Lesson 35 : Inverse Functions - Finding an Inverse Function, f^{-1}

Objective: Find the inverse of a given function.

Students will be able to:

- Find the inverse of a function algebraically.

Prerequisite Knowledge:

- Definition and properties of inverse function.

Lesson:

Steps for finding the Inverse of a Function

1. Rewrite $f(x)$ as y.
2. Switch x and y.
3. Solve for y.
4. Rewrite y as $f^{-1}(x)$.

I - Motivating Example(s):

Example: Find the inverse f^{-1} of the function $f(x) = \dfrac{1-2x}{5}$.

We replace $f(x)$ with y and proceed to switch x and y

$$\begin{aligned} y &= \frac{1-2x}{5} \\ x &= \frac{1-2y}{5} && \text{Switch } x \text{ and } y \\ 5x &= 1-2y && \text{Solve for } y \\ 5x-1 &= -2y \\ \frac{5x-1}{-2} &= y \\ y &= -\frac{5}{2}x + \frac{1}{2} \end{aligned}$$

We have $f^{-1}(x) = -\dfrac{5}{2}x + \dfrac{1}{2}$.

To verify this answer, we leave it as an exercise to the reader to check that $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f, and $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}. Note that since f and f^{-1} are both linear functions, the domain and range for each function is $(-\infty, \infty)$.

Example: Find the inverse g^{-1} of the function $g(x) = \dfrac{2x}{1-x}$.

Notice that the domain of g is $(-\infty, 1) \cup (1, \infty)$. One can verify graphically, that the range of g is $(-\infty, -2) \cup (-2, \infty)$.

To find $g^{-1}(x)$, we start by replacing $g(x)$ with y.

$$\begin{array}{rcll}
y & = & \dfrac{2x}{1-x} & \\
x & = & \dfrac{2y}{1-y} & \text{Switch } x \text{ and } y \\
x(1-y) & = & 2y & \text{Solve for } y\text{; clear denominator} \\
x - xy & = & 2y & \text{Distribute } x \\
x & = & xy + 2y & \text{Move } y \text{ terms to one side} \\
x & = & y(x+2) & \text{Factor out } y \\
y & = & \dfrac{x}{x+2} & \text{Divide by } x+2
\end{array}$$

We have $g^{-1}(x) = \dfrac{x}{x+2}$.

Notice that the domain of g^{-1} matches the range of g from earlier, $(-\infty, -2) \cup (-2, \infty)$. Again, we can use the graph of g^{-1} to verify that the range of g^{-1} also matches the domain of g, $(-\infty, 1) \cup (1, \infty)$.

We leave it as an exercise to show that $\left(g^{-1} \circ g\right)(x) = x$ and $\left(g \circ g^{-1}\right)(x) = x$.

II - Demo/Discussion Problems:

Find the inverse function for each of the following functions. Graph both the original function and your answer using Desmos to confirm your results and compare the domains and ranges for your pair of functions.

1. $f(x) = 3\sqrt{x} + 4$
2. $g(x) = 2\sqrt{x-1} - 4$
3. $h(x) = -x^2 - 1$, $x \geq 0$
4. $k(x) = -x^2 - 1$, $x \leq 0$
5. $\ell(x) = x^2 + 10x + 15$, where $x \geq -5$
6. $m(x) = x^2 + 10x + 15$, where $x \leq -5$

III - Practice Problems:

Find the inverse function for each of the following functions. Check your answer algebraically by finding $f \circ f^{-1}$ and $f^{-1} \circ f$. Graph both the original function and your answer using Desmos to confirm your results and compare the domains and ranges for your pair of functions.

1. $f(x) = 2 - 6x$
2. $f(x) = \dfrac{x-2}{3} + 4$
3. $f(x) = 1 - \dfrac{4+3x}{5}$
4. $f(x) = \sqrt{3x-1} + 5$
5. $f(x) = 2 - \sqrt{x-5}$
6. $f(x) = 3\sqrt{x-1} - 4$
7. $f(x) = 1 - 2\sqrt{2x+5}$
8. $f(x) = \sqrt[3]{3x-1}$
9. $f(x) = 3 - \sqrt[3]{x-2}$
10. $f(x) = 8(x-2)^3$
11. $f(x) = (x+3)^2 - 6,\ x \geq -3$
12. $f(x) = 2(x-1)^2 + 4,\ x < 1$
13. $f(x) = x^2 - 6x + 5,\ x \leq 3$
14. $f(x) = 4x^2 + 4x + 1,\ x < -1$
15. $f(x) = \dfrac{3}{4-x}$
16. $f(x) = \dfrac{x}{1-3x}$
17. $f(x) = \dfrac{2x-1}{3x+4}$
18. $f(x) = \dfrac{4x+2}{3x-6}$
19. $f(x) = \dfrac{-3x-2}{x+3}$
20. $f(x) = \dfrac{x-2}{2x-1}$