College Algebra

April 18, 2016

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Preface

About the Text

This text has been specifically structured with the MATH123 College Algebra student in mind. It is the result of a collaboration of several full and part-time faculty members at both Framingham State University and Framingham High School and has been made possible through a grant provided by the Educational Technology Office's Teaching with Technology Program at Framingham State University during the Spring of 2015. Specific details surrounding this Teaching with Technology grant application and anticipated outcomes may be obtained by emailing batchison@framingham.edu.

The majority of the content has been either taken verbatim or adapted from those *open educational resources* (OER) that have been generously provided by their respective authors and institutions via a Creative Commons license. Separate sections have been created only when no acceptable available resource could be located. A detailed list of each utilized resource can be found later on in this introduction. The collaborators are extremely grateful and appreciative to all OER authoring individuals and institutions for the contributions that have been utilized by this text and which are freely accessible to the greater academic community.

Additional Resources and Recommended Use

Additional areas of focus for the Teaching with Technology grant include:

- creation of concept summary and homework handouts.
- creation of an online repository for new and existing concept tutorial videos.
- utilization of the free online homework website www.myopenmath.org and graphing utility found at www.desmos.com.

By including access to these additional course enhancements, instructors may utilize this text as a secondary resource to their own course lecture notes and handouts, if they so choose. Students are strongly urged to take advantage of *all* course inclusions made available to them. This text should prove most useful as a resource, when additional explanations, examples and/or practice problems are desired.

A Note to the Student

Like any such undertaking, especially in its initial phase, the project's collaborators apologize for any unforeseen errors or formatting issues, and greatly appreciate the user's understanding with regards to such issues. As this text is very much a work-in-progress, any feedback on how it can be improved is very much appreciated and will be carefully considered.

Students wishing to provide feedback may either do so through their course instructor or by emailing batchison@framingham.edu.

Project Collaborators and Supporters (as of August 2015)

The following individuals have contributed in a variety of ways, including: consulting, content creation and instruction and assessment.

- Benjamin Atchison, Assistant Professor, Framingham State University
  Lead Instructor - Spring 2016

- Jeffrey Barden, Instructor, Framingham State University and Wentworth College

- Karen Dawe, Teacher and Instructor (College Tomorrow Summer Program), Framingham High School and Framingham State University

- Mark Morabito, Teacher and Instructor, Framingham High School and Framingham State University
  Lead Instructor - Fall 2015, Spring 2016

Additional support provided through the Framingham State University Educational Technology Office (ETO) and Whittemore Library, specifically:

- Stacy Cohen
- Millie Gonzalez
- Justin Lauzon
- Robin Robinson, ETO Director

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List of Utilized OERs

The following is a list of open educational resources whose content has been taken verbatim or, in most instances, adapted for the purposes of the creation of this text. Permission has been granted by the authors. The resource location and Creative Commons licensing has been included at the end of each item.

SZ *College Algebra*. Stitz, Carl and Zeager, Jeff. Third (corrected) Edition. 2013. http://www.stitz-zeager.com. CC-BY-NC-SA.

W *Beginning and Intermediate Algebra*. Wallace, Tyler. 2010. http://www.wallace.ccfaculty.org/book/book.html. CC-BY.

A more detailed itemization of this textbook's sections and corresponding attributed works will be included in a future edition of the text, and can be made available upon request.

Text Update - “To Do” List

As of April 18, 2016, the following chapters/sections have yet to be included in this version of the text.

- Applications of Quadratic Equations (Chapter 3)
- Inverse Functions (Chapter 4)
- Applications of Functions (Chapter 4)
- Sign Diagrams for Polynomials (Chapter 5)
- Rational Functions (Chapter 6)

The sections in Chapter 4 on Domain and Range and Transformations of Functions are also only partially completed at this time, since many of the graphs in these sections have been left for the reader to create directly in the text. Additionally, plans to further enhance existing sections of the text with more content and advanced examples are also ongoing. Answers to selected practice problems will also be included in the Fall 2016 version of the text.

Chapter 1

Linear Equations

1.1 Solving Linear Equations

1.1.1 One-Step Equations

Objective: Solve one-step linear equations by balancing using inverse operations

Solving linear equations is an important and fundamental skill in algebra. In algebra, we are often presented with a problem where the answer is known, but part of the problem is missing. The missing part of the problem is what we seek to find. An example of such a problem is shown below.

Example 1.1.

$$4x + 16 = -4$$

Notice the above problem has a missing part, or unknown, that is marked by x. If we are given that the solution to this equation is $x = -5$, it could be plugged into the equation, replacing the x with -5. This is shown in Example 1.2.

Example 1.2.

$$\begin{aligned} 4(-5) + 16 &= -4 && \text{Multiply } 4(-5) \\ -20 + 16 &= -4 && \text{Add } -20 + 16 \\ -4 &= -4 && \text{True!} \end{aligned}$$

Now the equation comes out to a true statement! Notice also that if another number, for example, $x = 3$, was plugged in, we would not get a true statement as seen in Example 1.3.

Example 1.3.

$$\begin{aligned} 4(3) + 16 &= -4 && \text{Multiply } 4(3) \\ 12 + 16 &= -4 && \text{Add } 12 + 16 \\ 28 &\neq -4 && \text{False!} \end{aligned}$$

Due to the fact that this is not a true statement, this demonstrates that $x = 3$ is not the solution. However, depending on the complexity of the problem, this "guess and check" method is not very efficient. Thus, we take a more algebraic approach to solving equations. Here we will focus on what are called "one-step equations" or equations that only require one step to solve. While these equations often seem very fundamental, it is important to master the pattern for solving these problems so we can solve more complex problems.

Addition Problems

To solve equations, the general rule is to do the opposite, as demonstrated in the following example.

Example 1.4.

$$\begin{aligned} x + 7 &= -5 && \text{The 7 is added to the } x \\ \underline{\mathbf{-7} \quad \mathbf{-7}} & && \text{Subtract 7 from both sides to get rid of it} \\ x &= -12 && \text{Our solution} \end{aligned}$$

It is important for the reader to recognize the benefit of checking an answer by plugging it back into the given equation, as we did with examples 1.2 and 1.3 above. This is a step that often gets overlooked by many individuals who may be eager to attempt the next problem. As is the case with most textbooks, we will often omit this step from this point forward, with the understanding that it will usually be an exercise that is left to the reader to verify the validity of each answer.

The same process is used in each of the following examples.

Example 1.5.

$\begin{array}{r} 4 + x = 8 \\ \underline{\mathbf{-4} \qquad \mathbf{-4}} \\ x = 4 \end{array}$	$\begin{array}{r} 7 = x + 9 \\ \underline{\mathbf{-9} \quad \mathbf{-9}} \\ -2 = x \end{array}$	$\begin{array}{r} 5 = 8 + x \\ \underline{\mathbf{-8} \quad \mathbf{-8}} \\ -3 = x \end{array}$

Table 1.1: Addition Examples

Subtraction Problems

In a subtraction problem, we get rid of negative numbers by adding them to both sides of the equation, as demonstrated in the following example.

Example 1.6.

$$\begin{array}{rl} x - 5 = 4 & \text{The 5 is negative, or subtracted from } x \\ \underline{\mathbf{+5} \quad \mathbf{+5}} & \text{Add 5 to both sides} \\ x = 9 & \text{Our solution} \end{array}$$

The same process is used in each of the following examples. Notice that each time we are getting rid of a negative number by adding.

In every example, we introduce the opposite operation of what is shown, in order to solve the given equation. This notion of opposites is more commonly referred to as an *inverse* operation. The inverse operation of addition is subtraction, and vice versa. Similarly, the inverse operation of multiplication is division, and vice versa, which we will see momentarily.

Example 1.7.

$\begin{array}{r} -6 + x = -2 \\ \underline{\mathbf{+6} \qquad \mathbf{+6}} \\ x = 4 \end{array}$	$\begin{array}{r} -10 = x - 7 \\ \underline{\mathbf{+7} \qquad \mathbf{+7}} \\ -3 = x \end{array}$	$\begin{array}{r} 5 = -8 + x \\ \underline{\mathbf{+8} \quad \mathbf{+8}} \\ 13 = x \end{array}$

Table 1.2: Subtraction Examples

Multiplication Problems

With a multiplication problem, we get rid of the number by dividing on both sides, as demonstrated in the following examples.

Example 1.8.

$$\begin{aligned} 4x &= 20 && \text{Variable is multiplied by } 4 \\ \overline{\mathbf{4}} \quad & \quad \overline{\mathbf{4}} && \text{Divide both sides by } 4 \\ x &= 5 && \text{Our solution} \end{aligned}$$

With multiplication problems it is very important that care is taken with signs. If x is multiplied by a negative then we will divide by a negative. This is shown in example 1.9.

Example 1.9.

$$\begin{aligned} -5x &= 30 && \text{Variable is multiplied by } -5 \\ \overline{\mathbf{-5}} \quad & \quad \overline{\mathbf{-5}} && \text{Divide both sides by } -5 \\ x &= -6 && \text{Our solution} \end{aligned}$$

The same process is used in each of the following examples. Notice how negative and positive numbers are handled as each problem is solved.

Example 1.10.

$8x = -24$	$-4x = -20$	$42 = 7x$
$\overline{\mathbf{8}} \qquad \overline{\mathbf{8}}$	$\overline{\mathbf{-4}} \qquad \overline{\mathbf{-4}}$	$\overline{\mathbf{7}} \qquad \overline{\mathbf{7}}$
$x = -3$	$x = 5$	$6 = x$

Table 1.3: Multiplication Examples

Division Problems

In division problems, we get rid of the denominator by multiplying on both sides, since multiplication is the opposite, or *inverse*, operation of division. This is demonstrated in the examples shown below.

Example 1.11.

$$\begin{aligned} \frac{x}{5} &= -3 && \text{Variable is divided by } 5 \\ (\mathbf{5})\frac{x}{5} &= -3(\mathbf{5}) && \text{Multiply both sides by } 5 \\ x &= -15 && \text{Our solution} \end{aligned}$$

Then we get our solution $x = -15$.

Example 1.12.

$\frac{x}{-7} = -2$	$\frac{x}{8} = 5$	$\frac{x}{-4} = 9$
$\mathbf{(-7)}\frac{x}{-7} = -2\mathbf{(-7)}$	$\mathbf{(8)}\frac{x}{8} = 5\mathbf{(8)}$	$\mathbf{(-4)}\frac{x}{-4} = 9\mathbf{(-4)}$
$x = 14$	$x = 40$	$x = -36$

Table 1.4: Division Examples

The process described above is fundamental to solving equations. Once this process is mastered, the problems we will see have several more steps. These problems may seem more complex, but the process and patterns used will remain the same.

World View Note: The study of algebra originally was called the "Cossic Art" from the Latin, the study of "things" (which we now call variables).

1.1.2 Two-Step Equations

Objective: Solve two-step equations by balancing and using inverse operations.

After mastering the technique for solving one-step equations, we are ready to consider two-step equations. As we solve two-step equations, the important thing to remember is that everything works backwards! When working with one-step equations, we learned that in order to clear a "plus five" in the equation, we would subtract five from both sides. We learned that to clear "divided by seven" we multiply by seven on both sides. The same pattern applies to the order of operations. When solving for our variable x, we use order of operations backwards as well. This means we will add or subtract first, then multiply or divide second (then exponents, and finally any parentheses or grouping symbols, but that's another lesson).

Example 1.13.

$$4x - 20 = -8$$

We have two numbers on the same side as the x. We need to move the 4 and the 20 to the other side. We know to move the 4 we need to divide, and to move the 20 we will add 20 to both sides. If order of operations is done backwards, we will add or subtract first. Therefore we will add 20 to both sides first. Once we are done with that, we will divide both sides by 4. The steps are shown below.

$$\begin{array}{rl}
4x - 20 = -8 & \text{Start by focusing on the subtract 20} \\
\underline{\mathbf{+20} \quad \mathbf{+20}} & \text{Add 20 to both sides} \\
4x = 12 & \text{Now we focus on the 4 multiplied by } x \\
\overline{\mathbf{4}} \quad\quad \overline{\mathbf{4}} & \text{Divide both sides by 4} \\
x = 3 & \text{Our solution}
\end{array}$$

Notice in our next example when we replace the x with 3 we get a true statement.

$$\begin{array}{rl}
4(3) - 20 = -8 & \text{Multiply } 4(3) \\
12 - 20 = -8 & \text{Subtract } 12 - 20 \\
-8 = -8 & \text{True!}
\end{array}$$

The same process is used to solve any two-step equation. Add or subtract first, then multiply or divide.

Example 1.14.

$$\begin{array}{rl}
5x + 7 = 7 & \text{Start by focusing on the plus } 7 \\
\underline{\mathbf{-7} \quad \mathbf{-\ 7}} & \text{Subtract } 7 \text{ from both sides} \\
5x = 0 & \text{Now focus on the multiplication by } 5 \\
\overline{\mathbf{5}} \qquad \overline{\mathbf{5}} & \text{Divide both sides by } 5 \\
x = 0 & \text{Our solution}
\end{array}$$

Notice the seven subtracted out completely! Many students get stuck on this point, do not forget that we have a number for “nothing left”, and that number is zero. With this in mind the process is almost identical to our first example.

A common error students make with two-step equations is with negative signs. Remember the sign always stays with the number. Consider the following example.

Example 1.15.

$$\begin{array}{rl}
4 - 2x = 10 & \text{Start by focusing on the positive } 4 \\
\underline{\mathbf{-4} \qquad \mathbf{-\ 4}} & \text{Subtract } 4 \text{ from both sides} \\
-2x = 6 & \text{Negative (subtraction) stays on the } 2x \\
\overline{\mathbf{-2}} \quad \overline{\mathbf{-2}} & \text{Divide by } -2 \\
x = -3 & \text{Our solution}
\end{array}$$

The same is true even if there is no apparent coefficient in front of the variable. The coefficient is 1 or -1 in this case. Consider the next example.

Example 1.16.

$$\begin{array}{rl}
8 - x = 2 & \text{Start by focusing on the positive } 8 \\
\underline{\mathbf{-8} \qquad \mathbf{-\ 8}} & \text{Subtract } 8 \text{ from both sides} \\
-x = -6 & \text{Negative(subtraction) stays on the } x \\
-1x = -6 & \text{Remember, no number in front of variable means } 1 \\
\overline{\mathbf{-1}} \quad \overline{\mathbf{-1}} & \text{Divide both sides by } -1 \\
x = 6 & \text{Our solution}
\end{array}$$

Solving two-step equations is a very important skill to master, as we study algebra. The first step is to add or subtract, the second is to multiply or divide. This pattern is seen in each of the following examples.

Example 1.17.

$$\begin{array}{rcl} -3x+7 & = & -8 \\ \underline{\mathbf{-7}} & & \underline{\mathbf{-7}} \\ -3x & = & -15 \\ \overline{\mathbf{-3}} & & \overline{\mathbf{-3}} \\ x & = & 5 \end{array}
\qquad
\begin{array}{rcl} -2+9x & = & 7 \\ \underline{\mathbf{+2}} & & \underline{\mathbf{+2}} \\ 9x & = & 9 \\ \overline{\mathbf{9}} & & \overline{\mathbf{9}} \\ x & = & 1 \end{array}
\qquad
\begin{array}{rcl} 8 & = & 2x+10 \\ \underline{\mathbf{-10}} & & \underline{\mathbf{-10}} \\ -2 & = & 2x \\ \overline{\mathbf{2}} & & \overline{\mathbf{2}} \\ -1 & = & x \end{array}$$

$$\begin{array}{rcl} 7-5x & = & 17 \\ \underline{\mathbf{-7}} & & \underline{\mathbf{-7}} \\ -5x & = & 10 \\ \overline{\mathbf{-5}} & & \overline{\mathbf{-5}} \\ x & = & -2 \end{array}
\qquad
\begin{array}{rcl} -5-3x & = & -5 \\ \underline{\mathbf{+5}} & & \underline{\mathbf{+5}} \\ -3x & = & 0 \\ \overline{\mathbf{-3}} & & \overline{\mathbf{-3}} \\ x & = & 0 \end{array}
\qquad
\begin{array}{rcl} -3 & = & \tfrac{x}{5}-4 \\ \underline{\mathbf{+4}} & & \underline{\mathbf{+4}} \\ & & \\ (\mathbf{5})(1) & = & \tfrac{x}{5}(\mathbf{5}) \\ 5 & = & x \end{array}$$

Table 1.5: Two-Step Equation Examples

As problems in algebra become more complex the process covered here will remain the same. In fact, as we solve problems like those in the next example, each one of them will have several steps to solve, but the last two steps will resemble solving a two-step equation. This is why it is very important to master two-step equations now!

Example 1.18.

$$3x^2+4-x=6 \qquad\qquad \frac{1}{x-8}+\frac{1}{x}=\frac{1}{3}$$

$$\sqrt{5x-5}+1=x \qquad\qquad \log_5(2x-4)=1$$

World View Note: Persian mathematician Omar Khayyam would solve algebraic problems geometrically by intersecting graphs rather than solving them algebraically.

1.1.3 General Equations

Objective: Solve general linear equations with variables on both sides.

Often as we are solving linear equations we will need to do some work to set them up into a form we are familiar with solving. This section will focus on manipulating an equation we are asked to solve in such a way that we can use our pattern for solving two-step equations to ultimately arrive at the solution.

One such issue that needs to be addressed is parentheses. Often the parentheses can get in the way of solving an otherwise easy problem. As you might expect we can get rid of the unwanted parentheses by using the distributive property. This is shown in the following example. Notice the first step is distributing, then it is solved like any other two-step equation.

Example 1.19.

$$\begin{array}{rl}
4(2x-6)=16 & \text{Distribute 4 through parentheses} \\
8x-24=16 & \text{Focus on the subtraction first} \\
\underline{\mathbf{+24}\ \ \mathbf{+24}} & \text{Add 24 to both sides} \\
8x=40 & \text{Now focus on the multiply by 8} \\
\overline{\mathbf{8}}\qquad\overline{\mathbf{8}} & \text{Divide both sides by 8} \\
x=5 & \text{Our solution}
\end{array}$$

Often after we distribute there will be some like terms on one side of the equation. Example 1.20 shows distributing to clear the parentheses and then combining like terms next. Notice we only combine like terms on the same side of the equation. Once we have done this, our next example solves just like any other two-step equation.

Example 1.20.

$$\begin{array}{rl}
3(2x-4)+9=15 & \text{Distribute the 3 through the parentheses}\\
6x-12+9=15 & \text{Combine like terms, } -12+9\\
6x-3=15 & \text{Focus on the subtraction first}\\
\underline{\mathbf{+3}\ \ \mathbf{+3}} & \text{Add 3 to both sides}\\
6x=18 & \text{Now focus on multiply by 6}\\
\overline{\mathbf{6}}\qquad\overline{\mathbf{6}} & \text{Divide both sides by 6}\\
x=3 & \text{Our solution}
\end{array}$$

A second type of problem that becomes a two-step equation after a bit of work is one where we see the variable on both sides. This is shown in the following example.

Example 1.21.

$$4x-6=2x+10$$

Notice here the x is on both the left and right sides of the equation. This can make it difficult to decide which side to work with. We fix this by moving one of the terms with x to the other side, much like we moved a constant term. It doesn't matter which term gets moved, $4x$ or $2x$, however, it would be the author's suggestion to move the smaller term (to avoid negative coefficients). For this reason we begin this problem by clearing the positive $2x$ by subtracting $2x$ from both sides.

$$\begin{array}{rl}
4x-6=2x+10 & \text{Notice the variable on both sides}\\
\underline{\boldsymbol{-2x}\qquad\boldsymbol{-\ 2x}}\qquad & \text{Subtract } 2x \text{ from both sides}\\
2x-6=10 & \text{Focus on the subtraction first}\\
\underline{\mathbf{+6}\ \ \mathbf{+6}} & \text{Add 6 to both sides}\\
2x=16 & \text{Focus on the multiplication by 2}\\
\overline{\mathbf{2}}\qquad\overline{\mathbf{2}} & \text{Divide both sides by 2}\\
x=8 & \text{Our solution}
\end{array}$$

The previous example shows the check on this solution. Here the solution is plugged into the x on both the left and right sides before simplifying.

Example 1.22.

$$\begin{array}{rl}
4(8)-6=2(8)+10 & \text{Multiply } 4(8) \text{ and } 2(8) \text{ first}\\
32-6=16+10 & \text{Add and Subtract}\\
26=26 & \text{True!}
\end{array}$$

The next example illustrates the same process with negative coefficients. Notice first the smaller term with the variable is moved to the other side, this time by adding because the coefficient is negative.

Example 1.23.

$$
\begin{array}{rl}
-3x + 9 = 6x - 27 & \text{Notice the variable on both sides, } -3x \text{ is smaller} \\
\underline{\boldsymbol{+3x \qquad + 3x}} & \text{Add } 3x \text{ to both sides} \\
9 = 9x - 27 & \text{Focus on the subtraction by 27} \\
\underline{\mathbf{+27 \qquad + 27}} & \text{Add 27 to both sides} \\
36 = 9x & \text{Focus on the multiplication by 9} \\
\overline{\mathbf{9}} \quad \overline{\mathbf{9}} & \text{Divide both sides by 9} \\
4 = x & \text{Our solution}
\end{array}
$$

Linear equations can become particularly interesting when the two processes are combined. In the following problems we have parentheses and the variable on both sides. Notice in each of the following examples we distribute, then combine like terms, then move the variable to one side of the equation.

Example 1.24.

$$
\begin{array}{rl}
2(x-5) + 3x = x + 18 & \text{Distribute the 2 through parentheses} \\
2x - 10 + 3x = x + 18 & \text{Combine like terms } 2x + 3x \\
5x - 10 = x + 18 & \text{Notice the variable is on both sides} \\
\underline{\boldsymbol{-x \qquad - x}} & \text{Subtract } x \text{ from both sides} \\
4x - 10 = 18 & \text{Focus on the subtraction of 10} \\
\underline{\mathbf{+10 \; + 10}} & \text{Add 10 to both sides} \\
4x = 28 & \text{Focus on multiplication by 4} \\
\overline{\mathbf{4}} \quad \overline{\mathbf{4}} & \text{Divide both sides by 4} \\
x = 7 & \text{Our solution}
\end{array}
$$

Sometimes we may have to distribute more than once to clear several parentheses. Remember to combine like terms after you distribute!

Example 1.25.

$3(4x-5)-4(2x+1)=5$	Distribute 3 and -4 through parentheses
$12x-15-8x-4=5$	Combine like terms $12x-8x$ and $-15-4$
$4x-19=5$	Focus on subtraction of 19
$\underline{\mathbf{+19 \quad +19}}$	Add 19 to both sides
$4x=24$	Focus on multiplication by 4
$\overline{\mathbf{4}} \quad \overline{\mathbf{4}}$	Divide both sides by 4
$x=6$	Our solution

This leads to a 5-step process to solve any linear equation. While all five steps aren't always needed, this can serve as a guide to solving equations.

1. Distribute through any parentheses.

2. Combine like terms on each side of the equation.

3. Get the variables on one side by adding or subtracting

4. Solve the remaining 2-step equation (add or subtract then multiply or divide)

5. Check your answer by plugging it back in for x to find a true statement. If your resulting statement is false, repeat the procedure, beginning with the first step.

The order of these steps is very important.

World View Note: The Chinese developed a method for solving equations that involved finding each digit one at a time about 2000 years ago!

We can see each of the above five steps worked through our next example.

Example 1.26.

$$\begin{array}{rl}
4(2x-6)+9=3(x-7)+8x & \text{Distribute 4 and 3 through parentheses}\\
8x-24+9=3x-21+8x & \text{Combine like terms } -24+9 \text{ and } 3x+8x\\
8x-15=11x-21 & \text{Notice the variable is on both sides}\\
\underline{\boldsymbol{-8x \qquad\quad -\ 8x}} & \text{Subtract } 8x \text{ from both sides}\\
-15=3x-21 & \text{Focus on subtraction of 21}\\
\underline{\boldsymbol{+21 \qquad +\ 21}} & \text{Add 21 to both sides}\\
6=3x & \text{Focus on multiplication by 3}\\
\overline{\mathbf{3}} \quad \overline{\mathbf{3}} & \text{Divide both sides by 3}\\
2=x & \text{Our solution}
\end{array}$$

Check:

$$\begin{array}{rl}
4[2(2)-6]+9=3[(2)-7]+8(2) & \text{Plug 2 in for each } x\text{. Multiply inside parentheses}\\
4[4-6]+9=3[-5]+8(2) & \text{Finish parentheses on left, multiply on right}\\
4[-2]+9=-15+8(2) & \text{Finish multiplication on both sides}\\
-8+9=-15+16 & \text{Add}\\
1=1 & \text{True!}
\end{array}$$

When we check our solution of $x = 2$ we found a true statement, $1 = 1$. Therefore, we know our solution $x = 2$ is the correct solution for the problem.

There are two special cases that can come up as we are solving these linear equations. The first is illustrated in the next two examples. Notice we start by distributing and moving the variables all to the same side.

Example 1.27.

$$\begin{array}{rl}
3(2x-5)=6x-15 & \text{Distribute 3 through parentheses}\\
6x-15=6x-15 & \text{Notice the variable on both sides}\\
\underline{\boldsymbol{-6x \qquad\quad -\ 6x}} & \text{Subtract } 6x \text{ from both sides}\\
-15=-15 & \text{Variable is gone! True!}
\end{array}$$

Here the variable subtracted out completely! We are left with a true statement, $-15 = -15$. If the variables subtract out completely and we are left with a true statement, this indicates that the equation is always true, no matter what x is. Thus, for our solution we say **all real numbers** or $\mathbb{R}$.

It is worth mentioning that in both the previous and following examples, we are still *solving* a given equation for all possible values of x. When the variable is eliminated entirely, this can sometimes be confused with *checking* a solution.

Example 1.28.

$$\begin{array}{rl} 2(3x-5)-4x=2x+7 & \text{Distribute 2 through parentheses} \\ 6x-10-4x=2x+7 & \text{Combine like terms } 6x-4x \\ 2x-10=2x+7 & \text{Notice the variable is on both sides} \\ \underline{\boldsymbol{-2x} \qquad \boldsymbol{-\ 2x}} \quad & \text{Subtract } 2x \text{ from both sides} \\ -10 \neq 7 & \text{Variable is gone! False!} \end{array}$$

Again, the variable subtracted out completely! However, this time we are left with a false statement, this indicates that the equation is never true, no matter what x is. Thus, for our solution we say **no solution** or $\varnothing$.

1.1.4 Equations Containing Fractions

Objective: Solve linear equations with rational coefficients by multiplying by the least common denominator to clear the fractions.

Often when solving linear equations we will need to work with an equation with fraction coefficients. We can solve these problems as we have in the past. This is demonstrated in our next example.

Example 1.29.

$$\begin{array}{rl} \frac{3}{4}x - \frac{7}{2} = \frac{5}{6} & \text{Focus on subtraction} \\ \underline{+\mathbf{\frac{7}{2}} \quad +\mathbf{\frac{7}{2}}} & \text{Add } \frac{7}{2} \text{ to both sides} \end{array}$$

Notice we will need to get a common denominator to add $\frac{5}{6} + \frac{7}{2}$. Notice we have a common denominator of 6. So we build up the denominator, $\frac{7}{2}\left(\frac{3}{3}\right) = \frac{21}{6}$, and we can now add the fractions:

$$\begin{array}{rl} \frac{3}{4}x - \frac{21}{6} = \frac{5}{6} & \text{Same problem, with common denominator 6} \\ \underline{+\mathbf{\frac{21}{6}} \quad +\mathbf{\frac{21}{6}}} & \text{Add } \frac{21}{6} \text{ to both sides} \\ \frac{3}{4}x = \frac{26}{6} & \text{Reduce } \frac{26}{6} \text{ to } \frac{13}{3} \\ \frac{3}{4}x = \frac{13}{3} & \text{Focus on multiplication by } \frac{3}{4} \end{array}$$

We can get rid of $\frac{3}{4}$ by dividing both sides by $\frac{3}{4}$.

Dividing by a fraction is the same as multiplying by the reciprocal, so we will multiply both sides by $\frac{4}{3}$.

$$\begin{array}{rl} \left(\mathbf{\frac{4}{3}}\right)\frac{3}{4}x = \frac{13}{3}\left(\mathbf{\frac{4}{3}}\right) & \text{Multiply by reciprocal} \\ x = \frac{52}{9} & \text{Our solution} \end{array}$$

While this process does help us arrive at the correct solution, the fractions can make the process quite difficult. This is why we have an alternate method for dealing with fractions - clearing fractions. Clearing fractions is nice as it gets rid of the fractions for the majority of the problem. We can easily clear the fractions by finding the LCD and multiplying each term by the LCD. This is shown in the next example, the same problem as our first example, but this time we will solve by clearing fractions.

Example 1.30.

$$\begin{array}{rl}
\frac{3}{4}x - \frac{7}{2} = \frac{5}{6} & \text{LCD} = 12, \text{multiply each term by 12} \\[1em]
\frac{\mathbf{(12)}3}{4}x - \frac{\mathbf{(12)}7}{2} = \frac{\mathbf{(12)}5}{6} & \text{Reduce each 12 with denominators} \\[1em]
\mathbf{(3)}3x - \mathbf{(6)}7 = \mathbf{(2)}5 & \text{Multiply out each term} \\
9x - 42 = 10 & \text{Focus on subtraction by 42} \\
\underline{\mathbf{+42} \quad \mathbf{+\ 42}} & \text{Add 42 to both sides} \\
9x = 52 & \text{Focus on multiplication by 9} \\
\overline{\mathbf{9}} \quad \overline{\mathbf{9}} & \text{Divide both sides by 9} \\
x = \frac{52}{9} & \text{Our solution}
\end{array}$$

The next example illustrates this as well. Notice the 2 isn't a fraction in the original equation, but to solve it we put the 2 over 1 to make it a fraction.

Example 1.31.

$$\begin{array}{rl}
\frac{2}{3}x - 2 = \frac{3}{2}x + \frac{1}{6} & \text{LCD} = 6, \text{multiply each term by 6} \\[1em]
\frac{\mathbf{(6)}2}{3}x - \frac{\mathbf{(6)}2}{1} = \frac{\mathbf{(6)}3}{2}x + \frac{\mathbf{(6)}1}{6} & \text{Reduce 6 with each denominator} \\[1em]
\mathbf{(2)}2x - \mathbf{(6)}2 = \mathbf{(3)}3x + \mathbf{(1)}1 & \text{Multiply out each term}
\end{array}$$

$$
\begin{array}{rl}
4x - 12 = 9x + 1 & \text{Notice variable on both sides} \\
\underline{\boldsymbol{-4x} \qquad \boldsymbol{-\ 4x}} & \text{Subtract } 4x \text{ from both sides} \\
-12 = 5x + 1 & \text{Focus on addition of 1} \\
\underline{\mathbf{-1} \qquad \mathbf{-\ 1}} & \text{Subtract 1 from both sides} \\
-13 = 5x & \text{Focus on multiplication of 5} \\
\overline{\mathbf{5}} \quad \overline{\mathbf{5}} & \text{Divide both sides by 5} \\
-\dfrac{13}{5} = x & \text{Our solution}
\end{array}
$$

We can use this same process if there are parenthesis in the problem. We will first distribute the coefficient in front of the parenthesis, then clear the fractions. This is seen in the following example.

Example 1.32.

$$
\begin{array}{rl}
\dfrac{3}{2}\left(\dfrac{5}{9}x + \dfrac{4}{27}\right) = 3 & \text{Distribute } \dfrac{3}{2} \text{ through parenthesis, reducing if possible} \\
\dfrac{5}{6}x + \dfrac{2}{9} = 3 & \text{LCD} = 18\text{, multiply each term by 18} \\
\dfrac{\mathbf{(18)}5}{6}x + \dfrac{\mathbf{(18)}2}{9} = \dfrac{\mathbf{(18)}3}{9} & \text{Reduce 18 with each denominator} \\
\mathbf{(3)}5x + \mathbf{(2)}2 = \mathbf{(18)}3 & \text{Multiply out each term} \\
15x + 4 = 54 & \text{Focus on addition of 4} \\
\underline{\mathbf{-4} \quad \mathbf{-\ 4}} & \text{Subtract 4 from both sides} \\
15x = 50 & \text{Focus on multiplication by 15} \\
\overline{\mathbf{15}} \quad \overline{\mathbf{15}} & \text{Divide both sides by 15, reduce on right side} \\
x = \dfrac{10}{3} & \text{Our solution}
\end{array}
$$

While the problem can take many different forms, the pattern to clear the fraction is the same, after distributing through any parentheses we multiply each term by the LCD and reduce. This will give us a problem with no fractions that is much easier to solve. The following example again illustrates this process.

Example 1.33.

$$
\begin{array}{rl}
\frac{3}{4}x-\frac{1}{2}=\frac{1}{3}(\frac{3}{4}x+6)-\frac{7}{2} & \text{Distribute } \frac{1}{3}\text{, reduce if possible}\\
\frac{3}{4}x-\frac{1}{2}=\frac{1}{4}x+2-\frac{7}{2} & \text{LCD} = 4\text{, multiply each term by } 4\\
\frac{\mathbf{(4)}3}{4}x-\frac{\mathbf{(4)}1}{2}=\frac{\mathbf{(4)}1}{4}x+\frac{\mathbf{(4)}2}{1}-\frac{\mathbf{(4)}7}{2} & \text{Reduce 4 with each denominator}\\
\mathbf{(1)}3x-\mathbf{(2)}1=\mathbf{(1)}1x+\mathbf{(4)}2-\mathbf{(2)}7 & \text{Multiply out each term}\\
3x-2=x+8-14 & \text{Combine like terms } 8-14\\
3x-2=x-6 & \text{Notice variable on both sides}\\
\underline{\boldsymbol{-x}\qquad\quad\boldsymbol{-x}} & \text{Subtract } x \text{ from both sides}\\
2x-2=-6 & \text{Focus on subtraction by 2}\\
\underline{\mathbf{+2}\quad\mathbf{+2}} & \text{Add 2 to both sides}\\
2x=-4 & \text{Focus on multiplication by 2}\\
\overline{\mathbf{2}}\qquad\overline{\mathbf{2}} & \text{Divide both sides by 2}\\
x=-2 & \text{Our solution}
\end{array}
$$

World View Note: The Egyptians were among the first to study fractions and linear equations. The most famous mathematical document from Ancient Egypt is the Rhind Papyrus where the unknown variable was called "heap"

1.2 Absolute Value Equations

Objective: Solve linear absolute value equations.

When solving equations with absolute value we can end up with more than one possible answer. This is because what is in the absolute value can be either negative or positive and we must account for both possibilities when solving equations. This is illustrated in the following example.

Example 1.34.

$$\begin{array}{rl} |x| = 7 & \text{Absolute value can be positive or negative} \\ x = 7 \text{ or } x = -7 & \text{Our solution} \end{array}$$

Notice that we have considered two possibilities, both the positive and negative. Either way, the absolute value of our number will be positive 7.

World View Note: The first set of rules for working with negatives came from 7th century India. However, in 1758, almost a thousand years later, British mathematician Francis Maseres claimed that negatives "darken the very whole doctrines of the equations and make dark of the things which are in their nature excessively obvious and simple."

When we have absolute values in our problem it is important to first isolate the absolute value, then remove the absolute value by considering both the positive and negative solutions. Notice in the next two examples, all the numbers outside of the absolute value are moved to the other side first before we remove the absolute value bars and consider both positive and negative solutions.

Example 1.35.

$$\begin{array}{rl} 5 + |x| = 8 & \text{Notice absolute value is not alone} \\ \underline{\mathbf{-5} \qquad\quad \mathbf{-5}} & \text{Subtract 5 from both sides} \\ |x| = 3 & \text{Absolute value can be positive or negative} \\ x = 3 \text{ or } x = -3 & \text{Our solution} \end{array}$$

Example 1.36.

$$\begin{array}{rl} -4|x| = -20 & \text{Notice absolute value is not alone} \\ \overline{\mathbf{-4}} \quad \overline{\mathbf{-4}} & \text{Divide both sides by } -4 \\ |x| = 5 & \text{Absolute value can be positive or negative} \\ x = 5 \text{ or } x = -5 & \text{Our solution} \end{array}$$

Notice we never combine what is inside the absolute value with what is outside the absolute value. This is very important as it will often change the final result to an incorrect solution. The next example requires two steps to isolate the absolute value. The idea is the same as a two-step equation, add or subtract, then multiply or divide.

Example 1.37.

$$
\begin{array}{rl}
5|x| - 4 = 26 & \text{Notice the absolute value is not alone} \\
\underline{\mathbf{+4 \ \ +4}} & \text{Add 4 to both sides} \\
5|x| = 30 & \text{Absolute value still not alone} \\
\overline{\mathbf{5}} \qquad \overline{\mathbf{5}} & \text{Divide both sides by 5} \\
|x| = 6 & \text{Absolute value can be positive or negative} \\
x = 6 \ \text{ or } \ x = -6 & \text{Our solution}
\end{array}
$$

Again we see the same process, get the absolute value alone first, then consider the positive and negative solutions. Often the absolute value will have more than just a variable in it. In this case we will have to solve the resulting equations when we consider the positive and negative possibilities. This is shown in the next example.

Example 1.38.

$$
\begin{array}{rl}
|2x - 1| = 7 & \text{Absolute value can be positive or negative} \\
2x - 1 = 7 \ \text{ or } \ 2x - 1 = -7 & \text{Two equations to solve}
\end{array}
$$

Now notice we have two equations to solve, each equation will give us a different solution. Both equations solve like any other two-step equation.

$$
\begin{array}{lcl}
2x - 1 = 7 & & 2x - 1 = -7 \\
\quad \underline{\mathbf{+1 \ \ +1}} & & \quad \underline{\mathbf{+1 \ \ \ +1}} \\
2x = 8 & \text{or} & 2x = -6 \\
\overline{\mathbf{2}} \quad \ \overline{\mathbf{2}} & & \overline{\mathbf{2}} \qquad \overline{\mathbf{2}} \\
x = 4 & & x = -3
\end{array}
$$

Thus, from our previous example we have two solutions, $x = 4$ or $x = -3$.

Again, it is important to remember that the absolute value must be alone first before we consider the positive and negative possibilities. This is illustrated below.

Example 1.39.

$$2 - 4|2x + 3| = -18$$

To get the absolute value alone we first need to get rid of the 2 by subtracting, then divide by -4. Notice we cannot combine the 2 and -4 because they are not like terms, the -4 has the absolute value connected to it. Also notice we do not distribute the -4 into the absolute value. This is because the numbers outside cannot be combined with the numbers inside the absolute value. Thus we get the absolute value alone in the following way:

$$\begin{array}{rl}
2 - 4|2x + 3| = -18 & \text{Notice absolute value is not alone} \\
\underline{\mathbf{-2} \qquad\qquad\quad \mathbf{-\ 2}} & \text{Subtract 2 from both sides} \\
-4|2x + 3| = -20 & \text{Absolute value still not alone} \\
\overline{\mathbf{-4}} \qquad \overline{\mathbf{-4}} & \text{Divide both sides by } -4 \\
|2x + 3| = 5 & \text{Absolute value can be positive or negative} \\
2x + 3 = 5 \text{ or } 2x + 3 = -5 & \text{Two equations to solve}
\end{array}$$

Now we just solve these two remaining equations to find our solutions.

$$\begin{array}{lcl}
2x + 3 = 5 & & 2x + 3 = -5 \\
\quad \underline{\mathbf{-3} \quad \mathbf{-\ 3}} & & \quad \underline{\mathbf{-3} \quad \mathbf{-\ 3}} \\
2x = 2 & \text{or} & 2x = -8 \\
\overline{\mathbf{2}} \quad \overline{\mathbf{2}} & & \overline{\mathbf{2}} \quad \overline{\mathbf{2}} \\
x = 1 & & x = -4
\end{array}$$

We now have our two solutions, $x = 1$ and $x = -4$.

As we are solving absolute value equations it is important to be aware of special cases. Remember the result of an absolute value must always be positive. Notice what happens in the next example.

Example 1.40.

$$\begin{array}{rl}
7 + |2x - 5| = 4 & \text{Notice absolute value is not alone} \\
\underline{\mathbf{-7} \qquad\qquad \mathbf{-\ 7}} & \text{Subtract 7 from both sides} \\
|2x - 5| = -3 & \text{Result of absolute value is negative!}
\end{array}$$

Notice the absolute value equals a negative number! This is impossible with absolute value. When this occurs we say there is **no solution** or $\varnothing$.

One other type of absolute value problem is when two absolute values are equal to each other. We still will consider both the positive and negative result, the difference here will be that we will have to distribute a negative into the second absolute value for the negative possibility.

Example 1.41.

$$\begin{array}{rll} & |2x-7| = |4x+6| & \text{Absolute value can be} \\ & & \quad \text{positive or negative} \\ & 2x-7 = 4x+6 & \text{Make second part of} \\ \text{or} & 2x-7 = -(4x+6) & \quad \text{second equation negative} \end{array}$$

Notice the first equation is the positive possibility and has no significant difference other than the missing absolute value bars. The second equation considers the negative possibility. For this reason we have a negative in front of the expression which will be distributed through the equation on the first step of solving. So we solve both these equations as follows:

$$\begin{array}{lcl} 2x-7 = 4x+6 & & 2x-7 = -(4x+6) \\ \underline{\boldsymbol{-2x} \qquad \boldsymbol{-2x}} & & 2x-7 = -4x-6 \\ -7 = 2x+6 & & \underline{\boldsymbol{+4x} \qquad \boldsymbol{+4x}} \\ \underline{\boldsymbol{-6} \qquad \boldsymbol{-6}} & \text{or} & 6x-7 = -6 \\ -13 = 2x & & \quad \underline{\boldsymbol{+7} \quad \boldsymbol{+7}} \\ \quad \overline{\mathbf{2}} \qquad \overline{\mathbf{2}} & & 6x = 1 \\ -\frac{13}{2} = x & & \overline{\mathbf{6}} \qquad \overline{\mathbf{6}} \\ & & x = \frac{1}{6} \end{array}$$

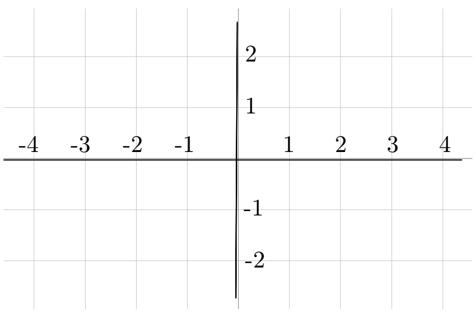
This gives us our two solutions, $x = -\frac{13}{2}$ or $x = \frac{1}{6}$.

1.3 Graphing Linear Equations

1.3.1 The Cartesian Plane

Objective: Locate and graph points using xy-coordinates

Often, to get an idea of the behavior of an equation we will make a picture that represents the solutions to the equations. A **graph** is simply a picture of the solutions to an equation. Before we spend much time on making a visual representation of an equation, we first have to understand the basis of graphing. Following is an example of what is called the coordinate plane.



The plane is divided into four *quadrants*, or sections, by a horizontal number line (x-axis) and a vertical number line (y-axis). The quadrants are numbered using the roman numerals I, II, III, and IV, beginning with the top-right quadrant (where both x and y are positive) and moving counter-clockwise.

Where the two lines, or axes, meet in the center is called the origin. This center origin is where $x = 0$ and $y = 0$.

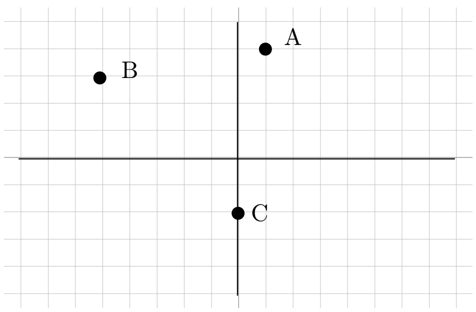
As we move to the right the numbers count up from zero, representing $x = 1, 2, 3\ldots$. To the left the numbers count down from zero, representing $x = -1, -2, -3, \ldots$. Similarly, as we move up the numbers count up from zero, $y = 1, 2, 3, \ldots$, and as we move down count down from zero, $y = -1, -2, -3\ldots$.

We can put dots on the graph which we will call points. Each point has an "address" that defines its location. The first number will be the value on the $x-$axis or horizontal number line. This is the distance the point moves left/right from the origin. The second number will represent the value on the $y-$axis or vertical number line. This is the distance the point moves up/down from the origin. The points are given as an ordered pair (x, y).

World View Note: Locations on the globe are given in the same manner, each number is a distance from a central point, the origin which is where the prime meridian and the equator. This “origin is just off the western coast of Africa.

The following example finds the address or coordinate pair for each of several points on the coordinate plane.

Example 1.42. Give the coordinates of each point.

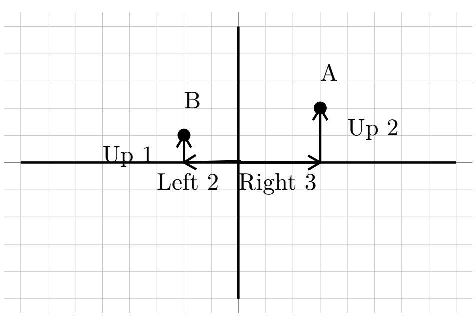


Tracing from the origin, point A is right 1, up 4. This becomes A$(1,4)$. Point B is left 5, up 3. Left is backwards or negative so we have B$(-5,3)$. C is straight down 2 units. There is no left or right. This means we go right zero so the point is C$(0,-2)$.

$$A(1,4), B(-5,3), C(0,-2) \qquad \text{Our solution}$$

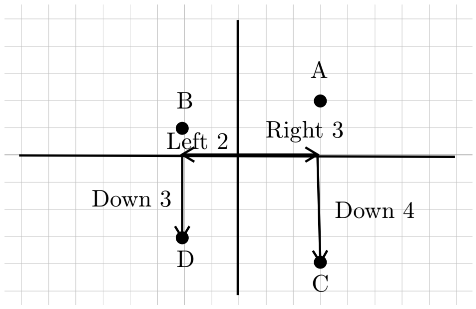
Just as we can give the coordinates for a set of points, we can take a set of points and plot them on the plane.

Example 1.43. Graph the points $A(3,2), B(-2,1), C(3,-4), D(-2,-3), E(-3,0), F(0,2), G(0,0)$



The first point, A is at $(3,2)$ this means $x = 3$ (right 3) and $y = 2$ (up 2). Following these instructions, starting from the origin, we get our point.

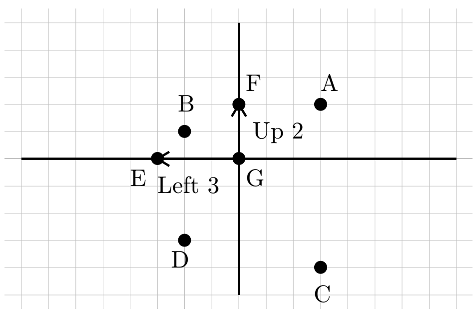
The second point, $B(-2,1)$, is left 2 (negative moves backwards), up 1. This is also illustrated on the graph.



The third point, $C(3,-4)$ is right 3, down 4 (negative moves backwards).

The fourth point, D $(-2,-3)$ is left 2, down 3 (both negative, both move backwards)

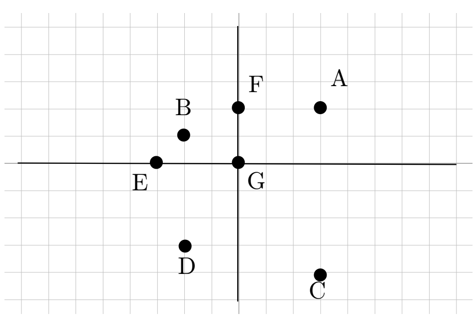
The last three points have zeros in them. We still treat these points just like the other points. If there is a zero there is just no movement.



Next is $E(-3,0)$. This is left 3 (negative is backwards), and up zero, right on the $x-$axis.

Then is $F(0,2)$. This is right zero, and up two, right on the $y-$axis.

Finally is $G(0,0)$. This point has no movement. Thus the point is right on the origin.



Our solution

1.3.2 Graphing Lines from Points

Objective: Graph lines using xy-coordinates.

The main purpose of graphs is not to plot random points, but rather to give a picture of the solutions to an equation. We may have an equation such as $y = 2x - 3$. We may be interested in what type of solution are possible in this equation. We can visualize the solution by making a graph of possible x and y combinations that make this equation a true statement. We will have to start by finding possible x and y combinations. We will do this using a table of values.

Example 1.44.

Graph $y = 2x - 3$ We make a table of values

x	y
-1	
0	
1	

We will test three values for x. Any three can be used

x	y
-1	-5
0	-3
1	-1

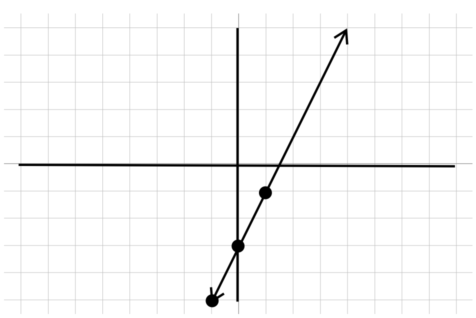
Evaluate each by replacing x with the given value

$x = -1 \qquad y = 2(-1) - 3 = -2 - 3 = -5$

$x = 0 \qquad y = 2(0) - 3 = 0 - 3 = -3$

$x = 1 \qquad y = 2(1) - 3 = 2 - 3 = -1$

$(-1, -5), (0, -3), (1, -1)$ These then become the points to graph on our equation



Plot each point.

Once the point are on the graph, connect the dots to make a line.

The graph is our solution.

What this line tells us is that any point on the line will work in the equation $y = 2x - 3$. For example, notice the graph also goes through the point $(2, 1)$.

If we use $x = 2$, we should get $y = 1$. Sure enough, $y = 2(2) - 3 = 4 - 3 = 1$, just as the graph suggests. Thus we have the line is a picture of all the solutions for $y = 2x - 3$. We can use this table of values method to draw a graph of any linear equation.

Example 1.45.

Graph $2x - 3y = 6$ We will use a table of values

x	y
-3	
0	
3	

We will test three values for x. Any three can be used .

$$\begin{aligned}
2(-3) - 3y &= 6 && \text{Substitute each value in for } x \text{ and solve for } y\\
-6 - 3y &= 6 && \text{Start with } x = -3\text{, multiply first}\\
\underline{\mathbf{+6} \qquad} & \underline{\;\mathbf{+\,6}} && \text{Add 6 to both sides}\\
-3y &= 12 && \text{Divide both sides by } -3\\
\overline{\mathbf{-3}} \quad & \overline{\mathbf{-3}} && \\
y &= -4 && \text{solution for } y \text{ when } x = -3\text{, add this to table}
\end{aligned}$$

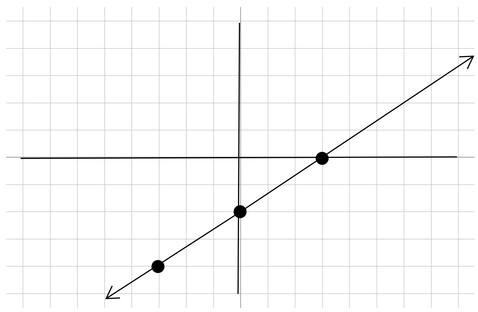
$$\begin{aligned}
2(0) - 3y &= 6 && \text{Next } x = 0\\
-3y &= 6 && \text{Multiplying clears the constant term}\\
\overline{\mathbf{-3}} \quad & \overline{\mathbf{-3}} && \text{Divide each side by } -3\\
y &= -2 && \text{solution for } y \text{ when } x = 0\text{, add this to table}
\end{aligned}$$

$$\begin{aligned}
2(3) - 3y &= 6 && \text{Next } x = 3\\
6 - 3y &= 6 && \text{Multiply}\\
\underline{\mathbf{-6} \qquad} & \underline{\;\mathbf{-\,6}} && \text{Subtract 9 from both sides}\\
-3y &= 0 && \text{Divide each side by } -3\\
\overline{\mathbf{-3}} \quad & \overline{\mathbf{-3}} && \\
y &= 0 && \text{solution for } y \text{ when } x = -3\text{, add this to table}
\end{aligned}$$

x	y
-3	-4
0	-2
3	0

Our completed table

$(-3,-4),(0,2),(3,0)$ Coordinate points from table



Graph points and connect dots

Our solution

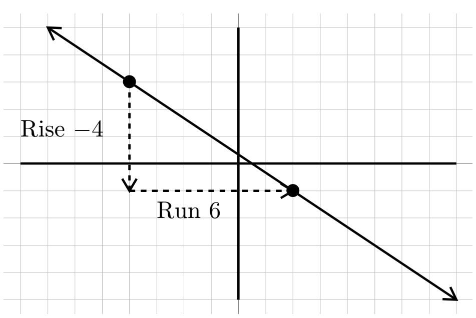
1.3.3 The Slope of a Line

Objective: Find the slope of a line given a graph or two points.

As we graph lines, we will want to be able to identify different properties of the lines we graph. One of the most important properties of a line is its slope. **Slope** is a measure of steepness. A line with a large slope, such as 25, is very steep. A line with a small slope, such as $\frac{1}{10}$ is very flat. We will also use slope to describe the direction of the line. A line that goes up from left to right will have a positive slope and a line that goes down from left to right will have a negative slope.

As we measure steepness we are interested in how fast the line rises compared to how far the line runs. For this reason we will describe slope as the fraction $\frac{\text{rise}}{\text{run}}$. Rise would be a vertical change, or a change in the y-values. Run would be a horizontal change, or a change in the x-values. So another way to describe slope would be the fraction $\frac{\text{change in } y}{\text{change in } x}$. It turns out that if we have a graph we can draw vertical and horiztonal lines from one point to another to make what is called a slope triangle. The sides of the slope triangle give us our slope. The following examples show graphs that we find the slope of using this idea.

Example 1.46.

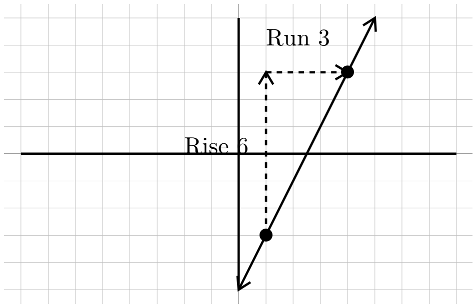


To find the slope of this line we will consider the rise, or vertical change and the run or horizontal change. Drawing these lines in makes a slope triangle that we can use to count from one point to the next the graph goes down 4, right 6. This is rise -4, run 6. As a fraction it would be, $\frac{-4}{6}$. Reduce the fraction to get $\frac{-2}{3}$.

$$\text{A slope of } -\frac{2}{3} \text{ is our solution.}$$

World View Note: When French mathematicians Rene Descartes and Pierre de Fermat first developed the coordinate plane and the idea of graphing lines (and other functions) the y-axis was not a vertical line!

Example 1.47.



To find the slope of this line, the rise is up 6, the run is right 3.

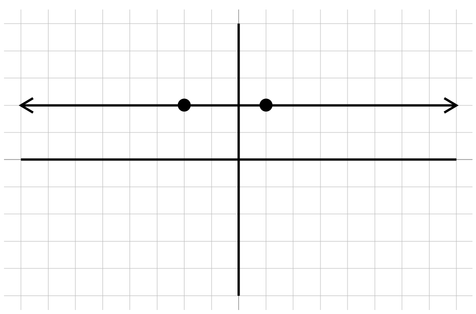
Our slope is then written as a fraction, $\frac{\text{rise}}{\text{run}}$ or $\frac{6}{3}$.

This fraction reduces to 2. This will be our slope.

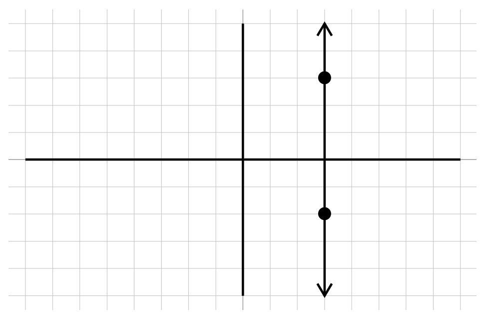
A slope of 2 is our solution.

There are two special lines that have unique slopes that we need to be aware of. They are illustrated in the following example.

Example 1.48.



In this graph there is no rise, but the run is 3 units. This slope becomes $\frac{0}{3} = 0$. This line, and all horizontal lines have a zero slope.



This line has a rise of 5, but no run. The slope becomes $\frac{5}{0}$ = undefined. This line, and all vertical lines, have no slope.

As you can see there is a big difference between having a zero slope and having no slope or undefined slope. Remember, slope is a measure of steepness. The first slope is not steep at all, in fact it is flat. Therefore it has a zero slope. The second slope can't get any steeper. It is so steep that there is no number large enough to express how steep it is. This is an undefined slope.

We can find the slope of a line through two points without seeing the points on a graph. We can do this using a slope formula. If the rise is the change in y

values, we can calculate this by subtracting the y values of a point. Similarly, if run is a change in the x values, we can calculate this by subtracting the x values of a point. In this way we get the following equation for slope.

$$\text{The slope of a line through } (x_1, y_1) \text{ and } (x_2, y_2) \text{ is } \frac{y_2 - y_1}{x_2 - x_1}.$$

When mathematicians began working with slope, it was called the modular slope. For this reason we often represent the slope with the variable m. Now we have the following for slope.

$$\boxed{\textbf{Slope} = \boldsymbol{m} = \frac{\textbf{rise}}{\textbf{run}} = \frac{\textbf{change in } \boldsymbol{y}}{\textbf{change in } \boldsymbol{x}} = \frac{\boldsymbol{y_2 - y_1}}{\boldsymbol{x_2 - x_1}}}$$

As we subtract the y values and the x values when calculating slope it is important we subtract them in the same order. This process is shown in the following examples.

Example 1.49.

$$\begin{array}{rl}
\text{Find the slope between} (-4, 3) \text{ and } (2, -9) & \text{Identify } x_1, y_1, x_2, y_2 \\
(x_1, y_1) \text{ and } (x_2, y_2) & \text{Use slope formula, } m = \dfrac{y_2 - y_1}{x_2 - x_1} \\
m = \dfrac{-9 - 3}{2 - (-4)} & \text{Simplify} \\
m = \dfrac{-12}{6} & \text{Reduce} \\
m = -2 & \text{Our solution}
\end{array}$$

Example 1.50.

$$\begin{array}{rl}
\text{Find the slope between} (4, 6) \text{ and } (2, -1) & \text{Identify } x_1, y_1, x_2, y_2 \\
(x_1, y_1) \text{ and } (x_2, y_2) & \text{Use slope formula, } m = \dfrac{y_2 - y_1}{x_2 - x_1} \\
m = \dfrac{-1 - 6}{2 - 4} & \text{Simplify} \\
m = \dfrac{-7}{-2} & \text{Reduce, dividing by } -1 \\
m = \dfrac{7}{2} & \text{Our solution}
\end{array}$$

We may come up against a problem that has a zero slope (horizontal line) or no slope (vertical line) just as with using the graphs.

Example 1.51.

$$\begin{aligned}
\text{Find the slope between } (-4,-1) \text{ and } (-4,-5) &\qquad \text{Identify } x_1, y_1, x_2, y_2\\
(x_1,y_1) \text{ and } (x_2,y_2) &\qquad \text{Use slope formula, } m = \frac{y_2-y_1}{x_2-x_1}\\
m = \frac{-5-(-1)}{-4-(-4)} &\qquad \text{Simplify}\\
m = \frac{-4}{0} &\qquad \text{Can't divide by zero}\\
\text{Slope } m \text{ is undefined} &\qquad \text{Our solution}
\end{aligned}$$

Example 1.52.

$$\begin{aligned}
\text{Find the slope between } (3,1) \text{ and } (-2,1) &\qquad \text{Identify } x_1, y_1, x_2, y_2\\
(x_1,y_1) \text{ and } (x_2,y_2) &\qquad \text{Use slope formula, } m = \frac{y_2-y_1}{x_2-x_1}\\
m = \frac{1-1}{-2-3} &\qquad \text{Simplify}\\
m = \frac{0}{-5} &\qquad \text{Reduce}\\
m = 0 &\qquad \text{Our solution}
\end{aligned}$$

Again, there is a big difference between no slope and a zero slope. Zero is an integer and it has a value, the slope of a flat horizontal line. No slope has no value, it is undefined, the slope of a vertical line.

Using the slope formula we can also find missing points if we know what the slope is. This is shown in the following two examples.

Example 1.53. Find the value of y between the points $(2, y)$ and (5, - 1) with slope -3.

$$\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{We will plug values into the slope formula}\\
-3 &= \frac{-1 - y}{5 - 2} && \text{Simplify}\\
-3 &= \frac{-1 - y}{3} && \text{Multiply both sides by 3}\\
-3(\mathbf{3}) &= \frac{-1 - y}{3}(\mathbf{3}) && \text{Simplify}\\
-9 &= -1 - y && \text{Add 1 to both sides}\\
\underline{+\mathbf{1}} & \quad \underline{+\mathbf{1}} && \\
-8 &= -y && \text{Divide both sides by } -1\\
\overline{-\mathbf{1}} & \quad \overline{-\mathbf{1}} && \\
8 &= y && \text{Our solution}
\end{aligned}$$

Example 1.54. Find the value of x between the points (- 3, 2) and $(x, 6)$ with slope $\frac{2}{5}$.

$$\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{We will plug values into slope formula}\\
\frac{2}{5} &= \frac{6 - 2}{x - (-3)} && \text{Simplify}\\
\frac{2}{5} &= \frac{4}{x + 3} && \text{Multiply both sides by } (x+3)\\
\frac{2}{5}(x + 3) &= 4 && \text{Multiply by 5 to clear fraction}\\
(\mathbf{5})\frac{2}{5}(x + 3) &= 4(\mathbf{5}) && \text{Simplify}\\
2(x + 3) &= 20 && \text{Distribute}\\
2x + 6 &= 20 && \\
\underline{-\mathbf{6}} & \quad \underline{-\mathbf{6}} && \text{Subtract 6 from both sides}\\
2x &= 14 && \text{Divide each side by 2}\\
\overline{\mathbf{2}} & \quad \overline{\mathbf{2}} && \\
x &= 7 && \text{Our solution}
\end{aligned}$$

1.4 The Two Forms of a Linear Equation

1.4.1 Slope-Intercept Form

Objective: Give the equation of a line with a known slope and y-intercept.

When graphing a line we found one method we could use is to make a table of values. However, if we can identify some properties of the line, we may be able to make a graph much quicker and easier. One such method is finding the slope and the y-intercept of the equation. The slope can be represented by m and the y-intercept, where it crosses the axis and $x = 0$, can be represented by $(0, b)$ where b is the value where the graph crosses the vertical y-axis. Any other point on the line can be represented by (x, y). Using this information we will look at the slope formula and solve the formula for y.

Example 1.55.

$$\begin{aligned}
m,\ (0,b),\ (x,y) & \qquad \text{Use the slope formula} \\
\frac{y-b}{x-0} = m & \qquad \text{Simplify} \\
\frac{y-b}{x} = m & \qquad \text{Multiply both sides by } x \\
y - b = mx & \qquad \text{Add } b \text{ to both sides} \\
\underline{+\boldsymbol{b} \quad +\boldsymbol{b}} & \\
y = mx + b & \qquad \text{Our solution}
\end{aligned}$$

This equation, $y = mx + b$ can be thought of as the equation of any line that as a slope of m and a y-intercept of b. This formula is known as the slope-intercept formula or equation.

$$\boxed{\text{Slope} - \text{intercept equation} : y = mx + b}$$

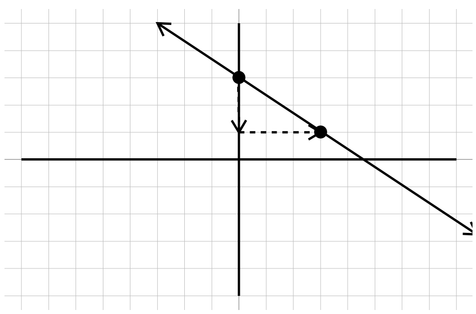
If we know the slope and the y-intercept we can easily find the equation that represents the line.

Example 1.56.

$$\begin{aligned} \text{Slope} = \frac{3}{4},\quad y-\text{intercept} = -3 &\qquad \text{Use the slope} - \text{intercept equation} \\ y = mx+b &\qquad m\,\text{is the slope},\quad b\,\text{is the}\,y-\text{intercept} \\ y = \frac{3}{4}x-3 &\qquad \text{Our solution} \end{aligned}$$

We can also find the equation by looking at a graph and finding the slope and y-intercept.

Example 1.57.



Identify the point where the graph crosses the y-axis (0,3).
This means the y-intercept is 3.

Identify one other point and draw a slope triangle to find the slope.

The slope is $m = -\frac{2}{3}$.

$$\begin{aligned} y &= mx+b &\qquad& \text{Slope-intercept equation} \\ y &= -\tfrac{2}{3}x+3 &\qquad& \text{Our solution} \end{aligned}$$

We can also move the opposite direction, using the equation identify the slope and y-intercept and graph the equation from this information. However, it will be important for the equation to first be in slope intercept form. If it is not, we will have to solve it for y so we can identify the slope and the y-intercept.

Example 1.58. Write the equation $2x = 4y = 6$ in slope-intercept form.

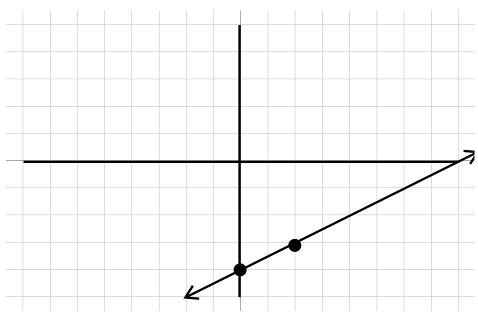
$$\begin{aligned} 2x-4y=6 &\qquad \text{Solve for}\,y \\ \underline{\boldsymbol{-2x}\qquad \boldsymbol{-\,2x}} &\qquad \text{Subtract}\,2x\,\text{from both sides} \\ -4y=-2x+6 &\qquad \text{Put}\,x\,\text{term first} \\ \overline{\boldsymbol{-4}}\quad \overline{\boldsymbol{-4}}\;\; \overline{\boldsymbol{-4}} &\qquad \text{Divide each term by}\,-4 \\ y=\frac{1}{2}x-\frac{3}{2} &\qquad \text{Our solution} \end{aligned}$$

Once we have an equation in slope-intercept form we can graph it by first plotting the y-intercept, then using the slope, finding a second point and connecting the dots.

Example 1.59. Graph $y = \frac{1}{2}x - 4$.

$$\begin{array}{rl} y = mx + b & \text{Slope} - \text{intercept equation} \\ m = \frac{1}{2},\ b = -4 & \text{Identify the slope, } m, \text{and the } y - \text{intercept, } b \end{array}$$

Now make the graph.



Starting with a point at the y-intercept of -4.

Then use the slope $\frac{\text{rise}}{\text{run}}$, so we will rise 1 unit and run 2 units to find the next point.

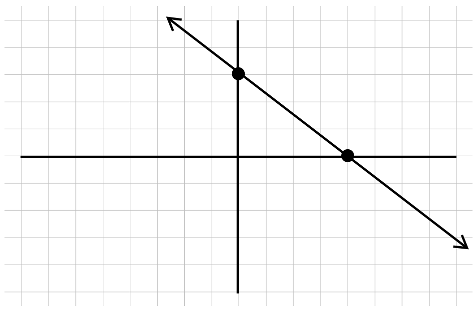
Once we have both points, connect the dots to get our graph.

World View Note: Before our current system of graphing, French Mathematician Nicole Oresme, in 1323 suggested graphing lines that would look more like a bar graph with a constant slope!

Example 1.60. Graph $3x + 4y = 12$.

$$\begin{array}{rl} 3x + 4y = 12 & \text{Not in slope-intercept form} \\ \underline{\mathbf{-3x} \qquad \mathbf{-3x}} & \text{Subtract } 3x \text{ from both sides} \\ 4y = -3x + 12 & \text{Put the } x \text{ term first} \\ \overline{\mathbf{4}} \qquad \overline{\mathbf{4}} \quad \overline{\mathbf{4}} & \text{Divide each term by 4} \\ y = -\frac{3}{4}x + 3 & \text{Now in slope} - \text{intercept form} \\ m = -\frac{3}{4},\ b = 3 & \text{Identify } m \text{ and } b \end{array}$$

Now make the graph.



Starting with a point at the y-intercept of 3.

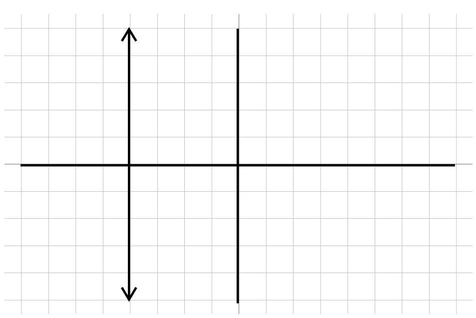
Then use the slope $\frac{\text{rise}}{\text{run}}$, but its negative so it will go downhill, so we will drop 3 units and run 4 units to find the next point.

Once we have both points, connect the dots to get our graph.

We want to be very careful not to confuse using slope to find the next point with use a coordinate such as $(4, -2)$ to find an individual point. Coordinates such as $(4, -2)$ start from the origin and move horizontally first, and vertically second. Slope starts from a point on the line that could be anywhere on the graph. The numerator is the vertical change and the denominator is the horizontal change.

Lines with zero slope or no slope can make a problem seem very different. Such lines are horizontal. A horizontal line will have a slope of zero which when multiplied by x gives zero. So the equation simply becomes $y = b$ or y is equal to the y-coordinate of the graph. If we have no slope, or a vertical line, the equation can't be written in slope intercept at all because the slope is undefined. There is no y in these equations. We will simply make x equal to the x-coordinate of the graph.

Example 1.61.



Give the equation of the line in the graph.

Because we have a vertical line and no slope there is no slope-intercept equation we can use.

Rather we make x equal to the x-coordinate of -4

Our solution is $x = -4$.

1.4.2 Point-Slope Form

Objective: Give the equation of a line with a known slope and point.

The slope-intercept form has the advantage of being simple to remember and use, however, it has one major disadvantage: we must know the y-intercept in order to use it! Generally we do not know the y-intercept, we only know one or more points (that are not the y-intercept). In these cases we can't use the slope intercept equation, so we will use a different, more flexible formula. If we let the slope of an equation be m, and a specific point on the line be (x_1, y_1), and any other point on the line be (x, y). We can use the slope formula to make a second equation.

Example 1.62.

$$\begin{aligned} m,\ (x_1, y_1),\ (x, y) & \qquad \text{Recall slope formula} \\ \frac{y_2 - y_1}{x_2 - x_1} = m & \qquad \text{Plug in values} \\ \frac{y - y_1}{x - x_1} = m & \qquad \text{Multiply both sides by} (x - x_1) \\ y - y_1 = m(x - x_1) & \qquad \text{Our equation} \end{aligned}$$

If we know the slope, m of an equation and any point on the line (x_1, y_1) we can easily plug these values into the equation above which will be called the point-slope formula or equation.

$$\boxed{\text{Point} - \text{slope equation} : y - y_1 = m(x - x_1)}$$

Example 1.63.

Write the equation of the line through the point $(3, -4)$ with a slope of $\frac{3}{5}$.

$$\begin{aligned} y - y_1 = m(x - x_1) & \qquad \text{Plug values into point} - \text{slope formula} \\ y - (-4) = \frac{3}{5}(x - 3) & \qquad \text{Simplify signs} \\ y + 4 = \frac{3}{5}(x - 3) & \qquad \text{Our solution} \end{aligned}$$

Often, we will prefer final answers be written in slope-intercept form. If the directions ask for the answer in slope-intercept form we will simply distribute the slope, then solve for y.

Example 1.64.

Write the equation of the line through the point $(-6, 2)$ with a slope of $-\frac{2}{3}$ in slope-intercept form.

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Plug values into point} - \text{slope formula} \\ y - 2 &= -\frac{2}{3}\,(x - (-6)) && \text{Simplify signs} \\ y - 2 &= -\frac{2}{3}(x + 6) && \text{Distribute slope} \\ y - 2 &= -\frac{2}{3}x - 4 && \text{Solve for } y \text{ by adding 2 to both sides} \\ \underline{\mathbf{+2}} &\quad \underline{\quad\;\; \mathbf{+\,2}} && \\ y &= -\frac{2}{3}x - 2 && \text{Our solution} \end{aligned}$$

An important thing to observe about the point slope formula is that the operation between the x's and y's is subtraction. This means when you simplify the signs you will have the opposite of the numbers in the point. We need to be very careful with signs as we use the point-slope formula.

In order to find the equation of a line we will always need to know the slope. If we don't know the slope to begin with we will have to do some work to find it first before we can get an equation.

Example 1.65.

Find the equation of the line through the points $(-2, 5)$ and $(4, -3)$.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{First we must find the slope} \\ m = \frac{-3 - 5}{4 - (-2)} &= \frac{-8}{6} = -\frac{4}{3} && \text{Plug values in slope formula and evaluate} \\ y - y_1 &= m(x - x_1) && \text{Use point} - \text{slope formula,} \\ & && \quad \text{plugging in slope and either point} \\ y - 5 &= -\frac{4}{3}(x - (-2)) && \text{Simplify signs} \\ y - 5 &= -\frac{4}{3}(x + 2) && \text{Our solution} \end{aligned}$$

Example 1.66.

Find the equation of the line through the points $(-3,4)$ and$(-1,-2)$ in slope-intercept form.

$$\begin{array}{rl}
m = \dfrac{y_2 - y_1}{x_2 - x_1} & \text{First we must find the slope} \\
m = \dfrac{-2-4}{-1-(-3)} = \dfrac{-6}{2} = -3 & \text{Plug values in slope formula and evaluate} \\
y - y_1 = m(x - x_1) & \text{Use point} - \text{slope formula,} \\
& \quad \text{plugging in slope and either point} \\
y - 4 = -3(x - (-3)) & \text{Simplify signs} \\
y - 4 = -3(x + 3) & \text{Distribute slope} \\
y - 4 = -3x - 9 & \text{Solve for } y \\
\underline{\mathbf{+4} \qquad\quad \mathbf{+\ 4}} & \text{Add 4 to both sides} \\
y = -3x - 5 & \text{Our solution}
\end{array}$$

Example 1.67.

Find the equation of the line through the points $(6,-2)$ and $(-4,1)$ in slope-intercept form.

$$\begin{array}{rl}
m = \dfrac{y_2 - y_1}{x_2 - x_1} & \text{First we must find the slope} \\
m = \dfrac{1-(-2)}{-4-6} = \dfrac{3}{-10} = -\dfrac{3}{10} & \text{Plug values into slope formula and evaluate} \\
y - y_1 = m(x - x_1) & \text{Use point} - \text{slope formula,} \\
& \quad \text{plugging in slope and either point} \\
y - (-2) = -\dfrac{3}{10}(x - 6) & \text{Simplify signs} \\
y + 2 = -\dfrac{3}{10}(x - 6) & \text{Distribute slope} \\
y + 2 = -\dfrac{3}{10}x + \dfrac{9}{5} & \text{Solve for } y\text{, by subtracting 2 from both sides} \\
\underline{\mathbf{-2} \qquad\qquad -\mathbf{\dfrac{10}{5}}} & \text{Use } \dfrac{10}{5} \text{ on right so we have a common denominator} \\
y = -\dfrac{3}{10}x - \dfrac{1}{5} & \text{Our solution}
\end{array}$$

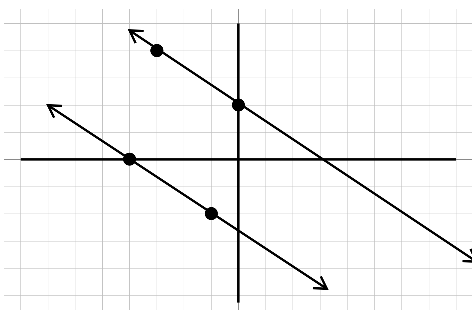
World View Note: The city of Königsberg (now Kaliningrad, Russia) had a river that flowed through the city breaking it into several parts. There were 7 bridges that connected the parts of the city. In 1735 Leonhard Euler considered the question of whether it was possible to cross each bridge exactly once and only once. It turned out that this problem was impossible, but the work laid the foundation of what would become graph theory.

1.5 Parallel and Perpendicular Lines

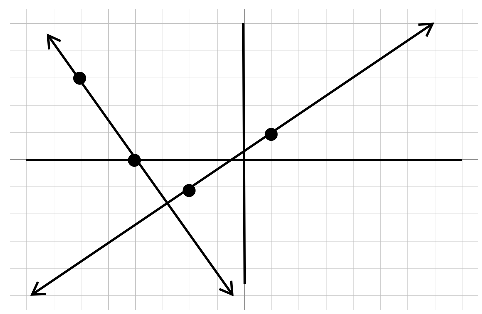
Objective: Identify the equation of a line given a parallel or perpendicular line.

There is an interesting connection between the slopes of lines that are parallel, as well as the slopes of lines that are perpendicular (meet at a right angle). This is shown in the following example.

Example 1.68.



The above graph has two parallel lines. The slope of the top line is down 2, run 3, or $-\frac{2}{3}$.
The slope of the bottom line is down 2, run 3 as well, or $-\frac{2}{3}$.



The above graph has two perpendicular lines. The slope of the flatter line is up 2, run 3 or $\frac{2}{3}$.
The slope of the steeper line is down 3, run 2, or $-\frac{3}{2}$.

As the first graph above illustrates, parallel lines have the same slope.

On the other hand, perpendicular lines are said to have *negative reciprocal* slopes. More precisely, if two lines with slopes m_1 and m_2 are known to be perpendicular, then $m_2 = -\dfrac{1}{m_1}$ (and so, $m_1 m_2 = -1$).

We can use these properties to make conclusions about parallel and perpendicular lines.

World View Note: Greek Mathematician Euclid lived around 300 BC and published a book titled, *The Elements*. In it is the famous parallel postulate which mathematicians have tried for years to drop from the list of postulates. The attempts have failed, yet all the work done has developed new types of geometries!

Example 1.69. Find the slope of a line parallel to $5y - 2x = 7$.

$5y - 2x = 7$	To find the slope we will put equation in slope − intercept form
$\underline{\ \mathbf{+2x} \quad \mathbf{+2x}}$	Add $2x$ to both sides
$5y = 2x + 7$	Put x term first
$\overline{\mathbf{5}} \quad \overline{\mathbf{5}} \quad \overline{\mathbf{5}}$	Divide each term by 5
$y = \frac{2}{5}x + \frac{7}{5}$	The slope is the coefficient of x
$m = \frac{2}{5}$	Slope of given line
	Parallel lines have the same slope
$m = \frac{2}{5}$	Our solution

Example 1.70. Find the slope of a line perpendicular to $3x - 4y = 2$.

$3x - 4y = 2$	To find slope we will put equation in slope − intercept form
$\underline{\mathbf{-3x} \qquad \mathbf{-3x}}$	Subtract $3x$ from both sides
$-4y = -3x + 2$	Put x term first
$\overline{\mathbf{-4}} \quad \overline{\mathbf{-4}} \quad \overline{\mathbf{-4}}$	Divide each term by -4
$y = \frac{3}{4}x - \frac{1}{2}$	The slope is the coefficient of x
$m = \frac{3}{4}$	Slope of given line
	Perpendicular lines have negative reciprocal slopes
$m = -\frac{4}{3}$	Our solution

Once we have a slope, it is possible to find the complete equation of the desired line, if we know one point on it.

