Lesson 58: Vertical Asymptotes

Objective: Identify one or more vertical asymptotes in the graph of a rational function.

Students will be able to:

- Identify infinite discontinuities and their corresponding vertical asymptotes in the graph of a rational function.

Prerequisite Knowledge:

- Equations of vertical lines.
- Factoring.
- The Rational Root Theorem.
- Polynomial and /or Synthetic Division.

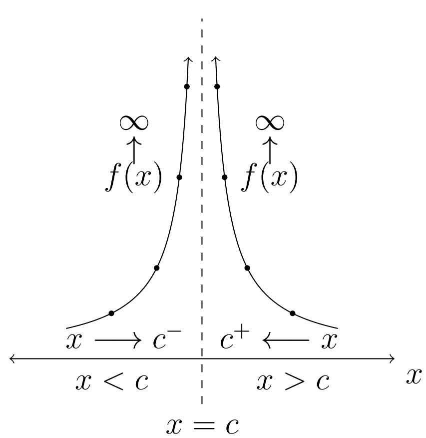
Lesson:

The central idea around a vertical asymptote, say $x = c$, is that as x approaches the value of c, either from the left or the right, the values for the corresponding function $f(x)$ will approach either ∞ or $-\infty$.

Approaching from the right: As $x \to c^+$, $f(x) \to \pm\infty$.

Approaching from the left: As $x \to c^-$, $f(x) \to \pm\infty$.

We should be clear here, in that when we say x approaches c *from the right,* what is meant is that we are evaluating the function at values of x that are getting arbitrarily close go c, but are all *greater* than c, i.e., $x > c$. This is precisely why we can write $x \to c^+$ in the statement above. The $+$ in the exponent signifies that $x > c$. The same can be said for when x approaches c from the left. The following graph further illustrates this point.



Our graph above shows that as x approaches c from either direction, the values for $f(x)$ approach $+\infty$. If, instead, we reflected the right-hand side of the graph across the $x-$axis, we would say that as $x \to c^+$, $f(x) \to -\infty$, since the right-hand side would now point downwards.

Up until this point, we have seen several examples of graphs of rational functions that contain vertical asymptotes. We are now ready to formally state the condition for the existence of a vertical asymptote.

> Let $f(x)$ be a rational function and let $g(x)$ represent the simplified expression for f. If $x = c$ is not in the domain of *both* f and g, then the graph of f will have a vertical asymptote at $x = c$.
>
> Alternatively, we could say that a vertical asymptote exists as a zero of a factor in the denominator of $f(x)$, *as long as that zero is it is not also in the numerator.* We will learn later that this would result in a hole.

II - Demo/Discussion Problems:

Identify the equation of any vertical asymptotes for the graph of each of the following rational functions. Use Desmos to verify your answers.

1. $f(x) = \dfrac{-2x+4}{x-5} = \dfrac{-2(x-2)}{x-5}$
2. $g(x) = \dfrac{x^2+25}{x^2-10x+25}$
3. $h(x) = \dfrac{x^2-9x+20}{x^2-3x-10}$

III - Practice Problems:

Identify the equation of any vertical asymptotes for the graph of each of the following rational functions. Use Desmos to verify your answers.

1. $a(x) = \dfrac{5x^2-1}{x+3}$
2. $g(x) = \dfrac{x^2-1}{x-4}$
3. $h(x) = \dfrac{x^2-4x-9}{x+2}$
4. $j(x) = \dfrac{18x^3-4x-1}{x^2-4}$
5. $k(x) = \dfrac{-x^2+4}{x-9}$
6. $p(x) = \dfrac{-x^2+4}{x^2+9}$
7. $q(x) = \dfrac{x^5}{x(x-5)}$
8. $r(x) = \dfrac{x^2-11x+30}{x^2+10x+24}$
9. $t(x) = \dfrac{15x^2-10}{5x-7}$