Lesson 47: Polynomial Division

Objective: Apply polynomial division.

Students will be able to:

- Divide polynomials of varying degrees.
- Correctly label a divisor, dividend, quotient, and remainder in a polynomial division equation.

Prerequisite Knowledge:

- Polynomial definition and terminology.
- Combining like terms.
- Distributive property.
- Properties of exponents.

Lesson:

Let's recall the terminology and format associated with division.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

Alternatively, multiplying both sides of the above equation by the divisor, we have the following.

$$\frac{\text{dividend}}{\cancel{\text{divisor}}} \cdot \cancel{\text{divisor}} = \text{quotient} \cdot \text{divisor} + \frac{\text{remainder}}{\cancel{\text{divisor}}} \cdot \cancel{\text{divisor}}$$
$$\text{dividend} = \text{quotient} \cdot \text{divisor} + \text{remainder}$$

The general process for division of polynomials follows closely with that for dividing integers.

General Steps for Polynomial (Long) Division

Let $D(x)$ and $d(x)$ represent two nonzero polynomial functions. The steps for simplifying the rational expression $\dfrac{D(x)}{d(x)}$ are as follows.

1. Divide the leading term of the dividend D by the leading term of the divisor d. Label the resulting term $a_n x^n$, and write it above the dividend. This will be the leading term of the quotient, $q(x)$.
2. Multiply $a_n x^n$ by the divisor, distribute, and simplify. Label this as $d_1(x)$ and write it directly below the dividend, D, making sure to align terms according to exponents.
3. Subtract the resulting terms from the dividend. Label the new expression D_1.

4. Repeat steps (1)-(3) for the divisor d and the new expression D_i until the degree of D_i is *less than* the degree of the divisor. Relabel the final new dividend as the remainder, $r(x)$. The entire polynomial expression appearing above the original dividend is the quotient, $q(x)$.

$$
\begin{array}{rl}
 & \underline{\quad q(x) \quad} \\
d(x)\,) & D(x) \\
- & \underline{d_1(x)} \\
 & \quad D_1(x) \\
 & - \ \underline{d_2(x)} \\
 & \qquad \ddots \\
 & \qquad - \ \underline{d_i(x)} \\
 & \qquad\quad D_i(x) = r(x)
\end{array}
$$

Step (3) often tends to pose the greatest challenge for students. It is important to keep in mind that we are are always subtracting the top term from the bottom term, which is why we must change the signs of the term(s) on the bottom. In most cases, we will need to utilize the distributive property.

I - Motivating Example(s):

Example: Divide $9x^5 + 6x^4 - 18x^3 - 24x^2$ by $3x^2$. Simplify and express your answer in the form

$$\frac{D(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

$$
\begin{array}{rllll}
 & 3x^3 & +\,2x^2 & -\,6x & -\,8 \\
3x^2\,) & 9x^5 + 6x^4 & -\,18x^3 & -\,24x^2 & \\
 & \underline{-\,9x^5} & & & \\
 & \quad 6x^4 & & & \\
 & \underline{-\,6x^4} & & & \\
 & & -\,18x^3 & & \\
 & & \underline{\ \ 18x^3} & & \\
 & & & -\,24x^2 & \\
 & & & \underline{\ \ 24x^2} & \\
 & & & \quad 0 &
\end{array}
$$

We set up our division process by first writing the dividend and the divisor in the appropriate locations. Next, we identify the leading term for our quotient, $3x^3$. Multiplying and subtracting produces our new expression, $D_1(x) = 6x^4$. While it is perfectly fine to carry down $-18x^3 - 24x^2$, it is not necessary until these terms play a role in the subtraction step.

Repeating our division steps gives us the second term in our quotient, $2x^2$. Multiplying, subtracting, and carrying down the next term gives us our new expression of $D_2(x) = -18x^3$. Repeating our steps again produces the third term in our quotient, $-6x$. Multiplying and subtracting produces another new expression, $D_3(x) = -24x^2$. Since the degree of D_3 equals that of our divisor, $d(x) = 3x^2$, we will need to apply our steps for division one final time. After our fourth and final round of steps, our new expression produces a remainder of $r(x) = 0$.

We express the results of our division in the required form as follows.

$$\frac{9x^5 + 6x^4 - 18x^3 - 24x^2}{3x^2} = 3x^3 + 2x^2 - 6x - 8 + \frac{0}{3x^2}$$

Example: Divide $3x^3 - 5x^2 - 32x + 7$ by $x - 4$. Simplify and express your answer in the form

$$\frac{D(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

$$
\begin{array}{rrrrr}
 & 3x^2 & +7x & -4 & \\
\cline{2-5}
x-4 \big) & 3x^3 & -5x^2 & -32x & +7 \\
 & -3x^3 & +12x^2 & & \\
\cline{2-3}
 & & 7x^2 & -32x & \\
 & & -7x^2 & +28x & \\
\cline{3-4}
 & & & -4x & +7 \\
 & & & 4x & -16 \\
\cline{4-5}
 & & & & -9
\end{array}
$$

Since our quotient, $3x^2 + 7x - 4$, is a trinomial, we must apply the steps for division three times.

