Lesson 29: Quadratic Inequalities

CC attribute: *College Algebra* by C. Stitz and J. Zeager.

Objective: Solve quadratic inequalities using a sign diagram.

Students will be able to:

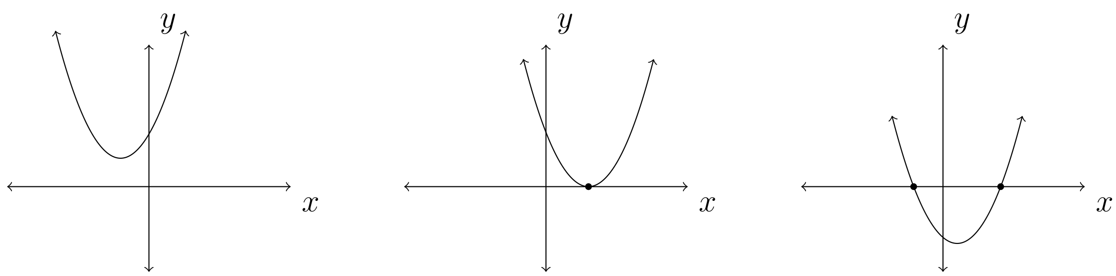
- Construct a sign diagram for a quadratic expression.
- Solve a quadratic inequality.
- Represent a solution to a quadratic inequality using interval notation.

Prerequisite Knowledge:

- Solving and factoring quadratic equations.
- Finding $x-$intercepts.
- Evaluating an expression for x.

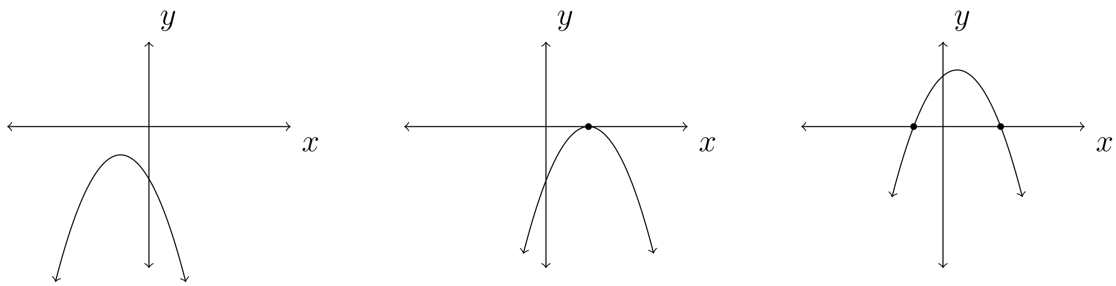
Lesson:

Recall that the *vertex form* for a quadratic equation is $y = a(x - h)^2 + k$, where $a \neq 0$ and (h, k) is the *vertex* of the corresponding parabola. If $a > 0$, then the parabola opens upward, and if $a < 0$, then the parabola opens downward. With any quadratic equation, we know that there are three possibilities for the number of roots, namely $0, 1$, or 2. Assuming $a > 0$, we illustrate these possibilities in the graphs below.



Notice also that each of these three graphs lie above the x-axis over different intervals. In the case of the parabola on the left, the entire graph lies above the x-axis, whereas the middle parabola lies above the x-axis everywhere *except* at its x-intercept (where $y = 0$). Even more interesting is the parabola on the right, which contains two *separate* intervals where its graph lies above the x-axis.

Considering the case where $a < 0$, we see three similar graphs as those appearing above, with the only major difference being the opening of each parabola downward instead of upward (when $a > 0$). When we consider again those intervals where each graph lies above the x-axis, each parabola exhibits a different behavior than those where $a > 0$.



Now, each of the first two graphs have no points that lie above the x-axis, whereas the last graph, on the right, lies above the x-axis over the interval that is between its x-intercepts.

Each of these six graphs exhibit all of the various possibilities for the *sign* of a quadratic expression $ax^2 + bx + c$, where $a \neq 0$. We can determine the general shape of the graph of a quadratic equation (or function) through identification of its roots and construction of a sign diagram. As a consequence, we will also use a sign diagram to solve a quadratic inequality.

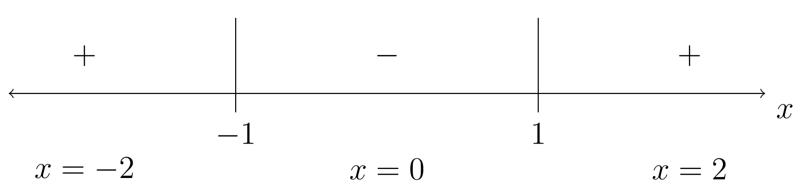
I - Motivating Example(s):

Example: Solve the quadratic inequality $x^2 - 1 > 0$.

Unlike with solving linear inequalities, one should not attempt to solve for the variable x, but rather set the given inequality equal to zero and attempt to *factor* the resulting expression on the other side. In doing this, we obtain $(x + 1)(x - 1) > 0$. Recalling that $x = \pm 1$ are roots of the given expression, we can therefore rule them out of our solution. Next, we will *test* the expression on the left by plugging in three values for x: one that is less than -1, one that is between -1 and 1, and one that is greater than 1.

Case	Test Value	Unsimplified	Simplified	Result
i	$x = -2$	$(-2 + 1)(-2 - 1)$	$(-)\cdot(-)$	$(+)$
ii	$x = 0$	$(0 + 1)(0 - 1)$	$(+)\cdot(-)$	$(-)$
iii	$x = 2$	$(2 + 1)(2 - 1)$	$(+)\cdot(+)$	$(+)$

Our end result can be summarized in the following *sign diagram*.



From our sign diagram, we can conclude that $x^2 - 1 > 0$ when $x < -1$ or $x > 1$. Using interval notation, our answer is $(-\infty, -1) \cup (1, \infty)$.

II - Demo/Discussion Problems:

Solve each of the following inequalities. Express your answers using interval notation.

1. $-x^2 + 4x + 5 \geq 0$
2. $x^2 - 17x \geq -60$
3. $x^2 + 4x + 1 > 0$
4. $-(x-1)^2 + 9 \geq 0$

Solve each of the following inequalities. Express your answers using interval notation.

5. $x^2 + 4x + 4 > 0$
6. $x^2 + 4x + 4 \geq 0$
7. $x^2 + 4x + 4 < 0$
8. $x^2 + 4x + 4 \leq 0$

III - Practice Problems:

Solve each of the following inequalities. Express your answers using interval notation.

1. $x^2 - 4 > 0$
2. $x^2 - 5x - 6 \leq 0$
3. $-(3x-2)(x+4) \geq 0$
4. $3x^2 + 2x - 8 > 0$
5. $2x^2 - 16 < 0$
6. $-x^2 - 13x + 30 \geq 0$
7. $x^2 - 5x + 6 > 0$
8. $2x^2 - 3x - 14 \leq 0$
9. $x^2 + 2x - 9 > 0$
10. $-2x^2 + 12x - 18 < 0$