Example 1.71. Find the equation of a line through $(4, -5)$ and parallel to $2x - 3y = 6$.

$$\begin{array}{rl}
2x - 3y = 6 & \text{We first need slope of parallel line} \\
\underline{\mathbf{-2x} \qquad\quad \mathbf{-\ 2x}} & \text{Subtract } 2x \text{ from each side} \\
-3y = -2x + 6 & \text{Put } x \text{ term first} \\
\overline{\mathbf{-3}} \quad \overline{\mathbf{-3}} \ \ \overline{\mathbf{-3}} & \text{Divide each term by } -3 \\
y = \dfrac{2}{3}x - 2 & \text{Identify the slope, the coefficient of } x \\
m = \dfrac{2}{3} & \text{Parallel lines have the same slope} \\
m = \dfrac{2}{3} & \text{We will use this slope and our point}(4, -5) \\
y - y_1 = m(x - x_1) & \text{Plug this information into point} - \text{slope formula} \\
y - (-5) = \dfrac{2}{3}(x - 4) & \text{Simplify signs} \\
y + 5 = \dfrac{2}{3}(x - 4) & \text{Our solution}
\end{array}$$

Example 1.72. Find the equation of the line through $(6, -9)$ perpendicular to $y = -\frac{3}{5}x + 4$ in slope-intercept form.

$$\begin{array}{rl}
y = -\dfrac{3}{5}x + 4 & \text{Identify the slope, coefficient of } x \\
m = -\dfrac{3}{5} & \text{Perpendicular lines have negative reciprocal slopes} \\
m = \dfrac{5}{3} & \text{We will use this slope and our point}(6, -9) \\
y - y_1 = m(x - x_1) & \text{Plug this information into point} - \text{slope formula} \\
y - (-9) = \dfrac{5}{3}(x - 6) & \text{Simplify signs} \\
y + 9 = \dfrac{5}{3}(x - 6) & \text{Distribute slope}
\end{array}$$

$$
\begin{array}{rl}
y + 9 = \dfrac{5}{3}x - 10 & \text{Solve for } y \\
\underline{\mathbf{-9} \qquad\quad \mathbf{-\ 9}} & \text{Subtract 9 from both sides} \\
y = \dfrac{5}{3}x - 19 & \text{Our solution}
\end{array}
$$

Zero slopes and undefined slopes may seem like opposites (one is a horizontal line, one is a vertical line). Because a horizontal line is perpendicular to a vertical line we can say that an undefined slope and a zero slope are actually perpendicular slopes!

Example 1.73. Find the equation of the line through (3, 4) perpendicular to $x = -2$.

$$
\begin{array}{rl}
x = -2 & \text{This equation has an undefined slope, a vertical line} \\
\text{Undefined slope} & \text{Perpendicular line then would have a zero slope} \\
m = 0 & \text{Use this and our point} (3, 4) \\
y - y_1 = m(x - x_1) & \text{Plug this information into point} - \text{slope formula} \\
y - 4 = 0(x - 3) & \text{Distribute slope} \\
y - 4 = 0 & \text{Solve for } y \\
\underline{\mathbf{+4} \ \ \mathbf{+\ 4}} & \text{Add 4 to each side} \\
y = 4 & \text{Our solution}
\end{array}
$$

Being aware that to be perpendicular to a vertical line means we have a horizontal line through a y value of 4, thus we could have jumped from this point right to the solution, $y = 4$.

1.6 Applications

1.6.1 Numbers and Geometry

Objective: Solve number and geometry problems by creating and solving a linear equation.

Word problems can be tricky. Often it takes a bit of practice to convert the English sentence into a mathematical sentence. This is what we will focus on here with some basic number problems, geometry problems, and parts problems.

A few important phrases are described below that can give us clues for how to set up a problem.

- **A number** (or unknown, an integer value, etc) often becomes our variable

- **Is** (or other forms of is: was, will be, are, etc) often represents equals (=)

  x is 5 becomes $x = 5$

- **More than** often represents addition and is usually built backwards, writing the second part plus the first

  Three more than a number becomes $x + 3$

- **Less than** often represents subtraction and is usually built backwards as well, writing the second part minus the first

  Four less than a number becomes $x - 4$

Using these key phrases we can take a number problem and set up and solve an equation.

Example 1.74. If 28 less than five times a certain number is 232. What is the number?

$$\begin{array}{rl} 5x - 28 & \text{Subtraction is built backwards, multiply the unknown by 5} \\ 5x - 28 = 232 & \text{“Is” translates to equals} \\ \underline{\mathbf{+28} \quad \mathbf{+\ 28}} & \text{Add 28 to both sides} \\ 5x = 260 & \text{The variable is multiplied by 5} \\ \overline{\mathbf{5}} \quad \overline{\mathbf{5}} & \text{Divide both sides by 5} \\ x = 52 & \text{The number is 52} \end{array}$$

This same idea can be extended to a more involved problem as shown in the next example.

Example 1.75. Fifteen more than three times a number is the same as ten less than six times the number. What is the number?

$$\begin{array}{rl} 3x + 15 & \text{First, addition is built backwards} \\ 6x - 10 & \text{Then, subtraction is also built backwards} \\ 3x + 15 = 6x - 10 & \text{“Is” between the parts tells us they must be equal} \\ \underline{\boldsymbol{-3x} \qquad \boldsymbol{-\ 3x}} & \text{Subtract } 3x, \text{ so variable is all on one side} \\ 15 = 3x - 10 & \text{Now we have a two} - \text{step equation} \\ \underline{\mathbf{+10} \qquad \mathbf{+\ 10}} & \text{Add 10 to both sides} \\ 25 = 3x & \text{The variable is multiplied by 3} \\ \overline{\mathbf{3}} \quad \overline{\mathbf{3}} & \text{Divide both sides by 3} \\ \dfrac{25}{3} = x & \text{Our number is } \dfrac{25}{3} \end{array}$$

Another type of number problem involves consecutive integers. **Consecutive integers** are whole numbers that come one after the other, such as 3, 4, 5. If we are looking for several consecutive integers it is important to first identify what they look like with variables, before we set up the equation. This is shown in the following example.

Example 1.76. The sum of three consecutive integers is 93. What are the integers?

$$\begin{array}{rl} \text{First } x & \text{Make the first number } x \\ \text{Second } x + 1 & \text{To get the next number we go up one or } +1 \\ \text{Third } x + 2 & \text{Add another } 1 (2 \text{ total}) \text{ to get the third} \end{array}$$

$$\begin{array}{rl}
F+S+T=93 & \text{First } (F) \text{ plus Second } (S) \text{ plus} \\
 & \quad \text{Third } (T) \text{ equals } 93 \\
(x)+(x+1)+(x+2)=93 & \text{Replace } F, S \text{ and } T \text{ with} \\
 & \quad \text{their respective expressions} \\
x+x+1+x+2=93 & \text{Here the parentheses aren't needed} \\
3x+3=93 & \text{Combine like terms } x+x+x \text{ and } 2+1 \\
\underline{\mathbf{-3} \quad \mathbf{-3}} & \text{Add 3 to both sides} \\
3x=90 & \text{The variable is multiplied by 3} \\
\overline{\mathbf{3}} \qquad \overline{\mathbf{3}} & \text{Divide both sides by 3} \\
x=30 & \text{Our solution for } x \\
\text{First is } 30 & \text{Replace } x \text{ in our original list with } 30 \\
\text{Second is } (30)+1=31 & \text{The numbers are } 30, 31, \text{ and } 32 \\
\text{Third is } (30)+2=32 &
\end{array}$$

Sometimes we will work with consecutive even or odd integers, rather than just consecutive integers. When we had consecutive integers, we only had to add 1 to get to the next integer so we had x, $x+1$, and $x+2$ for our first, second, and third integer respectively.

Sets of even (or odd) integers, however, are spaced apart by two. So if we want three consecutive even integers, if the first is x, the next integer would be $x+2$, then finally add two more to get the third, $x+4$. The same is true for consecutive odd integers, if the first is x, the next will be $x+2$, and the third would be $x+4$. It is important to note that we are still adding 2 and 4 even when the integers are odd. This is because the phrase "odd" is referring to our x, not to what is added to the integers. Consider the next two examples.

Example 1.77. The sum of three consecutive even integers is 246. What are the integers?

$$\begin{array}{rl}
\text{First } x & \text{Make the first } x \\
\text{Second } x+2 & \text{Even numbers, so we add 2 to get the next} \\
\text{Third } x+4 & \text{Add 2 more(4 total) to get the third} \\
F+S+T=246 & \text{Sum means add First}(F) \text{ plus} \\
 & \quad \text{Second}(S) \text{ plus Third}(T)
\end{array}$$

$$
\begin{array}{rl}
(x)+(x+2)+(x+4)=246 & \text{Replace each } F, S, \text{ and } T \text{ with} \\
 & \quad \text{their respective expressions} \\
x+x+2+x+4=246 & \text{Here the parentheses are not needed} \\
3x+6=246 & \text{Combine like terms } x+x+x \text{ and } 2+4 \\
\underline{\mathbf{-6 \quad -6}} & \text{Subtract 6 from both sides} \\
3x=240 & \text{The variable is multiplied by 3} \\
\overline{\mathbf{3}} \quad \overline{\mathbf{3}} & \text{Divide both sides by 3} \\
x=80 & \text{Our solution for } x \\
\text{First is } 80 & \text{Replace } x \text{ in the original list with 80} \\
\text{Second is } (80)+2=82 & \text{The numbers are 80, 82, and 84} \\
\text{Third is } (80)+4=84 &
\end{array}
$$

Example 1.78. Find three consecutive odd integers so that the sum of twice the first, the second and three times the third is 152.

$$
\begin{array}{rl}
\text{First } x & \text{Make the first } x \\
\text{Second } x+2 & \text{Odd numbers so we add 2 (same as even!)} \\
\text{Third } x+4 & \text{Add 2 more (4 total) to get the third} \\
2F+S+3T=152 & \text{Twice the first gives } 2F, \\
 & \quad \text{three times the third gives } 3T \\
2(x)+(x+2)+3(x+4)=152 & \text{Replace } F, S, \text{ and } T \text{ with} \\
 & \text{their respective expressions} \\
2x+x+2+3x+12=152 & \text{Distribute through parentheses} \\
6x+14=152 & \text{Combine like terms } 2x+x+3x \text{ and } 2+14 \\
\underline{\mathbf{-14 \quad -14}} & \text{Subtract 14 from both sides} \\
6x=138 & \text{Variable is multiplied by 6} \\
\overline{\mathbf{6}} \quad \overline{\mathbf{6}} & \text{Divide both sides by 6} \\
x=23 & \text{Our solution for } x \\
\text{First is } 23 & \text{Replace } x \text{ with 23 in the original list} \\
\text{Second is } (23)+2=25 & \text{The numbers are 23, 25, and 27} \\
\text{Third is } (23)+4=27 &
\end{array}
$$

When we started with our first, second, and third integers for both even and odd we had x, $x+2$, and $x+4$. The numbers added (2 and 4) do not change with successive odds or evens. It is our answer for x that will be odd or even.

Another example of translating English sentences to mathematical sentences comes from geometry. A well known property of triangles is that all three angles will always add to 180 degrees. For example, the first angle may be 50 degrees, the second 30 degrees, and the third 100 degrees. If you add these together, $50 + 30 + 100 = 180$. We can use this property to find angles of triangles.

World View Note: German mathematician Bernhart Thibaut in 1809 tried to prove that the angles of a triangle add to 180 degrees without using Euclid's parallel postulate (a point of much debate in mathematical history). He created a proof, but it was shown to contain an error.

Example 1.79. The second angle of a triangle is double the first. The third angle is 40 degrees less than the first. Find the measure of all three angles.

First angle is x	With nothing known about the first angle we label it x
Second angle is $2x$	The second angle is double the first
Third angle is $x - 40$	The third angle is $40°$ less than the first
$F + S + T = 180$	All three angles add to $180°$
$(x) + (2x) + (x - 40) = 180$	Replace F, S, and T with the labeled values
$x + 2x + x - 40 = 180$	Here the parentheses are not needed
$4x - 40 = 180$	Combine like terms, $x + 2x + x$
$\underline{\mathbf{+40} \quad \mathbf{+40}}$	Add $40°$ to both sides
$4x = 220$	The variable is multiplied by 4
$\overline{\mathbf{4}} \quad \overline{\mathbf{4}}$	Divide both sides by 4
$x = 55$	Our solution for x
First is 55	Replace x with $55°$ in the original list of angles
Second is $2(55) = 110$	Our angles are $55°$, $110°$, and $15°$
Third is $(55) - 40 = 15$	

Another geometry problem involves perimeter or the distance around an object. For example, consider a rectangle having a length of 8 units and a width of 3 units. There are two lengths and two widths in a rectangle (opposite sides) so we add $8 + 8 + 3 + 3 = 22$. As there are two lengths and two widths in a rectangle an alternative to find the perimeter of a rectangle is to use the formula $P = 2L + 2W$. So for the rectangle of length 8 units and width 3 units the formula would give, $P = 2(8) + 2(3) = 16 + 6 = 22$. With problems that we will consider here the formula $P = 2L + 2W$ will be used.

Example 1.80. The perimeter of a rectangle is 44 units. The width is 5 units less than double the length. Find the dimensions of the rectangle.

$$
\begin{array}{rl}
\text{Length } x & \text{We will make the length } x \\
\text{Width } 2x-5 & \text{Width is five less than two times the length} \\
P = 2L + 2W & \text{The formula for perimeter of a rectangle} \\
(44) = 2(x) + 2(2x-5) & \text{Replace } P, L, \text{and } W \text{ with labeled values} \\
44 = 2x + 4x - 10 & \text{Distribute through parentheses} \\
44 = 6x - 10 & \text{Combine like terms } 2x + 4x \\
\underline{\mathbf{+10 \quad\quad + 10}} & \text{Add 10 units to both sides} \\
54 = 6x & \text{The variable is multiplied by 6} \\
\overline{\mathbf{6}} \quad \overline{\mathbf{6}} & \text{Divide both sides by 6} \\
9 = x & \text{Our solution for } x \\
\text{Length is 9 units} & \text{Replace } x \text{ with 9 in the original list of sides} \\
\text{Width is } 2(9) - 5 = 13 \text{ units} &
\end{array}
$$

The dimensions of the rectangle are 9 units by 13 units.

We have seen that it is important to start by clearly labeling the variables in a short list before we begin to solve the problem. This is important in all word problems involving variables, not just consecutive integers or geometry problems. This is shown in the following example.

Example 1.81. A sofa and a love seat have a combined cost of \$444. The sofa costs double the love seat. How much do they each cost?

$$
\begin{array}{rl}
\text{Love seat cost is } x & \text{With no information known about the} \\
 & \quad \text{love seat, we label it } x \\
\text{Sofa cost is } 2x & \text{Sofa is double the love seat, so we multiply by 2}
\end{array}
$$

$$
\begin{array}{rl}
S + L = 444 & \text{Together they cost \$444, so we add} \\
(x) + (2x) = 444 & \text{Replace } S \text{ and } L \text{ with labeled values} \\
 & \text{Parentheses are not needed,} \\
3x = 444 & \quad \text{combine like terms } x + 2x \\
\overline{\mathbf{3}} \quad \overline{\mathbf{3}} & \text{Divide both sides by 3} \\
x = 148 & \text{Our solution for } x \\
\text{Love seat cost is } \$148 & \text{Replace } x \text{ with 148 in the original list} \\
\text{Sofa cost is } 2(148) = \$296 & \text{The love seat costs \$148 and the sofa costs \$296}
\end{array}
$$

Be careful on problems such as these. Many students see the phrase "double" and believe that means we only have to divide the \$444 by 2 and get \$222 for one or both of the prices. As you can see this will not work. By clearly labeling the variables in the original list we know exactly how to set up and solve these problems.

1.6.2 Age Problems

Objective: Solve age problems by creating and solving a linear equation.

Age problems present another application of linear equations. When we are solving age problems we generally will be comparing the age of two people both now and in the future (or past). Using the clues given in the problem we will be working to find their current age. There can be a lot of information in these problems and we can easily get lost in all the information. To help us organize and solve our problem we will fill out a three by three table for each problem. An example of the basic structure of the table is shown below.

	Age Now	Change
Person 1		
Person 2		

Structure of Age Table

Normally where we see "Person 1" and "Person 2" we will use the name of the person we are talking about. We will use this table to set up the following example.

Example 1.82. Adam is 20 years younger than Brian. In two years Brian will be twice as old as Adam. How old are they now?

	Age Now	+2
Adam		
Brian		

We use Adam and Brian for our persons. We use + 2 for change because the second phrase is two years in the future.

	Age Now	+2
Adam	$x-20$	
Brain	x	

Consider the "Now" part, Adam is 20 years younger than Brian.

We are given information about Adam, not Brian. So Brian is x now.

To show Adam is 20 years younger we subtract 20, Adam is $x-$ 20.

	Age Now	+2
Adam	$x-20$	$x-20+2$
Brian	x	$x+2$

Now the + 2 column is filled in. This is done by adding 2 to both Adam's and Brian's 'Age Now' column entry, as shown in the table.

	Age Now	+2
Adam	$x-20$	$x-18$
Brian	x	$x+2$

Combine like terms in Adam's future age : - 20 + 2.

This table is now completed, and we are ready to solve.

$$B=2A$$

Our equation comes from the future statement: Brian will be twice as old as Adam.

This means the younger, Adam, needs to be multiplied by 2.

$$(x+2)=2(x-18)$$

Replace B and A with the information in their future cells, Adam (A) is replaced with $x-$ 18 and Brian (B) is replaced with $x+$ 2.

This is the equation to solve!

$$x+2=2x-36$$

Distribute through parentheses

$$\underline{\boldsymbol{-x} \qquad \boldsymbol{-x}}$$

Subtract x from both sides to get variable on one side

$$2=x-36$$

Need to clear the $-$ 36

$$\underline{\mathbf{+36} \quad \mathbf{+36}}$$

Add 36 to both sides

$$38=x$$

Our solution for x

	Age now
Adam	$38 - 20 = 18$
Brian	38

The first column will help us answer the question.
Replace the x's with 38 and simplify.
Adam is 18 and Brian is 38.

Solving age problems can be summarized in the following five steps. These five steps are guidelines to help organize the problem we are trying to solve.

1. Fill in the 'Now' column. The age of the person we know nothing about is x.

2. Fill in the future/past column by adding/subtracting the change to the 'Now' column.

3. Make an equation for the relationship in the future. This is independent of the table.

4. Replace variables in the equation with information in the future/past cells of table.

5. Solve the equation for x, use the solution to answer the question.

These five steps are illustrated in the following example.

Example 1.83. Carmen is 12 years older than David. Five years ago the sum of their ages was 28. How old are they now?

	Age Now	-5
Carmen		
David		

Five years ago is -5 in the change column .

	Age Now	-5
Carmen	$x + 12$	
David	x	

Carmen is 12 years older than David. We don't know about David so his age is x.
Carmen's age is then $x+$ 12.

	Age Now	-5
Carmen	$x + 12$	$x + 12 - 5$
David	x	$x - 5$

Subtract 5 from 'Now' column to get the change.

	Age Now	-5
Carmen	$x+12$	$x+7$
David	x	$x-5$

Simplify by combining like terms, 12 - 5.
Our table is ready!

$C+D=28$	The sum of their ages will be 28.
	So we add C and D.
$(x+7)+(x-5)=28$	Replace C and D with the change cells .
$x+7+x-5=28$	Remove parentheses
$2x+2=28$	Combine like terms $x+x$ and $7-5$
$\underline{\mathbf{-2} \quad \mathbf{-2}}$	Subtract 2 from both sides
$2x=26$	Notice x is multiplied by 2
$\overline{\mathbf{2}} \quad \overline{\mathbf{2}}$	Divide both sides by 2
$x=13$	Our solution for x

	Age Now
Carmen	$13+12=25$
David	13

Replace x with 13 to answer the question. Carmen is 25 and David is 13.

Sometimes we are given the sum of two (or more) people's ages right now. These problems can be tricky. In this case we will write the sum above the 'Now' column and assign x to the first person's age now. The second person's age will then involve subtraction: (Total age) $-$ x. This is shown in the next example.

Example 1.84. The sum of the ages of Nicole and Kristen is 32. In two years Nicole will be three times as old as Kristen. How old are they now?

	Age Now	$+2$
Nicole	x	
Kristen	$32-x$	

The change is + 2 for two years in the future.
The first person's age is x.
The second person's age becomes 32 $-x$.

	Age Now	$+2$
Nicole	x	$x+2$
Kristen	$32-x$	$32-x+2$

Fill in the change column by adding 2 to each cell.

	Age Now	$+2$
Nicole	x	$x+2$
Kristen	$32-x$	$34-x$

Combine like terms $32+2$, our table is done!

$$N = 3K$$ Nicole is three times as old as Kristen

$$(x+2) = 3(34-x)$$ Replace variables with information in change cells

$$x + 2 = 102 - 3x$$ Distribute through parentheses

$$\underline{\mathbf{+3x} \qquad \mathbf{+\,3x}}$$ Add $3x$ to both sides so variable is only on one side

$$4x + 2 = 102$$ Solve the two $-$ step equation

$$\underline{\mathbf{-2} \quad \mathbf{-\,2}}$$ Subtract 2 from both sides

$$4x = 100$$ The variable is multiplied by 4

$$\overline{\mathbf{4}} \quad \overline{\mathbf{4}}$$ Divide both sides by 4

$$x = 25$$ Our solution for x

	Age Now
Nicole	25
Kristen	$32-25=7$

Plug 25 in for x in the 'Now' column.
Nicole is 25 and Kristen is 7.

A slight variation on age problems is to ask not how old the people are, but rather ask how long until we have some relationship about their ages. In this case we alter our table slightly. In the change column because we don't know the time to add or subtract we will use a variable, t, and add or subtract this from the 'Now' column. This is shown in the next example.

Example 1.85. Louise is 26 years old. Her daughter is 4 years old. In how many years will Louise be double her daughter's age?

	Age Now	$+t$
Louise	26	
Daughter	4	

As we are given their ages now, these numbers go into the table.
The change is unknown, so we write $+t$ for the change.

	Age Now	$+t$
Louise	26	$26+t$
Daughter	4	$4+t$

Fill in the change column by adding t to each person's age.
Our table is now complete.

$$
\begin{array}{rl}
L = 2D & \text{Louise will be double her daughter's age} \\
(26+t) = 2(4+t) & \text{Replace variables with information in change cells} \\
26 + t = 8 + 2t & \text{Distribute through parentheses} \\
\underline{\boldsymbol{-t} \qquad \boldsymbol{-\ t}} & \text{Subtract } t \text{ from both sides} \\
26 = 8 + t & \text{Now we have an 8 added to the } t \\
\underline{\mathbf{-8} \quad \mathbf{-\ 8}} \quad\;\; & \text{Subtract 8 from both sides} \\
18 = t & \text{In 18 years she will be double her daughter's age}
\end{array}
$$

Age problems have several steps to them. If we take the time to work through each of the steps carefully, however, keeping the information organized, the problems can be solved quite nicely.

World View Note: The oldest man in the world was Shigechiyo Izumi from Japan who lived to be 120 years, 237 days. His exact age, however, has been disputed.

1.6.3 Distance, Rate and Time

Objective: Solve distance problems by creating and solving a linear equation.

An application of linear equations can be found in distance problems. When solving distance problems we will use the relationship $rt = d$ or rate (speed) times time equals distance. For example, if a person were to travel 30 mph (miles per hour) for 4 hours. To find the total distance we would multiply rate times time or $(30)(4) = 120$. This person would travel a distance of 120 miles. The problems we will be solving here will require a few more steps than described above. So to keep the information in the problem organized we will use a table. The basic structure of the table is shown below.

	Rate	Time	Distance
Person 1			
Person 2			

Structure of Distance Problem

The third column, 'distance', will always be filled in by multiplying the 'rate' and 'time' columns together. If we are given a total distance of both persons or trips we will put this information below the distance column. We will now use this table to set up and solve the following example.

Example 1.86. Two joggers start from opposite ends of an 8 mile course running towards each other. One jogger is running at a rate of 4 mph, and the other is running at a rate of 6 mph. After how long will the joggers meet?

	Rate	Time	Distance
Jogger 1			
Jogger 2			

The basic table for the joggers, Jogger 1 and Jogger 2.

	Rate	Time	Distance
Jogger 1	**4**		
Jogger 2	**6**		

We are given the rates for each jogger.
These are added to the table.

	Rate	Time	Distance
Jogger 1	4	$\boldsymbol{t}$	
Jogger 2	6	$\boldsymbol{t}$	

We only know they both start and end at the same time. We use the variable t for both times.

	Rate	Time	Distance
Jogger 1	4	t	$\boldsymbol{4t}$
Jogger 2	6	t	$\boldsymbol{6t}$

The distance column is filled in by multiplying rate by time.

8 — We have **total distance**, 8 miles, under distance.

$$4t + 6t = 8$$

Add the entries in the distance column and set equal to total distance.

$$10t = 8$$

Combine like terms, $4t + 6t$.

$$\overline{\mathbf{10}} \quad \overline{\mathbf{10}}$$

Divide both sides by 10.

$$t = \frac{4}{5}$$

Our solution for t

The joggers will meet at $\frac{4}{5}$ hour, or 48 minutes.

As the example illustrates, once the table is filled in, the resulting equation is very easy to solve. This same process can be seen in the following example.

Example 1.87. Bob and Fred start from the same point and walk in opposite directions. Bob walks 2 miles per hour faster than Fred. After 3 hours they are 30 miles apart. How fast did each walk?

	Rate	Time	Distance
Bob		**3**	
Fred		**3**	

The basic table with given times filled in.
Both traveled 3 hours.

	Rate	Time	Distance
Bob	$\boldsymbol{r+2}$	3	
Fred	$\boldsymbol{r}$	3	

Bob walks 2 mph faster than Fred. We know nothing about Fred, so use r for his rate.

Bob's rate is $r + 2$, showing 2 mph faster.

	Rate	Time	Distance
Bob	$r+2$	3	$\boldsymbol{3r+6}$
Fred	r	3	$\boldsymbol{3r}$

Distance column is filled in by multiplying rate by time.

Be sure to distribute the $3\,(r + 2)$ for Bob.

30 — **Total distance** is put under distance .

$3r + 6 + 3r = 30$ — Add the entries in the distance column and set equal to total distance.

$6r + 6 = 30$ — Combine like terms, $3r + 3r$.

$\underline{\mathbf{-6 \quad -6}}$ — Subtract 6 from both sides .

$6r = 24$ — The variable is multiplied by 6.

$\overline{\mathbf{6}} \quad \overline{\mathbf{6}}$ — Divide both sides by 6.

$r = 4$ — Our solution for r

	Rate
Bob	$4 + 2 = 6$
Fred	4

To answer the question completely we plug 4 in for r in the table.

Bob traveled 6 miles per hour and Fred traveled 4 mph.

Some problems will require us to do a bit of work before we can just fill in the cells. One example of this is if we are given a total time, rather than the individual times like we had in the previous example. If we are given total time we will write this above the time column, use t for the first person's time, and make a subtraction problem, (Total) $-\ t$, for the second person's time. This is shown in the next example.

Example 1.88. Two campers left their campsite by canoe and paddled downstream at an average speed of 12 mph. They turned around and paddled back upstream at an average rate of 4 mph. The total trip took 1 hour. After how much time did the campers turn around downstream?

	Rate	Time	Distance
Down	**12**		
Up	**4**		

Basic table for down and upstream.
Given rates are filled in.

1

Total time is put above 'time' column .

	Rate	Time	Distance
Down	12	$\boldsymbol{t}$	
Up	4	$\mathbf{1-\boldsymbol{t}}$	

As we only have the total time, in the time down we have t, the time up becomes the difference, (total) $-\ t$.

	Rate	Time	Distance
Down	12	t	$\mathbf{12\boldsymbol{t}}$
Up	4	$1-t$	$\mathbf{4-4\boldsymbol{t}}$

Distance column is found by multiplying rate by time.
Be sure to distribute 4 $(1-t)$ for upstream.

$$12t = 4 - 4t$$

Since they cover the same distance, set values in last column equal to each other.

$$\underline{\mathbf{+4\boldsymbol{t}} \quad \mathbf{+\ 4\boldsymbol{t}}}$$

Add $4t$ to both sides so variable is only on one side.

$$16t = 4$$

Variable is multiplied by 16.

$$\overline{\mathbf{16}} \quad \overline{\mathbf{16}}$$

Divide both sides by 16.

$$t = \frac{1}{4}$$

Our solution

The campers turned around after $\frac{1}{4}$ hour, or 15 minutes.

Another type of a distance problem is that where one person catches up with another. Here a slower person has a head start and the faster person is trying to catch up with him or her. We want to know how long it will take for this to happen. Our strategy for this problem will be to use t for the faster person's time, and add the amount of time for the head start to obtain the slower person's time. This is shown in the next example.

Example 1.89. Mike leaves his house traveling 2 miles per hour. Joy leaves 6 hours later to catch up with him, traveling 8 miles per hour. How long will it take her to catch up with him?

	Rate	Time	Distance
Mike	**2**		
Joy	**8**		

Basic table for Mike and Joy.
The given rates are filled in.

	Rate	Time	Distance
Mike	2	$\boldsymbol{t+6}$	
Joy	8	$\boldsymbol{t}$	

We use t to represent the faster person‘s time.
Mike‘s time is $t + 6$, showing his 6 hour head start.

	Rate	Time	Distance
Mike	2	$t+6$	$\boldsymbol{2t+12}$
Joy	8	t	$\boldsymbol{8t}$

Distance column is found by multiplying the rate by time.
Be sure to distribute the $2\ (t + 6)$ for Mike.

$$2t + 12 = 8t$$ Since they cover the same distance, set values in last column equal to each other.

$$\underline{\boldsymbol{-2t} \qquad \boldsymbol{-\ 2t}}$$ Subtract $2t$ from both sides .

$$12 = 6t$$ The variable is multiplied by 6.

$$\overline{\boldsymbol{6}} \quad \overline{\boldsymbol{6}}$$ Divide both sides by 6.

$$2 = t$$ Our solution for t

Joy catches Mike after 2 hours.

World View Note: The 10,000 (or 10k) race is the longest standard track event. 10,000 meters is approximately 6.2 miles. The current (at the time of printing) world record for this race is held by Ethiopian Kenenisa Bekele with a time of 26 minutes, 17.53 seconds. That is a rate of 12.7 miles per hour!

As these example have shown, using a table can help keep all the given information organized, and consequently help find the equation we must solve. One final example illustrates this.

Example 1.90. On a 130 mile trip a car traveled at an average speed of 55 mph and then reduced its speed to 40 mph for the remainder of the trip. The trip took 2.5 hours. How long did the car travel at 40 mph?

	Rate	Time	Distance
Fast	**55**		
Slow	**40**		

Basic table for fast and slow speeds.

The given rates are filled in.

Total time is put above the 'Time' column.

2.5

	Rate	Time	Distance
Fast	55	$\boldsymbol{t}$	
Slow	40	$\boldsymbol{2.5 - t}$	

Since total time is given, we assign t for the time spent traveling 55mph.

The other time is the difference $2.5 - t$.

2.5

	Rate	Time	Distance
Fast	55	t	$\boldsymbol{55t}$
Slow	40	$2.5 - t$	$\boldsymbol{100 - 40t}$

130

Distance column is found by multiplying rate by time. Be sure to distribute 40 $(2.5 - t)$.

Total distance is put under 'Distance' column.

$$55t + 100 - 40t = 130$$

Add the entries in the distance column and set equal to total distance.

$$15t + 100 = 130$$

Combine like terms $55t - 40t$.

$$\underline{\mathbf{-100 \quad - 100}}$$

Subtract 100 from both sides .

$$15t = 30$$

The variable is multiplied by 30.

$$\overline{\mathbf{15}} \quad \overline{\mathbf{15}}$$

Divide both sides by 15.

$$t = 2$$

Our solution for t

	Time
Fast	2
Slow	$2.5 - 2 = 0.5$

To answer the question we plug 2 in for t

The car traveled 40 mph for 0.5 hours (30 minutes).

1.7 Linear Inequalities and Sign Diagrams

Objective: Solve, graph, and give interval notation for the solution to a linear inequality. Create a sign diagram to identify those intervals where a linear expression is positive or negative.

1.7.1 Linear Inequalities

When given a linear equation such as $x + 2 = 5$, one can solve to obtain *one* solution ($x = 3$). Although the method for solving an inequality is, in general, very similar to that for solving an equation, we will see that the solution to a inequality will usually include an entire range of values.

In order to solve any inequality, we must first understand the accompanying notation and respective terminology. Much of what will follow may seem familiar, as it was also introduced in the last chapter of Part I of this text. Students struggling with the notation may wish to review the earlier material in Part I regarding Number Sets.

Symbol	Meaning
$<$	less than
$>$	greater than
$\leq$	less than or equal to
$\geq$	greater than or equal to
$\neq$	not equal to

World View Note: English mathematician Thomas Harriot first used the above symbols in 1631. However, they were not immediately accepted as symbols such as $\sqsubset$ and $\sqsupset$ were already coined by another English mathematician, William Oughtred.

Notice that the "equals" symbol $=$ is not listed above, as we will be working with *inequalities*, rather than *equations*. It is also worth mentioning that there are several alternate ways of describing the same symbol. For example, the phrases "at most" or "no more than" can easily be interchanged with "less than or equal to", and similarly for "at least", "no less than", and "greater than or equal to". Because of this, one needs to use a bit of caution, when faced with any problem that is presented verbally.

Example 1.91.

$$2 < 5, \qquad 1 > -1, \qquad 5 \leq 10, \qquad 3 \leq 3,$$
$$7 \geq -2, \qquad 4 \geq 4, \qquad -1 \neq 1$$

The examples above, though true, do not contain a variable. We now will work with inequalities containing one (or more) variable(s). Following the previous sections of this chapter, we will first concern ourselves with linear inequalities, followed by basic absolute value inequalities. The solution to an inequality is the set of all real numbers that make the inequality true.

Example 1.92. Solve the linear inequality $x + 2 < 5$.

$$\begin{array}{rl} x + 2 < 5 & \\ \underline{\mathbf{-2}} \quad \underline{\mathbf{-2}} & \text{Subtract 2 from both sides} \\ x < 3 & \text{Our solution} \end{array}$$

Notice that we solve the previous inequality using the same method that one would use to solve the equation $x + 2 = 5$. Some differences will be seen later.

When describing the solution to a given inequality, it will often be useful to graph the solution on a number line and shade the section(s) of the number line that coincide with the solution set. The number line below illustrates our previous example.

$x < 3$

Note that an open circle is used to indicate that the value $x = 3$ is *not* a valid solution to the given inequality. A closed circle would therefore indicate that $x = 3$ is a valid solution. It is also a good idea to test a few values in order to check our work.

Check:

Test Location	Test Value	Unsimplified	Simplified	Result
Shaded region	$x = 1$	$1 + 2 < 5$	$3 < 5$	True
Boundary value	$x = 3$	$3 + 2 < 5$	$5 < 5$	False
Unshaded region	$x = 5$	$5 + 2 < 5$	$7 < 5$	False

A common misconception that many students have with an inequality such as $x < 3$ and is worth mentioning has to do with the values between $x = 2$ and $x = 3$. Although we have seen that x cannot equal 3 in the given inequality, this does not mean that the solution set has a largest value at $x = 2$ (the largest *integer* solution to the inequality). In fact, there are infinitely many *real-number* solutions to the inequality between the integers 2 and 3. For example, 2.5, 2.7, 2.9, 2.99, 2.999, and 2.9999999999999999 are all valid solutions to $x < 3$. Because of this, one could say that the inequality is *bounded above by* $x = 3$, but there is no *largest* solution that satisfies it.

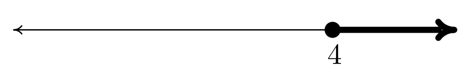
There are four primary ways of presenting the solution to an inequality:

1. In words (verbally): "All real numbers x greater than or equal to 4."

2. Using inequality (and set-builder) notation: $\{x|x \geq 4\}$.

3. Using interval notation: $[4, \infty)$.

4. Using real-number line notation (graphically):

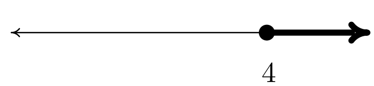


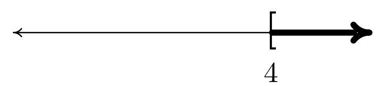
In many of our examples, it will be acceptable to exclude the set-builder notation $\{x| \quad \}$ altogether, and instead simply present the inequality $x \geq 4$. Still, it is important that students recognize the meaning behind the notation ("The set of all real numbers x such that...").

Recall that for interval notation we use brackets [or] to denote *inclusion* of a boundary value, and parentheses (and) to denote *exclusion.* This notation can therefore be interchanged with a closed circle (inclusion) or an open circle (exclusion), when graphing a given solution set on the real-number line. As a convention, from this point forward we will adopt brackets and parentheses instead of closed and open circles for graphical representations of solution sets, since it presents a nice connection between interval and real-number line notation. Both notations, however, are generally accepted. An example is shown below.

Example 1.93.

$x < 3$





$x \geq 4$

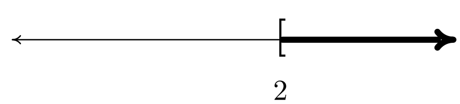
Next, we will solve and present the solution to a linear equality using all four presentation methods.

Example 1.94. Solve the linear inequality $4x - 3 \geq 5$.

$$\begin{array}{rcll}
4x - 3 & \geq & 5 & \\
\underline{\mathbf{+3}} & & \underline{\mathbf{+3}} & \text{Add 3 to both sides} \\
4x & \geq & 8 & \\
\overline{\mathbf{4}} & & \overline{\mathbf{4}} & \text{Divide both sides by 4} \\
x & \geq & 2 & \text{Our solution}
\end{array}$$

Our solution can be expressed as follows.

1. Verbally: "The set of all values of x that are greater than or equal to (at least) 2".
2. Inequality: $\{x | x \geq 2\}$
3. Interval: $[2, \infty)$
4. Real-number Line (Graphically):



<u>Check</u>:

Test Location	Test Value	Unsimplified	Simplified	Result
Shaded region	$x = 3$	$4(3) - 3 \geq 5$	$9 \geq 5$	True
Boundary value	$x = 2$	$4(2) - 3 \geq 5$	$5 \geq 5$	True
Unshaded region	$x = 0$	$4(0) - 3 \geq 5$	$-3 \geq 5$	False

Next, we would like to closely examine the impact that each of the four main operations ($+$, $-$, $\times$, $\div$) has on a given inequality. This will shed more light on one of the fundamental differences between solving an equation and solving an inequality. To demonstrate this, we will repeatedly use an obvious true statement, $4 < 10$.

Example 1.95.

Original Inequality: $4 < 10$

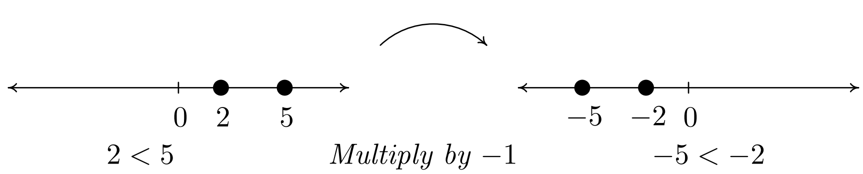
Action	Resulting Inequality	Outcome
Add 5	$9 < 15$	True
Subtract 5	$-1 < 5$	True
Add -3	$1 < 7$	True
Subtract -3	$7 < 13$	True

Note that since addition and subtraction are closely related, we see that the original inequality is also preserved when negative values are either added or subtracted. In other words, adding (or subtracting) -3 will also preserve the validity of the inequality. It is also worth noting that the action of adding -3 is analogous with that of subtracting 3, so there are no surprises. Later on, we will use the term *inverse* to describe the relationship between these two operations.

Original Inequality: $4 < 10$

Action	Resulting Inequality	Outcome
Multiply by 3	$12 < 30$	True
Divide by 2	$2 < 5$	True
Multiply by -3	$-12 < -30$	**False**
Divide by -2	$-2 < -5$	**False**

Here, we see that multiplication, and consequently division, by a negative value forces us to change the direction of the inequality ($-2 < -5$ changes to $-2 > -5$) in order to preserve its validity. This is best illustrated by the following diagram.



Note that as with addition and subtraction, the *inverse* relationship between the operations of multiplication and division is again at work, since for example, division by -2 is analogous to multiplication by $-1/2$.

We conclude our treatment of linear inequalities with a more complicated example. All our solution steps will be identical to those for solving a linear equation, with the only exception being those steps related to multiplication or division by a negative number.

Example 1.96. Solve the linear inequality $-1 - 2(x-3) \leq 5x - 9$.

$$
\begin{array}{rcll}
-1 - 2(x-3) & \leq & 5x - 9 & \\
-1 - 2x + 6 & \leq & 5x - 9 & \text{Distribute } -2 \\
5 - 2x & \leq & 5x - 9 & \text{Combine like terms} \\
\underline{\mathbf{-5}} & & \underline{\mathbf{-5}} & \text{Subtract 5 from both sides} \\
-2x & \leq & 5x - 14 & \\
\underline{\boldsymbol{-5x}} & & \underline{\boldsymbol{-5x}} & \text{Subtract } 5x \text{ from both sides} \\
-7x & \leq & -14 & \\
\overline{\mathbf{-7}} & & \overline{\mathbf{-7}} & \text{Divide both sides by } -7 \\
& x \geq 2 & & \text{Our solution}
\end{array}
$$

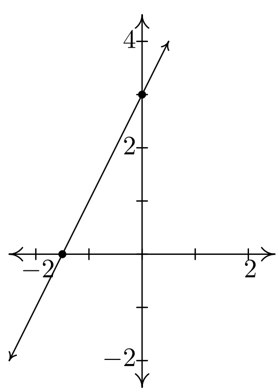
We leave it as an exercise to the reader to check that our solution is correct.

1.7.2 Introduction to Sign Diagrams

In a later chapter we will define a *function*, providing several examples of y as a function of x, and discuss in detail the processes associated with graphing certain families of functions. As both linears and quadratics (the next chapter) present the most basic examples of polynomials, we will take this opportunity to introduce a tool, called a sign diagram (or sign chart), that will be incredibly useful for graphing these and more complicated functions. For the sake of the mathematics, it should be noted that the usefulness of the sign diagram for graphing is a direct consequence of the *continuity* of a function and the *Intermediate Value Theorem* (IVT). The notion of continuity is one that will be studied more closely in subsequent courses (e.g. Calculus), and the IVT will be deferred to the later chapter on polynomials.

Example 1.97. Graph the linear equation $y = 2x + 3$.

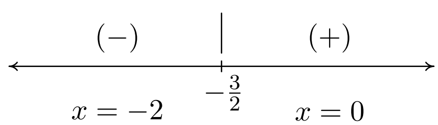
Our graph will have a y-intercept at the point $(0, 3)$. By setting $y = 0$, we obtain an x-intercept at the point $(-3/2, 0)$. We then obtain the following graph by plotting these two intercepts and connecting them.



When graphing any equation, it will be of particular interest to identify any x-intercepts on the graph. Though this will sometimes prove a daunting and even impossible task, as we have seen, it is relatively straightforward when faced with a linear equation. Recall that all lines which are not horizontal will have exactly one x-intercept. Horizontal lines will either have no x-intercepts or, in the case of the horizontal line $y = 0$, will have infinitely many x-intercepts. Once we know the x-intercept of the graph of our linear equation, we can easily determine the sign ($+$ or $-$) of the y-coordinate for every point to the left or right of our x-intercept. Since all lines are by their nature straight, this amounts to testing our equation, by plugging in a single *test value* for each interval on either side of our x-intercept.

In the case of our example, though we are free to choose any real-numbered test values we would like, we will make the more common selections of $x = -2$ and $x = 0$. Note that $x = -1$ would have been a perfectly fine value instead of $x = 0$, but it is often easier to plug $x = 0$ into a function than any other value. After plugging each test value into the equation, we determine the sign of the y-coordinate associated with $x = -2$ is negative $(-)$, since $2(-2) + 3 < 0$, and the sign of the y-coordinate associated with $x = 0$ is positive $(+)$, since $2(0) + 3 > 0$. Note that here we are **not** concerned with the actual values of the y-coordinates, just their respective signs. This point will be reiterated as we encounter more complicated mathematical expressions. The results of our calculations are presented on the real-number line shown below.

Example 1.98. Sign Diagram for $y = 2x + 3$.



Note that if constructed correctly, our sign diagram should be consistent with the graph of $y = 2x + 3$. Specifically, a $(+)$ corresponds to those points on the graph that sit *above* the x-axis, and a $(-)$ corresponds to those points that sit *below* the x-axis.

We now will summarize the steps for constructing a sign diagram for a linear equation (or function) with a nonzero slope.

1. If not provided, put the equation in slope-intercept form.

2. Determine the x-intercept of the graph of the equation. Mark this value on a real-number line by placing a symbol ($|$) directly above it that divides the line into two intervals.

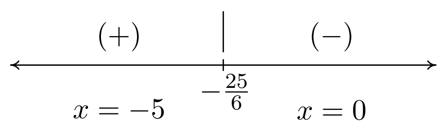
3. Identify a test value for each interval. Write your test values below their respective test intervals.

4. Determine the sign ($+$ or $-$) of the y-coordinate for each test value. Mark this on the real-number line by placing either a $+$ or $-$ above the interval.

Example 1.99. Construct a sign diagram for the linear equation $y = -12x - 50$.

By setting $y = 0$, we get $x = -\frac{50}{12} = -\frac{25}{6} = 4.1\overline{6}$. For test values, we will use $x = -5$ and $x = 0$.

Test Value	Resulting y-coordinate	Sign
$x = -5$	$-12(-5) - 50 = 60 - 50 > 0$	$(+)$
$x = 0$	$-12(0) - 50 = 0 - 50 < 0$	$(-)$



Note that in the instance of a horizontal line $m = 0$, our sign diagram will only require us to test a single value for the entire interval $(-\infty, \infty)$. It therefore suffices to just identify the sign of the y-intercept for the graph of our equation. Lastly, if the y-intercept is zero, then our sign diagram will have no test intervals to check, since all points on our graph will be of the form $(x, 0)$.

It is worth mentioning that here we have only sought to "set the table" for the construction of sign diagrams, using linear equations as a very basic introduction. Once we are exposed to more complicated equations and functions, such as quadratics in the next chapter, we will see how the construction of a sign diagram will become more involved. In short, more complicated examples will include more x-intercepts, which will result in more test intervals to check. The process, however, will essentially remain the same as we have outlined, and the resulting sign diagram will be critical in understanding the graph of a function.

1.8 Compound and Absolute Value Inequalities

1.8.1 Compound Inequalities

Objective: Solve, graph and give interval notation to the solution of compound inequalities and inequalities containing absolute values.

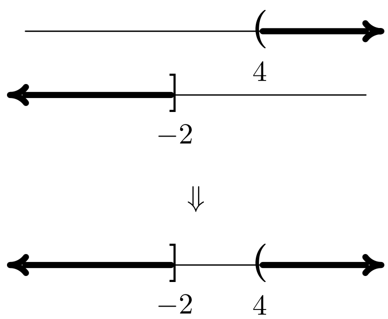
Several inequalities can be combined together to form what are called compound inequalities. There are three types of compound inequalities which we will investigate in this section.

The first type of a compound inequality is an OR inequality. For this type of inequality we want a true statement from either one inequality OR the other inequality OR both. When we are graphing these type of inequalities we will graph each individual inequality above the number line, then combine them together on the number line for our graph.

When we provide interval notation for our solution, if there are two different intervals to the graph we will put a $\cup$ (union) symbol between the two intervals.

Example 1.100. Solve each inequality, graph the solution, and provide the interval notation of your solution.

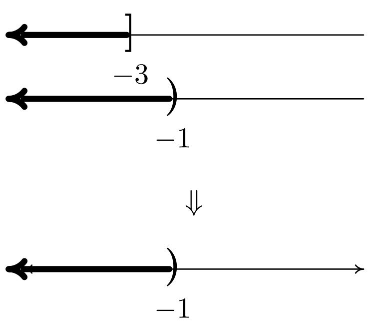
$$\begin{array}{lcll}
2x - 5 > 3 & \text{or} & 4 - x \geq 6 & \text{Solve each inequality} \\
\underline{\mathbf{+5}\ \ \mathbf{+5}} & & \underline{\mathbf{-4}\qquad \mathbf{-4}} & \text{Add or subtract first} \\
2x > 8 & \text{or} & -x \geq 2 & \text{Divide} \\
\overline{\mathbf{2}}\quad \overline{\mathbf{2}} & & \overline{\mathbf{-1}}\quad \overline{\mathbf{-1}} & \text{Dividing by negative flips sign} \\
x > 4 & \text{or} & x \leq -2 & \text{Graph the inequalities separately,} \\
 & & & \quad \text{then combine}
\end{array}$$



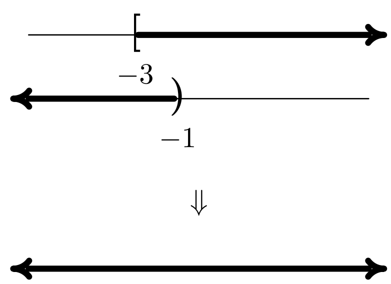
$x \leq -2$ or $x > 4$

World View Note: The symbol for infinity was first used by the Romans, although at the time the number was used for 1000. The Greeks also used the symbol for 10,000.

There are several different results that could result from an OR statement. The graphs could be pointing different directions, as in the graph above. The graphs could also be pointing in the same direction, as in the graph below on the left. Lastly, the graphs could be pointing in opposite directions, but overlapping, as in the graph below on the right. Notice how interval notation works for each of these cases.



$x \leq -3$ or $x < -1$



$x \geq -3$ or $x < -1$

As the graphs overlap, we take the largest graph ($x < -1$) for our solution.
Interval notation: $(-\infty, 1)$

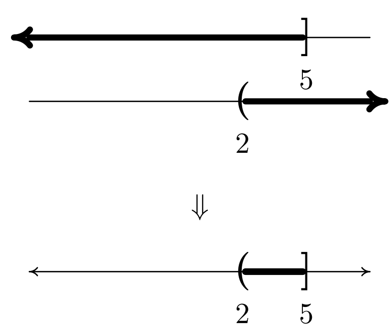
When the graphs are combined they cover the entire number line.

Interval notation: $(-\infty, \infty)$ or $\mathbb{R}$

The second type of compound inequality is an AND inequality. These inequalities require *both* statements to be true. If one is false, they both are false. When we graph these inequalities we can follow a similar process. First, graph both inequalities above the number line. This time, however, we will only consider where they overlap to the number line for our final graph. When our solution is given in interval notation it will be expressed in a manner very similar to single inequalities (there is a symbol that can be used for AND, the intersection, denoted by a $\cap$, but we will not use it here).

Example 1.101. Solve each inequality, graph the solution, and provide the interval notation of your solution.

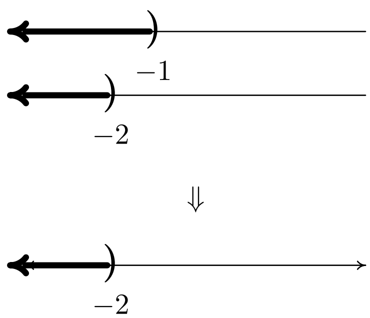
$$
\begin{array}{rclcrcll}
2x+8 & \geq & 5x-7 & \text{and} & 5x-3 & > & 3x+1 & \text{Move variables to one side} \\
\underline{\mathbf{-2x}} & & \underline{\mathbf{-\,2x}} & & \underline{\mathbf{-3x}} & & \underline{\mathbf{-\,3x}} & \\
8 & \geq & 3x-7 & \text{and} & 2x-3 & > & 1 & \text{Add 7 or 3 to both sides} \\
& & \underline{\mathbf{+7 \quad +\,7}} & & & & \underline{\mathbf{+3 \quad +\,3}} & \\
15 & \geq & 3x & \text{and} & 2x & > & 4 & \text{Divide} \\
\overline{\mathbf{3}} & & \overline{\mathbf{3}} & & \overline{\mathbf{2}} & & \overline{\mathbf{2}} & \\
5 & \geq & x & \text{and} & x & > & 2 & \text{Graph the inequalities} \\
& & & & & & & \quad\text{separately, then combine}
\end{array}
$$



$x > 2$ and $x \leq 5$

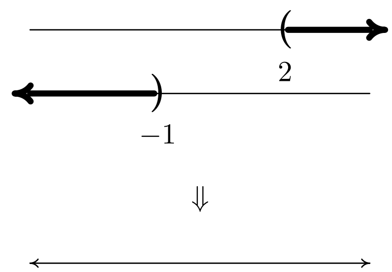
$(2, 5]$ Interval notation

Again, as we graph AND inequalities, only the overlapping parts of the individual graphs makes it to the final number line. There are also three different types of results we could get. The first is shown in the above example. The second occurs when the arrows both point in the same direction, as shown below on the left. The third occurs when the arrows point in opposite directions, but do not overlap, as shown below on the right. Notice how interval notation is expressed in each case.



$x < -1$ and $x < -2$

In this graph, the overlap is only the smaller graph ($x < -2$), so this is what makes it to the final number line.
Interval notation: $(-\infty, -2)$



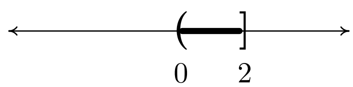
$x > 2$ and $x < -1$

In this graph there is no overlap of the parts. Because there is no overlap, no values make it to the final number line.
Interval notation: No solution or $\varnothing$

The third type of compound inequality is a special type of AND inequality. When our variable (or expression containing the variable) is between two numbers. We can write this as a single mathematical sentence with three parts, such as $5 < x \leq 8$, to show x is between 5 and 8 (or equal to 8). This type of inequality is often referred to as a *double inequality*, since it contains two inequalities. When solving these types of inequalities, as there are three parts to work with, in order to stay balanced we will do the same thing to all three parts (rather than just two sides), and eventually isolate the variable in the middle. The resulting graph will contain all values between the numbers on either side of the double inequality, with appropriate brackets on the ends.

Example 1.102. Solve each inequality, graph the solution, and provide the interval notation of your solution.

$$\begin{array}{rl}
-6 \leq -4x + 2 < 2 & \text{Subtract 2 from all three parts} \\
\underline{\mathbf{-2}} \qquad\quad \underline{\mathbf{-2}} \quad \underline{\mathbf{-2}} & \\
-8 \leq -4x < 0 & \text{Divide all three parts by } -4 \\
\overline{\mathbf{-4}} \quad \overline{\mathbf{-4}} \quad \overline{\mathbf{-4}} & \text{Dividing by a negative flips the symbols} \\
2 \geq x > 0 & \text{Flip entire statement so values get larger left to right} \\
0 < x \leq 2 & \text{Graph } x \text{ between 0 and 2}
\end{array}$$



$0 < x \leq 2$

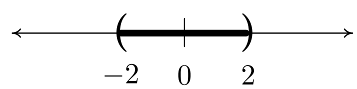
$(0, 2]$ Interval notation

1.8.2 Absolute Value Inequalities

When an inequality contains an absolute value we will have to remove the absolute value in order to graph the solution or provide interval notation. The way we remove the absolute value depends on the direction of the inequality symbol.

Consider $|x| < 2$.

Absolute value is defined as the distance from zero. Another way to read this inequality would be the distance that the variable x is from zero is less than 2. So on a number line we will shade all points that are less than 2 units away from zero.



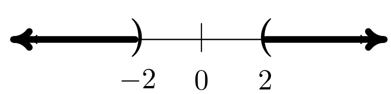
$|x| < 2$

$(-2, 2)$ Interval notation

This graph looks just like the graphs of the double (compound) inequalities! When the absolute value is **less than** a number we will remove the absolute value by changing the problem to a double inequality, with the negative value on the left and the positive value on the right. So $|x| < 2$ becomes $-2 < x < 2$, as the graph above illustrates.

Consider $|x| > 2$.

Similarly, another way to read this inequality would be the distance that x is from zero is greater than 2. So on the number line we shade all points that are more than 2 units away from zero.



$|x| > 2$

$(-\infty, -2) \cup (2, \infty)$ Interval notation

This graph looks just like the graphs of the OR compound inequalities! When the absolute value is **greater than** a number we will remove the absolute value by changing the problem to an OR inequality, the first inequality looking just like the problem with no absolute value, the second flipping the inequality symbol and changing the value to a negative. So $|x| > 2$ becomes $x > 2$ or $x < -2$, as the graph above illustrates.

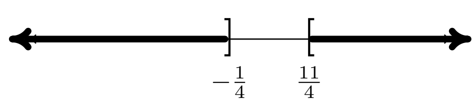
World View Note: The phrase "absolute value" comes from German mathematician Karl Weierstrass in 1876, though he used the absolute value symbol for complex numbers. The first known use of the symbol for integers comes from a 1939 edition of a college algebra text!

For all absolute value inequalities we can also express our answers in interval notation which is done the same way as for standard compound inequalities.

We can solve absolute value inequalities much like we solved absolute value equations. Our first step will be to isolate the absolute value. Next we will remove the absolute value by either making a double inequality if the absolute value is less than a number, or making an OR inequality if the absolute value is greater than a number. Then we will solve these inequalities. Remember, if we multiply or divide by a negative the inequality symbol(s) will switch directions!

Example 1.103. Solve, graph, and provide interval notation for the solution.

$$\begin{array}{cccl}
 & |4x-5| \geq 6 & & \text{Absolute value is greater, use OR} \\
4x-5 \geq 6 & \text{OR} & 4x-5 \leq -6 & \text{Solve} \\
\underline{\mathbf{+5} \quad \mathbf{+5}} & & \underline{\mathbf{+5} \quad \mathbf{+5}} & \text{Add } 5 \text{ to both sides} \\
4x \geq 11 & \text{OR} & 4x \leq -1 & \\
\overline{\mathbf{4}} \quad \overline{\mathbf{4}} & & \overline{\mathbf{4}} \quad \overline{\mathbf{4}} & \text{Divide both sides by } 4 \\
x \geq \dfrac{11}{4} & \text{OR} & x \leq -\dfrac{1}{4} & \text{Graph}
\end{array}$$

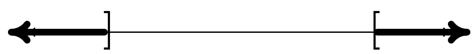


$|4x-5| \geq 6$

$$\left(-\infty, -\frac{1}{4}\right] \cup \left[\frac{11}{4}, \infty\right) \qquad \text{Interval notation}$$

Example 1.104. Solve, graph, and provide interval notation for the solution.

$$\begin{array}{rl}
-4-3|x| \leq -16 & \\
\underline{\mathbf{+4} \qquad\qquad \mathbf{+\,4}} & \text{Add } 4 \text{ to both sides} \\
-3|x| \leq -12 & \text{Divide both sides by } -3 \\
\overline{\mathbf{-3}} \quad \overline{\mathbf{-3}} & \text{Dividing by a negative switches the inequality} \\
|x| \geq 4 & \text{Absolute value is greater, use OR} \\
x \geq 4 \quad \text{OR} \quad x \leq -4 & \text{Graph}
\end{array}$$



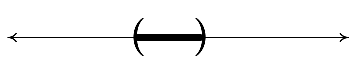
$-4-3|x| \leq -16$

$$(-\infty,-4] \cup [4,\infty) \qquad \text{Interval notation}$$

In the previous example, we cannot combine -4 and -3 because they are not like terms, the -3 has an absolute value attached. So we must first clear the -4 by adding 4, then divide by -3. The next example is similar.

Example 1.105. Solve, graph, and provide interval notation for the solution.

$$\begin{array}{rl}
9-2|4x+1| > 3 & \\
\underline{\mathbf{-9} \qquad\qquad\quad \mathbf{-\,9}} & \text{Subtract } 9 \text{ from both sides} \\
-2|4x+1| > -6 & \text{Divide both sides by } -2 \\
\overline{\mathbf{-2}} \qquad \overline{\mathbf{-2}} & \text{Dividing by negative switches the inequality} \\
|4x+1| < 3 & \text{Absolute value is less, use double inequality} \\
-3 < 4x+1 < 3 & \text{Solve} \\
\underline{\mathbf{-1}} \qquad \underline{\mathbf{-1}} \quad \underline{\mathbf{-1}} & \text{Subtract } 1 \text{ from all three parts} \\
-4 < 4x < 2 & \text{Divide all three parts by } 4 \\
\overline{\mathbf{4}} \quad \overline{\mathbf{4}} \quad \overline{\mathbf{4}} & \\
-1 < x < \dfrac{1}{2} & \text{Graph}
\end{array}$$



$9-2|4x+1| > 3$

$$\left(-1,\frac{1}{2}\right) \qquad \text{Interval notation}$$

In the previous example, we cannot distribute the -2 into the absolute value. We can never distribute or combine things outside the absolute value with what is inside the absolute value. Our only way to solve is to first isolate the absolute value by clearing the values around it, then convert to a compound inequality (either a double inequality or an OR inequality) and solve.

It is important to remember that as we are solving these equations, an absolute value is always positive. If we end up with an absolute value that is less than a negative number, then we will have no solution because the absolute value will always be positive, and therefore greater than a negative. Similarly, if an absolute value is greater than a negative, this will always happen. Here our answer will be all real numbers.

Example 1.106. Solve, graph, and provide interval notation for the solution.

$$\begin{array}{rl}
12+4|6x-1|<4 & \text{Subtract } 12 \text{ from both sides} \\
\underline{\mathbf{-12} \qquad\qquad \mathbf{-12}} & \\
4|6x-1|<-8 & \text{Divide both sides by } 4 \\
\overline{\mathbf{4}} \qquad \overline{\mathbf{4}} & \\
|6x-1|<-2 & \text{Absolute value cannot be less than a negative}
\end{array}$$

$$12+4|6x-1|<4$$

There is nothing to shade. So our answer is no solution or $\varnothing$.

Example 1.107. Solve, graph, and provide interval notation for the solution.

$$\begin{array}{rl}
5-6|x+7|\leq 17 & \\
\underline{\mathbf{-5} \qquad\qquad \mathbf{-5}} & \text{Subtract } 5 \text{ from both sides} \\
-6|x+7|\leq 12 & \text{Divide both sides by } -6 \\
\overline{\mathbf{-6}} \qquad \overline{\mathbf{-6}} & \text{Dividing by a negative flips the symbol} \\
|x+7|\geq -2 & \text{Absolute value is always greater than a negative}
\end{array}$$

$$5-6|x+7|\leq 17$$

We shade the entire real number line. Our answer is all real numbers or $\mathbb{R}$.

$$(-\infty,\infty) \qquad \text{Interval notation}$$

1.9 Practice Problems

1.9.1 Solving Linear Equations

One-Step Equations

Solve each equation.

1) $v + 9 = 16$
2) $14 = b + 3$
3) $x - 11 = -16$
4) $-14 = x - 18$
5) $30 = a + 20$
6) $-1 + k = 5$
7) $x - 7 = -26$
8) $-13 + p = -19$
9) $13 = n - 5$
10) $22 = 16 + m$
11) $340 = -17x$
12) $4r = -28$
13) $-9 = \frac{n}{12}$
14) $\frac{5}{9} = \frac{b}{9}$
15) $20v = -160$
16) $-20x = -80$
17) $340 = 20n$
18) $\frac{1}{2} = \frac{a}{8}$
19) $16x = 320$
20) $\frac{k}{13} = -16$
21) $-16 + n = -13$
22) $21 = x + 5$
23) $p - 8 = -21$
24) $m - 4 = -13$
25) $180 = 12x$
26) $3n = 24$
27) $20b = -200$
28) $-17 = \frac{x}{12}$
29) $\frac{r}{14} = \frac{5}{14}$
30) $n + 8 = 10$
31) $-7 = a + 4$
32) $v - 16 = -30$
33) $10 = x - 4$
34) $-15 = x - 16$
35) $13a = -143$
36) $-8k = 120$
37) $\frac{p}{20} = -12$
38) $-15 = \frac{x}{9}$
39) $9+m = -7$
40) $-19 = \frac{n}{20}$

Two-Step Equations

Solve each equation.

1) $5+\frac{n}{4}=4$
2) $-2=-2m+12$
3) $102=-7r+4$
4) $27=21-3x$
5) $-8n+3=-77$
6) $-4-b=8$
7) $0=-6v$
8) $-2+\frac{x}{2}=4$
9) $-8=\frac{x}{5}-6$
10) $-5=\frac{a}{4}-1$
11) $0=-7+\frac{k}{2}$
12) $-6=15+3p$
13) $-12+3x=0$
14) $-5m+2=27$
15) $24=2n-8$
16) $-37=8+3x$
17) $2=-12+2r$
18) $-8+\frac{n}{12}=-7$
19) $\frac{b}{3}+7=10$
20) $\frac{x}{1}-8=-8$
21) $152=8n+64$
22) $-11=-8+\frac{v}{2}$
23) $-16=8a+64$
24) $-2x-3=-29$
25) $56+8k=64$
26) $-4-3n=-16$
27) $-2x+4=22$
28) $67=5m-8$
29) $-20=4p+4$
30) $9=8+\frac{x}{6}$
31) $-5=3+\frac{n}{2}$
32) $\frac{m}{4}-1=-2$
33) $\frac{r}{8}-6=-5$
34) $-80=4x-28$
35) $-40=4n-32$
36) $33=3b+3$
37) $87=3-7v$
38) $3x-3=-3$
39) $-x+1=-11$
40) $4+\frac{a}{3}=1$

General Linear Equations

Solve each equation.

1) $2-(-3a-8)=1$

2) $2(-3n+8)=-20$

3) $-5(-4+2v)=-50$

4) $2-8(-4+3x)=34$

5) $66=6(6+5x)$

6) $32=2-5(-4n+6)$

7) $0=-8(p-5)$

8) $-55=8+7(k-5)$

9) $-2+2(8x-7)=-16$

10) $-(3-5n)=12$

11) $-21x+12=-6-3x$

12) $-3n-27=-27-3n$

13) $-1-7m=-8m+7$

14) $56p-48=6p+2$

15) $1-12r=29-8r$

16) $4+3x=-12x+4$

17) $20-7b=-12b+30$

18) $-16n+12=39-7n$

19) $-32-24v=34-2v$

20) $17-2x=35-8x$

21) $-2-5(2-4m)=33+5m$

22) $-25-7x=6(2x-1)$

23) $-4n+11=2(1-8n)+3n$

24) $-7(1+b)=-5-5b$

25) $-6v-29=-4v-5(v+1)$

26) $-8(8r-2)=3r+16$

27) $2(4x-4)=-20-4x$

28) $-8n-19=-2(8n-3)+3n$

29) $-a-5(8a-1)=39-7a$

30) $-4+4k=4(8k-8)$

31) $-57=-(-p+1)+2(6+8p)$

32) $16=-5(1-6x)+3(6x+7)$

33) $-2(m-2)+7(m-8)=-67$

34) $7=4(n-7)+5(7n+7)$

35) $50=8(7+7r)-(4r+6)$

36) $-8(6+6x)+4(-3+6x)=-12$

37) $-8(n-7)+3(3n-3)=41$

38) $-76=5(1+3b)+3(3b-3)$

39) $-61=-5(5r-4)+4(3r-4)$

40) $-6(x-8)-4(x-2)=-4$

41) $-2(8n-4)=8(1-n)$

42) $-4(1+a)=2a-8(5+3a)$

43) $-3(-7v+3)+8v=5v-4(1-6v)$

44) $-6(x-3)+5=-2-5(x-5)$

45) $-7(x-2)=-4-6(x-1)$

46) $-(n+8)+n=-8n+2(4n-4)$

47) $-6(8k+4)=-8(6k+3)-2$

48) $-5(x+7)=4(-8x-2)$

49) $-2(1-7p)=8(p-7)$

50) $8(-8n+4)=4(-7n+8)$

Equations Containing Fractions

Solve each equation.

1) $\frac{3}{5}(1+p)=\frac{21}{20}$

2) $-\frac{1}{2}=\frac{3}{2}k+\frac{3}{2}$

3) $0=-\frac{5}{4}(x-\frac{6}{5})$

4) $\frac{3}{2}n-\frac{8}{3}=-\frac{29}{12}$

5) $\frac{3}{4}-\frac{5}{4}m=\frac{113}{24}$

6) $\frac{11}{4}+\frac{3}{4}r=\frac{163}{32}$

7) $\frac{635}{72}=-\frac{5}{2}(-\frac{11}{4}+x)$

8) $-\frac{16}{9}=-\frac{4}{3}(\frac{5}{3}+n)$

9) $2b+\frac{9}{5}=-\frac{11}{5}$

10) $\frac{3}{2}-\frac{7}{4}v=-\frac{9}{8}$

11) $\frac{3}{2}(\frac{7}{3}n+1)=\frac{3}{2}$

12) $\frac{41}{9}=\frac{5}{2}(x+\frac{2}{3})-\frac{1}{3}x$

13) $-a-\frac{5}{4}(-\frac{8}{3}a+1)=-\frac{19}{4}$

14) $\frac{1}{3}(-\frac{7}{4}k+1)-\frac{10}{3}k=-\frac{13}{8}$

15) $\frac{55}{6}=-\frac{5}{2}(\frac{3}{2}p-\frac{5}{3})$

16) $-\frac{1}{2}(\frac{2}{3}x-\frac{3}{4})-\frac{7}{2}x=-\frac{83}{24}$

17) $\frac{16}{9}=-\frac{4}{3}(-\frac{4}{3}n-\frac{4}{3})$

18) $\frac{2}{3}(m+\frac{9}{4})-\frac{10}{3}=-\frac{53}{18}$

19) $-\frac{5}{8}=\frac{5}{4}(r-\frac{3}{2})$

20) $\frac{1}{12}=\frac{4}{3}x+\frac{5}{3}(x-\frac{7}{4})$

21) $-\frac{11}{3}+\frac{3}{2}b=\frac{5}{2}(b-\frac{5}{3})$

22) $\frac{7}{6}-\frac{4}{3}n=-\frac{3}{2}n+2(n+\frac{3}{2})$

23) $-(-\frac{5}{2}x-\frac{3}{2})=-\frac{3}{2}+x$

24) $-\frac{149}{16}-\frac{11}{3}r=-\frac{7}{4}r-\frac{5}{4}(-\frac{4}{3}r+1)$

25) $\frac{45}{16}+\frac{3}{2}n=\frac{7}{4}n-\frac{19}{16}$

26) $-\frac{7}{2}(\frac{5}{3}a+\frac{1}{3})=\frac{11}{4}a+\frac{25}{8}$

27) $\frac{3}{2}(v+\frac{3}{2})=-\frac{7}{4}v-\frac{19}{6}$

28) $-\frac{8}{3}-\frac{1}{2}x=-\frac{4}{3}x-\frac{2}{3}(-\frac{13}{4}x+1)$

29) $\frac{47}{9}+\frac{3}{2}x=\frac{5}{3}(\frac{5}{2}x+1)$

30) $\frac{1}{3}n+\frac{29}{6}=2(\frac{4}{3}n+\frac{2}{3})$

1.9.2 Absolute Value Equations

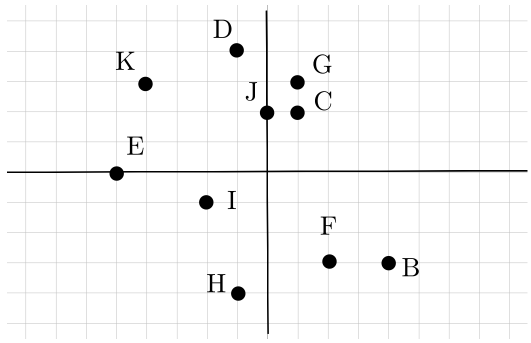
Solve each equation.

1) $|x| = 8$
2) $|n| = 7$
3) $|b| = 1$
4) $|x| = 2$
5) $|5 + 8a| = 53$
6) $|9n + 8| = 46$
7) $|3k + 8| = 2$
8) $|3 - x| = 6$
9) $|9 + 7x| = 30$
10) $|5n + 7| = 23$
11) $|8 + 6m| = 50$
12) $|9p + 6| = 3$
13) $|6 - 2x| = 24$
14) $|3n - 2| = 7$
15) $-7| - 3 - 3r| = -21$
16) $|2 + 2b| + 1 = 3$
17) $7| - 7x - 3| = 21$
18) $\frac{|-4-3n|}{4} = 2$
19) $\frac{|-4b-10|}{8} = 3$
20) $8|5p + 8| - 5 = 11$
21) $8|x + 7| - 3 = 5$
22) $3 - |6n + 7| = -40$
23) $5|3 + 7m| + 1 = 51$
24) $4|r + 7| + 3 = 59$
25) $3 + 5|8 - 2x| = 63$
26) $5 + 8| - 10n - 2| = 101$
27) $|6b - 2| + 10 = 44$
28) $7|10v - 2| - 9 = 5$
29) $-7 + 8| - 7x - 3| = 73$
30) $8|3 - 3n| - 5 = 91$
31) $|5x + 3| = |2x - 1|$
32) $|2 + 3x| = |4 - 2x|$
33) $|3x - 4| = |2x + 3|$
34) $|\frac{2x-5}{3}| = |\frac{3x+4}{2}|$
35) $|\frac{4x-2}{5}| = |\frac{6x+3}{2}|$
36) $|\frac{3x+2}{2}| = |\frac{2x-3}{3}|$

1.9.3 Graphing Linear Equations

The Cartesian Plane

State the coordinates of each point. 1)



Plot each point.

2) L$(-5, 5)$ K$(1, 0)$ J $(-3, 4)$
I$(-3, 0)$ H $(-4, 2)$ G$(4, -2)$
F$(-2, -2)$ E $(3, -2)$ D$(0,\ 3)$
C$(0, 4)$

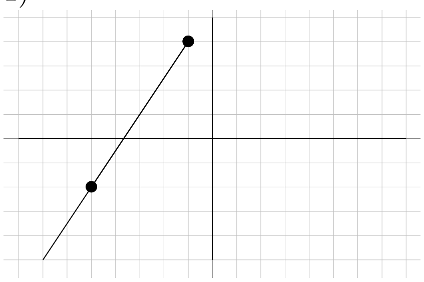
Graphing Lines from Points

Sketch the graph of each line.

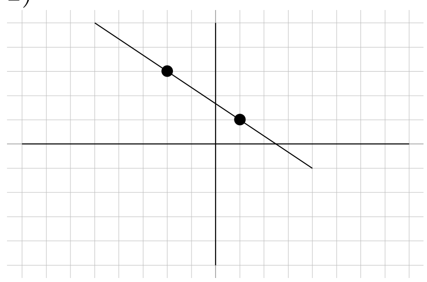
1) $y = -\frac{1}{4}x - 3$
2) $y = x - 1$
3) $y = -\frac{5}{4}x - 4$
4) $y = -\frac{3}{5}x + 1$
5) $y = -4x + 2$
6) $y = \frac{5}{3}x + 4$
7) $y = \frac{3}{2}x - 5$
8) $y = -x - 2$
9) $y = -\frac{4}{5}x - 3$
10) $y = \frac{1}{2}x$
11) $x + 5y = -15$
12) $8x - y = 5$
13) $4x + y = 5$
14) $3x + 4y = 16$
15) $2x - y = 2$
16) $7x + 3y = -12$
17) $x + y = -1$
18) $3x + 4y = 8$
19) $x - y = -3$
20) $9x - y = -4$

Find the slope of each line.

1)



2)



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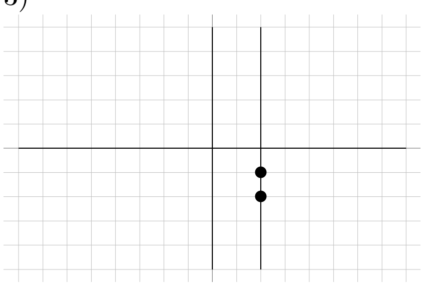

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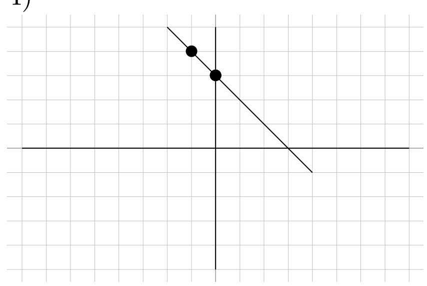
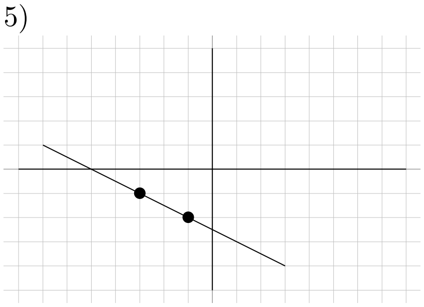
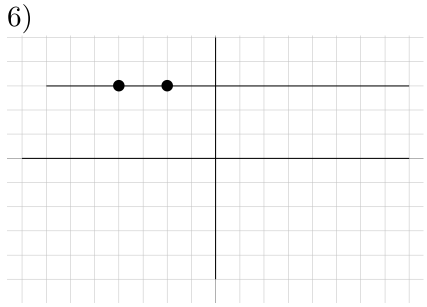


5)



6)



Find the slope of the line through each pair of points.

7) $(-2, 10), (-2, -15)$
8) $(1, 2), (-6, -14)$
9) $(-15, 10), (16, -7)$
10) $(13, -2), (7, 7)$
11) $(10, 18), (-11, -10)$
12) $(-3, 6), (-20, 13)$
13) $(-16, -14), (11, -14)$
14) $(13, 15), (2, 10)$
15) $(-4, 14), (-16, 8)$
16) $(9, -6), (-7, -7)$
17) $(12, -19), (6, 14)$
18) $(-16, 2), (15, -10)$
19) $(-5, -10), (-5, 20)$
20) $(8, 11), (-3, -13)$
21) $(-17, 19), (10, -7)$
22) $(11, -2), (1, 17)$
23) $(7, -14), (-8, -9)$
24) $(-18, -5), (14, -3)$
25) $(-5, 7), (-18, 14)$
26) $(19, 15), (5, 11)$

Find the value of x or y so that the line through the points has the given slope.

27) $(2, 6)$ and $(x, 2)$; slope : $\frac{4}{7}$
28) $(8, y)$ and $(-2, 4)$; slope : $-\frac{1}{5}$
29) $(-3, -2)$ and $(x, 6)$; slope : $-\frac{8}{5}$
30) $(-2, y)$ and $(2, 4)$; slope : $\frac{1}{4}$
31) $(-8, y)$ and $(-1, 1)$; slope : $\frac{6}{7}$
32) $(x, -1)$ and $(-4, 6)$; slope : $-\frac{7}{10}$
33) $(x, -7)$ and $(-9, -9)$; slope : $\frac{2}{5}$
34) $(2, -5)$ and $(3, y)$; slope : 6
35) $(x, 5)$ and $(8, 0)$; slope : $-\frac{5}{6}$
36) $(6, 2)$ and $(x, 6)$; slope : $-\frac{4}{5}$

1.9.4 The Two Forms of a Linear Equation

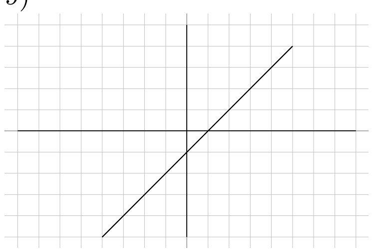
Slope-Intercept Form

Write the slope-intercept form of the equation of each line given the slope and the y-intercept.

