors.

This approach is similar to one that we have likely seen for identifying the constant term for a factored polynomial. To identify the constant term, a_0 of f , we would have

$$a_n x^n = c \cdot (\text{Constant Term 1})^{k_1} \cdot (\text{Constant Term 2})^{k_2} \cdot \dots \cdot (\text{Constant Term m})^{k_m}.$$

Example 29. Find the leading and constant terms for the given function, and use them to identify the end behavior and y -intercept of its graph.

$$f(x) = 3(-2x + 1)^2(x - 2)^2(x - 5)$$

First, we boldface the contributors for the leading term.

$$f(x) = \mathbf{3}(-\mathbf{2x} + 1)^2(\mathbf{x} - 2)^2(\mathbf{x} - 5)$$

This gives us the following.

$$\begin{aligned} a_n x^n &= 3(-2x)^2(x)^2(x) \\ &= 3(4x^2)x^3 \\ &= 12x^5 \end{aligned}$$

Next, we boldface the contributors for the constant term.

$$f(x) = \mathbf{3}(-2x + \mathbf{1})^2(x - \mathbf{2})^2(x - \mathbf{5})$$

This gives us the following.

$$\begin{aligned} a_0 &= 3(1)^2(-2)^2(-5) \\ &= 3(1)(4)(-5) \\ &= -60 \end{aligned}$$

Hence, we have that

$$f(x) = 12x^5 + \dots + (-60),$$

with middle terms unknown.

Since our degree, $n = 5$, is odd, and our leading coefficient, $a_n = 12$, is positive, we are in case III for end behavior.

$$\text{As } x \rightarrow -\infty, f(x) \rightarrow -\infty.$$

$$\text{As } x \rightarrow \infty, f(x) \rightarrow +\infty.$$

Our constant term also tells us that the graph of f has a y -intercept at $(0, -60)$.

It is natural to ask why the additional terms of a polynomial have no impact on its end behavior. To address this, let us consider factoring out the leading term from f , which will give us the following.

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0 \\ &= a_n x^n \left(1 + \frac{a_{n-1}}{a_n x} + \dots + \frac{a_2}{a_n x^{n-2}} + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n} \right) \end{aligned}$$

If we use $g(x)$ to denote the expression in parentheses,

$$g(x) = 1 + \frac{a_{n-1}}{a_n x} + \dots + \frac{a_2}{a_n x^{n-2}} + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n},$$

then

$$f(x) = a_n x^n \underbrace{\left(1 + \frac{a_{n-1}}{a_n x} + \dots + \frac{a_2}{a_n x^{n-2}} + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n} \right)}_{g(x)}$$

$$= a_n x^n \cdot g(x).$$

But recall that the end behavior of a polynomial is determined when $x \rightarrow \pm\infty$. So, as x gets large (either positively or negatively), with the exception of the first term, all subsequent terms in the expression for g will approach zero.

$$\text{As } x \rightarrow \pm\infty, \quad g(x) = 1 + \cancel{\frac{a_{n-1}}{a_n x}} \xrightarrow{0} \dots + \cancel{\frac{a_2}{a_n x^{n-2}}} \xrightarrow{0} + \cancel{\frac{a_1}{a_n x^{n-1}}} \xrightarrow{0} + \cancel{\frac{a_0}{a_n x^n}} \xrightarrow{0} 1.$$

Therefore, since $g(x)$ approaches 1, $f(x) = a_n x^n \cdot g(x)$ will approach its leading term, $a_n x^n$. Hence, we conclude that the end behavior of a polynomial f will coincide with the end behavior of its leading term.

Furthermore, for any polynomial $f(x)$, if we were to graph the two curves $y = f(x)$ and $y = a_n x^n$ using [Desmos](#) or another graphing utility, and continue to ‘zoom out’, the two graphs would become virtually indistinguishable from one another. We demonstrate this in our next example.

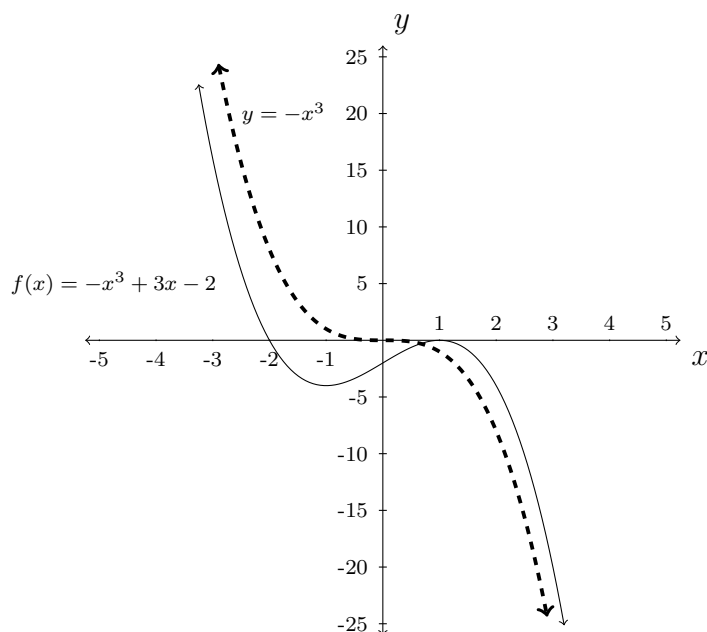
Example 30. Determine the end behavior of the polynomial function below, and graph both the function and its leading term on a single set of axes.

$$f(x) = -x^3 + 3x - 2$$

The leading term of f is $-x^3$, with odd degree, $n = 3$, and negative leading coefficient, $a_n = -1$. Hence, we are in case IV:

$$\text{As } x \rightarrow -\infty, \quad f(x) \rightarrow \infty.$$

$$\text{As } x \rightarrow \infty, \quad f(x) \rightarrow -\infty.$$



Local Behavior (L50)

Objective: Identify all real roots and their corresponding multiplicities for a polynomial function that is easily factorable.

In College Algebra and Precalculus, when we refer to the *local behavior* of a function f , we will be concerned with anything of interest in the interior of the graph of f , and not its end behavior. For polynomials, this is the x - and y -intercepts of the graph. These points coincide with when $f(x) = 0$ for any x -intercepts, and when $x = 0$ in the case of the y -intercept. In Calculus, local behavior will also include points where the graph changes inflection or achieves a local maximum or minimum value.

Since we should be very familiar with finding a y -intercept at this point, we will start with a simple example.

Example 31. Find the y -intercept for each of the following polynomials.

$$1. f(x) = 5x^3 - \frac{1}{2}x^2 + 6x - 18 \qquad 2. g(x) = \frac{1}{2}(x - 2)^2(x + 5)(x - 3)$$

1. $f(0) = -18$. Hence, the graph of f has a y -intercept at $(0, -18)$.
2. In the case of g , we have to identify the constant term a_0 in the expanded form of the polynomial. Recalling Example 29 from our last section, we can easily obtain this value without the need to expand g in its entirety.

$$\begin{aligned} g(0) &= \frac{1}{2}(0 - 2)^2(0 + 5)(0 - 3) \\ &= \frac{1}{2}(-2)^2(5)(-3) \\ &= \frac{1}{2}(4)(-15) \\ &= 2(-15) \\ &= -30 \end{aligned}$$

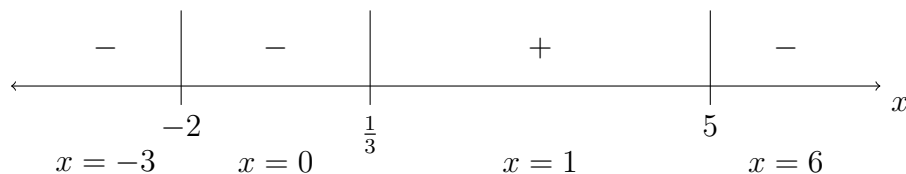
Hence, our y -intercept for the graph of g is $(0, -30)$.

Although it is certainly important to identify the y -intercept of any polynomial, the primary objective of this section will be finding the roots of a polynomial and classifying the respective x -intercepts of its graph. Since roots/ x -intercepts coincide with when a function equals zero, $f(x) = 0$, this section will depend heavily on working with a polynomial that is either in factored form or for which a complete factorization is easily obtainable. In a subsequent section of this chapter, we will see more a more advanced technique for finding a complete factorization of a polynomial, using polynomial division and the Rational Root Theorem.

