Lesson 56: Horizontal Asymptotes

Objective: Identify a horizontal asymptote in the graph of a rational function.

Students will be able to:

- Distinguish between and summarize the three cases for the end behavior of the graph of a rational function.
- Identify the equation of a horizontal asymptote in the graph of a rational function.
- Use appropriate notation to describe the end behavior of the graph of a rational function.

Prerequisite Knowledge:

- End behavior of polynomials, including degree and leading coefficient.

Lesson:

In this lesson, we will look at the end (or long run) behavior of the graph of a rational function f, as $x \to \pm\infty$.

Let $f(x) = \dfrac{p(x)}{q(x)}$ be a rational function with leading terms $a_n x^n$ and $b_m x^m$ of $p(x)$ and $q(x)$, respectively.

- If $n = m$, the graph of f will have a horizontal asymptote at $y = \dfrac{a_n}{b_m}$.
- If $n < m$, the graph of f will have a horizontal asymptote at $y = 0$.
- If $n > m$, the graph of f will not have a horizontal asymptote.

Since any polynomial is, by definition, also a rational function, we will begin by including the possibilities that $f(x) \to \infty$ or $f(x) \to -\infty$ for either the left (as $x \to -\infty$) or right (as $x \to \infty$) end behavior of the graph of a rational function f.

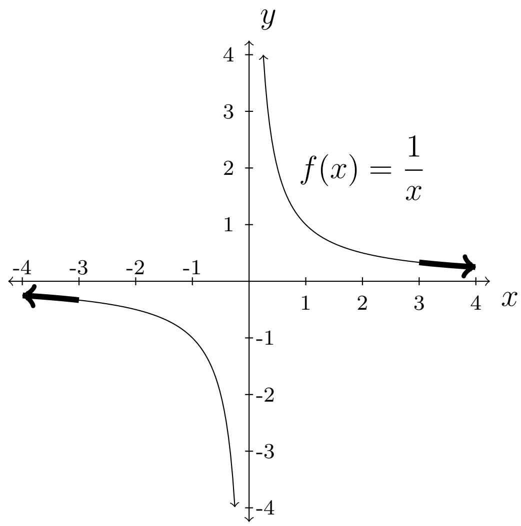
Recall that we used two aspects of a polynomial to identify the end behavior of its graph:

1. the parity of the degree (even or odd), and
2. the sign of the leading coefficient (positive or negative).

As with polynomials, we will use the degree and leading coefficient of both the numerator and denominator of a rational function f, to identify the end behavior of its graph.

I - Motivating Example(s):

Let us consider the graph of the reciprocal function $f(x) = \dfrac{1}{x}$, shown below.



This example presents us with the first instance in which a graph does not tend towards either ∞ or $-\infty$, but instead "levels off" as the values of x grow in either the positive (right) or negative (left) direction.

$$\text{As } x \to \infty,\ \ f(x) \to 0^+.$$

$$\text{As } x \to -\infty,\ \ f(x) \to 0^-.$$

Here, we use a $+$ or $-$ in the exponent to further describe how the tails of the graph approach 0, either from *above* $(+)$ or from *below* $(-)$. These identifiers can just as easily be omitted entirely, but provide a bit more insight into the graph of the function f. The tails of the graph are thickened for additional emphasis of this concept.

In fact, for any real number k, we can transform the graph above, by simply adding k to the function, to produce a new rational function whose graph levels off at k. The resulting graph represents a vertical shift of the graph of $\frac{1}{x}$ by k units. The shift is up when $k > 0$ and down when $k < 0$.

II - Demo/Discussion Problems:

Identify the equation of the horizontal asymptote, if one exists, for each of the following functions, and describe the end behavior of the function (As $x \to \pm\infty, \ldots$). Use Desmos to verify your answers.

1. $f(x) = \dfrac{-2x+4}{x-5}$
2. $g(x) = \dfrac{x-1}{x^2-4}$
3. $h(x) = \dfrac{x^2-4}{x-1}$
4. $j(x) = \dfrac{18x^3-4x-1}{6x^3-2x^2-8x}$

III - Practice Problems:

Identify the equation of the horizontal asymptote, if one exists, for each of the following functions, and describe the end behavior of the function (As $x \to \pm\infty, \ldots$). Use Desmos to verify your answers.

1. $a(x) = \dfrac{-x+4}{x-9}$
2. $b(x) = \dfrac{x^4 - x^3 - 2x - 19}{x^5 - 4}$
3. $c(x) = \dfrac{x^2 - 4}{5x^2 - 1}$
4. $d(x) = \dfrac{18x^7 - 4x - 1}{3x^7 - 2x^2 - 8x}$
5. $m(x) = \dfrac{18x^3 - 4x - 1}{3x^6 - 2x^2 - 8x}$
6. $n(x) = \dfrac{(x^2-9)(x-7)}{(x+3)(x-4)(2x-5)}$
7. $p(x) = \dfrac{-x^2+4}{x-9}$
8. $q(x) = \dfrac{x^3}{x^4 - 4}$
9. $r(x) = \dfrac{x^4 - 4}{x^3}$
10. $t(x) = \dfrac{15x - 10}{5x - 19}$