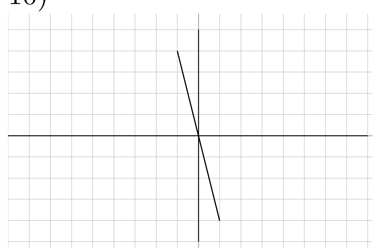
1) Slope = 2, y-intercept = 5
2) Slope = -6, y-intercept = 4
3) Slope = 1, y-intercept = -4
4) Slope = -1, y-intercept = -2
5) Slope = $-\frac{3}{4}$, y-intercept = -1
6) Slope = $-\frac{1}{4}$, y-intercept = 3
7) Slope = $\frac{1}{3}$, y-intercept = 1
8) Slope = $\frac{2}{5}$, y-intercept = 5

Write the slope-intercept form of the equation of each line.

9)



10)



Write each linear equation in slope-intercept form.

11) $x + 10y = -37$
12) $x - 10y = 3$
13) $2x + y = -1$
14) $6x - 11y = -70$
15) $7x - 3y = 24$
16) $4x + 7y = 28$
17) $x = -8$
18) $x - 7y = -42$
19) $y - 4 = -(x + 5)$
20) $y - 5 = \frac{5}{2}(x - 2)$
21) $y - 4 = 4(x - 1)$
22) $y - 3 = -\frac{2}{3}(x + 3)$
23) $y + 5 = -4(x - 2)$
24) $0 = x - 4$
25) $y + 1 = -\frac{1}{2}(x - 4)$
26) $y + 2 = \frac{6}{5}(x + 5)$

Sketch the graph of each line.

27) $y = \frac{1}{3}x + 4$
28) $y = -\frac{1}{5}x - 4$
29) $y = \frac{6}{5}x - 5$
30) $y = -\frac{3}{2}x - 1$
31) $y = \frac{3}{2}x$
32) $y = -\frac{3}{4}x + 1$
33) $x - y + 3 = 0$
34) $4x + 5 = 5y$
35) $-y - 4 + 3x = 0$
36) $-8 = 6x - 2y$
37) $-3y = -5x + 9$
38) $-3y = 3 - \frac{3}{2}x$

Point-Slope Form

Write the point-slope form of the equation of the line through the given point with the given slope.

1) through$(2,3)$, slope = undefined
2) through$(1,2)$, slope =undefined
3) through$(2,2)$, slope $= \frac{1}{2}$
4) through$(2,1)$, slope $= -\frac{1}{2}$
5) through$(-1,-5)$, slope $=9$
6) through$(2,-2)$, slope $= -2$
7) through$(-4,1)$, slope $= \frac{3}{4}$
8) through$(4,-3)$, slope $= -2$
9) through$(0,-2)$, slope $= -3$
10) through$(-1,1)$, slope $= 4$
11) through$(0,-5)$, slope $= -\frac{1}{4}$
12) through$(0,2)$, slope $= -\frac{5}{4}$
13) through$(-5,-3)$, slope $= \frac{1}{5}$
14) through$(-1,-4)$, slope $= -\frac{2}{3}$
15) through$(-1,4)$, slope $= -\frac{5}{4}$
16) through$(1,-4)$, slope $= -\frac{3}{2}$

Write the slope-intercept form of the equation of the line through the given point with the given slope.

17) through: $(-1,-5)$, slope $= 2$
18) through: $(2,-2)$, slope $= -2$
19) through: $(5,-1)$, slope $= -\frac{3}{5}$
20) through: $(-2,-2)$, slope $= -\frac{2}{3}$
21) through: $(-4,1)$, slope $= \frac{1}{2}$
22) through: $(4,-3)$, slope $= -\frac{7}{4}$
23) through: $(4,-2)$, slope $= -\frac{3}{2}$
24) through: $(-2,0)$, slope $= -\frac{5}{2}$
25) through: $(-5,-3)$, slope $= -\frac{2}{5}$
26) through: $(3,3)$, slope $= \frac{7}{3}$
27) through: $(2,-2)$, slope $= 1$
28) through: $(-4,-3)$, slope $=0$
29) through:$(-3,4)$, slope=undefined
30) through: $(-2,-5)$, slope $=2$
31) through: $(-4,2)$, slope $= -\frac{1}{2}$
32) through: $(5,3)$, slope $= \frac{6}{5}$

Write the point-slope form of the equation of the line through the given points.

33) through: $(-4,3)$ and $(-3,1)$

34) through: $(1,3)$ and $(-3,3)$

35) through: $(5,1)$ and $(-3,0)$

36) through: $(-4,5)$ and $(4,4)$

37) through: $(-4,-2)$ and $(0,4)$

38) through: $(-4,1)$ and $(4,4)$

39) through: $(3,5)$ and $(-5,3)$

40) through: $(-1,-4)$ and $(-5,0)$

41) through: $(3,-3)$ and $(-4,5)$

42) through: $(-1,-5)$ and $(-5,-4)$

Write the slope-intercept form of the equation of the line through the given points.

43) through: $(-5,1)$ and $(-1,-2)$

44) through: $(-5,-1)$ and $(5,-2)$

45) through: $(-5,5)$ and $(2,-3)$

46) through: $(1,-1)$ and $(-5,-4)$

47) through: $(4,1)$ and $(1,4)$

48) through: $(0,1)$ and $(-3,0)$

49) through: $(0,2)$ and $(5,-3)$

50) through: $(0,2)$ and $(2,4)$

51) through: $(0,3)$ and $(-1,-1)$

52) through: $(-2,0)$ and $(5,3)$

1.9.5 Parallel and Perpendicular Lines

Find the slope of a line parallel to each given line.

1) $y = 2x + 4$
2) $y = -\frac{2}{3}x + 5$
3) $y = 4x - 5$
4) $y = -\frac{10}{3}x - 5$
5) $x - y = 4$
6) $6x - 5y = 20$
7) $7x + y = -2$
8) $3x + 4y = -8$

Find the slope of a line perpendicular to each given line.

9) $x = 3$
10) $y = -\frac{1}{2}x - 1$
11) $y = -\frac{1}{3}x$
12) $y = \frac{4}{5}x$
13) $x - 3y = -6$
14) $3x - y = -3$
15) $x + 2y = 8$
16) $8x - 3y = -9$

Write the point-slope form of the equation of the line described.

17) through : $(2, 5)$, parallel to $x = 0$
18) through: $(5,\ 2)$, parallel to $y = \frac{7}{5}x + 4$
19) through : $(3, 4)$, parallel to $y = \frac{9}{2}x - 5$
20) through: $(1, -1)$, parallel to $y = -\frac{3}{4}x + 3$
21) through : $(2, 3)$, parallel to $y = \frac{7}{5}x + 4$
22) through : $(-1, 3)$, parallel to $y = -3x - 1$
23) through : $(4, 2)$, parallel to $x = 0$
24) through : $(1, 4)$, parallel to $y = \frac{7}{5}x + 2$
25) through: $(1,\ -5)$, perpendicular to $-x + y = 1$
26) through : $(1, -2)$, perpendicular to $-x + 2y = 2$
27) through : $(5, 2)$, perpendicular to $5x + y = -3$
28) through: $(1,\ 3)$, perpendicular to $-x + y = 1$
29) through : $(4, 2)$, perpendicular to $-4x + y = 0$
30) through: $(-3, -5)$, perpendicular to $3x + 7y = 0$
31) through : $(2, -2)$ perpendicular to $3y - x = 0$
32) through: $(-2, 5)$. perpendicular to $y - 2x = 0$

Write the slope-intercept form of the equation of the line described.

33) $\text{through} : (4, -3), \text{parallel to } y = -2x$

34) $\text{through} : (-5, 2), \text{parallel to } y = \frac{3}{5}x$

35) $\text{through} : (-3, 1), \text{parallel to } y = -\frac{4}{3}x - 1$

36) $\text{through} : (-4, 0), \text{parallel to } y = -\frac{5}{4}x + 4$

37) $\text{through} : (-4, -1), \text{parallel to } y = -\frac{1}{2}x + 1$

38) $\text{through} : (2, 3), \text{parallel to } y = \frac{5}{2}x - 1$

39) $\text{through} : (-2, -1), \text{parallel to } y = -\frac{1}{2}x - 2$

40) $\text{through} : (-5, -4), \text{parallel to } y = \frac{3}{5}x - 2$

41) $\text{through} : (4, 3), \text{perpendicular to } x + y = -1$

42) $\text{through} : (-3, -5), \text{perpendicular to } x + 2y = -4$

43) $\text{through} : (5, 2), \text{perpendicular to } x = 0$

44) $\text{through} : (5, -1), \text{perpendicular to } -5x + 2y = 10$

45) $\text{through} : (-2, 5), \text{perpendicular to } -x + y = -2$

46) $\text{through} : (2, -3), \text{perpendicular to } -2x + 5y = -10$

47) $\text{through} : (4, -3), \text{perpendicular to } -x + 2y = -6$

48) $\text{through} : (-4, 1), \text{perpendicular to } 4x + 3y = -9$

1.9.6 Applications

Numbers and Geometry

Solve.

1. When five is added to three more than a certain number, the result is 19. What is the number?

2. If five is subtracted from three times a certain number, the result is 10. What is the number?

3. When 18 is subtracted from six times a certain number, the result is -42. What is the number?

4. A certain number added twice to itself equals 96. What is the number?

5. A number plus itself, plus twice itself, plus 4 times itself, is equal to -104. What is the number?

6. Sixty more than nine times a number is the same as two less than ten times the number. What is the number?

7. Eleven less than seven times a number is five more than six times the number. Find the number.

8. Fourteen less than eight times a number is three more than four times the number. What is the number?

9. The sum of three consecutive integers is 108. What are the integers?

10. The sum of three consecutive integers is -126. What are the integers?

11. Find three consecutive integers such that the sum of the first, twice the second, and three times the third is -76.

12. The sum of two consecutive even integers is 106. What are the integers?

13. The sum of three consecutive odd integers is 189. What are the integers?

14. The sum of three consecutive odd integers is 255. What are the integers?

15. Find three consecutive odd integers such that the sum of the first, two times the second, and three times the third is 70.

16. The second angle of a triangle is the same size as the first angle. The third angle is 12 degrees larger than the first angle. How large are the angles?

17. Two angles of a triangle are the same size. The third angle is 12 degrees smaller than the first angle. Find the measure the angles.

18. Two angles of a triangle are the same size. The third angle is 3 times as large as the first. How large are the angles?

19. The third angle of a triangle is the same size as the first. The second angle is 4 times the third. Find the measure of the angles.

20. The second angle of a triangle is 3 times as large as the first angle. The third angle is 30 degrees more than the first angle. Find the measure of the angles.
21. The second angle of a triangle is twice as large as the first. The measure of the third angle is 20 degrees greater than the first. How large are the angles?
22. The second angle of a triangle is three times as large as the first. The measure of the third angle is 40 degrees greater than that of the first angle. How large are the three angles?
23. The second angle of a triangle is five times as large as the first. The measure of the third angle is 12 degrees greater than that of the first angle. How large are the angles?
24. The second angle of a triangle is three times the first, and the third is 12 degrees less than twice the first. Find the measures of the angles.
25. The second angle of a triangle is four times the first and the third is 5 degrees more than twice the first. Find the measures of the angles.
26. The perimeter of a rectangle is 150 cm. The length is 15 cm greater than the width. Find the dimensions.
27. The perimeter of a rectangle is 304 cm. The length is 40 cm longer than the width. Find the length and width.
28. The perimeter of a rectangle is 152 meters. The width is 22 meters less than the length. Find the length and width.
29. The perimeter of a rectangle is 280 meters. The width is 26 meters less than the length. Find the length and width.
30. The perimeter of a college basketball court is 96 meters and the length is 14 meters more than the width. What are the dimensions?
31. A mountain cabin on 1 acre of land costs $\$30,000$. If the land cost 4 times as much as the cabin, what was the cost of each?
32. A horse and a saddle cost \$5000. If the horse cost 4 times as much as the saddle, what was the cost of each?
33. A bicycle and a bicycle helmet cost \$240. How much did each cost, if the bicycle cost 5 times as much as the helmet?
34. Of 240 stamps that Harry and his sister collected, Harry collected 3 times as many as his sisters. How many did each collect?
35. If Mr. Brown and his son together had \$220, and Mr. Brown had 10 times as much as his son, how much money had each?
36. In a room containing 45 students there were twice as many girls as boys. How many of each were there?
37. Aaron had 7 times as many sheep as Beth, and both together had 608. How many sheep had each?

38. A man bought a cow and a calf for \$990, paying 8 times as much for the cow as for the calf. What was the cost of each?

39. Jamal and Moshe began a business with a capital of \$7500. If Jamal furnished half as much capital as Moshe, how much did each furnish?

40. A lab technician cuts a 12 inch piece of tubing into two pieces in such a way that one piece is 2 times longer than the other.

41. A 6 ft board is cut into two pieces, one twice as long as the other. How long are the pieces?

42. An eight ft board is cut into two pieces. One piece is 2 ft longer than the other. How long are the pieces?

43. An electrician cuts a 30 ft piece of wire into two pieces. One piece is 2 ft longer than the other. How long are the pieces?

44. The total cost for tuition plus room and board at State University is $\$2,584$. Tuition costs \$704 more than room and board. What is the tuition fee?

45. The cost of a private pilot course is $\$1,275$. The flight portion costs \$625 more than the groung school portion. What is the cost of each?

Age Problems

1. A boy is 10 years older than his brother. In 4 years he will be twice as old as his brother. Find the present age of each.
2. A father is 4 times as old as his son. In 20 years the father will be twice as old as his son. Find the present age of each.
3. Pat is 20 years older than his son James. In two years Pat will be twice as old as James. How old are they now?
4. Diane is 23 years older than her daughter Amy. In 6 years Diane will be twice as old as Amy. How old are they now?
5. Fred is 4 years older than Barney. Five years ago the sum of their ages was 48. How old are they now?
6. John is four times as old as Martha. Five years ago the sum of their ages was 50. How old are they now?
7. Tim is 5 years older than JoAnn. Six years from now the sum of their ages will be 79. How old are they now?
8. Jack is twice as old as Lacy. In three years the sum of their ages will be 54. How old are they now?
9. The sum of the ages of John and Mary is 32. Four years ago, John was twice as old as Mary. Find the present age of each.
10. The sum of the ages of a father and son is 56. Four years ago the father was 3 times as old as the son. Find the present age of each.
11. The sum of the ages of a china plate and a glass plate is 16 years. Four years ago the china plate was three times the age of the glass plate. Find the present age of each plate.
12. The sum of the ages of a wood plaque and a bronze plaque is 20 years. Four years ago, the bronze plaque was one-half the age of the wood plaque. Find the present age of each plaque.
13. Adam is now 34 years old, and Bryce is 4 years old. In how many years will Adam be twice as old as Bryce?
14. A man's age is 36 and that of his daughter is 3 years. In how many years will the man be 4 times as old as his daughter?
15. An Oriental rug is 52 years old and a Persian rug is 16 years old. How many years ago was the Oriental rug four times as old as the Persian Rug?
16. A log cabin quilt is 24 years old and a friendship quilt is 6 years old. In how may years will the log cabin quilt be three times as old as the friendship quilt?
17. The age of the older of two boys is twice that of the younger; 5 years ago it was three times that of the younger. Find the age of each.

18. A pitcher is 30 years old, and a vase is 22 years old. How many years ago was the pitcher twice as old as the vase?

19. Marge is twice as old as Consuelo. The sum of their ages seven years ago was 13. How old are they now?

20. The sum of Jason and Mandy's age is 35. Ten years ago Jason was double Mandy's age. How old are they now?

21. A silver coin is 28 years older than a bronze coin. In 6 years, the silver coin will be twice as old as the bronze coin. Find the present age of each coin.

22. A sofa is 12 years old and a table is 36 years old. In how many years will the table be twice as old as the sofa?

23. A limestone statue is 56 years older than a marble statue. In 12 years, the limestone will be three times as old as the marble statue. Find the present age of the statues.

24. A pewter bowl is 8 years old, and a silver bowl is 22 years old. In how many years will the silver bowl be twice the age of the pewter bowl?

25. Brandon is 9 years older than Ronda. In four years the sum of their ages will be 91. How old are they now?

26. A kerosene lamp is 95 years old, and an electric lamp is 55 years old. How many years ago was the kerosene lamp twice the age of the electric lamp?

27. A father is three times as old as his son, and his daughter is 3 years younger than the son. If the sum of their ages 3 years ago was 63 years, find the present age of the father.

28. The sum of Clyde and Wendy's age is 64. In four years, Wendy will be three times as old as Clyde. How old are they now?

29. The sum of the ages of two ships is 12 years. Two years ago, the age of the older ship was three times the age of the newer ship. Find the present age of each ship.

30. Chelsea's age is double Daniel's age. Eight years ago the sum of their ages was 32. How old are they now?

31. Ann is eighteen years older than her son. One year ago, she was three times as old as her son. How old are they now?

32. The sum of the ages of Kristen and Ben is 32. Four years ago Kristen was twice as old as Ben. How old are they both now?

33. A mosaic is 74 years older than the engraving. Thirty years ago, the mosaic was three times as old as the engraving. Find the present age of each.

34. The sum of the ages of Elli and Dan is 56. Four years ago Elli was 3 times as old as Dan. How old are they now?

35. A wool tapestry is 32 years older than a linen tapestry. Twenty years ago, the wool tapestry was twice as old as the linen tapestry. Find the present age of each.

36. Carolyn's age is triple her daughter's age. In eight years the sum of their ages will be 72. How old are they now?

37. Nicole is 26 years old. Emma is 2 years old. In how many years will Nicole be triple Emma's age?

38. The sum of the ages of two children is 16 years. Four years ago, the age of the older child was three times the age of the younger child. Find the present age of each child.

39. Mike is 4 years older than Ron. In two years, the sum of their ages will be 84. How old are they now?

40. A marble bust is 25 years old, and a terra-cotta bust is 85 years old. In how many years will the terra-cotta bust be three times as old as the marble bust?

Distance, Rate and Time

1. A is 60 miles from B. An automobile at A starts for B at the rate of 20 miles an hour at the same time that an automobile at B starts for A at the rate of 25 miles an hour. How long will it be before the automobiles meet?
2. Two automobiles are 276 miles apart and start at the same time to travel toward each other. They travel at rates differing by 5 miles per hour. If they meet after 6 hours, find the rate of each.
3. Two trains travel toward each other from points which are 195 miles apart. They travel at rate of 25 and 40 miles an hour respectively. If they start at the same time, how soon will they meet?
4. A and B start toward each other at the same time from points 150 miles apart. If A went at the rate of 20 miles an hour, at what rate must B travel if they meet in 5 hours?
5. A passenger and a freight train start toward each other at the same time from two points 300 miles apart. If the rate of the passenger train exceeds the rate of the freight train by 15 miles per hour, and they meet after 4 hours, what must the rate of each be?
6. Two automobiles started at the same time from a point, but traveled in opposite directions. Their rates were 25 and 35 miles per hour respectively. After how many hours were they 180 miles apart?
7. A man having ten hours at his disposal made an excursion, riding out at the rate of 10 miles an hour and returning on foot, at the rate of 3 miles an hour. Find the distance he rode.
8. A man walks at the rate of 4 miles per hour. How far can he walk into the country and ride back on a trolley that travels at the rate of 20 miles per hour, if he must be back home 3 hours from the time he started?
9. A boy rides away from home in an automobile at the rate of 28 miles an hour and walks back at the rate of 4 miles an hour. The round trip requires 2 hours. How far does he ride?
10. A motorboat leaves a harbor and travels at an average speed of 15 mph toward an island. The average speed on the return trip was 10 mph. How far was the island from the harbor if the total trip took 5 hours?
11. A family drove to a resort at an average speed of 30 mph and later returned over the same road at an average speed of 50 mph. Find the distance to the resort if the total driving time was 8 hours.
12. As part of his flight training, a student pilot was required to fly to an airport and then return. The average speed to the airport was 90 mph, and the average speed returning was 120 mph. Find the distance between the two airports if the total flying time was 7 hours.

13. A, who travels 4 miles an hour starts from a certain place 2 hours in advance of B, who travels 5 miles an hour in the same direction. How many hours must B travel to overtake A?

14. A man travels 5 miles an hour. After traveling for 6 hours another man starts at the same place, following at the rate of 8 miles an hour. When will the second man overtake the first?

15. A motorboat leaves a harbor and travels at an average speed of 8 mph toward a small island. Two hours later a cabin cruiser leaves the same harbor and travels at an average speed of 16 mph toward the same island. In how many hours after the cabin cruiser leaves will the cabin cruiser be alongside the motorboat?

16. A long distance runner started on a course running at an average speed of 6 mph. One hour later, a second runner began the same course at an average speed of 8 mph. How long after the second runner started will the second runner overtake the first runner?

17. A car traveling at 48 mph overtakes a cyclist who, riding at 12 mph, has had a 3 hour head start. How far from the starting point does the car overtake the cyclist?

18. A jet plane traveling at 600 mph overtakes a propeller-driven plane which has had a 2 hour head start. The propeller-driven plane is traveling at 200 mph. How far from the starting point does the jet overtake the propeller-driven plane?

19. Two men are traveling in opposite directions at the rate of 20 and 30 miles an hour at the same time and from the same place. In how many hours will they be 300 miles apart?

20. Running at an average rate of 8 m/s, a sprinter ran to the end of a track and then jogged back to the starting point at an average rate of 3 m/s. The sprinter took 55 s to run to the end of the track and jog back. Find the length of the track.

21. A motorboat leaves a harbor and travels at an average speed of 18 mph to an island. The average speed on the return trip was 12 mph. How far was the island from the harbor if the total trip took 5 h?

22. A motorboat leaves a harbor and travels at an average speed of 9 mph toward a small island. Two hours later a cabin cruiser leaves the same harbor and travels at an average speed of 18 mph toward the same island. In how many hours after the cabin cruiser leaves will the cabin cruiser be alongside the motorboat?

23. A jet plane traveling at 570 mph overtakes a propeller-driven plane that has had a 2 h head start. The propeller-driven plane is traveling at 190 mph. How far from the starting point does the jet overtake the propeller-driven plane?

24. Two trains start at the same time from the same place and travel in opposite directions. If the rate of one is 6 miles per hour more than the rate of the other and they are 168 miles apart at the end of 4 hours, what is the rate of each?

25. As part of flight training, a student pilot was required to fly to an airport and then return. The average speed on the way to the airport was 100 mph, and the average speed returning was 150 mph. Find the distance between the two airports if the total flight time was 5 h.

26. Two cyclists start from the same point and ride in opposite directions. One cyclist rides twice as fast as the other. In three hours they are 72 miles apart. Find the rate of each cyclist.

27. A car traveling at 56 mph overtakes a cyclist who, riding at 14 mph, has had a 3 h head start. How far from the starting point does the car overtake the cyclist?

28. Two small planes start from the same point and fly in opposite directions. The first plan is flying 25 mph slower than the second plane. In two hours the planes are 430 miles apart. Find the rate of each plane.

29. A bus traveling at a rate of 60 mph overtakes a car traveling at a rate of 45 mph. If the car had a 1 h head start, how far from the starting point does the bus overtake the car?

30. Two small planes start from the same point and fly in opposite directions. The first plane is flying 25 mph slower than the second plane. In 2 h, the planes are 470 mi apart. Find the rate of each plane.

31. A truck leaves a depot at 11 A.M. and travels at a speed of 45 mph. At noon, a van leaves the same place and travels the same route at a speed of 65 mph. At what time does the van overtake the truck?

32. A family drove to a resort at an average speed of 25 mph and later returned over the same road at an average speed of 40 mph. Find the distance to the resort if the total driving time was 13 h.

33. Three campers left their campsite by canoe and paddled downstream at an average rate of 10 mph. They then turned around and paddled back upstream at an average rate of 5 mph to return to their campsite. How long did it take the campers to canoe downstream if the total trip took 1 hr?

34. A motorcycle breaks down and the rider has to walk the rest of the way to work. The motorcycle was being driven at 45 mph, and the rider walks at a speed of 6 mph. The distance from home to work is 25 miles, and the total time for the trip was 2 hours. How far did the motorcycle go before if broke down?

35. A student walks and jogs to college each day. The student averages 5 km/hr walking and 9 km/hr jogging. The distance from home to college is 8 km, and the student makes the trip in one hour. How far does the student jog?

36. On a 130 mi trip, a car traveled at an average speed of 55 mph and then reduced its speed to 40 mph for the remainder of the trip. The trip took a total of 2.5 h. For how long did the car travel at 40 mph?

37. On a 220 mi trip, a car traveled at an average speed of 50 mph and then reduced its average speed to 35 mph for the remainder of the trip. The trip took a total of 5 h. How long did the car travel at each speed?

38. An executive drove from home at an average speed of 40 mph to an airport where a helicopter was waiting. The executive boarded the helicopter and flew to the corporate offices at and average speed of 60 mph. The entire distance was 150 mi. The entire trip took 3 h. Find the distance from the airport to the corporate offices.

1.9.7 Linear Inequalities and Sign Diagrams

Draw a graph for each inequality and provide interval notation.

1) $n > -5$
2) $n > 4$
3) $-2 \geq k$
4) $1 \geq k$
5) $5 \geq x$
6) $-5 < x$

Solve each inequality, graph each solution, and provide interval notation.

7) $\frac{x}{11} \geq 10$
8) $-2 \leq \frac{n}{13}$
9) $2 + r < 3$
10) $\frac{m}{5} \leq -\frac{6}{5}$
11) $8 + \frac{n}{3} \geq 6$
12) $11 > 8 + \frac{x}{2}$
13) $2 > \frac{a-2}{5}$
14) $\frac{v-9}{-4} \leq 2$
15) $-47 \geq 8 - 5x$
16) $\frac{6+x}{12} \leq -1$
17) $-2(3 + k) < -44$
18) $-7n - 10 \geq 60$
19) $18 < -2(-8 + p)$
20) $5 \geq \frac{x}{5} + 1$
21) $24 \geq -6(m - 6)$
22) $-8(n - 5) \geq 0$
23) $-r - 5(r - 6) < -18$
24) $-60 \geq -4(-6x - 3)$
25) $24 + 4b < 4(1 + 6b)$
26) $-8(2 - 2n) \geq -16 + n$
27) $-5v - 5 < -5(4v + 1)$
28) $-36 + 6x > -8(x + 2) + 4x$
29) $4 + 2(a + 5) < -2(-a - 4)$
30) $3(n + 3) + 7(8 - 8n) < 5n + 5 + 2$
31) $-(k - 2) > -k - 20$
32) $-(4 - 5p) + 3 \geq -2(8 - 5p)$

Construct a sign diagram for each of following graphs/linear equations referenced below.

33)-38): Graphs (1) through (6) on page 89.

39)-50): Linear equations (27) through (38) on page 91.

1.9.8 Compound and Absolute Value Inequalities

Compound Inequalities

Solve each compound inequality, graph its solution, and provide interval notation.

1) $\frac{n}{3} \leq -3$ or $-5n \leq -10$

2) $6m \geq -24$ or $m - 7 < -12$

3) $x + 7 \geq 12$ or $9x < -45$

4) $10r > 0$ or $r - 5 < -12$

5) $x - 6 < -13$ or $6x \leq -60$

6) $9 + n < 2$ or $5n > 40$

7) $\frac{v}{8} > -1$ and $v - 2 < 1$

8) $-9x < 63$ and $\frac{x}{4} < 1$

9) $-8 + b < -3$ and $4b < 20$

10) $-6n \leq 12$ and $\frac{n}{3} \leq 2$

11) $a + 10 \geq 3$ and $8a \leq 48$

12) $-6 + v \geq 0$ and $2v > 4$

13) $3 \leq 9 + x \leq 7$

14) $0 \geq \frac{x}{9} \geq -1$

15) $11 < 8 + k \leq 12$

16) $-11 \leq n - 9 \leq -5$

17) $-3 < x - 1 < 1$

18) $1 \leq \frac{p}{8} \leq 0$

19) $-4 < 8 - 3m \leq 11$

20) $3 + 7r > 59$ or $-6r - 3 > 33$

21) $-16 \leq 2n - 10 \leq -22$

22) $-6 - 8x \geq -6$ or $2 + 10x > 82$

23) $-5b + 10 \leq 30$ and $7b + 2 \leq -40$

24) $n + 10 \geq 15$ or $4n - 5 < -1$

25) $3x - 9 < 2x + 10$ and $5 + 7x \leq 10x - 10$

26) $4n + 8 < 3n - 6$ or $10n - 8 \geq 9 + 9n$

27) $-8 - 6v \leq 8 - 8v$ and $7v + 9 \leq 6 + 10v$

28) $5 - 2a \geq 2a + 1$ or $10a - 10 \geq 9a + 9$

29) $1 + 5k \leq 7k - 3$ or $k - 10 > 2k + 10$

30) $8 - 10r \leq 8 + 4r$ or $-6 + 8r < 2 + 8r$

31) $2x + 9 \geq 10x + 1$ and $3x - 2 < 7x + 2$

32) $-9m + 2 < -10 - 6m$ or $-m + 5 \geq 10 + 4m$

Absolute Value Inequalities

Solve each inequality, graph its solution, and provide interval notation.

1) $\|x\| < 3$	2) $\|x\| \leq 8$
3) $\|2x\| < 6$	4) $\|x+3\| < 4$
5) $\|x-2\| < 6$	6) $\|x-8\| < 12$
7) $\|x-7\| < 3$	8) $\|x+3\| \leq 4$
9) $\|3x-2\| < 9$	10) $\|2x+5\| < 9$
11) $1+2\|x-1\| \leq 9$	12) $10-3\|x-2\| \geq 4$
13) $6-\|2x-5\| \geq 3$	14) $\|x\| > 5$
15) $\|3x\| > 5$	16) $\|x-4\| > 5$
17) $\|x-3\| \geq 3$	18) $\|2x-4\| > 6$
19) $\|3x-5\| \geq 3$	20) $3-\|2-x\| < 1$
21) $4+3\|x-1\| \geq 10$	22) $3-2\|3x-1\| \geq -7$
23) $3-2\|x-5\| \leq -15$	24) $4-6\|-6-3x\| \leq -5$
25) $-2-3\|4-2x\| \geq -8$	26) $-3-2\|4x-5\| \geq 1$
27) $4-5\|-2x-7\| < -1$	28) $-2+3\|5-x\| \leq 4$
29) $3-2\|4x-5\| \geq 1$	30) $-2-3\|-3x-5\| \geq -5$
31) $-5-2\|3x-6\| < -8$	32) $6-3\|1-4x\| < -3$
33) $4-4\|-2x+6\| > -4$	34) $-3-4\|-2x-5\| \geq -7$
35) $\|-10+x\| \geq 8$	

Chapter 2

Systems of Linear Equations

2.1 Graphing

Objective: Solve systems of equations by graphing and identifying the point of intersection.

We have solved problems like $3x - 4 = 11$ by adding 4 to both sides and then dividing by 3 (solution is $x = 5$). We also have methods to solve equations with more than one variable in them. It turns out that to solve for more than one variable we will need the same number of equations as variables. For example, to solve for two variables such as x and y we will need two equations. When we have several equations we are using to solve, we call the equations a **system of equations**. When solving a system of equations we are looking for a solution that works in each equation simultaneously. This solution is usually given as an ordered pair (x, y). The following example illustrates a solution working in two equations.

Example 2.1. Show (x, y)=(2,1) is the solution to the system

$$3x - y = 5 \qquad x + y = 3$$

$(x, y) = (2, 1)$	Identify x and y from the ordered pair
$x = 2,\ y = 1$	Plug these values into each equation

$$\begin{aligned} 3(2) - (1) &= 5 && \text{First equation} \\ 6 - 1 &= 5 && \text{Evaluate} \\ 5 &= 5 && \text{True} \end{aligned}$$

$$\begin{aligned} (2) + (1) &= 3 && \text{Second equation, evaluate} \\ 3 &= 3 && \text{True} \end{aligned}$$

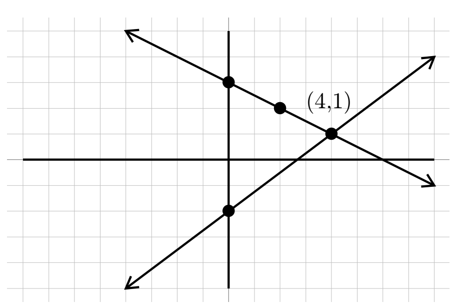
As we found a true statement for both equations we know (2,1) is the solution to the system. It is in fact the only combination of numbers that works in both equations. In this section, we will attempt to identify a (simultaneous) solution to two equations, if such a solution exists. It stands to reason that if we use points to describe the solution, we can use graphs to find the solution.

If the graph of a line is a picture of all the solutions to its equation, we can graph two lines on the same coordinate plane to see the solutions of both equations. In particular, we are interested in finding all points that are a solution for both equations. This will be the point(s) where the two lines intersect! If we can find the intersection of the lines we have found the solution that works in both equations.

Example 2.2. Solve the following system of equations.

$$\begin{aligned} y &= -\tfrac{1}{2}x + 3 \\ y &= \tfrac{3}{4}x - 2 \end{aligned} \qquad \text{First identify slopes and } y - \text{intercepts}$$

$$\begin{aligned} &\text{First Line}: \quad m = -\tfrac{1}{2}, \quad b = 3 \\ &\text{Second Line}: \quad m = \tfrac{3}{4}, \quad b = -2 \end{aligned} \qquad \text{Next graph both lines on the same plane}$$



To graph each equation, we start at the y-intercept and use the slope $\left(\frac{\text{rise}}{\text{run}}\right)$ to get the next point, then connect the dots.

Remember a line with a negative slope points downhill from left to right!

Find the intersection point, (4,1). This is our solution.

Often our equations won't be in slope-intercept form and we will have to solve both equations for y first so we can identify the slope and y-intercept.

Example 2.3. Solve the following system of equations.

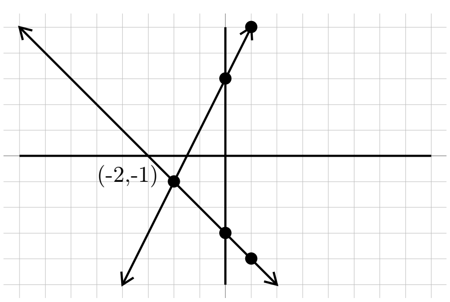
$$\begin{aligned} 6x - 3y &= -9 \\ 2x + 2y &= -6 \end{aligned} \qquad \text{Solve each equation for } y$$

$$\begin{array}{rcl} \begin{array}{r} 6x - 3y = -9 \\ \underline{\boldsymbol{-6x \qquad - 6x}} \end{array} & \begin{array}{r} 2x + 2y = -6 \\ \underline{\boldsymbol{-2x \qquad - 2x}} \end{array} & \text{Subtract } x \text{ terms} \end{array}$$

$$\begin{array}{lll} \dfrac{-3y}{\mathbf{-3}} = \dfrac{-6x}{\mathbf{-3}} \dfrac{-9}{\mathbf{-3}} & \dfrac{2y}{\mathbf{2}} = \dfrac{-2x}{\mathbf{2}} \dfrac{-6}{\mathbf{2}} & \begin{array}{l}\text{Rearrange equations} \\ \text{Divide by coefficient of } y\end{array} \end{array}$$

$$y = 2x + 3 \qquad y = -x - 3 \qquad \text{Identify slope and } y - \text{intercepts}$$

$$\begin{array}{ll} \text{First Line}: & m = \frac{2}{1}, \quad b = 3 \\ \text{Second Line}: & m = -\frac{1}{1}, \quad b = -3 \end{array} \qquad \text{Next graph both lines on the same plane}$$



To graph each equation, we start at the y-intercept and use the slope $\left(\frac{\text{rise}}{\text{run}}\right)$ to get the next point, then connect the dots.

Remember a line with a negative slope points downhill from left to right!

Find the intersection point, (-2,11). This is our solution.

As we are graphing our lines, it is possible to have one of two unexpected results. These are shown and discussed in the next two examples.

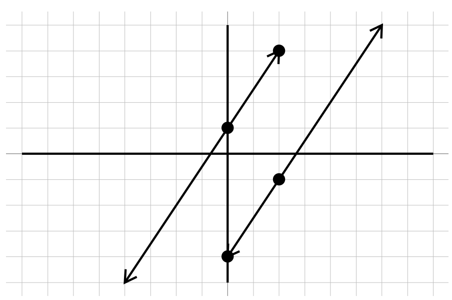
Example 2.4. Solve the following system of equations.

$$y = \frac{3}{2}x - 4 \qquad\qquad y = \frac{3}{2}x + 1$$

Identify the slope and $y-$intercept of each equation.

First Line : $m = \frac{3}{2}, \quad b = -4$
Second Line : $m = \frac{3}{2}, \quad b = 1$

Next graph both lines on the same plane



To graph each equation, we start at the y-intercept and use the slope $(\frac{\text{rise}}{\text{run}})$ to get the next point, then connect the dots.

The two lines do not intersect; they are parallel!

Since the lines do not intersect, we know that there is no point that will satisfy both equations.

There is no solution, or $\varnothing$.

Notice that we could also have recognized that both lines had the same slope. Remembering that parallel lines have the same slope one could conclude that there is no solution without having to graph the lines.

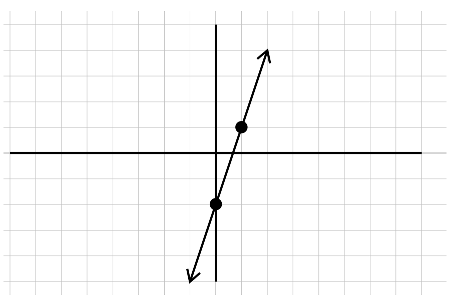
Example 2.5. Solve the following system of equations.

$$\begin{aligned} 2x - 6y &= 12 \\ 3x - 9y &= 18 \end{aligned} \qquad \text{Solve each equation for } y$$

$$\begin{array}{lll} 2x - 6y = 12 & 3x - 9y = 18 & \\ \underline{\mathbf{-2x} \qquad \mathbf{-\ 2x}} & \underline{\mathbf{-3x} \qquad \mathbf{-\ 3x}} & \text{Subtract } x \text{ terms} \end{array}$$

$$\begin{array}{lll} -6y = -2x + 12 & -9y = -3x + 18 & \text{Put } x \text{ terms first} \\ \overline{\mathbf{-6}} \quad \overline{\mathbf{-6}} \quad \overline{\mathbf{-6}} & \overline{\mathbf{-9}} \quad \overline{\mathbf{-9}} \quad \overline{\mathbf{-9}} & \text{Divide by coefficient of } y \\ y = \frac{1}{3}x - 2 & y = \frac{1}{3}x - 2 & \text{Identify the slopes and } y - \text{intercepts} \end{array}$$

$$\begin{array}{ll} \text{First Line}: & m = \frac{1}{3}, \quad b = -2 \\ \text{Second Line}: & m = \frac{1}{3}, \quad b = -2 \end{array} \qquad \text{Next graph both lines on the same plane}$$



To graph each equation, we start at the y-intercept and use the slope $\left(\frac{\text{rise}}{\text{run}}\right)$ to get the next point, then connect the dots.

Both equations are the same line!

As one line is directly on top of the other line, we can say that the lines "intersect" everywhere!

Here we say there are infinitely many solutions.

Notice that once we had both equations in slope-intercept form we could have recognized that the equations were the same. At this point one could state that there are infinitely many solutions without having to go through the work of graphing the equations.

World View Note: The Babylonians were the first to work with systems of equations with two variables. However, their work with systems was quickly passed by the Greeks who would solve systems of equations with three or four variables and around 300 A.D., developed methods for solving systems with any number of unknowns!

2.2 Substitution

Objective: Solve systems of equations using substitution.

Solving a system of equations by graphing has several limitations. First, it requires the graph to be perfectly drawn, if the lines are not straight we may arrive at the wrong answer. Second, graphing is not a great method to use if the answer is really large, over 100 for example, or a decimal, since a graph will not help us find an answer such as 3.2134. For these reasons we will rarely use graphs to solve a given system of equations. Instead, an algebraic approach will be used.

The first algebraic approach is called substitution. We will build the concepts of substitution through several examples, then end with a five-step process to solve problems using this method.

Example 2.6. Solve the following system of equations.

$$x = 5 \qquad\qquad y = 2x - 3$$

We already know x must equal 5, so we can substitute $x = 5$ into the other equation.

$y = 2(\mathbf{5}) - 3$	Evaluate: Multiply first
$y = 10 - 3$	Next subtract
$y = 7$	We now also have y
$(x, y) = (5, 7)$	Our solution

When we know what one variable equals we can plug that value (or expression) in for the variable in the other equation. It is very important that when we substitute, the substituted value goes in parentheses. The reason for this is shown in the next example.

Example 2.7. Solve the following system of equations.

$$2x - 3y = 7 \qquad y = 3x - 7$$

We begin by substituting $y = 3x - 7$ into the other equation.

$$\begin{array}{rl}
2x - 3(\mathbf{3x-7}) = 7 & \text{Solve for } x, \text{distributing } -3 \text{ first} \\
2x - 9x + 21 = 7 & \text{Combine like terms } 2x - 9x \\
-7x + 21 = 7 & \\
\underline{\mathbf{-21 \quad -21}} & \text{Subtract 21} \\
-7x = -14 & \\
\overline{\mathbf{-7}} \quad \overline{\mathbf{-7}} & \text{Divide by } -7 \\
x = 2 & \text{We now have our } x. \\
& \quad \text{Substitute back in equation to find } y \\
y = 3(\mathbf{2}) - 7 & \text{Evaluate: Multiply first} \\
y = 6 - 7 & \text{Next subtract} \\
y = -1 & \text{We now also have } y \\
(x, y) = (2, -1) & \text{Our solution}
\end{array}$$

By using the entire expression $3x - 7$ to replace y in the other equation we were able to reduce the system to a single linear equation which we can easily solve for our first variable. However, the "lone" variable (a variable with a coefficient of 1) is not always alone on one side of the equation. If this happens we can isolate the lone variable by solving for it.

Example 2.8. Solve the following system of equations.

$$3x + 2y = 1 \qquad \boldsymbol{x} - 5y = 6$$

The lone variable is x. Isolate the lone variable by adding $5y$ to both sides.

$$\begin{array}{rl}
x - 5y = 6 & \\
\underline{\mathbf{+5y \quad +5y}} & \\
x = 6 + 5y & \text{Substitute this into the untouched equation}
\end{array}$$

$$\begin{array}{rl}
3(\mathbf{6+5y}) + 2y = 1 & \text{Solve this equation, distributing 3 first} \\
18 + 15y + 2y = 1 & \text{Combine like terms } 15y + 2y
\end{array}$$

$$
\begin{array}{rl}
18 + 17y = 1 & \\
\underline{\mathbf{-18} \qquad \mathbf{-18}} & \text{Subtract 18 from both sides} \\
17y = -17 & \\
\overline{\mathbf{17}} \quad \overline{\mathbf{17}} & \text{Divide both sides by 17} \\
y = -1 & \text{We have our } y. \\
& \quad \text{Substitute back in equation to find } x \\
x = 6 + 5(\mathbf{-1}) & \text{Evaluate: Multiply first} \\
x = 6 - 5 & \text{Next subtract} \\
x = 1 & \text{We now also have } x \\
(x, y) = (1, -1) & \text{Our solution}
\end{array}
$$

The process in the previous example is known as solving by substitution. This process is described and illustrated in the following table which lists the five steps to solving by substitution.

Problem	$4x - 2y = 2$ $2x + y = -5$
1. Find the lone variable.	Lone variable is y, in the second equation: $2x + \boldsymbol{y} = -5$
2. Solve for the lone variable.	Subtract $2x$ from both sides. $\boldsymbol{y = -5 - 2x}$
3. Substitute into the untouched equation.	$4x - 2(\mathbf{-5 - 2}\boldsymbol{x}) = 2$
4. Solve.	$4x + 10 + 4x = 2$ $8x + 10 = 2$ $\underline{\mathbf{-10} \quad \mathbf{-10}}$ $8x = -8$ $\overline{\mathbf{8}} \quad \overline{\mathbf{8}}$ $\boldsymbol{x = -1}$
5. Plug into lone variable equation and evaluate.	$y = -5 - 2(\mathbf{-1})$ $y = -5 + 2$ $\boldsymbol{y = -3}$
Our solution	$(x, y) = (-1, -3)$

Sometimes we have several lone variables in a problem. In this case we will have the choice on which lone variable we wish to solve for, either will give the same final result.

Example 2.9. Solve the following system of equations.

$$x + y = 5 \qquad\qquad x - y = -1$$

$\boldsymbol{x} + y = 5$	Find the lone variable: x in the first or second equation,or y in the first equation. We will choose x in the first equation.
$\boldsymbol{x} + y = 5$	Solve for the lone variable x
$\underline{\boldsymbol{-y} \quad \boldsymbol{-y}}$	Subtract y from both sides
$x = 5 - y$	Plug into the untouched equation, the second equation
$\boldsymbol{(5 - y)} - y = -1$	Combine like terms. Parentheses may be removed
$5 - 2y = -1$	
$\underline{\boldsymbol{-5} \qquad \boldsymbol{-5}}$	Subtract 5 from both sides
$-2y = -6$	
$\overline{\boldsymbol{-2}} \quad \overline{\boldsymbol{-2}}$	Divide both sides by -2
$y = 3$	We have our y!
$x = 5 - (3)$	Plug into lone variable equation and evaluate
$x = 2$	Now we have our x
$(x, y) = (2, 3)$	Our solution

Just as with graphing it is possible to have no solution $\varnothing$ (parallel lines) or infinite solutions (same line) with the substitution method. While we won't have a parallel line or the same line to look at and conclude if it is one or the other, the process takes an interesting turn as shown in the following example.

Example 2.10. Solve the following system of equations.

$$y + 4 = 3x \qquad\qquad 2y - 6x = -8$$

$\boldsymbol{y} + 4 = 3x$	Find the lone variable: y in the first equation

$$
\begin{array}{rl}
y+4=3x & \text{Solve for the lone variable } y \\
\underline{\mathbf{-4 \quad -4}} & \text{Subtract 4 from both sides} \\
y=3x-4 & \text{Plug into second equation} \\
2(\boldsymbol{3x-4})-6x=-8 & \text{Solve, distribute through parentheses} \\
6x-8-6x=-8 & \text{Combine like terms } 6x-6x \\
-8=-8 & \text{Variables are gone!}
\end{array}
$$

Since we are left with a true statement ($-8=-8$), we conclude that there are infinitely many solutions.

Because we had a true statement, and no variables, we know that anything that works in the first equation, will also work in the second equation. However, we do not always end up with a true statement.

Example 2.11. Solve the following system of equations.

$$6x-3y=-9 \qquad\qquad -2x+y=5$$

$$
\begin{array}{rl}
-2x+\boldsymbol{y}=5 & \text{Find the lone variable: } y \text{ in the second equation} \\
-2x+y=5 & \text{Solve for the lone variable} \\
\underline{\boldsymbol{+2x \qquad +2x}} & \text{Add } 2x \text{ to both sides} \\
y=5+2x & \text{Plug into untouched equation} \\
6x-3(\boldsymbol{5+2x})=-9 & \text{Solve, distribute through parentheses} \\
6x-15-6x=-9 & \text{Combine like terms } 6x-6x \\
-15\neq-9 & \text{Variables are gone!}
\end{array}
$$

Since we are left with a false statement ($-15\neq-9$) and no variables, we know that nothing will work in both equations and we may conclude that there are no solutions, or $\varnothing$.

World View Note: French mathematician Rene Descartes wrote a book which included an appendix on geometry. It was in this book that he suggested using letters from the end of the alphabet for unknown values. This is why often we are solving for the variables x, y, and z.

One more question needs to be considered: what if there is no lone variable? If there is no lone variable substitution can still work, we will just have to select one variable to solve for, and introduce fractions.

Example 2.12. Solve the following system of equations.

$$5x - 6y = -14 \qquad\qquad -2x + 4y = 12$$

There is no lone variable, so we will solve the first equation for x.

$$\begin{array}{rl}
5\boldsymbol{x} - 6y = -14 & \text{Solve for our variable } x \\
\underline{\boldsymbol{+6y} \quad \boldsymbol{+6y}} & \text{Add } 6y \text{ to both sides} \\
5x = -14 + 6y & \\
\overline{\mathbf{5}} \quad\ \ \overline{\mathbf{5}} \quad \overline{\mathbf{5}} & \text{Divide each term by } 5 \\
x = \dfrac{-14}{5} + \dfrac{6y}{5} & \text{Plug into untouched equation} \\
-2\left(\dfrac{-14}{5} + \dfrac{6y}{5}\right) + 4y = 12 & \text{Solve, distribute through parentheses} \\
\dfrac{28}{5} - \dfrac{12y}{5} + 4y = 12 & \text{Clear fractions by multiplying by } 5 \\
\dfrac{28\mathbf{(5)}}{5} - \dfrac{12y\mathbf{(5)}}{5} + 4y\mathbf{(5)} = 12\mathbf{(5)} & \text{Reduce fractions and multiply} \\
28 - 12y + 20y = 60 & \text{Combine like terms } -12y + 20y \\
28 + 8y = 60 & \\
\underline{\mathbf{-28} \qquad\quad \mathbf{-\ 28}} & \text{Subtract } 28 \text{ from both sides} \\
8y = 32 & \\
\overline{\mathbf{8}} \quad\ \ \overline{\mathbf{8}} & \text{Divide both sides by } 8 \\
y = 4 & \text{We have our } y \\
x = -\dfrac{14}{5} + \dfrac{6(4)}{5} & \text{Plug into lone variable equation, multiply} \\
x = -\dfrac{14}{5} + \dfrac{24}{5} & \text{Add fractions} \\
x = \dfrac{10}{5} & \text{Reduce fraction} \\
x = 2 & \text{Now we have our } x \\
(x, y) = (2, 4) & \text{Our solution}
\end{array}$$

Using the fractions does make the problem a bit trickier. This is why we have yet another method for solving systems of equations that will be discussed in the next section.

2.3 Addition/Elimination

Objective: Solve systems of equations using the addition/elimination method.

When solving systems we have found that graphing is very limited when solving equations. We then considered a second method known as substitution. This is probably the most used idea in solving systems in various areas of algebra. However, substitution can get ugly if we don't have a lone variable. This leads us to our second method for solving systems of equations. This method is known as either Elimination or Addition. We will set up the process in the following examples, then define the five step process we can use to solve by elimination.

Example 2.13. Solve the following system of equations.

$$3x - 4y = 8 \qquad 5x + 4y = -24$$

$3x - 4y = 8$	Notice opposite signs in front of y
$+\quad 5x + 4y = -24$	Add columns to eliminate y
$8x \quad = -16$	Solve for x
$\overline{\mathbf{8}} \quad \overline{\mathbf{8}}$	Divide by 8
$x = -2$	We have our x!
$5(-2) + 4y = -24$	Plug into either original equation
$-10 + 4y = -24$	Simplify
$\mathbf{+10} \quad \mathbf{+10}$	Add 10 to both sides
$4y = -14$	
$\overline{\mathbf{4}} \quad \overline{\mathbf{4}}$	Divide by 4
$y = -\dfrac{7}{2}$	Now we have our y!
$(x, y) = \left(-2, -\dfrac{7}{2}\right)$	Our solution

In the previous example one variable had opposites in front of it, $-4y$ and $4y$. Adding these together eliminated the y completely. This allowed us to solve for the x. This is the idea behind the addition method. However, generally we won't have opposites in front of one of the variables. In this case we will manipulate the equations to get the opposites we want by multiplying one or both equations (on both sides!). This is shown in the next example.

Example 2.14. Solve the following system of equations.

$$-6x + 5y = 22 \qquad 2x + 3y = 2$$

Notice that we can obtain "opposite" coefficients (one positive and one negative) in front of x by multiplying both sides of the second equation by 3.

$$\begin{array}{rl}
3(2x + 3y) = (2)3 & \text{Distribute to get new second equation} \\
6x + 9y = \ \ 6 & \text{New second equation} \\
\underline{-6x + 5y = 22} & \text{Add equations to eliminate } x \\
14y = 28 & \\
\overline{\mathbf{14}} \qquad \overline{\mathbf{14}} & \text{Divide both sides by } 14 \\
y = 2 & \text{We have our } y! \\
2x + 3(2) = 2 & \text{Plug into one of the original equations} \\
2x + 6 = 2 & \text{Simplify} \\
\underline{\mathbf{-6} \ \ \mathbf{-6}} & \text{Subtract } 6 \text{ from both sides} \\
2x = -4 & \\
\overline{\mathbf{2}} \qquad \overline{\mathbf{2}} & \text{Divide both sides by } 2 \\
x = -2 & \text{We also have our } x! \\
(x, y) = (-2, 2) & \text{Our solution}
\end{array}$$

When we looked at the x terms, $-6x$ and $2x$ we decided to multiply the $2x$ by 3 to get the opposites we were looking for. What we are looking for with our opposites is the least common multiple (LCM) of the coefficients. We also could have solved the above problem by looking at the terms with y, $5y$ and $3y$. The LCM of 3 and 5 is 15. So we would want to multiply both equations, the $5y$ by 3, and the $3y$ by -5 to get opposites, $15y$ and $-15y$. This illustrates an important point: for some problems we will have to multiply both equations by a constant (on both sides) to get the opposites we are looking for.

Example 2.15. Solve the following system of equations.

$$3x + 6y = -9 \qquad\qquad 2x + 9y = -26$$

Here, we can obtain opposite coefficients in front of y by finding the least common multiple (LCM) of 6 and 9, which is 18. We will therefore multiply both sides of both equations by the appropriate values to get $18y$ and $-18y$.

$$\begin{array}{rl}
3(3x + 6y) = (-9)3 & \text{Multiply the first equation by } 3 \\
9x + 18y = -27 & \\
 & \\
-2(2x + 9y) = (-26)(-2) & \text{Multiply the second equation by } -2 \\
-4x - 18y = 52 & \\
 & \\
9x + 18y = -27 & \text{Add two new equations together} \\
\underline{-4x - 18y = 52} & \quad \text{to eliminate } y \\
5x = 25 & \\
\overline{\mathbf{5}} \quad \overline{\mathbf{5}} & \text{Divide both sides by } 5 \\
x = 5 & \text{We have our solution for } x \\
3(5) + 6y = -9 & \text{Plug into either original equation} \\
15 + 6y = -9 & \text{Simplify} \\
\underline{\mathbf{-15} \qquad \mathbf{-\ 15}} & \text{Subtract } 15 \text{ from both sides} \\
6y = -24 & \\
\overline{\mathbf{6}} \quad \overline{\mathbf{6}} & \text{Divide both sides by } 6 \\
y = -4 & \text{Now we have our solution for } y \\
(x, y) = (5, -4) & \text{Our solution}
\end{array}$$

As we get started, it is important for each problem that all variables and constants are aligned before we begin multiplying and adding equations. This is illustrated in the next example which includes the five steps we will go through to solve a problem using elimination.

Problem	$2x-5y=-13$ $-3y+4=-5x$
1. Line up the variables and constants.	Rearrange the second equation $2x-5y=-13$ $5x-3y=-4$
2. Multiply to get opposites (use LCM).	First Equation : multiply by -5 $-5(2x-5y)=(-13)(-5)$ $-10x+25y=65$ Second Equation : multiply by 2 $2(5x-3y)=(-4)2$ $10x-6y=-8$
3. Add equations to eliminate a variable.	$-10x+25y=65$ $\underline{10x\ -\ 6y=-8}$ $19y=57$
4. Solve.	$19y=57$ $\overline{\mathbf{19}}\quad\overline{\mathbf{19}}$ $y=3$
5. Plug back into either of the given equations and solve.	$2x-5(3)=-13$ $2x-15=-13$ $\underline{\mathbf{+15}\ \ \mathbf{+15}}$ $2x=2$ $\overline{\mathbf{2}}\quad\overline{\mathbf{2}}$ $x=1$
Solution	$(x,y)=(1,3)$

World View Note: The famous mathematical text, *The Nine Chapters on the Mathematical Art*, which was printed around 179 AD in China describes a formula very similar to Gaussian elimination which is very similar to the addition method.

Just as with graphing and substitution, it is possible to have no solution or infinitely many solutions with elimination. If the variables all disappear from our problem, a true statement will always indicate infinitely many solutions and a false statement will always indicate no solutions.

Example 2.16. Solve the following system of equations.

$$2x - 5y = 3 \qquad\qquad -6x + 15y = -9$$

In order to obtain opposite coefficients in front of x, multiply the first equation by 3.

$$\begin{aligned} 3(2x - 5y) &= (3)3 \\ 6x - 15y &= 9 \qquad \text{Distribute} \end{aligned}$$

$$\begin{array}{rl} 6x - 15y = 9 & \\ \underline{-6x + 15y = -9} & \text{Add equations together} \\ 0 = 0 & \text{True statement} \end{array}$$

Since we are left with a true statement, we conclude that there are infinitely many solutions.

Example 2.17. Solve the following system of equations.

$$4x - 6y = 8 \qquad\qquad 6x - 9y = 15$$

Here, we will seek to obtain opposite coefficients for x. This means we must find the LCM of 4 and 6, which is 12. We will multiply both sides of both equations by the appropriate values in order to get $12x$ and $-12x$.

$$\begin{aligned} 3(4x - 6y) &= (8)3 \qquad \text{Multiply first equation by } 3 \\ 12x - 18y &= 24 \end{aligned}$$

$$\begin{aligned} -2(6x - 9y) &= (15)(-2) \qquad \text{Multiply second equation by } -2 \\ -12x + 18y &= -30 \end{aligned}$$

$$\begin{array}{rl} 12x - 18y = 24 & \\ \underline{-12x + 18y = -30} & \text{Add both new equations together} \\ 0 \neq -6 & \text{False statement} \end{array}$$

Since we are left with a false statement, we conclude that there are no solutions, or $\varnothing$.

We have now covered three different methods that can be used to solve a system of two equations with two variables: graphing, substitution, and addition/elimination. While all three can be used to solve any system, graphing works well for small integer solutions. Substitution works best when we have a lone variable, and addition/elimination works best when the other two methods fail. As each method has its own strengths, it is important that students become familiar with all three methods.

2.4 Three Variables

Objective: Solve systems of equations with three variables using addition/elimination.

Recall that the graph of an equation containing two variables is a (two-dimensional) line. If we increase the number of variables in an equation to three, then the resulting graph will be a three-dimensional plane. This particular section deals with solving a system of equations containing three variables. Whereas the solution for a system of *two* equations is the set of points where their respective *lines* intersect, the solution for a system of *three* equations will be the set of points where all three respective *planes* intersect. Although we do not intend to undertake the arduous task of graphing even a single equation containing three variables in this setting, the visual is sometimes helpful in justifying a particular outcome, and is often critical to understanding in more advanced mathematics courses such as multivariate calculus and linear algebra.

The method for solving a system of equations with three (or more) variables is very similar to that for solving a system with two variables. When we had two variables we reduced the system down to one equation with one variable (by either substitution or addition/elimination). With three variables we will reduce the system down to one equation with two variables (usually by addition/elimination), which we can then solve by either substitution or addition/elimination.

To reduce from three variables down to two it is very important to keep the work organized by lining up the variables vertically and using enough space to carefully keep track of everything. We will use addition/elimination with two equations to eliminate one variable. This new equation we will call (A). Then we will use a different pair of equations and use addition/elimination to eliminate the **same** variable. This second new equation we will call (B). Once we have done this we will have a system of two equations, (A) and (B), with the same two variables that we can solve using either method, substitution or elimination, depending on the context of the problem. This is demonstrated in the following examples.

Example 2.18. Solve the following system of equations.

$$\begin{aligned} 3x + 2y \ - z &= -1 \\ -2x - 2y + 3z &= 5 \\ 5x + 2y \ - z &= 3 \end{aligned}$$

Our strategy will be to first eliminate y using two different pairs of equations from those provided above.

$3x + 2y \ - z = -1$	Using the first two equations,
$\underline{-2x - 2y + 3z = \ \ 5}$	Add
$x + 2z = \ \ 4$	Call this equation (A)
$-2x - 2y + 3z = 5$	Using the second two equations
$\underline{5x + 2y \ - \ z = 3}$	Add
$3x + 2z = 8$	Call this equation (B)
$x + 2z = 4$	Equation (A)
$3x + 2z = 8$	Equation (B)
$-1(x + 2z) = (4)(-1)$	Multiply equation (A) by -1
$-x - 2z = -4$	Simplify
$-x - 2z = -4$	
$\underline{3x + 2z = \ \ 8}$	Add the two equations
$2x = 4$	
$\overline{\mathbf{2}} \quad \overline{\mathbf{2}}$	Divide by 2
$x = 2$	We now have x!
	Plug x into either (A) or (B)
$(2) + 2z = 4$	We will use (A)
$\underline{\mathbf{-2} \qquad \mathbf{-\ 2}}$	Subtract 2
$2z = 2$	
$\overline{\mathbf{2}} \quad \overline{\mathbf{2}}$	Divide by 2
$z = 1$	We now have z!
	Plug x and z into any of the original equations

$$
\begin{array}{rl}
3(2)+2y-(1)=-1 & \text{We will use the first equation} \\
 & \text{Simplify; reduce and combine constant terms} \\
2y+5=-1 & \text{Solve for } y \\
\underline{\mathbf{-5} \quad \mathbf{-5}} & \text{Subtract } 5 \\
2y=-6 & \\
\overline{\mathbf{2}} \qquad \overline{\mathbf{2}} & \text{Divide by } 2 \\
y=-3 & \text{We now have } y! \\
(x,y,z)=(2,-3,1) & \text{Our solution}
\end{array}
$$

As we are solving for $x, y,$ and z we will have an ordered triplet (x, y, z) instead of just the ordered pair (x, y). In the previous problem, y was easily eliminated using the addition method. Sometimes, however, we may have to do a bit of work to eliminate a variable. Just as with the addition of two equations, we may have to multiply the equations by a constant on both sides in order to get the opposites we want and eliminate the variable. As we do this, remember that it is important to eliminate the **same variable each time**, using two **different** pairs of equations.

Example 2.19. Solve the following system of equations.

$$
\begin{aligned}
4x-3y+2z&=-29 \\
6x+2y-z&=-16 \\
-8x-y+3z&=23
\end{aligned}
$$

Notice that no variable will easily eliminate. Although we are free to choose any variable to eliminate, we will choose x here. Remember, we will be eliminating x *twice*, using two different equations each time.

$$
\begin{array}{rl}
4x-3y+2z=-29 & \text{Begin with the first two equations} \\
6x+2y-z=-16 & \text{The LCM of 4 and 6 is 12}
\end{array}
$$

We will multiply both sides of the first equation by 3 to obtain $12x$. Similarly, we will multiply both sides of the second equation by -2 to obtain $-12x$.

$$3(4x - 3y + 2z) = (-29)3 \qquad \text{Multiply the first equation by } 3$$
$$12x - 9y + 6z = -87$$

$$-2(6x + 2y - z) = (-16)(-2) \qquad \text{Multiply the second equation by } -2$$
$$-12x - 4y + 2z = 32$$

$$\begin{array}{rl} 12x - 9y + 6z = -87 & \\ \underline{-12x - 4y + 2z = \ \ 32} & \text{Add these two equations together} \\ -13y + 8z = -55 & \text{Call this equation (A)} \end{array}$$

Next, we will use a different pair of equations.

$$\begin{array}{rl} 6x + 2y - z = -16 & \text{Now use the second pair of equations} \\ -8x - y + 3z = \ \ 23 & \text{The LCM of 6 and } -8 \text{ is } 24 \end{array}$$

Now, we will multiply both sides of the first equation by 4 to obtain $24x$, and both sides of the second equation by 3 to obtain $-24x$.

$$4(6x + 2y - z) = (-16)4 \qquad \text{Multiply the first equation by } 4$$
$$24x + 8y - 4z = -64$$

$$3(-8x - y + 3z) = (23)3 \qquad \text{Multiply the second equation by } 3$$
$$-24x - 3y + 9z = 69$$

$$\begin{array}{rl} 24x + 8y - 4z = -64 & \\ \underline{-24x - 3y + 9z = \ \ 69} & \text{Add these two equations together} \\ 5y + 5z = \ \ 5 & \text{Call this equation (B)} \end{array}$$

Now, using equations (A) and (B), we will solve the given system.

$$\begin{aligned} -13y + 8z &= -55 && \text{Equation (A)} \\ 5y + 5z &= \ \ 5 && \text{Equation (B)} \end{aligned}$$

$$\begin{array}{rl} 5y + 5z = 5 & \text{Solve equation (B) for } z \\ \underline{\boldsymbol{-5y} \qquad \boldsymbol{-\ 5y}} & \text{Subtract } 5y \\ 5z = 5 - 5y & \\ \overline{\mathbf{5}} \quad \overline{\mathbf{5}} \quad \overline{\mathbf{5}} & \text{Divide both sides by } 5 \\ z = 1 - y & \text{Equation for } z \end{array}$$

Next, substitute z into equation (A).

$$\begin{array}{rl} -13y + 8\boldsymbol{(1 - y)} = -55 & \text{Simplify} \\ -13y + 8 - 8y = -55 & \text{Distribute} \\ -21y + 8 = -55 & \text{Combine like terms} \\ \underline{\boldsymbol{-8} \quad \boldsymbol{-\ 8}} & \text{Subtract } 8 \\ -21y = -63 & \\ \overline{\mathbf{-21}} \quad \overline{\mathbf{-21}} & \text{Divide by } -21 \\ y = 3 & \text{We have our } y! \end{array}$$

Now plug y into the equation for z.

$$\begin{array}{ll} z = 1 - (3) & \text{Evaluate} \\ z = -2 & \text{We have } z! \end{array}$$

Now, we can find x from one of our original equations. We will use the first equation.

$$\begin{array}{rl} 4x - 3(3) + 2(-2) = -29 & \text{Simplify} \\ 4x - 13 = -29 & \text{Combine like terms} \\ \underline{\mathbf{+13} \quad \mathbf{+\ 13}} & \text{Add } 13 \\ 4x = -16 & \\ \overline{\mathbf{4}} \qquad \overline{\mathbf{4}} & \text{Divide by } 4 \\ x = -4 & \text{We have our } x! \end{array}$$

$$(x, y, z) = (-4, 3, -2) \qquad \text{Our solution}$$

World View Note: Around 250, *The Nine Chapters on the Mathematical Art* were published in China. This book had 246 problems, and Chapter eight was about solving systems of equations. One particular problem had four equations with five variables!

Just as with two variables and two equations, we can have special cases come up with three variables and three equations. Specifically, it is possible to encounter a system of equations that has infinitely many solutions, or none at all. The way we handle such systems is identical to that for a system containing only two equations/variables.

Example 2.20. Solve the following system of equations.

$$\begin{aligned} 5x - 4y + 3z &= -4 \\ -10x + 8y - 6z &= \ 8 \\ 15x - 12y + 9z &= -12 \end{aligned}$$

Again, we will choose to eliminate x.

$$\begin{aligned} 5x - 4y + 3z &= -4 && \text{Begin with the first two equations} \\ -10x + 8y - 6z &= \ 8 && \text{The LCM of 5 and } -10 \text{ is 10} \end{aligned}$$

We will multiply both sides of the first equation by 2 to obtain $10x$. Since the second equation contains $-10x$, we do not need to multiply it by a constant.

$$\begin{aligned} 2(5x - 4y + 3z) &= -4(2) && \text{Multiply the first equation by } 2 \\ 10x - 8y + 6z &= -8 \end{aligned}$$

$$\begin{aligned} 10x - 8y + 6z &= -8 \\ \underline{-10x + 8y - 6 = \ 8} & && \text{Add the two equations} \\ 0 &= 0 && \text{A true statement} \end{aligned}$$

Since we are left with a true statement, we conclude that there are infinitely many solutions to the first two equations.

Remember, that our usual procedure requires us to eliminate a variable (x in this case) *twice*, using two different equations each time. Even though we have concluded that there are infinitely many simultaneous solutions to the first two equations, we still must consider two different equations. In this particular example, we will obtain the same outcome by choosing *any* two

equations, and it is left as an exercise for the reader to show this.
Hint: What do you notice about the set of coefficients for each equation, in relation to each of the other two equations? Do you think our results are related to this?

Once we have eliminated the same variable *twice* and drawn the same conclusions as above, we can conclude that there are infinitely many simultaneous solutions (x, y, z) to all three equations, i.e., the entire system.

There are, in fact, cases where two equations will share infinitely many solutions, but the entire system of equations might *fail* to have any simultaneous solutions. This is why it is critical that we not rush to an incorrect conclusion. These more subtle cases will usually be treated in detail in a multivariate calculus or a linear algebra course.

Our last example demonstrates the only time when it is permissible to eliminate a variable from two equations in our system once.

Example 2.21. Solve the following system of equations.

$$\begin{aligned} 3x - 4y + z &= 2 \\ -9x + 12y - 3z &= -5 \\ 4x - 2y - z &= 3 \end{aligned}$$

Here, it will be slightly easier to eliminate z.

$$\begin{aligned} 3x - 4y + z &= 2 && \text{Begin with the first two equations} \\ -9x + 12y - 3z &= -5 && \text{The LCM of 1 and } -3 \text{ is } 3 \end{aligned}$$

We will multiply both sides of the first equation by 3 to obtain $3z$. Since the second equation contains $-3z$, we do not need to multiply it by a constant.

$$\begin{aligned} &3(3x - 4y + z) = (2)3 && \text{Multiply the first equation by } 3 \\ &9x - 12y + 3z = 6 \end{aligned}$$

$$\begin{array}{rl} 9x - 12y + 3z = 6 & \\ \underline{-9x + 12y - 3z = -5} & \text{Add the two equations} \\ 0 \neq 1 & \text{A false statement} \end{array}$$

Since we are left with a false statement, we conclude that there are no solutions to the given system.

Remember, that our usual procedure requires us to eliminate a variable (z in this case) *twice*, using two different equations each time. In this particular case we need only eliminate the variable once. Since we obtained a false statement, which implies that there can be no solution to the first *two* equations in the system, it will be impossible to obtain a simultaneous solution to *all three* equations.

Equations with three (or more) variables are no more difficult to attempt to solve than those containing two variables, if we are careful to keep our information organized and eliminate the same variable twice, each time using two different pairs of equations. As with many problems, it is possible to solve each system several different ways. We can use different pairs of equations or eliminate variables in different orders. But as long as our information is organized and our algebra is correct, we should always arrive at the same conclusion.

2.5 Applications

2.5.1 Value Problems

Objective: Solve value and mixture problems by setting up a system of equations.

One application related to solving systems of equations is known as a value problem. Value problems are problems where each variable has a value attached to it. For example, if our variable is the number of nickels in a person's pocket, those nickels would have a value of five cents each. We will use a table to help us set up and solve value problems. The basic structure of the table is shown below.

	Number	Value	Total
Item 1			
Item 2			
Total			

The first column in the table is used for the number of things we have. Quite often, this will contain our variables. The second column is used for the value each item has. The third column is used for the total value, which we calculate by multiplying the number by the value. For example, if we have 7 dimes, each with a value of 10 cents, the total value is $7 \cdot 10 = 70$ cents. The last row of the table is for totals. We will only use the third row (also marked total) for the totals that are given to us. This means that this row may sometimes contain blank cells. Once the table is filled in we can easily make equations by adding each column and setting it equal to the total at the bottom of the column. This is shown in the following example.

Example 2.22. In a Luke's bank there are 11 coins that have a value of \$1.85. The coins are either quarters or dimes. How many of each coin does Luke have?

	Number	Value	Total
Quarter	q	25	
Dime	d	10	
Total			

Using the value table, use q for quarters, d for dimes.

Each quarter's value is 25 cents, each dime's is 10 cents.

	Number	Value	Total
Quarter	q	25	$25q$
Dime	d	10	$10d$
Total			

Multiply the number by the value to get the totals.

We have a total of 11 coins. This is the number total (first column). We also know that we have \$1.85. This is the money total (last column).

	Number	Value	Total
Quarter	q	25	$25q$
Dime	d	10	$10d$
Total	11		185

Write the final total in cents (185) because 25 and 10 represent cents.

Next, we add the first and last columns to obtain our system of equations.

$$q + d = 11 \qquad\qquad 25q + 10d = 185$$

We may either solve by substitution or addition/elimination. Here, we will use addition/elimination.

$$\begin{aligned} -10(q+d) &= (11)(-10) && \text{Multiply first equation by } -10 \\ -10q - 10d &= -110 && \text{Simplify, distribute and multiply} \end{aligned}$$

$$\begin{array}{rl} -10q - 10d = -110 & \\ \underline{25q + 10d = 185} & \text{Add equations together} \\ 15q = 75 & \\ \overline{\mathbf{15}} \quad \overline{\mathbf{15}} & \text{Divide both sides by } 15. \\ q = 5 & \text{We have our } q \end{array}$$

The number of quarters is $q = 5$. We can plug this back into one of our original equations to obtain d. For simplicity, we will use the first equation.

$$\begin{array}{rl} (5) + d = 11 & \text{Solve for } d \\ \underline{\mathbf{-5} \quad \mathbf{-5}} & \text{Subtract } 5 \\ d = 6 & \text{We have our } d \end{array}$$

The number of dimes is $d = 6$. Luke's bank contains 6 dimes and 5 quarters.

World View Note: American coins are the only coins that do not state the value of the coin. On the back of the dime it says "one dime" (not 10 cents). On the back of the quarter it says "one quarter" (not 25 cents). On the penny it says "one cent" (not 1 cent). The rest of the world (Euros, Yen, Pesos, etc.) all write the value as a number so people who don't speak the language can easily use the coins.

Ticket sales also have a value. Often different types of tickets sell for different prices (values). These problems can be solved in much the same way.

Example 2.23. There were 41 tickets sold for an event. Tickets for children cost \$1.50 and tickets for adults cost \$2.00. Ticket sales for the event totaled \$73.50. How many of each type of ticket were sold?

We will use c to denote the number of child tickets sold and a to denote the number of adult tickets sold.

	Number	Value	Total
Child	c	1.5	
Adult	a	2	
Total			

Child tickets have a value of \$1.50, adult tickets have a value of \$2.00 (we can drop the zeros after the decimal point).

	Number	Value	Total
Child	c	1.5	$1.5c$
Adult	a	2	$2a$
Total			

Multiply the number by the value to get the totals.

We have a total of 41 tickets sold. This is the number total (first column). We also know that we have \$73.50. This is the money total (last column).

	Number	Value	Total
Child	c	1.5	$1.5c$
Adult	a	2	$2a$
Total	41		73.5

Write the ticket values in dollars as 1.5 and 2.

Next, we add the first and last columns to obtain our system of equations.

$$c + a = 41 \qquad 1.5c + 2a = 73.5$$

We may either solve by substitution or addition/elimination. Here, we will use substitution.

$$\begin{array}{rl}
c + a = 41 & \text{Solve for } a \\
\underline{\boldsymbol{-c} \quad\; \boldsymbol{-c}} & \text{Subtract } c \\
a = 41 - c & \text{Equation for } a \\
1.5c + 2\boldsymbol{(41 - c)} = 73.5 & \text{Substitute into untouched equation} \\
1.5c + 82 - 2c = 73.5 & \text{Distribute} \\
-0.5c + 82 = 73.5 & \text{Combine like terms} \\
\underline{\boldsymbol{-82} \quad \boldsymbol{-82}} & \text{Subtract } 82 \text{ from both sides} \\
-0.5c = -8.5 & \\
\overline{\boldsymbol{-0.5}} \quad \overline{\boldsymbol{-0.5}} & \text{Divide both sides by } -0.5 \\
c = 17 & \text{We have } c
\end{array}$$

The number of child tickets sold is $c = 17$. We can plug this back into our equation for a.

$$\begin{array}{ll}
a = 41 - (17) & \text{Simplify} \\
a = 24 & \text{We have our } a
\end{array}$$

The number of adult tickets sold is $a = 24$. We conclude that there were 17 child tickets and 24 adult tickets sold for the event.

Some problems will not give us the total number of items we have. Instead they will provide a relationship between the items. Here we will have statements such as "There are twice as many dimes as nickels". While it is clear that we need to multiply one variable by 2, it may not be clear which variable gets multiplied. Generally the equations are determined by working backwards from the actual sentence. If there are twice as many dimes, than we multiply the other variable (nickels) by two. So the equation would be $d = 2n$. This type of problem is worked out in the next example.

Example 2.24. Bryce has a collection made up of 5-cent stamps and 8-cent stamps. There are three times as many 8-cent stamps as 5-cent stamps. The total value of all the stamps is \$3.48. How many of each stamp does Bryce have?

We will use f to denote the number of 5-cent stamps and e to denote the number of 8-cent stamps in the collection.

	Number	Value	Total
Five	f	5	$5f$
Eight	$3f$	8	$24f$
Total			348

List the value of each stamp in the value column.

	Number	Value	Total
Five	f	5	$5f$
Eight	e	8	$8e$
Total			

Multiply the number by the value to get the totals.

	Number	Value	Total
Five	f	5	$5f$
Eight	e	8	$8e$
Total			348

The final total was 338 (written in cents).

We do not know the total number, this is left blank.

Recall that there are 3 times as many 8-cent stamps as there are 5-cent stamps. This enables us to establish the following equation.

$$e = 3f$$

Adding up the last (total) column gives us our second equation.

$$5f + 8e = 348$$

We will now use substitution to solve for e and f.

$$\begin{aligned}
5f + 8(\boldsymbol{3f}) &= 348 && \text{Substitute first equation in second} \\
5f + 24f &= 348 && \text{Simplify, multiply} \\
29f &= 348 && \text{Combine like terms} \\
\overline{\mathbf{29}} \quad & \quad \overline{\mathbf{29}} && \text{Divide both sides by } 29. \\
f &= 12 && \text{We have } f
\end{aligned}$$

There are $f = 12$ 5-cent stamps.

$$\begin{aligned}
e &= 3(12) && \text{Plug into first equation} \\
e &= 36 && \text{We have } e
\end{aligned}$$

There are $e = 36$ 8-cent stamps. We conclude that Bryce has twelve 5-cent stamps and thirty-six 8-cent stamps in his collection.

The same process for solving value problems can be applied to solving interest problems. For these problems we will make a slight adjustment to our table titles.

	Invest	Rate	Interest
Account 1			
Account 2			
Total			

Our first column represents the amount invested in each account. The second column represents the interest rate earned (written as a decimal - move the decimal point two spaces to the left), and the last column represents the amount of interest earned. Just as before, we multiply the investment amount by the interest rate to find the final column, the interest earned. This is demonstrated in the following example.

Example 2.25. Nicki invests \$4000 in two accounts, one at 6% interest, the other at 9% interest for one year. At the end of the year she had earned \$270 in interest. How much did Nicki invest in each account?

Let x represent the amount invested in the first account and y represent the amount invested in the second account.

	Invest	Rate	Interest
Account 1	x	0.06	
Account 2	y	0.09	
Total			

Fill in the interest rates as decimals.

	Invest	Rate	Interest
Account 1	x	0.06	$0.06x$
Account 2	y	0.09	$0.09y$
Total			

Multiply the investment by the rate to get the interest.

	Invest	Rate	Interest
Account 1	x	0.06	$0.06x$
Account 2	y	0.09	$0.09y$
Total	4000		270

The total investment is \$4000. The total interest is \$270.