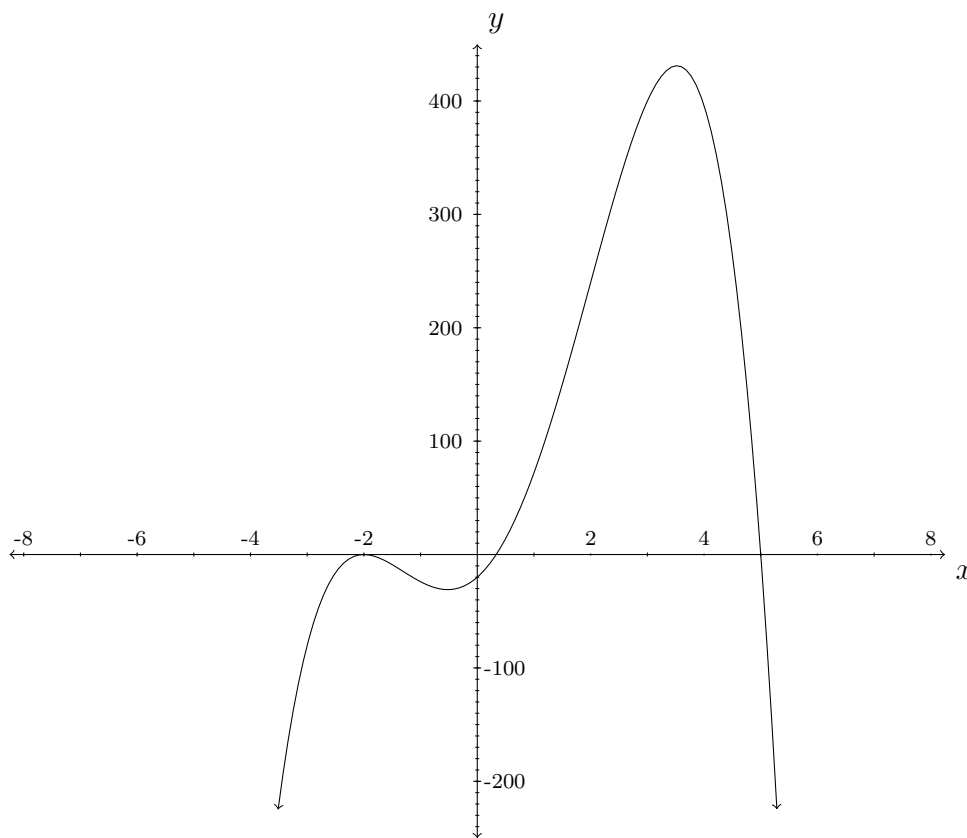
We begin our exploration of x -intercepts by revisiting a past example.

Example 32. Find all roots of the polynomial function $h(x) = (x + 2)^2(3x - 1)(5 - x)$, and graph h using [Desmos](#) or another graphing utility. For each root, identify whether the graph of h crosses over or turns around at the corresponding x -intercept.

For this example, we will first recall the work done Example 8, where we identified the roots of h to be $x = -2, \frac{1}{3}$, and 5, as well as the following sign diagram.



Our graph of h is shown below.



Based upon our picture, we see that the graph of h crosses over the x -axis at $x = \frac{1}{3}$ and $x = 5$. The graph turns around at $x = -2$.

In our last example, we have included our sign diagram to point out a connection. Our diagram confirms the nature of each x -intercept without the need to graph h , since both sides of our *turnaround point* $x = -2$ show the *same* sign (either $++$ or $--$). Similarly, the signs *change* from either positive to negative ($+|-$) or negative to positive ($-|+$) for each of our *crossover points*.

In fact, this idea of turnaround and crossover points can be parsed down to one basic concept, known as the *multiplicity* of a root. We define the multiplicity of a root below, followed immediately by an example for clarification.

Suppose f is a polynomial function with real root $x = c$. For some positive integer k , if $(x - c)^k$ is a factor of f but $(x - c)^{k+1}$ is not, then we say $x = c$ is a root of f having associated multiplicity k .

Example 33. Determine the set of roots and corresponding multiplicities for the following functions.

$$1. f(x) = x^6 - 2x^5 - 15x^4 \qquad 2. g(x) = (x - 6)^5(x + 2)^2(x^2 + 1)$$

1. Factoring f gives us the following.

$$\begin{aligned} f(x) &= x^6 - 2x^5 - 15x^4 \\ &= x^4(x^2 - 2x - 15) \\ &= x^4(x - 5)(x + 3) \end{aligned}$$

We then can easily see that f has a root at $x = 0$ with multiplicity four, and roots at $x = 5$ and $x = -3$, each with multiplicity one.

2. Since g is already factored, we see that $x = 6$ is a root having multiplicity five, and $x = -2$ is a root having multiplicity two. The factor of $x^2 + 1$ is meant to throw us off, since its roots are the imaginary numbers $\pm i$.

Another way of describing the multiplicity k of a root $x = c$ is that k represents the maximum number of factors of $(x - c)$ that divide the polynomial f (with a remainder of 0). That is,

$$f(x) = (x - c)^k \cdot q(x),$$

where $(x - c)$ is *not* a factor of the quotient $q(x)$.

If we apply this idea to g in our last example, we see that although $(x - 6)^4$ divides our polynomial,

$$g(x) = (x - 6)^4 \cdot \underbrace{(x - 6)(x + 2)^2(x^2 + 1)}_{q(x)},$$

the value of four does not represent the *maximum* number of factors of $(x - 6)$ that divide g :

$$g(x) = (x - 6)^5 \cdot \underbrace{(x + 2)^2(x^2 + 1)}_{q(x)}.$$

At this point, we are ready to highlight the importance of multiplicities in graphing polynomials.

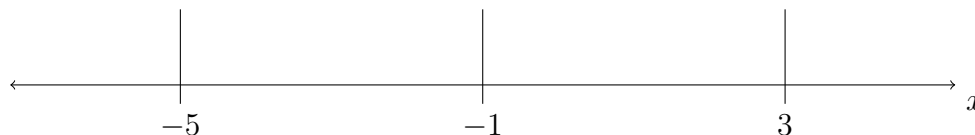
Let f be a polynomial function with a real root at $x = c$ having multiplicity k .

- If k is *even*, the corresponding x -intercept $(c, 0)$ is a *turnaround point*. In other words, the graph of f touches and rebounds from the x -axis at $(c, 0)$, leaving the y -values to maintain the same sign on either side of the root $x = c$.
- If k is *odd*, the corresponding x -intercept $(c, 0)$ is a *crossover point*. In other words, the graph of f crosses through the x -axis at $(c, 0)$, leaving the y -values to change signs on either side of the root $x = c$.

Combining this new notion about multiplicities of roots with all that we have already learned about polynomials will enable us to quickly identify all important aspects of a particular polynomial function, culminating in a sketch of its graph. We capitalize on this in our next example.

Example 34. Construct a sign diagram for the factored polynomial $f(x) = -(x-3)^2(x+1)(x+5)^2$.

The dividers for our sign diagram come from the set of roots of f , namely $\{-5, -1, 3\}$.



Instead of assigning test values, however, we will use both multiplicities and end behavior to determine our various signs.

First, we identify the leading term of f .

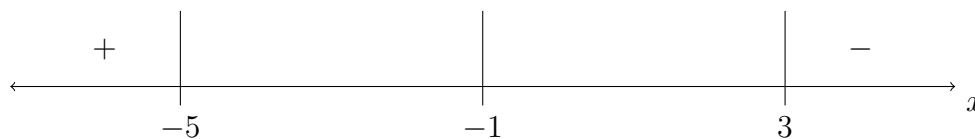
$$a_n x^n = -(x)^2(x)(x)^2 = -x^5$$

Since $a_n < 0$ and $n = 5$ is odd, our end behavior follows case IV:

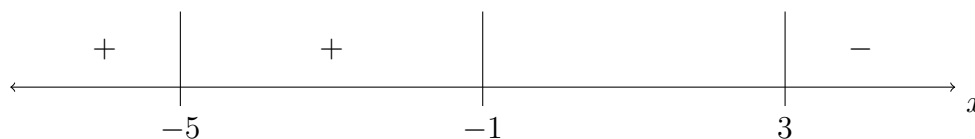
$$\text{As } x \rightarrow -\infty, f(x) \rightarrow \infty.$$

$$\text{As } x \rightarrow \infty, f(x) \rightarrow -\infty.$$

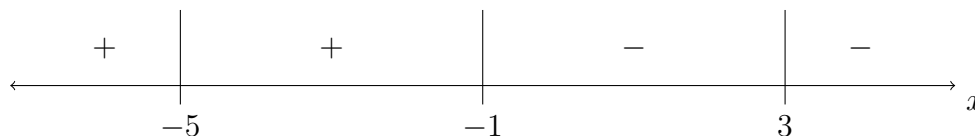
This tells us that our diagram will begin with a positive sign and end with a negative sign.



