Lesson 34 : Inverse Functions - Definition and the HLT

Objective: Understand the definition of an inverse function and graphical implications. Determine whether a function is invertible.

Students will be able to:

- Define an inverse function.
- Obtain the graph of an inverse function from the graph of a function.
- Use the Horizontal Line Test (HLT) to determine if a function is invertible.

Prerequisite Knowledge:

- Plotting points on the Cartesian plane.
- Graphing horizontal lines.
- Graph functions by creating a table.

Lesson:

One often considers the operations of addition and subtraction to be "opposites" of one another, and similarly for multiplication and division. The reason for this, naturally, is because each of these operations "undoes" the other. In mathematics, since the term "opposite" can take on different meanings, we instead consider addition and subtraction (or multiplication and division) to be *inverse operations* of one another. This notion of an inverse can be applied to entire functions, which we will now discuss.

We start by analyzing a very basic function which is reversible, a linear function. Consider the function $f(x) = 3x + 4$. Thinking of f as a process, we start with an input x and apply two steps, in order:

1. multiply by 3
2. add 4.

To reverse this process, we seek a function g which will undo each of these steps, by taking the output from f, $3x + 4$, and returning the original input x. If we think of the real-world reversible two-step process of first putting on socks then putting on shoes, to reverse the process, we first take off the shoes, and then we take off the socks. In much the same way, the function g should undo the last step of f first. That is, the function g should:

1. subtract 4, then
2. divide by 3.

Following this procedure, we get $g(x) = \dfrac{x-4}{3}$.

Now we can test our function to see if it conceptually agrees with our "feet, socks, and shoes" analogy. Just as in the first part of the process we began with our bare feet and ended up in shoes, the reverse process brings us back, in the end, to our bare feet. We can see if this holds for f and g by using what we already know about functions.

For example, if $x = 5$, then

$$f(5) = 3(5) + 4 = 15 + 4 = 19.$$

Substituting the output 19 from f as our new input for g, we get our original input for f.

$$g(19) = \frac{19-4}{3} = \frac{15}{3} = 5$$

To check that g does this for all x in the domain of f (not just a single value), we will need to find and simplify the composite function $(g \circ f)(x) = g(f(x))$.

$$g(f(x)) = g(3x+4) = \frac{(3x+\cancel{4}) - \cancel{4}}{3} = \frac{\cancel{3}x}{\cancel{3}} = x$$

Not only does g "undo" f, but f also undoes g, which we can verify by once again looking at a composite function. This time we will find and simplify $(f \circ g)(x) = f(g(x))$.

$$f(g(x)) = f\left(\frac{x-4}{3}\right) = \cancel{3}\left(\frac{x-4}{\cancel{3}}\right) + 4 = (x - \cancel{4}) + \cancel{4} = x$$

Two functions f and g which are related in this manner are defined to be *inverse functions*, or simply *inverses*, of each other. More precisely, using the language of function composition, two functions f and g are said to be inverses if both:

- $g(f(x)) = x$ for all x in the domain of f, and
- $f(g(x)) = x$ for all x in the domain of g.

We say that a function f is *invertible* if an inverse function of f exists. If two functions g and f are inverses of each other, then we denote this by $g(x) = f^{-1}(x)$, and similarly $f(x) = g^{-1}(x)$. This notation can be a bit "gnarly" at first, since an inverse function f^{-1} of f must not be confused with the reciprocal function, $1/f$. The primary difference between these two functions is that a reciprocal function satisfies the property that