Next, we add the first and last columns to obtain our system of equations.

$$x + y = 4000 \qquad 0.06x + 0.09y = 270$$

We may either solve by substitution or addition/elimination. Here, we will use addition/elimination.

$$-0.06(x + y) = (4000)(-0.06) \quad \text{Multiply first equation by } -0.06$$
$$-0.06x - 0.06y = -240 \quad \text{Simplify, distribute and multiply}$$

$$\begin{array}{rl} -0.06x - 0.06y = -240 & \\ \underline{0.06x + 0.09y = 270} & \text{Add equations together} \\ 0.03y = 30 & \\ \overline{\mathbf{0.03}} \quad \overline{\mathbf{0.03}} & \text{Divide both sides by } 0.03 \\ y = 1000 & \text{We have } y \end{array}$$

Nicki invested \$1000 at an interest rate of 9%. We can plug this back into one of our original equations to obtain x. For simplicity, we will use the first equation.

$$\begin{array}{rl} x + 1000 = 4000 & \text{Solve for } x \\ \underline{\mathbf{-1000} \quad \mathbf{-1000}} & \text{Subtract 1000 from both sides} \\ x = 3000 & \text{We have } x \end{array}$$

Nicki invested \$3000 at an interest rate of 6%. We conclude that Nicki invested \$3000 and \$100 at interest rates of 6% and 9%, respectively.

The same process can be used to find an unknown interest rate.

Example 2.26. Whit invests \$5000 in one account and \$8000 in an account paying 4% more in interest. He earned \$1230 in interest after one year. At what rates did Whit invest?

In the previous example, we used x and y to represent investment amounts. Here, we will use x and y to represent interest rates for each investment. Since we are told that the second account pays 4% more than the first account in interest, we may immediately replace y with $x + 0.04$ (since $y = x + 0.04$).

	Invest	Rate	Interest
Account 1	5000	x	
Account 2	8000	$x + 0.04$	
Total			

Our investment table. Make sure to write all rates as decimals!

	Invest	Rate	Interest
Account 1	5000	x	$5000x$
Account 2	8000	$x + 0.04$	$8000x + 320$
Total			

Multiply to fill in interest column.
Make sure to distribute, $8000\,(x + 0.04)$.

	Invest	Rate	Interest
Account 2	5000	x	$5000x$
Account 2	8000	$x + 0.04$	$8000x + 320$
Total			1230

Total interest was \$1230.

Adding up the last column gives our equation.

$$5000x + 8000x + 320 = 1230$$

$$
\begin{array}{rl}
5000x + 8000x + 320 = 1230 & \text{Solve for } x \\
13000x + 320 = 1230 & \text{Combine like terms} \\
\underline{\mathbf{-320 \quad -320}} & \text{Subtract } 320 \text{ from both sides} \\
13000x = \quad 910 & \\
\overline{\mathbf{13000}} \quad \overline{\mathbf{13000}} & \text{Divide both sides by } 13000. \\
x = 0.07 & \text{We have our } x
\end{array}
$$

The first account had an interest rate of $x = 7\%$. Plugging this back into our equation for y gives us $y = (0.07) + 0.04 = 0.11$. The second account has an interest rate of $y = 11\%$.

We conclude that Whit invested \$5000 and \$8000 at interest rates of 7% and 11%, respectively.

2.5.2 Mixture Problems

One application related to solving systems of equations is known as a mixture problem. Mixture problems are problems where two different solutions are mixed together, resulting in a new final solution. We will use the following table to help us solve mixture problems.

	Amount	Part	Total
Item 1			
Item 2			
Final			

The first column represents the amount (usually a volume or weight) of each solution that we have. The second column, labeled "part", represents the percentage (written as a decimal) of the particular item that is used in each solution. If we are mixing prices we will put them in this column. Then we can multiply the amount by the part to find the total. Once we have the total, we can obtain an equation(s) by adding the amount and/or total columns, which will enable us to solve the problem.

Basic mixture problems can have either one or two variables. We will start with a mixture problem containing one variable.

Example 2.27. A chemist has 70 mL of a 50% methane solution. How much of an 80% solution must he add so that the final solution is 60% methane?

	Amount	Part	Total
Start	70	0.5	
Add	x	0.8	
Final			

Set up the mixture table .

We start the problem with 70 mL, but do not know how much of the other solution is added. This amount is denoted by x above. The respective parts for each solution (entered as decimals) are 0.5 and 0.8.

	Amount	Part	Total
Start	70	0.5	
Add	x	0.8	
Final	$70 + x$	0.6	

Add the amount column to get final amount.

The part for the final amount is 0.6, since we want the final solution to be 60% methane.

	Amount	Part	Total
Start	70	0.5	35
Add	x	0.8	$0.8x$
Final	$70 + x$	0.6	$42 + 0.6x$

Multiply the amount by the part to get the total.

Be sure to distribute for the last row : $(70 + x)\ 0.6$.

Next we add the Total column to obtain the following equation.

$$35 + 0.8x = 42 + 0.6x$$

$35 + 0.8x = 42 + 0.6x$	Solve for x
$\underline{\boldsymbol{-0.6x \qquad\quad -\ 0.6x}}$	Subtract $-0.6x$ from both sides
$35 + 0.2x = 42$	
$\underline{\boldsymbol{-35 \qquad\quad -\ 35}}$	Subtract 35 from both sides
$0.2x = 7$	
$\overline{\mathbf{0.2}} \quad \overline{\mathbf{0.2}}$	Divide both sides by 0.2
$x = 35$	We have our x

We conclude that 35 mL of the 80% methane solution must be added.

The same process can be used if the starting and final amount have a price attached to them, rather than a percentage.

Example 2.28. A coffee mix that sells for \$2.50 is to be made by mixing two types of coffee. The cafe plans to use 40 mL of the first type of coffee, which costs \$3.00. How much of the second type of coffee, costing \$1.50, should the cafe mix with the first?

	Amount	Part	Total
Start	40	3	
Add	x	1.5	
Final			

Set up mixture table .

We know the starting amount is 40 mL. The amount that is added is unknown; label it as x. The costs for each type of coffee, \$3 and \$1.5, are entered in the Part column.

	Amount	Part	Total
Start	40	3	
Add	x	1.5	
Final	$40+x$	2.5	

Add the amounts to get the final amount.

We want this final amount to sell for \$2.50.

	Amount	Part	Total
Start	40	3	120
Add	x	1.5	$1.5x$
Final	$40+x$	2.5	$100+2.5x$

Multiply the amount by the part to get the total.
Be sure to distribute on the last row $(40 + x)\ 2.5$.

Next we add the Total column to obtain the following equation.

$$120 + 1.5x = 100 + 2.5x$$

$$\begin{array}{rll}
120 + 1.5x = 100 + 2.5x & & \text{Solve for } x \\
\underline{\mathbf{-1.5}\boldsymbol{x} \qquad \mathbf{-\ 1.5}\boldsymbol{x}} & & \text{Subtract } 1.5x \text{ from both sides} \\
120 = 100 + x & & \\
\underline{\mathbf{-100} \quad \mathbf{-\ 100}} & & \text{Subtract } 100 \text{ from both sides} \\
20 = x & & \text{We have our } x
\end{array}$$

We conclude that 20 mL of the \$1.50 coffee must be added.

World View Note: Brazil is the world's largest coffee producer, producing 2.59 million metric tons of coffee a year! That is more than three times the amount that the second-largest producer, Vietnam, produces!

The previous examples illustrate how we can construct a table in order to obtain a related equation and solve a mixture problem. But we are also interested in *systems* of equations, with two or more unknowns. The following example is one such problem.

Example 2.29. Farmer Reeb has two types of milk, one that is 24% butterfat and another which is 18% butterfat. How much of each should he use to end up with 42 gallons of 20% butterfat?

Here, we don't know either amount of milk. Label them as x and y, respectively. The final amount of milk is entered in the table below as 42.

	Amount	Part	Total
Milk 1	x	0.24	
Milk 2	y	0.18	
Final	42	0.2	

Fill in the Part column with the percentage of each type of milk included in the final solution.

	Amount	Part	Total
Milk 1	x	0.24	$0.24x$
Milk 2	y	0.18	$0.18y$
Final	42	0.2	8.4

Multiply the amount by the part to get the total.

Adding up the Amount column provides us with our first equation. Adding up the Total column gives our second equation. Both are stated below.

$$x + y = 42 \qquad 0.24x + 0.18y = 8.4$$

We will use addition/elimination to solve the new system of equations.

$$-0.18(x + y) = (42)(-0.18) \qquad \text{Multiply the first equation by } -0.18$$
$$-0.18x - 0.18y = -7.56 \qquad \text{Simplify; distribute and multiply}$$

$$\begin{array}{rl} -0.18x - 0.18y = -7.56 & \\ \underline{0.24x + 0.18y = 8.4} & \text{Add the equations together} \\ 0.06x = 0.84 & \\ \overline{\mathbf{0.06}} \quad \overline{\mathbf{0.06}} & \text{Divide both sides by } 0.06. \\ x = 14 & \text{We have our } x \end{array}$$

We know that $x = 14$ gal of the 24% butterfat milk should be added. Next, we can plug 14 into either of the two equations in our system to find y. For simplicity, we will use the first equation.

$$\begin{array}{rl} (14) + y = 42 & \text{Solve for } y \\ \underline{\mathbf{-14} \qquad \mathbf{-14}} & \text{Subtract } 14 \text{ from both sides} \\ y = 28 & \text{We have our } y \end{array}$$

So $y = 28$ gal of the 18% butterfat milk should be added. We conclude that farmer Reeb should add 14 gallons of the 24% milk and 28 gallons of the 18% milk in order to end up with 42 gallons of a 20% butterfat milk.

The same process can be used to solve mixtures of prices with two unknowns.

Example 2.30. In a candy shop, chocolate which sells for \$4 a pound is mixed with nuts which are sold for \$2.50 a pound to form a chocolate-nut candy which sells for \$3.50 a pound. How much of each are used to make 30 pounds (lbs) of the mixture?

As we fill in the mixture table below, we will use c and n to represent the amounts of chocolate and nuts, respectively.

	Amount	Part	Total
Chocolate	c	4	
Nut	n	2.5	
Final	30	3.5	

We know the final amount (30) and price (\$3.5); include these in the table.

	Amount	Part	Total
Chocolate	c	4	$4c$
Nut	n	2.5	$2.5n$
Final	30	3.5	105

Multiply the amount by the part to get the total.

Adding up the Amount and the Total columns gives us our system of equations, shown below.

$$c + n = 30 \qquad 4c + 2.5n = 105$$

We will solve this system using substitution.

$$\begin{array}{rl}
c + n = 30 & \text{Solve for } c \\
\underline{\boldsymbol{-n \quad -n}} & \text{Subtract } n \text{ from both sides} \\
c = 30 - n & \text{Equation for } c \\
\\
4(\mathbf{30} - \boldsymbol{n}) + 2.5n = 105 & \text{Substitute into untouched equation} \\
120 - 4n + 2.5n = 105 & \text{Distribute} \\
120 - 1.5n = 105 & \text{Combine like terms} \\
\underline{\mathbf{-120 \qquad\quad - 120}} & \text{Subtract } 120 \text{ from both sides} \\
-1.5n = -15 & \\
\overline{\mathbf{-1.5}} \quad \overline{\mathbf{-1.5}} & \text{Divide both sides by } -1.5 \\
n = 10 & \text{We have our } n
\end{array}$$

We know that $n = 10$ lbs of nuts must be added. Next, we can plug 10 into our equation for c.

$$\begin{aligned} c &= 30 - (10) && \text{Simplify} \\ c &= 20 && \text{We have our } c \end{aligned}$$

So $c = 20$ lbs of chocolate must be added. We conclude that 10 pounds of nuts and 20 pounds of chocolate must be mixed in order to obtain a 30-pound chocolate-nut candy mixture that sells for \$3.50 a pound.

With mixture problems we often are mixing with a pure solution or using water which contains none of the chemical we are interested in. For pure solutions, the percentage is 100% (or 1 in the table). For water, the percentage is 0% (or 0). This following example solves one such problem.

Example 2.31. A solution of pure antifreeze is mixed with water to make a 65% antifreeze solution. How much of each item should be used to make 70 L?

Here, we will use a and w to denote the amounts of pure antifreeze and water, respectively, that are added.

	Amount	Part	Final
Antifreeze	a	1	
Water	w	0	
Total	70	0.65	

Antifreeze is pure, 100% or 1 in our table, written as a decimal.

Water contains no antifreeze; its percentage is 0.

	Amount	Part	Total
Antifreeze	a	1	a
Water	w	0	0
Final	70	0.65	45.5

Multiply across to find totals

Adding up the Amount and the Total columns gives us our system of equations, shown below.

$$a + w = 70 \qquad\qquad a = 45.5$$

Since our second equation gives us a, we need only plug this into our first equation to find w.

$$\begin{array}{rl} (45.5) + w = 70 & \text{Solve for } w \\ \underline{\mathbf{-45.5 \qquad - 45.5}} & \text{Subtract } 45.5 \text{ from both sides} \\ w = 24.5 & \text{We have our } w \end{array}$$

We quickly conclude that $a = 45.5$ L pure antifreeze and $w = 24.5$ L of water must be used in order to produce a 70 L mixture that is 65% antifreeze.

2.6 Matrix Notation

Objective: Represent a system of linear equations as an augmented matrix. Solve a system of linear equations using matrix row reduction.

In this section, we will solve systems of linear equations using matrices and row operations. The first step will be to represent a system as an augmented matrix, as in the following example.

Example 2.32.

System	Augmented Matrix
$\begin{aligned} x - 2y + z &= 7 \\ 3x - 5y + z &= 14 \\ 2x - 2y - z &= 3 \end{aligned}$	$\left[\begin{array}{ccc\|c} 1 & -2 & 1 & 7 \\ 3 & -5 & 1 & 14 \\ 2 & -2 & -1 & 3 \end{array}\right]$

In our example the entries in the first three columns of the matrix are given by the coefficients of each of the variables in their corresponding equations; the first column contains the coefficients of x, the second column contains the coefficients of y, and the third the coefficients of z. The last column of the matrix will always contain the constant term from each equation, and is separated from the coefficient columns by a vertical line. Each row of the matrix should also match its respective equation in the ordered system.

The following row operations may be used to reduce an augmented matrix.

1. Interchange two rows.
2. Multiply all entries of a row by a nonzero constant.
3. Add one row to another row.

Furthermore, multiple row operations may be used in combination, as our first example will demonstrate.

Initially, our goal will be to transform (or reduce) the given augmented matrix using the row operations specified above into a matrix in *triangular form*. A matrix obtained from our original matrix that is in triangular form will have a solution that equals the solution for our original matrix, but which will be easier to identify.

World View Note: Transforming a matrix to triangular form is commonly referred to as *Gaussian elimination*, named after the German mathematician Carl Friedrich Gauss (1777-1855).

We will now use the specified row reduction operations to transform our given matrix to a matrix in triangular form.

Example 2.33.

Original Matrix $\left[\begin{array}{rrr|r} 1 & -2 & 1 & 7 \\ 3 & -5 & 1 & 14 \\ 2 & -2 & -1 & 3 \end{array}\right]$

Multiply Row 1 by -3 and add to Row 2 (replacing Row 2)
Symbolic: R2+(-3)R1 $\Rightarrow$ R2 $\left[\begin{array}{rrr|r} 1 & -2 & 1 & 7 \\ 0 & 1 & -2 & -7 \\ 2 & -2 & -1 & 3 \end{array}\right]$

Multiply Row 1 by -2 and add to Row 3
Symbolic: R3+(-2)R1 $\Rightarrow$ R3 $\left[\begin{array}{rrr|r} 1 & -2 & 1 & 7 \\ 0 & 1 & -2 & -7 \\ 0 & 2 & -3 & -11 \end{array}\right]$

Multiply Row 2 by -2 and add to Row 3
Symbolic: R3+(-2)R2 $\Rightarrow$ R3 $\left[\begin{array}{rrr|r} 1 & -2 & 1 & 7 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 1 & 3 \end{array}\right]$

The new matrix is now in triangular form, with resulting system of equations listed below.

$$\begin{aligned} x - 2y + z &= 7 \\ y - 2z &= -7 \\ z &= 3 \end{aligned}$$

At this point, we can easily solve the new system by first substituting $z = 3$ into the second equation to find y, and then substituting both known values for z and y into the first equation to find x. This results in the following solution, which the reader can easily verify.

$$(x, y, z) = (2, -1, 3)$$

It is also worth mentioning that, just as with every problem we have encountered, it is straightforward to check whether a particular answer is correct. In our previous example, this will amount to plugging $(x, y, z) = (2, -1, 3)$ into each equation and simplifying. Although this can be a tedious process, it is important to do every so often, in order to ensure accuracy. In the previous example, we see below that the answer checks out.

$$\begin{array}{rcccccc} x - 2y + z = 7 & = & 2 - 2(-1) + 3 & = & 2 + 2 + 3 & = & 7 \\ 3x - 5y + z = 14 & = & 3(2) - 5(-1) + 3 & = & 6 + 5 + 3 & = & 14 \\ 2x - 2y - z = 3 & = & 2(2) - 2(-1) - 3 & = & 4 + 2 - 3 & = & 3 \end{array}$$

The last matrix obtained in the previous example is said to be in **row echelon form**. A matrix is in row echelon form if the following conditions are satisfied.

1. Any row consisting entirely of zeros (if any exist) is listed at the bottom of the matrix.

2. The first coefficient entry of any nonzero row (i.e., a row that does not consist entirely of zeros) is 1. We will call such an entry a "leading one".

3. The leading ones indent. In other words, the column number for the leading ones increases from left to right as the row numbers increase from top to bottom.

In fact, if we continue to apply the permissible row operations to the row echelon form of a matrix, we can obtain a matrix in which all the columns that contain a leading one will have zeros elsewhere. This particular type of matrix is known as the ***reduced*** **row echelon form** of a matrix.

Continuing with our previous example, we will obtain the reduced row echelon form for our original augmented matrix.

Example 2.34.

Row Echelon Form

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 7 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Multiply Row 2 by 2 and
add to Row 1
Symbolic: R1+(2)R2 $\Rightarrow$ R1

$$\left[\begin{array}{rrr|r} 1 & 0 & -3 & -7 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Multiply Row 3 by 2 and
add to Row 2
Symbolic: R2+(2)R3 $\Rightarrow$ R2

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Multiply Row 3 by 3 and
add to Row 1
Symbolic: R1+(3)R3 $\Rightarrow$ R1

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Our resulting equations are shown below.

$$\begin{aligned} x &= 2 \\ y &= -1 \\ z &= 3 \end{aligned}$$

Consequently, no additional work is needed to obtain our solution!

This example helps demonstrate the benefit to solving a given system of equations by row reducing its corresponding augmented matrix. And, although the row echelon form was certainly helpful in completing our task, by continuing our row reduction to obtain the *reduced* row echelon form of the matrix we completely eliminated the requirement to directly solve any equations.

This is because the applied row operations have done the work of solving the equations for us. In fact, throughout our reduction process, it would not be difficult for us to "translate" each step into an application of the addition/elimination procedure learned earlier in the chapter. So, although row reducing an augmented matrix may appear somewhat as 'mathematical magic', it is nothing more than a prescribed arithmetic manipulation of coefficients and constants to achieve a solution to a system of equations.

We continue with our next example.

Example 2.35.

System	Augmented Matrix
$\begin{aligned} x+y+z&=3\\ 2x+y+4z&=8\\ x+2y-z&=1 \end{aligned}$	$\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 3\\ 2 & 1 & 4 & 8\\ 1 & 2 & -1 & 1 \end{array}\right]$
Multiply Row 1 by -2 and add to Row 2 Symbolic: R2+(-2)R1 $\Rightarrow$ R2	$\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 3\\ 0 & -1 & 2 & 2\\ 1 & 2 & -1 & 1 \end{array}\right]$
Multiply Row 1 by -1 and add to Row 3 Symbolic: R3+(-1)R1 $\Rightarrow$ R3	$\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 3\\ 0 & -1 & 2 & 2\\ 0 & 1 & -2 & -2 \end{array}\right]$
Add Row 2 to Row 3 Symbolic: R3+R2 $\Rightarrow$ R3	$\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 3\\ 0 & -1 & 2 & 2\\ 0 & 0 & 0 & 0 \end{array}\right]$
Add Row 2 to Row 1 Symbolic: R1+R2 $\Rightarrow$ R1	$\left[\begin{array}{ccc\|c} 1 & 0 & 3 & 5\\ 0 & -1 & 2 & 2\\ 0 & 0 & 0 & 0 \end{array}\right]$
Multiply Row 2 by -1 Symbolic: (-1)R2 $\Rightarrow$ R2	$\left[\begin{array}{ccc\|c} 1 & 0 & 3 & 5\\ 0 & 1 & -2 & -2\\ 0 & 0 & 0 & 0 \end{array}\right]$

Our last matrix is in reduced row echelon form, since the row containing all zeros occurs at the bottom and the two columns that contain leading ones also contain zeros elsewhere. The resulting system of equations is shown below.

$$\begin{aligned} x+3z &= 5\\ y-2z &= -2\\ 0 &= 0 \end{aligned}$$

The last equation in our system $(0 = 0)$ above can be interpreted to mean that the variable z in this example is an *independent variable*. In other words, we are free to choose any real number for z (since $0 = 0$ is a true statement). On the other hand, the variables x and y in this case are *dependent variables*, since they depend on the choice of z. Specifically, solving for both x and y, we get $x = 5 - 3z$ and $y = -2 + 2z$. Since we are free to choose any value for z, we may conclude that there are infinitely many solutions to the given system of equations. Moreover, a solution to the given system must be of the following form.

$$(x, y, z) = (5 - 3z, -2 + 2z, z)$$

Furthermore, we may once again check that our solution makes sense by plugging it back into the original system.

$$\begin{aligned}
x + y + z &= (5 - 3z) + (-2 + 2z) + z \\
&= 5 - 3z - 2 + 2z + z \\
&= (5 - 2) + (-3z + 2z + z) \\
&= 3
\end{aligned}$$

$$\begin{aligned}
2x + y + 4z &= 2(5 - 3z) + (-2 + 2z) + 4z \\
&= 10 - 6z - 2 + 2z + 4z \\
&= (10 - 2) + (-6z + 2z + 4z) \\
&= 8
\end{aligned}$$

$$\begin{aligned}
x + 2y - z &= (5 - 3z) + 2(-2 + 2z) - z \\
&= 5 - 3z - 4 + 4z - z \\
&= (5 - 4) + (-3z + 4z - z) \\
&= 1
\end{aligned}$$

For out last example, we will work with a system of equations that will have no solution.

Example 2.36.

System	Augmented Matrix
$\begin{aligned} x + y + 3z &= 2 \\ 3x + 4y + 10z &= 5 \\ x + 2y + 4z &= 3 \end{aligned}$	$\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 2 \\ 3 & 4 & 10 & 5 \\ 1 & 2 & 4 & 3 \end{array}\right]$
Multiply Row 1 by -3 and add to Row 2 Symbolic: R2+(-3)R1 $\Rightarrow$ R2	$\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & 4 & 3 \end{array}\right]$
Multiply Row 1 by -1 and add to Row 3 Symbolic: R3+(-1)R1 $\Rightarrow$ R3	$\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{array}\right]$
Multiply Row 2 by -1 and add to Row 3 Symbolic: R3+(-1)R2 $\Rightarrow$ R3	$\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{array}\right]$

The resulting matrix is in row echelon form, but not reduced row echelon form. Notice that the last row of our matrix has corresponding equation $0 = 2$, which is false. Since our row reduction has resulted in a false statement, we conclude that the given system of equations has no solution. Therefore, we have no need to continue row reducing in order to obtain the reduced row echelon form.

We have now seen three examples of how matrices can be used to solve a system of equations containing three variables: one example with a single solution, one with infinitely many solutions, and one with no solution. Naturally, we can apply this approach to simpler systems, containing just two variables/equations, as well as to more complicated systems.

2.7 Practice Problems

2.7.1 Graphing

Solve each system by graphing.

1)
$y = -x + 1$
$y = -5x - 3$

2)
$y = -\frac{5}{4}x - 2$
$y = -\frac{1}{4}x + 2$

3)
$y = -3$
$y = -x - 4$

4)
$y = -x - 2$
$y = \frac{2}{3}x + 3$

5)
$y = -\frac{3}{4}x + 1$
$y = -\frac{3}{4}x + 2$

6)
$y = 2x + 2$
$y = -x - 4$

7)
$y = \frac{1}{3}x + 2$
$y = -\frac{5}{3}x - 4$

8)
$y = 2x - 4$
$y = \frac{1}{2}x + 2$

9)
$y = \frac{5}{3}x + 4$
$y = -\frac{2}{3}x - 3$

10)
$y = \frac{1}{2}x + 4$
$y = \frac{1}{2}x + 1$

11)
$x + 3y = -9$
$5x + 3y = 3$

12)
$x + 4y = -12$
$2x + y = 4$

13)
$x - y = 4$
$2x + y = -1$

14)
$6x + y = -3$
$x + y = 2$

15)
$2x + 3y = -6$
$2x + y = 2$

16)
$3x + 2y = 2$
$3x + 2y = -6$

17)
$2x + y = 2$
$x - y = 4$

18)
$x + 2y = 6$
$5x - 4y = 16$

19)
$2x + y = -2$
$x + 3y = 9$

20)
$x - y = 3$
$5x + 2y = 8$

21)
$9y + 6x = 36$
$3y - 6x = -12$

22)
$-2y + x = 4$
$2 = -x + \frac{1}{2}y$

23)
$2x - y = -1$
$3 = -2x - y$

24)
$-2y = -4 - x$
$-2y = -5x + 4$

25)
$3 + y = -x$
$-4 - 6x = -y$

26)
$16 = -x - 4y$
$-2x = -4 - 4y$

27)
$-y + 7x = 4$
$-y + 7x = 3$

28)
$-4 + y = x$
$x + 2 = -y$

29)
$-12 + x = 4y$
$12 - 5x = 4y$

30)
$-5x + 1 = -y$
$-y + x = -3$

2.7.2 Substitution

Solve each system by substitution.

1)
$y = -3x$
$y = 6x - 9$

2)
$y = x + 5$
$y = -2x - 4$

3)
$y = -2x - 9$
$y = 2x - 1$

4)
$y = -6x + 3$
$y = 6x + 3$

5)
$y = 6x + 4$
$y = -3x - 5$

6)
$y = 3x + 13$
$y = -2x - 22$

7)
$y = 3x + 2$
$y = -3x + 8$

8)
$y = -2x - 9$
$y = -5x - 21$

9)
$y = 2x - 3$
$y = -2x + 9$

10)
$y = 7x - 24$
$y = -3x + 16$

11)
$y = 6x - 6$
$-3x - 3y = -24$

12)
$-x + 3y = 12$
$y = 6x + 21$

13)
$y = -6$
$3x - 6y = 30$

14)
$6x - 4y = -8$
$y = -6x + 2$

15)
$y = -5$
$3x + 4y = -17$

16)
$7x + 2y = -7$
$y = 5x + 5$

17)
$-2x + 2y = 18$
$y = 7x + 15$

18)
$y = x + 4$
$3x - 4y = -19$

19)
$y = -8x + 19$
$-x + 6y = 16$

20)
$y = -2x + 8$
$-7x - 6y = -8$

21)
$7x - 2y = -7$
$y = 7$

22)
$x - 2y = -13$
$4x + 2y = 18$

23)
$x - 5y = 7$
$2x + 7y = -20$

24)
$3x - 4y = 15$
$7x + y = 4$

25)
$-2x - y = -5$
$x - 8y = -23$

26)
$6x + 4y = 16$
$-2x + y = -3$

27)
$-6x + y = 20$
$-3x - 3y = -18$

28)
$7x + 5y = -13$
$x - 4y = -16$

29)
$3x + y = 9$
$2x + 8y = -16$

30)
$-5x - 5y = -20$
$-2x + y = 7$

31)
$2x + y = 2$
$3x + 7y = 14$

32)
$2x + y = -7$
$5x + 3y = -21$

33)
$x + 5y = 15$
$-3x + 2y = 6$

34)
$2x + 3y = -10$
$7x + y = 3$

35)
$-2x + 4y = -16$
$y = -2$

36)
$-2x + 2y = -22$
$-5x - 7y = -19$

37)	38)	39)	40)
$-6x + 6y = -12$	$-8x + 2y = -6$	$2x + 3y = 16$	$-x - 4y = -14$
$8x - 3y = 16$	$-2x + 3y = 11$	$-7x - y = 20$	$-6x + 8y = 12$

2.7.3 Addition/Elimination

Solve each system by elimination.

1)
$4x + 2y = 0$
$-4x - 9y = -28$

2)
$-7x + y = -10$
$-9x - y = -22$

3)
$-9x + 5y = -22$
$9x - 5y = 13$

4)
$-x - 2y = -7$
$x + 2y = 7$

5)
$-6x + 9y = 3$
$6x - 9y = -9$

6)
$5x - 5y = -15$
$5x - 5y = -15$

7)
$4x - 6y = -10$
$4x - 6y = -14$

8)
$-3x + 3y = -12$
$-3x + 9y = -24$

9)
$-x - 5y = 28$
$-x + 4y = -17$

10)
$-10x - 5y = 0$
$10x + 10y = 30$

11)
$2x - y = 5$
$5x + 2y = -28$

12)
$-5x + 6y = -17$
$x - 2y = 5$

13)
$10x + 6y = 24$
$-6x + y = 4$

14)
$x + 3y = -1$
$10x + 6y = -10$

15)
$2x + 4y = 24$
$4x - 12y = 8$

16)
$-6x + 4y = 12$
$12x + 6y = 18$

17)
$-7x + 4y = -4$
$10x - 8y = -8$

18)
$-6x + 4y = 4$
$-3x - y = 26$

19)
$5x + 10y = 20$
$-6x - 5y = -3$

20)
$-9x - 5y = -19$
$3x - 7y = -11$

21)
$-7x - 3y = 12$
$-6x - 5y = 20$

22)
$-5x + 4y = 4$
$-7x - 10y = -10$

23)
$9x - 2y = -18$
$5x - 7y = -10$

24)
$3x + 7y = -8$
$4x + 6y = -4$

25)
$9x + 6y = -21$
$-10x - 9y = 28$

26)
$-4x - 5y = 12$
$-10x + 6y = 30$

27)
$-7x + 5y = -8$
$-3x - 3y = 12$

28)
$8x + 7y = -24$
$6x + 3y = -18$

29)
$-8x - 8y = -8$
$10x + 9y = 1$

30)
$-7x + 10y = 13$
$4x + 9y = 22$

31)
$9y = 7 - x$
$-18y + 4x = -26$

32)
$21 = -9x + 12y$
$\frac{4}{3}y + \frac{7}{3}x = -1$

33)
$0 = 9x + 5y$
$y = \frac{2}{7}x$

34)
$-6 - 42y = -12x$
$x - \frac{7}{2}y = \frac{1}{2}$

2.7.4 Three Variables

Solve each of the following systems of equation.

1)
$a - 2b + c = 5$
$2a + b - c = -1$
$3a + 3b - 2c = -4$

2)
$2x + 3y = z - 1$
$3x = 8z - 1$
$5y + 7z = -1$

3)
$3x + y - z = 11$
$x + 3y = z + 13$
$x + y - 3z = 11$

4)
$x + y + z = 2$
$6x - 4y + 5z = 31$
$5x + 2y + 2z = 13$

5)
$x + 6y + 3z = 4$
$2x + y + 2z = 3$
$3x - 2y + z = 0$

6)
$x - y + 2z = -3$
$x + 2y + 3z = 4$
$2x + y + z = -3$

7)
$x + y + z = 6$
$2x - y - z = -3$
$x - 2y + 3z = 6$

8)
$x + y - z = 0$
$x + 2y - 4z = 0$
$2x + y + z = 0$

9)
$x + y - z = 0$
$x - y - z = 0$
$x + y + 2z = 0$

10)
$x + 2y - z = 4$
$4x - 3y + z = 8$
$5x - y = 12$

11)
$-2x + y - 3z = 1$
$x - 4y + z = 6$
$4x + 16y + 4z = 24$

12)
$4x + 12y + 16z = 4$
$3x + 4y + 5z = 3$
$x + 8y + 11z = 1$

13)
$2x + y - 3z = 0$
$x - 4y + z = 0$
$4x + 16y + 4z = 0$

14)
$4x + 12y + 16z = 0$
$3x + 4y + 5z = 0$
$x + 8y + 11z = 0$

15)
$3x + 2y + 2z = 3$
$x + 2y - z = 5$
$2x - 4y + z = 0$

16)
$p + q + r = 1$
$p + 2q + 3r = 4$
$4p + 5q + 6r = 7$

17)
$x - 2y + 3z = 4$
$2x - y + z = -1$
$4x + y + z = 1$

18)
$x + 2y - 3z = 9$
$2x - y + 2z = -8$
$3x - y - 4z = 3$

19)
$x - y + 2z = 0$
$x - 2y + 3z = -1$
$2x - 2y + z = -3$

20)
$4x - 7y + 3z = 1$
$3x + y - 2z = 4$
$4x - 7y + 3z = 6$

21)
$4x - 3y + 2z = 40$
$5x + 9y - 7z = 47$
$9x + 8y - 3z = 97$

22)
$3x + y - z = 10$
$8x - y - 6z = -3$
$5x - 2y - 5z = 1$

23)
$3x + 3y - 2z = 13$
$6x + 2y - 5z = 13$
$5x - 2y - 5z = -1$

24)
$2x - 3y + 5z = 1$
$3x + 2y - z = 4$
$4x + 7y - 7z = 7$

25)
$3x - 4y + 2z = 1$
$2x + 3y - 3z = -1$
$x + 10y - 8z = 7$

26)
$2x + y = z$
$4x + z = 4y$
$y = x + 1$

27)
$m + 6n + 3p = 8$
$3m + 4n = -3$
$5m + 7n = 1$

28)
$3x + 2y = z + 2$
$y = 1 - 2x$
$3z = -2y$

29)
$2w - 2x - 2y + 2z = 10$
$w + x + y + z = -5$
$3w + 2x + 2y + 4z = -11$
$w + 3x - 2y + 2z = -6$

30)
$w - 2x + 3y - z = 8$
$w - x - y + z = 4$
$w + x + y + z = 22$
$w - x + y + z = 14$

31)
$w + x + y + z = 2$
$w + 2x + 2y + 4z = 1$
$w - x + y + z = 6$
$w - 3x - y + z = 2$

32)
$w + x - y + z = 0$
$-w + 2x + 2y + z = 5$
$w - 3x - y + z = 4$
$2w - x - y + 3z = 7$

2.7.5 Applications

Value Problems

Solve each value problem.

1) A collection of dimes and quaters is worth \$15.25. There are 103 coins in all. How many of each is there?

2) A collection of half dollars and nickels is worth \$13.40. There are 34 coins in all. How many are there?

3) The attendance at a school concert was 578. Admission was \$2.00 for adults and \$1.50 for children. The total receipts were \$985.00. How many adults and how many children attended?

4) A purse contains \$3.90 made up of dimes and quarters. If there are 21 coins in all, how many dimes and how many quarters were there?

5) A boy has \$2.25 in nickels and dimes. If there are twice as many dimes as nickels, how many of each kind has he?

6) \$3.75 is made up of quarters and half dollars. If the number of quarters exceeds the number of half dollars by 3, how many coins of each denomination are there?

7) A collection of 27 coins consisting of nickels and dimes amounts to \$2.25. How many coins of each kind are there?

8) \$3.25 in dimes and nickels, were distributed among 45 boys. If each received one coin, how many received dimes and how many received nickels?

9) There were 429 people at a play. Admission was \$1 each for adults and 75-cents each for children. The receipts were \$372.50. How many children and how many adults attended?

10) There were 200 tickets sold for a women's basketball game. Tickets for students were 50-cents each and for adults 75-cents each. The total amount of money collected was \$132.50. How many of each type of ticket was sold?

11) There were 203 tickets sold for a volleyball game. For activity-card holders, the price was \$1.25 each and for noncard holders the price was \$2 each. The total amount of money collected was \$310. How many of each type of ticket was sold?

12) At a local ball game the hotdogs sold for \$2.50 each and the hamburgers sold for \$2.75 each. There were 131 total sandwiches sold for a total value of \$342. How many of each sandwich was sold?

13) At a recent Vikings game \$445 in admission tickets was taken in. The cost of a student ticket was \$1.50 and the cost of a non-student ticket was \$2.50. A total of 232 tickets were sold. How many students and how many non-students attended the game?

14) A bank contains 27 coins in dimes and quarters. The coins have a total value of \$4.95. Find the number of dimes and quarters in the bank.

15) A coin purse contains 18 coins in nickels and dimes. The coins have a total value of \$1.15. Find the number of nickels and dimes in the coin purse.

16) A business executive bought 40 stamps for \$9.60. The purchase included 25-cents stamps and 20-cents stamps. How many of each type of stamp were bought?

17) A postal clerk sold some 15-cents stamps and some 25-cents stamps. Altogether, 15 stamps were sold for a total cost of \$3.15. How many of each type of stamps were sold?

18) A drawer contains 15-cents stamps and 18-cents stamps. The number of 15-cents stamps is four less than three times the number of 18-cents stamps. The total value of all the stamps is \$1.29. How many 15-cents stamps are in the drawer?

19) The total value of dimes and quarters in a bank is \$6.05. There are six more quarters than dimes. Find the number of each type of coin in the bank.

20) A child's piggy bank contains 44 coins in quarters and dimes. The coins have a total value of \$8.60. Find the number of quarters in the bank.

21) A coin bank contains nickels and dimes. The number of dimes is 10 less than twice the number of nickels. The total value of all the coins is \$2.75. Find the number of each type of coin in the bank.

22) A total of 26 bills are in a cash box. Some of the bills are one dollar bills, and the rest are five dollar bills. The total amount of cash in the box is \$50. Find the number of each type of bill in the cash box.

23) A bank teller cashed a check for \$200 using twenty dollar bills and ten dollar bills. In all, twelve bills were handed to the customer. Find the number of twenty dollar bills and the number of ten dollar bills.

24) A collection of stamps consists of 22-cents stamps and 40-cents stamps. The number of 22-cents stamps is three more than four times the number of 40-cents stamps. The total value of the stamps is \$8.34. Find the number of 22-cents stamps in the collection.

25) A total of \$27000 is invested, part of it at 12% and the rest at 13%. The total interest after one year is \$3385. How much was invested at each rate?

26) A total of \$50000 is invested, part of it at 5% and the rest at 7.5%. The total interest after one year is \$3250. How much was invested at each rate?

27) A total of \$9000 is invested, part of it at 10% and the rest at 12%. The total interest after one year is \$1030. How much was invested at each rate?

28) A total of \$18000 is invested, part of it at 6% and the rest at 9%. The total interest after one year is \$1248. How much was invested at each rate?

29) An inheritance of \$10000 is invested in 2 ways, part at 9.5% and the remainder at 11%. The combined annual interest was \$1038.50. How much was invested at each rate?

30) Kerry earned a total of \$900 last year on his investments. If \$7000 was invested at a certain rate of return and \$9000 was invested in a fund with a rate that was 2% higher, find the two rates of interest.

31) Jason earned \$256 interest last year on his investments. If \$1600 was invested at a certain rate of return and \$2400 was invested in a fund with a rate that was double the rate of the first fund, find the two rates of interest.

32) Millicent earned \$435 last year in interest. If \$3000 was invested at a certain rate of return and \$4500 was invested in a fund with a rate that was 2% lower, find the two rates of interest.

33) A total of \$8500 is invested, part of it at 6% and the rest at 3.5%. The total interest after one year is \$385. How much was invested at each rate?

34) A total of \$12000 was invested, part of it at 9% and the rest at 7.5%. The total interest after one year is \$1005. How much was invested at each rate?

35) A total of \$15000 is invested, part of it at 8% and the rest at 11%. The total interest after one year is \$1455. How much was invested at each rate?

36) A total of \$17500 is invested, part of it at 7.25% and the rest at 6.5%. The total interest after one year is \$1227.50. How much was invested at each rate?

37) A total of \$6000 is invested, part of it at 4.25% and the rest at 5.75%. The total interest after one year is \$300. How much was invested at each rate?

38) A total of \$14000 is invested, part of it at 5.5% and the rest at 9%. The total interest after one year is \$910. How much was invested at each rate?

39) A total of \$11000 is invested, part of it at 6.8% and the rest at 8.2%. The total interest after one year is \$797. How much was invested at each rate?

40) An investment portfolio earned \$2010 in interest last year. If \$3000 was invested at a certain rate of return and \$24000 was invested in a fund with a rate that was 4% lower, find the two rates of interest.

41) Samantha earned \$1480 in interest last year on her investments. If \$5000 was invested at a certain rate of return and \$11000 was invested in a fund with a rate that was two-thirds the rate of the first fund, find the two rates of interest.

42) A man has \$5.10 in nickels, dimes, and quarters. There are twice as many nickels as dimes and 3 more dimes than quarters. How many coins of each kind were there?

43) 30 coins having a value of \$3.30 consists of nickels, dimes and quarters. If there are twice as many quarters as dimes, how many coins of each kind were there?

44) A bag contains nickels, dimes and quarters having a value of \$3.75. If there are 40 coins in all and 3 times as many dimes as quarters, how many coins of each kind were there?

Mixture Problems

Solve each mixture problem.

1) A tank contains 8000 liters of a solution that is 40% acid. How much water should be added to make a solution that is 30% acid?

2) How much antifreeze should be added to 5 quarts of a 30% mixture of antifreeze to make a solution that is 50% antifreeze?

3) Of 12 pounds of salt water 10% is salt; of another mixture 3% is salt. How many pounds of the second should be added to the first in order to get a mixture of 5% salt?

4) How much alcohol must be added to 24 gallons of a 14% solution of alcohol in order to produce a 20% solution?

5) How many pounds of a 4% solution of borax must be added to 24 pounds of a 12% solution of borax to obtain a 10% solution of borax?

6) How many grams of pure acid must be added to 40 grams of a 20% acid solution to make a solution which is 36% acid?

7) A 100 LB bag of animal feed is 40% oats. How many pounds of oats must be added to this feed to produce a mixture which is 50% oats?

8) A 20 oz alloy of platinum that costs \$220 per ounce is mixed with an alloy that costs \$400 per ounce. How many ounces of the \$400 alloy should be used to make an alloy that costs \$300 per ounce?

9) How many pounds of tea that cost \$4.20 per pound must be mixed with 12 lb of tea that cost \$2.25 per pound to make a mixture that costs \$3.40 per pound?

10) How many liters of a solvent that costs \$80 per liter must be mixed with 6 L of a solvent that costs \$25 per liter to make a solvent that costs \$36 per liter?

11) How many kilograms of hard candy that cost \$7.50 per kilogram must be mixed with 24 kg of jelly beans that cost \$3.25 per kilogram to make a mixture that sells for \$4.50 per kilogram?

12) How many kilograms of soil supplement that costs \$7.00 per kilogram must be mixed with 20 kg of aluminum nitrate that costs \$3.50 per kilogram to make a fertilizer that costs \$4.50 per kilogram?

13) How many pounds of lima beans that cost 90-cents per pound must be mixed with 16 lb of corn that cost 50-cents per pound to make a mixture of vegetables that costs 65-cents per pound?

14) How many liters of a blue dye that costs \$1.60 per liter must be mixed with 18 L of a dye that costs \$2.50 per liter to make a mixture that costs \$1.90 per liter?

15) Solution A is 50% acid and solution B is 80% acid. How much of each should be used to make 100cc. of a solution that is 68% acid?

16) A certain grade of milk contains 10% butter fat and a certain grade of cream 60% butter fat. How many quarts of each must be taken so as to obtain a mixture of 100 quarts that will be 45% butter fat?

17) A farmer has some cream which is 21% butterfat and some which is 15% butter fat. How many gallons of each must be mixed to produce 60 gallons of cream which is 19% butterfat?

18) A syrup manufacturer has some pure maple syrup and some which is 85% maple syrup. How many liters of each should be mixed to make 150L which is 96% maple syrup?

19) A chemist wants to make 50ml of a 16% acid solution by mixing a 13% acid solution and an 18% acid solution. How many milliliters of each solution should the chemist use?

20) A hair dye is made by blending 7% hydrogen peroxide solution and a 4% hydrogen peroxide solution. How many milliliters of each are used to make a 300 ml solution that is 5% hydrogen peroxide?

21) A paint that contains 21% green dye is mixed with a paint that contains 15% green dye. How many gallons of each must be used to make 60 gal of paint that is 19% green dye?

22) A candy mix sells for \$2.20 per kilogram. It contains chocolates worth \$1.80 per kilogram and other candy worth \$3.00 per kilogram. How much of each are in 15 kilograms of the mixture?

23) To make a weed and feed mixture, the Green Thumb Garden Shop mixes fertilizer worth \$4.00/lb. with a weed killer worth \$8.00/lb. The mixture will cost \$6.00/lb. How much of each should be used to prepare 500 lb. of the mixture?

24) A grocer is mixing 40 cent per lb. coffee with 60 cent per lb. coffee to make a mixture worth 54-cents per lb. How much of each kind of coffee should be used to make 70 lb. of the mixture?

25) A grocer wishes to mix sugar at 9-cents per pound with sugar at 6-cents per pound to make 60 pounds at 7-cents per pound. What quantity of each must he take?

26) A high-protein diet supplement that costs \$6.75 per pound is mixed with a vitamin supplement that costs \$3.25 per pound. How many pounds of each should be used to make 5 lb of a mixture that costs \$4.65 per pound?

27) A goldsmith combined an alloy that costs \$4.30 per ounce with an alloy that costs \$1.80per ounce. How many ounces of each were used to make a mixture of 200 oz costing \$2.50 per ounce?

28) A grocery store offers a cheese and fruit sampler that combines cheddar cheese that costs \$8 per kilogram with kiwis that cost \$3 per kilogram. How many kilograms of each were used to make a 5 kg mixture that costs \$4.50 per kilogram?

29) The manager of a garden shop mixes grass seed that is 60% rye grass with 70 lb of grass seed that is 80% rye grass to make a mixture that is 74% rye grass. How much of the 60% mixture is used?

30) How many ounces of water evaporated from 50 oz of a 12% salt solution to produce a 15% salt solution?

31) A caterer made an ice cream punch by combining fruit juice that cost \$2.25 per gallon with ice cream that costs \$3.25 per gallon. How many gallons of each were used to make 100 gal of punch costing \$2.50 per pound?

32) A clothing manufacturer has some pure silk thread and some thread that is 85% silk. How many kilograms of each must be woven together to make 75 kg of cloth that is 96% silk?

33) A carpet manufacturer blends two fibers, one 20% wool and the second 50% wool. How many pounds of each fiber should be woven together to produce 600 lb of a fabric that is 28% wool?

34) How many pounds of coffee that is 40% java beans must be mixed with 80 lbs of coffee that is 30% java beans to make a coffee blend that is 32% java beans?

35) The manager of a specialty food store combined almonds that cost \$4.50 per pound with walnuts that cost \$2.50 per pound. How many pounds of each were used to make a 100 lb mixture that cost \$3.24 per pound?

36) A tea that is 20% jasmine is blended with a tea that is 15% jasmine. How many pounds of each tea are used to make 5 lb of tea that is 18% jasmine?

37) How many ounces of dried apricots must be added to 18 oz of a snack mix that contains 20% dried apricots to make a mixture that is 25% dried apricots?

38) How many milliliters of pure chocolate must be added to 150 ml of chocolate topping that is 50% chocolate to make a topping that is 75% chocolate?

39) How many ounces of pure bran flakes must be added to 50 oz of cereal that is 40% bran flakes to produce a mixture that is 50% bran flakes?

40) A ground meat mixture is formed by combining meat that costs \$2.20 per pound with meat that costs \$4.20 per pound. How many pounds of each were used to make a 50 lb mixture that costs \$3.00 per pound?

41) How many grams of pure water must be added to 50 g of pure acid to make a solution that is 40% acid?

42) A lumber company combined oak wood chips that cost \$3.10 per pound with pine wood chips that cost \$2.50 per pound. How many pounds of each were used to make an 80 lb mixture costing \$2.65 per pound?

43) How many ounces of pure water must be added to 50 oz of a 15% saline solution to make a saline solution that is 10% salt?

2.7.6 Matrix Notation

Construct an augmented matrix for each of the systems of equations referenced below. Then row reduce your matrix to its row echelon form and determine if the given system has (1) no solution, (2) infinitely many solutions, or (3) exactly one solution. If one solution exists, determine the reduced row echelon form for your matrix and use it to find the solution to the given system.

1) - 5): Systems (1) through (5) on page 160.

6) - 10): Systems (11) through (15) on page 160.

11) - 20): Systems (1) through (10) on page 161.

21) - 24): Systems (29) through (32) on page 162.

Chapter 3

Quadratic Equations

3.1 Introduction

3.1.1 Quadratic Equations

Objective: Recognize and classify a quadratic equation algebraically and graphically

A quadratic equation is an equation of the form:

$$y = ax^2 + bx + c$$

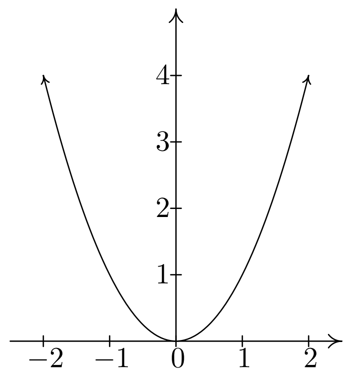
where the *coefficients* of a,b, and c are real numbers and $a \neq 0$. This form is most commonly referred to as the *standard form* of a quadratic. We call a the *leading coefficient*, ax^2 the *leading term* (also known as the *quadratic term*), bx the *linear term* and c the *constant term* of the equation. The quadratic term ax^2, must have a nonzero coefficient in order for the equation to be a quadratic (otherwise y would be linear, in slope-intercept form). The most fundamental quadratic equation is $y = x^2$ and its graph, like all quadratics, is known as a *parabola*.

Example 3.1. $y = x^2$

From the standard form, since $a > 0$, the graph opens upwards and is said to be *concave up*.

As a result, there is a minimum point, known as the *vertex*, located at the origin, $(0, 0)$.

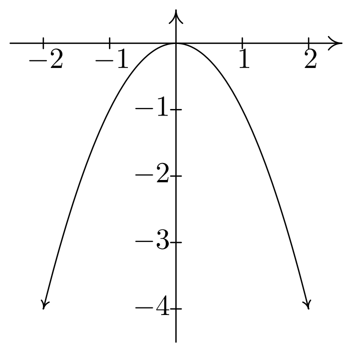
Notice the symmetry over the y-axis.



Example 3.2. $y = -x^2$

Since $a = -1$, the graph opens downward or we say that it is *concave down.*

Every parabola with a negative leading coefficient ($a < 0$) will be concave down with a maximum value at its vertex.



The graph above has the same vertex as that in the previous example, but is a reflection of the previous graph about the y-axis. This flip of the graph is known as a transformation and will be discussed in the next chapter.

Aside from the shape and concavity, there is little else that the standard form immediately provides for graphing a quadratic. Additional aspects related to graphing quadratics will be covered a bit later in the chapter. Following this introduction, we will primarily focus on factoring quadratics from standard form. With all of the algebraic material that will follow, however, it will help to have a graphical sense of a quadratic equation.

3.1.2 Introduction to Vertex Form and Graphing

Objective: Recognize and utilize the vertex form to graph a quadratic

The most useful form for graphing a quadratic equation is the *vertex form*. A quadratic equation is said to be in vertex form if it is represented as

$$y = a(x-h)^2 + k,$$

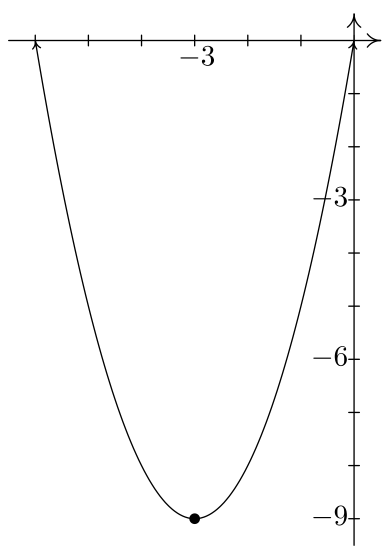
where h and k are real numbers.

It is important to note that the value a appearing above is also the leading coefficient from the standard form for a quadratic. Later, we will see the relationships between the coefficients a, b, and c in the standard form with h and k above.

When $a = 1$, the graph of a quadratic equation given in vertex form can be represented as a *shift*, or translation, of the original or "parent equation" $y = x^2$ presented earlier. The vertex form, unlike the standard form, allows us to immediately identify the vertex of the resulting parabola, which will be the point (h, k).

Next, we will see a few examples of quadratics in vertex form, the last of which is a bit surprising.

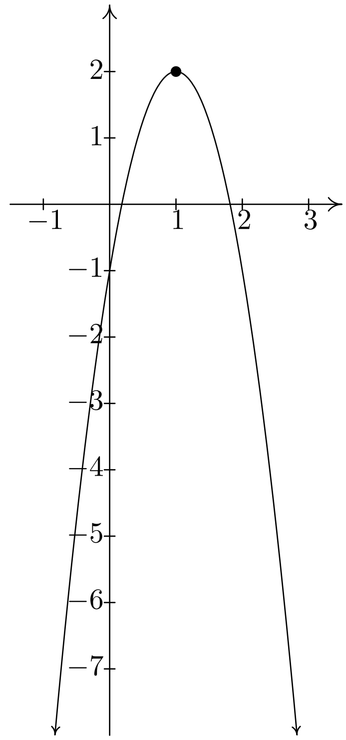
Example 3.3. $y = (x+3)^2 - 9$



The vertex is at $(-3, -9)$ and the graph can be realized as the graph of $y = x^2$ shifted left 3 units and down 9 units from the origin.

Since our graph is concave up there will be two x-intercepts as the function opens upward from below the x-axis.

Example 3.4. $y = -3(x-1)^2 + 2$



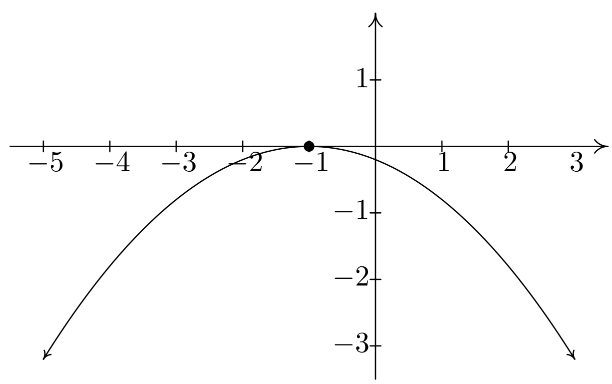
The vertex is at $(1, 2)$ and represents a translation of the vertex for the graph of $y = x^2$ right 1 unit and up 2 units.

This graph is also concave down, since the leading coefficient $a = -3$ is less than zero.

Moreover, since $|a| > 1$, the shape of the graph is narrower than those which we have seen thus far.

Just like the previous example, this graph will have two x-intercepts as its vertex is above the x-axis and it opens downward.

Example 3.5. $y = -\frac{1}{5}(x+1)^2$



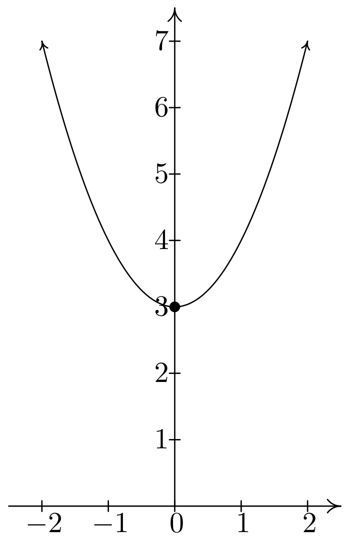
The vertex is at $(-1, 0)$ and represents a translation of the vertex for the graph of $y = x^2$ left 1 unit.

There is no vertical shift, since there is no addition of a constant outside of the given expression.

Our graph is concave down and is much wider than any example we have seen thus far. This is on account of the fact that a is both negative and $|a| < 1$.

The following example shows an equation represented in both vertex and standard forms.

Example 3.6. $y = x^2 + 3$



The vertex is at $(0, 3)$ and our graph is a shift of the graph of $y = x^2$ up 3 units.

Since our graph is concave up with a vertex above the x-axis, there will be no real x-intercepts.

Notice that there is no horizontal shift because no number has been added or subtracted to x prior to it being squared.

This final example above may be recognized as a quadratic equation in standard form, where $b = 0$. Since there is no linear term, this quadratic is also in vertex form.

More generally, the graph of any equation of the form

$$y = ax^2 + c$$

has a y-intercept and vertex at $(0, c)$, since the resulting parabola represents a only a vertical shift of the graph of $y = x^2$ by c units and no horizontal shift.

3.2 Factoring

3.2.1 Greatest Common Factor

Objective: Find the greatest common factor of a polynomial and factor it out of the expression.

In order to discuss the factorization methods of this section, it will be necessary to introduce some of the terminology a bit early. In particular, in this section we will be working with *polynomial expressions*. While most of our work will be with polynomials containing a single variable, it will be helpful to see a few examples of polynomials that contain two (or more) variables.

Both linear and quadratic expressions of a variable x are basic examples of polynomials. A more general description of a polynomial in terms of the variable x is

$$a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0,$$

where n is a positive integer and $a_0, a_1, \ldots, a_{n-1}, a_n$ represent real coefficients ($a_n \neq 0$).

A basic interpretation of this description is a sum of n terms, each containing a real coefficient (possibly equal to 0), where the associated power of the variable is a positive integer (or possibly 0, in the case of the constant term $a_0 = a_0x^0$).

The expression $8x^4 - 12x^3 + 32x$ would be an example of a polynomial, in which the power n (known as the *degree* of the polynomial) equals 4, and the coefficients are as follows.

$$a_4 = 8, \quad a_3 = -12, \quad a_2 = 0, \quad a_1 = 32, \quad a_0 = 0$$

If we inserted another variable(s) into each of the terms of our expression, we could create a polynomial expression in terms of two (or more) variables. An example of this would be

$$8x^4y - 12x^3y^2 + 32x.$$

While there is much more that we could say about this important concept of algebra, we will postpone a more in-depth treatment of polynomials until a later chapter, and move on to the topic of factorization.

Factoring a polynomial could be considered as the "opposite" action of multiplying (or expanding) polynomials together. In working with polynomial expressions, there are many benefits to identifying both its expanded and factored forms. Specifically, we will use factored polynomials to help us solve equations, learn behaviors of graphs, and understand more complicated rational expressions. Because so many concepts in algebra depend on being able to factor polynomials, it is critical that we establish strong factorization skills.

In this first part of the section, we will focus on factoring using the greatest common factor or GCF of a polynomial. When multiplying polynomials, we employ the distributive property, as demonstrated below.

$$4x^2(2x^2 - 3x + 8) = 8x^4 - 12x^3 + 32x$$

Here, we will work with the same expression, but with a backwards approach, starting with the expanded form and obtaining one that is partially (or completely) factored.

We will start with $8x^2 - 12x^3 + 32x$ and try and work backwards to reach $4x^2(2x - 3x + 8)$.

To do this we have to be able to first identify what the GCF of a polynomial is. We will first introduce this concept by finding the GCF of a set of integers. To find a GCF of two or more integers, we must find the largest integer d that divides nicely into each of the given integers. Alternatively stated, d should be the largest factor of each of the integers in our set. This can often be determined with quick "mental math", as shown in the following example.

Example 3.7. Find the GCF of 15, 24, and 27.

$$\frac{15}{3} = 5, \quad \frac{24}{3} = 6, \quad \frac{27}{3} = 9 \qquad \text{Each of the numbers can be divided by 3}$$
$$\text{GCF} = 3 \qquad \text{Our solution}$$

When there are variables in our problem we can first find the GCF of the numbers, then we can identify any variables that appear in every term and factor them out, taking the smallest exponent in each case. This is shown in the next example.

Example 3.8. Find the GCF of $24x^4y^2z$, $18x^2y^4$, and $12x^3yz^5$.

$$\frac{24}{6} = 4, \quad \frac{18}{6} = 3, \quad \frac{12}{6} = 2 \qquad \text{Each number can be divided by 6}$$

$$x^2y \qquad x \text{ and } y \text{ appear in all three terms, taking the lowest exponent for each variable}$$

$$\text{GCF} = 6x^2y \qquad \text{Our solution}$$

To factor out a GCF from a polynomial we first need to identify the GCF of all the terms, this is the part that goes in front of the parentheses, then we divide each term by the GCF in order to determine what should appear inside of the parentheses. This is demonstrated in the following examples.

Example 3.9. Find and factor out the GCF of the given polynomial expression.

$$4x^2 - 20x + 16 \qquad \text{GCF is 4, divide each term by 4}$$

$$\frac{4x^2}{4} = x^2, \quad \frac{-20x}{4} = -5x, \quad \frac{16}{4} = 4 \qquad \text{This is what is left inside the parentheses}$$

$$4(x^2 - 5x + 4) \qquad \text{Our solution}$$

With factoring we can always check our solutions by expanding or multiplying out the answer. As in the example above, this usually will involve some form of the distributive property. Our end result upon checking should match the original expression.

Example 3.10. Find and factor out the GCF of the given polynomial expression.

$$25x^4 - 15x^3 + 20x^2 \qquad \text{GCF is } 5x^2\text{, divide each term by } 5x^2$$

$$\frac{25x^4}{5x^2} = 5x^2, \quad \frac{-15x^3}{5x^2} = -3x, \quad \frac{20x^2}{5x^2} = 4$$

This is what is left inside the parentheses.

$$5x^2(5x^2 - 3x + 4) \qquad \text{Our solution}$$

Example 3.11. Find and factor out the GCF of the given polynomial expression.

$$3x^3y^2z + 5x^4y^3z^5 - 4xy^4 \qquad \text{GCF is } xy^2\text{, divide each term by } xy^2$$

$$\frac{3x^3y^2z}{xy^2} = 3x^2z, \quad \frac{5x^4y^3z^5}{xy^2} = 5x^3yz^5, \quad \frac{-4xy^4}{xy^2} = -4y^2$$

This is what is left inside the parentheses.

$$xy^2(3x^2z + 5x^3yz^5 - 4y^2) \qquad \text{Our solution}$$

World View Note: The first recorded algorithm for finding the greatest common factor comes from Greek mathematician Euclid around the year 300 B.C.!

Example 3.12. Find and factor out the GCF of the given polynomial expression.

$$21x^3 + 14x^2 + 7x \qquad \text{GCF is } 7x\text{, divide each term by } 7x$$

$$\frac{21x^3}{7x} = 3x^2, \quad \frac{14x^2}{7x} = 2x, \quad \frac{7x}{7x} = 1$$

This is what is left inside the parentheses.

$$7x(3x^2 + 2x + 1) \qquad \text{Our solution}$$

It is important to note that in the previous example, the GCF of $7x$ was also one of the original terms. Dividing this term by the GCF left us with 1. A common mistake is to try to factor out the $7x$ and leave a value of zero. Factoring, however, will never make terms disappear completely. Any (nonzero) number or term that is divided by itself will always equal 1. Therefore, we must make certain to not forget to include a 1 in our solution.

Often the line showing the division is not written in the work of factoring the GCF, and we will simply identify the GCF and put it in front of the parentheses. This step is one that will eventually be understood, and can therefore be omitted once the skill has been mastered. The following two examples demonstrate this.

Example 3.13. Find and factor out the GCF of the given polynomial expression.

$$12x^5y^2 - 6x^4y^4 + 8x^3y^5$$

Notice, the GCF is $2x^3y^2$. Write $2x^3y^2$ in front of the parentheses and divide each term by it, writing the resulting terms inside the parentheses.

$$2x^3y^2(6x^2 - 3xy^2 + 4y^3) \qquad \text{Our solution}$$

Example 3.14. Find and factor out the GCF of the given polynomial expression.

$$18a^4b^3 - 27a^3b^3 + 9a^2b^3$$

Notice, the GCF is $9a^2b^3$. Write $9a^2b^3$ in front of the parentheses and divide each term by it, writing the resulting terms inside the parentheses.

$$9a^2b^3(2a^2 - 3a + 1) \qquad \text{Our solution}$$

Again, in the previous problem, when dividing $9a^2b^3$ by itself, the resulting term is 1, not zero. Be very careful that each term is accounted for in your final solution, and never forget that we can easily check our answers by expanding.

3.2.2 Grouping

Objective: Factor polynomials with four terms using grouping.

The first thing we will always do when factoring is try to factor out a GCF. This GCF is often a *monomial* (a single term) like in the expression $5xy+10xz$. Here, the GCF is the monomial $5x$, so we would have $5x(y+2z)$ as our answer. However, a GCF does not have to be a monomial. It could, in fact, be a *binomial* and contain two terms. To see this, consider the following two examples.

Example 3.15. Find and factor out the GCF of the given expression.

$$\begin{aligned} 3ax - 7bx & \quad \text{Both terms have } x \text{ in common, factor it out} \\ x(3a - 7b) & \quad \text{Our solution} \end{aligned}$$

Now we will work with the same expression, replacing x with $(2a + 5b)$.

Example 3.16. Find and factor out the GCF of the given expression.

$$\begin{aligned} 3a(2a + 5b) - 7b(2a + 5b) & \quad \text{Both terms have } (2a + 5b) \text{ in common,} \\ & \quad \quad \text{factor it out} \\ (2a + 5b)(3a - 7b) & \quad \text{Our solution} \end{aligned}$$

In the same way that we factored out a GCF of x we can factor out a GCF which is a binomial, such as $(2a + 5b)$ in the example above. This process can be extended to factoring expressions in which there is either no apparent GCF or there is more factoring that can be done after the GCF has been factored. At this point, we will introduce another useful factorization strategy, known as *grouping*. Grouping is typically employed when faced with an expression containing four terms.

Throughout this section, it is important to reinforce the fact that factoring is essentially expansion (multiplication) done in reverse. Therefore, we will first look at problem which requires us to multiply two expressions, and then try to reverse the process.

Example 3.17. Write the expanded form for the given expression.

$(2a+3)(5b+2)$	Distribute $(2a+3)$ into second parentheses
$5b(2a+3)+2(2a+3)$	Distribute each monomial
$10ab+15b+4a+6$	Our solution

Our solution above has four terms in it. We arrived at this solution by focusing on the two parts, $5b(2a+3)$ and $2(2a+3)$.

When attempting to factor by grouping, we will always divide an expression into two parts, or groups: group one will contain the first two terms of our expression and group two will contain the last two terms. Then we can identify and factor the GCF out of each group. In doing this, our hope is that what is left over in each group will be the same expression. If the resulting expressions match, we can then factor out this matching GCF from both of our designated groups, writing what is left in a new set of parentheses.

Although the description of this method can sound rather complicated, the next few examples will help to clear up any lingering questions. We will start by working through the last example in reverse, factoring instead of multiplying.

Example 3.18. Factor the given expression.

$10ab+15b+4a+6$	Split expression into two groups
$\boxed{10ab+15b}\boxed{+4a+6}$	GCF on left is $5b$, on the right is 2
$\boxed{5b(2a+3)}\boxed{+2(2a+3)}$	$(2a+3)$ appears twice! Factor out this GCF
$(2a+3)(5b+2)$	Our solution

The key for grouping to be successful is for the two binomials to match exactly, once the GCF has been factored out of both groups. If there is any difference between the two binomials, we either have to do some adjusting or we cannot factor by grouping. Consider the following example.

Example 3.19. Factor the given expression.

$$6x^2 + 9xy - 14x - 21y \qquad \text{Split expression into two groups}$$

$$\boxed{6x^2 + 9xy \,|\, -14x - 21y} \qquad \text{GCF on left is } 3x\text{, on right is } 7$$

$$\boxed{3x(2x + 3y) \,|\, +7(-2x - 3y)} \qquad \text{The signs in the parentheses do not match!}$$

When the signs on both terms do not match, we can easily make them match by factoring a negative out of the GCF on the right side. Instead of 7 we will use -7. This will change the signs inside the second set of parentheses.

$$\boxed{3x(2x + 3y) \,|\, -7(2x + 3y)} \qquad (2x + 3y) \text{ appears twice! Factor out this GCF}$$

$$(2x + 3y)(3x - 7) \qquad \text{Our solution}$$

It will often be easy to recognize if we will need to factor out a negative sign when grouping. Specifically, if the first term of the first binomial is positive, the first term of the second binomial will also need to be positive. Similarly, if the first term of the first binomial is negative, the first term of the second binomial will also need to be negative.

Example 3.20. Factor the given expression.

$$5xy - 8x - 10y + 16 \qquad \text{Split the expression into two groups}$$

$$\boxed{5xy - 8x \,|\, -10y + 16} \qquad \text{GCF on left is } x\text{, on right we need to factor out a negative, we will use } -2$$

$$\boxed{x(5y - 8) \,|\, -2(5y - 8)} \qquad (5y - 8) \text{ appears twice! Factor out this GCF}$$

$$(5y - 8)(x - 2) \qquad \text{Our solution}$$

Occasionally, when factoring out a GCF from either group, it will appear as though there is nothing that can be factored out. In this case a GCF of either 1 or -1 should be used. Often this will be all that is required, in order to match up the two binomials.

Example 3.21. Factor the given expression.

$12ab - 14a - 6b + 7$	Split the expression into two groups
$\boxed{12ab - 14a} \boxed{-6b + 7}$	GCF on left is $2a$, on right use GCF of -1
$\boxed{2a(6b - 7)} \boxed{-1(6b - 7)}$	$(6b - 7)$ appears twice! Factor out this GCF
$(6b - 7)(2a - 1)$	Our solution

Example 3.22. Factor the given expression.

$6x^3 - 15x^2 + 2x - 5$	Split expression into two groups
$\boxed{6x^3 - 15x^2} \boxed{+2x - 5}$	GCF on left is $3x^2$, on right use GCF of 1
$\boxed{3x^2(2x - 5)} \boxed{+1(2x - 5)}$	$(2x - 5)$ appears twice! Factor out this GCF
$(2x - 5)(3x^2 + 1)$	Our solution

When grouping, the selection or assignment of terms for each group can also be an area of concern. In particular, if after factoring out the GCF from the preassigned groups, the binomials do not match *and* cannot be adjusted as in the previous examples, a change in the group assignments may be necessary. In the next example we will demonstrate this by eventually moving the second term to the end of the given expression, to see if grouping may still be used.

Example 3.23. Factor the given expression.

$4a^2 - 21b^3 + 6ab - 14ab^2$	Split the expression into two groups
$\boxed{4a^2 - 21b^3} \boxed{+6ab - 14ab^2}$	GCF on left is 1, on right is $2ab$
$\boxed{1(4a^2 - 21b^3)} \boxed{+2ab(3 - 7b)}$	Binomials do not match! Move second term to end

$4a^2 + 6ab - 14ab^2 - 21b^3$	Start over, split expression into two groups
$\boxed{4a^2 + 6ab \mid -14ab^2 - 21b^3}$	GCF on left is $2a$, on right is $-7b^2$
$\boxed{2a(2a+3b) \mid -7b^2(2a+3b)}$	$(2a+3b)$ appears twice! Factor out this GCF
$(2a+3b)(2a-7b^2)$	Our solution

When rearranging terms the expression might still appear to be out of order. Sometimes after factoring out the GCF the resulting binomials appear "backwards". There are two scenarios where this can happen: one with addition and one with subtraction. In the first scenario, if the binomials are say $(a+b)$ and $(b+a)$, then we do not have to do any extra work. This is because addition is a *commutative* operation. This means that the sum of two terms is the same, regardless of their order. For example, $5+3=3+5=8$.

Example 3.24. Factor the given expression.

$7 + y - 3xy - 21x$	Split the expression into two groups
$\boxed{7 + y \mid -3xy - 21x}$	GCF on left is 1, on the right is $-3x$
$\boxed{1(7+y) \mid -3x(y+7)}$	$y+7$ and $7+y$ are equal, use either one
$(y+7)(1-3x)$	Our solution

In the second scenario, if the binomials contain subtraction, then we need to be a bit more careful. For example, if the binomials are $(a-b)$ and $(b-a)$, we will factor a negative sign out of either group (usually the second). Notice what happens when we factor out a -1 in the following example.

Example 3.25. Factor the given expression.

$(b-a)$	Factor out a -1
$-1(-b+a)$	Resulting binomial contains addition, we may switch the order
$-1(a-b)$	The order of the subtraction has been switched!

Generally we will not show all of the steps in the previous example when simplifying. Instead, we will simply factor out a negative sign and switch the order of the subtraction to make the resulting binomials. As with previous concepts, this omission should only be made by the student when the skill has been mastered. We conclude our discussion of grouping with one final example.

Example 3.26. Factor the given expression.

$8xy - 12y + 15 - 10x$	Split the expression into two groups
$\boxed{8xy - 12y} \boxed{15 - 10x}$	GCF on left is $4y$, on right is 5
$\boxed{4y(2x-3)} \boxed{+5(3-2x)}$	Need to switch order, Factor negative sign out of second binomial
$\boxed{4y(2x-3)} \boxed{-5(2x-3)}$	$(2x-3)$ appears twice! Factor out this GCF
$(2x-3)(4y-5)$	Our solution

World View Note: Sofia Kovalevskaya of Russia was the first woman on the editorial staff of a mathematical journal in the late 19^{th} century. She also did research the rotation of the rings of Saturn.

3.2.3 Trinomials with Leading Coefficient $a = 1$

Objective: Factor trinomials where the coefficient of x^2 is one.

Factoring polynomial expressions that contain three terms, or *trinomials*, is the most essential factorization skill to algebra. Consequently, it is also the most important factorization skill to master. Again, since factoring is basically multiplication performed in reverse, we will start with a multiplication example and look at how we can reverse the process.

Example 3.27. Write the expanded form for the given expression.

$$\begin{array}{rl} (x+6)(x-4) & \text{Distribute } (x+6) \text{ through second parentheses} \\ x(x+6)-4(x+6) & \text{Distribute each monomial through parentheses} \\ x^2+6x-4x-24 & \text{Combine like terms} \\ x^2+2x-24 & \text{Our solution} \end{array}$$

Notice that if we reverse the last three steps of the previous example, the process looks like grouping. This is because it is grouping! In the second-to-last line, the GCF of the first two terms is x and the GCF of the last two terms is -4. In this manner, we will factor trinomials by writing them as a polynomial containing four terms, and then factor by grouping. This is demonstrated in the following example, which is the previous one done in reverse.

Example 3.28. Factor the given expression.

$$\begin{array}{rl} x^2+2x-24 & \text{split middle (linear) term into} +6x-4x \\ x^2+6x-4x-24 & \text{Grouping : GCF on left is } x, \text{ on right is } -4 \\ x(x+6)-4(x+6) & (x+6) \text{ appears twice, factor out this GCF} \\ (x+6)(x-4) & \text{Our solution} \end{array}$$

The trick to make these problems work resides in how we split the middle (or linear) term. Why did we choose $+6x - 4x$ and not $+5x - 3x$? The reason is because $6x - 4x$ is the only combination that will allow grouping to work! So how do we know what is the one combination that we need? To find the correct way to split the middle term we will use what is called the *ac*-method. Later, we will discuss why it is called the *ac*-method.

The idea behind the *ac*-method is that we must find a pair of numbers that *multiply* to get the last (or constant) term in the expression and *add* to get the coefficient of the middle (or linear) term. In the previous example, we would want two numbers whose product is -24 and sum is $+2$. The only numbers that can do this are 6 and -4 ($6 \cdot -4 = -24$ and $6 + (-4) = 2$). This method is demonstrated in the next few examples.

Example 3.29. Factor the given expression.

$$\begin{aligned} x^2 + 9x + 18 &\quad \text{Need to multiply to } 18, \text{ add to } 9 \\ x^2 + 6x + 3x + 18 &\quad \text{Use } \ 6 \text{ and } 3, \text{ split the middle term} \\ x(x+6) + 3(x+6) &\quad \text{Factor by grouping} \\ (x+6)(x+3) &\quad \text{Our solution} \end{aligned}$$

Example 3.30. Factor the given expression.

$$\begin{aligned} x^2 - 4x + 3 &\quad \text{Need to multiply to } 3, \text{ add to } -4 \\ x^2 - 3x - x + 3 &\quad \text{Use } \ -3 \text{ and } -1, \text{ split the middle term} \\ x(x-3) - 1(x-3) &\quad \text{Factor by grouping} \\ (x-3)(x-1) &\quad \text{Our solution} \end{aligned}$$

Example 3.31. Factor the given expression.

$$\begin{aligned} x^2 - 8x - 20 &\quad \text{Need to multiply to } -20, \text{ add to } -8 \\ x^2 - 10x + 2x - 20 &\quad \text{Use } \ -10 \text{ and } 2, \text{ split the middle term} \\ x(x-10) + 2(x-10) &\quad \text{Factor by grouping} \\ (x-10)(x+2) &\quad \text{Our solution} \end{aligned}$$

Often when factoring we are faced with an expression containing two variables. These expressions are treated just like those containing only one variable. As in the next example, we will still use the coefficients to decide how to split the linear term.

Example 3.32. Factor the given expression.

$$\begin{aligned} a^2 - 9ab + 14b^2 &\quad \text{Need to multiply to } 14, \text{ add to } -9 \\ a^2 - 7ab - 2ab + 14b^2 &\quad \text{Use } \ -7 \text{ and } -2, \text{ split the middle term} \\ a(a-7b) - 2b(a-7b) &\quad \text{Factor by grouping} \\ (a-7b)(a-2b) &\quad \text{Our solution} \end{aligned}$$

As the past few examples has shown, it is very important to be aware of negatives in finding the right pair of numbers used to split the linear term. Consider the following example, done *incorrectly*, ignoring negative signs.

Example 3.33. Factor the given expression.

$$\begin{array}{rl} x^2+5x-6 & \text{Need to multiply to } 6, \text{ add to } 5 \\ x^2+2x+3x-6 & \text{Use } 2 \text{ and } 3, \text{ split the middle term} \\ x(x+2)+3(x-2) & \text{Factor by grouping} \\ ??? & \text{Binomials do not match!} \end{array}$$

Because we did not consider the negative sign with the constant term of -6 to find our pair of numbers, the binomials did not match and grouping was unsuccessful. Now we show factorization done correctly.

Example 3.34. Factor the given expression.

$$\begin{array}{rl} x^2+5x-6 & \text{Need to multiply to } -6, \text{ add to } 5 \\ x^2+6x-x-6 & \text{Use } 6 \text{ and } -1, \text{ split the middle term} \\ x(x+6)-1(x+6) & \text{Factor by grouping} \\ (x+6)(x-1) & \text{Our solution} \end{array}$$

At this point, one might notice a shortcut for factoring such expressions. Once we identify the two numbers that are used to split the linear term, these will be the two numbers in each of our factors! In the previous example, the numbers used to split the linear term were 6 and -1, our factors turned out to be $(x+6)(x-1)$.

This shortcut will not always work out, as we will see momentarily. We can use it, however, when we have a leading coefficient of $a=1$ for our quadratic term ax^2, which has been the case for all of the trinomials we have factored thus far. This shortcut is employed in the next few examples.

Example 3.35. Factor the given expression.

$$\begin{array}{rl} x^2-7x-18 & \text{Need to multiply to } -18, \text{ add to } -7 \\ & \text{Use } -9 \text{ and } 2, \text{ write the factors} \\ (x-9)(x+2) & \text{Our solution} \end{array}$$

Example 3.36. Factor the given expression.

$$
\begin{array}{rl}
m^2 - mn - 30n^2 & \text{Need to multiply to } -30, \text{ add to } -1 \\
& \text{Use } 5 \text{ and } -6, \text{ write the factors} \\
& \text{Do not forget second variable!} \\
(m+5n)(m-6n) & \text{Our solution}
\end{array}
$$

It is also certainly possible to have a trinomial that does not factor using the *ac*-method. If there is no combination of numbers that multiplies and adds to the correct numbers, then we say that we cannot factor the polynomial "nicely", or easily. Later on in the chapter, we will learn of some other methods and terminology for factoring quadratic expressions of this type. The next example is of a quadratic expression that is not easily factorable.