Furthermore, the multiplicity of the root $x = -5$ is two, which is even. So, our diagram must contain the same signs on either side of $x = -5$, namely two positive signs.



Applying this same idea to both of our remaining roots, we get the following diagram, and are done!



In fact, we can take this last example further, and easily sketch a graph of our polynomial. All that remains is to identify the y -intercept.

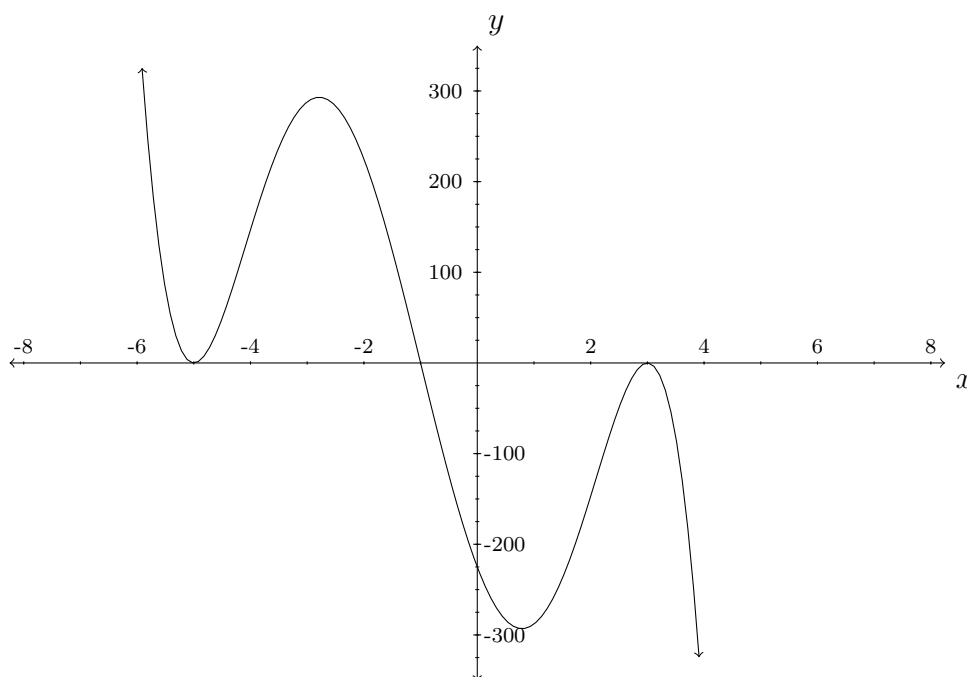
Example 35. Sketch a graph of the factored polynomial $f(x) = -(x - 3)^2(x + 1)(x + 5)^2$, making sure to identify a clearly defined scale and any x - and y -intercepts.

Since the roots, $x = 3$ and $x = -5$ have even multiplicities, their corresponding x -intercepts will be turnaround points. Our sign diagram confirms this, and further shows that the intercept at $x = -5$ will be a local minimum ($+|+$), whereas the intercept at $x = 3$ will be a local maximum ($-|-$). On the other hand, the root at $x = -1$ has an odd multiplicity, and the x -intercept at $x = -1$ will be a crossover point.

For a y -intercept, we evaluate the function at $x = 0$.

$$\begin{aligned} f(0) &= -(0 - 3)^2(0 + 1)(0 + 5)^2 \\ &= -(9)(1)(25) \\ &= -225 \end{aligned}$$

Since our y -intercept is a large negative value, we will have to shrink our scale for the y -axis accordingly.



This last example achieves what we have sought after since beginning the chapter. In it, we have identified the end behavior and all intercepts of a completely factored polynomial. We have further used both a sign diagram and a multiplicity argument in order to graph the given function.

What remains is to take a closer look at more challenging polynomials, for which a factorization may not be given or even easily identifiable.

The Rational Root Theorem (L51)

Objective: Apply the Rational Root Theorem to determine a set of possible rational roots for and a factorization of a given polynomial.

The Rational Root Theorem is used to identify a list of all possible rational roots for a given polynomial.

Rational Root Theorem: Suppose $f(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ is a polynomial of degree n with $n \geq 1$, and a_0, a_1, \dots, a_n are integers. If r is a rational root of f , then r is of the form $\pm \frac{p}{q}$, where p is a factor of the constant term a_0 , and q is a factor of the leading coefficient a_n .

The Rational Root Theorem gives us a list of numbers to test as roots of a given polynomial using synthetic division, which is a nicer approach than simply guessing at possible roots. If none of the numbers in the list turn out to be roots, then either the polynomial has no real roots at all, or all of the real roots will be irrational numbers.

Example 36. Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. Use the Rational Root Theorem to list all of the possible rational roots of f .

To generate a complete list of rational roots, we need to take each of the factors of the constant term, $a_0 = -3$, and divide them by each of the factors of the leading coefficient $a_4 = 2$.

The factors of -3 are ± 1 and ± 3 . Since the Rational Root Theorem tacks on a \pm anyway, for the moment, we consider only the positive factors 1 and 3. The factors of 2 are 1 and 2, so the Rational Root Theorem gives the list $\{\pm \frac{1}{1}, \pm \frac{1}{2}, \pm \frac{3}{1}, \pm \frac{3}{2}\}$ or $\{\pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \pm 3\}$.

Additionally, we can evaluate f at each of the eight potential rational roots in our list, to see if any of them are indeed roots. Starting with ± 1 , we see that

$$f(1) = 2 + 4 - 1 - 6 - 3 = -4 \neq 0 \quad \text{and} \quad f(-1) = 2 - 4 - 1 + 6 - 3 = 0.$$

Hence, we can conclude that $x = -1$ is a root of f and $x = 1$ is not. Using synthetic division, we can then divide f by the linear factor $x + 1$ as follows.

$$\begin{array}{r|rrrrr} -1 & 2 & 4 & -1 & -6 & -3 \\ & & -2 & -2 & 3 & 3 \\ \hline & 2 & 2 & -3 & -3 & 0 \end{array}$$

We can then begin to factor f ,

$$2x^4 + 4x^3 - x^2 - 6x - 3 = (x + 1)(2x^3 + 2x^2 - 3x - 3)$$

The resulting quotient polynomial is then factorable by grouping,

$$2x^3 + 2x^2 - 3x - 3 = (2x^2 - 3)(x + 1).$$

Factoring out a 2 from the expression $2x^2 - 3$, allows us to factor it as the difference of two squares,

$$\begin{aligned} 2x^2 - 3 &= 2 \left(x^2 - \frac{3}{2} \right) \\ &= 2 \left(x - \sqrt{\frac{3}{2}} \right) \left(x + \sqrt{\frac{3}{2}} \right) \\ &= 2 \left(x - \frac{\sqrt{6}}{2} \right) \left(x + \frac{\sqrt{6}}{2} \right) \end{aligned}$$

So, a complete factorization for f would be

$$2x^4 + 4x^3 - x^2 - 6x - 3 = 2 \left(x - \frac{\sqrt{6}}{2} \right) \left(x + \frac{\sqrt{6}}{2} \right) (x + 1)^2,$$

and the set of real roots for f is $\left\{ -1, \pm \frac{\sqrt{6}}{2} \right\}$.

Graphing Summary (L52)

Objective: Graph a polynomial function in its entirety.

At this point, we have addressed all key features of polynomials individually. This section pulls each of these aspects together, for a detailed analysis of a polynomial, culminating in a complete sketch of its graph. Along the way, we will need to address each of the following aspects for our polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$. It is important to note that there is no universally accepted order to this checklist.

- Find the y -intercept of the graph of f , $(0, f(0)) = (0, a_0)$.
- Use the degree n and leading coefficient a_n to determine the end behavior of the graph of f .
- Identify a complete factorization of f , and use it to find any x -intercepts of the graph of f . Using multiplicities, classify each x -intercept as a crossover or turnaround (“bounce”) point.
- Using the x -intercepts, construct a sign diagram for f .

In each polynomial we encounter, we will carefully examine the function, making sure not to omit any of the checklist items above and to compare each item to those that precede it along the way for accuracy. Although the process will take some time, if we are thorough, our end result should be a complete, accurate sketch of the given polynomial.

Example 37. Sketch a complete graph of $f(x) = 14x^4 - 17x^3 - 6x^2 + 7x + 2$.

We will start with the y -intercept, which is $(0, 2)$.

Next, we see that f has even degree and positive leading coefficient. So, the tails of the graph of f both point upwards. In other words, as $x \rightarrow \pm\infty$, $f(x) \rightarrow \infty$.

