Lesson 32: Function Arithmetic

Objective: Add, subtract, multiply, and divide functions.

Students will be able to:

- Evaluate functions that are added, subtracted, multiplied, or divided by substituting a value into each function, then applying the operation and simplifying.
- Apply the four basic operations to functions of the same variable.

Prerequisite Knowledge:

- Order of operations.
- Evaluating functions.
- Parentheses and grouping.

Lesson:

The notation for the four basic function operations is as follows.

$$\begin{array}{rl} \text{Addition} & (f+g)(x) = f(x) + g(x) \\ \text{Subtraction} & (f-g)(x) = f(x) - g(x) \\ \text{Multiplication} & (f \cdot g)(x) = f(x) \cdot g(x) \\ \text{Division} & \left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} \end{array}$$

I - Motivating Example(s):

Example: Find $f + g$, where $f(x) = x^2 - x - 2$ and $g(x) = x + 1$.

$$\begin{array}{cl} (f+g)(x) & \text{Consider the problem} \\ f(x) + g(x) & \text{Rewrite as a sum of two functions} \\ (x^2 - x - 2) + (x + 1) & \text{Substitute functions, inserting parentheses} \\ x^2 - x - 2 + x + 1 & \text{Simplify; remove the parentheses} \\ x^2 - x + x - 2 + 1 & \text{Combine like terms} \\ (f+g)(x) = x^2 - 1 & \text{Our solution} \\ = (x-1)(x+1) & \text{Our solution in factored form} \end{array}$$

Generally, either form (expanded or factored) would be considered acceptable.

Example: Find $g - f$, where $f(x) = x^2 - x - 2$ and $g(x) = x + 1$.

$(g-f)(x)$	Consider the problem
$g(x) - f(x)$	Rewrite as a difference of two functions
$(x+1) - (x^2 - x - 2)$	Substitute functions, inserting parentheses
$x + 1 - x^2 + x + 2$	Simplify; distribute the negative sign
$-x^2 + x + x + 1 + 2$	Combine like terms
$(g-f)(x) = -x^2 + 2x + 3$	Our solution
$= -(x-3)(x+1)$	Our solution in factored form

Example: Find $h \cdot k$, where $h(x) = 3x^2 - 4x$ and $k(x) = x - 2$.

$(h \cdot k)(x)$	Consider the problem
$h(x) \cdot k(x)$	Rewrite as a product of two functions
$(3x^2 - 4x)(x-2)$	Substitute functions, inserting parentheses
$3x^3 - 6x^2 - 4x^2 + 8x$	Expand by distributing
$3x^3 - 10x^2 + 8x$	Combine like terms
$(h \cdot k)(x) = 3x^3 - 10x^2 + 8x$	Our solution
$= x(3x-4)(x-2)$	Our solution in factored form

Example: Find $\dfrac{g}{f}$, where $f(x) = x^2 - x - 2$ and $g(x) = x + 1$.

$\left(\dfrac{g}{f}\right)(x)$	Consider the problem
$\dfrac{g(x)}{f(x)}$	Rewrite as a quotient of two functions
$\dfrac{x+1}{x^2 - x - 2}$	Substitute functions, parentheses unnecessary
$\dfrac{x+1}{(x+1)(x-2)}$	Factor (if possible)
$x \neq -1 \quad \text{and} \quad x \neq 2$	Restrict denominator: $g(x) \neq 0$
$\dfrac{\cancel{x+1}}{\cancel{(x+1)}(x-2)}$	Simplify: reduce $\dfrac{x+1}{x+1}$
$\left(\dfrac{g}{f}\right)(x) = \dfrac{1}{x-2}, \quad x \neq -1$	Our solution with added restriction

II - Demo/Discussion Problems:

Use $h(x) = 2x - 4$ and $k(x) = -3x + 1$ to find each of the following values for $x = 3$ in two ways:

a. Evaluate both h and k at $x = 3$, then combine and simplify the two values accordingly.

b. Find a simplified expression for the desired combined function, then evaluate it at $x = 3$.

1. $(h + k)(3)$
2. $(h \cdot k)(3)$
3. $(h - k)(3)$
4. $(k - h)(3)$
5. $\left(\dfrac{h}{k}\right)(3)$
6. $\left(\dfrac{k}{h}\right)(3)$

III - Practice Problems:

In each problem, use the pair of functions f and g to find the following values, if they exist.

- $(f + g)(2)$
- $(f - g)(-1)$
- $(g - f)(1)$
- $(fg)\left(\dfrac{1}{2}\right)$
- $\left(\dfrac{f}{g}\right)(0)$
- $\left(\dfrac{g}{f}\right)(-2)$

1. $f(x) = 3x + 1 \quad g(x) = 4 - x$
2. $f(x) = x^2 \quad g(x) = -2x + 1$
3. $f(x) = x^2 - x \quad g(x) = 12 - x^2$
4. $f(x) = 2x^3 \quad g(x) = -x^2 - 2x - 3$
5. $f(x) = \sqrt{x + 3} \quad g(x) = 2x - 1$
6. $f(x) = \sqrt{4 - x} \quad g(x) = \sqrt{x + 2}$
7. $f(x) = 2x \quad g(x) = \dfrac{1}{2x + 1}$
8. $f(x) = x^2 \quad g(x) = \dfrac{3}{2x - 3}$
9. $f(x) = x^2 \quad g(x) = \dfrac{1}{x^2}$
10. $f(x) = x^2 + 1 \quad g(x) = \dfrac{1}{x^2 + 1}$

In each problem, use the pair of functions f and g to find the domain of the indicated function then find and simplify an expression for it.

- $(f + g)(x)$
- $(f - g)(x)$
- $(fg)(x)$
- $\left(\dfrac{f}{g}\right)(x)$

11. $f(x) = 2x + 1 \quad g(x) = x - 2$
12. $f(x) = 1 - 4x \quad g(x) = 2x - 1$
13. $f(x) = x^2 \quad g(x) = 3x - 1$
14. $f(x) = x^2 - x \quad g(x) = 7x$
15. $f(x) = x^2 - 4 \quad g(x) = 3x + 6$
16. $f(x) = -x^2 + x + 6 \quad g(x) = x^2 - 9$
17. $f(x) = \dfrac{x}{2} \quad g(x) = \dfrac{2}{x}$
18. $f(x) = x - 1 \quad g(x) = \dfrac{1}{x - 1}$
19. $f(x) = x \quad g(x) = \sqrt{x + 1}$
20. $f(x) = g(x) = \sqrt{x - 5}$