Example 3.37. Factor the given expression.

$$
\begin{array}{rl}
x^2 + 2x + 6 & \text{Need to multiply to } 6, \text{ add to } 2 \\
1 \cdot 6 \text{ and } 2 \cdot 3 & \text{Only possibilities to multiply to } 6, \text{ none add to } 2 \\
\text{Not easily factorable} & \text{Our solution}
\end{array}
$$

Later, we will discover that the quadratic expression above cannot be factored over the real numbers. In other words, there exist no real numbers r and s such that

$$x^2 + 2x + 6 = (x - r)(x - s)$$

Such expressions are said to be *irreducible over the reals*, and any factorization will require us to use *complex* numbers. Complex numbers will be discussed later on in the chapter.

When factoring any expression, it is important to not forget about first identifying a GCF of all the given terms. If all the terms in an expression have a common factor we will want to first factor out the GCF before using any other method.

Example 3.38. Factor the given expression.

$$
\begin{array}{rl}
3x^2 - 24x + 45 & \text{GCF of all terms is } 3, \text{ factor this out first} \\
3(x^2 - 8x + 15) & \text{Need to multiply to } 15, \text{ add to } -8 \\
& \text{Use } -5 \text{ and } -3, \text{ write the factors} \\
3(x-5)(x-3) & \text{Our solution}
\end{array}
$$

Again it is important to comment on the shortcut of jumping right to the factors, this only works if the leading coefficient $a = 1$. In the example above, we applied the shortcut only *after* we factored out a GCF of 3. Next, we will look at how this process changes when $a \neq 1$.

World View Note: The first person to use letters for unknown values was Francois Vieta in 1591 in France. He used vowels to represent variables, just as basic codes often use letters to encrypt a message.

3.2.4 Trinomials with Leading Coefficient $a \neq 1$

Objective: Factor trinomials using the ac-method when the coefficient of x^2 is not one.

When factoring trinomials we used the ac-method to split the middle (or linear) term and then factor by grouping. The ac-method gets its name from the general trinomial expression, $ax^2 + bx + c$, where a, b, and c are the leading coefficient, linear coefficient, and constant term, respectively.

World View Note: It was French philosopher Rene Descartes who first used letters from the beginning of the alphabet (a, b, c) to represent values we know and letters from the end (x, y, z) to represent values we don't know and must solve for.

The ac-method is named as such because we will use the product $a \cdot c$ to help find out what two numbers we will need for grouping later on. Previously, we always found two numbers whose product was equal to c, since the leading coefficient a was 1 in our expression (so $ac = 1c = c$). Now we will be working with trinomials where $a \neq 1$, so we will need to identify two numbers that multiply to ac and add to b. Aside from this adjustment, the process will be the same as before.

Example 3.39. Factor the given expression.

$$\begin{array}{rl}
3x^2 + 11x + 6 & \text{Multiply to } ac \text{ or } (3)(6) = 18, \text{ add to } 11 \\
3x^2 + 9x + 2x + 6 & \text{The numbers are } 9 \text{ and } 2, \text{ split the linear term} \\
3x(x+3) + 2(x+3) & \text{Factor by grouping} \\
(x+3)(3x+2) & \text{Our solution}
\end{array}$$

When $a = 1$, we were able to use a shortcut, using the numbers that split the linear term for our factors. The previous example illustrates an important point: the shortcut does not work when $a \neq 1$. Therefore, we must go through all the steps of grouping in order to factor the expression.

Example 3.40. Factor the given expression.

$$\begin{array}{rl}
8x^2-2x-15 & \text{Multiply to } ac \text{ or } (8)(-15)=-120, \text{ add to } -2\\
8x^2-12x+10x-15 & \text{The numbers are } -12 \text{ and } 10, \text{ split the linear term}\\
4x(2x-3)+5(2x-3) & \text{Factor by grouping}\\
(2x-3)(4x+5) & \text{Our solution}
\end{array}$$

Example 3.41. Factor the given expression.

$$\begin{array}{rl}
10x^2-27x+5 & \text{Multiply to } ac \text{ or } (10)(5)=50, \text{ add to } -27\\
10x^2-25x-2x+5 & \text{The numbers are } -25 \text{ and } -2, \text{ split the linear term}\\
5x(2x-5)-1(2x-5) & \text{Factor by grouping}\\
(2x-5)(5x-1) & \text{Our solution}
\end{array}$$

The same process will work for trinomials containing two variables.

Example 3.42. Factor the given expression.

$$\begin{array}{rl}
4x^2-xy-5y^2 & \text{Multiply to } ac \text{ or } (4)(-5)=-20, \text{ add to } -1\\
4x^2+4xy-5xy-5y^2 & \text{The numbers are } 4 \text{ and } -5, \text{ split the middle term}\\
4x(x+y)-5y(x+y) & \text{Factor by grouping}\\
(x+y)(4x-5y) & \text{Our solution}
\end{array}$$

As always, when factoring we will first look for a GCF before using any other method, including the ac-method. Factoring out the GCF first also has the added bonus of making the coefficients smaller, so other methods become easier.

Example 3.43. Factor the given expression.

$$\begin{array}{rl}
18x^3+33x^2-30x & \text{GCF is } 3x, \text{ factor this out first}\\
3x(6x^2+11x-10) & \text{Multiply to } ac \text{ or} (6)(-10)=-60, \text{ add to } 11\\
3x(6x^2+15x-4x-10) & \text{The numbers are } 15 \text{ and } -4, \text{ split the linear term}\\
3x[3x(2x+5)-2(2x+5)] & \text{Factor by grouping}\\
3x(2x+5)(3x-2) & \text{Our solution}
\end{array}$$

As was the case with trinomials when $a = 1$, not all trinomials can be factored easily. If there are no combinations that multiply and add correctly, then we can say the trinomial is not easily factorable. In such cases, the expression will require a new method of factorization, and may even be shown to be irreducible over the real numbers (the factorization will require complex numbers). We will encounter such expressions and learn how to properly handle them before the end of this chapter. We conclude this section with one such example.

Example 3.44. Factor the given expression.

$$\begin{array}{rl} 3x^2 + 2x - 7 & \text{Multiply to } ac \text{ or } (3)(-7) = -21, \text{ add to } 2 \\ -3(7) \text{ and } -7(3) & \text{Only two ways to multiply to } -21, \text{ neither adds to } 2 \\ \text{Not easily factorable} & \text{Our solution} \end{array}$$

It turns out that the previous example *is* factorable over the real numbers, but we will postpone this discovery until later.

3.3 Solving by Factoring

Objective: Solve quadratic equations by factoring and using the zero factor property.

When solving linear equations such as $2x-5=21$ we can solve for the variable directly by adding 5 and dividing by 2 to get 13. When working with quadratic equations (or higher degree polynomials), however, we cannot simply isolate the variable as we did with linear equations. One property that we can use to solve for the variable is known as the zero factor property.

$$\textbf{Zero Factor Property : If } \boldsymbol{ab=0} \textbf{ then either } \boldsymbol{a=0} \textbf{ or } \boldsymbol{b=0}.$$

The zero factor property tells us that if the product of two factors is zero, then one of the factors must be zero. We can use this property to help us solve factored polynomials as in the following example.

Example 3.45. Solve the given equation for all possible values of x.

$$\begin{array}{rl}
(2x-3)(5x+1)=0 & \text{One factor must be zero} \\
2x-3=0 \text{ or } 5x+1=0 & \text{Set each factor equal to zero} \\
\underline{\mathbf{+3 \quad +3}} \qquad \underline{\mathbf{-1 \quad -1}} & \text{Solve each equation} \\
2x=3 \quad \text{or} \quad 5x=-1 & \\
\overline{\mathbf{2}} \quad \overline{\mathbf{2}} \qquad \overline{\mathbf{5}} \quad \overline{\mathbf{5}} & \\
x=\dfrac{3}{2} \text{ or } -\dfrac{1}{5} & \text{Our solution}
\end{array}$$

For the zero factor property to work we must have factors to set equal to zero. This means if an expression is not already factored, we must first factor it.

Example 3.46. Solve the given equation for all possible values of x.

$$\begin{array}{rl}
4x^2+x-3=0 & \text{Factor using the } ac\text{-method,} \\
 & \quad \text{multiply to } -12, \text{ add to } 1 \\
4x^2-3x+4x-3=0 & \text{The numbers are } -3 \text{ and } 4, \text{ split the linear term} \\
x(4x-3)+1(4x-3)=0 & \text{Factor by grouping}
\end{array}$$

$$\begin{array}{rl}
(4x-3)(x+1)=0 & \text{One factor must be zero} \\
4x-3=0 \text{ or } x+1=0 & \text{Set each factor equal to zero} \\
\underline{\mathbf{+3 \quad +3}} \qquad \underline{\mathbf{-1 \quad -1}} & \text{Solve each equation} \\
4x=3 \quad \text{or} \quad x=-1 & \\
\overline{\mathbf{4}} \quad \overline{\mathbf{4}} & \\
x=\frac{3}{4} \text{ or } -1 & \text{Our solution}
\end{array}$$

Another important aspect of the zero factor property is that before we factor, our equation must equal zero. If it does not, we must move terms around so it does equal zero. Although it is not necessary, it will generally be easier to keep our leading term ax^2 positive.

Example 3.47. Solve the given equation for all possible values of x.

$$\begin{array}{rl}
x^2=8x-15 & \text{Set equal to zero by moving terms to the left} \\
x^2-8x+15=0 & \text{Factor using the } ac\text{-method,} \\
 & \quad \text{multiply to 15, add to } -8 \\
(x-5)(x-3)=0 & \text{The numbers are } -5 \text{ and } -3 \\
x-5=0 \text{ or } x-3=0 & \text{Set each factor equal to zero} \\
\underline{\mathbf{+5 \quad +5}} \qquad \underline{\mathbf{+3 \quad +3}} & \text{Solve each equation} \\
x=5 \text{ or } 3 & \text{Our solution}
\end{array}$$

Example 3.48. Solve the given equation for all possible values of x.

$$\begin{array}{rl}
(x-7)(x+3)=-9 & \text{Not equal to zero, multiply first} \\
x^2-7x+3x-21=-9 & \text{Combine like terms} \\
x^2-4x-21=-9 & \text{Move } -9 \text{ to other side so equation equals zero} \\
\underline{\mathbf{+9 \quad +9}} & \\
x^2-4x-12=0 & \text{Factor using the } ac\text{-method,} \\
 & \quad \text{multiply to } -12, \text{ add to } -4 \\
(x-6)(x+2)=0 & \text{The numbers are 6 and } -2 \\
x-6=0 \text{ or } x+2=0 & \text{Set each factor equal to zero} \\
\underline{\mathbf{+6 \quad +6}} \qquad \underline{\mathbf{-2 \quad -2}} & \text{Solve each equation} \\
x=6 \text{ or } -2 & \text{Our solution}
\end{array}$$

Example 3.49. Solve the given equation for all possible values of x.

$$
\begin{array}{rl}
3x^2+4x-5=7x^2+4x-14 & \text{Set equal to zero by} \\
 & \quad\text{moving terms to the right} \\
0=4x^2-9 & \text{Factor using difference of squares} \\
0=(2x+3)(2x-3) & \text{One factor must be zero} \\
2x+3=0 \;\text{ or }\; 2x-3=0 & \text{Set each factor equal to zero} \\
\underline{\mathbf{-3\ \ -3}} \qquad \underline{\mathbf{+3\ \ +3}} & \text{Solve each equation} \\
2x=-3 \quad\text{or}\quad 2x=3 & \\
\overline{\mathbf{2}} \quad \overline{\mathbf{2}} \qquad \overline{\mathbf{2}} \quad \overline{\mathbf{2}} & \\
x=-\dfrac{3}{2} \;\text{ or }\; \dfrac{3}{2} & \text{Our solution}
\end{array}
$$

Most quadratic equations will have two unique real solutions. It is possible, however, to have only one real solution as the next example illustrates.

Example 3.50. Solve the given equation for all possible values of x.

$$
\begin{array}{rl}
4x^2=12x-9 & \text{Set equal to zero by moving terms to left} \\
4x^2-12x+9=0 & \text{Factor using the } ac\text{-method,} \\
 & \quad\text{multiply to 36, add to } -12 \\
4x^2-6x-6x+9=0 & \text{Use } -6 \text{ and } -6, \text{ split the linear term} \\
2x(2x-3)-3(2x-3)=0 & \text{Factor by grouping} \\
(2x-3)^2=0 & \text{A perfect square!} \\
2x-3=0 & \text{Set this factor equal to zero} \\
\underline{\mathbf{+3\ \ +3}} & \text{Solve the equation} \\
2x=3 & \\
\overline{\mathbf{2}} \quad \overline{\mathbf{2}} & \\
x=\dfrac{3}{2} & \text{Our solution}
\end{array}
$$

As always, it will be important to factor out the GCF first if we have one. This GCF is also a factor, and therefore must also be set equal to zero using the zero factor property. The next example illustrates this.

Example 3.51. Solve the given equation for all possible values of x.

$$
\begin{array}{rl}
4x^2 = 8x & \text{Set equal to zero by moving the terms to left} \\
 & \text{Be careful, } 4x^2 \text{ and } 8x \text{ are not like terms!} \\
4x^2 - 8x = 0 & \text{Factor out the GCF of } 4x \\
4x(x-2) = 0 & \text{One factor must be zero} \\
4x = 0 \text{ or } x - 2 = 0 & \text{Set each factor equal to zero} \\
\overline{\mathbf{4}} \quad \overline{\mathbf{4}} \qquad \underline{\mathbf{+2} \quad \mathbf{+2}} & \text{Solve each equation} \\
x = 0 \text{ or } 2 & \text{Our solution}
\end{array}
$$

If our polynomial is not a quadratic, as in the next example, we may end up with more than two solutions.

Example 3.52. Solve the given equation for all possible values of x.

$$
\begin{array}{rl}
2x^3 - 14x^2 + 24x = 0 & \text{Factor out the GCF of } 2x \\
2x(x^2 - 7x + 12) = 0 & \text{Factor with } ac\text{-method,} \\
 & \quad \text{multiply to } 12, \text{ add to } -7 \\
2x(x-3)(x-4) = 0 & \text{The numbers are } -3 \text{ and } -4 \\
2x = 0 \text{ or } x - 3 = 0 \text{ or } x - 4 = 0 & \text{Set each factor equal to zero} \\
\overline{\mathbf{2}} \quad \overline{\mathbf{2}} \qquad \underline{\mathbf{+3} \quad \mathbf{+3}} \qquad \underline{\mathbf{+4} \quad \mathbf{+4}} & \text{Solve each equation} \\
x = 0 \quad \text{or} \quad 3 \quad \text{or} \quad 4 & \text{Our solution}
\end{array}
$$

Example 3.53. Solve the given equation for all possible values of x.

$$
\begin{array}{rl}
6x^2 + 21x - 27 = 0 & \text{Factor out the GCF of } 3 \\
3(2x^2 + 7x - 9) = 0 & \text{Factor with } ac\text{-method,} \\
 & \quad \text{multiply to } -18, \text{ add to } 7 \\
3(2x^2 + 9x - 2x - 9) = 0 & \text{The numbers are } 9 \text{ and } -2 \\
3[x(2x+9) - 1(2x+9)] = 0 & \text{Factor by grouping} \\
3(2x+9)(x-1) = 0 & \text{One factor must be zero}
\end{array}
$$

$$\begin{array}{llll}
3=0 \text{ or } 2x+9=0 \text{ or } x-1=0 & & & \text{Set each factor equal to zero} \\
\mathbf{3 \neq 0} \qquad \underline{\mathbf{-9 \quad -9}} \qquad \underline{\mathbf{+1 \quad +1}} & & & \text{Solve each equation} \\
\qquad 2x=-9 \quad \text{or} \quad x=1 & & & \\
\qquad \overline{\mathbf{2}} \quad\ \ \overline{\mathbf{2}} & & & \\
\qquad\qquad x=-\dfrac{9}{2} \text{ or } 1 & & & \text{Our solution}
\end{array}$$

In the previous example, the GCF did not have a variable in it. When we set this factor equal to zero we got a false statement. No solutions come from this factor. We can only disregard setting the GCF factor equal to zero if it is a constant.

Just as not all polynomials can be easily factored, all equations cannot be easily solved by factoring. If an equation does not factor easily, we will have to solve it using another method. These other methods are saved for another section.

World View Note: While factoring works great to solve quadratic equations, Tartaglia, in 16^{th} century Italy, developed a method to solve cubic equations. He kept his method a secret until another mathematician, Cardan, talked him out of his secret and published the results. To this day the formula is known as Cardan's Formula.

It is a common question to ask if it is permissible to get rid of the square on the variable x^2 by taking the square root of both sides of the equation. Although it is sometimes possible, there are a few properties of square roots that we have not covered yet, and thus it is more common to inadvertently break a rule of roots that we may not yet be aware of. Because of this, we will postpone a discussion of roots until we see how they can be employed properly to solve quadratic equations. For now, we will advise to **never** take the square root of both sides of an equation!

3.4 Vertex Form and Graphing

3.4.1 Vertex Form

Objective: Express a quadratic equation in vertex form.

Recall the two forms of a quadratic equation, shown below. In both forms, assume $a \neq 0$.

Standard Form: $y = ax^2 + bx + c$, where a, b, and c are real numbers

Vertex Form: $y = a(x - h)^2 + k$, where a, h, and k are real numbers

Unlike the standard form, a quadratic equation written in vertex form allows for immediate recognition of the vertex (h, k), which will always coincide with either a maximum (if $a < 0$) or a minimum (if $a > 0$) on the accompanying graph, called a parabola. Additionally, using the vertex form, we can easily identify the *axis of symmetry* for the parabola, which is a vertical line $x = h$ that passes through the x-coordinate of the vertex and "splits" the graph into two identical halves.

When graphing parabolas, it will help to think of the axis of symmetry as a vertical line over which either half of the graph could be "folded", to produce the other half. This will allow us to reflect (by symmetry) any point on the parabola to the other side of the axis of symmetry, and identify another point on the graph. As a result, both points will have the same y-coordinate, and will be (horizontally) equidistant from the axis of symmetry. By reflecting points about the axis of symmetry, we can graph not just one, but two points on the graph, for every single value of x that we plug into the given equation.

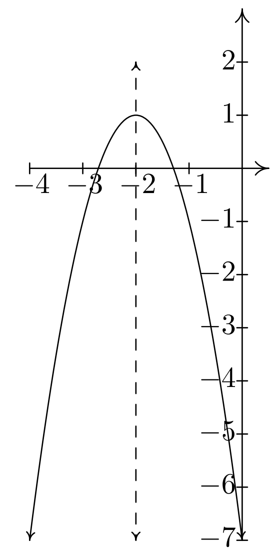
Example 3.54. Consider $y = -2(x + 2)^2 + 1$.

In this example we can see immediately that the vertex is at $(-2, 1)$. It is important that we not overlook the negative value for h. The axis of symmetry, passes through the x-coordinate for the vertex, $x = -2$.

Now to find more points on the parabola we can plug in $x = -1$. We can see that $y = -2(-1 + 2)^2 + 1 = -1$, so $(-1, -1)$ is a point on our parabola.

Since the point we just located sits one unit to the right of the axis of symmetry, we also know that the point $(-3, -1)$, sitting one to the left of the axis of symmetry will also be a point on our graph. We can always check this by plugging $x = -3$ into the equation and solving for y.

In a similar fashion we can plug in $x = 0$, a coordinate that is two units to the right of the axis of symmetry and get a y-coordinate of -7. Thus an x-coordinate two units left of the axis, $x = -4$, will also yield a y-coordinate of -7. The accompanying graph shows our parabola, with the axis of symmetry appearing as a dashed vertical line at $x = -2$.



We began the discussion of vertex form in the introductory section of this chapter. It follows naturally to learn how to transform a quadratic equation that is given in standard form into one written in vertex form.

If $y = ax^2 + bx + c$ $(a \neq 0)$, we can identify the x-coordinate for the vertex (and consequently the equation for the axis of symmetry) using the following formula.

$$h = -\frac{b}{2a}$$

After identifying h, we can determine based upon the sign of the leading coefficient a whether the vertex will be a maximum (if a is negative, $a < 0$) or a minimum (if a is positive, $a > 0$). The equation for the vertical line $x = h$ will be our axis of symmetry.

Finally, we know that the y-coordinate for our vertex must occur somewhere on the axis of symmetry. This can easily be found by plugging $x = h$ back into the given equation for our quadratic, and simplifying to find the y-coordinate, which we will relabel as k.

Once we have h and k, we can use them, along with a, to write the vertex form for our quadratic,

$$y = a(x-h)^2 + k.$$

The following examples will clearly demonstrate this process.

Example 3.55. Identify the vertex and axis of symmetry for the parabola represented by the given quadratic equation.

$$\begin{aligned}
y &= x^2 + 8x - 12 && \text{Given an equation in standard form} \\
a = 1, \quad b = 8, \quad c &= -12 && \text{Identify } a, b, \text{ and } c \\
h = -\frac{b}{2a} = -\frac{8}{2(1)} &= -4 && \text{Identify } h \\
x &= -4 && \text{Use } h \text{ for axis of symmetry, a vertical line} \\
k = (-4)^2 + 8(-4) &- 12 && \text{Plug in } h \text{ to find } k \\
k = 16 - 32 - 12 &= -28 && \text{Simplify} \\
(-4, &-28) && \text{Write the vertex as an ordered pair } (h, k)
\end{aligned}$$

Example 3.56. Identify the vertex and axis of symmetry for the parabola represented by the given quadratic equation.

$$\begin{aligned}
y &= -3x^2 + 6x - 1 && \text{Given an equation in standard form} \\
a = -3, \quad b = 6, \quad c &= -1 && \text{Identify } a, b, \text{ and } c \\
h = -\frac{b}{2a} = -\frac{6}{2(-3)} &= 1 && \text{Identify } h \\
x &= 1 && \text{Use } h \text{ for axis of symmetry, a vertical line} \\
k = -3(1)^2 + 6(1) &- 1 && \text{Plug in } h \text{ to find } k \\
k = -3 + 6 - 1 &= 2 && \text{Simplify} \\
(1, &2) && \text{Write the vertex as an ordered pair } (h, k)
\end{aligned}$$

Example 3.57. Identify the vertex and axis of symmetry for the parabola represented by the given quadratic equation.

$$\begin{aligned}
y &= -x^2 - 12 && \text{Given an equation in standard form} \\
a = -1, \quad b = 0, \quad c &= -12 && \text{Identify } a, b, \text{ and } c \\
h = -\frac{b}{2a} = -\frac{0}{2(-1)} &= 0 && \text{Identify } h
\end{aligned}$$

$$
\begin{aligned}
x &= 0 && \text{Use } h \text{ for axis of symmetry, a vertical line} \\
k &= -(0)^2 - 12 && \text{Plug in } h \text{ to find } k \\
k &= 0 - 12 = -12 && \text{Simplify} \\
&(0, -12) && \text{Write the vertex as an ordered pair } (h, k)
\end{aligned}
$$

There is a more algebraic (and complicated) method of transforming a quadratic equation given in standard form into one that is in vertex form, known as *completing the square*. This method will be explained in detail towards the end of the chapter.

We will also see how the vertex form can be particularly useful when solving a quadratic equation, in order to identify the x-intercepts of the corresponding parabola. Solving a quadratic equation using the vertex form is know as the method of *extracting square roots*, and will be seen once we have had a thorough discussion of square roots, as well as complex numbers.

3.4.2 Graphing Quadratics

Objective: Graph quadratic equations using the vertex, x-intercepts, and y-intercept.

Up until now, we have discussed the general shape of the graph of a quadratic equation (known as a *parabola*), but have only seen a few examples. Furthermore, most of our examples have only identified the vertex of the parabola, and perhaps an $x-$ or $y-$intercept of the graph. Although these examples have been able to show us the general shape of each graph (where it is centered, whether it opens up or down, whether it is narrow or wide), our steps for obtaining each graph have not followed a standard procedure. Here, we will define that procedure more precisely, and provide a few examples for reinforcement.

One way that we can always build a picture of the general shape of a graph is to make a table of values, as we will do in our first example.

Example 3.58. Sketch a graph of the quadratic equation $y = x^2 - 4x + 3$ by making a table of values and plotting points on the graph.

We will test five values to get an idea of the shape of the graph.

x	0	1	2	3	4
y					

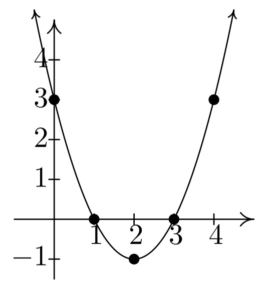
$$
\begin{aligned}
y = (0)^2 + 4(0) + 3 = 0 - 0 + 3 &= 3 && \text{Plug in 0 for } x \text{ and evaluate.}\\
y = (1)^2 - 4(1) + 3 = 1 - 4 + 3 &= 0 && \text{Plug in 1 for } x \text{ and evaluate.}\\
y = (2)^2 - 4(2) + 3 = 4 - 8 + 3 &= -1 && \text{Plug in 2 for } x \text{ and evaluate.}\\
y = (3)^2 - 4(3) + 3 = 9 - 12 + 3 &= 0 && \text{Plug in 3 for } x \text{ and evaluate.}\\
y = (4)^2 - 4(4) + 3 = 16 - 16 + 3 &= 3 && \text{Plug in 4 for } x \text{ and evaluate.}
\end{aligned}
$$

Our completed table is below. Plot the points on the $xy-$plane.

x	0	1	2	3	4
y	3	0	-1	0	3

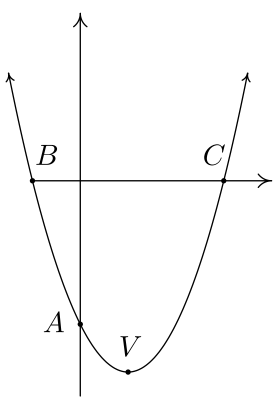
Plot the points $(0,3),(1,0),(2,-1),(3,0)$, and $(4,3)$.

Connect the dots with a smooth curve.



World View Note: The first major female mathematician was Hypatia of Egypt who was born around 370 A.D.. She studied conic sections. The parabola is one type of conic section.

The above method to graph a parabola works for any equation, however, it can be very difficult to find a sufficient collection of points in order to identify the overall shape of the complete graph. For this reason, we will now formally identify several key points on the graph of a parabola, which will enable us to always determine a complete graph. These points are the $y-$intercept, $x-$intercepts, and the vertex (h,k).



Point A: y-intercept; where the graph crosses the vertical y-axis (when $x=0$).

Points B and C: x-intercepts; where the graph crosses the horizontal x-axis (when $y=0$)

Point V: vertex (h,k); The point of the minimum (or maximum) value, where the graph changes direction.

We will use the following method to find each of the key points on our parabola.

Steps for graphing a quadratic in standard form, $y = ax^2 + bx + c$.

1. Identify and plot the vertex: $h = -\dfrac{b}{2a}$. Plug h into the equation to find k. Resulting point is (h, k).

2. Identify and plot the y-intercept: Set $x = 0$ and solve. The y-intercept will correspond to the constant term c. Resulting point is $(0, c)$.

3. Identify and plot the x-intercept(s): Set $y = 0$ and solve for x. Depending on the expression, we will end up with zero, one or two $x-$intercepts.

   Important: Up until now, we have only discussed how to solve a quadratic equation for x by factoring. If an expression is not easily factorable, we may not be able to identify the x-intercepts. Soon, we will learn of two additional methods for finding x-intercepts, which will prove especially useful, when an equation is not easily factorable.

After ploting these points we can connect them with a smooth curve to find a complete sketch of our parabola!

Example 3.59. Provide a complete sketch of the equation $y = x^2+4x+3$.

$$\begin{aligned}
y &= x^2 + 4x + 3 && \text{Find the key points}\\
h &= -\frac{4}{2(1)} = -\frac{4}{2} = -2 && \text{To find the vertex, use } h = -\frac{b}{2a}\\
k &= (-2)^2 + 4(-2) + 3 && \text{Plug } h \text{ into the equation to find } k\\
k &= 4 - 8 + 3 && \text{Evaluate}\\
k &= -1 && \text{The } y\text{-coordinate of the vertex}\\
&(-2, -1) && \text{Vertex as a point}\\
y &= 3 && (0, c) \text{ is the } y\text{-intercept}
\end{aligned}$$

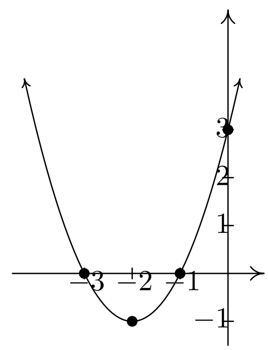
$$
\begin{array}{rl}
0 = x^2 + 4x + 3 & \text{To find the } x\text{-intercept we solve the equation} \\
0 = (x+3)(x+1) & \text{Factor} \\
x + 3 = 0 \text{ and } x + 1 = 0 & \text{Set each factor equal to zero} \\
\underline{\mathbf{-3 \quad -3}} \qquad \underline{\mathbf{-1 \quad -1}} & \text{Solve each equation} \\
x = -3 \quad \text{and} \quad x = -1 & \text{Our } x\text{-intercepts}
\end{array}
$$

Graph the y-intercept at $(0, 3)$,

the x-intercepts at $(-3, 0)$ and $(-1, 0)$,

and the vertex at $(-2, -1)$.

Connect the dots with a smooth curve in a 'U'-shape to get our parabola.

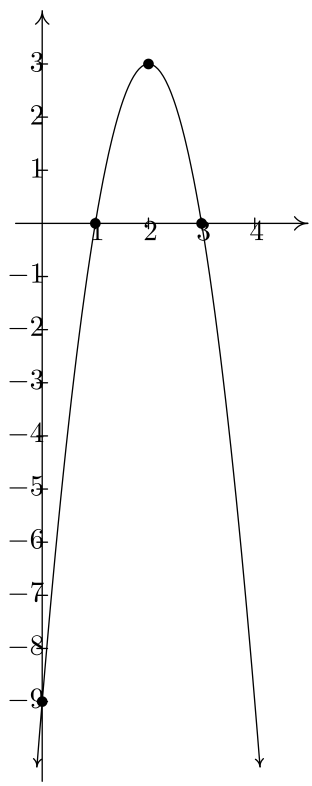


Remember that if $a > 0$, then our parabola will open upwards, as in the previous example. In our next example, $a < 0$, and the resulting parabola will open downwards.

Example 3.60. Provide a complete sketch of the equation $y = -3x^2 + 12x - 9$.

$$
\begin{array}{rl}
y = -3x^2 + 12x - 9 & \text{Find key points} \\
h = -\dfrac{12}{2(-3)} = -\dfrac{12}{-6} = 2 & \text{To find the vertex, use } h = -\dfrac{b}{2a} \\
k = -3(2)^2 + 12(2) - 9 & \text{Plug } h \text{ into the equation to find } k \\
k = -3(4) + 24 - 9 & \text{Evaluate} \\
k = 3 & \text{The } y\text{-coordinate of the vertex} \\
(2, 3) & \text{Vertex as a point} \\
\\
y = -9 & (0, c) \text{ is the } y\text{-intercept}
\end{array}
$$

$$
\begin{array}{rl}
0 = -3x^2 + 12x - 9 & \text{To find the } x\text{-intercept we solve the equation} \\
0 = -3(x^2 - 4x + 3) & \text{Factor out GCF first} \\
0 = -3(x - 3)(x - 1) & \text{Factor remaining trinomial} \\
x - 3 = 0 \text{ and } x - 1 = 0 & \text{Set each factor with a variable equal to zero} \\
\underline{\mathbf{+3 \quad +3}} \qquad \underline{\mathbf{+1 \quad +1}} & \text{Solve each equation} \\
x = 3 \quad \text{and} \quad x = 1 & \text{Our } x\text{-intercepts}
\end{array}
$$



Graph the y-intercept at $(0, -9)$,

the x-intercepts at $(3, 0)$ and $(1, 0)$,

and the vertex at $(2, 3)$.

Connect the dots with a smooth curve in a 'U'-shape to get our parabola.

Remember that the graph of any quadratic is a parabola with the same 'U'-shape (opening up or down). If we plot our points and we cannot connect them in the correct 'U'-shape, then one or more of our points is likely to be incorrect. If this happens, a simple check of our calculations should identify where any mistakes were made! Each of our examples have involved quadratics that were easily factorable. Although we can still graph quadratics such as $y = x^2 - 3$ without actually identifying the $x-$intercepts, being able to identify them by solving $x^2 - 3 = 0$ and other more involved quadratic equations for x is a skill that we will eventually come to master.

3.5 Square Roots and the Imaginary Number i

3.5.1 Square Roots

Objective: Simplify expressions with square roots.

Recall that we define a radical (or n^{th} root) as follows.

$$\sqrt[n]{a} = a^{1/n},$$

where a is a nonnegative real number and n a positive integer.

We refer to n as the *index* of the radical and a as the *radicand*. Square roots (when $n = 2$) are the most common type of radical used in mathematics. A square root "un-squares" a number. In other words, if $a^2 = b$, then $\sqrt[2]{b} = a$. This relationship between a square and a square root is similar to the relationship between multiplication and division, as well as the relationship between addition and subtraction. In each case, the two operations are said to be *inverse* operations of each other. The idea behind inverses and the notion of an inverse function is one that will be discussed in detail in a later chapter.

Note that although we have written the index of 2 for the square root of b in the previous paragraph, in general, the index of a square root is usually omitted ($\sqrt[2]{b} = \sqrt{b}$). Using numbers, since $5^2 = 25$ we say the square root of 25 is 5, and write $\sqrt{25} = 5$.

While a great deal more could be said about radicals and how they fit in with the properties of exponents, for now we will focus our attention on properly working with expressions that contain a square root.

World View Note: The radical sign, when first used was an R with a line through the tail, similar to our prescription symbol today. The R came from the Latin, "radix", which can be translated as "source" or "foundation". It wasn't until the 16^{th} century that our current symbol was first used in Germany. Even then, it was just a check mark with no bar over the numbers!

The following example gives several square roots.

Example 3.61.

$\sqrt{0}=0$	$\sqrt{121}=11$
$\sqrt{1}=1$	$\sqrt{625}=25$
$\sqrt{4}=2$	$\sqrt{-81}=$ Undefined

The final example of $\sqrt{-81}$ is currently considered to be undefined, since the square root of a negative number does not equal a real number. This is because if we square a positive or a negative number, the answer will be positive, not to mention that $0^2=0$. Thus we can only take square roots of nonnegative numbers (positive numbers or zero). In the second part of this section, we will define a method we can use to work with and evaluate negative square roots. For now we will simply say they are undefined.

Not all numbers have a "nice" (or *rational*) square root. For example, if we found $\sqrt{8}$ on our calculator, the answer would be 2.828427124746190097..., and even this number is a rounded approximation of the square root. To be as accurate as possible, we will never use the calculator to find decimal approximations of square roots. Instead we will express roots in simplest radical form. We will do this using a property known as the product rule of radicals (in this case, square roots).

$$\textbf{Product Rule of Square Roots : } \boldsymbol{\sqrt{a\cdot b}=\sqrt{a}\cdot\sqrt{b}}$$

More generally,

$$\textbf{Product Rule of Radicals : } \boldsymbol{\sqrt[n]{a\cdot b}=\sqrt[n]{a}\cdot\sqrt[n]{b}}$$

We can use the product rule of square roots to simplify an expression such as $\sqrt{180}=\sqrt{36\cdot 5}$ by splitting it into two roots, $\sqrt{36}\cdot\sqrt{5}$, and simplifying the first root, $6\sqrt{5}$. The trick in this process is being able to recognize that an expression like $\sqrt{180}$ may be rewritten as $\sqrt{36\cdot 5}$, since $180=36\cdot 5$. In the case of $\sqrt{8}$, we may write $\sqrt{8}=\sqrt{4}\cdot\sqrt{2}=2\sqrt{2}$.

There are several ways of applying the product rule of square roots. The most common and, with a bit of practice, fastest method is to find perfect squares that divide nicely into the radicand (the number under the radical). This is demonstrated in the next example.

Example 3.62. Completely simplify the given radical.

$$\begin{array}{rl}
\sqrt{75} & \text{75 is divisible by 25, a perfect square} \\
\sqrt{25\cdot 3} & \text{Split into factors} \\
\sqrt{25}\cdot\sqrt{3} & \text{Product rule, take the square root of 25} \\
5\sqrt{3} & \text{Our solution}
\end{array}$$

If there is a coefficient in front of the radical to begin with, the problem merely becomes a big multiplication problem, as seen in the next example.

Example 3.63. Completely simplify the given radical.

$$\begin{array}{rl}
5\sqrt{63} & \text{63 is divisible by 9, a perfect square} \\
5\sqrt{9\cdot 7} & \text{Split into factors} \\
5\sqrt{9}\cdot\sqrt{7} & \text{Product rule, take the square root of 9} \\
5\cdot 3\sqrt{7} & \text{Multiply coefficients} \\
15\sqrt{7} & \text{Our solution}
\end{array}$$

As we simplify radicals using this method it is important to be sure our final answer can not be simplified further, as seen in the next example.

Example 3.64. Completely simplify the given radical.

$$\begin{array}{rl}
\sqrt{72} & \text{72 is divisible by 9, a perfect square} \\
\sqrt{9\cdot 8} & \text{Split into factors} \\
\sqrt{9}\cdot\sqrt{8} & \text{Product rule, take the square root of 9} \\
3\sqrt{8} & \text{But 8 is also divisible by a perfect square, 4} \\
3\sqrt{4\cdot 2} & \text{Split into factors} \\
3\sqrt{4}\cdot\sqrt{2} & \text{Product rule, take the square root of 4} \\
3\cdot 2\sqrt{2} & \text{Multiply} \\
6\sqrt{2} & \text{Our solution}
\end{array}$$

The previous example could also have been done in fewer steps if we had noticed that $72 = 36\cdot 2$, but often it can take longer to discover the larger perfect square than to simplify in several steps.

Variables often are part of the radicand as well. When taking the square roots of one (or more) variable(s), we can divide the associated exponent of the variable by two, and write the new exponent outside of the root. For example, $\sqrt{x^{10}} = x^5$. This follows from a familiar property of exponents, shown below.

$$(x^m)^n = x^{mn}$$

Applying this to a square root, we have

$$\sqrt{x^m} = (x^m)^{1/2} = x^{m/2}.$$

So, $\sqrt{x^{10}} = x^{10/2} = x^5$. This makes sense, since

$$\begin{aligned}
\left(x^5\right)^2 &= x^5 \cdot x^5 \\
&= \underbrace{x \cdot x \cdot \ldots \cdot x}_{\text{10 times}} \\
&= x^{10} \\
&= x^{5 \cdot 2}.
\end{aligned}$$

In summary, when squaring, we multiply the exponent by two. So, when taking a square root, we divide the exponent by two. The following example demonstrates this property.

Example 3.65. Completely simplify the given radical.

$-5\sqrt{18x^4y^6z^{10}}$	18 is divisible by 9, a perfect square
$-5\sqrt{9 \cdot 2x^4y^6z^{10}}$	Split into factors
$-5\sqrt{9} \cdot \sqrt{2} \cdot \sqrt{x^4} \cdot \sqrt{y^6} \cdot \sqrt{z^{10}}$	Product rule applied to all parts
$-5 \cdot 3x^2y^3z^5\sqrt{2}$	Simplify roots, divide exponents by 2
$-15x^2y^3z^5\sqrt{2}$	Multiply coefficients, Our solution

We can't always nicely divide the exponent on a variable by two, since sometimes we will have a positive remainder. If there is a positive remainder, this means the remainder is left inside the radical, and the whole number portion (or quotient) represents the exponent that should appear outside of the radical. The next example demonstrates this.

Example 3.66. Completely simplify the given radical.

$$\sqrt{20x^5y^9z^6} \qquad \text{20 is divisible by 4, a perfect square}$$

$$\sqrt{4\cdot 5x^5y^9z^6} \qquad \text{Split into factors}$$

$$\sqrt{4}\cdot\sqrt{5}\cdot\sqrt{x^5}\cdot\sqrt{y^9}\cdot\sqrt{z^6} \qquad \text{Simplify, divide exponents by 2}$$

Remainder is left inside

$$2x^2y^4z^3\sqrt{5xy} \qquad \text{Our solution}$$

If we focus on the variable y in the previous example, when we divide the exponent 9 by 2, we get a quotient of 4 and a remainder of 1 ($9 = 2\cdot 4+1$). Consequently, $\sqrt{y^9} = y^4\sqrt{y}$. This same idea also applies to x above, since the exponent 5 is odd and therefore will have a remainder of 1. Since the exponent for z is even, it is divisible by 2, and so the radical in our final answer does not contain z.

3.5.2 Introduction to Complex Numbers

Objective: Add, subtract, multiply, rationalize, and simplify expressions using complex numbers.

World View Note: Initially, the primary purpose for mathematics was counting. Consequently, concepts related to negatives, zero, fractions or irrational numbers did not initially accompany the establishment of many early number systems. The ancient Egyptians, however, quickly developed the need for a "part", and so they developed a new type of number, the ratio or fraction. The Ancient Greeks did not believe in irrational numbers (people were killed for believing otherwise). The Mayans of Central America later realized the number zero when they found use for it as a placeholder. Ancient Chinese mathematicians also established negative numbers.

In mathematics, when the current number system does not provide the tools to solve the problems the culture is working with, we tend to develop new ways for solving the problem. Throughout history, this has been the case with the need for a number that represents nothing (0), smaller than zero (negatives), between integers (fractions), and between fractions (irrational numbers). This is also the case for square roots of negative numbers. To work with the square root of a negative number, mathematicians have defined what we now know as imaginary and complex numbers.

$$\textbf{Imaginary Number } \boldsymbol{i}: \ \boldsymbol{i^2 = -1} \ (\textbf{thus } \boldsymbol{i = \sqrt{-1}})$$

Examples of imaginary numbers include $3i, -6i, \frac{3}{5}i$ and $3i\sqrt{5}$. A *complex number* is one that contains both a real and imaginary part, such as $2+5i$.

Complex Number: $\boldsymbol{a+bi},$ where $\boldsymbol{a}$ and $\boldsymbol{b}$ are real numbers, $\boldsymbol{i = \sqrt{-1}}$

With this definition, the square root of a negative number will no longer be considered undefined. We now will be able to perform basic operations with the square root of a negative number. First we will consider powers of imaginary numbers. We will do this by manipulating our definition of $i^2 = -1$. If we multiply both sides of the definition by i, the equation becomes $i^3 = -i$. Then if we multiply both sides of the equation again by i, the equation becomes $i^4 = -i^2 = -(-1) = 1$, or simply $i^4 = 1$. Multiplying again by i gives $i^5 = i$. One more time gives $i^6 = i^2 = -1$.

This pattern continues, and we can see a cycle forming. Specifically, as the exponents on i increase, our simplified value for i^n will cycle through the simplified values for i, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$. As there are 4 different possible answers in this cycle, if we divide the exponent n by 4 and consider the remainder, we can easily simplify any power of i by knowing the following four values:

Cyclic Property of Powers of i

$$\begin{aligned} i^0 &= 1 \\ i^1 &= i \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 = i^0 &= 1 \end{aligned}$$

Example 3.67. Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl} i^{35} & \text{Divide exponent by 4} \\ 8R3 & \text{Use remainder as exponent for } i \\ i^3 & \text{Simplify} \\ -i & \text{Our solution} \end{array}$$

Example 3.68. Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl} i^{124} & \text{Divide exponent by 4} \\ 31R0 & \text{Use remainder as exponent for } i \\ i^0 & \text{Simplify} \\ 1 & \text{Our solution} \end{array}$$

When performing the basic mathematical operations (addition, subtraction, multiplication, division) we may treat i just like any other variable. This means that when adding and subtracting complex numbers we may simply combine like terms.

Example 3.69. Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl} (2+5i)+(4-7i) & \text{Combine like terms, } 2+4 \text{ and } 5i-7i \\ 6-2i & \text{Our solution} \end{array}$$

It is important to recognize what operation we are applying. A common mistake in the previous example is to view the parentheses and think that one must distribute. The previous example, however, requires addition. So we simply add (or combine) the like terms.

For problems involving subtraction the idea is the same, but we must first remember to distribute the negative to each term in the parentheses.

Example 3.70. Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{aligned} (4 - 8i) - (3 - 5i) &\quad \text{Distribute the negative} \\ 4 - 8i - 3 + 5i &\quad \text{Combine like terms, } 4 - 3 \text{ and } -8i + 5i \\ 1 - 3i &\quad \text{Our solution} \end{aligned}$$

Addition and subtraction may also appear in a single problem.

Example 3.71. Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{aligned} (5i) - (3 + 8i) + (-4 + 7i) &\quad \text{Distribute the negative} \\ 5i - 3 - 8i - 4 + 7i &\quad \text{Combine like terms, } 5i - 8i + 7i \text{ and } -3 - 4 \\ -7 + 4i &\quad \text{Our solution} \end{aligned}$$

Multiplying two (or more) complex numbers is similar to the multiplication of two binomials with one key exception. In each problem, we will want to simplify our final answer so that it does not contain any power of i greater than or equal to 2. This will always enable us to write our answer in the standard form of $a + bi$. We now show this in general below, remembering that $i^2 = -1$.

$$\begin{aligned} (c + di)(g + hi) &\quad \text{Expand} \\ cg + chi + dgi + dhi^2 &\quad \text{Simplify, } i^2 = -1 \\ cg + chi + dgi - dh &\quad \text{Combine like terms} \\ (cg - dh) + (ch + dg)i &\quad \text{Our solution, in standard form} \end{aligned}$$

Here, $cg - dh$ represents the real part a and $ch + dg$ represents the imaginary part b of our resulting complex number $a + bi$.

Next we will see several examples to reinforce the concept. We will begin with the product of two imaginary numbers.

Example 3.72. Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl} (3i)(7i) & \text{Multiply, } 3\cdot 7 \text{ and } i \cdot i \\ 21i^2 & \text{Simplify, } i^2 = -1 \\ 21(-1) & \text{Multiply} \\ -21 & \text{Our solution} \end{array}$$

Example 3.73. Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl} 5i(3i-7) & \text{Distribute} \\ 15i^2 - 35i & \text{Simplify, } i^2 = -1 \\ 15(-1) - 35i & \text{Multiply} \\ -15 - 35i & \text{Our solution} \end{array}$$

Example 3.74. Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl} (2-4i)(3+5i) & \text{Expand} \\ 6 + 10i - 12i - 20i^2 & \text{Simplify, } i^2 = -1 \\ 6 + 10i - 12i - 20(-1) & \text{Multiply} \\ 6 + 10i - 12i + 20 & \text{Combine like terms } 6+20 \text{ and } 10i - 12i \\ 26 - 2i & \text{Our solution} \end{array}$$

Example 3.75. Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl} (3i)(6i)(2-3i) & \text{Multiply first two monomials} \\ 18i^2(2-3i) & \text{Simplify, } i^2 = -1 \\ 18(-1)(2-3i) & \text{Multiply} \\ -18(2-3i) & \text{Distribute} \\ -36 + 54i & \text{Our solution} \end{array}$$

Notice that in the previous example we chose to simplify i^2 before distributing. This could also have been done *after* distributing $18i^2$ through $(2-3i)$. The resulting expression of $36i^2 - 54i^3$ will then simplify to match our solution above.

Recall that when squaring a binomial such as $(a-b)^2$, we must be careful to expand *completely*, and not forget the inner and outer terms of the product.

$$\begin{aligned}(a-b)^2 &= (a-b)(a-b)\\ &= a^2 - ab - ab + b^2\\ &= a^2 - 2ab + b^2\end{aligned}$$

The next example demonstrates this using complex numbers.

Example 3.76. Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl} (4-5i)^2 & \text{Rewrite as a product of two binomials}\\ (4-5i)(4-5i) & \text{Expand}\\ 16-20i-20i+25i^2 & \text{Simplify, } i^2=-1\\ 16-20i-20i-25 & \text{Combine like terms}\\ -9-40i & \text{Our solution}\end{array}$$

When simplifying rational expressions (fractions) that contain imaginary or complex numbers in a denominator, we will employ the same strategy as that which is used for eliminating square roots from a denominator. This is a logical progression, since we defined i so that $i^2 = \sqrt{-1}$. We refer to this strategy as *rationalizing the denominator*, since the end result will be an expression in which the denominator is a rational number (it contains no radicals).

As we did with complex multiplcation, we will first demonstrate the technique generally, followed by several examples.

$$\begin{array}{rl} \dfrac{c+di}{g+hi} & \text{Multiply top and bottom by } g-hi\\ \dfrac{c+di}{g+hi}\cdot\left(\dfrac{g-hi}{g-hi}\right) & \text{Expand numerator and denominator}\\ \dfrac{cg-chi+dgi-dhi^2}{g^2-ghi+ghi-h^2i^2} & \text{Simplify, } i^2=-1\end{array}$$

$$\frac{cg - chi + dgi + dh}{g^2 - \cancel{ghi} + \cancel{ghi} + h^2} \quad \text{Combine like terms in top and bottom}$$

$$\frac{(cg + dh) + (dg - ch)i}{g^2 + h^2} \quad \text{Rewrite as } a + bi$$

$$\left(\frac{cg + dh}{g^2 + h^2}\right) + \left(\frac{dg - ch}{g^2 + h^2}\right) i \quad \text{Our solution, in standard form}$$

Here, $\frac{cg+dh}{g^2+h^2}$ represents the real part a and $\frac{dg-ch}{g^2+h^2}$ represents the imaginary part b of our resulting complex number $a + bi$. Remember that c, d, g and h all represent real numbers, so our denominator $g^2 + h^2$ is also a real number.

As shown above, the expression that we will typically choose to rationalize with (in this case $g - hi$) is known as the *complex conjugate* to the original denominator $(g + hi)$. When multiplying two complex numbers that are conjugates to one another, the resulting product in our denominator (g^2+h^2) should have no imaginary part.

For our first example, we will start with a denominator which only contains an imaginary part, $0+bi$. In this case, although the complex conjugate would equal $0 - bi$, we only need to multiply the numerator and denominator by i, since multiplying by $-bi$ would result in an eventual cancellation of $-b$ from the entire expression.

Example 3.77. Write the given expression as $a + bi$, where a and b are real numbers.

$$\frac{7 + 3i}{-5i} \quad \text{A monomial in denominator, multiply by } i$$

$$\frac{7 + 3i}{-5i}\left(\frac{i}{i}\right) \quad \text{Distribute } i \text{ in numerator}$$

$$\frac{7i + 3i^2}{-5i^2} \quad \text{Simplify } i^2 = -1$$

$$\frac{7i+3(-1)}{-5(-1)}$$ Multiply

$$\frac{7i-3}{5}$$ Simplify, split up fraction

$$\frac{7i}{5}-\frac{3}{5}$$ Rewrite as $a+bi$

$$-\frac{3}{5}+\frac{7}{5}i$$ Our solution

As shown in the previous example, a solution for such problems can be written several different ways, for example $\frac{-3+7i}{5}$ or $-\frac{3}{5}+\frac{7}{5}i$. Although both answers are generally accepted, we will keep our final answers consistent with the definition of a complex number, $a+bi =$(Real part) + (Imaginary part)i.

Example 3.78. Write the given expression as $a+bi$, where a and b are real numbers.

$$\frac{2-6i}{4+8i}$$ Binomial in denominator, multiply by conjugate, $4-8i$

$$\frac{2-6i}{4+8i}\left(\frac{4-8i}{4-8i}\right)$$ Expand the numerator, denominator is a difference of two squares

$$\frac{8-16i-24i+48i^2}{16-64i^2}$$ Simplify $i^2=-1$

$$\frac{8-16i-24i+48(-1)}{16-64(-1)}$$ Multiply

$$\frac{8-16i-24i-48}{16+64}$$ Combine like terms

$$\frac{-40-40i}{80}$$ Reduce, factor out 40 and divide

$$\frac{-1-i}{2}$$ Rewrite as $a+bi$

$$-\frac{1}{2}-\frac{1}{2}i$$ Our solution

By rewriting $\sqrt{-1}$ as i, we can now simplify square roots with negatives underneath. We will use the product rule and simplify the negative as a factor of negative one. This is shown in the following examples.

Example 3.79. Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl} \sqrt{-16} & \text{Consider the negative as a factor of } -1 \\ \sqrt{-1 \cdot 16} & \text{Take each root, square root of } -1 \text{ is } i \\ 4i & \text{Our solution} \end{array}$$

Example 3.80. Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl} \sqrt{-24} & \text{Find perfect square factors . Factor out } -1 \\ \sqrt{-1 \cdot 4 \cdot 6} & \text{Square root of } -1 \text{ is } i\text{, square root of 4 is 2} \\ 2i\sqrt{6} & \text{Move } i \text{ over} \\ (2\sqrt{6})i & \text{Our solution} \end{array}$$

When simplifying complex radicals, it is important that we take the -1 out of the radical (as an i) before we combine radicals.

Notice also that in the previous example our final answer is $(2\sqrt{6})i$ and not $2\sqrt{6i}$. Although the parentheses are not technically needed, they are included because there is a subtle mathematical difference between these two values, since having i *underneath* a square root ($\sqrt{6i}$) is not equivalent to having it *beside* the square root ($\sqrt{6}i$). This common mistake can be easily avoided by taking care not to extend the square root too far when writing our final answer. The parentheses are simply an added precaution. The same care must be made in order to distinguish an expression like $\sqrt{6}x$ from $\sqrt{6x}$.

Example 3.81. Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl} \sqrt{-6}\sqrt{-3} & \text{Simplify the negatives, bringing } i \text{ out of radicals} \\ (i\sqrt{6})(i\sqrt{3}) & \text{Multiply, } i^2 = -1 \\ -\sqrt{18} & \text{Simplify the radical} \\ -\sqrt{9 \cdot 2} & \text{Take square root of 9} \\ -3\sqrt{2} & \text{Our solution} \end{array}$$

Lastly, when reducing fractions that involve i, as is often the case, we must take extra care to properly simplify and avoid any common mistakes. This is demonstrated in the following example.

Example 3.82. Write the given expression as $a + bi$, where a and b are real numbers.

$\dfrac{-15 - \sqrt{-200}}{20}$	We will simplify the radical first
$\sqrt{-200}$	Find perfect square factors . Factor out -1
$\sqrt{-1 \cdot 100 \cdot 2}$	Take square root of -1 and 100
$10i\sqrt{2}$	Put this back into original expression
$\dfrac{-15 - 10i\sqrt{2}}{20}$	Factor out 5 and divide
$\dfrac{-3 - 2i\sqrt{2}}{4}$	Simplify answer, split up fraction
$-\dfrac{3}{4} - \dfrac{2i\sqrt{2}}{4}$	Reduce, move i to side
$-\dfrac{3}{4} - \dfrac{\sqrt{2}}{2}i$	Our solution

By using $i = \sqrt{-1}$ we will be able to simplify expressions and solve problems that we could not before. In the next few sections, we will see how this will enable us to better understand quadratic equations and their graphs.

3.6 Solving by Extracting Square Roots

Objective: Find the zeros of a quadratic in vertex form by extracting square roots.

Up until now, when attempting to solve an equation such as $x^2 - 4 = 0$, we have had no choice but to factor the expression on the left and set each factor equal to zero.

Example 3.83. Solve the given equation for all possible values of x.

$$\begin{aligned} x^2 - 4 &= 0 && \text{Factor, difference of two squares} \\ (x-2)(x+2) &= 0 && \text{Use Zero Factor Property} \\ x - 2 = 0 \text{ or } x + 2 &= 0 && \text{Solve} \\ x = 2 \ \text{ or } \ &-2 && \text{Our solution} \end{aligned}$$

The values $x = 2$ and $x = -2$ are known as the *zeros* or *roots* of the equation $y = x^2 - 4$. Observe that the graphical interpretation of a zero is an $x-$intercept (when $y = 0$). In this case, the $x-$intercepts of the resulting parabola are at $(2, 0)$ and $(-2, 0)$.

We will now introduce a new technique for identifying the zeros of a quadratic equation, known as the method of *extracting square roots.* The method of extracting square roots will only be employed once we have identified the vertex form for a given quadratic, $y = a(x-h)^2 + k$. The general steps for the method are shown below, and the requirement of the vertex form will be essential.

Example 3.84. Determine the zeros of the quadratic equation $y = ax^2 + bx + c$, where $a \neq 0$.

First obtain the vertex form: $h = -\dfrac{b}{2a}$, set $x = h$ to find k.

$$\begin{aligned} a(x-h)^2 + k &= 0 && \text{Vertex form} \\ \underline{\boldsymbol{-k} \quad} & \underline{\boldsymbol{-k}} && \text{Subtract } k \text{ from both sides} \\ a(x-h)^2 &= -k \\ \overline{\boldsymbol{a}} \qquad & \overline{\boldsymbol{a}} && \text{Divide both sides by } a \\ (x-h)^2 &= -\frac{k}{a} \end{aligned}$$

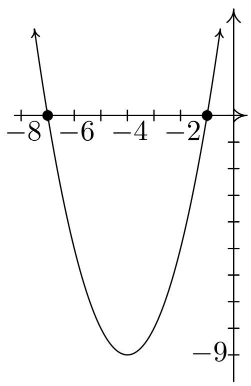
$$
\begin{aligned}
\sqrt{(x-h)^2} &= \pm\sqrt{-\frac{k}{a}} && \text{Take square root of both sides} \\
& && \quad \text{to extract radicand, } x-h \\
x-h &= \pm\sqrt{-\frac{k}{a}} \\
\underline{+\boldsymbol{h}} \quad & \quad \underline{+\boldsymbol{h}} && \text{Add } h \text{ to both sides} \\
x &= h \pm \sqrt{-\frac{k}{a}} && \text{Our solution}
\end{aligned}
$$

In the previous example, there are two important points to consider. First is the introduction of the square root into the equation. This step is the reason behind the name of the method, and its success hinges upon the fact that the vertex form contains a single instance of the variable x. Unlike with the vertex form, if we were to introduce a square root directly to the equation $ax^2+bx+c=0$ (using the standard form), we would immediately reach a dead end, and be unable to simplify the resulting equation. This is primarily because we cannot combine the "unlike" terms ax^2 and bx, and we cannot split up sums (and differences) of terms underneath a square root.

Additionally, it is critical that we include a '$\pm$' on the right side of the equation once the square root has been introduced. The justification for this follows from the fact that there are always two values (one positive and one negative) that will equal the value underneath a square root (assuming that value is nonzero, since $\sqrt{0}=0$). For example, $\sqrt{4}=\pm 2$ and $\sqrt{-9}=\pm 3i$.

We now present a few examples that demonstrate the method, as well as some of the possibilities for the number of zeros, and consequently, the number of $x-$intercepts of the corresponding graph.

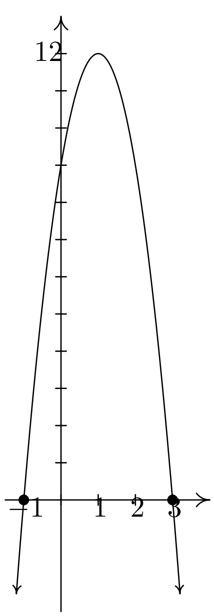
Example 3.85. Use the method of extracting square roots to find the zeros of the equation $y = (x+4)^2 - 9$.



$$
\begin{array}{rcll}
0 &=& (x+4)^2 - 9 & \text{Set equal to zero and solve} \\
\underline{\mathbf{+9}} & & \underline{\mathbf{+9}} & \text{Isolate the square} \\
9 &=& (x+4)^2 & \\
\pm\sqrt{9} &=& \sqrt{(x+4)^2} & \text{Square root both sides} \\
\pm 3 &=& x+4 & \text{Solve for } x \\
\underline{\mathbf{-4}} & & \underline{\mathbf{-4}} & \text{Subtract } 4 \\
x &=& \pm 3 - 4 & \text{Two solutions} \\
x = 3-4 &\Rightarrow& x = -1 & \text{One solution} \\
x = -3-4 &\Rightarrow& x = -7 & \text{The other solution}
\end{array}
$$

Our zeros are $x = -7$ and $x = -1$. The corresponding $x-$intercepts are at the points $(-7, 0)$ and $(-1, 0)$.

Example 3.86. Use the method of extracting square roots to find the zeros of the equation $y = -3(x-1)^2 + 12$.



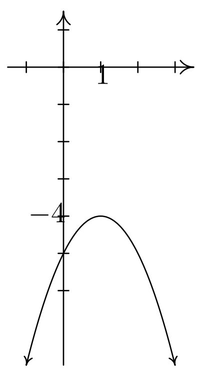
$$
\begin{array}{rcll}
0 &=& -3(x-1)^2 + 12 & \text{Set equal to zero and solve} \\
\underline{\mathbf{-12}} & & \underline{\mathbf{-12}} & \text{Subtract } 12 \\
-12 &=& -3(x-1)^2 & \text{Isolate the square,} \\
\overline{\mathbf{-3}} & & \overline{\mathbf{-3}} & \quad \text{divide both sides by } -3 \\
4 &=& (x-1)^2 & \\
\pm\sqrt{4} &=& \sqrt{(x-1)^2} & \text{Square root both sides} \\
\pm 2 &=& x-1 & \text{Solve for } x \\
\underline{\mathbf{+1}} & & \underline{\mathbf{+1}} & \text{Add } 1 \\
x &=& \pm 2 + 1 & \text{Two solutions} \\
x = 1-2 &\Rightarrow& x = -1 & \text{One solution} \\
x = 1+2 &\Rightarrow& x = 3 & \text{The other solution}
\end{array}
$$

Our two zeros are $x = -1$ and $x = 3$.

In some cases, the introduction of a square root results in an imaginary number. This scenario coincides with our corresponding parabola having no $x-$intercepts. In the previous example, if we were to change the sign of k from $+12$ to -12, the corresponding parabola would still open downwards, while having a vertex at $(1, -12)$, located below the $x-$axis. This will result in the appearance of a $\sqrt{-4} = 2i$, rather than a $\sqrt{4}$, in our solution. Consequently, there will be no real zeros for the equation and no $x-$intercepts on its graph.

We conclude this section with a final example, which will also result in no real zeros.

Example 3.87. Use the method of extracting square roots to find the zeros of the equation $y = -1(x-1)^2 - 4$.



$$\begin{aligned}
0 &= -1(x-1)^2 - 4 && \text{Set equal to zero and solve}\\
\underline{\mathbf{+4}} & \qquad\qquad\quad \underline{\mathbf{+4}} && \text{Add } 4\\
4 &= -1(x-1)^2 && \text{Isolate the square,}\\
\overline{\mathbf{-1}} & \quad \overline{\mathbf{-1}} && \quad \text{divide both sides by } -1\\
-4 &= (x-1)^2 && \\
\pm\sqrt{-4} &= \sqrt{(x-1)^2} && \text{Square root both sides}\\
\pm 2i &= x - 1 && \text{Solve for } x\\
\underline{\mathbf{+1}} & \quad \underline{\mathbf{+1}} && \text{Add } 1\\
x &= \pm 2i + 1 && \text{Two solutions}\\
x &= 1 - 2i && \text{One solution}\\
x &= 1 + 2i && \text{The other solution}
\end{aligned}$$

Once again, the negative appearing under the square root results in two complex zeros (no real zeros). Graphically, the function never touches or crosses the $x-$axis.

3.7 Completing the Square

Objective: Solve quadratic equations by completing the square.

In this section, we will introduce a method for obtaining the vertex form of a quadratic function from the standard form, without having to rely on the vertex formula $h = -\dfrac{b}{2a}$. This method is known as *completing the square*. To complete the square, and convert a quadratic expression ax^2+bx+c from standard form to the vertex form $a(x-h)^2+k$ (without our prior knowledge of the relationship between h, a and b), we will first start by considering the expression ax^2+bx.

Observe that if a quadratic is of the form x^2+bx+c, and *is* a perfect square, the constant term, c, can be found by the formula $\left(\frac{1}{2}\cdot b\right)^2$. This is shown in the following examples. In each example, we will find the number needed to complete the perfect square, and then factor it.

Example 3.88. Identify the constant term c that is needed to factor the given trinomial as a perfect square.

$$\begin{array}{rl} x^2+8x+c & c = \left(\dfrac{1}{2}\cdot b\right)^2 \text{ and our } b = 8 \\ \left(\dfrac{1}{2}\cdot 8\right)^2 = 4^2 = 16 & \text{The necessary constant term is } 16 \\ x^2+8x+16 & \text{Our desired trinomial; factor} \\ (x+4)^2 & \text{Our solution} \end{array}$$

Example 3.89. Identify the constant term c that is needed to factor the given trinomial as a perfect square.

$$\begin{array}{rl} x^2-7x+c & c = \left(\dfrac{1}{2}\cdot b\right)^2 \text{ and our } b = 7 \\ \left(\dfrac{1}{2}\cdot 7\right)^2 = \left(\dfrac{7}{2}\right)^2 = \dfrac{49}{4} & \text{The necessary constant term is } \dfrac{49}{4} \\ x^2-7x+\dfrac{49}{4} & \text{Our desired trinomial; factor} \\ \left(x-\dfrac{7}{2}\right)^2 & \text{Our solution} \end{array}$$

Example 3.90. Identify the constant term c that is needed to factor the given trinomial as a perfect square.

$x^2 + \frac{5}{3}x + c$	$c = \left(\frac{1}{2} \cdot b\right)^2$ and our $b = \frac{5}{3}$
$\left(\frac{1}{2} \cdot \frac{5}{3}\right)^2 = \left(\frac{5}{6}\right)^2 = \frac{25}{36}$	The necessary constant term is $\frac{25}{36}$
$x^2 + \frac{5}{3}x + \frac{25}{36}$	Our desired trinomial; factor
$\left(x + \frac{5}{6}\right)^2$	Our solution

The process demonstrated in the previous examples may be used to obtain the vertex form of a quadratic. The following set of steps describes the process used to complete the square. Since all three of the previous examples contained a leading coefficient of $a = 1$, an example where $a \neq 1$ has been included below to illustrate the special care that must be taken in this case.

Expression	$3x^2 + 18x - 6$
1. Separate constant term from variables	$(3x^2 + 18x) - 6$
2. Factor out a from each term in parentheses	$3(x^2 + 6x) - 6$
3. Determine value to complete the square: $\left(\frac{1}{2} \cdot b\right)^2$	$\left(\frac{1}{2} \cdot 6\right)^2 = 3^2 = 9$
4. Add & subtract value to expression in parentheses	$3(x^2 + 6x + 9 - 9) - 6$
5. Separate subtracted value from other three terms, making sure to multiply by a	$3(x^2 + 6x + 9) - 3(9) - 6$
6. Combine constant terms outside parentheses	$3(x^2 + 6x + 9)^2 - 27 - 6$
7. Factor remaining trinomial	$3(x + 3)^2 - 33$

World View Note: The Chinese in 200 B.C. were the first known cultural group to use a method similar to completing the square, but their method was only used to calculate positive roots. The advantage of this method is it can be used to solve any quadratic equation. The following examples show how completing the square can give us rational solutions, irrational solutions, and even complex solutions.

Example 3.91. Use the method of completing the square to solve the given equation.

$$\begin{aligned}
4x^2 + 40x + 51 &= 0 && \text{Equation in standard form} \\
(4x^2 + 40x) + 51 &= 0 && \text{Separate constant term} \\
4(x^2 + 10x) + 51 &= 0 && \text{Factor out } a \\
\left(\frac{1}{2} \cdot 10\right)^2 &= 5^2 = 25 && \text{Complete the square : find } \left(\frac{1}{2} \cdot b\right)^2 \\
4(x^2 + 10x + 25 - 25) + 51 &= 0 && \text{Add and subtract } 25 \text{ inside parentheses} \\
4(x^2 + 10x + 25) - 4(25) + 51 &= 0 && \text{Separate trinomial} \\
4(x^2 + 10x + 25)^2 - 100 + 51 &= 0 && \text{Simplify: combine constant terms,} \\
& && \quad \text{factor trinomial} \\
4(x+5)^2 - 49 &= 0 && \text{Solve by extracting square roots} \\
(x+5)^2 &= \frac{49}{4} && \text{Isolate the square} \\
\sqrt{(x+5)^2} &= \pm\sqrt{\frac{49}{4}} && \text{Square root both sides} \\
x + 5 &= \pm\frac{7}{2} && \text{Subtract } 5 \text{ from both sides} \\
\underline{\mathbf{-5}} \quad & \quad \underline{\mathbf{-5}} && \\
x &= -5 \pm \frac{7}{2} && \\
x &= -\frac{17}{2} \text{ or } -\frac{3}{2} && \text{Our solution}
\end{aligned}$$

Example 3.92. Use the method of completing the square to solve the given equation.

$$
\begin{aligned}
x^2 - 3x - 2 &= 0 && \text{Equation in standard form} \\
(x^2 - 3x) - 2 &= 0 && \text{Separate constant term} \\
& && \text{Leading coefficient is } a = 1 \\
\left(\frac{1}{2} \cdot -3\right)^2 = \left(-\frac{3}{2}\right)^2 &= \frac{9}{4} && \text{Complete the square : find } \left(\frac{1}{2} \cdot b\right)^2 \\
\left(x^2 - 3x + \frac{9}{4} - \frac{9}{4}\right) - 2 &= 0 && \text{Add and subtract } \frac{9}{4} \text{ inside parentheses} \\
\left(x^2 - 3x + \frac{9}{4}\right) - \frac{9}{4} - 2 &= 0 && \text{Separate trinomial} \\
\left(x^2 - 3x + \frac{9}{4}\right) - \frac{9}{4} - 2 &= 0 && \text{Simplify: combine constant terms,} \\
& && \quad \text{factor trinomial} \\
\left(x - \frac{3}{2}\right)^2 - \frac{17}{4} &= 0 && \text{Solve by extracting square roots} \\
\left(x - \frac{3}{2}\right)^2 &= \frac{17}{4} && \text{Isolate the square} \\
\sqrt{\left(x - \frac{3}{2}\right)^2} &= \pm\sqrt{\frac{17}{4}} && \text{Square root both sides} \\
x - \frac{3}{2} &= \pm\,\frac{\sqrt{17}}{2} && \text{Reduce square root;} \\
\underline{+\mathbf{\frac{3}{2}}} \qquad & \quad \underline{+\mathbf{\frac{3}{2}}} && \text{Add } \frac{3}{2} \text{ to both sides} \\
x = \frac{3}{2} &\pm \frac{\sqrt{17}}{2} \\
x = \frac{3+\sqrt{17}}{2} \text{ or } &\frac{3-\sqrt{17}}{2} && \text{Our solution}
\end{aligned}
$$

As the previous example shows, completing the square when $a = 1$ can be seen as slightly easier than when $a \neq 1$. Our last example demonstrates how we can more also handle the case when $a \neq 1$ early on in our solution, by simply dividing the equation by a.

Example 3.93. Use the method of completing the square to solve the given equation.

$$\begin{array}{rl}
\dfrac{3x^2}{\mathbf{3}} - \dfrac{2x}{\mathbf{3}} + \dfrac{7}{\mathbf{3}} = \dfrac{0}{\mathbf{3}} & \text{Equation in standard form; Divide both sides by 3} \\[2ex]
x^2 - \dfrac{2}{3}x + \dfrac{7}{3} = 0 & \text{Resulting equation has } a = 1 \\[2ex]
\left(\dfrac{1}{2} \cdot -\dfrac{2}{3}\right)^2 = \left(-\dfrac{1}{3}\right)^2 = \dfrac{1}{9} & \text{Complete the square: find } \left(\dfrac{1}{2} \cdot b\right)^2 \\[2ex]
x^2 - \dfrac{2}{3}x + \dfrac{1}{9} - \dfrac{1}{9} + \dfrac{7}{3} = 0 & \text{Add and subtract } \dfrac{1}{9} \text{ to left side} \\[2ex]
\left(x^2 - \dfrac{2}{3}x + \dfrac{1}{9}\right) - \dfrac{1}{9} + \dfrac{7}{3} = 0 & \text{Combine constant terms by} \\
 & \quad \text{obtaining a common denominator} \\[2ex]
-\dfrac{1}{9} + \dfrac{7}{3} = -\dfrac{1}{9} + \dfrac{21}{9} = \dfrac{20}{9} & \\[2ex]
\left(x^2 - \dfrac{2}{3}x + \dfrac{1}{9}\right) + \dfrac{20}{9} = 0 & \text{Factor trinomial} \\[2ex]
\left(x - \dfrac{1}{3}\right)^2 + \dfrac{20}{9} = 0 & \text{Solve by extracting square roots} \\[2ex]
\left(x - \dfrac{1}{3}\right)^2 = -\dfrac{20}{9} & \text{Isolate the square} \\[2ex]
\sqrt{\left(x - \dfrac{1}{3}\right)^2} = \pm\sqrt{\dfrac{-20}{9}} & \text{Square root both sides} \\[2ex]
x - \dfrac{1}{3} = \dfrac{\pm 2i\sqrt{5}}{3} & \text{Simplify the radical} \\
\underline{+\dfrac{\mathbf{1}}{\mathbf{3}} \qquad +\dfrac{\mathbf{1}}{\mathbf{3}}} & \text{Add } \dfrac{1}{3} \text{ to both sides} \\[2ex]
x = \dfrac{1}{3} \pm \dfrac{2\sqrt{5}}{3}i & \text{Our solution}
\end{array}$$

As we mentioned earlier, completing the square is simply an alternative method to the vertex formula for converting a quadratic expression from standard form into vertex form. Still, as many of the previous examples have demonstrated, we will often need to work with fractions and be comfortable finding common denominators when solving quadratic equations using this method. Although this can be intimidating, with enough practice, one should be able to easily solve almost any quadratic equation by completing the square.

In the next section, we will present one final method for determining the zeros of a quadratic.

3.8 The Quadratic Formula and the Discriminant

Objective: Solve quadratic equations by using the quadratic formula and use the discriminant to determine the number of real zeros of a quadratic.

Recall that the general from of a quadratic equation is $y = ax^2 + bx + c$, where $a \neq 0$. We are now ready to solve the general equation $ax^2+bx+c = 0$ for x by completing the square, which we show in the following example.

Example 3.94. Solve the equation $ax^2 + bx + c = 0$ for all values of x using the method of completing the square.

$$
\begin{aligned}
ax^2 + bx + c &= 0 && \text{Divide each term by } a \\
x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 && \text{Separate constant term } \frac{c}{a} \\
\left(x^2 + \frac{b}{a}x\right) + \frac{c}{a} &= 0 && \text{Complete the square} \\
\left(\frac{1}{2}\cdot\frac{b}{a}\right)^2 = \left(\frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} && \text{Add and subtract } \frac{b^2}{4a^2} \text{ inside parentheses} \\
\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + \frac{c}{a} &= 0 && \text{Separate trinomial} \\
\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a^2} + \frac{c}{a} &= 0 && \text{Simplify:} \\
-\frac{b^2}{4a^2} + \frac{c}{a}\left(\frac{4a}{4a}\right) = -\frac{b^2}{4a^2} + \frac{4ac}{4a^2} &= -\frac{b^2-4ac}{4a^2} && \text{(1) Combine constant terms} \\
\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) &= \left(x + \frac{b}{2a}\right)^2 && \text{(2) Factor trinomial} \\
\left(x + \frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2} &= 0 && \text{Now solve by extracting square roots} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2-4ac}{4a^2} && \text{Isolate the square}
\end{aligned}
$$

$$\sqrt{\left(x+\frac{b}{2a}\right)^2} = \pm\sqrt{\frac{b^2-4ac}{4a^2}} \qquad \text{Square root both sides}$$

$$x+\frac{b}{2a} = \frac{\pm\sqrt{b^2-4ac}}{2a} \qquad \text{Subtract } \frac{b}{2a} \text{ from both sides}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} \qquad \text{Write as single fraction}$$

$$x = \frac{-b \pm\sqrt{b^2-4ac}}{2a} \qquad \text{Our solution}$$

This solution is a very important one to us. Since we solved a *general* equation by completing the square, we can now use this formula to solve any quadratic equation. Once we identify what $a, b,$ and c are, we can substitute those values into the equation $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ and simplify in order to find our solution to the given quadratic. This formula is known as the *quadratic formula*. We call the expression underneath the square root, $b^2 - 4ac$, the *discriminant* of the quadratic equation $ax^2 + bx + c = 0$, and will see its importance later on in the section.

Quadratic Formula: The solutions to $ax^2 + bx + c = 0$ are given by the formula $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

Discriminant: The discriminant of a quadratic equation $ax^2 + bx + c = 0$ is the expression $b^2 - 4ac$

We can use the quadratic formula to solve any quadratic, this is shown in the following examples. Notice that we focus on calculating the discriminant first, and that it will have a major impact on the type of solutions that we receive.

Example 3.95. Solve the given equation for all values of x.

$$x^2+3x+2=0 \qquad \text{Identify } a, b, \text{ and } c$$

$$a=1,\ b=3,\ c=2 \qquad \text{Use quadratic formula}$$

$$x = \frac{-3 \pm\sqrt{3^2-4(1)(2)}}{2(1)} \qquad \text{Substitute } a, b, \text{ and } c \text{ without simplifying}$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{-3 \pm \sqrt{9-8}}{2} \qquad \text{Simplify}$$

$$x = \frac{-3 \pm \sqrt{1}}{2} \qquad \text{Discriminant is } 1$$

$$x = \frac{-3 \pm 1}{2} \qquad \text{Evaluate} \pm\text{; write as two equations}$$

$$x = \frac{-3+1}{2} \text{ or } \frac{-3-1}{2} \qquad \text{Simplify}$$

$$x = \frac{-2}{2} \text{ or } \frac{-4}{2}$$

$$x = -1 \text{ or } -2 \qquad \text{Our solutions}$$

Notice that the previous equation resulted in two real solutions. This is directly related to the discriminant being positive (in this case, 1). If the discriminant had been zero, then we would not have had anything underneath the square root, meaning that the plus or minus ($\pm$) would have had no effect on the rest of the procedure. Consequently, we would have only had one real solution. Furthermore, since the discriminant was a perfect square, we actually could have factored our quadratic from the start.

$$x^2 + 3x + 2 = (x+1)(x+2)$$

It is important to mention that when solving using the quadratic formula, we must remember to first set the given equation equal to zero and make sure the quadratic is in standard form.