Since f is degree-4, contains more than four terms, and is not of quadratic type, we will apply the Rational Root Theorem. In this case, our set of possible rational roots is

$$\left\{ \pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{7}, \pm \frac{1}{14}, \pm \frac{2}{7} \right\}$$

Fortunately, we see that $f(1) = 14 - 17 - 6 + 7 + 2 = 0$. So, $x - 1$ is a factor of f . Dividing, we get:

$$\begin{array}{r} x-1 \overline{) \begin{array}{r} 14x^3 - 3x^2 - 9x - 2 \\ 14x^4 - 17x^3 - 6x^2 + 7x + 2 \\ -14x^4 + 14x^3 \\ \hline -3x^3 - 6x^2 \\ 3x^3 - 3x^2 \\ \hline -9x^2 + 7x \\ 9x^2 - 9x \\ \hline -2x + 2 \\ 2x - 2 \\ \hline 0 \end{array}} \end{array} \quad \begin{array}{r} 1 \left| \begin{array}{rrrrr} 14 & -17 & -6 & 7 & 2 \\ & 14 & -3 & -9 & -2 \\ \hline 14 & -3 & -9 & -2 & 0 \end{array} \right. \end{array}$$

So, $f(x) = (x - 1)(14x^3 - 3x^2 - 9x - 2)$. Applying the Rational Root Theorem a second time, we can see that $x = 1$ is also a root of the cubic factor of f , since $14 - 3 - 9 - 2 = 0$. Again, we can divide to factor f further.

$$\begin{array}{r} x-1 \overline{) \begin{array}{r} 14x^2 + 11x + 2 \\ 14x^3 - 3x^2 - 9x - 2 \\ -14x^3 + 14x^2 \\ \hline 11x^2 - 9x \\ -11x^2 + 11x \\ \hline 2x - 2 \\ -2x + 2 \\ \hline 0 \end{array}} \end{array} \quad \begin{array}{r} 1 \left| \begin{array}{rrrr} 14 & -3 & -9 & -2 \\ & 14 & 11 & 2 \\ \hline 14 & 11 & 2 & 0 \end{array} \right. \end{array}$$

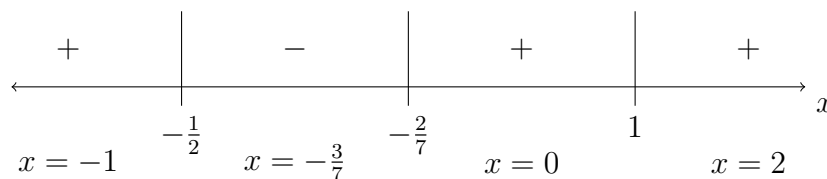
So, $f(x) = (x - 1)^2(14x^2 + 11x + 2)$. Factoring the remaining quadratic, we have

$$f(x) = (x - 1)^2(7x + 2)(2x + 1),$$

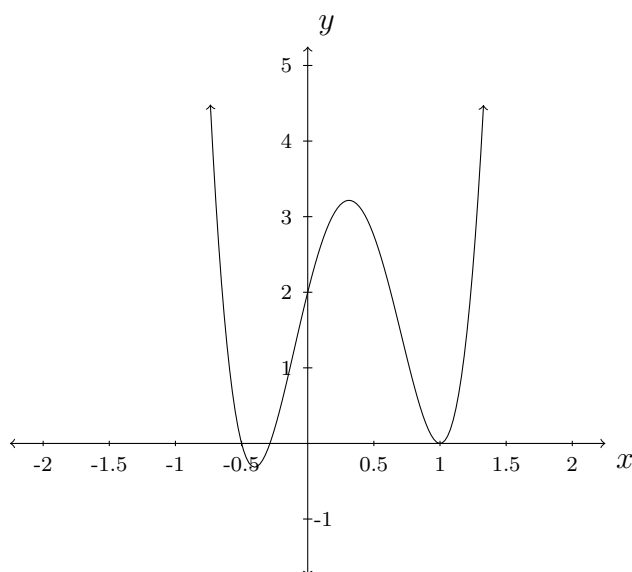
with accompanying set of roots $\{1, -\frac{1}{2}, -\frac{2}{7}\}$.

Using multiplicities, we conclude that the x -intercept $(1, 0)$ is a turnaround point, and the intercepts $(-\frac{1}{2}, 0)$ and $(-\frac{2}{7}, 0)$ are crossover points.

Though not necessary for graphing, a sign diagram confirms our end and local behavior findings.



Putting all of this information together results in the following graph.



Polynomial Inequalities (L53)

Objective: Solve a polynomial inequality by constructing a sign diagram.

Example 38. Solve the polynomial inequality

$$x^4 + 6x^2 - 15x \leq x^4 + 2x^3 - 7x^2.$$

Just as with quadratic inequalities, we begin by setting one side equal to zero. This gives us

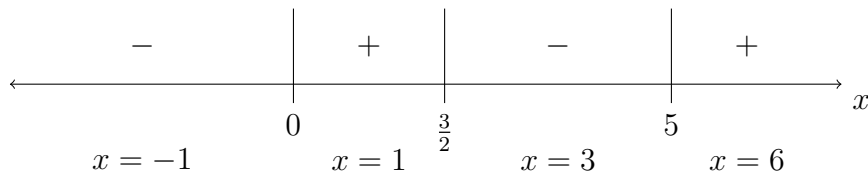
$$2x^3 - 13x^2 + 15x \geq 0.$$

In order to construct a sign diagram, we must find a factorization and identify the roots of the left-hand side of our inequality.

$$2x^3 - 13x^2 + 15x = 2x \left(x - \frac{3}{2} \right) (x - 5)$$

So the dividers in our diagram will be the roots $x = 0, \frac{3}{2}$, and 5. Below is a chart for testing the intervals in our sign diagram, as well as the end result.

<u>Interval</u>	<u>Test Value</u>	<u>Signs</u>	<u>Result</u>
$(-\infty, 0)$	$x = -1$	$(-)(-)(-)$	$-$
$(0, \frac{3}{2})$	$x = 1$	$(+)(-)(-)$	$+$
$(\frac{3}{2}, 5)$	$x = 3$	$(+)(+)(-)$	$-$
$(5, \infty)$	$x = 6$	$(+)(+)(+)$	$+$



So, using our diagram as an aide, we see that the solution to the inequality

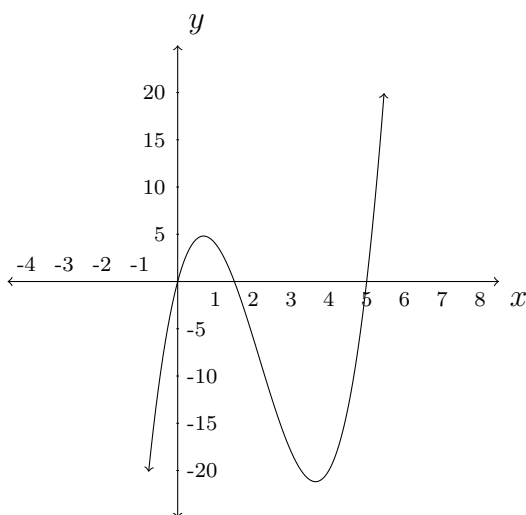
$$2x^3 - 13x^2 + 15x \geq 0,$$

as well as our original inequality

$$x^4 + 6x^2 - 15x \leq x^4 + 2x^3 - 7x^2,$$

will be

$$\left[0, \frac{3}{2}\right] \cup [5, \infty).$$



Since our given inequality was inclusive (\leq or \geq), we include the corresponding endpoints in our answer.

We can verify that our answer is correct by comparing it to the graph of the function

$$f(x) = 2x^3 - 13x^2 + 15x,$$

which lies above (or on) the x -axis over the intervals in our answer.

Practice Problems

Introduction and Terminology

Identify the degree, set of coefficients, leading coefficient, leading term and constant term for each of the polynomials listed. Classify each polynomial by both degree and number of nonzero terms. If it is not already provided, write the polynomial in descending-power order.

1. $f(x) = -2x^3 - 1$
2. $f(x) = -2x^4 + 4x + 1$
3. $f(x) = 40 - x^3$
4. $f(x) = (x - 1)^2$
5. $f(x) = 32x^5 + x^2 + x$
6. $f(x) = 4x^2 - 3x^4$
7. $f(x) = -2x^4 - 4x^2 - 6x - 8$
8. $f(x) = 5x + 3x^2 + x^3 + \sqrt{3}$
9. $f(x) = \frac{1}{2}x^4 - 5x^2 - \frac{1}{2}$
10. $f(x) = 12 - 6x + 3x^2 - 2x^3 - x^6$
11. $f(x) = -3x^4 - 12x^3 + x - 13$

Sign Diagrams

Construct a sign diagram for the factored polynomial functions below. Use [Desmos](#) to graph each function and check the accuracy of your diagram. Identify the interval(s) where the function is positive and where it is negative.

