Lesson 51: The Rational Root Theorem

Objective: Apply the Rational Root Theorem to determine a set of possible rational roots for and a factorization of a given polynomial.

Students will be able to:

- Identify a set of possible rational roots for a given polynomial.
- Test the values from a set of possible rational roots to determine if they are actual roots of a given polynomial.

Prerequisite Knowledge:

- Polynomial definition and terminology.
- Evaluate a function for a given x.
- Synthetic (or polynomial) division.

Lesson:

The Rational Root Theorem is used to identify a list of all possible rational roots for a given polynomial.

> **Rational Root Theorem:** Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial of degree n with $n \geq 1$, and a_0, a_1, $\ldots a_n$ are integers. If r is a rational root of f, then r is of the form $\pm\frac{p}{q}$, where p is a factor of the constant term a_0, and q is a factor of the leading coefficient a_n.

The Rational Root Theorem gives us a list of numbers to test as roots of a given polynomial using synthetic division, which is a nicer approach than simply guessing at possible roots. If none of the numbers in the list turn out to be roots, then either the polynomial has no real roots at all, or all of the real roots will be irrational numbers.

I - Motivating Example(s):

Example: Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. Use the Rational Root Theorem to list all of the possible rational roots of f.

To generate a complete list of rational roots, we need to take each of the factors of the constant term, $a_0 = -3$, and divide them by each of the factors of the leading coefficient $a_4 = 2$.

The factors of -3 are ± 1 and ± 3. Since the Rational Root Theorem tacks on a $\pm$ anyway, for the moment, we consider only the positive factors 1 and 3. The factors of 2 are 1 and 2, so the Rational Root Theorem gives the list $\left\{\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{3}{1}, \pm\frac{3}{2}\right\}$ or $\left\{\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 3\right\}$.

Additionally, we can evaluate f at each of the eight potential rational roots in our list, to see if any of them are indeed roots. Starting with ± 1, we see that

$$f(1) = 2 + 4 - 1 - 6 - 3 = -4 \neq 0 \qquad \text{and} \qquad f(-1) = 2 - 4 - 1 + 6 - 3 = 0.$$

Hence, we can conclude that $x = -1$ is a root of f and $x = 1$ is not. Using synthetic division, we can then divide f by the linear factor $x + 1$ as follows.

$$\begin{array}{r|rrrrr} -1 & 2 & 4 & -1 & -6 & -3 \\ & & -2 & -2 & 3 & 3 \\ \hline & 2 & 2 & -3 & -3 & \mathbf{0} \end{array}$$

We can then begin to factor f,

$$2x^4 + 4x^3 - x^2 - 6x - 3 = (x+1)(2x^3 + 2x^2 - 3x - 3)$$

The resulting quotient polynomial is then factorable by grouping,

$$2x^3 + 2x^2 - 3x - 3 = (2x^2 - 3)(x+1).$$

Factoring out a 2 from the expression $2x^2 - 3$, allows us to factor it as the difference of two squares,

$$\begin{aligned} 2x^2 - 3 &= 2\left(x^2 - \frac{3}{2}\right) \\ &= 2\left(x - \sqrt{\frac{3}{2}}\right)\left(x + \sqrt{\frac{3}{2}}\right) \\ &= 2\left(x - \frac{\sqrt{6}}{2}\right)\left(x + \frac{\sqrt{6}}{2}\right) \end{aligned}$$

So, a complete factorization for f would be

$$2x^4 + 4x^3 - x^2 - 6x - 3 = 2\left(x - \frac{\sqrt{6}}{2}\right)\left(x + \frac{\sqrt{6}}{2}\right)(x+1)^2,$$

and the set of real roots for f is $\left\{-1, \pm\frac{\sqrt{6}}{2}\right\}$.

II - Demo/Discussion Problems:

Use the Rational Root Theorem to identify a set of possible rational roots for each of the polynomial functions below. Evaluate the function at at least two of your possible roots, in order to determine if they are actual roots of the polynomial. If successful, divide your polynomial by the respective factor. Use Desmos to help determine the actual set of real roots.

1. $f(x) = x^4 - 9x^2 - 4x + 12$
2. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
3. $f(x) = 6x^3 + 19x^2 - 6x - 40$

III - Practice Problems:

Use the Rational Root Theorem to identify a set of possible rational roots for each of the polynomial functions below. Evaluate the function at $x = 1$. If $x = 1$ is a real root, divide the polynomial by $x - 1$ and factor the resulting quotient. If $x = 1$ is not a real root, evaluate the function at at least one of your remaining possible roots, in order to determine if they are actual roots of the polynomial. If successful, divide your polynomial by the respective factor and factor the remaining quotient. Use Desmos to help determine the actual set of real roots.

1. $f(x) = x^3 - 2x^2 - 5x + 6$
2. $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$
3. $f(x) = x^5 - x^4 - 37x^3 + 37x^2 + 36x - 36$
4. $f(x) = 3x^3 + 3x^2 - 11x - 10$

Use the Rational Root Theorem to identify a set of possible rational roots for each of the polynomial functions below. Evaluate the function at at least two of your possible roots, in order to determine if they are actual roots of the polynomial. If successful, divide your polynomial by the respective factor. Use Desmos to help determine the actual set of real roots.

5. $f(x) = x^4 - 2x^3 + 5x^2 - 8x + 4$
6. $f(x) = x^3 + 4x^2 - 11x + 6$
7. $f(x) = -2x^3 + 19x^2 - 49x + 20$
8. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$