Example 3.96. Solve the given equation for all values of x.

$$25x^2 = 30x + 11 \qquad \text{First set equal to zero}$$

$$25x^2 - 30x - 11 = 0 \qquad \text{Identify } a, b, \text{ and } c$$

$$a = 25, \; b = -30, \; c = -11 \qquad \text{Use quadratic formula}$$

$$x = \frac{30 \pm \sqrt{(-30)^2 - 4(25)(-11)}}{2(25)} \qquad \text{Substitute } a, b, \text{ and } c \text{ without simplifying}$$

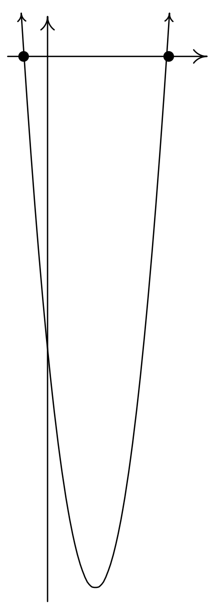
$$x = \frac{30 \pm \sqrt{(-30)^2 - 4(25)(-11)}}{2(25)}$$

$$x = \frac{30 \pm \sqrt{900 + 1100}}{50} \quad \text{Simplify}$$

$$x = \frac{30 \pm \sqrt{2000}}{50} \quad \text{Discriminant is } 2000$$

$$x = \frac{30 \pm 20\sqrt{5}}{50} \quad \text{Divide each term by 10}$$

$$x = \frac{3 \pm 2\sqrt{5}}{5} \quad \text{Our solutions}$$



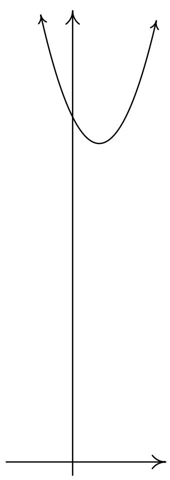
In each of the previous two examples the discriminant was positive, and consequently, there were two real solutions. Graphically, quadratics with a positive discriminant will intersect the x-axis at two distinct points.

The included graph shows the two real solutions to $25x^2 - 30x - 11 = 0$. This example demonstrates the importance of our efforts to relate an algebraic solution to a graphical representation, in order to help internalize the meaning behind the quadratic formula.

World View Note: Indian mathematician Brahmagupta gave the first explicit formula for solving quadratics in 628. However, at that time mathematics was not done with variables and symbols, so the formula he gave was, "To the absolute number multiplied by four times the square, add the square of the middle term; the square root of the same, less the middle term, being divided by twice the square is the value." This would translate to $\frac{\sqrt{4ac+b^2}-b}{2a}$ as the solution to the equation $ax^2 + bx = c$.

Example 3.97. Solve the given equation for all values of x.

$3x^2+4x+8=2x^2+6x-5$	First set equation equal to zero
$x^2-2x+13=0$	Identify $a, b,$ and c
$a=1,\ b=-2,\ c=13,$	Use quadratic formula
$x=\dfrac{2\pm\sqrt{(-2)^2-4(1)(13)}}{2(1)}$	Substitute $a, b,$ and c without simplifying
$x=\dfrac{2\pm\sqrt{4-52}}{2}$	Simplify
$x=\dfrac{2\pm\sqrt{-48}}{2}$	Discriminant is -48
$x=\dfrac{2\pm 4i\sqrt{3}}{2}$	Simplify: reduce radical, divide by 2
$x=1\pm 2i\sqrt{3}$	Our solutions



The previous example has two complex solutions that are not real. Consequently, we see that graphically our parabola has no $x-$intercepts. This results from the discriminant being negative, -48 in this case.

When using the quadratic formula, it is possible to *not* obtain two unique real (or complex) solutions. If the discriminant under the square root simplifies to zero, we can end up with only one real solution.

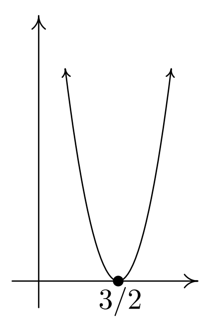
As it turns out, this single solution will coincide with the vertex of our parabola, (h, k). Recalling that $h=-\frac{b}{2a}$, we can verify that this result makes sense, when we consider that a discriminant of zero will eliminate the term $\pm\frac{\sqrt{b^2-4ac}}{2a}$ from our quadratic formula completely. What we are left with is precisely h.

Our next example will result in a single real solution, and will coincide to a parabola that touches the x-axis exactly once, at its vertex.

Example 3.98. Solve the given equation for all values of x.

$$
\begin{aligned}
4x^2 - 12x + 9 &= 0 && \text{Identify } a, b, \text{ and } c \\
a = 4,\ b = -12,\ c &= 9, && \text{Use quadratic formula} \\
x &= \frac{12 \pm \sqrt{(-12)^2 - 4(4)(9)}}{2(4)} && \text{Substitute } a, b, \text{ and } c \text{ without simplifying} \\
x &= \frac{12 \pm \sqrt{144 - 144}}{8} && \text{Simplify} \\
x &= \frac{12 \pm \sqrt{0}}{8} && \text{Discriminant is zero} \\
x &= \frac{12 \pm 0}{8} && \text{We get one real solution} \\
x &= \frac{12}{8} && \text{Reduce fraction} \\
x &= \frac{3}{2} && \text{Our solution}
\end{aligned}
$$

A graph of our resulting parabola, shown below, confirms our previous result.



If a term is absent from our quadratic, we can still use the quadratic formula and simply use zero in place of the missing coefficient. The order of terms, however, is still important. If, for example, the linear term was absent, we would use $b = 0$. And, if the constant term is missing, we would use $c = 0$.

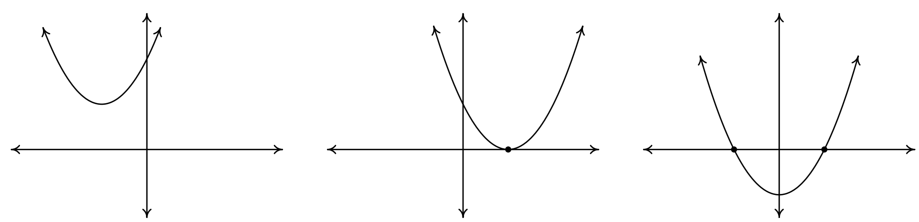
It is necessary that we take extra precautions when using the quadratic formula, since one false step can lead to a substantial amount of time lost. Taking the time to write the quadratic in standard form, set equal to zero, and identify the correct values for a,b, and c is crucial to the success of the quadratic formula.

Example 3.99. Solve the given equation for all values of x.

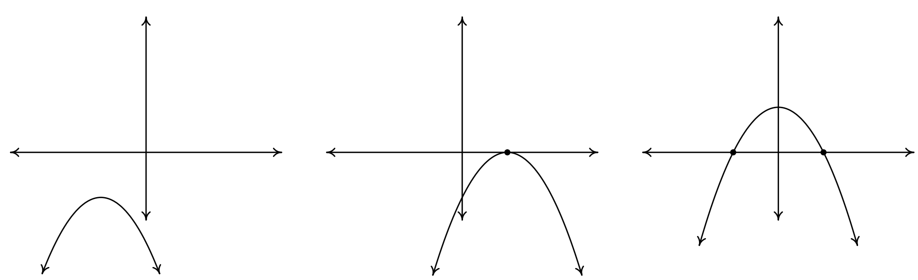
$$\begin{aligned} 3x^2+7&=0 && \text{Identify } a, b, \text{ and } c \\ a=3,\ b=0\ (\text{missing term}),\ c&=7 && \text{Use quadratic formula} \\ x&=\frac{-0\pm\sqrt{0^2-4(3)(7)}}{2(3)} && \text{Substitute } a, b, \text{ and } c \text{ without simplifying} \\ x&=\frac{\pm\sqrt{-84}}{6} && \text{Simplify; discriminant is } -84 \\ x&=\frac{\pm 2i\sqrt{21}}{6} && \text{Reduce radical and divide by } 2 \\ x&=\frac{\pm i\sqrt{21}}{3} && \text{Our solutions} \end{aligned}$$

We leave it as an exercise to the reader to graph the corresponding parabola and confirm that our solution is correct. Remember, the fact that we have two imaginary solutions means that our parabola should have no $x-$intercepts.

As we have seen in the previous examples, the discriminant determines the nature and quantity of the solutions of the quadratic formula. The following collection of graphs summarizes both the graphical and algebraic consequences for each type of discriminant (negative, zero, or positive).



Negative Disciminant	Zero Disciminant	Positive Disciminant
$b^2 - 4ac < 0$	$b^2 - 4ac = 0$	$b^2 - 4ac > 0$
Zero Real Solutions	One Real Solution	Two Real Solutions



We have now outlined three different methods to use to solve a quadratic equation: factoring, extracting square roots, and using the quadratic formula. It is important to be familiar with all three methods, since each has its advantages.

The following table suggests a procedure to help determine which method might be best to use for solving a given quadratic equation.

1. If we can easily factor, solve by factoring	$x^2 - 5x + 6 = 0$ $(x-2)(x-3) = 0$ $x = 2$ or $x = 3$
2. If $a = 1$ and b is even, complete the square (or use the vertex formula) and extract square roots	$x^2 + 2x - 4 = 0$ $\left(\frac{1}{2} \cdot 2\right)^2 = 1^2 = 1$ $(x^2 + 2x + 1) - 1 - 4 = 0$ $(x+1)^2 - 5 = 0$ $(x+1)^2 = 5$ $x + 1 = \pm\sqrt{5}$ $x = -1 \pm \sqrt{5}$
3. As a last resort, apply the quadratic formula	$x^2 - 3x + 4 = 0$ $x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(4)}}{2(1)}$ $x = \frac{3 \pm i\sqrt{7}}{2}$

The above table is merely a suggestion for approaching quadratic equations. Recall that completing the square and extracting square roots, as well as the quadratic formula may always be used to solve any quadratic, but often may not be the most efficient or "clean" method. Factoring can be very efficient but only works if the given equation can be factored.

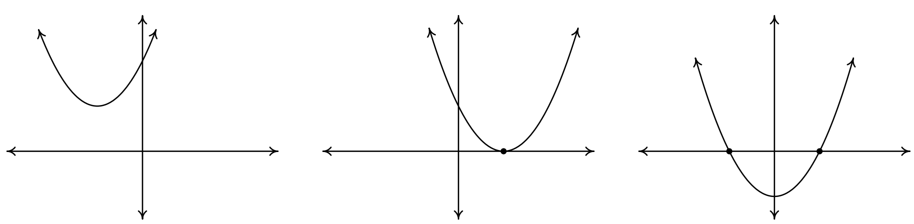
3.9 Applications

This section is not yet complete.

3.10 Quadratic Inequalities and Sign Diagrams

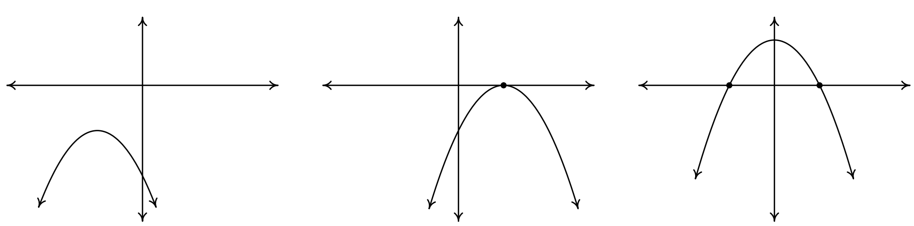
Objective: Solve and give interval notation for the solution to a quadratic inequality. Create a sign diagram to identify those intervals where a quadratic expression is positive or negative.

Recall that the *vertex form* for a quadratic equation is $y = a(x-h)^2 + k$, where $a \neq 0$ and (h, k) represents the *vertex* of the corresponding graph, called a *parabola*. If $a > 0$, then the parabola opens upward, and if $a < 0$, then the parabola opens downward. With any quadratic equation, we have seen that there are three possibilities for the number of *zeros* or *roots* of the equation $(0, 1, \text{ or } 2)$. Assuming $a > 0$, we illustrate these possibilities in the graphs below.



Notice also that each of these three graphs lie above the x-axis over different intervals. In the case of the parabola on the left, the entire graph lies above the x-axis, whereas the middle parabola lies above the x-axis everywhere *except* at its x-intercept (where $y = 0$). Even more interesting is the parabola on the right, which contains two *separate* intervals where its graph lies above the x-axis.

Considering the case where $a < 0$, we see three similar graphs as those appearing above, with the only major difference being the opening of each parabola downward instead of upward (when $a > 0$). When we consider again those intervals where each graph lies above the x-axis, each parabola below exhibits a different behavior than those where $a > 0$.



Now, each of the first two graphs have no points that lie above the x-axis, whereas the last graph, on the right, lies above the x-axis over the interval that is between its x-intercepts.

Each of these six graphs above exhibit all of the various possibilities for the *sign* of a quadratic expression $ax^2 + bx + c$, where $a \neq 0$. As was the case with linears in the previous chapter, we can determine the general shape of the graph of a quadratic equation (or function) through identification of its zeros and construction of a sign diagram. As a consequence, we will also see the care that must be taken when asked to solve a quadratic inequality.

Let us begin with what should be a familiar example, $y = x^2 - 1$, which we can recall has a factorization of $y = (x+1)(x-1)$.

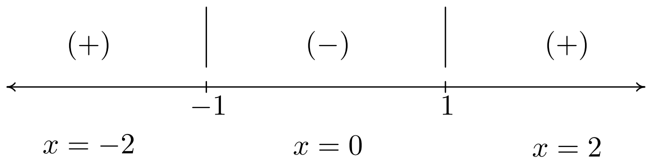
Example 3.100. Solve the quadratic inequality $x^2 - 1 < 0$.

As has often been the case, our first instinct is to add 1 to both sides of the given inequality, obtaining $x^2 < 1$. Our next guess is most likely to take a square root of both sides of the given inequality. Here, however, is where we encounter a common "pitfall", which begs the question: how does one handle radicals and inequalities?

The answer is that unlike with solving linear inequalities, one should not attempt to solve for the variable x, but rather set the given inequality equal to zero and attempt to *factor* the resulting expression on the other side. In doing this, we obtain $(x+1)(x-1) < 0$. Recalling that $x = \pm 1$ are zeros of the given expression, we can therefore rule them out of our solution. Next, we will *test* the expression on the left by plugging in three values for x:
(i) $x < -1$, (ii) $-1 < x < 1$, and (iii) $x > 1$.

Case	Test Value	Unsimplified	Simplified	Result
i	$x = -2$	$(-2+1)(-2-1)$	$(-)\cdot(-)$	$(+)$
ii	$x = 0$	$(0+1)(0-1)$	$(+)\cdot(-)$	$(-)$
iii	$x = 2$	$(2+1)(2-1)$	$(+)\cdot(+)$	$(+)$

Our end result can be summarized in the following *sign diagram.*



From our sign diagram, we can conclude that $x^2 - 1 < 0$ when $-1 < x < 1$, or using interval notation, $(-1, 1)$.

Example 3.101. Solve the inequality $x^2 \geq 1$.

Here, we need only subtract -1 from both sides of the inequality, to obtain $x^2 - 1 \geq 0$. After factoring the left-hand side, We may then use the sign diagram from our previous example. Our solution set will be the *union* of two intervals, $(-\infty, -1] \cup [1, \infty)$.

Example 3.102. Solve the inequality $-(x-1)^2 + 9 \geq 0$.

Notice that the left-hand side of our inequality is in vertex form. So we will draw upon our knowledge of the graph of $y = -(x-1)^2 + 9$ later on to confirm our answer.

We start by expanding the left-hand side to obtain

$$-(x^2 - 2x + 1) + 9 \geq 0,$$

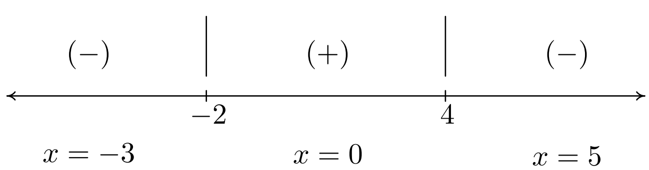
which reduces to

$$-x^2 + 2x + 8 \geq 0.$$

After factoring, we obtain

$$-(x+2)(x-4) \geq 0.$$

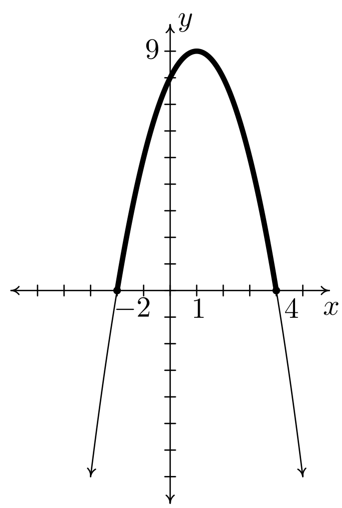
Since both $x = -2$ and $x = 4$ are zeros of the left-hand side, for our sign diagram, we will therefore test $x = -3$, $x = 0$, and $x = 5$. It is important to not overlook the negative sign that appears in front of our inequality when testing our values. Our results are shown below.



From our sign diagram, we can determine that

$$-(x-1)^2+9 \geq 0 \text{ when } -2 \leq x \leq 4.$$

Again, the vertex form $y = -(x-1)^2+9$ confirms this, since the corresponding parabola will have a vertex of $(1,9)$, which lies above the x-axis, and will open downward, as the leading coefficient $a = -1$ is negative. This implies that there will be two x-intercepts, which we found to be at the points $(-2,0)$ and $(4,0)$. Hence the graph will be nonnegative over an interval between (and including) the x-intercepts. To reinforce this, we provide the graph below, highlighting the portion that coincides with our desired interval.



In our next example, we will touch upon the notion of the *multiplicity* of a zero for a given equation/function, and how it affects the graph.

Example 3.103. Solve the inequality $x^2+4x > -4$.

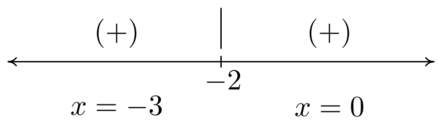
Setting the right-hand side to zero gives us

$$x^2+4x+4 > 0.$$

Factoring, we then have

$$(x+2)^2 > 0.$$

Hence, we have only one zero for the left-hand side ($x = -2$), which means that there are only two intervals to test.



Our solution set may be represented as the inequality $x \neq -2$, or as the union of intervals $(-\infty, -2) \cup (-2, \infty)$.

Notice that $x = -2$ was a zero in each of the last two examples. In the first example, a change in sign occurred (negative to positive) as the values of x increased from one side of our zero to the other. In the second example, however, both the values below and above $x = -2$ yield positive signs.

This result has to do with the number of factors of $(x + 2)$ appearing in our expression. This number is known as the *multiplicity* of the zero $x = -2$. Briefly stated, the *parity* of a zero's multiplicity (whether the number of factors is even or odd) will determine whether or not the sign of the given expression on either side of the zero remains the same or changes. This notion will be quite useful when graphing complicated functions, and will be revisited in the chapter on polynomial functions.

Example 3.104. Solve the inequality $x^2 + 4x < -4$.

Since we have only switched the direction of our inequality in the last example, we may conclude that the inequality has no solution set, represented by the empty set, $\varnothing$.

Up until this point, all of our examples have reduced to expressions that can easily be factored. As this is often not the case for quadratic expressions, we will now attempt to solve some more challenging inequalities.

Example 3.105. Solve the inequality $x^2 - x + 1 > 0$.

After brief inspection, we see that the expression on the left-hand side is not easily factorable. At this point, in order to determine if any real zeros exist for $x^2 - x + 1$, we have a few methods to choose from. We will use the quadratic formula, shown below.

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$x = \frac{1}{2} \pm \frac{\sqrt{-3}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

Since we are left with a negative under the square root, we conclude that the given expression has no real zeros. Hence, the corresponding parabola will have no x-intercepts. Note: A slightly quicker method would have been to simply calculate the discriminant of $(-1)^2 - 4(1)(1) = -3 < 0$.

As our leading coefficient $a = 1$ in the above expression is positive, we know that the corresponding parabola will open upward. Using this information, along with the fact that there are no x-intercepts, we may conclude that the entire parabola must lie above the x-axis. Hence, our solution set is all real numbers, $(-\infty, \infty)$.

Example 3.106. Solve the inequality $x^2 > 4x - 1$.

Setting the right-hand side to zero, we have

$$x^2 - 4x + 1 > 0.$$

Although we could again resort to the quadratic formula, we will instead identify the vertex form of the expression on the left, shown below.

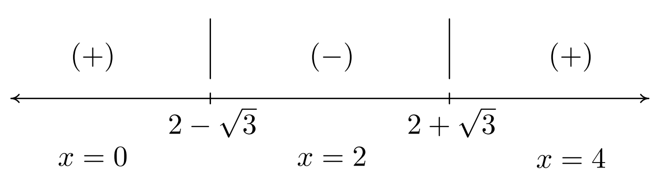
$$h = -\frac{-4}{2(1)} = 2 \qquad\qquad k = 2^2 - 4(2) + 1 = -3$$

$$x^2 - 4x + 1 = (x - 2)^2 - 3$$

So, setting $(x-2)^2 - 3$ equal to zero and extracting square roots, we obtain two real zeros at $x = 2 \pm \sqrt{3}$. It then follows that

$$x^2 - 4x + 1 = \left(x - (2 - \sqrt{3})\right)\left(x - (2 + \sqrt{3})\right).$$

Since we have two real zeros, we will construct a sign diagram, using test values on either side of $2 - \sqrt{3} \approx 0.27$ and $2 + \sqrt{3} \approx 3.73$. Our results are shown below.



Note that since we already obtained the vertex of $(2, -3)$, we have chosen $x = 2$ as a test value for our middle interval.

From the above diagram, we conclude that $x^2 > 4x - 1$ precisely on the union of intervals $(-\infty, 2 - \sqrt{3}) \cup (2 + \sqrt{3}, \infty)$.

3.11 Practice Problems

3.11.1 Introduction

Quadratic Equations

After simplifying, classify each equation as linear, quadratic, or neither. If the equation is a quadratic, then specify whether it is concave up or down.

1) $y = x^2 + 9$
2) $y = 5 - 2x + x^2$
3) $y = x + 6 - 3x$
4) $y = 5x + x^2 - 3x - 3x^2$
5) $y = -5x + 3 + 2x - 3x^2 + 6$
6) $y = 3x^2 + -x + x - 3x^2 + 6$
7) $y = (x - 1)(x + 2) + 3$
8) $y = (x - 5)(2x + 3) - 2(x - 3)$
9) $y = (x - 4)(x + 4) - (x + 1)^2$
10) $y = (2x-4)(x-1)-2(x+3)^2+3x^2$

Vertex Form of a Quadratic

Identify the vertex and concavity (concave up or down) of each quadratic.

1) $y = (x - 3)^2 + 4$
2) $y = (x - 2)^2 + 5$
3) $y = 6(x + 3)^2 + 4$
4) $y = -2(x - 3)^2 + 4$
5) $y = -2(x - 1)^2 - 7$
6) $y = -(x + 1)^2$
7) $y = 7x^2 + 4$
8) $y = -\frac{1}{23}(x - 8)^2 + 5$
9) $y = x^2 + 4$
10) $y = 5x^2 + 23$

3.11.2 Factoring

Greatest Common Factor

Factor the common factor out of each expression.

1) $4 + 8b^2$
2) $x - 5$
3) $45x^2 - 25$
4) $-n - 2n^2$
5) $56 - 35p$
6) $50x - 80y$
7) $7ab - 35a^2b$
8) $27x^2y^5 - 72x^3y^2$
9) $-3a^2b + 6a^3b^2$
10) $8x^3y^2 + 4x^3$
11) $-5x^2 - 5x^3 - 15x^4$
12) $-32n^9 + 32n^6 + 40n^5$
13) $20x^4 - 30x + 30$
14) $21p^6 + 30p^2 + 27$
15) $28m^4 + 40m^3 + 8$
16) $-10x^4 + 20x^2 + 12x$
17) $30b^9 + 5ab - 15a^2$
18) $27y^7 + 12y^2x + 9y^2$
19) $-48a^2b^2 - 56a^3b - 56a^5b$
20) $30m^6 + 15mn^2 - 25$
21) $20x^8y^2z^2 + 15x^5y^2z + 35x^3y^3z$
22) $3p + 12q - 15q^2r^2$
23) $50x^2y + 10y^2 + 70xz^2$
24) $30y^4z^3x^5 + 50y^4z^5 - 10y^4z^3x$
25) $30qpr - 5qp + 5q$
26) $28b + 14b^2 + 35b^3 + 7b^5$
27) $-18n^5 + 3n^3 - 21n + 3$
28) $30a^8 + 6a^5 + 27a^3 + 21a^2$
29) $-40x^{11} - 20x^{12} + 50x^{13} - 50x^{14}$
30) $-24x^6 - 4x^4 + 12x^3 + 4x^2$
31) $-32mn^8 + 4m^6n + 12mn^4 + 16mn$
32) $-10y^7 + 6y^{10} - 4y^{10}x - 8y^8x$

Grouping

Factor each expression completely.

1) $40r^3 - 8r^2 - 25r + 5$
2) $35x^3 - 10x^2 - 56x + 16$
3) $3n^3 - 2n^2 - 9n + 6$
4) $14v^3 + 10v^2 - 7v - 5$
5) $15b^3 + 21b^2 - 35b - 49$
6) $6x^3 - 48x^2 + 5x - 40$
7) $3x^3 + 15x^2 + 2x + 10$
8) $28p^3 + 21p^2 + 20p + 15$
9) $35x^3 - 28x^2 - 20x + 16$
10) $7n^3 + 21n^2 - 5n - 15$
11) $7xy - 49x + 5y - 35$
12) $42r^3 - 49r^2 + 18r - 21$
13) $32xy + 40x^2 + 12y + 15x$
14) $15ab - 6a + 5b^3 - 2b^2$
15) $16xy - 56x + 2y - 7$
16) $3mn - 8m + 15n - 40$
17) $2xy - 8x^2 + 7y^3 - 28y^2x$
18) $5mn + 2m - 25n - 10$
19) $40xy + 35x - 8y^2 - 7y$
20) $8xy + 56x - y - 7$
21) $32uv - 20u + 24v - 15$
22) $4uv + 14u^2 + 12v + 42u$
23) $10xy + 30 + 25x + 12y$
24) $24xy + 25y^2 - 20x - 30y^3$
25) $3uv + 14u - 6u^2 - 7v$
26) $56ab + 14 - 49a - 16b$
27) $16xy - 3x - 6x^2 + 8y$

TRINOMIALS WITH LEADING COEFFICIENT $a = 1$

Factor each expression completely.

1) $p^2 + 17p + 72$
2) $x^2 + x - 72$
3) $n^2 - 9n + 8$
4) $x^2 + x - 30$
5) $x^2 - 9x - 10$
6) $x^2 + 13x + 40$
7) $b^2 + 12b + 32$
8) $b^2 - 17b + 70$
9) $x^2 + 3x - 70$
10) $x^2 + 3x - 18$
11) $n^2 - 8n + 15$
12) $a^2 - 6a - 27$
13) $p^2 + 15p + 54$
14) $p^2 + 7p - 30$
15) $n^2 - 15n + 56$
16) $m^2 - 15mn + 50n^2$
17) $u^2 - 8uv + 15v^2$
18) $m^2 - 3mn - 40n^2$
19) $m^2 + 2mn - 8n^2$
20) $x^2 + 10xy + 16y^2$
21) $x^2 - 11xy + 18y^2$
22) $u^2 - 9uv + 14v^2$
23) $x^2 + xy - 12y^2$
24) $x^2 + 14xy + 45y^2$
25) $x^2 + 4xy - 12y^2$
26) $4x^2 + 52x + 168$
27) $5a^2 + 60a + 100$
28) $5n^2 - 45n + 40$
29) $6a^2 + 24a - 192$
30) $5v^2 + 20v - 25$
31) $6x^2 + 18xy + 12y^2$
32) $5m^2 + 30mn - 90n^2$
33) $6x^2 + 96xy + 378y^2$
34) $6m^2 - 36mn - 162n^2$

Trinomials with Leading Coefficient $a \neq 1$

Factor each expression completely.

1) $7x^2 - 48x + 36$
2) $7n^2 - 44n + 12$
3) $7b^2 + 15b + 2$
4) $7v^2 - 24v - 16$
5) $5a^2 - 13a - 28$
6) $5n^2 - 7n - 24$
7) $2x^2 - 5x + 2$
8) $3r^2 - 4r - 4$
9) $2x^2 + 19x + 35$
10) $7x^2 + 29x - 30$
11) $2b^2 - b - 3$
12) $5k^2 - 26k + 24$
13) $5k^2 + 13k + 6$
14) $3r^2 + 16r + 21$
15) $3x^2 - 17x + 20$
16) $3u^2 + 13uv - 10v^2$
17) $3x^2 + 17xy + 10y^2$
18) $7x^2 - 2xy - 5y^2$
19) $5x^2 + 28xy - 49y^2$
20) $5u^2 + 31uv - 28v^2$
21) $6x^2 - 39x - 21$
22) $10a^2 - 54a - 36$
23) $21k^2 - 87k - 90$
24) $21n^2 + 45n - 54$
25) $14x^2 - 60x + 16$
26) $4r^2 + r - 3$
27) $6x^2 + 29x + 20$
28) $6p^2 + 11p - 7$
29) $4k^2 - 17k + 4$
30) $4r^2 + 3r - 7$
31) $4x^2 + 9xy + 2y^2$
32) $4m^2 + 6mn + 6n^2$
33) $4m^2 - 9mn - 9n^2$
34) $4x^2 - 6xy + 30y^2$
35) $4x^2 + 13xy + 3y^2$
36) $18u^2 - 3uv - 36v^2$
37) $12x^2 + 62xy + 70y^2$
38) $16x^2 + 60xy + 36y^2$
39) $24x^2 - 52xy + 8y^2$
40) $12x^2 + 50xy + 28y^2$

3.11.3 Solving by Factoring

Set each of the following expressions equal to zero and solve for the given variable.

1) - 15): Expressions (1) through (15) on page 256.

16) - 30): Expressions (1) through (15) on page 257.

31) - 40): Expressions (21) through (30) on page 257.

3.11.4 Vertex Form and Graphing

Vertex Form of a Quadratic

Identify whether the quadratic is in vertex form, standard form, or both. If it is in vertex form, then identify the vertex (h, k).

1) $y = (x - 12)^2 + 5$
2) $y = -3(x - 3)^2 + 5$
3) $y = x^2 + 8$
4) $y = 2(x - 4)^2$
5) $y = -4(x - 1)^2 + 2$
6) $y = -5(x - 7)^2$
7) $y = x^2 + 3x + 4$
8) $y = x^2 - 1$
9) $y = x^2 - 3$
10) $y = (x - 1)^2 - 3$
11) $y = (x - 1)^2$
12) $y = x^2$

Each quadratic equation below has been given in standard form. Rewrite each equation in vertex form.

13) $y = x^2 + 2x - 1$
14) $y = -3x^2 - 12x - 5$
15) $y = 3x^2 + 12x - 1$
16) $y = x^2 + 2x$
17) $y = x^2 + 6$
18) $y = -5x^2 - 40x$
19) $y = x^2 + 8x$
20) $y = x^2$
21) $y = x^2 + 4x - 2$
22) $y = x^2 + 16x - 2$
23) $y = 4x^2 + 10x$

Graphing Quadratics

Find the vertex and intercepts of the following quadratics. Use this information to graph the resulting parabola.

1) $y = x^2 - 2x - 8$
2) $y = x^2 - 2x - 3$
3) $y = 2x^2 - 12x + 10$
4) $y = 2x^2 - 12x + 16$
5) $y = -2x^2 + 12x - 18$
6) $y = -2x^2 + 12x - 10$
7) $y = -3x^2 + 24x - 45$
8) $y = -3x^2 + 12x - 9$
9) $y = -x^2 + 4x + 5$
10) $y = -x^2 + 4x - 3$
11) $y = -x^2 + 6x - 5$
12) $y = -2x^2 + 16x - 30$
13) $y = -2x^2 + 16x - 24$
14) $y = 2x^2 + 4x - 6$
15) $y = 3x^2 + 12x + 9$
16) $y = 5x^2 + 30x + 45$
17) $y = 5x^2 - 40x + 75$
18) $y = 5x^2 + 20x + 15$
19) $y = -5x^2 - 60x - 175$
20) $y = -5x^2 + 20x - 15$

3.11.5 Square Roots and the Imaginary Number i

Square Roots

Simplify each of the following square roots completely.

1) $\sqrt{245}$	2) $\sqrt{125}$
3) $\sqrt{36}$	4) $\sqrt{196}$
5) $\sqrt{12}$	6) $\sqrt{72}$
7) $3\sqrt{12}$	8) $5\sqrt{32}$
9) $6\sqrt{128}$	10) $7\sqrt{128}$
11) $-8\sqrt{392}$	12) $-7\sqrt{63}$
13) $\sqrt{192n}$	14) $\sqrt{343b}$
15) $\sqrt{196v^2}$	16) $\sqrt{100n^3}$
17) $\sqrt{252x^2}$	18) $\sqrt{200a^3}$
19) $-\sqrt{100k^4}$	20) $-4\sqrt{175p^4}$
21) $-7\sqrt{64x^4}$	22) $-2\sqrt{128n}$
23) $-5\sqrt{36m}$	24) $8\sqrt{112p^2}$
25) $\sqrt{45x^2y^2}$	26) $\sqrt{72a^3b^4}$
27) $\sqrt{16x^3y^3}$	28) $\sqrt{512a^4b^2}$
29) $\sqrt{320x^4y^4}$	30) $\sqrt{512m^4n^3}$
31) $6\sqrt{80xy^2}$	32) $8\sqrt{98mn}$
33) $5\sqrt{245x^2y^3}$	34) $2\sqrt{72x^2y^2}$
35) $-2\sqrt{180u^3v}$	36) $-5\sqrt{72x^3y^4}$
37) $-8\sqrt{180x^4y^2z^4}$	38) $6\sqrt{50a^4bc^2}$
39) $2\sqrt{80hj^4k}$	40) $-\sqrt{32xy^2z^3}$
41) $-4\sqrt{54mnp^2}$	42) $-8\sqrt{32m^2p^4q}$

Intro to Complex Numbers

Rewrite each of the following complex numbers in the form $a+bi$, where a and b are real numbers and $i=\sqrt{-1}$.

1) $3-(-8+4i)$
2) $(3i)-(7i)$
3) $(7i)-(3-2i)$
4) $5+(-6-6i)$
5) $(-6i)-(3+7i)$
6) $(-8i)-(7i)-(5-3i)$
7) $(3-3i)+(-7-8i)$
8) $(-4-i)+(1-5i)$
9) $(i)-(2+3i)-6$
10) $(5-4i)+(8-4i)$
11) $(6i)(-8i)$
12) $(3i)(-8i)$
13) $(-5i)(8i)$
14) $(8i)(-4i)$
15) $(-7i)^2$
16) $(-i)(7i)(4-3i)$
17) $(6+5i)^2$
18) $(8i)(-2i)(-2-8i)$
19) $(-7-4i)(-8+6i)$
20) $(3i)(-3i)(4-4i)$
21) $(-4+5i)(2-7i)$
22) $-8(4-8i)-2(-2-6i)$
23) $(-8-6i)(-4+2i)$
24) $(-6i)(3-2i)-(7i)(4i)$
25) $(1+5i)(2+i)$
26) $(-2+i)(3-5i)$
27) $\dfrac{-9+5i}{i}$
28) $\dfrac{-3+2i}{-3i}$
29) $\dfrac{-10-9i}{6i}$
30) $\dfrac{-4+2i}{3i}$
31) $\dfrac{-3-6i}{4i}$
32) $\dfrac{-5+9i}{9i}$
33) $\dfrac{10-i}{-i}$
34) $\dfrac{10}{5i}$
35) $\dfrac{4i}{-10+i}$
36) $\dfrac{9i}{1-5i}$
37) $\dfrac{8}{7-6i}$
38) $\dfrac{4}{4+6i}$
39) $\dfrac{7}{10-7i}$
40) $\dfrac{9}{-8-6i}$
41) $\dfrac{5i}{-6-i}$
42) $\dfrac{8i}{6-7i}$
43) $\sqrt{-81}$
44) $\sqrt{-45}$
45) $\sqrt{-10}\sqrt{-2}$
46) $\sqrt{-12}\sqrt{-2}$
47 $\frac{3+\sqrt{-27}}{6}$
48) $\frac{-4-\sqrt{-8}}{-4}$
49) $\frac{8-\sqrt{-16}}{4}$
50) $\frac{6+\sqrt{-32}}{4}$
51) i^{73}
52) i^{251}
53) i^{48}
54) i^{68}
55) i^{62}
56) i^{181}
57) i^{154}
58) i^{51}

3.11.6 Solving by Extracting Square Roots

Find the x-intercepts of each quadratic by setting $y = 0$ and using the method of extracting square roots.

1) $y = (x - 12)^2 - 5$
2) $y = -3(x - 3)^2 + 30$
3) $y = x^2 - 16$
4) $y = 2(x - 4)^2 - 200$
5) $y = -4(x - 1)^2 + 20$
6) $y = -2(x - 7)^2 + 50$
7) $y = -4(x + 6)^2 + 8$
8) $y = x^2 - 4$
9) $y = (x - 4)^2 - 9$
10) $y = (x - 1)^2 - 25$
11) $y = (x + 2)^2 + 16$
12) $y = 9(x - 11)^2 - 81$

3.11.7 Completing the Square

Find the value that completes the square and then rewrite the given expression as a perfect square.

1) $x^2 - 30x +$ _______
2) $a^2 - 24a +$ _______
3) $m^2 - 36m +$ _______
4) $x^2 - 34x +$ _______
5) $x^2 - 15x +$ _______
6) $r^2 - \frac{1}{9}r +$ _______
7) $y^2 - y +$ _______
8) $p^2 - 17p +$ _______

Solve each equation by completing the square.

9) $x^2 - 16x + 55 = 0$
10) $n^2 - 8n - 12 = 0$
11) $v^2 - 8v + 45 = 0$
12) $b^2 + 2b + 43 = 0$
13) $6x^2 + 12x + 63 = 0$
14) $3x^2 - 6x + 47 = 0$
15) $5k^2 - 10k + 48 = 0$
16) $8a^2 + 16a - 1 = 0$
17) $x^2 + 10x - 57 = 4$
18) $p^2 - 16p - 52 = 0$
19) $n^2 - 16n + 67 = 4$
20) $m^2 - 8m - 3 = 6$
21) $2x^2 + 4x + 38 = -6$
22) $6r^2 + 12r - 24 = -6$
23) $8b^2 + 16b - 37 = 5$
24) $6n^2 - 12n - 14 = 4$
25) $x^2 = -10x - 29$
26) $v^2 = 14v + 36$
27) $n^2 = -21 + 10n$
28) $a^2 - 56 = -10a$
29) $3k^2 + 9 = 6k$
30) $5x^2 = -26 + 10x$
31) $2x^2 + 63 = 8x$
32) $5n^2 = -10n + 15$
33) $p^2 - 8p = -55$
34) $x^2 + 8x + 15 = 8$
35) $7n^2 - n + 7 = 7n + 6n^2$
36) $n^2 + 4n = 12$
37) $13b^2 + 15b + 44 = -5 + 7b^2 + 3b$
38) $-3r^2 + 12r + 49 = -6r^2$
39) $5x^2 + 5x = -31 - 5x$
40) $8n^2 + 16n = 64$
41) $v^2 + 5v + 28 = 0$
42) $b^2 + 7b - 33 = 0$
43) $7x^2 - 6x + 40 = 0$
44) $4x^2 + 4x + 25 = 0$
45) $k^2 - 7k + 50 = 3$
46) $a^2 - 5a + 25 = 3$
47) $5x^2 + 8x - 40 = 8$
48) $2p^2 - p + 56 = -8$
49) $m^2 = -15 + 9m$
50) $n^2 - n = -41$
51) $8r^2 + 10r = -55$
52) $3x^2 - 11x = -18$
53) $5n^2 - 8n + 60 = -3n + 6 + 4n^2$
54) $4b^2 - 15b + 56 = 3b^2$
55) $-2x^2 + 3x - 5 = -4x^2$
56) $10v^2 - 15v = 27 + 4v^2 - 6v$

3.11.8 The Quadratic Formula and the Discriminant

Use the discriminant in order to determine the number of real roots for each equation. If an equation is shown to have one (or two) real root(s), set $y = 0$ and use the quadratic formula to find them.

1) $y = x^2 + 2x - 1$
2) $y = -3x^2 - 12x - 5$
3) $y = 3x^2 + 12x - 1$
4) $y = x^2 + 2x$
5) $y = x^2 + 6$
6) $y = -5x^2 - 40x$
7) $y = x^2 + 8x$
8) $y = x^2$
9) $y = x^2 + 4x - 2$
10) $y = x^2 + 16x - 2$
11) $y = 4x^2 + 10x$

Solve each equation using the quadratic formula.

12) $4a^2 + 6 = 0$
13) $3k^2 + 2 = 0$
14) $2x^2 - 8x - 2 = 0$
15) $6n^2 - 1 = 0$
16) $2m^2 - 3 = 0$
17) $5p^2 + 2p + 6 = 0$
18) $3r^2 - 2r - 1 = 0$
19) $2x^2 - 2x - 15 = 0$
20) $4n^2 - 36 = 0$
21) $3b^2 + 6 = 0$
22) $v^2 - 4v - 5 = -8$
23) $2x^2 + 4x + 12 = 8$
24) $2a^2 + 3a + 14 = 6$
25) $6n^2 - 3n + 3 = -4$
26) $3k^2 + 3k - 4 = 7$
27) $4x^2 - 14 = -2$
28) $7x^2 + 3x - 16 = -2$
29) $4n^2 + 5n = 7$
30) $2p^2 + 6p - 16 = 4$
31) $m^2 + 4m - 48 = -3$
32) $3n^2 + 3n = -3$
33) $3b^2 - 3 = 8b$
34) $2x^2 = -7x + 49$
35) $3r^2 + 4 = -6r$
36) $5x^2 = 7x + 7$
37) $6a^2 = -5a + 13$
38) $8n^2 = -3n - 8$
39) $6v^2 = 4 + 6v$
40) $2x^2 + 5x = -3$
41) $x^2 = 8$
42) $4a^2 - 64 = 0$
43) $2k^2 + 6k - 16 = 2k$
44) $4p^2 + 5p - 36 = 3p^2$
45) $12x^2 + x + 7 = 5x^2 + 5x$
46) $-5n^2 - 3n - 52 = 2 - 7n^2$
47) $7m^2 - 6m + 6 = -m$
48) $7r^2 - 12 = -3r$
49) $3x^2 - 3 = x^2$
50) $2n^2 - 9 = 4$
51) $6b^2 = b^2 + 7 - b$

3.11.9 Applications

3.11.10 Quadratic Inequalities and Sign Diagrams

Construct a sign diagram for each of the following expressions/equations. Then using interval notation, describe the set of values for which the given expression is greater than or equal to zero.

1) - 5): Expressions (1) through (5) on page 256.

6) - 10): Expressions (1) through (5) on page 257.

11) - 15): Expressions (21) through (25) on page 257.

16) - 20): Equations (1) through (5) on page 264.

Chapter 4

Introduction to Functions

4.1 Notation and Basic Examples

4.1.1 Definitions and the Vertical Line Test

Objective: Identify functions and use correct notation to evaluate functions at specific values.

A **relation** R is a set of points in the xy-plane. A relation in which each x-coordinate is paired with exactly one y-coordinate is said to describe y as a **function** of x. Relations which represent functions of x will often be denoted by f, or $f(x)$, rather than R. The set of all x-coordinates of the points in a function f is called the **domain** of f, and the set of all y-coordinates of the points in f is called the **range** of f.

Example 4.1. The following examples represent relations. Examples (5) and (6) also represent y as a *function* of x, $y = f(x)$, since each $x-$coordinate is paired with exactly one $y-$coordinate.

1. $\{(1, 1), (2, -3), (2, 0), (0, 3), (-2, 1/2)\}$
2. $\{(x, y) \mid x > 3 \text{ and } y \leq 2\}$
3. $x^2 + y^2 = 9$
4. $x = y^2$
5. $y = x^2$
6. $y = 3 - 2x$

Alternatively, one can define a function as a rule that assigns to each element of one set (the domain) exactly one element of a second set (the range). This definition is essentially the same as that given above, but avoids the term "relation" entirely. In each definition, however, the critical phrase that cannot be overlooked is "*exactly one*". This means that the first four relations given above cannot represent y as a function of x, since, for example, the third relation contains the points $(0, 3)$ and $(0, -3)$. On the other hand, each of the last two relations above can be considered to represent y as a function of x. Furthermore, their graphs should also look familiar, since they represent a quadratic equation ($y = x^2$) and a linear equation ($y = -2x+3$).

In each of the last two examples above, we refer to the variable x as the **independent variable**, since we are free to choose any real number for x. We consequently refer to y as the **dependent variable**, since its value depends on the choice of value for x. One can also more simply refer to x as the *input* of the function and y as the *output*. This terminology naturally lends itself to what is the standard function notation of $f(x)$, read as "f of x". In the following example, we will use the given function to complete a table of values for x and $f(x)$. Each pair $(x, f(x))$ corresponds to a point (x, y) on the graph of f.

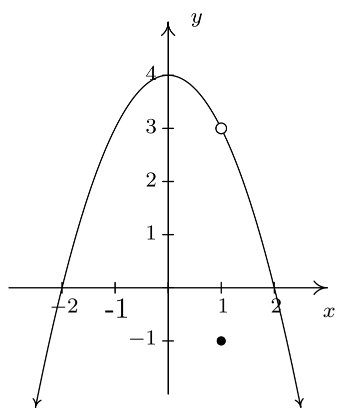
Example 4.2. $f(x) = x^2 - 4x - 5$

x	$f(x)$
-2	$(-2)^2 - 4(-2) - 5 = 7$
-1	$(-1)^2 - 4(-1) - 5 = 0$
0	$(0)^2 - 4(0) - 5 = -5$
1	$(1)^2 - 4(1) - 5 = -8$
2	$(2)^2 - 4(2) - 5 = -9$

To complete each row of the table, we simply substitute the specified value for x into the given equation and simplify. So, if we wanted to complete another row in the table, we could substitute $x = 3$ into the equation to obtain $f(3) = (3)^2 - 4(3) - 5 = -8$.

In the previous example, the y-coordinates for the relation $y = x^2 - 4x - 5$ are represented by $f(x)$, or more simply $y = f(x)$. It is important to note that the parentheses in function notation do not represent multiplication. This is a common misconception among students. Instead, one should consider the parentheses as an identifier, enclosing the value of x that the rule f is applied to. This will be especially important as we discuss composite functions later in the chapter.

In the following examples we will answer a variety of questions related to functions and their graphs. First, we will consider the case where we are presented with the graph of a particular function and asked to identify specific values of x or $f(x)$ from it.



The graph of f

In our first scenario, we will be provided with an input x and asked to find the output $f(x)$. To find an output when given a specific input, locate the input value on the x-axis and follow the vertical line (above and below) the input value until it intersects, or "hits", the graph. The corresponding $y-$coordinate for the point of intersection will be the desired output, $y = f(x)$.

Example 4.3. Use the graph of f provided to find the desired outputs.

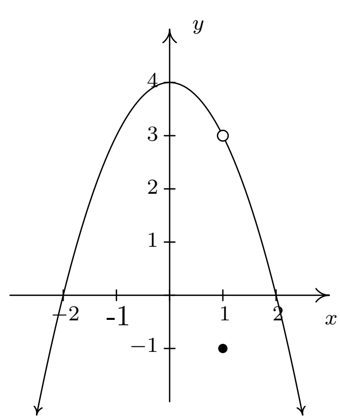
$$\begin{aligned} f(2) &= ? && \text{What is } y \text{ when } x = 2? \\ f(2) &= 0 && \text{Our answer} \end{aligned}$$

$$\begin{aligned} f(0) &= ? && \text{What is } y \text{ when } x = 0? \\ f(0) &= 4 && \text{Our answer} \end{aligned}$$

$$\begin{aligned} f(1) &= ? && \text{What is } y \text{ when } x = 1? \\ f(1) &= -1 && \text{Our answer} \end{aligned}$$

It is important to point out that many students will misinterpret the last example and incorrectly conclude that $f(1) = 3$, since the open circle at $(1, 3)$ appears to coincide with the rest of the graph of f. An *open* circle, however, is used to identify a *break* in the graph of f, also known as a point of *discontinuity*. In fact, the given function is not defined at $(1, 3)$, but rather at the **solid** (or *closed*) point $(1, -1)$. Hence, we get a corresponding value of $y = -1$ for $f(x)$.

Next, we will be provided with an output y and asked to find all corresponding inputs x such that $f(x) = y$. To find all possible inputs, we will make a simple adjustment to the method used in the previous example. Now, we will locate the output value on the y-axis and follow the horizontal line (left and right) of the output value until it intersects, or "hits", the graph. All corresponding $x-$coordinates for the points of intersection will represent the set of all values of x such that $f(x)$ equals our given output y and should be included as part of our final answer.



The graph of f

Example 4.4. Use the graph above to find all possible inputs that correspond to the specified output.

$$\begin{aligned} \text{Find } x \text{ where } f(x) = 0. &\quad \text{Which inputs for } x \text{ have an output of } y = 0? \\ x = -2, 2 &\quad \text{Our answers} \\ \text{Find } x \text{ where } f(x) = 3. &\quad \text{Which inputs for } x \text{ have an output of } y = 3? \\ x = -1 &\quad \text{Our answer; We should not include } x = 1. \end{aligned}$$

Similarly, if we were also asked to find all possible x such that $f(x) = -1$, then we would end up with three values, since there are three points that intersect the horizontal line $y = -1$, namely $x \approx -2.2$, $x = 1$, and $x \approx 2.2$.

There are four major representations of functions: verbal (in words), numerical (using a table), symbolic (with an algebraic expression), and visual (with a graph). In many cases, we will be asked to identify one representation of y as a function of x when given a different representation. The next two examples demonstrate this.

Example 4.5. Provide the symbolic form for each of the following verbal descriptions of a function.

1. Add 2 to a value and then take the square root of the resulting value.

$$\text{Our answer} \qquad f(x) = \sqrt{2 + x}$$

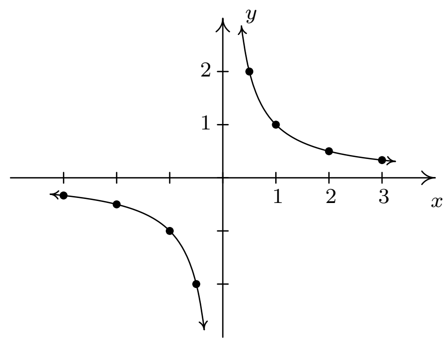
2. Take the square root of a value and then add 2 to the resulting value.

$$\text{Our answer} \qquad g(x) = \sqrt{x} + 2$$

Note: It is often beneficial to rewrite $g(x)$ in the previous example as $g(x) = 2 + \sqrt{x}$, so as not to accidentally extend the radical to include the $+2$.

Example 4.6. Provide a graphical representation for the function given by the following table of values.

x	$f(x)$
-3	-1/3
-2	-1/2
-1	-1
-1/2	-2
1/2	2
1	1
2	1/2
3	1/3

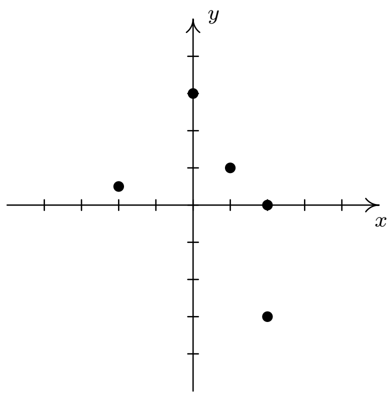


Since there are often advantages to working with either symbolic or graphical representations of functions, we will focus our attention on working with these two representations. One major test that is used to determine whether or not a graph of a relation represents y as a function of x is known as the Vertical Line Test. We will now state the Vertical Line Test as a mathematical theorem and then demonstrate its use.

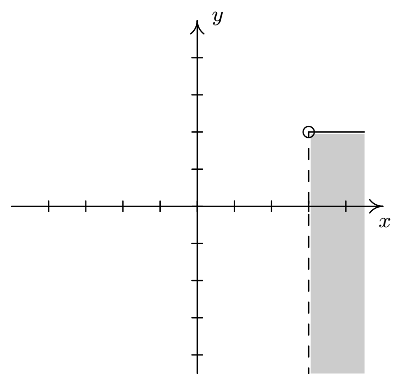
Vertical Line Test: A set of points in the xy-plane represents y as a function of x if and only if no two points lie on the same vertical line.

Alternatively stated, if a graph is known to represent y as a function of x, then there can be no vertical line that intersects the graph in more than one point. Conversely, if a known graph has the property that no vertical line intersects it in more than one point, then the given graph represents y as a function of x.

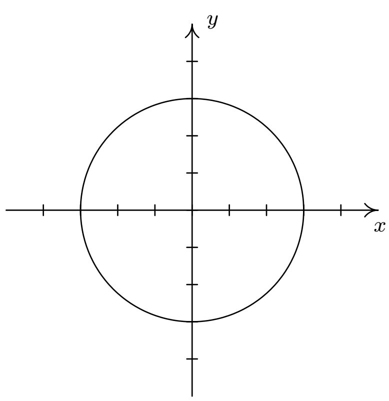
Example 4.7. Use the Vertical Line Test to determine whether or not each of the following graphs represent y as a function of x.



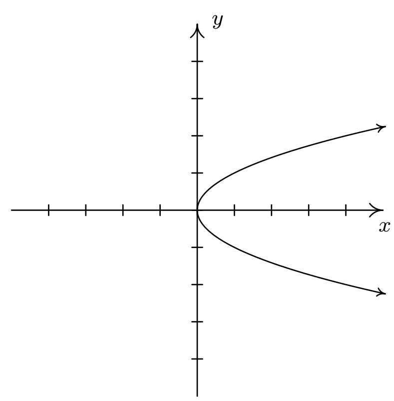
$\{(1,1),(2,-3),(2,0),(0,3),(-2,1/2)\}$



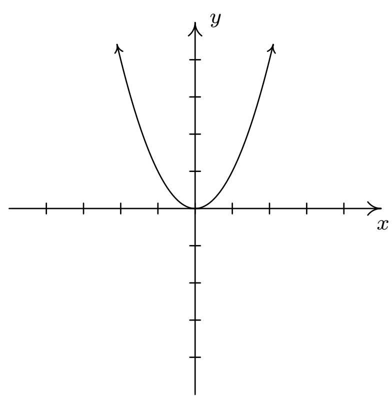
$\{(x,y) \mid x>3 \text{ and } y\le 2\}$



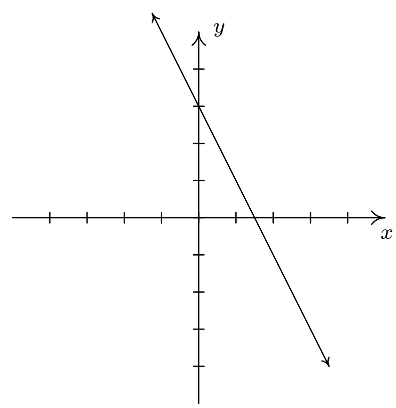
$x^2+y^2=9$



$x=y^2$



$y = x^2$



$y = 3 - 2x$

In each example, we utilize the Vertical Line Test by “slicing” through each graph with several vertical lines, located at various values along the x-axis. Consequently, we can see that each of the first four examples at the beginning of the section *do not* represent y as a function of x, since in each case there exists at least one vertical line that intersects the graph in two (or possibly more) points. The last two examples *do* represent y as a function of x, since no such vertical line exists. As a result, we say that the first four examples *fail* the Vertical Line Test, and the last two examples *pass* the Vertical Line Test.

4.1.2 Evaluating Functions

Another function-related skill we will want to quickly master is evaluating functions at certain values of the independent variable (usually x). This is accomplished by substituting the specified value into the function for x and simplifying the resulting expression to find $f(x)$. This idea of "plugging in" values of x to find $f(x)$ is demonstrated in the following examples.

Example 4.8. Find $f(-2)$, where $f(x) = 3x^2 - 4x$.

$$\begin{aligned} f(x) &= 3x^2 - 4x && \text{Evaluate; Substitute } -2 \text{ for each } x \\ f(-2) &= 3(-2)^2 - 4(-2) && \text{Simplify using order of operations; exponent first} \\ f(-2) &= 3(4) - 4(-2) && \text{Multiply} \\ f(-2) &= 12 + 8 && \text{Add} \\ f(-2) &= 20 && \text{Our solution} \end{aligned}$$

Example 4.9. Find $h(4)$, where $h(x) = 3^{2x-6}$.

$$\begin{aligned} h(x) &= 3^{2x-6} && \text{Evaluate; Substitute } 4 \text{ for } x \\ h(4) &= 3^{2(4)-6} && \text{Simplify exponent, multiplying first} \\ h(4) &= 3^{8-6} && \text{Subtract in exponent} \\ h(4) &= 3^2 && \text{Evaluate exponent} \\ h(4) &= 9 && \text{Our solution} \end{aligned}$$

Example 4.10. Find $k(-7)$, where $k(a) = 2|a+4|$.

$$\begin{aligned} k(a) &= 2|a+4| && \text{Evaluate; Substitute } -7 \text{ for } a \\ k(-7) &= 2|-7+4| && \text{Simplify, add inside absolute value} \\ k(-7) &= 2|-3| && \text{Evaluate absolute value} \\ k(-7) &= 2(3) && \text{Multiply} \\ k(-7) &= 6 && \text{Our solution} \end{aligned}$$

As the previous examples show, a function can take many different forms, but the method to evaluate the function is always the same: replace each instance of the variable with the specified value and simplify.

We can also substitute entire expressions into functions using this same process. This idea is known as a *composition* of two functions or expressions, and will be formally outlined in a later section. We present the following two examples as a preview of this concept.

Example 4.11. Find $g(3x)$, where $g(x) = x^4 + 1$.

$$
\begin{aligned}
g(x) &= x^4 + 1 && \text{Replace } x \text{ in the function with } (3x)\\
g(3x) &= (3x)^4 + 1 && \text{Simplify exponent}\\
g(3x) &= 81x^4 + 1 && \text{Our solution}
\end{aligned}
$$

Example 4.12. Find $p(t+1)$, where $p(t) = t^2 - t$.

$$
\begin{aligned}
p(t) &= t^2 - t && \text{Replace each } t \text{ in } p(t) \text{ with } (t+1)\\
p(t+1) &= (t+1)^2 - (t+1) && \text{Simplify; square binomial}\\
p(t+1) &= t^2 + 2t + 1 - (t+1) && \text{Distribute negative sign}\\
p(t+1) &= t^2 + 2t + 1 - t - 1 && \text{Combine like terms}\\
p(t+1) &= t^2 + t && \text{Our solution}\\
p(t+1) &= t(t+1) && \text{Our solution in factored form}
\end{aligned}
$$

As is the case with each of the previous examples, it is important to keep in mind that each expression (or function) will often use the same variable. Hence, it is critical that we recognize that each variable must be replaced by whatever expression appears in parentheses.

So far, all of the previous examples have shown how to find an output when given a specific input. Next, we will demonstrate how one can also algebraically find which input(s) yield a required output.

Example 4.13. Given $f(x) = x^2 + 3x + 5$, find all x such that $f(x) = 5$.

$$
\begin{aligned}
f(x) &= x^2 + 3x + 5 && \text{Substitute 5 in for } f(x)\\
5 &= x^2 + 3x + 5 && \text{Solve for } x \text{ by factoring}\\
0 &= x^2 + 3x && \text{Set equal to 0}\\
0 &= x(x+3) && \text{Factor}\\
x = 0 &\text{ or } x = -3 && \text{Our solutions}
\end{aligned}
$$

The above answer can be verified by checking. When we input $x = 0$ into the function, we simplify to find that $f(0) = 5$. Similarly, we see that when $x = -3$, $f(-3) = 5$.

Example 4.14. Given $h(x) = 4x - 1$, find all x such that $h(x) = -3$.

$$\begin{aligned} h(x) &= 4x - 1 && \text{Substitute } -3 \text{ for } h(x) \\ -3 &= 4x - 1 && \text{Solve for } x \\ -2 &= 4x && \text{Divide} \\ x &= -\frac{1}{2} && \text{Our solution} \end{aligned}$$

It is important that we become comfortable with function notation and how to use it, as we begin to transition to more advanced algebraic concepts.

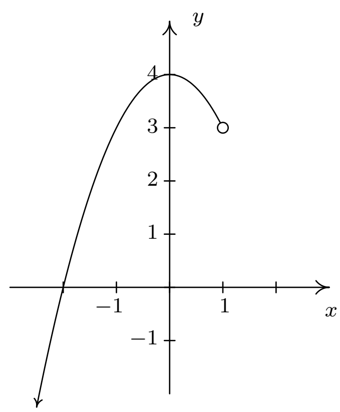
4.2 Domain and Range

4.2.1 Identifying Domain and Range Graphically

Objective: Identify the domain and range of a function that is described either graphically or algebraically. Establish several fundamental functions and identify their domain and range.

In this section, we will first discuss how one can identify the domain and range of a function using its graph. Later, we will explore finding the domain of a function using algebraic methods. As finding the range of a function using algebraic methods can often prove quite challenging, we will postpone this topic for a later section. We conclude the section with an introduction to a few of the most fundamental and foundational functions in algebra.

Example 4.15. Find the domain and range of the function f whose graph is given below.

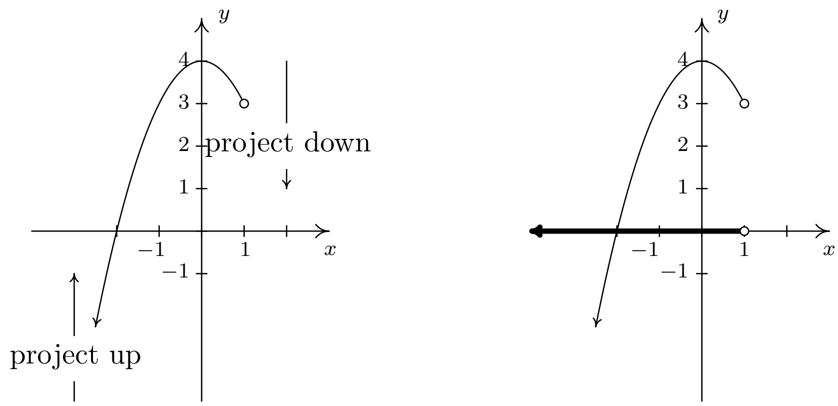


The graph of f

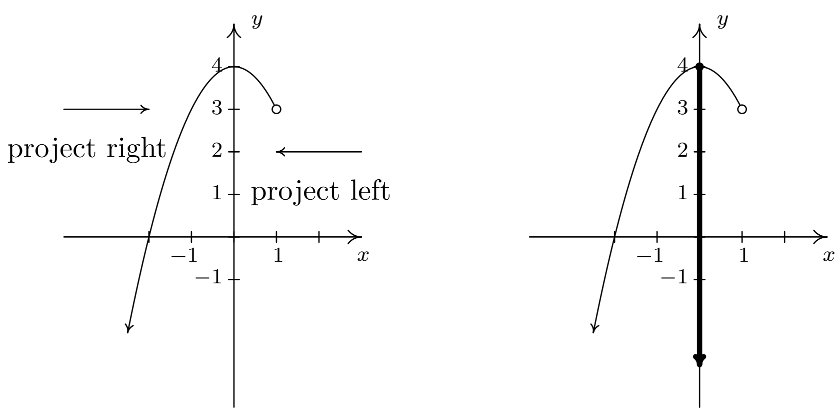
To determine the domain and range of f, we need to determine which x and y-values respectively occur as coordinates of points on the given graph.

To find the domain, it will be helpful to imagine collapsing the curve onto the x-axis and determining the portion of the x-axis that gets covered. This is often described as **projecting** the curve onto the x-axis.

Before we project, we need to pay attention to two subtle notations on the graph: the arrowhead on the lower left corner of the graph indicates that the graph continues to curve downwards to the left forever; and the open circle at $(1, 3)$ indicates that the point $(1, 3)$ is *not* on the graph, but all the points on the curve leading up to $(1, 3)$ are on the graph.



We see from the figure that if we project the graph of f to the x-axis, we get all real numbers less than 1. Using interval notation, we write the domain of f as $(-\infty, 1)$.
To determine the range of f, we use a similar method, projecting the curve onto the y-axis as follows.



Note that even though there is an open circle at $(1, 3)$, we still include the y value of 3 in our range, since the point $(-1, 3)$ is on the graph of f. We also include $y = 4$ in our answer, since the point $(0, 4)$ is also on our graph. Consequently, the range of f is all real numbers less than or equal to 4, or $(-\infty, 4]$.

4.2.2 Identifying Domain and Range Algebraically

When trying to identify the domain of a function that has been described algebraically or whose graph is not known, we will often need to consider what is *not* permissible for the function, then exclude any values of x that will make the function undefined from the interval $(-\infty, \infty)$. What is left will be our domain. With virtually every algebraic function, this amounts to avoiding the following situations.

- Negatives under an even radical $\left(\sqrt{\ }\ ,\ \ \sqrt[4]{\ }\ ,\ \ \sqrt[6]{\ }\ ,\ \ldots\right)$
- Zero in a denominator

In the previous three chapters, we dealt exclusively with linear and quadratic equations. While both linears and quadratics also represent y as a function of x, they are also included in a much larger family of functions known as *polynomials*. Polynomial functions, which are discussed in detail in the next chapter, are functions of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x + a_1 x + a_0,$$

where each of the coefficients a_i represent real numbers (with $a_n \neq 0$) and n represents a positive integer. Since polynomials contain no radicals or variables in a denominator, we can immediately conclude that their domain will always be all real numbers, or $(-\infty, \infty)$. We reiterate this with our first example.

Example 4.16. Find the domain of $f(x) = \frac{1}{3}x^2 - x$.

$f(x) = \dfrac{1}{3}x^2 - x$	No radicals or variables in a denominator
	No values of x need to be excluded
All real numbers or $(-\infty, \infty)$	Our solution

Our next example will be of a *rational function*, which is defined as a ratio of two polynomial functions. We will explore rational functions and their graphs in a later chapter. Since rational functions usually include expressions in a denominator, their domains will often require us to exclude one or more values of x.

Example 4.17. Find the domain of the function $f(x) = \dfrac{3x-1}{x^2+x-6}$.

$$\begin{aligned} f(x) &= \frac{3x-1}{x^2+x-6} && \text{Cannot have zero in a denominator} \\ x^2+x-6 &\neq 0 && \text{Solve by factoring} \\ (x+3)(x-2) &\neq 0 && \text{Set each factor not equal to zero} \\ x+3\neq 0 \text{ and } x-2 &\neq 0 && \text{Solve each inequality} \\ x &\neq -3, 2 && \text{Our solution as an inequality} \\ (-\infty,-3)\cup(-3,2)\cup(2,\infty) & && \text{Our solution using interval notation} \end{aligned}$$

The notation in the previous example tells us that x can be any value except for -3 and 2. If x were to equal one of those two values, our expression in the denominator would reduce to zero and the function would consequently be undefined. Furthermore, although one can easily see that $x = \frac{1}{3}$ will make the numerator equal zero, since $x = \frac{1}{3}$ does not coincide with the two values obtained above (either -3 or 2), we should not exclude it from our domain.

This example further illustrates that whenever we are finding the domain of a rational function, we need not be concerned at all with the numerator, and instead must restrict our domain to exclude any value for x that would make the *denominator* equal to zero.

For our final two examples, we will introduce a square root in our function, first in the numerator and later in the denominator.

Example 4.18. Find the domain of $f(x) = \sqrt{-2x+3}$.

$$\begin{aligned} f(x) &= \sqrt{-2x+3} && \text{Even radical; cannot have negative underneath} \\ -2x+3 &\geq 0 && \text{Set greater than or equal to zero and solve} \\ -2x &\geq -3 && \text{Remember to switch direction of inequality} \\ x \leq \frac{3}{2} \text{ or } \left(-\infty, \frac{3}{2}\right] & && \text{Our solution as an inequality or an interval} \end{aligned}$$

The notation in the above example states that our variable can be $\frac{3}{2}$ or any real number less than $\frac{3}{2}$. But any number greater than $\frac{3}{2}$ would make the function undefined.

Example 4.19. Find the domain of $m(x) = \dfrac{-x}{\sqrt{7x-3}}$.

The even radical tells us that we cannot have a negative value underneath. But also, the denominator cannot equal zero. This results in two inequalities.

$$7x - 3 \geq 0 \ \text{ AND } \ 7x - 3 \neq 0$$

Solving for x, we get the following.

$$x \geq \frac{3}{7} \ \text{ AND } \ x \neq \frac{3}{7}$$

Our final solution is $x > \frac{3}{7}$, or $\left(\frac{3}{7}, \infty\right)$ as an interval. This represents the intersection of both inequalities above.

The previous two examples can be generalized as follows.

- In instances where the given function is a square root (or even radical), to find the domain we may set up and solve an inequality in which the entire expression underneath is set ≥ 0.
- In instances where the numerator of a given function is a polynomial and the denominator is a square root (or even radical), to find the domain we may set up and solve an inequality in which the expression underneath is set > 0 (strictly positive).

Since these two cases certainly do not handle every possible function than we may encounter, one should always be cautious when attempting to find the domain of any function.

4.2.3 Ten Fundamental Functions

This subsection will be completed at a later date. Extra pages have been included for instructors and students to list their own fundamental functions. This list will most likely include one or more of the following functions.

1. $f(x) = x$
2. $g(x) = x^2$
3. $h(x) = \sqrt{x}$
4. $k(x) = \sqrt{9 - x^2}$
5. $\ell(x) = |x|$
6. $m(x) = x^3$
7. $n(x) = \sqrt[3]{x}$
8. $p(x) = \dfrac{1}{x}$
9. $r(x) = 2^x$
10. $s(x) = \log_2(x)$

4.3 Combining Functions

4.3.1 Function Arithmetic

Objective: Combine functions using sum, difference, product, quotient and composition.

In this section, we demonstrate how two (or more) functions can be combined to create new functions. This is accomplished using five common operations: the four basic arithmetic operations of addition, subtraction, multiplication and division, and a fifth operation that we will establish later in the section, known as a *composition*.

The notation for the four basic functions is as follows.

$$\begin{array}{rl}
\text{Addition} & (f+g)(x) = f(x) + g(x) \\
\text{Subtraction} & (f-g)(x) = f(x) - g(x) \\
\text{Multiplication} & (f \cdot g)(x) = f(x)g(x) \\
\text{Division} & \left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}, \text{ where } g(x) \neq 0
\end{array}$$

As we will see in the next few examples, when applying the specified operations, one must be careful to completely simplify, by distributing and combining like terms where it is necessary. We will demonstrate this for each operation, highlighting the most critical steps in the process.

Example 4.20. Find $f+g$, where $f(x) = x^2 - x - 2$ and $g(x) = x + 1$.

$$\begin{array}{rl}
(f+g)(x) & \text{Consider the problem} \\
f(x) + g(x) & \text{Rewrite as a sum of two functions} \\
(x^2 - x - 2) + (x + 1) & \text{Substitute functions, inserting parentheses} \\
x^2 - x - 2 + x + 1 & \text{Simplify; remove the parentheses} \\
x^2 - x + x - 2 + 1 & \text{Combine like terms} \\
(f+g)(x) = x^2 - 1 & \text{Our solution} \\
= (x-1)(x+1) & \text{Our solution in factored form}
\end{array}$$

We include the factored form of $f+g$ in the previous example to reinforce the methods of factorization learned in an earlier chapter. Generally, either form (expanded or factored) would be considered acceptable.

Although the parentheses are not entirely necessary in our first example, we have included them nevertheless, to reinforce that each operation is applied to an *entire* function or expression. This will become more apparent in our next example (subtraction), when we will need to distribute a negative sign.

Example 4.21. Find $g - f$, where $f(x) = x^2 - x - 2$ and $g(x) = x + 1$.

$(g - f)(x)$	Consider the problem
$g(x) - f(x)$	Rewrite as a difference of two functions
$(x + 1) - (x^2 - x - 2)$	Substitute functions, inserting parentheses
$x + 1 - x^2 + x + 2$	Simplify; distribute the negative sign
$-x^2 + x + x + 1 + 2$	Combine like terms
$(g - f)(x) = -x^2 + 2x + 3$	Our solution
$= -(x - 3)(x + 1)$	Our solution in factored form

Example 4.22. Find $(h \cdot k)(x)$, where $h(x) = 3x^2 - 4x$ and $k(x) = x - 2$.

$(h \cdot k)(x)$	Consider the problem
$h(x) \cdot k(x)$	Rewrite as a product of two functions
$(3x^2 - 4x)(x - 2)$	Substitute functions, inserting parentheses
$3x^3 - 6x^2 - 4x^2 + 8x$	Expand by distributing
$3x^3 - 10x^2 + 8x$	Combine like terms
$(h \cdot k)(x) = 3x^3 - 10x^2 + 8x$	Our solution
$= x(3x - 4)(x - 2)$	Our solution in factored form

Example 4.23. Find $\left(\frac{g}{f}\right)(x)$, where $f(x) = x^2 - x - 2$ and $g(x) = x + 1$.

$\left(\frac{g}{f}\right)(x)$	Consider the problem
$\frac{g(x)}{f(x)}$	Rewrite as a quotient of two functions
$\frac{x + 1}{x^2 - x - 2}$	Substitute functions, parentheses unnecessary

$$\frac{x+1}{(x+1)(x-2)} \qquad \text{Factor (if possible)}$$

$$x \neq -1 \quad \text{and} \quad x \neq 2 \qquad \text{Restrict denominator: } g(x) \neq 0$$

$$\frac{\cancel{x+1}}{\cancel{(x+1)}(x-2)} \qquad \text{Simplify: reduce } \frac{x+1}{x+1}$$

$$\left(\frac{g}{f}\right)(x) = \frac{1}{x-2}, \quad x \neq -1 \qquad \text{Our solution with added restriction}$$

The previous example presents us with a new precautionary measure that we must be careful not to overlook. This has to do with the simplification of g/f and the requirement that we include the necessary restriction of $x \neq -1$. Although the *domain* of the resulting quotient is still $x \neq -1, 2$, we have included $x \neq -1$ as part of our final answer, since the simplified expression allows us to easily determine that x cannot equal 2, but fails to carry through the additional restriction.

In general, whenever we simplify any function, we must be careful to insure that the domain of the resulting expression will be in agreement with the initial *unsimplified* expression. In the chapter on rational functions, we will see the graphical consequence that arises when the restriction $x \neq -1$ is overlooked.

Thus far, we have sought to create new functions by combining two functions f and g accordingly, keeping the variable x in place throughout. We could, however, just as easily evaluate the functions $f+g$, $f-g$, $f \cdot g$, and f/g at certain values of x. We do this in our next example.

Example 4.24. Find $(h \cdot k)(5)$, where $h(x) = 2x - 4$ and $k(x) = -3x + 1$.

$$h(x) = 2x - 4 \text{ and } k(x) = -3x + 1 \qquad \text{Evaluate each function at } 5$$

$$h(5) = 2(5) - 4 = 6 \qquad \text{Evaluate } h \text{ at } 5$$

$$k(5) = -3(5) + 1 = -14 \qquad \text{Evaluate } k \text{ at } 5$$

$$\begin{aligned} (h \cdot k)(5) &= (h(5)) \cdot (k(5)) && \text{Multiply the two results} \\ &= (6)(-14) \\ &= -84 && \text{Our solution} \end{aligned}$$

The clear advantage to this process is that the simplification can be substantially easier when the variable has been replaced with a constant. One major disadvantage, however, is that our end result represents only a single value, instead of an entire function. Particularly in situations where the resulting function is not demanded, students will likely find it more efficient to use this approach when evaluating $f+g$, $f-g$, $f \cdot g$ and f/g at a specified value.

4.3.2 Composite Functions

In addition to the four basic arithmetic operations $(+, -, \cdot, \div)$, we will now discuss a fifth operation, known as a *composition* and denoted by $\circ$ (not to be confused with a product, $\cdot$). The result of a composition is called a *composite function* and is defined as follows.

$$\boldsymbol{(f \circ g)(x) = f(g(x))}$$

The notation $(f \circ g)(x)$ above should always be interpreted as "f of g of x". In this situation, we consider g to be the *inner* function, since it is being substituted into f for x. Consequently, we refer to f as the *outer* function.

Similarly, if we reversed the order of the two functions f and g, then the resulting composite function $(g \circ f)(x) = g(f(x))$ will have inner function f and outer function g, and should be interpreted as "g of f of x". As we will see, one should never assume that the two composite functions $f \circ g$ and $g \circ f$ will be equal.

The idea behind a composition, though relatively simple, can often pose a formidable challenge at first. We will begin by evaluating a composite function at a single value. This is accomplished by first evaluating the inner function at the specified value, and then substituting ("plugging in") the corresponding *output* into the outer function.

Example 4.25. Find $(f \circ g)(3)$, where $f(x) = x^2 - 2x + 1$ and $g(x) = x - 5$.

$(f \circ g)(3) = f(g(3))$	Rewrite $f \circ g$ as inner and outer functions
$g(3) = (3) - 5 = -2$	Evaluate inner function at $x = 3$
	Use output of -2 as input for f
$f(-2) = (-2)^2 - 2(-2) + 1$	Evaluate outer function at $x = -2$
$= 4 + 4 + 1$	Simplify
$(f \circ g)(3) = 9$	Our solution

We can also identify a composite function in terms of the variable. In the next example, we will substitute the inner function into the outer function for every instance of the variable and then simplify. This approach is often referred to as the "inside-out" approach by some instructors.

Example 4.26. Find $(f \circ g)(x)$, where $f(x) = x^2 - x$ and $g(x) = x + 3$.

$(f \circ g)(x) = f(g(x))$	Rewrite $f \circ g$ as inner and outer functions
	Our inner function is $g(x) = x + 3$
$f(x+3)$	Replace each x in f with $(x+3)$
	Make sure to include parentheses!
$(x+3)^2 - (x+3)$	Simplify; expand binomial
$(x^2 + 6x + 9) - (x+3)$	Distribute negative
$x^2 + 6x + 9 - x - 3$	Combine like terms
$(f \circ g)(x) = x^2 + 5x + 6$	Our solution
$= (x+3)(x+2)$	Our solution in factored form

It is important to reiterate that $(f \circ g)(x)$ usually will *not* equal $(g \circ f)(x)$ as the next example shows. Again, we will take the "inside-out" approach, where the inner function is now f and the outer function is g.

Example 4.27. Find $(g \circ f)(x)$, where $f(x) = x^2 - x$ and $g(x) = x + 3$.

$(g \circ f)(x) = g(f(x))$	Rewrite $g \circ f$ as inner and outer functions
	Our inner function is $f(x) = x^2 - x$
$g(x^2 - x)$	Replace each x in f with $(x^2 - x)$
$(x^2 - x) + 3$	Simplify; remove parentheses
$(g \circ f)(x) = x^2 - x + 3$	Our solution

Notice that a simple calculation of the discriminant,

$$b^2 - 4ac = (-1)^2 - 4(1)(3) = -11 < 0,$$

tells us that the resulting composite function is irreducible (not factorable) over the real numbers.

We close this section by demonstrating the "outside-in" approach to finding a composite function $f \circ g$. The idea behind this approach is to *first* rewrite the outer function f by its given expression, replacing each instance of the variable with the general $g(x)$. To see that this will yield the same result as the "inside-out" approach, we will revisit example 4.26 above.

Example 4.28. Find $(f \circ g)(x)$, where $f(x) = x^2 - x$ and $g(x) = x + 3$.

$(f \circ g)(x) = f(g(x))$	Rewrite $f \circ g$ as inner and outer functions
	Our outer function is $f(x) = x^2 - x$
$[g(x)]^2 - [g(x)]$	Replace each x in f with $g(x)$
$(x+3)^2 - (x+3)$	Replace each $g(x)$ by $x + 3$
	Make sure to include parentheses!
$(x^2 + 6x + 9) - (x + 3)$	Simplify; expand binomial
$x^2 + 6x + 9 - x - 3$	Distribute negative
$x^2 + 5x + 6$	Combine like terms
$(f \circ g)(x) = x^2 + 5x + 6$	Our solution
$= (x+3)(x+2)$	Our solution in factored form

World View Note: The term "function" came from Gottfried Wilhelm Leibniz, a German mathematician from the late 17th century.