1. $f(x) = x^3(x - 2)(x + 2)$
2. $g(x) = (x^2 + 1)(1 - x)$
3. $h(x) = x(x - 3)^2(x + 3)$
4. $k(x) = (3x - 4)^3$
5. $\ell(x) = (x^2 + 2)(x^2 + 3)$
6. $m(x) = -2(x + 7)^2(1 - 2x)^2$
7. $f(x) = (x^2 - 1)(x + 4)$
8. $g(x) = (x^2 - 1)(x^2 - 16)$
9. $h(x) = -2x^3(3x - 1)(2 - x)$
10. $k(x) = (x^2 - 4x + 1)(x + 2)^2$

Factoring

Some Special Cases

Completely factor each of the following polynomial expressions.

1. $2x^2 - 11x + 15$
2. $5n^3 + 7n^2 - 6n$
3. $54u^3 - 16$
4. $54 - 128x^3$
5. $n^2 - n$
6. $2x^4 - 21x^2 - 11$
7. $24az - 18ah + 60yz - 45yh$
8. $5u^2 - 9uv + 4v^2$
9. $16x^2 + 48xy + 36y^2$
10. $-2x^3 + 128y^3$
11. $20uv - 60u^3 - 5xv + 15xu^2$
12. $2x^3 + 5x^2y + 3y^2x$

Quadratic Type

Completely factor each of the following polynomials over the real numbers and identify the set of all real roots.

1. $x^4 + 13x^2 + 40$
2. $x^4 - 5x^2 + 4$
3. $x^4 - 17x^2 + 16$
4. $x^4 - 3x^2 - 40$
5. $3x^4 - 32x^2 + 45$
6. $x^4 + x^2 - 12$
7. $x^4 - 3x^2 - 10$
8. $x^6 - 82x^3 + 81$
9. $8x^4 + 2x^2 - 3$
10. $2x^4 - 19x^2 + 9$

Division

Polynomial (Long) Division

Use polynomial long division to divide and simplify each of the given expressions. Express each answer in the form below.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

1. $\frac{20x^4 + x^3 + 2x^2}{4x^3}$	11. $\frac{x^2 + 13x + 32}{x + 5}$	21. $\frac{x^3 - x^2 - 16x + 8}{x - 4}$
2. $\frac{5x^4 + 45x^3 + 4x^2}{9x}$	12. $\frac{x^2 - 10x + 16}{x - 7}$	22. $\frac{x^2 - 10x + 22}{x - 4}$
3. $\frac{20x^4 + x^3 + 40x^2}{10x}$	13. $\frac{x^2 - 2x - 89}{x - 10}$	23. $\frac{x^3 - 16x^2 + 71x - 56}{x - 8}$
4. $\frac{3x^3 + 4x^2 + 2x}{8x}$	14. $\frac{x^2 + 4x - 26}{x + 7}$	24. $\frac{x^3 - 4x^2 - 6x + 4}{x - 1}$
5. $\frac{12x^4 + 24x^3 + 3x^2}{6x}$	15. $\frac{x^2 - 4x - 38}{x - 8}$	25. $\frac{8x^3 - 66x^2 + 12x + 37}{x - 8}$
6. $\frac{5x^4 + 16x^3 + 16x^2}{4x}$	16. $\frac{x^2 - 4}{x - 2}$	26. $\frac{3x^2 + 9x - 9}{3x - 3}$
7. $\frac{10x^4 + 50x^3 + 2x^2}{10x^2}$	17. $\frac{x^3 + 15x^2 + 49x - 55}{x + 7}$	27. $\frac{2x^2 - 5x - 8}{2x + 3}$
8. $\frac{3x^4 + 18x^3 + 27x^2}{9x^2}$	18. $\frac{x^3 - 26x - 41}{x + 4}$	28. $\frac{3x^2 - 32}{3x - 9}$
9. $\frac{x^2 - 2x - 71}{x + 8}$	19. $\frac{3x^3 + 9x^2 - 64x - 68}{x + 6}$	29. $\frac{4x^2 - 23x - 38}{4x + 5}$
10. $\frac{x^2 - 3x - 53}{x - 9}$	20. $\frac{9x^3 + 45x^2 + 27x - 5}{9x + 9}$	30. $\frac{2x^3 + 21x^2 + 25x}{2x + 3}$

31. $\frac{4x^3 - 21x^2 + 6x + 19}{4x + 3}$

32. $\frac{8x^3 - 57x^2 + 42}{8x + 7}$

33. $\frac{2x^3 + 12x^2 + 4x - 37}{2x + 6}$

34. $\frac{45x^2 + 56x + 19}{9x + 4}$

35. $\frac{10x^2 - 32x + 9}{10x - 2}$

36. $\frac{4x^2 - x - 1}{4x + 3}$

37. $\frac{27x^2 + 87x + 35}{3x + 8}$

38. $\frac{4x^2 - 33x + 28}{4x - 5}$

39. $\frac{48x^2 - 70x + 16}{6x - 2}$

40. $\frac{12x^3 + 12x^2 - 15x - 4}{2x + 3}$

41. $\frac{24x^3 - 38x^2 + 29x - 60}{4x - 7}$

Synthetic Division

Use synthetic division to divide and simplify each of the given expressions. Express each answer in the form below.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

1. $\frac{x^4 - 4x^3 + 2x^2 - x + 1}{x + 2}$	7. $\frac{12x^4 - x^3 + x^2 - 3x + 1}{x + 2}$
2. $\frac{x^4 - 2x^3 + 7x^2 - 6x + 3}{x - 2}$	8. $\frac{3x^4 + 3x^3 + 13x^2 - 4x + 14}{x + 1}$
3. $\frac{2x^4 - 2x^3 - 10x^2 + 1}{x + 2}$	9. $\frac{1x^4 - 3x^3 + 5x^2 - 14x + 2}{x - 2}$
4. $\frac{5x^4 - 2x^3 + 4x^2 - 5x}{x - 1}$	10. $\frac{2x^4 - 2x + 1}{x + 3}$
5. $\frac{-x^4 - x^3 + x^2 + x + 1}{x + 5}$	11. $\frac{x^4 - 3x - 4}{x - 3}$
6. $\frac{x^4 - 3x^3 + 2x^2 - x + 1}{x - 4}$	12. $\frac{x^4 - 4x^3 + 13x^2 - 5x + 7}{x - 4}$

Use synthetic division to divide and simplify each of the given expression from Exercises 9-41.

End Behavior

Determine the end behavior of each of the following functions. Write your answers as mathematical sentences. Graph each function on [Desmos](#) to check your answers.

1. $f(x) = -2x^3 + 4x + 1$

2. $g(x) = 32x^5 + x^2 + 15$

3. $h(x) = -3x^4 + 4x^2$

4. $k(x) = 15x^4 - 32x^2 - x - 14$

5. $\ell(x) = x^5 + 40$

6. $m(x) = 5x^5 + 3x^2 + x + 14$

7. $n(x) = 123x^4 - 7x^3 - 5x^2 - 3x + 1$

8. $p(x) = x^3 - 1$

9. $q(x) = -23x^6 + x^3 + x^2 + x + 1$

Identify the degree, leading coefficient, and constant term of each polynomial function below. Use the degree and leading coefficient to identify the end behavior of the graph of each function. Write your answers as mathematical sentences. Graph each function on [Desmos](#) to check your answers.

10. $f(x) = x^3(x - 2)(x + 2)$

11. $g(x) = (x^2 + 1)(1 - x)$

12. $h(x) = x(x - 3)^2(x + 3)$

13. $k(x) = (3x - 4)^3$

14. $\ell(x) = (x^2 + 2)(x^2 + 3)$

15. $m(x) = -2(x + 7)^2(1 - 2x)^2$

16. $f(x) = (x^2 - 1)(x + 4)$

17. $g(x) = (x^2 - 1)(x^2 - 16)$

18. $h(x) = -2x^3(3x - 1)(2 - x)$

19. $k(x) = (x^2 - 4x + 1)(x + 2)^2$

Local Behavior

Determine the set of roots and corresponding multiplicities for the following functions. In each case, classify the corresponding x -intercept as either a turnaround or crossover point. Use [Desmos](#) to check your answers.

