Lesson 23: Complex Numbers

Objective: Simplify expressions involving complex numbers.

Students will be able to:

- Define the form of a complex number.
- Simplify square roots with negative radicands.
- Add, subtract, multiply, rationalize, and simplify expressions using complex numbers.

Prerequisite Knowledge:

- Properties of exponents.
- Combining like terms.
- Polynomial arithmetic.

Lesson:

To work with the square root of a negative number, mathematicians have defined what we now know as imaginary and complex numbers.

Imaginary Number $i: \; i^2 = -1$ (thus $i = \sqrt{-1}$)

Examples of imaginary numbers include $3i$, $-6i$, $\frac{3}{5}i$, and $3\sqrt{5}i$. A *complex number* is one that contains both a real and imaginary part, such as $2+5i$.

Complex Number: $a + bi$, where a and b are real numbers, $i = \sqrt{-1}$

With this definition, the square root of a negative number will no longer be considered undefined. We now will be able to perform basic operations with the square root of a negative number.

First, we consider powers of the imaginary number i. As the exponents of i^n increase, our simplified value for i^n will cycle through the simplified values for i, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$. As there are 4 different possible answers in this cycle, if we divide the exponent n by 4 and consider the remainder, we can easily simplify any power of i by knowing the following four values:

Cyclic Property of Powers of i

$$\begin{aligned} i^0 &= 1 \\ i^1 &= i \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 = i^0 &= 1 \end{aligned}$$

I - Motivating Example(s):

Example: Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl} (2+5i)+(4-7i) & \text{Combine like terms, } 2+4 \text{ and } 5i-7i \\ 6-2i & \text{Our solution} \end{array}$$

Example: Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl} (2-4i)(3+5i) & \text{Expand} \\ 6+10i-12i-20i^2 & \text{Simplify, } i^2=-1 \\ 6+10i-12i-20(-1) & \text{Multiply} \\ 6+10i-12i+20 & \text{Combine like terms } 6+20 \text{ and } 10i-12i \\ 26-2i & \text{Our solution} \end{array}$$

Example: Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl} \dfrac{2-6i}{4+8i} & \text{Binomial in denominator,} \\ & \quad \text{multiply by conjugate, } 4-8i \\ \dfrac{2-6i}{4+8i}\left(\dfrac{4-8i}{4-8i}\right) & \text{Expand the numerator,} \\ & \quad \text{denominator is a difference of two squares} \\ \dfrac{8-16i-24i+48i^2}{16-64i^2} & \text{Simplify } i^2=-1 \\ \dfrac{8-16i-24i+48(-1)}{16-64(-1)} & \text{Multiply} \\ \dfrac{8-16i-24i-48}{16+64} & \text{Combine like terms} \\ \dfrac{-40-40i}{80} & \text{Reduce, factor out 40 and divide} \\ \dfrac{-1-i}{2} & \text{Rewrite as } a+bi \\ -\dfrac{1}{2}-\dfrac{1}{2}i & \text{Our solution} \end{array}$$

II - Demo/Discussion Problems:

Rewrite each of the following complex numbers in the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$.

1. $\sqrt{-16}$
2. $\sqrt{-24}$
3. $\sqrt{-6}\sqrt{3}$
4. $\dfrac{-15 - \sqrt{-200}}{20}$
5. i^{35}
6. i^{124}
7. $(4 - 8i) - (3 - 5i)$
8. $5i - (3 + 8i) + (-4 + 7i)$
9. $(3i)(7i)$
10. $5i(3i - 7)$
11. $(3i)(6i)(2 - 3i)$
12. $(4 - 5i)^2$
13. $\dfrac{7 + 3i}{-5i}$

III - Practice Problems:

Rewrite each of the following complex numbers in the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$.

1. $\sqrt{-81}$
2. $\sqrt{-45}$
3. $\sqrt{-10}\sqrt{-2}$
4. $\sqrt{-12}\sqrt{-2}$
5. $\dfrac{3 + \sqrt{-27}}{6}$
6. $\dfrac{-4 - \sqrt{-8}}{-4}$
7. $\dfrac{8 - \sqrt{-16}}{4}$
8. $\dfrac{6 + \sqrt{-32}}{4}$
9. i^{73}
10. i^{251}
11. i^{48}
12. i^{68}
13. i^{62}
14. i^{181}
15. i^{154}
16. i^{51}
17. $3 - (-8 + 4i)$
18. $3i - (7i)$
19. $7i - (3 - 2i)$
20. $5 + (-6 - 6i)$
21. $-6i - (3 + 7i)$
22. $-8i - 7i - (5 - 3i)$
23. $(3 - 3i) + (-7 - 8i)$
24. $(-4 - i) + (1 - 5i)$
25. $-6 + i - (2 + 3i)$
26. $(5 - 4i) + (8 - 4i)$
27. $(6i)(-8i)$
28. $(3i)(-8i)$
29. $(-5i)(8i)$
30. $(8i)(-4i)$
31. $(-7i)^2$
32. $(-i)(7i)(4 - 3i)$
33. $(6 + 5i)^2$
34. $(8i)(-2i)(-2 - 8i)$
35. $(-7 - 4i)(-8 + 6i)$
36. $(3i)(-3i)(4 - 4i)$
37. $(-4 + 5i)(2 - 7i)$
38. $-8(4 - 8i) - 2(-2 - 6i)$
39. $(-8 - 6i)(-4 + 2i)$
40. $(-6i)(3 - 2i) - (7i)(4i)$
41. $(1 + 5i)(2 + i)$
42. $(-2 + i)(3 - 5i)$
43. $\dfrac{-9 + 5i}{i}$
44. $\dfrac{-3 + 2i}{-3i}$
45. $\dfrac{-10 - 9i}{6i}$
46. $\dfrac{-4 + 2i}{3i}$
47. $\dfrac{-3 - 6i}{4i}$
48. $\dfrac{-5 + 9i}{9i}$
49. $\dfrac{10 - i}{-i}$
50. $\dfrac{10}{5i}$
51. $\dfrac{4i}{-10 + i}$
52. $\dfrac{9i}{1 - 5i}$
53. $\dfrac{8}{7 - 6i}$
54. $\dfrac{4}{4 + 6i}$
55. $\dfrac{7}{10 - 7i}$
56. $\dfrac{9}{-8 - 6i}$
57. $\dfrac{5i}{-6 - i}$
58. $\dfrac{8i}{6 - 7i}$