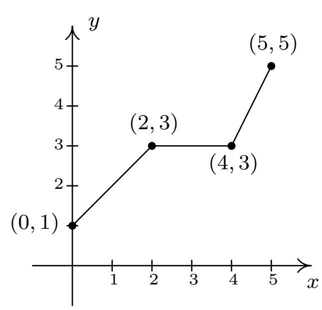
4.4 Transformations

Note: This section has been taken in its entirety from Carl Stitz and Jeff Zeager's *College Algebra* text. As of this version of the text, it remains largely unedited. In many of the examples that follow, graphs have been intentionally left blank for the student to complete directly in the text, so as to further reinforce each concept.

4.4.1 Shifts

Objective: Graph functions by translating up, down, left or right

In this section, we study how the graphs of functions change, or **transform**, when certain specialized modifications are made to their formulas. The transformations we will study fall into three broad categories: shifts, reflections and scalings, and we will present them in that order. Suppose the graph below is the complete graph of a function f.

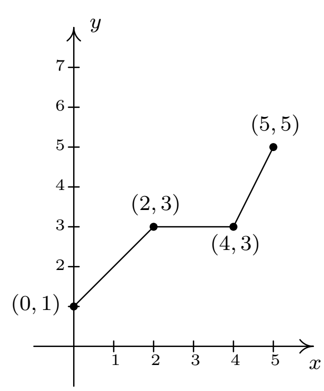


$y = f(x)$

The Fundamental Graphing Principle for Functions says that for a point (a, b) to be on the graph, $f(a) = b$. In particular, we know $f(0) = 1$, $f(2) = 3$, $f(4) = 3$ and $f(5) = 5$. Suppose we wanted to graph the function defined by the formula $g(x) = f(x) + 2$. Let's take a minute to remind ourselves of what g is doing. We start with an input x to the function f and we obtain the output $f(x)$. The function g takes the output $f(x)$ and adds 2 to it. In order to graph g, we need to graph the points $(x, g(x))$. How are we to find the values for $g(x)$ without a formula for $f(x)$? The answer is that we don't need a *formula* for $f(x)$, we just need the *values* of $f(x)$. The values of $f(x)$ are the y values on the graph of $y = f(x)$. For example, using the points indicated on the graph of f, we can make the following table.

x	$(x, f(x))$	$f(x)$	$g(x) = f(x) + 2$	$(x, g(x))$
0	$(0, 1)$	1	3	$(0, 3)$
2	$(2, 3)$	3	5	$(2, 5)$
4	$(4, 3)$	3	5	$(4, 5)$
5	$(5, 5)$	5	7	$(5, 7)$

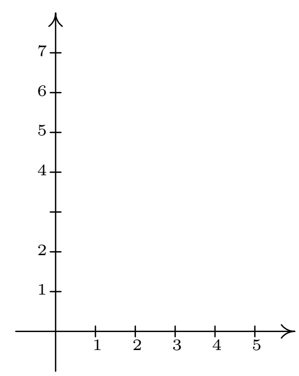
In general, if (a, b) is on the graph of $y = f(x)$, then $f(a) = b$, so $g(a) = f(a) + 2 = b + 2$. Hence, $(a, b + 2)$ is on the graph of g. In other words, to obtain the graph of g, we add 2 to the y-coordinate of each point on the graph of f. Geometrically, adding 2 to the y-coordinate of a point moves the point 2 units above its previous location. Adding 2 to every y-coordinate on a graph *en masse* is usually described as 'shifting the graph up 2 units'. Notice that the graph retains the same basic shape as before, it is just 2 units above its original location. In other words, we connect the four points we moved in the same manner in which they were connected before. We have the results side-by-side below.



$y = f(x)$

shift up 2 units

add 2 to each y-coordinate



$y = g(x) = f(x) + 2$

You'll note that the domain of f and the domain of g are the same, namely $[0, 5]$, but that the range of f is $[1, 5]$ while the range of g is $[3, 7]$. In general, shifting a function vertically like this will leave the domain unchanged, but could very well affect the range. You can easily imagine what would happen if we wanted to graph the function $j(x) = f(x) - 2$. Instead of adding 2 to each of the y-coordinates on the graph of f, we'd be subtracting 2. Geometrically, we would be moving the graph down 2 units. We leave it to the reader to verify that the domain of j is the same as f, but the range of j is $[-1, 3]$. What we have discussed is generalized in the following theorem.

Theorem 4.1. Vertical Shifts Suppose f is a function and k is a positive number.

- To graph $y = f(x) + k$, shift the graph of $y = f(x)$ up k units by adding k to the y-coordinates of the points on the graph of f.
- To graph $y = f(x) - k$, shift the graph of $y = f(x)$ down k units by subtracting k from the y-coordinates of the points on the graph of f.

The key to understanding Theorem 4.1 and, indeed, all of the theorems in this section comes from an understanding of the Fundamental Graphing Principle for Functions. If (a, b) is on the graph of f, then $f(a) = b$. Substituting $x = a$ into the equation $y = f(x) + k$ gives $y = f(a) + k = b + k$. Hence, $(a, b + k)$ is on the graph of $y = f(x) + k$, and we have the result. In the language of 'inputs' and 'outputs', Theorem 4.1 can be paraphrased as "Adding to, or subtracting from, the *output* of a function causes the graph to shift up or down, respectively." So what happens if we add to or subtract from the *input* of the function?

Keeping with the graph of $y = f(x)$ above, suppose we wanted to graph $g(x) = f(x + 2)$. In other words, we are looking to see what happens when we add 2 to the input of the function. It is worth noting that $f(x + 2)$ and $f(x) + 2$ are, in general, wildly different algebraic animals. We will see momentarily that their geometry is also dramatically different. Let's try to generate a table of values of g based on those we know for f. We quickly find that we run into some difficulties.

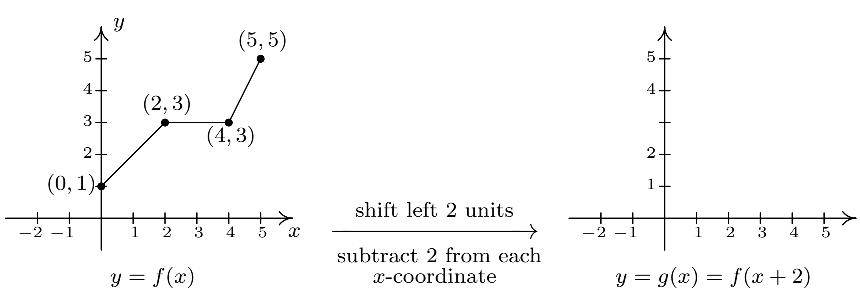
x	$(x, f(x))$	$f(x)$	$g(x) = f(x + 2)$	$(x, g(x))$
0	$(0, 1)$	1	$f(0 + 2) = f(2) = 3$	$(0, 3)$
2	$(2, 3)$	3	$f(2 + 2) = f(4) = 3$	$(2, 3)$
4	$(4, 3)$	3	$f(4 + 2) = f(6) = ?$	
5	$(5, 5)$	5	$f(5 + 2) = f(7) = ?$	

When we substitute $x = 4$ into the formula $g(x) = f(x + 2)$, we are asked to find $f(4 + 2) = f(6)$ which doesn't exist because the domain of f is only $[0, 5]$. The same thing happens when we attempt to find $g(5)$. What we need here is a new strategy. We know, for instance, $f(0) = 1$. To determine the corresponding point on the graph of g, we need to figure out what value of x we must substitute into $g(x) = f(x+2)$ so that the quantity $x+2$, works out to be 0. Solving $x+2 = 0$ gives $x = -2$, and $g(-2) = f((-2)+2) = f(0) = 1$ so $(-2, 1)$ is on the graph of g. To use the fact $f(2) = 3$, we set $x + 2 = 2$

to get $x = 0$. Substituting gives $g(0) = f(0+2) = f(2) = 3$. Continuing in this fashion, we get

x	$x+2$	$g(x) = f(x+2)$	$(x, g(x))$
-2	0	$g(-2) = f(0) = 1$	$(-2, 1)$
0	2	$g(0) = f(2) = 3$	$(0, 3)$
2	4	$g(2) = f(4) = 3$	$(2, 3)$
3	5	$g(3) = f(5) = 5$	$(3, 5)$

In summary, the points $(0, 1)$, $(2, 3)$, $(4, 3)$ and $(5, 5)$ on the graph of $y = f(x)$ give rise to the points $(-2, 1)$, $(0, 3)$, $(2, 3)$ and $(3, 5)$ on the graph of $y = g(x)$, respectively. In general, if (a, b) is on the graph of $y = f(x)$, then $f(a) = b$. Solving $x+2 = a$ gives $x = a-2$ so that $g(a-2) = f((a-2)+2) = f(a) = b$. As such, $(a-2, b)$ is on the graph of $y = g(x)$. The point $(a-2, b)$ is exactly 2 units to the *left* of the point (a, b) so the graph of $y = g(x)$ is obtained by shifting the graph $y = f(x)$ to the left 2 units, as pictured below.



Note that while the ranges of f and g are the same, the domain of g is $[-2, 3]$ whereas the domain of f is $[0, 5]$. In general, when we shift the graph horizontally, the range will remain the same, but the domain could change. If we set out to graph $j(x) = f(x-2)$, we would find ourselves *adding* 2 to all of the x values of the points on the graph of $y = f(x)$ to effect a shift to the *right* 2 units. Generalizing these notions produces the following result.

Theorem 4.2. Horizontal Shifts Suppose f is a function and h is a positive number.

- To graph $y = f(x + h)$, shift the graph of $y = f(x)$ left h units by subtracting h from the x-coordinates of the points on the graph of f.
- To graph $y = f(x - h)$, shift the graph of $y = f(x)$ right h units by adding h to the x-coordinates of the points on the graph of f.

In other words, Theorem 4.2 says that adding to or subtracting from the *input* to a function amounts to shifting the graph left or right, respectively. Theorems 4.1 and 4.2 present a theme which will run common throughout the section: changes to the outputs from a function affect the y-coordinates of the graph, resulting in some kind of vertical change; changes to the inputs to a function affect the x-coordinates of the graph, resulting in some kind of horizontal change.

4.4.2 Reflections

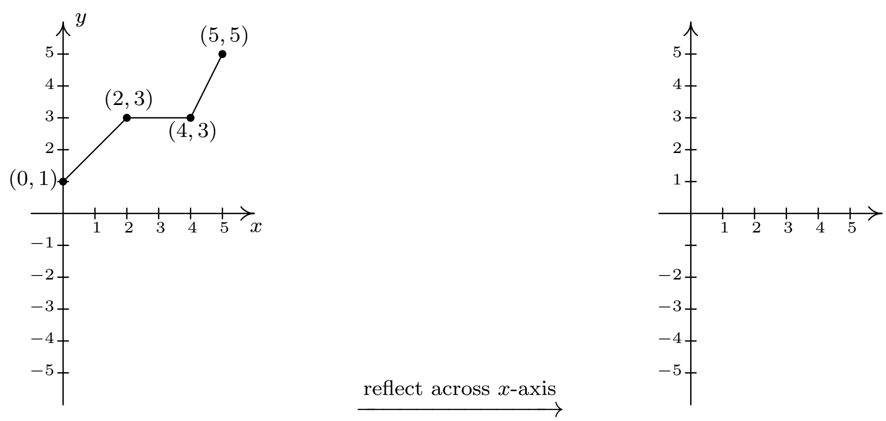
Objective: Graph functions by reflecting vertically or horizontally

We now turn our attention to reflections. As we know from graphing quadratics, multiplying the output from a function by -1 reflects its graph across the x-axis. Alternatively, it can be verified that multiplying the input to a function by -1 reflects the graph across the y-axis.

Theorem 4.3. Reflections Suppose f is a function.

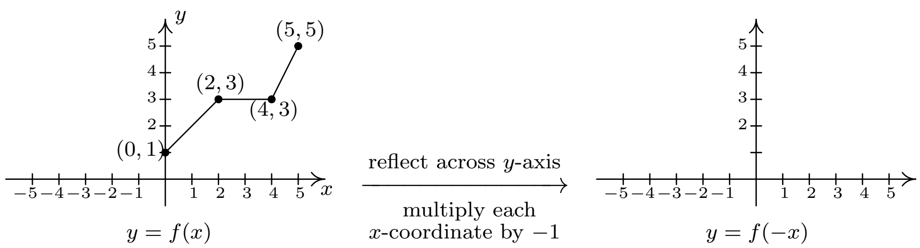
- To graph $y = -f(x)$, reflect the graph of $y = f(x)$ across the x-axis by multiplying the y-coordinates of the points on the graph of f by -1.
- To graph $y = f(-x)$, reflect the graph of $y = f(x)$ across the y-axis by multiplying the x-coordinates of the points on the graph of f by -1.

Applying Theorem 4.3 to the graph of $y = f(x)$ given at the beginning of the section, we can graph $y = -f(x)$ by reflecting the graph of f about the x-axis



$y = f(x)$ multiply each y-coordinate by -1 $y = -f(x)$

By reflecting the graph of f across the y-axis, we obtain the graph of $y = f(-x)$.



With the addition of reflections, it is now more important than ever to consider the order of transformations, as the next example illustrates.

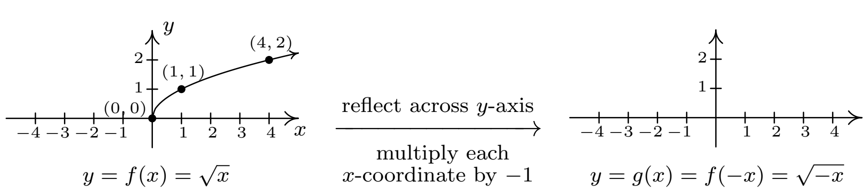
Example 4.29. Let $f(x) = \sqrt{x}$. Use the graph of f to graph the following functions. Also, state their domains and ranges.

1. $g(x) = \sqrt{-x}$
2. $j(x) = \sqrt{3-x}$
3. $m(x) = 3 - \sqrt{x}$

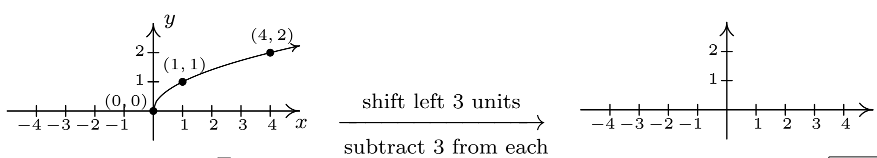
Solution.

1. Looking at a $\sqrt{-x}$ may seem unacceptable, especially after we disallowed negatives from the radicands of even roots having to make those solutions be non-real.

   However, we must remember that x is a variable, and as such, the quantity $-x$ isn't always negative. For example, if $x = -4$, $-x = 4$, thus $\sqrt{-x} = \sqrt{-(-4)} = 2$ is perfectly well-defined. To find the domain analytically, we set $-x \geq 0$ which gives $x \leq 0$, so that the domain of g is $(-\infty, 0]$. Since $g(x) = f(-x)$, Theorem 4.3 tells us that the graph of g is the reflection of the graph of f across the y-axis. We accomplish this by multiplying each x-coordinate on the graph of f by -1, so that the points $(0,0)$, $(1,1)$, and $(4,2)$ move to $(0,0)$, $(-1,1)$, and $(-4,2)$, respectively. Graphically, we see that the domain of g is $(-\infty, 0]$ and the range of g is the same as the range of f, namely $[0,\infty)$.



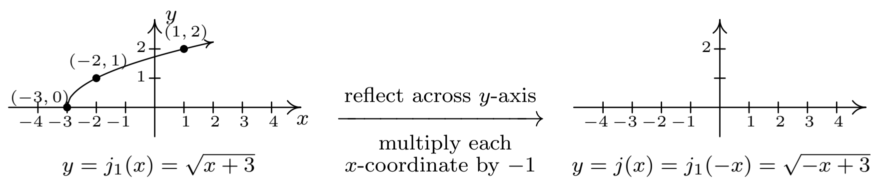
2. To determine the domain of $j(x) = \sqrt{3-x}$, we solve $3 - x \geq 0$ and get $x \leq 3$, or $(-\infty, 3]$. To determine which transformations we need to apply to the graph of f to obtain the graph of j, we rewrite $j(x) = \sqrt{-x+3} = f(-x+3)$. Comparing this formula with $f(x) = \sqrt{x}$, we see that not only are we multiplying the input x by -1, which results in a reflection across the y-axis, but also we are adding 3, which indicates a horizontal shift to the left. Does it matter in which order we do the transformations? If so, which order is the correct order? Let's consider the point $(4, 2)$ on the graph of f. We refer to the discussion leading up to Theorem 4.2. We know $f(4) = 2$ and wish to find the point on $y = j(x) = f(-x+3)$ which corresponds to $(4, 2)$. We set $-x+3 = 4$ and solve. Our first step is to subtract 3 from both sides to get $-x = 1$. Subtracting 3 from the x-coordinate 4 is shifting the point $(4, 2)$ to the left. From $-x = 1$, we then multiply (or divide) both sides by -1 to get $x = -1$. Multiplying the x-coordinate by -1 corresponds to reflecting the point about the y-axis. Hence, we perform the horizontal shift first, then follow it with the reflection about the y-axis. Starting with $f(x) = \sqrt{x}$, we let $j_1(x)$ be the intermediate function which shifts the graph of f 3 units to the left, $j_1(x) = f(x+3)$.



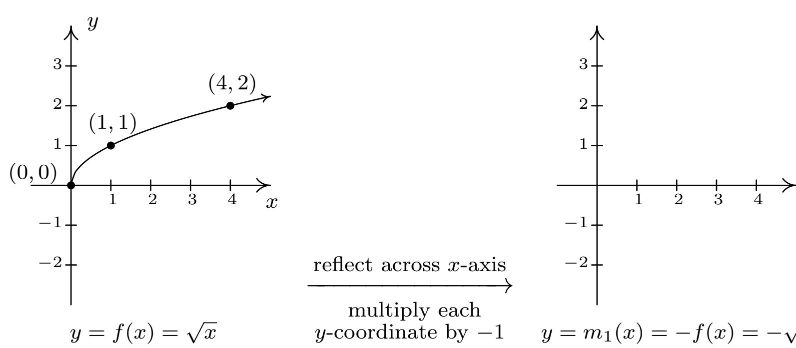
$y = f(x) = \sqrt{x}$ $\qquad$ $y = j_1(x) = f(x+3) = \sqrt{x+3}$

To obtain the function j, we reflect the graph of j_1 about y-axis. Theorem 4.3 tells us we have $j(x) = j_1(-x)$. Putting it all together, we have $j(x) = j_1(-x) = f(-x+3) = \sqrt{-x+3}$, which is what we want.[1] From the graph, we confirm the domain of j is $(-\infty, 3]$ and we get that the range is $[0, \infty)$.

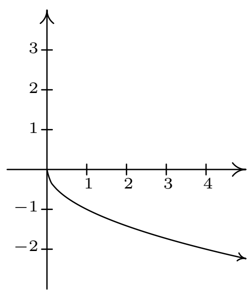
[1] If we had done the reflection first, then $j_1(x) = f(-x)$. Following this by a shift left would give us $j(x) = j_1(x+3) = f(-(x+3)) = f(-x-3) = \sqrt{-x-3}$ which isn't what we want. However, if we did the reflection first and followed it by a shift to the right 3 units, we would have arrived at the function $j(x)$. We leave it to the reader to verify the details.



3. The domain of m works out to be the domain of f, $[0, \infty)$. Rewriting $m(x) = -\sqrt{x} + 3$, we see $m(x) = -f(x) + 3$. Since we are multiplying the output of f by -1 and then adding 3, we once again have two transformations to deal with: a reflection across the x-axis and a vertical shift. To determine the correct order in which to apply the transformations, we imagine trying to determine the point on the graph of m which corresponds to $(4, 2)$ on the graph of f. Since in the formula for $m(x)$, the input to f is just x, we substitute to find $m(4) = -f(4) + 3 = -2 + 3 = 1$. Hence, $(4, 1)$ is the corresponding point on the graph of m. If we closely examine the arithmetic, we see that we first multiply $f(4)$ by -1, which corresponds to the reflection across the x-axis, and then we add 3, which corresponds to the vertical shift. If we define an intermediate function $m_1(x) = -f(x)$ to take care of the reflection, we get

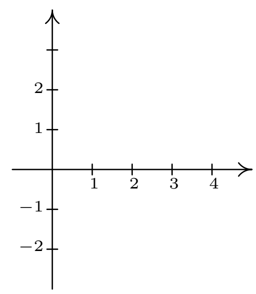


To shift the graph of m_1 up 3 units, we set $m(x) = m_1(x) + 3$. Since $m_1(x) = -f(x)$, when we put it all together, we get $m(x) = m_1(x) + 3 = -f(x) + 3 = -\sqrt{x} + 3$. We see from the graph that the range of m is $(-\infty, 3]$.



$y = m_1(x) = -\sqrt{x}$

$$\xrightarrow[\text{add 3 to each } y\text{-coordinate}]{\text{shift up 3 units}}$$



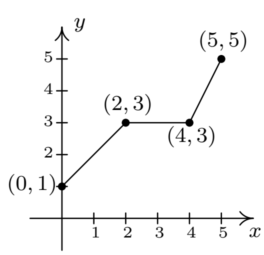
$y = m(x) = m_1(x) + 3 = -\sqrt{x} + 3$

4.4.3 Scalings

Objective: Graph functions by stretching or shrinking

We now turn our attention to our last class of transformations known as **scalings**. A thorough discussion of scalings can get complicated because they are not as straight-forward as the previous transformations. A quick review of what we've covered so far, namely vertical shifts, horizontal shifts and reflections, will show you why those transformations are known as **rigid transformations**. Simply put, they do not change the *shape* of the graph, only its position and orientation in the plane. If, however, we wanted to make a new graph twice as tall as a given graph, or one-third as wide, we would be changing the shape of the graph. This type of transformation is called **non-rigid** for obvious reasons. Not only will it be important for us to differentiate between modifying inputs versus outputs, we must also pay close attention to the magnitude of the changes we make. As you will see shortly, the Mathematics turns out to be easier than the associated grammar.

Suppose we wish to graph the function $g(x) = 2f(x)$ where $f(x)$ is the function whose graph is given at the beginning of the section. From its graph, we can build a table of values for g as before.

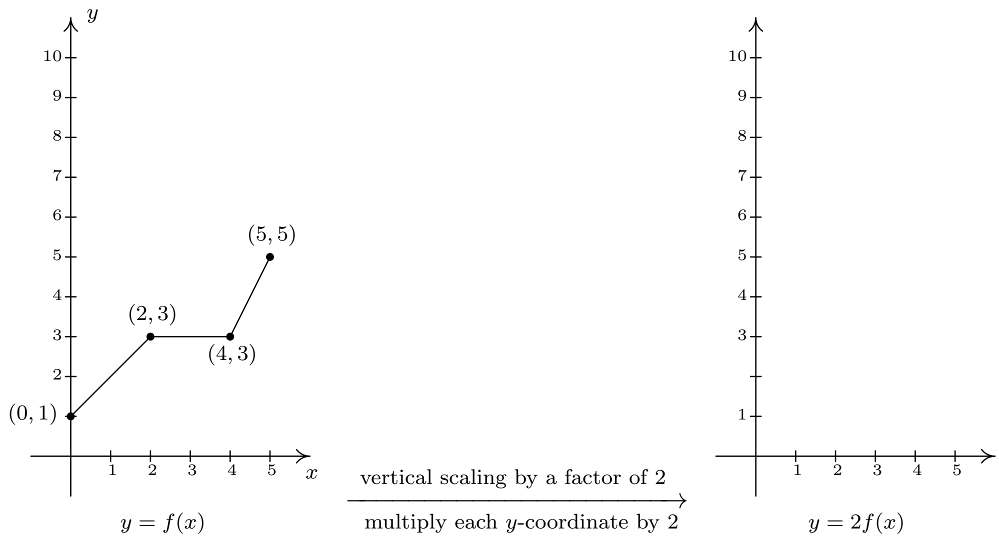


$y = f(x)$

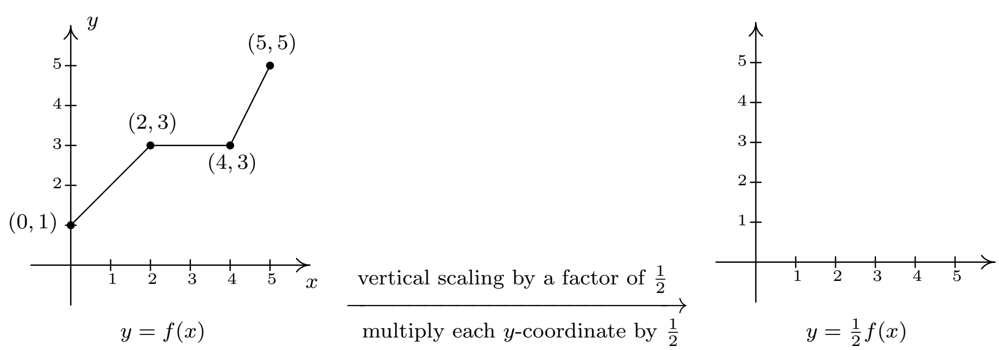
x	$(x, f(x))$	$f(x)$	$g(x) = 2f(x)$	$(x, g(x))$
0	$(0, 1)$	1	2	$(0, 2)$
2	$(2, 3)$	3	6	$(2, 6)$
4	$(4, 3)$	3	6	$(4, 6)$
5	$(5, 5)$	5	10	$(5, 10)$

In general, if (a, b) is on the graph of f, then $f(a) = b$ so that $g(a) = 2f(a) = 2b$ puts $(a, 2b)$ on the graph of g. In other words, to obtain the graph of g, we multiply all of the y-coordinates of the points on the graph of f by 2. Multiplying all of the y-coordinates of all of the points on the graph of f by 2 causes what is known as a 'vertical scaling[2] by a factor of 2', and the results are given on the next page.

[2]Also called a 'vertical stretching', 'vertical expansion' or 'vertical dilation' by a factor of 2.



If we wish to graph $y = \frac{1}{2}f(x)$, we multiply the all of the y-coordinates of the points on the graph of f by $\frac{1}{2}$. This creates a 'vertical scaling[3] by a factor of $\frac{1}{2}$' as seen below.



These results are generalized in the following theorem.

[3]Also called 'vertical shrinking', 'vertical compression' or 'vertical contraction' by a factor of 2.

Theorem 4.4. Vertical Scalings. Suppose f is a function and $a > 0$. To graph $y = af(x)$, multiply all of the y-coordinates of the points on the graph of f by a. We say the graph of f has been vertically scaled by a factor of a.

- If $a > 1$, we say the graph of f has undergone a vertical stretching (expansion, dilation) by a factor of a.
- If $0 < a < 1$, we say the graph of f has undergone a vertical shrinking (compression, contraction) by a factor of $\frac{1}{a}$.

A few remarks about Theorem 4.4 are in order. First, a note about the verbiage. To the authors, the words 'stretching', 'expansion', and 'dilation' all indicate something getting bigger. Hence, 'stretched by a factor of 2' makes sense if we are scaling something by multiplying it by 2. Similarly, we believe words like 'shrinking', 'compression' and 'contraction' all indicate something getting smaller, so if we scale something by a factor of $\frac{1}{2}$, we would say it 'shrinks by a factor of 2' - not 'shrinks by a factor of $\frac{1}{2}$'. This is why we have written the descriptions 'stretching by a factor of a' and 'shrinking by a factor of $\frac{1}{a}$' in the statement of the theorem. Second, in terms of inputs and outputs, Theorem 4.4 says multiplying the *outputs* from a function by positive number a causes the graph to be vertically scaled by a factor of a. It is natural to ask what would happen if we multiply the *inputs* of a function by a positive number. This leads us to our last transformation of the section.

Referring to the graph of f given at the beginning of this section, suppose we want to graph $g(x) = f(2x)$. In other words, we are looking to see what effect multiplying the inputs to f by 2 has on its graph. If we attempt to build a table directly, we quickly run into the same problem we had in our discussion leading up to Theorem 4.2, as seen in the table below.

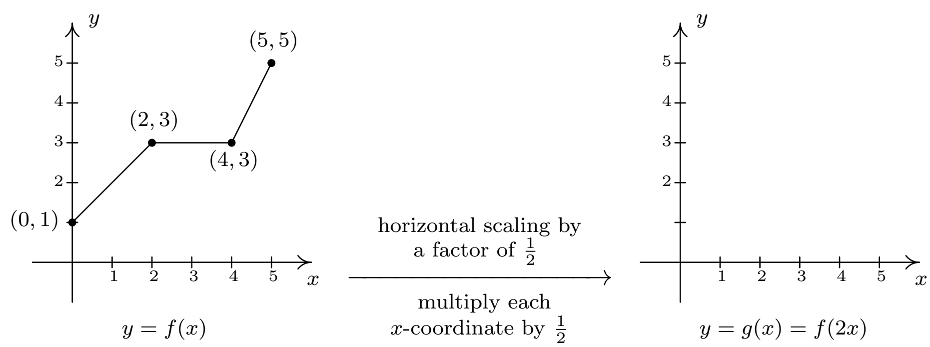
x	$(x, f(x))$	$f(x)$	$g(x) = f(2x)$	$(x, g(x))$
0	$(0,1)$	1	$f(2 \cdot 0) = f(0) = 1$	$(0,1)$
2	$(2,3)$	3	$f(2 \cdot 2) = f(4) = 3$	$(2,3)$
4	$(4,3)$	3	$f(2 \cdot 4) = f(8) = ?$	
5	$(5,5)$	5	$f(2 \cdot 5) = f(10) = ?$	

We solve this problem in the same way we solved this problem before. For example, if we want to determine the point on g which corresponds to the point $(2,3)$ on the graph of f, we set $2x = 2$ so that $x = 1$. Substituting

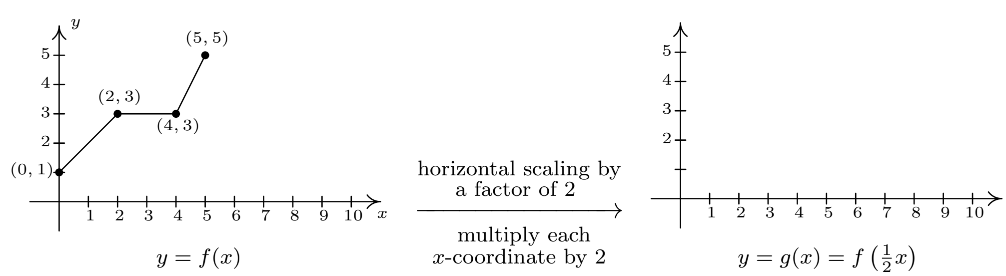
$x = 1$ into $g(x)$, we obtain $g(1) = f(2 \cdot 1) = f(2) = 3$, so that $(1,3)$ is on the graph of g. Continuing in this fashion, we obtain the following table.

x	$2x$	$g(x) = f(2x)$	$(x, g(x))$
0	0	$g(0) = f(0) = 1$	$(0,0)$
1	2	$g(1) = f(2) = 3$	$(1,3)$
2	4	$g(2) = f(4) = 3$	$(2,3)$
$\frac{5}{2}$	5	$g\left(\frac{5}{2}\right) = f(5) = 5$	$\left(\frac{5}{2}, 5\right)$

In general, if (a,b) is on the graph of f, then $f(a) = b$. Hence $g\left(\frac{a}{2}\right) = f\left(2 \cdot \frac{a}{2}\right) = f(a) = b$ so that $\left(\frac{a}{2}, b\right)$ is on the graph of g. In other words, to graph g we divide the x-coordinates of the points on the graph of f by 2. This results in a horizontal scaling[4] by a factor of $\frac{1}{2}$.



If, on the other hand, we wish to graph $y = f\left(\frac{1}{2}x\right)$, we end up multiplying the x-coordinates of the points on the graph of f by 2 which results in a horizontal scaling[5] by a factor of 2, as demonstrated below.



[4]Also called 'horizontal shrinking', 'horizontal compression' or 'horizontal contraction' by a factor of 2.

[5]Also called 'horizontal stretching', 'horizontal expansion' or 'horizontal dilation' by a factor of 2.

We have the following theorem.

Theorem 4.5. Horizontal Scalings. Suppose f is a function and $b > 0$. To graph $y = f(bx)$, divide all of the x-coordinates of the points on the graph of f by b. We say the graph of f has been horizontally scaled by a factor of $\frac{1}{b}$.

- If $0 < b < 1$, we say the graph of f has undergone a horizontal stretching (expansion, dilation) by a factor of $\frac{1}{b}$.
- If $b > 1$, we say the graph of f has undergone a horizontal shrinking (compression, contraction) by a factor of b.

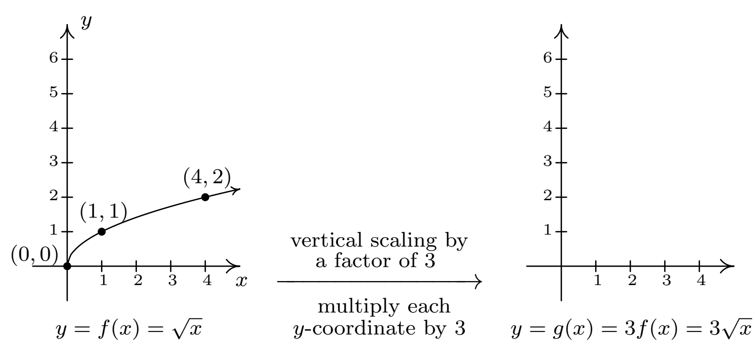
Theorem 4.5 tells us that if we multiply the input to a function by b, the resulting graph is scaled horizontally by a factor of $\frac{1}{b}$ since the x-values are divided by b to produce corresponding points on the graph of $y = f(bx)$. The next example explores how vertical and horizontal scalings sometimes interact with each other and with the other transformations introduced in this section.

Example 4.30. Let $f(x) = \sqrt{x}$. Use the graph of f to graph the following functions. Also, state their domains and ranges.

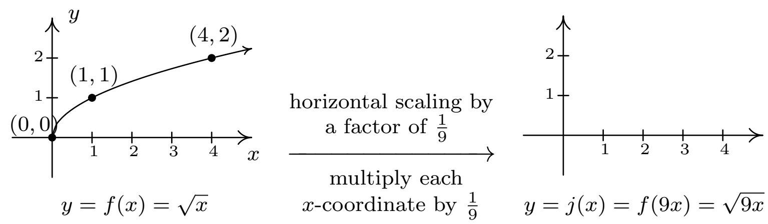
1. $g(x) = 3\sqrt{x}$
2. $j(x) = \sqrt{9x}$

Solution.

1. First we note that the domain of g is $[0, \infty)$ for the usual reason. Next, we have $g(x) = 3f(x)$ so by Theorem 4.4, we obtain the graph of g by multiplying all of the y-coordinates of the points on the graph of f by 3. The result is a vertical scaling of the graph of f by a factor of 3. We find the range of g is also $[0, \infty)$.



2. To determine the domain of j, we solve $9x \geq 0$ to find $x \geq 0$. Our domain is once again $[0, \infty)$. We recognize $j(x) = f(9x)$ and by Theorem 4.5, we obtain the graph of j by dividing the x-coordinates of the points on the graph of f by 9. From the graph, we see the range of j is also $[0, \infty)$.



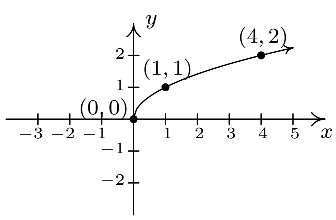
4.4.4 Transformations Summary

Objective: Summarize the various transformations that can be applied to a particular function and outline a strategy for graphing functions using a sequence of transformations.

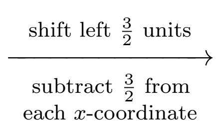
We will begin to summarize the concepts outlined in this section by extending the previous example.

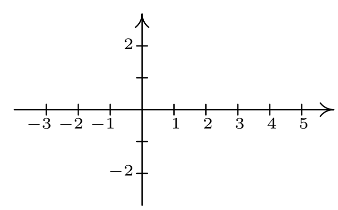
Example 4.31. Let $f(x) = \sqrt{x}$. Use the graph of f to graph the function $m(x) = 1 - \sqrt{\frac{x+3}{2}}$. Also, state the domain and range.

Solution. Solving $\frac{x+3}{2} \geq 0$ gives $x \geq -3$, so the domain of m is $[-3, \infty)$. To take advantage of what we know of transformations, we rewrite $m(x) = -\sqrt{\frac{1}{2}x + \frac{3}{2}} + 1$, or $m(x) = -f\left(\frac{1}{2}x + \frac{3}{2}\right) + 1$. Focusing on the inputs first, we note that the input to f in the formula for $m(x)$ is $\frac{1}{2}x + \frac{3}{2}$. Multiplying the x by $\frac{1}{2}$ corresponds to a horizontal stretching by a factor of 2, and adding the $\frac{3}{2}$ corresponds to a shift to the left by $\frac{3}{2}$. As before, we resolve which to perform first by thinking about how we would find the point on m corresponding to a point on f, in this case, $(4, 2)$. To use $f(4) = 2$, we solve $\frac{1}{2}x + \frac{3}{2} = 4$. Our first step is to subtract the $\frac{3}{2}$ (the horizontal shift) to obtain $\frac{1}{2}x = \frac{5}{2}$. Next, we multiply by 2 (the horizontal stretching) and obtain $x = 5$. We define two intermediate functions to handle first the shift, then the stretching. In accordance with Theorem 4.2, $m_1(x) = f\left(x + \frac{3}{2}\right) = \sqrt{x + \frac{3}{2}}$ will shift the graph of f to the left $\frac{3}{2}$ units.



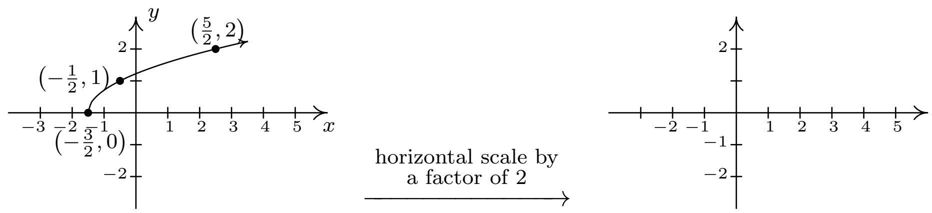
$y = f(x) = \sqrt{x}$





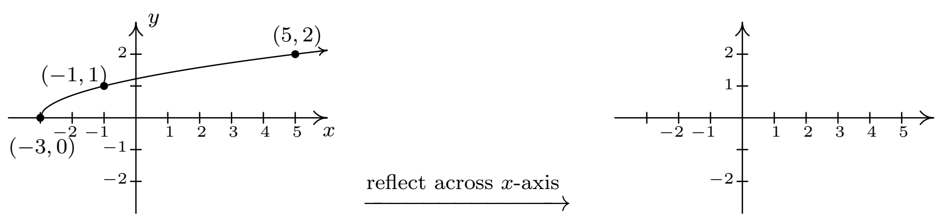
$y = m_1(x) = f\left(x + \frac{3}{2}\right) = \sqrt{x + \frac{3}{2}}$

Next, $m_2(x) = m_1\left(\frac{1}{2}x\right) = \sqrt{\frac{1}{2}x + \frac{3}{2}}$ will, according to Theorem 4.5, horizontally stretch the graph of m_1 by a factor of 2.



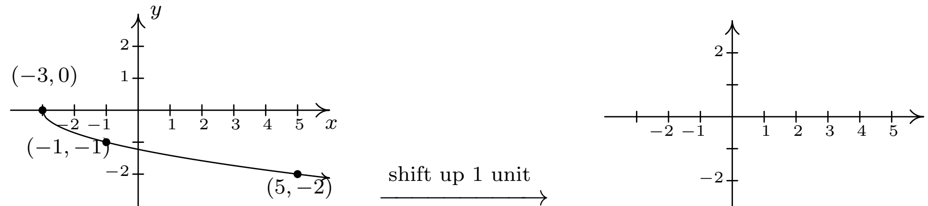
$y = m_1(x) = \sqrt{x + \frac{3}{2}}$ — multiply each x-coordinate by 2 — $y = m_2(x) = m_1\left(\frac{1}{2}x\right) = \sqrt{\frac{1}{2}x + \frac{3}{2}}$

We now examine what's happening to the outputs. From $m(x) = -f\left(\frac{1}{2}x + \frac{3}{2}\right) + 1$, we see that the output from f is being multiplied by -1 (a reflection about the x-axis) and then a 1 is added (a vertical shift up 1). As before, we can determine the correct order by looking at how the point $(4, 2)$ is moved. We already know that to make use of the equation $f(4) = 2$, we need to substitute $x = 5$. We get $m(5) = -f\left(\frac{1}{2}(5) + \frac{3}{2}\right) + 1 = -f(4) + 1 = -2 + 1 = -1$. We see that $f(4)$ (the output from f) is first multiplied by -1 then the 1 is added meaning we first reflect the graph about the x-axis then shift up 1. Theorem 4.3 tells us $m_3(x) = -m_2(x)$ will handle the reflection.



$y = m_2(x) = \sqrt{\frac{1}{2}x + \frac{3}{2}}$ — multiply each y-coordinate by -1 — $y = m_3(x) = -m_2(x) = -\sqrt{\frac{1}{2}x + \frac{3}{2}}$

Finally, to handle the vertical shift, Theorem 4.1 gives $m(x) = m_3(x) + 1$, and we see that the range of m is $(-\infty, 1]$.



$y = m_3(x) = -m_2(x) = -\sqrt{\frac{1}{2}x + \frac{3}{2}}$ — add 1 to each y-coordinate — $y = m(x) = m_3(x) + 1 = -\sqrt{\frac{1}{2}x + \frac{3}{2}} + 1$

Some comments about Example 4.31 are in order. First, recalling the properties of radicals from Intermediate Algebra, we know that the functions g and j are the same, since j and g have the same domains and $j(x) = \sqrt{9x} = \sqrt{9}\sqrt{x} = 3\sqrt{x} = g(x)$. (We invite the reader to verify that all of the points we plotted on the graph of g lie on the graph of j and vice-versa.) Hence, for $f(x) = \sqrt{x}$, a vertical stretch by a factor of 3 and a horizontal shrinking by a factor of 9 result in the same transformation. While this kind of phenomenon is not universal, it happens commonly enough with some of the families of functions studied in College Algebra that it is worthy of note. Secondly, to graph the function m, we applied a series of four transformations. While it would have been easier on the authors to simply inform the reader of which steps to take, we have strived to explain why the order in which the transformations were applied made sense. We generalize the procedure in the theorem below.

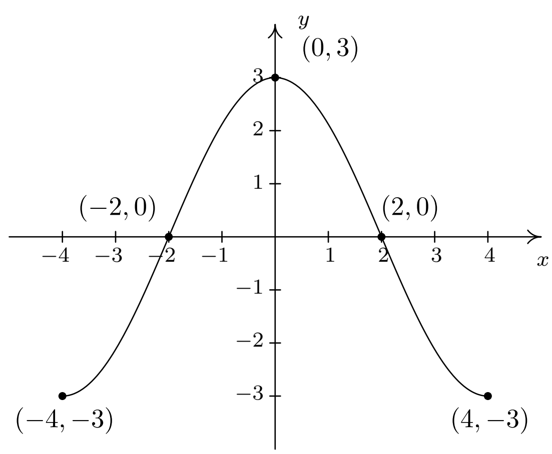
Theorem 4.6. Transformations. Suppose f is a function. If $A \neq 0$ and $B \neq 0$, then to graph

$$g(x) = Af(Bx + H) + K$$

1. Subtract H from each of the x-coordinates of the points on the graph of f. This results in a horizontal shift to the left if $H > 0$ or right if $H < 0$.
2. Divide the x-coordinates of the points on the graph obtained in Step 1 by B. This results in a horizontal scaling, but may also include a reflection about the y-axis if $B < 0$.
3. Multiply the y-coordinates of the points on the graph obtained in Step 2 by A. This results in a vertical scaling, but may also include a reflection about the x-axis if $A < 0$.
4. Add K to each of the y-coordinates of the points on the graph obtained in Step 3. This results in a vertical shift up if $K > 0$ or down if $K < 0$.

Theorem 4.6 can be established by generalizing the techniques developed in this section. Suppose (a, b) is on the graph of f. Then $f(a) = b$, and to make good use of this fact, we set $Bx + H = a$ and solve. We first subtract the H (causing the horizontal shift) and then divide by B. If B is a positive number, this induces only a horizontal scaling by a factor of $\frac{1}{B}$. If $B < 0$, then we have a factor of -1 in play, and dividing by it induces a reflection about the y-axis. So we have $x = \frac{a-H}{B}$ as the input to g which corresponds to the input $x = a$ to f. We now evaluate $g\left(\frac{a-H}{B}\right) = Af\left(B \cdot \frac{a-H}{B} + H\right) + K = Af(a) + K = Ab + K$. We notice that the output from f is first multiplied by A. As with the constant B, if $A > 0$, this induces only a vertical scaling. If $A < 0$, then the -1 induces a reflection across the x-axis. Finally, we add K to the result, which is our vertical shift. A less precise, but more intuitive way to paraphrase Theorem 4.6 is to think of the quantity $Bx + H$ is the 'inside' of the function f. What's happening inside f affects the inputs or x-coordinates of the points on the graph of f. To find the x-coordinates of the corresponding points on g, we undo what has been done to x in the same way we would solve an equation. What's happening to the output can be thought of as things happening 'outside' the function, f. Things happening outside affect the outputs or y-coordinates of the points on the graph of f. Here, we follow the usual order of operations agreement: we first multiply by A then add K to find the corresponding y-coordinates on the graph of g.

Example 4.32. Below is the complete graph of $y = f(x)$. Use it to graph $g(x) = \frac{4-3f(1-2x)}{2}$.



Solution. We use Theorem 4.6 to track the five 'key points' $(-4,-3)$, $(-2,0)$, $(0,3)$, $(2,0)$ and $(4,-3)$ indicated on the graph of f to their new locations. We first rewrite $g(x)$ in the form presented in Theorem 4.6, $g(x) = -\frac{3}{2}f(-2x+1)+2$. We set $-2x+1$ equal to the x-coordinates of the key points and solve. For example, solving $-2x+1 = -4$, we first subtract 1 to get $-2x = -5$ then divide by -2 to get $x = \frac{5}{2}$. Subtracting the 1 is a horizontal shift to the left 1 unit. Dividing by -2 can be thought of as a two step process: dividing by 2 which compresses the graph horizontally by a factor of 2 followed by dividing (multiplying) by -1 which causes a reflection across the y-axis. We summarize the results in the table on the next page.

$(a, f(a))$	a	$-2x+1=a$	x
$(-4,-3)$	-4	$-2x+1=-4$	$x=\frac{5}{2}$
$(-2,0)$	-2	$-2x+1=-2$	$x=\frac{3}{2}$
$(0,3)$	0	$-2x+1=0$	$x=\frac{1}{2}$
$(2,0)$	2	$-2x+1=2$	$x=-\frac{1}{2}$
$(4,-3)$	4	$-2x+1=4$	$x=-\frac{3}{2}$

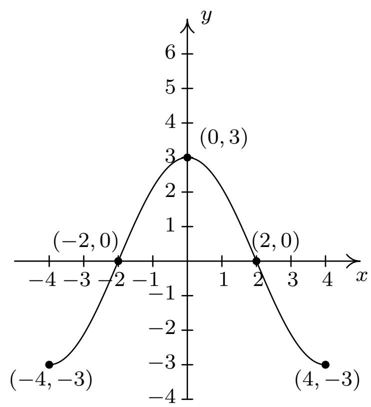
Next, we take each of the x values and substitute them into $g(x) = -\frac{3}{2}f(-2x+1)+2$ to get the corresponding y-values. Substituting $x=\frac{5}{2}$, and using the fact that $f(-4)=-3$, we get

$$g\left(\frac{5}{2}\right) = -\frac{3}{2}f\left(-2\left(\frac{5}{2}\right)+1\right)+2 = -\frac{3}{2}f(-4)+2 = -\frac{3}{2}(-3)+2 = \frac{9}{2}+2 = \frac{13}{2}$$

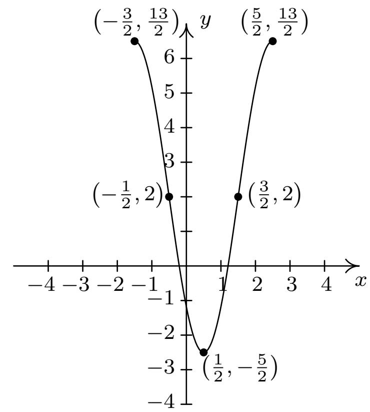
We see that the output from f is first multiplied by $-\frac{3}{2}$. Thinking of this as a two step process, multiplying by $\frac{3}{2}$ then by -1, we have a vertical stretching by a factor of $\frac{3}{2}$ followed by a reflection across the x-axis. Adding 2 results in a vertical shift up 2 units. Continuing in this manner, we get the table below.

x	$g(x)$	$(x, g(x))$
$\frac{5}{2}$	$\frac{13}{2}$	$\left(\frac{5}{2}, \frac{13}{2}\right)$
$\frac{3}{2}$	2	$\left(\frac{3}{2}, 2\right)$
$\frac{1}{2}$	$-\frac{5}{2}$	$\left(\frac{1}{2}, -\frac{5}{2}\right)$
$-\frac{1}{2}$	2	$\left(-\frac{1}{2}, 2\right)$
$-\frac{3}{2}$	$\frac{13}{2}$	$\left(-\frac{3}{2}, \frac{13}{2}\right)$

To graph g, we plot each of the points in the table above and connect them in the same order and fashion as the points to which they correspond. Plotting f and g side-by-side gives



graph of f



graph of g

The reader is encouraged to graph the series of functions which shows the gradual transformation of the graph of f into the graph of g. We have outlined the sequence of transformations in the above exposition; all that remains is to plot the five intermediate stages.

Our last example turns the tables and asks for the formula of a function given a desired sequence of transformations. If nothing else, it is a good review of function notation.

Example 4.33. Let $f(x) = x^2$. Find and simplify the formula of the function $g(x)$ whose graph is the result of f undergoing a vertical shift up 2 units, a reflection across the x-axis, a horizontal shift right 1 unit, and a horizontal stretching by a factor of 2.

Solution. We build up to a formula for $g(x)$ using intermediate functions as we've seen in previous examples. We let g_1 take care of our first step. From our experience with vertical shifts we can see that $g_1(x) = f(x)+2 = x^2+2$. Next, we reflect the graph of g_1 about the x-axis: $g_2(x) = -g_1(x) = -\left(x^2+2\right) = -x^2 - 2$. We shift the graph to the right 1 unit by a horizontal shift where we set $g_3(x) = g_2(x-1) = -(x-1)^2 - 2 = -x^2 + 2x - 3$. Finally, we induce a horizontal stretch by a factor of 2 to scale $g(x)$ and get $g(x) = g_3\left(\frac{1}{2}x\right) = -\left(\frac{1}{2}x\right)^2 + 2\left(\frac{1}{2}x\right) - 3$ which yields $g(x) = -\frac{1}{4}x^2 + x - 3$.

4.5 Piecewise-Defined and Absolute Value Functions

4.5.1 Piecewise-Defined Functions

Objective: Evaluate a piecewise-defined function at a point, find all solutions for an equation involving a piecewise-defined function and graph a piecewise-defined function.

A *piecewise-defined* (or simply, a *piecewise*) function is a function that is defined in pieces. More precisely, a piecewise-defined function is a function that is presented using one or more expressions, each defined over non-intersecting intervals. An example of a piecewise-defined function is shown below.

$$f(x) \;=\; \begin{cases} 2x-1 & \text{if } \; x>0 \\ x^2-1 & \text{if } \; x\leq 0 \end{cases}$$

To evaluate a piecewise-defined function at a particular value of the variable, we must first compare our value to the various intervals (or domains) applied to each piece, and then substitute our value into the piece that coincides with the correct domain. For example, since $x=1$ is greater than zero, we would use the expression $2x-1$ to evaluate $f(1)$,

$$f(1) = 2(1) - 1 = 2 - 1 = 1.$$

Similarly, since $x=-1$ is less than zero, we would use the expression x^2-1 to evaluate $f(-1)$,

$$f(-1) = (-1)^2 - 1 = 1 - 1 = 0.$$

Below is a table of points obtained from the piecewise-defined function f above.

Example 4.34.

$$f(x) \ = \ \begin{cases} 2x-1 & \text{if } \ x>0 \\ x^2-1 & \text{if } \ x\leq 0 \end{cases}$$

x	$f(x)$
2	$2(2)-1=3$
1	$2(1)-1=1$
0	$(0)^2-1=-1$
-1	$(-1)^2-1=0$
-2	$(-2)^2-1=3$

We have included an extra line between the values of $x = 0$ and $x = 1$ in the table above, in order to emphasize the changeover from one piece of our function $(2x-1)$ to another (x^2-1). The value of $x = 0$ is very important, since it is an endpoint for the two domains of our function, $(0, \infty)$ and $(-\infty, 0]$. A common misconception among students is to evaluate $f(0)$ at both $2x-1$ and x^2-1 because it seems to "straddle" both individual domains. And although the values for both pieces are equal at $x = 0$,

$$2(0)-1=-1=0^2-1$$

this will often not be the case. Regardless, we must be careful to *always* associate $x = 0$ with x^2-1, since it is contained in our second piece's domain $(0 \leq 0)$ and not in our first. Our next example demonstrates what can happen with a piecewise function, if one mishandles such values of x.

Example 4.35.

$$g(x) \ = \ \begin{cases} 2x+1 & \text{if } \ x>0 \\ x^2-1 & \text{if } \ x\leq 0 \end{cases}$$

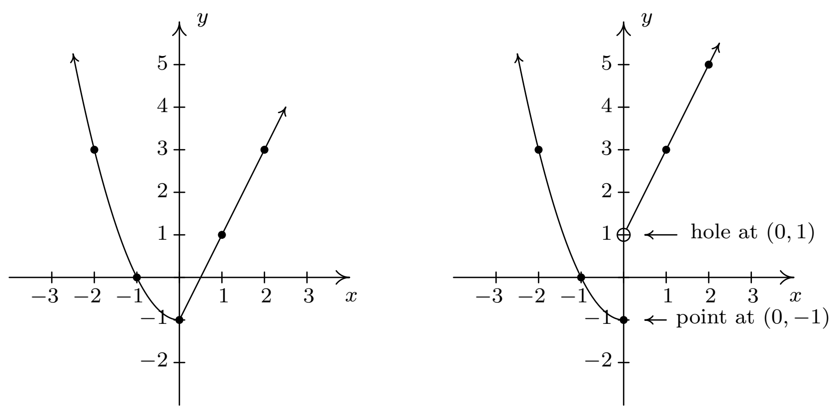
x	$g(x)$
2	$2(2)+1=5$
1	$2(1)+1=3$
0	$(0)^2-1=-1$
-1	$(-1)^2-1=0$
-2	$(-2)^2-1=3$

In the previous example, we can see that both pieces for $g(x)$ do not "match up", since the values we obtain for both pieces at $x = 0$ do not agree,

$$g(0) = 0^2 - 1 = -1 \qquad \text{but} \qquad 2(0) + 1 = 1.$$

Remember that when evaluating any function at a value of x in its domain, we should always only ever get a *single value* for $g(x)$, since this is how we defined a function earlier in the chapter. Furthermore, if we were to associate two values ($g(0) = \pm 1$) to $x = 0$, our graph would consequently contain points at $(0, -1)$ and $(0, 1)$, and therefore fail the Vertical Line Test.

When we consider the graphs of both f and g, since both pieces of f seem to "match up" at $x = 0$, we will see that the graph of f will be one *continuous* graph, with no breaks or separations appearing. On the other hand, since both pieces of g do not "match up" at $x = 0$, we will see that the graph of g will contain a break at $x = 0$, known as a *discontinuity* in the graph. The formal definition of a *continuous function* is one that is usually reserved for a follow-up course to Algebra (either Precalculus or Calculus). Both graphs are shown below.



The graph of f The graph of g

Notice that in order for us to have a *complete* sketch of the graph of g, we have evaluated *both* pieces of g at $x = 0$, so that we can properly identify the *point* at the end of the quadratic piece $x^2 - 1$ and the *hole* at the end of the linear piece, $2x + 1$. In general, whenever faced with the task of graphing a piecewise-defined function, one should always make sure to identify exactly where each piece of the graph starts and stops, even if a location corresponds to a hole, i.e., a coordinate pair that is not actually a point on the graph.

We can also observe, both from how our functions are defined (algebraically) and from their graphs that the domain of both f and g is all real numbers, or $(-\infty, \infty)$. To identify the range of each function, we can project each of our graphs onto the $y-$axis. In doing so, we obtain a range of $[-1, \infty)$ for both f and g. Notice that although both functions produce distinctly different graphs, their range is coincidentally the same, since the quadratic piece $x^2 - 1$ begins at the same minimum value ($y = -1$) for each graph.

As we have already discussed evaluating piecewise-defined functions at a value of x, we will now address the issue of solving an equation that involves a piecewise function for all possible values of x. We will do this, once again, using our functions f and g from before.

For some constant k, to find all x such that $f(x) = k$, we will use the strategy outlined below, which will be the same for any piecewise-defined function.

- Set each separate piece equal to k and solve for x.

- Compare your answers for x to the domain applied to each piece. Only keep those solutions that coincide with the specified domain.

We illustrate this approach by finding all possible zeros (or roots) of both f and g.

Example 4.36. Find the set of all zeros of

$$f(x) \;=\; \begin{cases} 2x - 1 & \text{if } \; x > 0 \\ x^2 - 1 & \text{if } \; x \leq 0 \end{cases}$$

.

$$\begin{aligned} f(x) &= 0 && \text{Apply to each piece separately} \\ \\ 2x - 1 &= 0, \; x > 0 && \text{First piece; solve for } x \\ x &= \frac{1}{2}, \; x > 0 && \text{One solution; coincides with domain} \end{aligned}$$

$$
\begin{aligned}
x^2 - 1 = 0\ , \quad x \le 0 & \quad \text{Second piece; solve for } x \\
(x-1)(x+1) = 0,\ x \le 0 & \quad \text{Solve by factoring} \\
x = \pm 1,\ x \le 0 & \quad \text{Two potential solutions} \\
x = -1,\ x \le 0 & \quad \text{Exclude } x = 1; \text{does not coincide with domain}
\end{aligned}
$$

$$
f(x) = 0 \text{ when } x = -1,\ \frac{1}{2} \qquad \text{Our answer}
$$

Example 4.37. Find the set of all zeros of

$$
g(x) \ = \ \begin{cases} 2x+1 & \text{if } \ x > 0 \\ x^2 - 1 & \text{if } \ x \le 0 \end{cases}
$$

.

$$
g(x) = 0 \qquad \text{Apply to each piece separately}
$$

$$
\begin{aligned}
2x + 1 = 0,\ x > 0 & \quad \text{First piece; solve for } x \\
x = -\frac{1}{2},\ x > 0 & \quad \text{Invalid solution; does not coincide with domain}
\end{aligned}
$$

$$
\begin{aligned}
x^2 - 1 = 0\ , \quad x \le 0 & \quad \text{Second piece; solve for } x \\
(x-1)(x+1) = 0,\ x \le 0 & \quad \text{Solve by factoring} \\
x = \pm 1,\ x \le 0 & \quad \text{Two potential solutions} \\
x = -1,\ x \le 0 & \quad \text{Exclude } x = 1; \text{does not coincide with domain}
\end{aligned}
$$

$$
g(x) = 0 \text{ when } x = -1 \qquad \text{Our answer}
$$

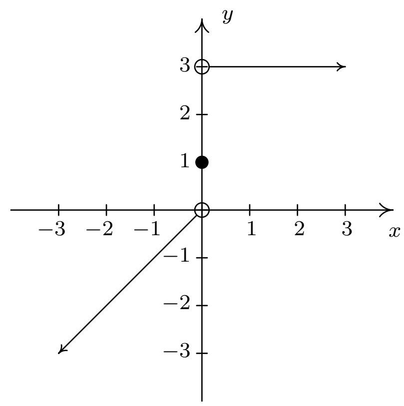
Each of the previous examples can also be confirmed by the graphs that we obtained earlier.

For our next example, we will graph a piecewise function that consists of three pieces.

Example 4.38.

$$h(x) = \begin{cases} 3 & \text{if } x > 0 \\ 1 & \text{if } x = 0 \\ x & \text{if } x < 0 \end{cases}$$

x	$h(x)$
2	3
1	3
0	1
-1	-1
-2	-2



The graph of h

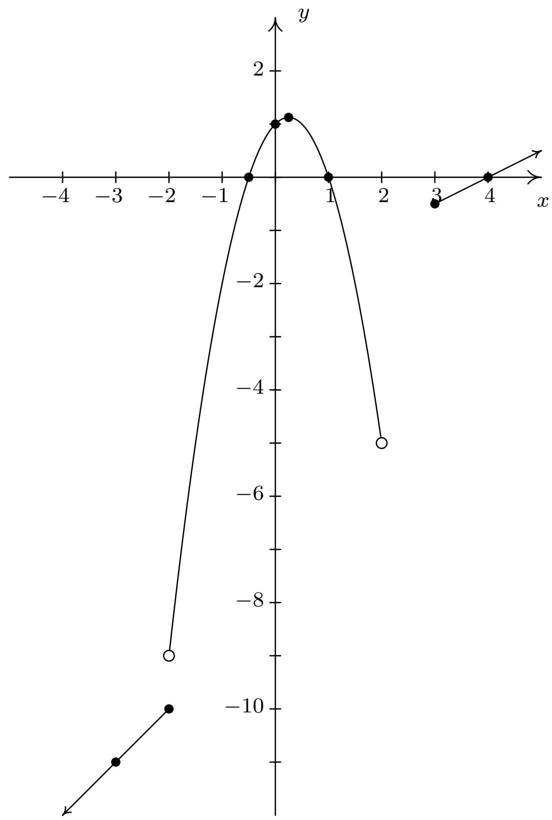
Here, we see that our graph consists of three pieces, one of which is a single point at $(0, 1)$. We can also once again determine both algebraically and graphically that our domain is $(-\infty, \infty)$. Using our graph, we obtain a range of $(-\infty, 0) \cup \{1\} \cup \{3\}$. Our complete graph also contains holes at $(0, 3)$ and $(0, 0)$.

We can easily identify all three of the coordinate pairs associated with $x = 0$ (two holes and one point) by evaluating all three pieces at $x = 0$. To reinforce this concept, we will present another example of a piecewise function that consists of three pieces.

Example 4.39.

$$f(x) = \begin{cases} \frac{x}{2} - 2 & \text{if } x \geq 3 \\ -2x^2 + x + 1 & \text{if } -2 < x < 2 \\ x - 8 & \text{if } x \leq -2 \end{cases}$$

x	$f(x)$
4	0
3	$-\frac{1}{2}$
1	0
$\frac{1}{4}$	$\frac{9}{8}$
0	1
$-\frac{1}{2}$	0
-1	-2
-2	-10
-3	-11



The graph of f

In the previous example, we see that there is a "gap" in our domain between the $x-$coordinates of 2 and 3. Hence, our domain is $(-\infty, 2) \cup [3, \infty)$. From our graph, we see that our range also contains a gap between the $y-$coordinates of -10 and -9. Hence, our range is $(-\infty, -10] \cup (-9, \infty)$.

In our example we have have also identified several other essential coordinate pairs that should be included in our graph. We will now list each pair below, as well as the piece that is used to obtain it. We include the function f, once again, for reinforcement.

$$f(x) \; = \; \begin{cases} \frac{x}{2} - 2 & \text{if } \; x \geq 3 \\ -2x^2 + x + 1 & \text{if } \; -2 < x < 2 \\ x - 8 & \text{if } \; x \leq -2 \end{cases}$$

- A $y-$intercept at $(0, 1)$ from our second piece
- An $x-$intercept at $(4, 0)$ from our first piece
- Two $x-$intercepts at $(1, 0)$ and $\left(-\frac{1}{2}, 0\right)$ from our second piece
- A vertex at $\left(\frac{1}{4}, \frac{9}{8}\right)$ from our second piece
- An endpoint at $\left(3, -\frac{1}{2}\right)$ from our first piece
- An endpoint at $(-2, -10)$ from our third piece
- Two holes at $(-2, -9)$ and $(2, -5)$ from our second piece.

Lastly, we have included the point at $(-3, -11)$, to help identify the slope of the third piece of our graph.

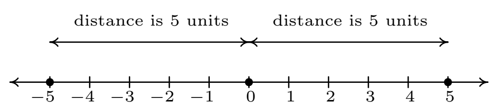
Although this example may first appear to be quite complicated, when considered on the level of each individual piece, we see that our training in the chapters leading up to this section has adequately prepared us to handle these, as well as more challenging piecewise-defined functions that we will eventually encounter.

4.5.2 Absolute Value Functions

Graphing Absolute Value Equations

Objective: Graph absolute values from a given equation

There are a few ways to describe what is meant by the absolute value $|x|$ of a real number x. A common description is that $|x|$ represents the distance from the number x to 0 on the real number line. So, for example, $|5| = 5$ and $|-5| = 5$, since each is 5 units away from 0 on the real number line.



Another way to define an absolute value is by the equation $|x| = \sqrt{x^2}$. Using this definition, we have

$$|5| = \sqrt{(5)^2} = \sqrt{25} = 5 \qquad \text{and} \qquad |-5| = \sqrt{(-5)^2} = \sqrt{25} = 5.$$

The long and short of both of these descriptions is that $|x|$ takes negative real numbers and assigns them to their positive counterparts, while it leaves positive real numbers (and zero) alone. This last description is the one we shall adopt, and is summarized in the following definition.

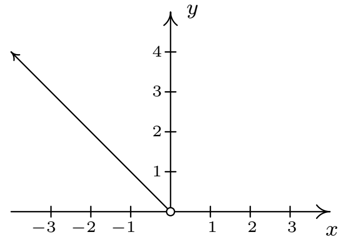
Definition. The **absolute value** of a real number x, denoted $|x|$, is given by

$$|x| = \begin{cases} -x, & \text{if} \quad x < 0 \\ x, & \text{if} \quad x \geq 0 \end{cases}$$

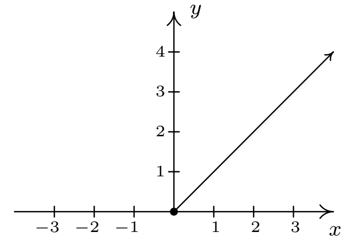
Notice that we have defined $|x|$ using a piecewise-defined function. To check that this definition agrees with what we previously understood to be the absolute value of x, observe that since $5 \geq 0$, to find $|5|$ we use the rule $|x| = x$, so $|5| = 5$. Similarly, since $-5 < 0$, we use the rule $|x| = -x$, so that $|-5| = -(-5) = 5$.

We will now graph some functions that contain an absolute value. Our strategy is to use our knowledge of the absolute value coupled with what we now know about graphing linear functions and piecewise-defined functions.

Example 4.40. Sketch a complete graph of $f(x) = |x|$.

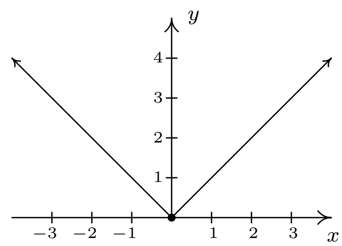


$f(x) = |x|,\ x < 0$



$f(x) = |x|,\ x \geq 0$

Notice that we have a hole at $(0, 0)$ in the graph when $x < 0$. As we have seen before, this is due to the fact that the points on $y = -x$ approach $(0, 0)$ as the x-values approach 0. Since x is required to be strictly less than zero on this interval, we include a hole at the origin. Notice, however, that when $x \geq 0$, we get to include the point at $(0, 0)$, which effectively fills in the hole from our first piece. Our final graph is shown below.

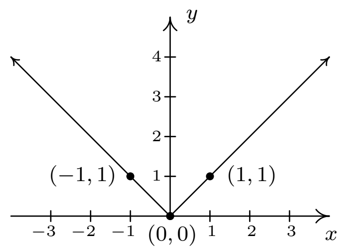


$f(x) = |x|$

By projecting our graph onto the $x-$axis, we see that the domain of $f(x) = |x|$ is $(-\infty, \infty)$, as expected. Projecting onto the $y-$axis gives us our range of $[0, \infty)$. Our function is also increasing over the interval $[0, \infty)$ and decreasing over the interval $(-\infty, 0]$. We can also say that the graph of f has an absolute minimum at $y = 0$, since this coordinate coincides with the (absolute) lowest point on the graph, which occurs at the origin. From our graph, we can further conclude that there is no absolute maximum value of f, since the y values on the graph extend infinitely upwards.

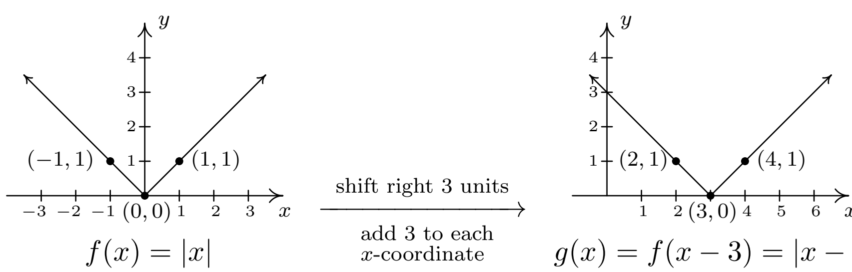
Example 4.41. Use the graph of $f(x) = |x|$ to graph the function $g(x) = |x - 3|$.

We begin by graphing $f(x) = |x|$ and labeling three reference points: $(-1, 1)$, $(0, 0)$ and $(1, 1)$.



$f(x) = |x|$

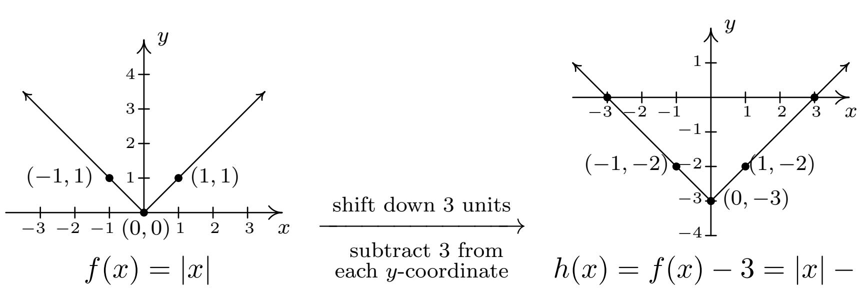
Since $g(x) = |x-3| = f(x-3)$, we will add 3 to each of the $x-$coordinates of the points on the graph of $y = f(x)$ to obtain the graph of $y = g(x)$. This shifts the graph of $y = f(x)$ to the *right* by 3 units and moves the points $(-1, 1)$ to $(2, 1)$, $(0, 0)$ to $(3, 0)$ and $(1, 1)$ to $(4, 1)$. Connecting these points in the classic '$\vee$' fashion produces the graph of $y = g(x)$.



$f(x) = |x|$ $\qquad$ $g(x) = f(x-3) = |x-3|$

Example 4.42. Use the graph of $f(x) = |x|$ to graph the function $h(x) = |x| - 3$.

Since $h(x) = |x| - 3 = f(x) - 3$, we will subtract 3 from each of the $y-$coordinates of the points on the graph of $y = f(x)$ to obtain the graph of $y = h(x)$. This shifts the graph of $y = f(x)$ *down* by 3 units and moves the points $(-1, 1)$ to $(-1, -2)$, $(0, 0)$ to $(0, -3)$ and $(1, 1)$ to $(1, -2)$. Connecting these points with the '$\vee$' shape produces our graph of $y = h(x)$.



$f(x) = |x|$ $\qquad$ $h(x) = f(x) - 3 = |x| - 3$

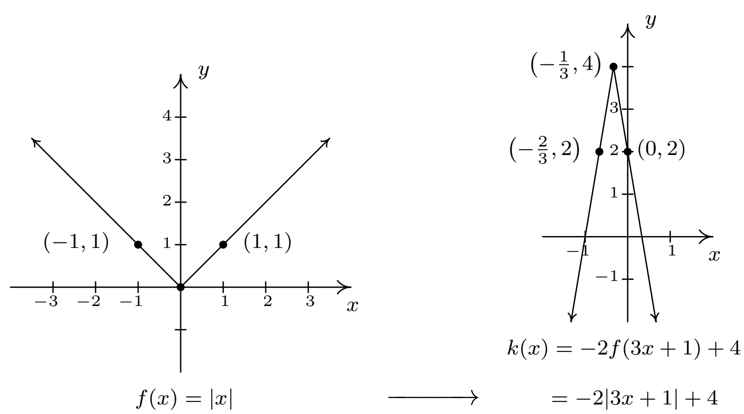
Example 4.43. Use the graph of $f(x) = |x|$ to graph the function $k(x) = 4 - 2|3x+1|$.

Notice that

$$k(x) = 4 - 2|3x+1| = 4 - 2f(3x+1) = -2f(3x+1) + 4.$$

First, we will determine the corresponding transformations resulting from inside of the absolute value. Instead of $|x|$, we have $|3x+1|$, so we must first subtract 1 from each of the $x-$coordinates of points on the graph of $y = f(x)$, then divide each of those new values by 3. This corresponds to a horizontal shift left by 1 unit followed by a horizontal shrink by a factor of 3. These transformations move the points $(-1,1)$ to $\left(-\frac{2}{3},1\right)$, $(0,0)$ to $\left(-\frac{1}{3},0\right)$ and $(1,1)$ to $(0,1)$.

Next, we will determine the corresponding transformations resulting from what appears outside of the absolute value. We must first multiply each $y-$coordinate of our new points by -2 and then *add* 4. Geometrically, this corresponds to a vertical *stretch* by a factor of 2, a reflection across the $x-$axis and finally, a vertical shift *up* by 4 units. The resulting transformations move the points $\left(-\frac{2}{3},1\right)$ to $\left(-\frac{2}{3},2\right)$, $\left(-\frac{1}{3},0\right)$ to $\left(-\frac{1}{3},4\right)$ and $(0,1)$ to $(0,2)$. Connecting our final points with the usual '∨' shape produces the graph of $y = k(x)$, shown below.



Recognizing Absolute Value Functions as Piecewise-Defined

Objective: Express an absolute value function as a piecewise-defined function (without an absolute value).

By definition, we know that

$$|x| = \begin{cases} -x, & \text{if} \quad x < 0 \\ x, & \text{if} \quad x \geq 0 \end{cases}$$

If $m \neq 0$ and b is a real number, we may generalize the definition above as follows.

$$|mx+b| = \begin{cases} -(mx+b), & \text{if} \quad mx+b < 0 \\ mx+b\ , & \text{if} \quad mx+b \geq 0 \end{cases} = \begin{cases} -mx-b, & \text{if} \quad mx+b < 0 \\ mx+b, & \text{if} \quad mx+b \geq 0 \end{cases}$$

Notice that since we have never specified whether m is positive or negative above, it would not be wise to attempt to simplify either inequality in our new definition. Once we are given a value for m, as in our next example, we will be able to simplify our piecewise representation completely.

Example 4.44. Express $g(x) = |x-3|$ as a piecewise-defined function.

$$g(x) = |x-3| = \begin{cases} -(x-3), & \text{if} \quad x-3 < 0 \\ (x-3), & \text{if} \quad (x-3) \geq 0 \end{cases}$$

Simplifying, we get

$$g(x) = \begin{cases} -x+3, & \text{if} \quad x < 3 \\ x-3, & \text{if} \quad x \geq 3 \end{cases}$$

Our piecewise answer above should begin to make sense, when one considers the graph of g as a horizontal shift of $y = |x|$ to the right by 3 units.

Example 4.45. Express $h(x) = |x| - 3$ as a piecewise-defined function.

Since the variable within the absolute value remains unchanged, the domains for each piece in our resulting function will not change. Instead, we need only subtract 3 from each piece of our answer. Thus, we get the following representation.

$$h(x) = \begin{cases} -x - 3, & \text{if} \quad x < 0 \\ x - 3, & \text{if} \quad x \geq 0 \end{cases}$$

Similarly, this answer again seems reasonable, as the graph of $h(x) = |x| - 3$ represents a vertical shift of $y = |x|$ down by 3 units.

Example 4.46. Express $k(x) = 4 - 2|3x + 1|$ as a piecewise-defined function and identify any $x-$ and $y-$intercepts on its graph. Determine the domain and range of $k(x)$.

We set $k(x) = 0$ to find any zeros: $4 - 2|3x + 1| = 0$.
Isolating the absolute value gives us $|3x + 1| = 2$, so either

$$3x + 1 = 2 \qquad \text{or} \qquad 3x + 1 = -2.$$

This results in $x = \frac{1}{3}$ or $x = -1$, so our $x-$intercepts are $\left(\frac{1}{3}, 0\right)$ and $(-1, 0)$.

For our $y-$intercept, substituting $x = 0$ into $k(x)$ gives us

$$y = k(0) = 4 - 2|3(0) + 1| = 2.$$

So our y-intercept is at $(0, 2)$.
Rewriting the expression for k as a piecewise function gives us the following.

$$\begin{aligned} k(x) &= \begin{cases} 4 - 2[-(3x + 1)], & \text{if} \quad 3x + 1 < 0 \\ 4 - 2(3x + 1), & \text{if} \quad 3x + 1 \geq 0 \end{cases} \\ &= \begin{cases} 4 + 6x + 2, & \text{if} \quad 3x < -1 \\ 4 - 6x - 2, & \text{if} \quad 3x \geq -1 \end{cases} \\ &= \begin{cases} 6x + 6, & \text{if} \quad x < -\frac{1}{3} \\ -6x + 2, & \text{if} \quad x \geq -\frac{1}{3} \end{cases} \end{aligned}$$

Either algebraically, or using the graph of k from page 328, we see that the domain of k is $(-\infty, \infty)$ while the range is $(-\infty, 4]$.

4.6 Practice Problems

4.6.1 Notation and Basic Examples

In Exercises 1 - 10, determine whether or not the relation represents y as a function of x.

1. $\{(-3,9),\ (-2,4),\ (-1,1),\ (0,0),\ (1,1),\ (2,4),\ (3,9)\}$

2. $\{(-3,0),(1,6),(2,-3),(4,2),(-5,6),(4,-9),(6,2)\}$

3. $\{(-3,0),(-7,6),(5,5),(6,4),(4,9),(3,0)\}$

4. $\{(1,2),(4,4),(9,6),(16,8),(25,10),(36,12),\ldots\}$

5. $\{(x,y)\,|\,x$ is an odd integer, and y is an even integer$\}$

6. $\{(x,1)\,|\,x$ is an irrational number$\}$

7. $\{(1,0),\ (2,1),\ (4,2),\ (8,3),\ (16,4),\ (32,5),\ \ldots\}$

8. $\{\ldots,\ (-3,9),\ (-2,4),\ (-1,1),\ (0,0),\ (1,1),\ (2,4),\ (3,9),\ \ldots\}$

9. $\{(-2,y)\,|\,-3 < y < 4\}$

10. $\{(x,3)\,|\,-2 \le x < 4\}$

In Exercises 11 - 30, determine whether or not the relation represents y as a function of x.