1. $f(x) = x^3(x - 2)(x + 2)$

2. $g(x) = (x^2 + 1)(1 - x)$

3. $h(x) = x(x - 3)^2(x + 3)$

4. $k(x) = (3x - 4)^3$

5. $\ell(x) = (x^2 + 2)(x^2 + 3)$

6. $m(x) = -2(x + 7)^2(1 - 2x)^2$

7. $f(x) = (x^2 - 1)(x + 4)$

8. $g(x) = (x^2 - 1)(x^2 - 16)$

9. $h(x) = -2x^3(3x - 1)(2 - x)$

10. $k(x) = (x^2 - 4x + 1)(x + 2)^2$

11. $f(x) = \frac{1}{2}(x - 2)^2(x + 5)(x - 3)$

12. $g(x) = (x + 2)^2(3x - 1)(5 - x)$

The Rational Root Theorem

Use the Rational Root Theorem to identify a set of possible rational roots for each of the polynomial functions below. Evaluate the function at $x = 1$. If $x = 1$ is a real root, divide the polynomial by $x - 1$ and factor the resulting quotient. If $x = 1$ is not a real root, evaluate the function at at least one of your remaining possible roots, in order to determine if they are actual roots of the polynomial. If successful, divide your polynomial by the respective factor and factor the remaining quotient. Use [Desmos](#) to help determine the actual set of real roots.

1. $f(x) = x^3 - 2x^2 - 5x + 6$

2. $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$

3. $f(x) = x^5 - x^4 - 37x^3 + 37x^2 + 36x - 36$

4. $f(x) = 3x^3 + 3x^2 - 11x - 10$

Use the Rational Root Theorem to identify a set of possible rational roots for each of the polynomial functions below. Evaluate the function at at least two of your possible roots, in order to determine if they are actual roots of the polynomial. If successful, divide your polynomial by the respective factor. Use [Desmos](#) to help determine the actual set of real roots.

5. $f(x) = x^4 - 2x^3 + 5x^2 - 8x + 4$
6. $f(x) = x^3 + 4x^2 - 11x + 6$
7. $f(x) = -2x^3 + 19x^2 - 49x + 20$
8. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
9. $f(x) = x^4 - 9x^2 - 4x + 12$
10. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
11. $f(x) = 6x^3 + 19x^2 - 6x - 40$

Graphing Summary

Factor each polynomial below, and sketch a complete graph of the function, making sure to have a clearly defined scale and label any intercepts. Use [Desmos](#) to compare your results.

- | | |
|--------------------------------------|----------------------------------|
| 1. $f(x) = -17x^3 + 5x^2 + 34x - 10$ | 7. $f(x) = x^6 - 6x^3 - 16$ |
| 2. $f(x) = x^4 - 9x^2 + 14$ | 8. $f(x) = 2x^6 - 7x^3 + 5$ |
| 3. $f(x) = 3x^4 - 14x^2 - 5$ | 9. $f(x) = -x^3 + 7x^2 - x + 7$ |
| 4. $f(x) = 2x^4 - 7x^2 + 6$ | 10. $f(x) = 3x^4 - 5x^3 - 12x^2$ |
| 5. $f(x) = x^5 - 2x^4 - x + 2$ | 11. $f(x) = 2x^3 - 5x^2 - x$ |
| 6. $f(x) = 2x^5 + 3x^4 - 32x - 48$ | 12. $f(x) = -x^4 - 2x^2 + 15$ |

Get a complete factorization of each polynomial below by first dividing the function by $x - 1$. Then sketch a graph of the function, making sure to have a clearly defined scale and label any intercepts. Use [Desmos](#) to compare your results.

13. $f(x) = x^3 - 2x^2 - 5x + 6$
14. $f(x) = x^3 + 4x^2 - 11x + 6$
15. $f(x) = x^5 - x^4 - 37x^3 + 37x^2 + 36x - 36$
16. $f(x) = x^4 - 2x^3 + 5x^2 - 8x + 4$

Use the Rational Root Theorem and polynomial division to get a complete factorization of each polynomial function below. Then sketch a graph of the function, making sure to have a clearly defined scale and label any intercepts. Use [Desmos](#) to compare your results.

17. $f(x) = x^4 - 9x^2 - 4x + 12$
18. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
19. $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$
20. $f(x) = 3x^3 + 3x^2 - 11x - 10$
21. $f(x) = 6x^3 + 19x^2 - 34x - 40$
22. $f(x) = -2x^3 + 19x^2 - 49x + 20$
23. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
24. $f(x) = x^4 + 4x^3 - x - 4$
25. $f(x) = 2x^3 - 5x^2 - 52x + 60$
26. $f(x) = -x^3 - x^2 + 39x + 45$
27. $f(x) = -2x^4 + 7x^3 + 17x^2 - 28x - 36$
28. $f(x) = x^7 - 5x^6 - 24x^5 + 120x^4 - 25x^3 + 125x^2$

Polynomial Inequalities

Solve each polynomial inequality below, expressing your answers using interval notation. Use [Desmos](#) to help confirm that each answer is correct.

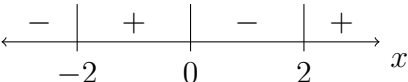
1. $x^4 + x^2 \geq 6$
2. $x^4 - 9x^2 \leq 4x - 12$
3. $4x^3 \geq 3x + 1$
4. $x^4 \leq 16 + 4x - x^3$
5. $3x^2 + 2x < x^4$
6. $\frac{x^3 + 2x^2}{2} < x + 2$
7. $\frac{x^3 + 20x}{8} \geq x^2 + 2$
8. $19x^2 + 20 > 2x^3 + 49x$
9. $x^3 < 4x^2$
10. $x^3 - 7x^2 \leq 12x - 84$
11. $(x - 1)^2 \geq 4$
12. $2x^4 > 5x^2 + 3$

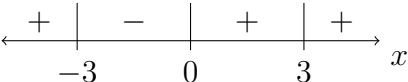
Selected Answers

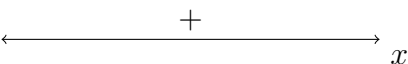
Introduction and Terminology

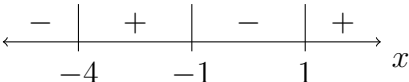
1. $n = 3$, $a_n = -2$, $a_n x^n = -2x^3$, $a_0 = -1$, $\{-2, 0, 0, -1\}$
3. $n = 3$, $a_n = -1$, $a_n x^n = -1x^3$, $a_0 = 40$, $\{-1, 0, 0, 40\}$
5. $n = 5$, $a_n = 32$, $a_n x^n = 32x^5$, $a_0 = 0$, $\{32, 0, 0, 1, 1, 0\}$
7. $n = 4$, $a_n = -2$, $a_n x^n = -2x^4$, $a_0 = -8$, $\{-2, 0, -4, -6, -8\}$
9. $n = 4$, $a_n = \frac{1}{2}$, $a_n x^n = \frac{1}{2}x^4$, $a_0 = -\frac{1}{2}$, $\{\frac{1}{2}, 0, -5, 0, -\frac{1}{2}\}$
11. $n = 4$, $a_n = -3$, $a_n x^n = -3x^4$, $a_0 = -13$, $\{-3, -12, 0, 1, -13\}$

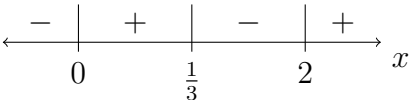
Sign Diagrams

1. 

3. 

5. 

7. 

9. 

Factoring

Some Special Cases

1. $(2x - 5)(x - 3)$

3. $2(3x - 2)(9x^2 + 6x + 4)$

5. $n(n - 1)$

7. $3(2a + 15y)(4z - 3h)$

9. $4(2x + 3y)^2$

11. $-5(4u - x)(3u^2 - v)$

Quadratic Type

1. $(x^2 + 8)(x^2 + 5)$

3. $(x - 1)(x + 1)(x - 4)(x + 4)$

5. $(3x^2 - 5)(x^2 - 9) = 3(x - \frac{\sqrt{15}}{3})(x + \frac{\sqrt{15}}{3})(x - 3)(x + 3)$

7. $(x^2 - 5)(x^2 + 2) = (x - \sqrt{5})(x + \sqrt{5})(x^2 + 2)$

9. $(2x^2 - \frac{1}{2})(4x^2 + 3) = 2(x - \frac{\sqrt{2}}{2})(x + \frac{\sqrt{2}}{2})(4x^2 + 3)$