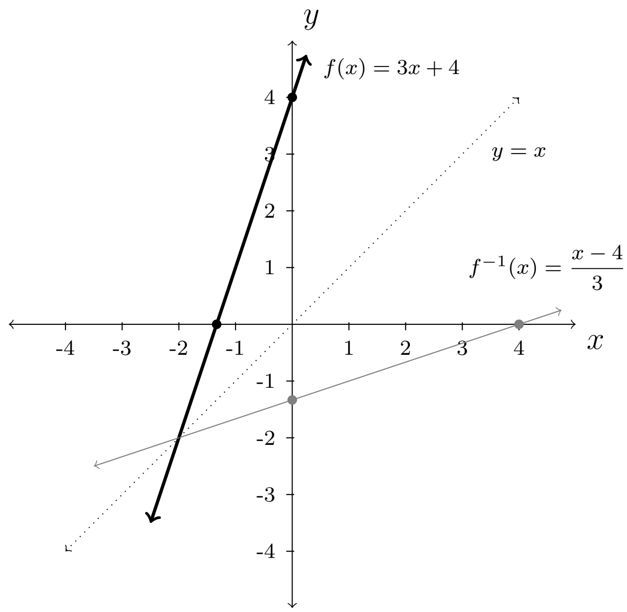
$$f(x) \cdot (1/f)(x) = 1,$$

whereas for inverses,

$$(f \circ f^{-1})(x) = x \quad \text{and} \quad (f^{-1} \circ f)(x) = x.$$

Properties of Inverse Functions:
Let f and f^{-1} be inverse functions of one another.

- The range of f is the domain of f^{-1} and the domain of f is the range of f^{-1}.
- $f(a) = b$ if and only if $f^{-1}(b) = a$.
- The point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}.



Graphically, we can identify one-to-one functions using the following test.

The Horizontal Line Test (HLT):
A function f is one-to-one if and only if no horizontal line intersects the graph of f more than once.

We say that the graph of a function **passes** the Horizontal Line Test if no horizontal line intersects the graph more than once; otherwise, we say the graph of the function **fails** the Horizontal Line Test.

Lastly, we have argued that if f is invertible, then f must be one-to-one, since otherwise the reflection of the graph of $y = f(x)$ about the line $y = x$ will fail the Vertical Line Test. It turns out that being one-to-one is also enough to guarantee invertibility of a function f. To see this, we can think of f as the set of ordered pairs which constitute its graph. If switching the x- and y-coordinates of the points results in a function (i.e., passes the VLT), then f is invertible and we have found the graph of its inverse, f^{-1}. This is precisely what the Horizontal Line Test does for us: it checks to see whether or not a set of points describes x as a function of y.

We can now summarize our results.

Equivalent Conditions for Invertibility:
Suppose f is a function. The following statements are equivalent.

- f is invertible (f^{-1} exists).
- f is one-to-one.
- The graph of f passes the Horizontal Line Test.

II - Demo/Discussion Problems:

Graph each function. Use the HLT to determine if the given function is invertible. If so, graph both the function and its inverse on the same set of axes. Identify at least three reference points for your function and its inverse.

1. $f(x) = 4x - 3$
2. $g(x) = \frac{1}{2}|x|$
3. $h(x) = (x+1)^2 - 4$
4. $k(x) = x^3$
5. $\ell(x) = \sqrt{x} - 5$
6. $m(x) = \dfrac{1}{x-2}$
7. $n(x) = x^2 - 10x,\ x \geq 5$
8. $p(x) = 3(x+4)^2 - 5,\ x \leq -4$

III - Practice Problems:

Graph each function in Desmos. Use the HLT to determine if the given function is invertible. If so, graph both the function and its inverse on the same set of axes. Identify at least three reference points for your function and its inverse.

1. $f(x) = 2 - 6x$
2. $f(x) = \dfrac{x-2}{3} + 4$
3. $f(x) = 1 - \dfrac{4+3x}{5}$
4. $f(x) = \sqrt{3x-1} + 5$
5. $f(x) = 2 - \sqrt{x-5}$
6. $f(x) = 3\sqrt{x-1} - 4$
7. $f(x) = 1 - 2\sqrt{2x+5}$
8. $f(x) = \sqrt[3]{3x-1}$
9. $f(x) = 3 - \sqrt[3]{x-2}$
10. $f(x) = x^2 - 6x + 5,\ x \leq 3$
11. $f(x) = 4x^2 + 4x + 1,\ x < -1$
12. $f(x) = \dfrac{3}{4-x}$
13. $f(x) = \dfrac{x}{1-3x}$
14. $f(x) = \dfrac{2x-1}{3x+4}$
15. $f(x) = \dfrac{4x+2}{3x-6}$
16. $f(x) = \dfrac{-3x-2}{x+3}$
17. $f(x) = \dfrac{x-2}{2x-1}$