In this second example, our remainder is the constant term $r(x) = -9$, which has one degree less than our linear divisor, $d(x) = x - 4$.

Our answer is $\dfrac{3x^3 - 5x^2 - 32x + 7}{x - 4} = 3x^2 + 7x - 4 + \dfrac{-9}{x - 4}$.

II - Demo/Discussion Problems:

Use polynomial long division to divide and simplify each of the given expressions. Express each answer in the form below.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

1. $\dfrac{8x^3 + 4x^2 - 2x + 6}{4x^2}$
2. $\dfrac{n^2 + 7n + 15}{n + 4}$
3. $\dfrac{x^3 - 46x + 22}{x + 7}$
4. $\dfrac{6x^3 - 8x^2 + 10x + 103}{4 + 2x}$
5. $\dfrac{2x^3 - 4x + 42}{x + 3}$

III - Practice Problems:

Use polynomial long division to divide and simplify each of the given expressions. Express each answer in the form below.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

1. $\dfrac{20x^4 + x^3 + 2x^2}{4x^3}$
2. $\dfrac{5x^4 + 45x^3 + 4x^2}{9x}$
3. $\dfrac{20x^4 + x^3 + 40x^2}{10x}$
4. $\dfrac{3x^3 + 4x^2 + 2x}{8x}$
5. $\dfrac{12x^4 + 24x^3 + 3x^2}{6x}$
6. $\dfrac{5x^4 + 16x^3 + 16x^2}{4x}$
7. $\dfrac{10x^4 + 50x^3 + 2x^2}{10x^2}$
8. $\dfrac{3x^4 + 18x^3 + 27x^2}{9x^2}$
9. $\dfrac{x^2 - 2x - 71}{x + 8}$
10. $\dfrac{x^2 - 3x - 53}{x - 9}$
11. $\dfrac{x^2 + 13x + 32}{x + 5}$
12. $\dfrac{x^2 - 10x + 16}{x - 7}$
13. $\dfrac{x^2 - 2x - 89}{x - 10}$
14. $\dfrac{x^2 + 4x - 26}{x + 7}$
15. $\dfrac{x^2 - 4x - 38}{x - 8}$
16. $\dfrac{x^2 - 4}{x - 2}$
17. $\dfrac{x^3 + 15x^2 + 49x - 55}{x + 7}$
18. $\dfrac{x^3 - 26x - 41}{x + 4}$
19. $\dfrac{3x^3 + 9x^2 - 64x - 68}{x + 6}$
20. $\dfrac{9x^3 + 45x^2 + 27x - 5}{9x + 9}$
21. $\dfrac{x^3 - x^2 - 16x + 8}{x - 4}$
22. $\dfrac{x^2 - 10x + 22}{x - 4}$
23. $\dfrac{x^3 - 16x^2 + 71x - 56}{x - 8}$
24. $\dfrac{x^3 - 4x^2 - 6x + 4}{x - 1}$
25. $\dfrac{8x^3 - 66x^2 + 12x + 37}{x - 8}$
26. $\dfrac{3x^2 + 9x - 9}{3x - 3}$
27. $\dfrac{2x^2 - 5x - 8}{2x + 3}$
28. $\dfrac{3x^2 - 32}{3x - 9}$
29. $\dfrac{4x^2 - 23x - 38}{4x + 5}$
30. $\dfrac{2x^3 + 21x^2 + 25x}{2x + 3}$
31. $\dfrac{4x^3 - 21x^2 + 6x + 19}{4x + 3}$
32. $\dfrac{8x^3 - 57x^2 + 42}{8x + 7}$
33. $\dfrac{2x^3 + 12x^2 + 4x - 37}{2x + 6}$
34. $\dfrac{45x^2 + 56x + 19}{9x + 4}$
35. $\dfrac{10x^2 - 32x + 9}{10x - 2}$
36. $\dfrac{4x^2 - x - 1}{4x + 3}$
37. $\dfrac{27x^2 + 87x + 35}{3x + 8}$
38. $\dfrac{4x^2 - 33x + 28}{4x - 5}$
39. $\dfrac{48x^2 - 70x + 16}{6x - 2}$
40. $\dfrac{12x^3 + 12x^2 - 15x - 4}{2x + 3}$
41. $\dfrac{24x^3 - 38x^2 + 29x - 60}{4x - 7}$