Division

Polynomial (Long) Division

1.
$$\begin{array}{r} 5x + \frac{1}{4} \\ 4x^3 \overline{) 20x^4 + x^3 + 2x^2} \\ \underline{-20x^4} \\ x^3 \\ \underline{-x^3} \\ 2x^2 \end{array}$$

$$\begin{array}{r}
 3. \quad \frac{2x^3 + \frac{1}{10}x^2 + 4x}{10x) \overline{20x^4 + x^3 + 40x^2}} \\
 \underline{-20x^4} \\
 x^3 \\
 \underline{-x^3} \\
 40x^2 \\
 \underline{-40x^2} \\
 0
 \end{array}$$

$$\begin{array}{r}
 5. \quad \frac{2x^3 + 4x^2 + \frac{1}{2}x}{6x) \overline{12x^4 + 24x^3 + 3x^2}} \\
 \underline{-12x^4} \\
 24x^3 \\
 \underline{-24x^3} \\
 3x^2 \\
 \underline{-3x^2} \\
 0
 \end{array}$$

$$\begin{array}{r}
 7. \quad \frac{x^2 + 5x + \frac{1}{5}}{10x^2) \overline{10x^4 + 50x^3 + 2x^2}} \\
 \underline{-10x^4} \\
 50x^3 \\
 \underline{-50x^3} \\
 2x^2 \\
 \underline{-2x^2} \\
 0
 \end{array}$$

$$\begin{array}{r}
 9. \quad \frac{x - 10}{x + 8) \overline{x^2 - 2x - 71}} \\
 \underline{-x^2 - 8x} \\
 -10x - 71 \\
 \underline{10x + 80} \\
 9
 \end{array}$$

$$\begin{array}{r}
 11. \quad \frac{x + 8}{x + 5) \overline{x^2 + 13x + 32}} \\
 \underline{-x^2 - 5x} \\
 8x + 32 \\
 \underline{-8x - 40} \\
 -8
 \end{array}$$

$$\begin{array}{r}
 13. \quad \frac{x + 8}{x - 10) \overline{x^2 - 2x - 89}} \\
 \underline{-x^2 + 10x} \\
 8x - 89 \\
 \underline{-8x + 80} \\
 -9
 \end{array}$$

$$\begin{array}{r}
 15. \quad \frac{x + 4}{x - 8) \overline{x^2 - 4x - 38}} \\
 \underline{-x^2 + 8x} \\
 4x - 38 \\
 \underline{-4x + 32} \\
 -6
 \end{array}$$

$$\begin{array}{r}
 17. \quad \frac{x^2 + 8x - 7}{x + 7) \overline{x^3 + 15x^2 + 49x - 55}} \\
 \underline{-x^3 - 7x^2} \\
 8x^2 + 49x \\
 \underline{-8x^2 - 56x} \\
 -7x - 55 \\
 \underline{7x + 49} \\
 -6
 \end{array}$$

$$\begin{array}{r}
 19. \quad \frac{3x^2 - 9x - 10}{x + 6) \overline{3x^3 + 9x^2 - 64x - 68}} \\
 \underline{-3x^3 - 18x^2} \\
 -9x^2 - 64x \\
 \underline{9x^2 + 54x} \\
 -10x - 68 \\
 \underline{10x + 60} \\
 -8
 \end{array}$$

$$\begin{array}{r}
 21. \quad \frac{x^2 + 3x - 4}{x - 4) \overline{x^3 - x^2 - 16x + 8}} \\
 \underline{-x^3 + 4x^2} \\
 3x^2 - 16x \\
 \underline{-3x^2 + 12x} \\
 -4x + 8 \\
 \underline{4x - 16} \\
 -8
 \end{array}$$

$$\begin{array}{r}
 23. \quad \quad \quad x^2 - 8x + 7 \\
 x - 8 \overline{) \quad x^3 - 16x^2 + 71x - 56} \\
 \underline{- x^3 + 8x^2} \\
 \phantom{x - 8 \overline{) \quad}} - 8x^2 + 71x \\
 \phantom{x - 8 \overline{) \quad}} \underline{8x^2 - 64x} \\
 \phantom{x - 8 \overline{) \quad}} 7x - 56 \\
 \phantom{x - 8 \overline{) \quad}} \underline{- 7x + 56} \\
 \phantom{x - 8 \overline{) \quad}} 0
 \end{array}$$

$$\begin{array}{r}
 25. \quad \quad \quad 8x^2 - 2x - 4 \\
 x - 8 \overline{) \quad 8x^3 - 66x^2 + 12x + 37} \\
 \underline{- 8x^3 + 64x^2} \\
 \phantom{x - 8 \overline{) \quad}} - 2x^2 + 12x \\
 \phantom{x - 8 \overline{) \quad}} \underline{2x^2 - 16x} \\
 \phantom{x - 8 \overline{) \quad}} - 4x + 37 \\
 \phantom{x - 8 \overline{) \quad}} \underline{4x - 32} \\
 \phantom{x - 8 \overline{) \quad}} 5
 \end{array}$$

$$\begin{array}{r}
 27. \quad \quad \quad x - 4 \\
 2x + 3 \overline{) \quad 2x^2 - 5x - 8} \\
 \underline{- 2x^2 - 3x} \\
 \phantom{2x + 3 \overline{) \quad}} - 8x - 8 \\
 \phantom{2x + 3 \overline{) \quad}} \underline{8x + 12} \\
 \phantom{2x + 3 \overline{) \quad}} 4
 \end{array}$$

$$\begin{array}{r}
 29. \quad \quad \quad x - 7 \\
 4x + 5 \overline{) \quad 4x^2 - 23x - 38} \\
 \underline{- 4x^2 - 5x} \\
 \phantom{4x + 5 \overline{) \quad}} - 28x - 38 \\
 \phantom{4x + 5 \overline{) \quad}} \underline{28x + 35} \\
 \phantom{4x + 5 \overline{) \quad}} - 3
 \end{array}$$

$$\begin{array}{r}
 31. \quad \quad \quad x^2 - 6x + 6 \\
 4x + 3 \overline{) \quad 4x^3 - 21x^2 + 6x + 19} \\
 \underline{- 4x^3 - 3x^2} \\
 \phantom{4x + 3 \overline{) \quad}} - 24x^2 + 6x \\
 \phantom{4x + 3 \overline{) \quad}} \underline{24x^2 + 18x} \\
 \phantom{4x + 3 \overline{) \quad}} 24x + 19 \\
 \phantom{4x + 3 \overline{) \quad}} \underline{- 24x - 18} \\
 \phantom{4x + 3 \overline{) \quad}} 1
 \end{array}$$

$$\begin{array}{r}
 33. \quad \quad \quad x^2 + 3x - 7 \\
 2x + 6 \overline{) \quad 2x^3 + 12x^2 + 4x - 37} \\
 \underline{- 2x^3 - 6x^2} \\
 \phantom{2x + 6 \overline{) \quad}} 6x^2 + 4x \\
 \phantom{2x + 6 \overline{) \quad}} \underline{- 6x^2 - 18x} \\
 \phantom{2x + 6 \overline{) \quad}} - 14x - 37 \\
 \phantom{2x + 6 \overline{) \quad}} \underline{14x + 42} \\
 \phantom{2x + 6 \overline{) \quad}} 5
 \end{array}$$

$$\begin{array}{r}
 35. \quad \quad \quad x - 3 \\
 10x - 2 \overline{) \quad 10x^2 - 32x + 9} \\
 \underline{- 10x^2 + 2x} \\
 \phantom{10x - 2 \overline{) \quad}} - 30x + 9 \\
 \phantom{10x - 2 \overline{) \quad}} \underline{30x - 6} \\
 \phantom{10x - 2 \overline{) \quad}} 3
 \end{array}$$

$$\begin{array}{r}
 37. \quad \quad \quad 9x + 5 \\
 3x + 8 \overline{) \quad 27x^2 + 87x + 35} \\
 \underline{- 27x^2 - 72x} \\
 \phantom{3x + 8 \overline{) \quad}} 15x + 35 \\
 \phantom{3x + 8 \overline{) \quad}} \underline{- 15x - 40} \\
 \phantom{3x + 8 \overline{) \quad}} - 5
 \end{array}$$

$$\begin{array}{r}
 39. \quad \quad \quad 8x - 9 \\
 6x - 2 \overline{) \quad 48x^2 - 70x + 16} \\
 \underline{- 48x^2 + 16x} \\
 \phantom{6x - 2 \overline{) \quad}} - 54x + 16 \\
 \phantom{6x - 2 \overline{) \quad}} \underline{54x - 18} \\
 \phantom{6x - 2 \overline{) \quad}} - 2
 \end{array}$$

$$\begin{array}{r}
 41. \quad \quad \quad 6x^2 + x + 9 \\
 4x - 7 \overline{) \quad 24x^3 - 38x^2 + 29x - 60} \\
 \underline{- 24x^3 + 42x^2} \\
 \phantom{4x - 7 \overline{) \quad}} 4x^2 + 29x \\
 \phantom{4x - 7 \overline{) \quad}} \underline{- 4x^2 + 7x} \\
 \phantom{4x - 7 \overline{) \quad}} 36x - 60 \\
 \phantom{4x - 7 \overline{) \quad}} \underline{- 36x + 63} \\
 \phantom{4x - 7 \overline{) \quad}} 3
 \end{array}$$

