Lesson 46: Factoring Polynomials of Quadratic Type

Objective: Recognize and factor a polynomial expression of quadratic type.

Students will be able to:

- Identify polynomial expressions of quadratic type.
- Apply the appropriate substitution, $y = x^n$, to obtain a factorable quadratic polynomial in terms of y.
- Find the set of roots of a polynomial expression of quadratic type.

Prerequisite Knowledge:

- Properties of exponents.
- Factoring a difference of squares.
- The ac-method for factoring quadratics.

Lesson:

Recall that a quadratic expression in terms of a variable x is an expression of the form

$$ax^2 + bx + c.$$

If y is any algebraic expression, we say that the expression

$$ay^2 + by + c$$

is an expression of *quadratic type.*

In just about every case we will see, we will consider y as a power of x, $y = x^n$, so that our expression of quadratic type will appear as follows.

Quadratic Type:

$$\boxed{ax^{2n} + bx^n + c \;=\; a\,[x^n]^2 + b\,[x^n] + c}$$

If $y = x^3$, then the expression

$$ay^2 + by + c = ax^6 + bx^3 + c$$

would be an expression of quadratic type.

Similarly, if $y = x^4$, then the expression

$$ay^2 + by + c = ax^8 + bx^4 + c$$

would be an expression of quadratic type.

In each of these last two examples, notice the exponential pattern, where the middle term has an exponent that is half that of the leading term's. This will always be apparent, as long as the middle coefficient b is nonzero.

By viewing certain expressions as quadratic type, we can often apply more familiar methods, such as the $ac-$method, to obtain a complete factorization.

I - Motivating Example(s):

Example: If we let $y = x^2$, then the difference of fourth powers $x^4 - 16$ can be rewritten as a difference of squares, $y^2 - 4^2$, leading us to the complete factorization over the real numbers shown below.

$$\begin{aligned} x^4 - 16 &= \left(x^2\right)^2 - 4^2 \\ &= y^2 - 4^2, \quad y = x^2 \\ &= (y+4)(y-4) \\ &= \left(x^2+4\right)\left(x^2-4\right) \\ &= \left(x^2+4\right)(x+2)(x-2) \end{aligned}$$

Example: The trinomial expression $x^4 + 2x^2 - 24$ exhibits quadratic type characteristics, since the degree of four is double the exponent appearing in the middle term. Consequently, we will let $y = x^2$ and rewrite the expression in terms of y.

$$y^2 + 2y - 24$$

Applying the $ac-$method, we see the following.

$$\begin{aligned} y^2 + 2y - 24 &= y^2 + 6y - 4y - 24 \\ &= y(y+6) - 4(y+6) \\ &= (y+6)(y-4) \end{aligned}$$

Substituting back for x, we have $(x^2+6)(x^2-4)$. The first factor is a sum of squares, which is irreducible over the reals. The second factor of $x^2 - 4$ is a difference of perfect squares, which we know is factorable as $(x+2)(x-2)$. Our final factorization is

$$x^4 + 2x^2 - 24 = \left(x^2+6\right)(x+2)(x-2).$$

II - Demo/Discussion Problems:

Completely factor each of the following polynomials over the real numbers and identify the set of all real roots.

1. $x^4 - 12x^2 + 27$
2. $x^8 + 2x^4 - 24$
3. $x^6 + 2x^3 - 24$
4. $x^4 - 49$
5. $x^6 - 4x^3 - 5$

III - Practice Problems:

Completely factor each of the following polynomials over the real numbers and identify the set of all real roots.

1. $x^4 + 13x^2 + 40$
2. $x^4 - 5x^2 + 4$
3. $x^4 - 17x^2 + 16$
4. $x^4 - 3x^2 - 40$
5. $3x^4 - 32x^2 + 45$
6. $x^4 + x^2 - 12$
7. $x^4 - 3x^2 - 10$
8. $x^6 - 82x^3 + 81$
9. $8x^4 + 2x^2 - 3$
10. $2x^4 - 19x^2 + 9$