Lesson 50: Polynomial Local Behavior; Roots and Multiplicities

Objective: Identify all real roots and their corresponding multiplicities for a polynomial function.

Students will be able to:

- Identify the multiplicity of a root of a polynomial f, and use it to describe the nature of the graph of f at the corresponding $x-$intercept.

Prerequisite Knowledge:

- Factoring.
- Properties of exponents.

Lesson:

In this lesson, we seek to classify $x-$intercepts for the graph of a polynomial as either "turnaround" or "crossover" points. This can be parsed down to one basic concept, known as the *multiplicity* of a root, defined as follows.

Definition: Suppose f is a polynomial function with real root $x = c$. For some positive integer k, if $(x-c)^k$ is a factor of f but $(x-c)^{k+1}$ is not, then we say $x = c$ is a root of f having associated multiplicity k.

Another way of describing the multiplicity k of a root $x = c$ is that k represents the maximum number of factors of $(x-c)$ that divide the polynomial f (with a remainder of 0). That is,

$$f(x) = (x-c)^k \cdot q(x),$$

where $(x-c)$ is *not* a factor of the quotient $q(x)$.

The multiplicity of a root of a polynomial tells us the following information about the corresponding $x-$intercept.

Let f be a polynomial function with a real root at $x = c$ having multiplicity k.

- If k is *even*, the corresponding $x-$intercept $(c, 0)$ is a *turnaround point*. In other words, the graph of f touches and rebounds from the x-axis at $(c, 0)$, leaving the $y-$values to maintain the same sign on either side of the root $x = c$.
- If k is *odd*, the corresponding $x-$intercept $(c, 0)$ is a *crossover point*. In other words, the graph of f crosses through the x-axis at $(c, 0)$, leaving the $y-$values to change signs on either side of the root $x = c$.

I - Motivating Example(s):

Example: Determine the set of roots and corresponding multiplicities for the following functions. In each case, classify the corresponding $x-$intercept as either a turnaround or crossover point.

1. $f(x) = x^6 - 2x^5 - 15x^4$
2. $g(x) = (x-6)^5(x+2)^2(x^2+1)$

1. Factoring f gives us the following.

$$\begin{aligned} f(x) &= x^6 - 2x^5 - 15x^4 \\ &= x^4(x^2 - 2x - 15) \\ &= x^4(x-5)(x+3) \end{aligned}$$

   We then can easily see that f has a root at $x = 0$ with multiplicity four, and roots at $x = 5$ and $x = -3$, each with multiplicity one. Hence, the graph of f has a turnaround point at the origin and crossover points at the $x-$intercepts $(-3, 0)$ and $(5, 0)$.

2. Since g is already factored, we see that $x = 6$ is a root having multiplicity five, and $x = -2$ is a root having multiplicity two. The factor of $x^2 + 1$ is meant to throw us off, since its roots are the imaginary numbers $\pm i$. Hence, the graph of g has a crossover point at $x-$intercept $(6, 0)$ and a turnaround point at $x-$intercept $(-2, 0)$.

II - Demo/Discussion Problems:

Determine the set of roots and corresponding multiplicities for the following functions. In each case, classify the corresponding $x-$intercept as either a turnaround or crossover point. Use Desmos to check your answers.

1. $f(x) = \frac{1}{2}(x-2)^2(x+5)(x-3)$
2. $g(x) = (x+2)^2(3x-1)(5-x)$
3. $h(x) = -(x-3)^2(x+1)(x+5)^2$

III - Practice Problems:

Determine the set of roots and corresponding multiplicities for the following functions. In each case, classify the corresponding $x-$intercept as either a turnaround or crossover point. Use Desmos to check your answers.

1. $f(x) = x^3(x-2)(x+2)$
2. $g(x) = (x^2+1)(1-x)$
3. $h(x) = x(x-3)^2(x+3)$
4. $k(x) = (3x-4)^3$
5. $\ell(x) = (x^2+2)(x^2+3)$
6. $m(x) = -2(x+7)^2(1-2x)^2$
7. $f(x) = (x^2-1)(x+4)$
8. $g(x) = (x^2-1)(x^2-16)$
9. $h(x) = -2x^3(3x-1)(2-x)$
10. $k(x) = (x^2-4x+1)(x+2)^2$