Synthetic Division

$$\begin{array}{r}
1. \quad -2 \left| \begin{array}{rrrrr} 1 & -4 & 2 & -1 & 1 \\ & -2 & 12 & -28 & 58 \\ \hline & 1 & -6 & 14 & -29 & \mathbf{59} \end{array} \right. \\
\\
3. \quad -2 \left| \begin{array}{rrrrr} 2 & -2 & -10 & 0 & 1 \\ & -4 & 12 & -4 & 8 \\ \hline & 2 & -6 & 2 & -4 & \mathbf{9} \end{array} \right. \\
\\
5. \quad -5 \left| \begin{array}{rrrrrr} -1 & -1 & 1 & 1 & 1 \\ & 5 & -20 & 95 & -480 \\ \hline & -1 & 4 & -19 & 96 & \mathbf{-479} \end{array} \right. \\
\\
7. \quad -2 \left| \begin{array}{rrrrr} 12 & -1 & 1 & -3 & 1 \\ & -24 & 50 & -102 & 210 \\ \hline & 12 & -25 & 51 & -105 & \mathbf{211} \end{array} \right. \\
\\
9. \quad 2 \left| \begin{array}{rrrrr} 1 & -3 & 5 & -14 & 2 \\ & 2 & -2 & 6 & -16 \\ \hline & 1 & -1 & 3 & -8 & \mathbf{-14} \end{array} \right. \\
\\
11. \quad 3 \left| \begin{array}{rrrrr} 1 & 0 & 0 & -3 & -4 \\ & 3 & 9 & 27 & 72 \\ \hline & 1 & 3 & 9 & 24 & \mathbf{68} \end{array} \right.
\end{array}$$

End Behavior

1. As $x \rightarrow -\infty$, $f(x) \rightarrow \infty$. As $x \rightarrow \infty$, $f(x) \rightarrow -\infty$.
3. As $x \rightarrow -\infty$, $h(x) \rightarrow -\infty$. As $x \rightarrow \infty$, $h(x) \rightarrow -\infty$.
5. As $x \rightarrow -\infty$, $\ell(x) \rightarrow -\infty$. As $x \rightarrow \infty$, $\ell(x) \rightarrow \infty$.
7. As $x \rightarrow -\infty$, $n(x) \rightarrow \infty$. As $x \rightarrow \infty$, $n(x) \rightarrow \infty$.
9. As $x \rightarrow -\infty$, $q(x) \rightarrow -\infty$. As $x \rightarrow \infty$, $q(x) \rightarrow -\infty$.
11. $a_n x^n = -1x^3$, $a_0 = 1$, As $x \rightarrow -\infty$, $f(x) \rightarrow \infty$. As $x \rightarrow \infty$, $f(x) \rightarrow -\infty$.
13. $a_n x^n = 27x^3$, $a_0 = -64$, As $x \rightarrow -\infty$, $k(x) \rightarrow -\infty$. As $x \rightarrow \infty$, $k(x) \rightarrow \infty$.
15. $a_n x^n = -8x^4$, $a_0 = -98$, As $x \rightarrow -\infty$, $m(x) \rightarrow -\infty$. As $x \rightarrow \infty$, $m(x) \rightarrow -\infty$.
17. $a_n x^n = 1x^4$, $a_0 = 16$, As $x \rightarrow -\infty$, $g(x) \rightarrow \infty$. As $x \rightarrow \infty$, $g(x) \rightarrow \infty$.
19. $a_n x^n = 1x^4$, $a_0 = 2$, As $x \rightarrow -\infty$, $k(x) \rightarrow \infty$. As $x \rightarrow \infty$, $k(x) \rightarrow \infty$.

Local Behavior

1. $x_1 = 0, k_1 = 3$, Crossover; $x_2 = 2, k_2 = 1$, Crossover; $x_3 = -2, k_3 = 1$, Crossover
3. $x_1 = 0, k_1 = 1$, Crossover; $x_2 = 3, k_2 = 2$, Turnaround; $x_3 = -3, k_3 = 1$, Crossover
5. No real roots
7. $x_1 = 1, k_1 = 1$, Crossover; $x_2 = -1, k_2 = 1$, Crossover; $x_3 = -4, k_3 = 1$, Crossover
9. $x_1 = 0, k_1 = 3$, Crossover; $x_2 = \frac{1}{3}, k_2 = 1$, Crossover; $x_3 = 2, k_3 = 1$, Crossover
11. $x_1 = 2, k_1 = 2$, Turnaround; $x_2 = -5, k_2 = 1$, Crossover; $x_3 = 3, k_3 = 1$, Crossover

The Rational Root Theorem

1. $f(x) = (x - 1)(x + 2)(x - 3)$
List of possible rational roots: $\{\pm 6, \pm 3, \pm 2, \pm 1\}$
3. $f(x) = (x - 1)^2(x + 1)(x - 6)(x + 6)$
List of possible rational roots: $\{\pm 36, \pm 18, \pm 12, \pm 9, \pm 6, \pm 4, \pm 3, \pm 2, \pm 1\}$
5. $f(x) = (x^2 + 4)(x - 1)^2$
List of possible rational roots: $\{\pm 4, \pm 2, \pm 1\}$
7. $f(x) = -2(x - \frac{1}{2})(x - 4)(x - 5)$
List of possible rational roots: $\{\pm 20, \pm 10, \pm 5, \pm 4, \pm \frac{5}{2}, \pm 2, \pm 1, \pm \frac{1}{2}\}$
9. $f(x) = (x - 1)(x + 2)^2(x - 3)$
List of possible rational roots: $\{\pm 12, \pm 6, \pm 4, \pm 3, \pm 2, \pm 1\}$
11. $f(x) = 6(x + 2)(x - \frac{4}{3})(x + \frac{5}{2})$
List of possible rational roots:
 $\{\pm 40, \pm 20, \pm \frac{40}{3}, \pm 10, \pm 8, \pm \frac{20}{3}, \pm 5, \pm 4, \pm \frac{10}{3}, \pm \frac{8}{3}, \pm \frac{5}{2}, \pm 2, \pm \frac{5}{3}, \pm \frac{4}{3}, \pm 1, \pm \frac{5}{6}, \pm \frac{2}{3}, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}\}$

Graphing Summary

1. $f(x) = -17(x - \frac{5}{17})(x + \sqrt{2})(x - \sqrt{2})$
3. $f(x) = (3x^2 + 1)(x + \sqrt{5})(x - \sqrt{5})$
5. $f(x) = (x^2 + 1)(x - 1)(x + 1)(x - 2)$
7. $f(x) = (x^3 - 8)(x^3 + 2) = (x - 2)(x^2 + 2x + 4)(x + \sqrt[3]{2})(x^2 - \sqrt[3]{2}x + \sqrt[3]{4})$
9. $f(x) = -(x^2 + 1)(x - 7)$
11. $f(x) = 2x\left(x - \frac{5}{4} + \frac{\sqrt{33}}{4}\right)\left(x - \frac{5}{4} - \frac{\sqrt{33}}{4}\right)$
13. $f(x) = (x - 1)(x + 2)(x - 3)$
15. $f(x) = (x - 1)^2(x + 1)(x - 6)(x + 6)$
17. $f(x) = (x - 1)(x + 2)^2(x - 3)$

19. $f(x) = 2(x+1)\left(x - \frac{1}{2}\right)(x + \sqrt{3})(x - \sqrt{3})$

21. $f(x) = 6(x+2)\left(x - \frac{4}{3}\right)\left(x + \frac{5}{2}\right)$

23. $f(x) = 36\left(x - \frac{1}{2}\right)^2\left(x + \frac{1}{3}\right)^2$

25. $f(x) = 2(x-6)\left(x + \frac{7}{4} + \frac{\sqrt{129}}{4}\right)\left(x + \frac{7}{4} - \frac{\sqrt{129}}{4}\right)$

27. $f(x) = -2(x+1)(x-2)(x+2)\left(x - \frac{9}{2}\right)$

Polynomial Inequalities

1. $(-\infty, \sqrt{2}] \cup [\sqrt{2}, \infty)$

3. $[1, \infty)$

5. $(-\infty, 0) \cup (2, \infty)$

7. $[4, \infty)$

9. $(-\infty, 4)$

11. $(-\infty, -1] \cup [3, \infty)$