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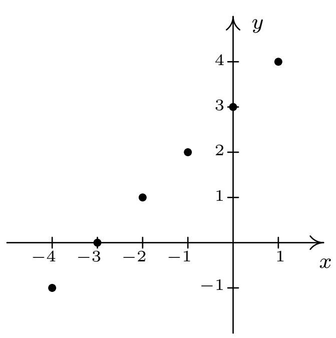

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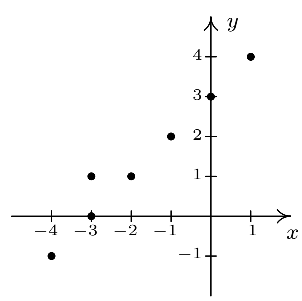

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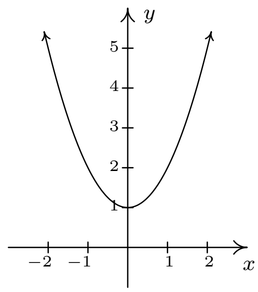

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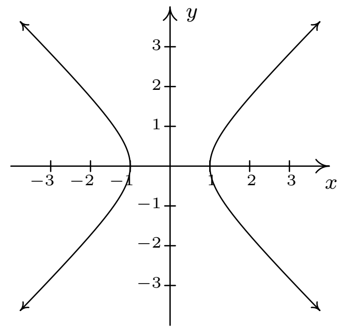

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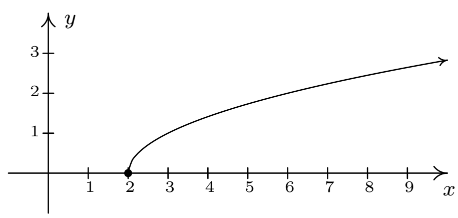

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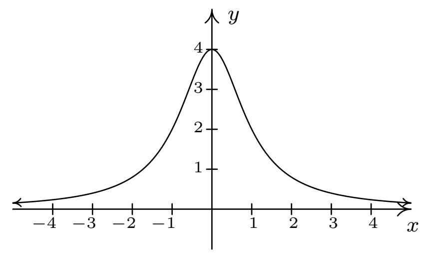

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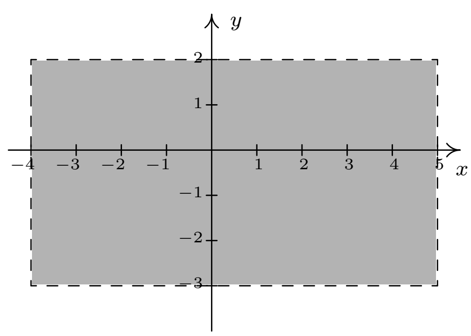

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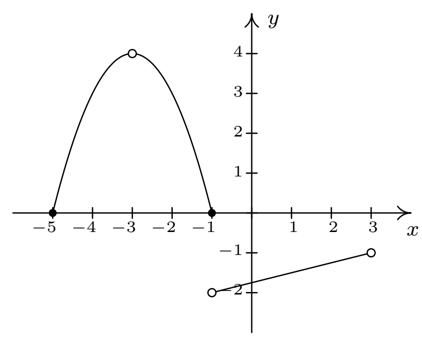

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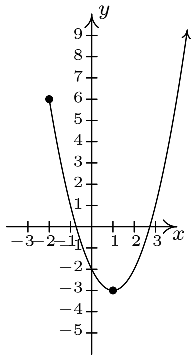

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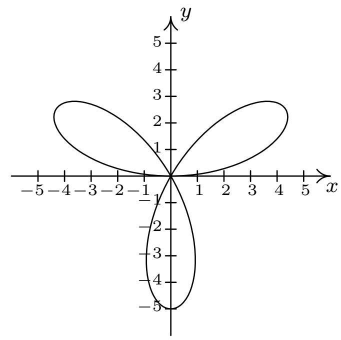

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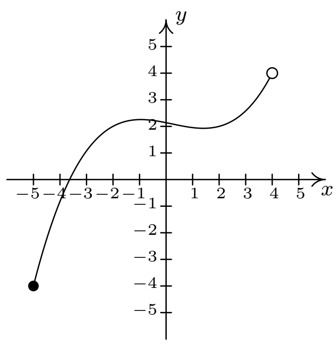

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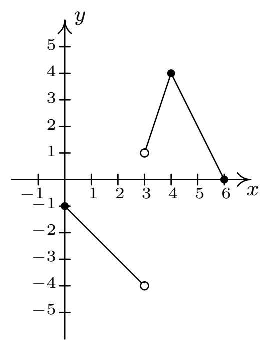

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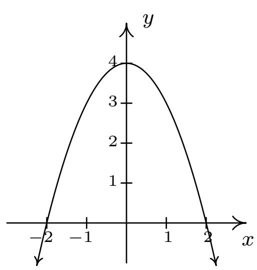

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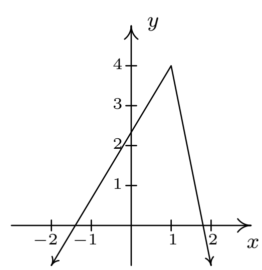

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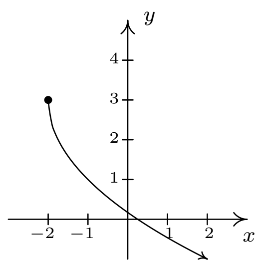

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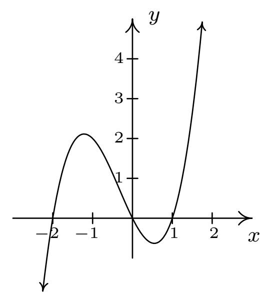

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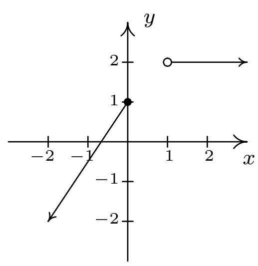

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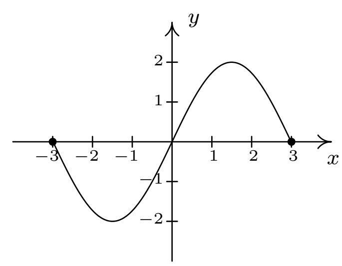

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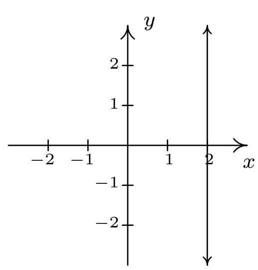

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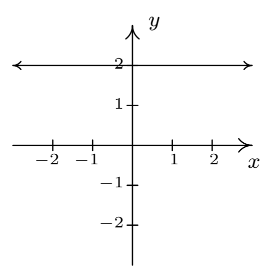


In Exercises 31 - 39, determine whether or not the equation represents y as a function of x.

31. $y = x^3 - x$
32. $y = \sqrt{x-2}$
33. $3x + 2y = 6$
34. $x^2 - y^2 = 1$
35. $y = \dfrac{x}{x^2 - 9}$
36. $x = -6$
37. $x = y^2 + 4$
38. $y = x^2 + 4$
39. $x^2 + y^2 = 4$

In Exercises 40 - 49, find an expression for $f(x)$.

40. f is a function that takes a real number x and performs the following three steps in the order given: (1) multiply by 2; (2) add 3; (3) divide by 4.

41. f is a function that takes a real number x and performs the following three steps in the order given: (1) add 3; (2) multiply by 2; (3) divide by 4.

42. f is a function that takes a real number x and performs the following three steps in the order given: (1) divide by 4; (2) add 3; (3) multiply by 2.

43. f is a function that takes a real number x and performs the following three steps in the order given: (1) multiply by 2; (2) add 3; (3) take the square root.

44. f is a function that takes a real number x and performs the following three steps in the order given: (1) add 3; (2) multiply by 2; (3) take the square root.

45. f is a function that takes a real number x and performs the following three steps in the order given: (1) add 3; (2) take the square root; (3) multiply by 2.

46. f is a function that takes a real number x and performs the following three steps in the order given: (1) take the square root; (2) subtract 13; (3) make the quantity the denominator of a fraction with numerator 4.

47. f is a function that takes a real number x and performs the following three steps in the order given: (1) subtract 13; (2) take the square root; (3) make the quantity the denominator of a fraction with numerator 4.

48. f is a function that takes a real number x and performs the following three steps in the order given: (1) take the square root; (2) make the quantity the denominator of a fraction with numerator 4; (3) subtract 13.

49. f is a function that takes a real number x and performs the following three steps in the order given: (1) make the quantity the denominator of a fraction with numerator 4; (2) take the square root; (3) subtract 13.

In Exercises 50 - 57, use the given function f to find and simplify the following:

- $f(1)$
- $f(-3)$
- $f\left(\frac{3}{2}\right)$
- $f(4x)$
- $4f(x)$
- $f(-x)$
- $f(x-4)$
- $f(x)-4$
- $f\left(x^2\right)$

50. $f(x) = 2x + 1$

51. $f(x) = 3 - 4x$

52. $f(x) = 2 - x^2$

53. $f(x) = x^2 - 3x + 2$

54. $f(x) = \sqrt{x-1}$

55. $f(x) = \dfrac{x}{x-1}$

56. $f(x) = 6$

57. $f(x) = 0$

In Exercises 58 - 65, use the given function f to find and simplify the following:

- $f(2)$
- $f(-2)$
- $f(2a)$
- $2f(a)$
- $f(a+2)$
- $f(a)+f(2)$
- $f\left(\frac{2}{a}\right)$
- $\frac{f(a)}{2}$
- $f(a+h)$

58. $f(x) = 2x - 5$
59. $f(x) = 5 - 2x$
60. $f(x) = 2x^2 - 1$
61. $f(x) = 3x^2 + 3x - 2$
62. $f(x) = \sqrt{2x+1}$
63. $f(x) = 1$
64. $f(x) = \dfrac{x}{2}$
65. $f(x) = \dfrac{2}{x}$

In Exercises 66 - 73, use the given function f to find $f(0)$ and solve $f(x) = 0$

66. $f(x) = 2x - 1$
67. $f(x) = 3 - \frac{2}{5}x$
68. $f(x) = 2x^2 - 6$
69. $f(x) = x^2 - x - 12$
70. $f(x) = \sqrt{x+4}$
71. $f(x) = \sqrt{1-2x}$
72. $f(x) = \dfrac{3}{4-x}$
73. $f(x) = \dfrac{3x^2 - 12x}{4 - x^2}$

4.6.2 Domain and Range

For each of Exercises 11 - 30 from page 331, find the domain and range of those relations which represent y as a function of x. Hint: There are exactly 15 relations which represent y as a function of x, i.e., that pass the Vertical Line Test.

In Exercises 1 - 24, find the domain of the function.

1. $f(x) = x^4 - 13x^3 + 56x^2 - 19$
2. $f(x) = x^2 - 4$
3. $f(x) = \dfrac{x-2}{x+1}$
4. $f(x) = \dfrac{3x}{x^2+x-2}$
5. $f(x) = \dfrac{2x}{x^2+3}$
6. $f(x) = \dfrac{2x}{x^2-3}$
7. $f(x) = \dfrac{x+4}{x^2-36}$
8. $f(x) = \dfrac{x-2}{x-2}$
9. $f(x) = \sqrt{3-x}$
10. $f(x) = \sqrt{2x+5}$
11. $f(x) = 9x\sqrt{x+3}$
12. $f(x) = \dfrac{\sqrt{7-x}}{x^2+1}$
13. $f(x) = \sqrt{6x-2}$
14. $f(x) = \dfrac{6}{\sqrt{6x-2}}$
15. $f(x) = \sqrt[3]{6x-2}$
16. $f(x) = \dfrac{6}{4-\sqrt{6x-2}}$
17. $f(x) = \dfrac{\sqrt{6x-2}}{x^2-36}$
18. $A(x) = \sqrt{x-7} + \sqrt{9-x}$
19. $s(t) = \dfrac{t}{t-8}$
20. $Q(r) = \dfrac{\sqrt{r}}{r-8}$
21. $p(n) = \dfrac{n}{\sqrt{n-8}}$
22. $g(v) = \dfrac{1}{4 - \frac{1}{v^2}}$
23. $T(t) = \dfrac{\sqrt{t}-8}{5-t}$
24. $u(w) = \dfrac{w-8}{5-\sqrt{w}}$

4.6.3 Combining Functions

In Exercises 1 - 10, use the pair of functions f and g to find the following values if they exist.

- $(f+g)(2)$
- $(f-g)(-1)$
- $(g-f)(1)$
- $(fg)\left(\frac{1}{2}\right)$
- $\left(\frac{f}{g}\right)(0)$
- $\left(\frac{g}{f}\right)(-2)$

1. $f(x) = 3x+1 \quad g(x) = 4-x$
2. $f(x) = x^2 \quad g(x) = -2x+1$
3. $f(x) = x^2-x \quad g(x) = 12-x^2$
4. $f(x) = 2x^3 \quad g(x) = -x^2-2x-3$
5. $f(x) = \sqrt{x+3} \quad g(x) = 2x-1$
6. $f(x) = \sqrt{4-x} \quad g(x) = \sqrt{x+2}$
7. $f(x) = 2x \quad g(x) = \dfrac{1}{2x+1}$
8. $f(x) = x^2 \quad g(x) = \dfrac{3}{2x-3}$
9. $f(x) = x^2 \quad g(x) = \dfrac{1}{x^2}$
10. $f(x) = x^2+1 \quad g(x) = \dfrac{1}{x^2+1}$

In Exercises 11 - 20, use the pair of functions f and g to find the domain of the indicated function then find and simplify an expression for it.

- $(f+g)(x)$
- $(f-g)(x)$
- $(fg)(x)$
- $\left(\frac{f}{g}\right)(x)$

11. $f(x) = 2x+1 \quad g(x) = x-2$
12. $f(x) = 1-4x \quad g(x) = 2x-1$
13. $f(x) = x^2 \quad g(x) = 3x-1$
14. $f(x) = x^2-x \quad g(x) = 7x$
15. $f(x) = x^2-4 \quad g(x) = 3x+6$
16. $f(x) = -x^2+x+6 \quad g(x) = x^2-9$
17. $f(x) = \dfrac{x}{2} \quad g(x) = \dfrac{2}{x}$
18. $f(x) = x-1 \quad g(x) = \dfrac{1}{x-1}$
19. $f(x) = x \quad g(x) = \sqrt{x+1}$
20. $f(x) = g(x) = \sqrt{x-5}$

In Exercises 21 - 32, let f be the function defined by

$$f = \{(-3,4), (-2,2), (-1,0), (0,1), (1,3), (2,4), (3,-1)\}$$

and let g be the function defined

$$g = \{(-3,-2), (-2,0), (-1,-4), (0,0), (1,-3), (2,1), (3,2)\}$$

. Compute the indicated value if it exists.

21. $(f+g)(-3)$
22. $(f-g)(2)$
23. $(fg)(-1)$
24. $(g+f)(1)$
25. $(g-f)(3)$
26. $(gf)(-3)$
27. $\left(\frac{f}{g}\right)(-2)$
28. $\left(\frac{f}{g}\right)(-1)$
29. $\left(\frac{f}{g}\right)(2)$
30. $\left(\frac{g}{f}\right)(-1)$
31. $\left(\frac{g}{f}\right)(3)$
32. $\left(\frac{g}{f}\right)(-3)$

In Exercises 33 - 40, use the given pair of functions to find the following values if they exist.

- $(g \circ f)(0)$
- $(f \circ g)(-1)$
- $(f \circ f)(2)$
- $(g \circ f)(-3)$
- $(f \circ g)\left(\frac{1}{2}\right)$
- $(f \circ f)(-2)$

33. $f(x) = x^2$, $g(x) = 2x+1$
34. $f(x) = 4-x$, $g(x) = 1-x^2$
35. $f(x) = 4-3x$, $g(x) = |x|$
36. $f(x) = |x-1|$, $g(x) = x^2-5$
37. $f(x) = 4x+5$, $g(x) = \sqrt{x}$
38. $f(x) = \sqrt{3-x}$, $g(x) = x^2+1$
39. $f(x) = \dfrac{3}{1-x}$, $g(x) = \dfrac{4x}{x^2+1}$
40. $f(x) = \dfrac{x}{x+5}$, $g(x) = \dfrac{2}{7-x^2}$

In Exercises 41 - 52, use the given pair of functions to find and simplify expressions for the following functions and state the domain of each using interval notation.

- $(g \circ f)(x)$
- $(f \circ g)(x)$
- $(f \circ f)(x)$

41. $f(x) = 2x + 3$, $g(x) = x^2 - 9$
42. $f(x) = x^2 - x + 1$, $g(x) = 3x - 5$
43. $f(x) = x^2 - 4$, $g(x) = |x|$
44. $f(x) = 3x - 5$, $g(x) = \sqrt{x}$
45. $f(x) = |x + 1|$, $g(x) = \sqrt{x}$
46. $f(x) = 3 - x^2$, $g(x) = \sqrt{x + 1}$
47. $f(x) = |x|$, $g(x) = \sqrt{4 - x}$
48. $f(x) = x^2 - x - 1$, $g(x) = \sqrt{x - 5}$
49. $f(x) = 3x - 1$, $g(x) = \dfrac{1}{x + 3}$
50. $f(x) = \dfrac{3x}{x - 1}$, $g(x) = \dfrac{x}{x - 3}$
51. $f(x) = \dfrac{x}{2x + 1}$, $g(x) = \dfrac{2x + 1}{x}$
52. $f(x) = \dfrac{2x}{x^2 - 4}$, $g(x) = \sqrt{1 - x}$

In Exercises 53 - 62, write the given function as a composition of two or more non-identity functions. (There are several correct answers, so check your answer using function composition.)

53. $p(x) = (2x + 3)^3$
54. $P(x) = \left(x^2 - x + 1\right)^5$
55. $h(x) = \sqrt{2x - 1}$
56. $H(x) = |7 - 3x|$
57. $r(x) = \dfrac{2}{5x + 1}$
58. $R(x) = \dfrac{7}{x^2 - 1}$
59. $q(x) = \dfrac{|x| + 1}{|x| - 1}$
60. $Q(x) = \dfrac{2x^3 + 1}{x^3 - 1}$
61. $v(x) = \dfrac{2x + 1}{3 - 4x}$
62. $w(x) = \dfrac{x^2}{x^4 + 1}$
63. Let $g(x) = -x$, $h(x) = x + 2$, $j(x) = 3x$ and $k(x) = x - 4$. In what order must these functions be composed with $f(x) = \sqrt{x}$ to create $F(x) = 3\sqrt{-x + 2} - 4$?
64. What linear functions could be used to transform $f(x) = x^3$ into $F(x) = -\frac{1}{2}(2x - 7)^3 + 1$? What is the proper order of composition?

In Exercises 65 - 73, let f be the function defined by

$$f = \{(-3,4),(-2,2),(-1,0),(0,1),(1,3),(2,4),(3,-1)\}$$

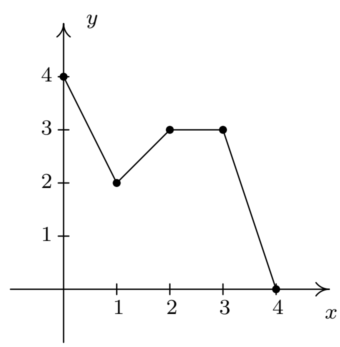
and let g be the function defined

$$g = \{(-3,-2),(-2,0),(-1,-4),(0,0),(1,-3),(2,1),(3,2)\}$$

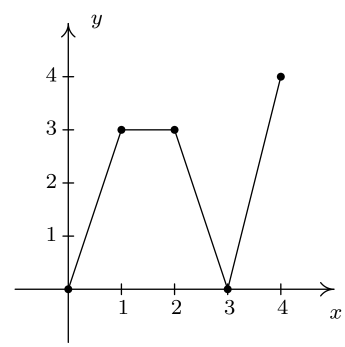
. Find the value if it exists.

65. $(f \circ g)(3)$
66. $f(g(-1))$
67. $(f \circ f)(0)$
68. $(f \circ g)(-3)$
69. $(g \circ f)(3)$
70. $g(f(-3))$
71. $(g \circ g)(-2)$
72. $(g \circ f)(-2)$
73. $g(f(g(0)))$

In Exercises 74 - 79, use the graphs of $y = f(x)$ and $y = g(x)$ below to find the function value.



$y = f(x)$



$y = g(x)$

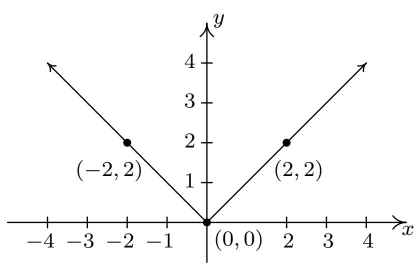
74. $(g \circ f)(1)$
75. $(f \circ g)(3)$
76. $(g \circ f)(2)$
77. $(f \circ g)(0)$
78. $(f \circ f)(1)$
79. $(g \circ g)(1)$

4.6.4 Transformations

Suppose $(2, -3)$ is on the graph of $y = f(x)$. In Exercises 1 - 17, use the given point to find a point on the graph of the given transformed function.

1. $g(x) = f(x) + 3$
2. $g(x) = f(x + 3)$
3. $g(x) = f(x) - 1$
4. $g(x) = f(x - 1)$
5. $g(x) = 3f(x)$
6. $g(x) = f(3x)$
7. $g(x) = -f(x)$
8. $g(x) = f(-x)$
9. $g(x) = f(x-3)+1$
10. $g(x) = 2f(x + 1)$
11. $g(x) = 10 - f(x)$
12. $g(x) = 3f(2x) - 1$
13. $g(x) = \frac{1}{2}f(4 - x)$
14. $g(x) = 5f(2x) + 3$
15. $g(x) = 2f(1 - x) - 1$
16. $g(x) = \dfrac{f(3x) - 1}{2}$
17. $g(x) = \dfrac{4 - f(3x - 1)}{7}$

The complete graph of $f(x) = |x|$ is given below. In Exercises 18 - 26, use it to graph the given transformed function.

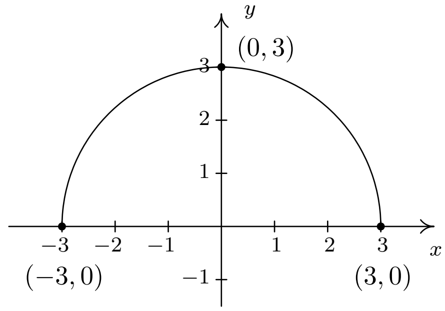


The graph for Ex. 18 - 26

18. $g(x) = f(x) + 1$
19. $g(x) = f(x) - 2$
20. $g(x) = f(x + 1)$
21. $g(x) = f(x - 2)$
22. $g(x) = 2f(x)$
23. $g(x) = f(2x)$
24. $g(x) = 2 - f(x)$
25. $g(x) = f(2 - x)$
26. $g(x) = 2-f(2-x)$

27. Some of the answers to Exercises 18 - 26 above should be the same. Which ones match up? What properties of the graph of $y = f(x)$ contribute to the duplication?

The complete graph of $f(x) = \sqrt{9-x^2}$ is given below. In Exercises 28 - 38, use it to graph the given transformed function.



The graph for Ex. 28 - 38

28. $g(x) = f(x) + 3$

29. $h(x) = f(x) - \frac{1}{2}$

30. $j(x) = f\left(x - \frac{2}{3}\right)$

31. $a(x) = f(x + 4)$

32. $b(x) = f(x+1) - 1$

33. $c(x) = \frac{3}{5}f(x)$

34. $d(x) = -2f(x)$

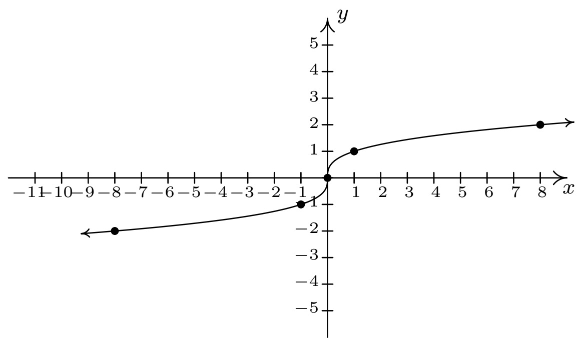
35. $k(x) = f\left(\frac{2}{3}x\right)$

36. $m(x) = -\frac{1}{4}f(3x)$

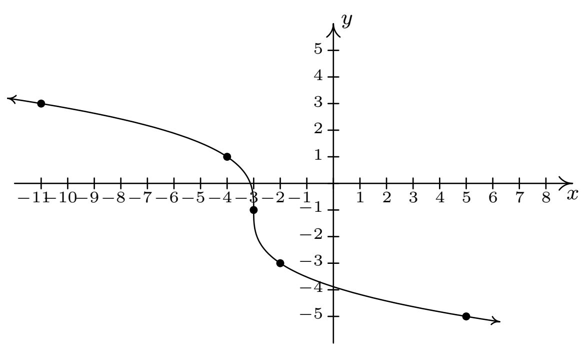
37. $n(x) = 4f(x - 3) - 6$

38. $p(x) = 4 + f(1 - 2x)$

39. The graphs of $y = f(x) = \sqrt[3]{x}$ and $y = g(x)$ are shown below. Find a formula for g based on transformations of the graph of f. Check your answer by confirming that the points shown on the graph of g satisfy the equation $y = g(x)$.



$y = \sqrt[3]{x}$



$y = g(x)$

Let $f(x) = \sqrt{x}$. Find a formula for a function g whose graph is obtained from f from the given sequence of transformations.

40. (1) shift right 2 units; (2) shift down 3 units
41. (1) shift down 3 units; (2) shift right 2 units
42. (1) reflect across the x-axis; (2) shift up 1 unit
43. (1) shift up 1 unit; (2) reflect across the x-axis
44. (1) shift left 1 unit; (2) reflect across the y-axis; (3) shift up 2 units
45. (1) reflect across the y-axis; (2) shift left 1 unit; (3) shift up 2 units
46. (1) shift left 3 units; (2) vertical stretch by a factor of 2; (3) shift down 4 units
47. (1) shift left 3 units; (2) shift down 4 units; (3) vertical stretch by a factor of 2
48. (1) shift right 3 units; (2) horizontal shrink by a factor of 2; (3) shift up 1 unit
49. (1) horizontal shrink by a factor of 2; (2) shift right 3 units; (3) shift up 1 unit

4.6.5 Piecewise-Defined and Absolute Value Functions

Piecewise-Defined Functions

1. Let $f(x) = \begin{cases} x+5 & \text{if} \quad x \leq -3 \\ \sqrt{9-x^2} & \text{if} \quad -3 < x \leq 3 \\ -x+5 & \text{if} \quad x > 3 \end{cases}$

   Compute the following function values.

   (a) $f(-4)$ (b) $f(-3)$ (c) $f(3)$

   (d) $f(3.1)$ (e) $f(-3.01)$ (f) $f(2)$

2. Let $f(x) = \begin{cases} x^2 & \text{if} \quad x \leq -1 \\ \sqrt{1-x^2} & \text{if} \quad -1 < x \leq 1 \\ x & \text{if} \quad x > 1 \end{cases}$

   Compute the following function values.

   (a) $f(4)$ (b) $f(-3)$ (c) $f(1)$

   (d) $f(0)$ (e) $f(-1)$ (f) $f(-0.99)$

In Exercises 3 - 9, find all possible x such that $f(x) = 0$. Then sketch the graph of the given piecewise-defined function. Use your graph to identify the domain and range of each function.

3. $f(x) = \begin{cases} 4-x & \text{if} \quad x \leq 3 \\ 2 & \text{if} \quad x > 3 \end{cases}$

4. $f(x) = \begin{cases} x^2 & \text{if} \quad x \leq 0 \\ 2x & \text{if} \quad x > 0 \end{cases}$

5. $f(x) = \begin{cases} -3 & \text{if} \quad x < 0 \\ 2x-3 & \text{if} \quad 0 \leq x \leq 3 \\ 3 & \text{if} \quad x > 3 \end{cases}$

6. $f(x) = \begin{cases} x^2-4 & \text{if} \quad x \leq -2 \\ 4-x^2 & \text{if} \quad -2 < x < 2 \\ x^2-4 & \text{if} \quad x \geq 2 \end{cases}$

7. $f(x) = \begin{cases} -2x - 4 & \text{if} \quad x < 0 \\ 3x & \text{if} \quad x \geq 0 \end{cases}$

8. $f(x) = \begin{cases} x^2 & \text{if} \quad x \leq -2 \\ 3 - x & \text{if} \quad -2 < x < 2 \\ 4 & \text{if} \quad x \geq 2 \end{cases}$

9. $f(x) = \begin{cases} \dfrac{1}{x} & \text{if} \quad -6 < x < -1 \\ x & \text{if} \quad -1 < x < 1 \\ \sqrt{x} & \text{if} \quad 1 < x < 9 \end{cases}$

Absolute Value Functions

In Exercises 10 - 18, find the zeros of each function and the x- and y-intercepts of each graph, if any exist. Then graph the given absolute value function and express it as a piecewise-defined function. Use the graph to determine the domain and range of each function.

10. $f(x) = |x + 4|$

11. $f(x) = |x| + 4$

12. $f(x) = |4x|$

13. $f(x) = |2x - 5|$

14. $f(x) = |-2x + 5|$

15. $f(x) = 2|x - \frac{5}{2}|$

16. $f(x) = -3|x|$

17. $f(x) = 3|x+4|-4$

18. $f(x) = \dfrac{1}{3}|2x - 1|$

Chapter 5

Polynomial Functions

5.1 Notation and Basic Examples

Objective: Understand component parts of a polynomial and classify polynomials by degree and number of terms.

Recall that a polynomial in terms of a variable x is an equation (or function) of the form

$$y = a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0,$$

where each *coefficient*, a_i, is a real number ($a_n \neq 0$) and the exponent, or *degree* of the polynomial, n, is a positive integer.

Examples of polynomials include: $y = x^2+5$, $y = x$ and $y = -3x^7+4x^3-5x$. Before classifying polynomials, we will take a moment to establish some key terminology. For our general polynomial above, the

degree	is	n
coefficients	are	$a_n, a_{n-1}, \ldots, a_1, a_0$
leading coefficient	is	a_n
leading term	is	$a_n x^n$
constant term	is	$a_0 x^0 = a_0$.

A concrete example will help to clarify each of these terms.

Example 5.1. Identify the degree, leading coefficient, leading term and constant term for the given polynomial.

$$y = -19x^5 + 4x^4 - 6x + 21$$

The degree of this polynomial is $n = 5$, since five is the greatest exponent.

The leading term, which is the term that contains the greatest exponent (degree) is $a_n x^n = -19x^5$.

The leading coefficient is the real number being multiplied by x in the leading term, namely $a_n = -19$.

The constant term is $a_0 = 21$, which also represents the $y-$intercept for the graph of the given polynomial, just as it did in the chapter on quadratics.

The complete list of coefficients for the given polynomial is

$$a_5 = -19,\ a_4 = 4,\ a_3 = 0,\ a_2 = 0,\ a_1 = -6 \text{ and } a_0 = 21.$$

It is important to point out the fact that the previous example contains no *cubic* or *quadratic* terms, since the respective coefficients are both zero. This example demonstrates that not every polynomial will contain a nonzero coefficient for every term. As another example, the *power function* $f(x) = x^{10}$ is also characterized as a polynomial having degree $n = 10$, leading coefficient $a_{10} = 1$, and trailing coefficients $a_i = 0$ for $i = 9, 8, \ldots, 1, 0$.

Before we can identify and begin to classify a polynomial, we may need to simplify the given expression for x, by distributing and combining all like terms. The general form of a polynomial should be reminiscent of the standard form of a quadratic, will possibly more terms. Hence the name "polynomial", meaning "many terms".

The following example shows how to identify a polynomial after the necessary simplification has taken place.

Example 5.2. Identify the degree, leading coefficient, leading term and constant term for the given polynomial function.

$$\begin{aligned} f(x) &= 3(x+1)(x-1)+4x^3+2x+3 \\ &= 3(x^2-1)+4x^3+2x+3 \\ &= 3x^2-3+4x^3+2x+3 \\ &= 4x^3+3x^2+2x \end{aligned}$$

Upon simplying, we see that f has degree $n=3$, since three is the greatest exponent.

The leading term is $4x^3$ with a leading coefficient of $a_n = 4$.

Since no constant term is listed, $a_0 = 0$ is our constant term.

Now that we can identify the essential components of a polynomial, we will categorize polynomials based upon their degree, as well as the number of terms, after all necessary simplification.

Types of Polynomials

Degree	Type	Example
0	Constant*	-1
1	Linear	$2x+\sqrt{5}$
2	Quadratic	$5x^2-32x+2$
3	Cubic	$(-1/2)x^3$
4	Quartic	$-3x^4+2x^2+3x+1$
5	Quintic	$-2x^5$
6 or more	n^{th} Degree	$-2x^7+52x^6+12$

*Note: A constant function is generally not considered a polynomial, since the degree $n=0$ and is not positive.

One point of note in the table above is the occurrence of both rational and irrational coefficients $\left(-1/2 \text{ and } \sqrt{5}\right)$. The appearance of such coefficients is permissible in polynomials, since our coefficients a_i are simply required to be real numbers. A coefficient containing the imaginary number $i=\sqrt{-1}$, on the other hand, is not permitted.

Polynomial Characterizations by Number of Nonzero Terms

Number of Terms	Name	Example
1	Monomial	$4x^5$
2	Binomial	$2x^3 + 1$
3	Trinomial	$-23x^{18} + 4x^2 + 3x$
4	Tetranomial	$-23x^{18} + 4x^2 + 3x + 1$
5 or more	Polynomial	$-2x^4 + x^3 + 15x^2 - 41x + 12$

Example 5.3. Describe the type and characterization (number of terms) of the polynomial given below.

$$y = -19x^5 + 4x^4 - 6x + 21$$

Polynomials are typically named by their degree first and then their number of terms. The polynomial above is a *quintic tetranomial*; quintic because it is degree five and tetranomial since it contains four terms.

Example 5.4. Describe the type and characterization (number of terms) of the polynomial given below.

$$y = x^3 + x^2$$

The polynomial above is a *cubic binomial*, since it has degree three and contains two terms.

Example 5.5. Describe the type and characterization (number of terms) of the polynomial given below.

$$y = 21x^4 + 12x^2 - 3x^2 - 9x^2 - 22x^4$$

Upon simplifying, we see that the given polynomial reduces to $y = -x^4$. As a result, our polynomial is a quartic (degree four) monomial (one term).

This section has "set the table" for the basic terminology that will be used throughout the chapter. In the next section, we will review two additional factoring techniques which will be necessary for working with certain polynomials, and provide a brief summary of all factoring methods that have been discussed in this text. Once we have finished our review of factoring, we will be ready to begin the natural (albeit lengthy) method of analyzing and graphing a polynomial function.

5.2 Factoring

5.2.1 Special Products

Objective: Identify and factor special products including a difference of squares, perfect squares, and sum and difference of cubes.

When factoring polynomials there are a few special products that, if we can recognize them, we can easily break down. The first is one we have seen before. When multiplying, we found that the product of a sum and difference of the same two terms results in the difference of two squares. Here we will use this special product to help us factor.

$$\textbf{Difference of two Squares : } \boldsymbol{a^2 - b^2 = (a+b)(a-b)}$$

Consequently, if we faced with the difference of two squares, we may conclude that they will always factor to the sum and difference of their square roots. Our first four examples demonstrate this fact.

Example 5.6. Factor the given expression completely over the real numbers.

$$\begin{aligned} x^2 - 16 &\quad \text{Difference of two perfect squares;} \\ &\quad \quad \text{the square roots are } x \text{ and } 4 \\ (x+4)(x-4) &\quad \text{Our solution} \end{aligned}$$

Example 5.7. Factor the given expression completely over the real numbers.

$$\begin{aligned} 9a^2 - 25b^2 &\quad \text{Difference of two perfect squares;} \\ &\quad \quad \text{the square roots are } 3a \text{ and } 5b \\ (3a+5b)(3a-5b) &\quad \text{Our solution} \end{aligned}$$

When factoring a difference of two squares, $a^2 - b^2$, the previous technique is often overlooked whenever one (or both) of the given terms is not a *perfect* square. The following examples demonstrate that the same method may still be employed in such situations.

Example 5.8. Factor the given expression completely over the real numbers.

$$\begin{aligned} x^2 - 24 &\quad \text{Difference of two squares;} \\ &\quad \quad \text{the square roots are } x \text{ and } 2\sqrt{6} \\ \left(x+2\sqrt{6}\right)\left(x-2\sqrt{6}\right) &\quad \text{Our solution} \end{aligned}$$

Example 5.9. Factor the given expression completely over the real numbers.

$$\begin{array}{rl} 2x^2-5 & \text{Difference of two squares;} \\ & \quad \text{the square roots are } \sqrt{2}x \text{ and } \sqrt{5} \\ \left(\sqrt{2}x+\sqrt{5}\right)\left(\sqrt{2}x-\sqrt{5}\right) & \text{Our solution} \end{array}$$

It is important to note that, unlike differences, a *sum* of squares will never factor over the real numbers. Such expressions are only factorable over the complex numbers. Hence, we say that they are *irreducible* over the reals. This can be seen in our next example, where we will attempt to employ the $ac-$method to factor.

Example 5.10. Factor the given expression completely over the real numbers.

$$\begin{array}{rl} x^2+36 & \text{No linear term; introduce } 0x \\ x^2+0x+36 & \text{Multiply to } ac \text{ or } 36, \text{ add to } 0 \end{array}$$

Our choices are: $1 \cdot 36$, $2 \cdot 18$, $3 \cdot 12$, $4 \cdot 9$ and $6 \cdot 6$.

Since there are no combinations that multiply to 36 and add to 0, we conclude that the given expression is irreducible over the reals.

Notice that x^2+36 does, however, factor over the complex numbers.

$$\begin{aligned} x^2+36 &= x^2-(-36) \\ &= x^2-\left(\sqrt{36}\cdot\sqrt{-1}\right)^2 \\ &= x^2-(6i)^2 \\ &= (x-6i)(x+6i) \end{aligned}$$

We can make further sense of this result by recalling the methods from the chapter on quadratics. For example, since the discriminant of x^2+36 is $b^2-4ac = 0^2-4(1)(36) < 0$, we know that the expression has no real zeros. Hence, any factorization must contain imaginary numbers.

Since, for graphing purposes, we will primarily be concerned with factorizations over the real numbers, we may conclude that such expressions are always irreducible over the real numbers,and therefore cannot be factored. We present the general factorization over the complex numbers below.

$$\textbf{Sum of Squares : } \boldsymbol{a^2 + b^2 = (a + bi)(a - bi)}$$

A special case where we can interpret an expression as a difference of two squares comes from the factorization of a difference of 4^{th} (fourth) powers. Since the square root of a fourth power is, in fact, a square $\left(\sqrt{a^4} = a^2\right)$, we can factor a difference of fourth powers just like we factor a difference of squares. This will yield two factors, one which will be an irreducible sum of squares. The other factor will be a difference of squares, which we can factor further. The next two examples demonstrate this.

Example 5.11. Factor the given expression completely over the real numbers.

$$\begin{array}{rl} a^4 - b^4 & \text{Difference of squares with roots } a^2 \text{ and } b^2 \\ (a^2 + b^2)(a^2 - b^2) & \text{The first factor is irreducible;} \\ & \quad \text{the second is a difference of squares} \\ (a^2 + b^2)(a + b)(a - b) & \text{Our solution} \end{array}$$

Example 5.12. Factor the given expression completely over the real numbers.

$$\begin{array}{rl} x^4 - 16 & \text{Difference of squares with roots } x^2 \text{ and } 4 \\ (x^2 + 4)(x^2 - 4) & \text{The first factor is irreducible;} \\ & \text{the second is a difference of squares} \\ (x^2 + 4)(x + 2)(x - 2) & \text{Our solution} \end{array}$$

Such expressions as the previous two examples will be classified in the next subsection as expressions of *quadratic type*, since, for example, we may consider $x^4 - 16$ as $(x^2)^2 + 0(x^2) + 16$. Replacing x^2 with y gives us $y^2 + 0y + 16$, which we could consider as a quadratic in terms of y.

Another factoring technique involves recognizing an entire expression as a perfect square.

$$\textbf{Perfect Square : } \boldsymbol{a^2 + 2ab + b^2 = (a+b)^2}$$

A perfect square can be difficult to recognize at first glance, but if we use the $ac-$method to produce two of the same number, the resulting factorization will be a perfect square. In this case, we can just factor by identifying the square roots of the first and last terms and using the sign from the middle term. This is demonstrated in the following examples.

Example 5.13. Factor the given expression completely over the real numbers.

$$x^2 - 6x + 9 \qquad \text{Multiply to } ac \text{ or } 9, \text{ add to } -6$$

The numbers we need are -3 and -3. We have a perfect square. Use the square roots of $a = 1$ and $c = 9$ and the negative sign from the linear term.

$$(x-3)^2 \qquad \text{Our solution}$$

Example 5.14. Factor the given expression completely over the real numbers.

$$4x^2 + 20xy + 25y^2 \qquad \text{Multiply to } ac \text{ or } 100, \text{ add to } 20$$

The numbers we need are 10 and 10. We have a perfect square. Use the square roots of $a = 4$ and $c = 25$ and the positive sign from the middle term.

$$(2x+5y)^2 \qquad \text{Our solution}$$

World View Note: The first known record of work with polynomials comes from the Chinese around 200 BC. Problems would be written as "three sheaves of a good crop, two sheaves of a mediocre crop, and one sheaf of a bad crop sold for 29 dou". This would be interpreted as the polynomial (trinomial) $3x + 2y + z = 29$.

Another factoring shortcut involves sums and differences of cubes. Both sums and differences of cubes have very similar factorizations.

$$\textbf{Sum of Cubes}: \boldsymbol{a^3 + b^3 = (a+b)(a^2 - ab + b^2)}$$

$$\textbf{Difference of Cubes}: \boldsymbol{a^3 - b^3 = (a-b)(a^2 + ab + b^2)}$$

Comparing the formulas one may notice that the only difference resides in the signs between the terms. One way to remember these two formulas is to think of SOAP. Here, S stands for the 'same sign' as the given expression. If we have a sum of cubes, we add first. For a difference of cubes, we subtract first. O stands for the 'opposite sign'. If we have a sum of cubes, then the second sign is subtraction. For a difference of cubes, we would have addition for the second sign. Finally, AP stands for 'always positive', since both formulas end with addition. The following examples show the factorization for a sum or difference of cubes.

Example 5.15. Factor the given expression completely over the real numbers.

$$\begin{array}{rl} m^3 - 27 & \text{Desired cube roots are } m \text{ and } 3 \\ (m \ ? \ 3)(m^2 \ ? \ 3m \ ? \ 9) & \text{Use SOAP to fill in signs} \\ (m-3)(m^2+3m+9) & \text{Our solution} \end{array}$$

Example 5.16. Factor the given expression completely over the real numbers.

$$\begin{array}{rl} 125p^3 + 8r^3 & \text{Desired cube roots are } 5p \text{ and } 2r \\ (5p \ ? \ 2r)(25p^2 \ ? \ 10pr \ ? \ 4r^2) & \text{Use SOAP to fill in signs} \\ (5p+2r)(25p^2-10pr+4r^2) & \text{Our solution} \end{array}$$

The previous example illustrates an important point. When we identify the first and last terms of the trinomial in our factorization, we must square the cube roots $5p$ and $2r$. In situations such as this, one must not forget to square both the coefficient and the variable.

Often after factoring a sum or difference of cubes, one will look to factor the resulting trinomial (our second factor) further. As a general rule, this factor should always be irreducible over the reals (unless there is a GCF which should have been factored out initially).

The following table summarizes all of the special factorizations that we have discussed thus far in the section.

Factoring Special Products

Difference of Squares	$a^2 - b^2 = (a+b)(a-b)$
Sum of Squares	$a^2 + b^2$ (irreducible over the reals)
Perfect Square	$a^2 + 2ab + b^2 = (a+b)^2$
Sum of Cubes	$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
Difference of Cubes	$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

As always, when factoring special products it is important to check for a GCF first. Only after checking for a GCF should we identify and factor the special product. We demonstrate this in the last three examples.

Example 5.17. Factor the given expression completely over the real numbers.

$72x^2 - 2$	GCF is 2
$2(36x^2 - 1)$	Difference of squares; square roots are $6x$ and 1
$2(6x+1)(6x-1)$	Our solution

Example 5.18. Factor the given expression completely over the real numbers.

$48x^2y - 24xy + 3y$	GCF is $3y$
$3y(16x^2 - 8x + 1)$	Multiply to ac or 16, add to 8
	The numbers are 4 and 4; we have a perfect square
$3y(4x-1)^2$	Our solution

Example 5.19. Factor the given expression completely over the real numbers.

$128a^4b^2 + 54ab^5$	GCF is $2ab^2$
$2ab^2(64a^3 + 27b^3)$	Sum of cubes; cube roots are $4a$ and $3b$
$2ab^2(4a+3b)(16a^2 - 12ab + 9b^2)$	Our solution

5.2.2 Quadratic Type

Objective: Identify, factor, and solve equations of quadratic type by factoring and extracting square roots.

Recall that $x^2 + 2x - 24 = 0$ can be factored into $(x+6)(x-4) = 0$. Also, using the method of extracting square roots, the solutions to $x^2 - 4 = 0$ can be found to be $x = 2$ and $x = -2$. We will apply both of these ideas, factoring and extracting square roots, to find the zeros of a special type of higher degree polynomial, which we will classify as a polynomial of *quadratic type.*

The idea behind polynomials of quadratic type is best illustrated with an example. Notice that the degree 4 polynomial equation $x^4+2x^2-24 = 0$ can be made to look like the quadratic equation $y^2 + 2y - 24 = 0$, if we were to replace x^2 with y. This is possible, since the degree of the leading term (4) is double that of the middle term (2). Thus, the equation $x^4+2x^2-24 = 0$ may be rewritten as $(x^2)^2 + 2(x^2) - 24 = 0$. Replacing x^2 with the new variable y helps us recognize that we may be able to factor the given equation just like a quadratic.

$$
\begin{aligned}
x^4 + 2x^2 - 24 &= 0 && \text{Initial equation}\\
(x^2)^2 + 2(x^2) - 24 &= 0 && \text{Rewritten as quadratic type}\\
y^2 + 2y - 24 &= 0 && \text{Change of variable to } y\\
(y+6)(y-4) &= 0 && \text{Factor}\\
(x^2+6)(x^2-4) &= 0 && \text{Replace variable } y \text{ with } x^2
\end{aligned}
$$

Working with this same structure, we can generate other polynomials that are of quadratic type and can also be factored. For example, $x^6+2x^3-24 = 0$ may be written as $(x^3)^2+2(x^3)-24 = 0$ and $x^8+2x^4-24 = 0$ may be written as $(x^4)^2 + 2(x^4) - 24 = 0$. We can then factor and extract square roots to find all possible solutions to the given equation. The next few examples demonstrate this technique. We begin by finishing up our first example.

Example 5.20. Solve the given equation for all possible values of x.

$$\begin{array}{rl}
x^4 + 2x^2 - 24 = 0 & \text{Quadratic type;} \\
& \quad \text{degree of } x^4 \text{ is twice degree of } x^2 \\
(x^2+6)(x^2-4) = 0 & \text{Resulting factorization} \\
x^2 + 6 = 0 \quad \text{or} \quad x^2 - 4 = 0 & \text{Set each factor equal to 0 and solve} \\
\sqrt{x^2} = \sqrt{-6} \qquad \sqrt{x^2} = \sqrt{4} & \text{Extract square roots} \\
x = \pm\sqrt{-6} \qquad x = \pm\sqrt{4} & \text{Even roots require a } (\pm) \\
x = \pm\sqrt{6}i \quad \text{or} \quad x = \pm 2 & \text{Our solutions}
\end{array}$$

Notice that we obtain two real solutions and two imaginary solutions in the example above.

Example 5.21. Solve the given equation for all possible values of x.

$$\begin{array}{rl}
x^4 - 12x^2 + 27 = 0 & \text{Quadratic type;} \\
& \quad \text{degree of } x^4 \text{ is twice degree of } x^2 \\
(x^2-3)(x^2-9) = 0 & \text{Resulting factorization} \\
x^2 - 3 = 0 \quad \text{or} \quad x^2 - 9 = 0 & \text{Set each factor equal to 0 and solve} \\
\sqrt{x^2} = \sqrt{3} \qquad \sqrt{x^2} = \sqrt{9} & \text{Extract square roots} \\
x = \pm\sqrt{3} \qquad x = \pm\sqrt{9} & \text{Even roots require a } (\pm) \\
x = \pm\sqrt{3} \quad \text{or} \quad x = \pm 3 & \text{Our solutions}
\end{array}$$

Unlike the first example, here we have obtained four real solutions. In the next example we will see a higher degree polynomial equation that still is of quadratic type. Recall that when extracting roots for a binomial factor having an odd degree, we will obtain just one solution, and there will be no need for a $(\pm)$ sign.

Example 5.22. Solve the given equation for all possible values of x.

$$\begin{array}{rl}
x^6 - 4x^3 - 5 = 0 & \text{Quadratic type;} \\
& \quad \text{degree of } x^6 \text{ is twice degree of } x^3 \\
(x^3-5)(x^3+1) = 0 & \text{Resulting factorization} \\
x^3 - 5 = 0 \quad \text{or} \quad x^3 + 1 = 0 & \text{Set each factor equal to 0 and solve} \\
\sqrt[3]{x^3} = \sqrt[3]{5} \qquad \sqrt[3]{x^3} = \sqrt[3]{-1} & \text{Extract cube roots} \\
x = \sqrt[3]{5} \quad \text{or} \quad x = -1 & \text{Our solutions; odd roots do not require a } (\pm)
\end{array}$$

Our last example is of an equation of quadratic type, that is also a difference of two squares. Our solution will reinforce the first part of this section.

Example 5.23. Solve the given equation for all possible values of x.

$x^4 - 49 = 0$	Quadratic type and difference of two squares
$(x^2 + 7)(x^2 - 7) = 0$	Resulting factorization
$x^2 + 7 = 0 \quad \text{or} \quad x^2 - 7 = 0$	Set each factor equal to 0 and solve
$\sqrt{x^2} = \sqrt{-7} \qquad \sqrt{x^2} = \sqrt{7}$	Extract square roots
$x = \pm\sqrt{-7} \qquad x = \pm\sqrt{7}$	Even roots require a $(\pm)$
$x = \pm\sqrt{7}i \quad \text{or} \quad x = \pm\sqrt{7}$	Our solutions (two real solutions)

5.2.3 Factoring Summary

Objective: Identify and use the correct method to factor various polynomials.

In this subsection, we will summarize the many factoring methods we have seen thus far. An important part of the process for solving any polynomial equation is the identification of the number of terms in the simplified equation. For any equation we will always try to factor out a GCF first.

Factoring Summary

- **GCF** - Always look for a GCF first!
- **2 terms** - Sum or difference of squares or cubes.
  $a^2 - b^2 = (a+b)(a-b)$
  $a^2 + b^2$ irreducible over the reals
  $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
  $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
- **3 terms** - Factor; watch for a perfect square.
  $ax^2 + bx + c$ Multiply to ac and add to b
  $a^2 + 2ab + b^2 = (a+b)^2$
- **4 terms** - Grouping
- **Special case** - Quadratic type (used in cases with polynomials having even degree and containing 2 or 3 terms)

We will employ a few of the techniques summarized above in order to factor each of the following examples. Here the emphasis will be on which strategy to use, rather than the steps that follow.

Example 5.24. Factor the given expression completely over the real numbers.

$$\begin{array}{rl} 4x^2 + 56xy + 196y^2 & \text{Factor out a GCF of } 4 \\ 4(x^2 + 14xy + 49y^2) & \text{3 terms, try } ac\text{-method} \end{array}$$

We need two terms to multiply to ac, or 49, and add to 14. Use 7 and 7. We have a perfect square.

$$4(x+7y)^2 \qquad \text{Our solution}$$

Example 5.25. Factor the given expression completely over the real numbers.

$5x^2y + 15xy - 35x^2 - 105x$	Factor out a GCF of $5x$
$5x(xy + 3y - 7x - 21)$	4 terms, try factor by grouping
$5x[y(x + 3) - 7(x + 3)]$	We have a common factor of $(x + 3)$
$5x(x + 3)(y - 7)$	Our solution

Example 5.26. Factor the given expression completely over the real numbers.

$100x^2 - 400$	Factor out a GCF of 100
$100(x^2 - 4)$	We have a difference of squares
$100(x + 2)(x - 2)$	Our solution

Example 5.27. Factor the given expression completely over the real numbers.

$108x^3y^2 - 39x^2y^2 + 3xy^2$	Factor out a GCF of $3xy^2$
$3xy^2(36x^2 - 13x + 1)$	3 terms, try ac-method

We need two terms to multiply to ac, or 36, and add to -13.
Use -9 and -4.

$3xy^2(36x^2 - 9x - 4x + 1)$	Split middle term
$3xy^2[9x(4x - 1) - 1(4x - 1)]$	Factor by grouping
$3xy^2(4x - 1)(9x - 1)$	Our solution

World View Note: Variables originated in ancient Greece where Aristotle would use a single capital letter to represent a number.

Example 5.28. Factor the given expression completely over the real numbers.

$5 + 625y^3$	Factor out a GCF of 5
$5(1 + 125y^3)$	2 terms, a sum of cubes
$5(1 + 5y)(1 - 5y + 25y^2)$	Our solution

It is important to be comfortable and confident not just with using all the factoring methods, but also on deciding which method is best to use. This is why practice is so important.

5.3 Division

5.3.1 Polynomial Division

Objective: Divide polynomials using long division.

Dividing polynomials is a process very similar to long division of whole numbers. But before we look at that, we will first want to be able to master dividing a polynomial by a monomial. The way we do this is very similar to distributing, but the operation we distribute is the division, dividing each term by the monomial and reducing the resulting expression. This is shown in the following examples.

Example 5.29. Divide and simplify the given expression.

$$\frac{9x^5+6x^4-18x^3-24x^2}{3x^2} \qquad \text{Divide each term in the numerator by } 3x^2$$

$$\frac{9x^5}{3x^2}+\frac{6x^4}{3x^2}-\frac{18x^3}{3x^2}-\frac{24x^2}{3x^2} \qquad \text{Reduce each fraction, subtracting exponents}$$

$$3x^3+2x^2-6x-8 \qquad \text{Our solution}$$

Example 5.30. Divide and simplify the given expression.

$$\frac{8x^3+4x^2-2x+6}{4x^2} \qquad \text{Divide each term in the numerator by } 4x^2$$

$$\frac{8x^3}{4x^2}+\frac{4x^2}{4x^2}-\frac{2x}{4x^2}+\frac{6}{4x^2} \qquad \text{Reduce each fraction, subtracting exponents}$$

Remember that negative exponents are moved to a denominator, $x^{-n}=\frac{1}{x^n}$.

$$2x+1-\frac{1}{2x}+\frac{3}{2x^2} \qquad \text{Our solution}$$

The previous example illustrates that sometimes a division of polynomials will not produce a polynomial, but will contain fractions (also known as *rational expressions*). It is important that we remember to correctly reduce such expressions.

Additionally, another interesting aspect of the previous example is the reduction of the second term $\frac{4x^2}{4x^2}$. Remember that this ratio reduces to 1, and not 0.

Unlike the previous examples, long division is required when we divide a polynomial by more than just a single term.
Long division with polynomials works very similar to long division involving whole numbers and it can be employed for a divisor having any degree.
For clarity, an example with whole numbers is provided in order to review the (general) steps that will also be used for polynomial long division.

Example 5.31. Divide 631 by 4.

$$\begin{array}{rl}
4\,\overline{)\,6\;3\;1} & \text{Divide }6\text{ by }4:\;6/4 = 1\ldots \\
\mathbf{1} & \\
4\,\overline{)\,6\;3\;1} & \text{Multiply this number by divisor}: 1\cdot 4 = 4 \\
\underline{\mathbf{-4}} & \text{Subtract 4 from 6} \\
\mathbf{2\;3} & \text{Move 3 down}
\end{array}$$

$$\begin{array}{rl}
1\;\mathbf{5} & \text{Repeat, divide 23 by 4}: 23/4 = 5\ldots \\
4\,\overline{)\,6\;3\;1} & \\
\underline{-4} & \\
23 & \text{Multiply this number by divisor}: 5\cdot 4 = 20 \\
\underline{\mathbf{-20}} & \text{Subtract 20 from 23} \\
\mathbf{3\;1} & \text{Move 1 down}
\end{array}$$

$$\begin{array}{rl}
1\;5\;\mathbf{7} & \text{Repeat, divide 31 by 4}: 31/4 = 7\ldots \\
4\,\overline{)\,6\;3\;1} & \\
\underline{-4} & \\
23 & \\
\underline{-20} & \\
31 & \text{Multiply this number by divisor}: 7\cdot 4 = 28 \\
\underline{\mathbf{-28}} & \text{Subtract 28 from 31} \\
\mathbf{3} & \text{Our remainder}
\end{array}$$

$$\begin{array}{rl}
157\,\dfrac{3}{4} & \text{Our solution}
\end{array}$$

Note in the previous example that we write our remainder as a fraction, next to the quotient of 157, since it is technically still being divided by the divisor of 4. This same idea will be employed for remainders when dividing polynomials.

One way of summarizing our result is as follows.

$$\frac{631}{4} = 157 + \frac{3}{4}$$

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

Just like dividing numbers, this same process will be used to divide polynomials. The only difference is we will replace the word "number" with "term" throughout the process.

Dividing Polynomials

1. Divide the leading term by the leading term of the divisor.
2. Multiply the resulting term by the divisor.
3. Subtract the terms carefully with the bottom terms, making sure to change signs.
4. Bring down the next term.
5. Repeat steps (1)-(4) until the degree of the remainder is *less than* the degree of the divisor.

Step (3) above often tends to pose the greatest challenge for students. It is important to keep in mind that we are are always subtracting the top term from the bottom term, which is why we must change the signs of the term(s) on the bottom. In most cases, we will need to utilize the distributive property. We now present an example of the subtraction step.

Example 5.32. Subtract $-12x^2+7x$ from $4x^2-12x$.

$$\begin{array}{rl} 14x^2-12x & \text{Line up like terms} \\ \underline{-(-12x^2+7x)} & \text{Subtract, making sure to distribute the negative} \end{array}$$

This is equivalent to the following.

$$\begin{array}{rl} 14x^2-12x & \\ \underline{+12x^2\ -\ 7x} & \text{Combine like terms} \\ 26x^2-19x & \text{Our solution} \end{array}$$

World View Note: Paolo Ruffini was an Italian Mathematician of the early 19$^{\text{th}}$ century. In 1809 he was the first to describe a process called synthetic division which we will later see can be used to as an alternative to polynomial long division.

We are now ready for our first example of polynomial long division.

Example 5.33. Divide and simplify the given expression.

$$\frac{3x^3-5x^2-32x+7}{x-4} \qquad \text{Rewrite problem in long division format}$$

$$x-4\ \overline{)\ 3x^3-5x^2-32x+7} \qquad \text{Divide leading term by } x: \frac{3x^3}{x}=3x^2$$

$$\begin{array}{rl} \mathbf{3x^2}\qquad\qquad\qquad & \\ x-4\ \overline{)\ 3x^3-5x^2-32x+7} & \text{Multiply by the divisor}: 3x^2(x-4)=3x^3-12x^2 \\ \underline{\mathbf{-3x^3+12x^2}}\qquad\qquad & \text{Subtract, changing signs} \\ \mathbf{7x^2-32x}\quad & \text{Bring down the next term, } -32x \end{array}$$

$$\begin{array}{rl}
\begin{array}{r}
3x^2 + \mathbf{7x} \\
x-4 \overline{)\ 3x^3 - 5x^2 - 32x + 7} \\
\underline{-3x^3 + 12x^2} \qquad\qquad \\
7x^2 - 32x \quad \\
\underline{\mathbf{-7x^2 + 28x}} \quad \\
\mathbf{-4x + 7}
\end{array}
&
\begin{array}{l}
\text{Repeat, divide new leading term by } x: \dfrac{7x^2}{x} = 7x \\
\\
\\
\text{Multiply by the divisor}: 7x(x-4) = 7x^2 - 28x \\
\text{Subtract, changing signs} \\
\text{Bring down the next term, } 7
\end{array}
\end{array}$$

$$\begin{array}{rl}
\begin{array}{r}
3x^2 + 7x\mathbf{-4} \quad \\
x-4 \overline{)\ 3x^3 - 5x^2 - 32x + 7} \\
\underline{-3x^3 + 12x^2} \qquad\qquad \\
7x^2 - 32x \quad \\
\underline{-7x^2 + 28x} \quad \\
-4x + 7 \\
\underline{\mathbf{+4x - 16}} \\
\mathbf{-9}
\end{array}
&
\begin{array}{l}
\text{Repeat, divide new leading term by } x: \dfrac{-4x}{x} = -4 \\
\\
\\
\\
\\
\text{Multiply by the divisor}: -4(x-4) = -4x + 16 \\
\text{Subtract, changing signs} \\
\text{Our remainder}
\end{array}
\end{array}$$

$$3x^2 + 7x - 4 - \frac{9}{x-4} \qquad \text{Our solution}$$

Remember that since our final remainder of -9 has a degree of zero, which is less than the degree of our divisor, $x - 4$, we know from step (4) that our division is complete.

If we chose to follow the same format as was earlier described, one could write the result of the previous example as follows.

$$\frac{3x^3 - 5x^2 - 32x + 7}{x-4} = 3x^2 + 7x - 4 + \frac{-9}{x-4}$$

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

We continue with another example.

Example 5.34. Divide and simplify the given expression.

$$\frac{6x^3 - 8x^2 + 10x + 103}{2x + 4} \quad \text{Rewrite problem in long division format}$$

$$2x + 4 \; \overline{)\; 6x^3 - 8x^2 + 10x + 103} \quad \text{Divide leading term by } 2x : \frac{6x^3}{2x} = 3x^2$$

$$\begin{array}{rll}
& \mathbf{3x^2} & \\
2x + 4 & \overline{)\; 6x^3 - 8x^2 + 10x + 103} & \text{Multiply term by divisor} : 3x^2(2x + 4) = 6x^3 + 12x^2 \\
& \underline{\mathbf{-6x^3 - 12x^2}} & \text{Subtract, changing signs} \\
& \mathbf{-20x^2 + 10x} & \text{Bring down the next term, } 10x
\end{array}$$

$$\begin{array}{rll}
& 3x^2 \mathbf{-10x} & \\
2x + 4 & \overline{)\; 6x^3 - 8x^2 + 10x + 103} & \text{Repeat, divide new leading term by } 2x : \frac{-20x^2}{2x} = -10x \\
& \underline{-6x^3 - 12x^2} & \text{Multiply this term by divisor} : -10x(2x + 4) = -20x^2 - 40x \\
& -20x^2 + 10x & \\
& \underline{\mathbf{+20x^2 + 40x}} & \text{Subtract, changing signs} \\
& \mathbf{50x + 103} & \text{Bring down the next term, } 103
\end{array}$$

$$\begin{array}{rll}
& 3x^2 - 10x \mathbf{+25} & \\
2x + 4 & \overline{)\; 6x^3 - 8x^2 + 10x + 103} & \text{Repeat, divide new leading term by } 2x : \frac{50x}{2x} = 25 \\
& \underline{-6x^3 - 12x^2} & \\
& -20x^2 + 10x & \\
& \underline{+20x^2 + 40x} & \\
& 50x + 103 & \text{Multiply this term by divisor} : 25(2x + 4) = 50x + 100 \\
& \underline{\mathbf{-50x - 100}} & \text{Subtract, changing signs} \\
& 3 & \text{Our remainder}
\end{array}$$

$$3x^2 - 10x + 25 + \frac{3}{2x + 4} \quad \text{Our solution}$$

Again, we can summarize the last example by

$$\frac{6x^3 - 8x^2 + 10x + 103}{2x + 4} = 3x^2 - 10x + 25 + \frac{3}{2x + 4}$$

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

In both of the previous examples the terms in the dividend had decreasing exponents, all of which included a nonzero coefficient. In other words, no term was skipped over, from the leading term (a cubic, degree of 3) down to the constant term, $a_0 = a_0 x^0$.

In polynomial long division, it is essential that both the dividend and divisor are written in what is commonly referred to as *descending power order*, in which no exponent is overlooked or omitted. If a polynomial is not given in descending power order, we must make certain to rewrite it correctly before beginning our long division.

Additionally, any polynomial that appears to skip one (or more) term(s) should be rewritten, with the missing term(s) in place, accompanied by zero for the coefficient. The inclusion of such a term is known as a *placeholder* for the polynomial, and is incredibly important for a successful long division.

Our last example demonstrates the importance of these preliminary steps.

Example 5.35. Divide and simplify the given expression.

$$\frac{2x^4 + 42x - 4x^2}{x^2 + 3x} \qquad \text{Reorder dividend; need } x^3 \text{ term, add } 0x^3$$

$$x^2+3x\ \overline{)\ 2x^4+\mathbf{0x^3}-4x^2+42x} \qquad \text{Divide the front terms : } \frac{2x^4}{x^2}=2x^2$$

$$\begin{array}{lll}
 & \mathbf{2x^2} & \\
x^2+3x & \overline{)\ 2x^4+0x^3-4x^2+42x} & \text{Multiply this term by divisor : } 2x^2(x^2+3x)=2x^4+6x^3 \\
 & \underline{\mathbf{-2x^4-6x^3}} & \text{Subtract, changing terms} \\
 & \quad\quad \mathbf{-6x^3-4x^2} & \text{Bring down the next term, } -4x^2
\end{array}$$

$$\begin{array}{lll}
 & 2x^2\mathbf{-6x} & \\
x^2+3x & \overline{)\ 2x^4+0x^3-4x^2+42x} & \text{Repeat, divide new leading term by } x^2 : \frac{-6x^3}{x^2}=-6x \\
 & \underline{-2x^4-6x^3} & \\
 & \quad\quad -6x^3-4x^2 & \text{Multiply this term by divisor : } -6x(x^2+3x)=-6x^3-18x^2 \\
 & \quad\quad \underline{\mathbf{+6x^3+18x^2}} & \text{Subtract, changing signs} \\
 & \quad\quad\quad\quad \mathbf{14x^2+42x} & \text{Bring down the next term, } 42x
\end{array}$$

$$\begin{array}{lll}
 & 2x^2-6x\ \mathbf{+14} & \\
x^2+3x & \overline{)\ 2x^4+0x^3-4x^2+42x} & \text{Repeat, divide new leading term by } x^2 : \frac{14x^2}{x^2}=14 \\
 & \underline{-2x^4-6x^3} & \\
 & \quad\quad -6x^3-4x^2 & \\
 & \quad\quad \underline{+6x^3+18x^2} & \\
 & \quad\quad\quad\quad 14x^2+42x & \text{Multiply this term by the divisor : } 14(x^2+3x)=14x^2+42x \\
 & \quad\quad\quad\quad \underline{\mathbf{-14x^2-42x}} & \text{Subtract, changing signs} \\
 & \quad\quad\quad\quad\quad\quad 0 & \text{No remainder} \\
 & & \\
 & \quad\quad\quad\quad 2x^2-6x+14 & \text{Our solution}
\end{array}$$

So we have,

$$\frac{2x^4-4x^2+42x}{x^2+3x} = 2x^2-6x+14$$

It is important to take a moment to check each problem, to verify that the exponents decrease incrementally and that none are skipped.
This final example also illustrates that, just as with classic numerical long division, sometimes our remainder will be zero.

5.3.2 Synthetic Division

Objective: Perform synthetic division only when dividing by a linear term.

Next, we will introduce a method of division that can be used to streamline the polynomial division process and is often preferred over the more traditional long division method. This method, known as *synthetic division*, although quick, is only useful when the divisor is linear. Specifically, we will require our divisor to be of the form $x - c$.

For our first example, we will divide $x^3 + 4x^2 - 5x - 14$ by $x - 2$, which one can check will produce a quotient of $x^2 + 6x + 7$ and a remainder of zero using polynomial long division.

$$\frac{x^3 + 4x^2 - 5x - 14}{x - 2} \;=\; x^2 + 6x + 7$$

The method of synthetic division focuses primarily on the coefficients of both the divisor and dividend. We must still pay careful attention, however, to the powers of our exponents, which will serve as placeholders throughout the process. To start the process, we will write our coefficients in what we will refer to as a **synthetic division tableau** prior to dividing.

To divide $x^3 + 4x^2 - 5x - 14$ by $x - 2$, we first write 2 in the place of the divisor since 2 is zero of the factor $x - 2$ and we write the coefficients of $x^3 + 4x^2 - 5x - 14$ in for the dividend. As our next step, we 'bring down' the first coefficient of the dividend. We will then multiply and add repeatedly, as demonstrated below.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & & & \\ \hline & & & & \end{array} \qquad\qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & & & \\ \hline & 1 & & & \end{array}$$

Next, take the 2 from the divisor and multiply by the 1 that was brought down to get 2. Write this underneath the 4, then add to get 6.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & 6 & & \end{array}$$

Now multiply the 2 from the divisor by the 6 to get 12, and add it to the -5 to get 7.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & 7 & \end{array}$$

Finally, multiply the 2 in the divisor by the 7 to get 14, and add it to the -14 to get 0.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \boxed{0} \end{array}$$

The first three numbers in the last row of our tableau will be the coefficients of the desired quotient polynomial. Remember, we started with a third degree polynomial and divided by a first degree polynomial, so the quotient will be a second degree polynomial. Hence the quotient is $x^2 + 6x + 7$. The number in the box represents the remainder, which is zero in this case.

Due in large part to its speed, synthetic division is often a 'tool of choice' for dividing polynomials by divisors of the form $x - c$. It is important to reiterate that synthetic division will *only* work for these kinds of divisors (linear divisors with leading coefficient 1), and we will need to use polynomial long division for divisors having degree larger than 1.

Another observation worth mentioning is that when a polynomial (of degree at least 1) is divided by $x - c$, the result will be a quotient polynomial of exactly one less degree than the original polynomial. This is a direct result of the divisor being a linear expression.

For a more complete understanding of the relationship between long and synthetic division, students are encouraged to trace each step in synthetic division back to its corresponding step in long division.

We conclude this section with three examples using synthetic division. We will summarize each example using the form below.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

Example 5.36. Use synthetic division to perform the following polynomial division. Find the quotient and the remainder polynomials.

$$\frac{5x^3 - 2x^2 + 1}{x - 3}$$

When setting up the synthetic division tableau, we need to enter 0 for the coefficient of x in the dividend as a placeholder, just like in polynomial division. Doing so gives us the following tableau.

$$\begin{array}{r|rrrr} 3 & 5 & -2 & 0 & 1 \\ & \downarrow & 15 & 39 & 117 \\ \hline & 5 & 13 & 39 & \boxed{118} \end{array}$$

Since the dividend was a third degree polynomial, the quotient is a quadratic polynomial with coefficients 5, 13 and 39. Our quotient is then $q(x) = 5x^2 + 13x + 39$ and the remainder is $r(x) = 118$.

Putting this all together, we have the following equation.

$$\frac{5x^3 - 2x^2 + 1}{x - 3} = 5x^2 + 13x + 39 + \frac{118}{x - 3}$$

Example 5.37. Use synthetic division to perform the following polynomial division. Find the quotient and the remainder polynomials.

$$\frac{x^3 + 8}{x + 2}$$

For this division, since we have a factor of $x + 2$, we must use the zero of $x = -2$ to begin.
Here, we will once again stress that it is critical to take the time in order to ensure we have set the synthetic division tableau up correctly at the onset

of the problem. Failure to do so will result in an incorrect answer, as well as a considerable amount time spent re-doing the problem.

$$\begin{array}{r|rrrr} -2 & 1 & 0 & 0 & 8 \\ & \downarrow & -2 & 4 & -8 \\ \hline & 1 & -2 & 4 & \boxed{0} \end{array}$$

We then obtain a quotient of $q(x) = x^2 - 2x + 4$ and remainder of $r(x) = 0$. This gives us the following equation.

$$\frac{x^3 + 8}{x + 2} = x^2 - 2x + 4$$

This answer is a great reminder of the factoring rules for cubic polynomials that we outlined earlier in the chapter.

Example 5.38. Use synthetic division to perform the following polynomial division. Find the quotient and the remainder polynomials.

$$\frac{4 - 8x - 12x^2}{2x - 3}$$

To divide $4 - 8x - 12x^2$ by $2x - 3$, two things must be done. First, we write the dividend in descending powers of x as $-12x^2 - 8x + 4$. Second, since synthetic division works only for factors of the form $x - c$, we factor $2x - 3$ as $2\left(x - \frac{3}{2}\right)$. Our strategy is to first divide $-12x^2 - 8x + 4$ by 2, to get $-6x^2 - 4x + 2$. Next, we divide by $\left(x - \frac{3}{2}\right)$. The tableau becomes

$$\begin{array}{r|rrr} \frac{3}{2} & -6 & -4 & 2 \\ & \downarrow & -9 & -\frac{39}{2} \\ \hline & -6 & -13 & \boxed{-\frac{35}{2}} \end{array}$$

From this, we get a quotient of $q(x) = -6x - 13$ and a remainder of $r(x) = -\frac{35}{2}$. This gives us the following equation.

$$\frac{-6x^2 - 4x + 2}{\left(x - \frac{3}{2}\right)} = -6x - 13 - \frac{\left(\frac{35}{2}\right)}{\left(x - \frac{3}{2}\right)}$$

Multiplying both sides by of our equation by $\frac{2}{2}$ and distributing gives us our desired answer.

$$\frac{-12x^2 - 8x + 4}{2x - 3} = -6x - 13 - \frac{35}{2x - 3}$$

Note that we could also multiply both sides of our last equation by $2x - 3$ to obtain the following equation.

$$-12x^2 - 8x + 4 = (2x - 3)(-6x - 13) - 35$$

While both of the forms above are certainly equivalent, the previous one may remind us of the familiar classic division algorithm for integers, shown below.

$$\text{dividend} = (\text{divisor})\cdot(\text{quotient}) + \text{remainder}$$

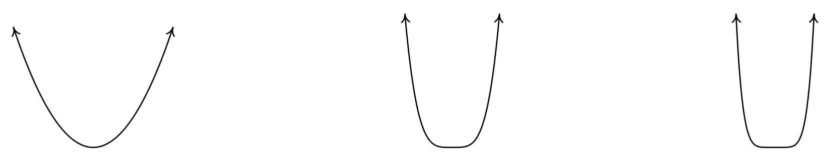
The first form, however, will be particularly useful when we graph more complicated rational functions in the next chapter.

5.4 Graphing

5.4.1 End Behavior

Objective: Determine the end behavior of polynomial functions for graphing purposes.

Now that we have learned how to classify polynomials, we are ready to see how the leading term (the leading coefficient and degree) will affect the graph. Below are the graphs of $y = x^2$, $y = x^4$ and $y = x^6$, side-by-side. We have omitted the axes to allow you to see that as the exponent increases, the 'bottom' becomes 'flatter' and the 'sides' become 'steeper'.



$y = x^2$ $\qquad$ $y = x^4$ $\qquad$ $y = x^6$

All of these functions are *even*, since $f(x) = -f(x)$, and consequently, their graphs are symmetric about the $y-$axis. It is no coincidence that each of their degrees are even. The symmetry of these graphs is important, but we want to explore a different, yet equally important feature of these functions which can be seen graphically – their **end behavior**.

The end behavior of a function identifies what is happening to the function's values (the y-values) as the x-values approach the extreme left (written $x \to -\infty$) and the extreme right (written $x \to \infty$) of the $x-$axis.

For example, given $f(x) = x^2$, as $x \to -\infty$, we imagine substituting $x = -100$, $x = -1000$, etc., into f to get $f(-100) = 10000$, $f(-1000) = 1000000$, and so on. Thus, the function's values are becoming larger and larger positive numbers (without bound). To describe this behavior, we write the following.

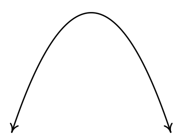
$$\text{As } x \to -\infty,\ f(x) \to \infty.$$

If we study the behavior of f as x gets large in the positive direction, we obtain the following result.

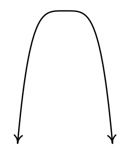
$$\text{As } x \to \infty,\ f(x) \to \infty.$$

Each of the three equations above produce graphs that are also concave up, all with even degree, and consequently exhibit the same end-behavior.

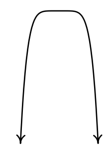
Below are the graphs of $y = -x^2$, $y = -x^4$ and $y = -x^6$, side-by-side. Each of these graphs is a reflection of the previous three graphs, respectively, over the $x-$axis. Hence, each graph is concave down, with end behavior described below.



$y = -x^2$



$y = -x^4$



$y = -x^6$

As $x \to -\infty$, $f(x) \to -\infty$.
As $x \to \infty$, $f(x) \to -\infty$.

Remember that when determining the end behavior for these three functions, our last operation is multiplication by -1, since we have $-x^2$ and not $(-x)^2$, for example. This will always produce a negative value as $x \to \pm\infty$.

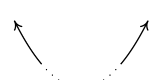
The following table summarizes our findings for the end behavior of a power function having a degree of n, where n represents a positive even integer.

End Behavior of $f(x) = ax^n$, where $n > 0$ is even.

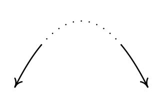
Suppose $f(x) = ax^n$ where $a \neq 0$ is a real number and n, the degree, is an even integer. The end behavior of the graph of $y = f(x)$ matches one of the following.

- For $a > 0$, as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to \infty$
- For $a < 0$, as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to -\infty$

Graphically:



$a > 0$

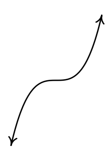


$a < 0$

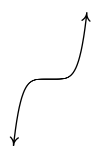
It is important to note that what we have discussed thus far only pertains to the *end* behavior (or 'tails') of the graph of a particular function, and does not relate to identifying what happens in its interior, which we will refer to as its *local* behavior. The local behavior of a polynomial function is tied closely to its zeros, and will be discussed shortly.

We now turn our attention to functions of the form $f(x) = x^n$, where $n \geq 3$ is an odd integer. We omit the function $f(x) = x$, having degree 1 because it is just the linear identity function (whose graph splits the first and third quadrants) and has already been covered extensively.

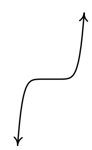
Below we have graphed $y = x^3$, $y = x^5$, and $y = x^7$. The 'flattening' and 'steepening' that we saw with the even-powered examples before presents itself here as well, but all of these equations are of odd-degree. Recall that each of these functions is *odd*, and their graphs are symmetric about the origin.



$y = x^3$



$y = x^5$

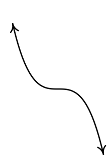


$y = x^7$

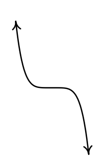
Determining the end behavior is addressed in the same manner as before, which can be confirmed by looking at the graphs.

$$\text{As } x \to -\infty,\ f(x) \to -\infty.$$
$$\text{As } x \to \infty,\ f(x) \to \infty.$$

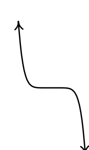
Below we have shown $y = -x^3$, $y = -x^5$, and $y = -x^7$, whose graphs are simply reflections of the previous three graphs over the $x-$axis.



$y = -x^3$



$y = -x^5$



$y = -x^7$

The end behavior is given below.

$$\text{As } x \to -\infty,\ f(x) \to \infty.$$
$$\text{As } x \to \infty,\ f(x) \to -\infty.$$

As with power functions $f(x) = ax^n$ having a positive even degree n, we can also generalize the end behavior when n is odd.

End Behavior of $f(x) = ax^n$, where $n > 0$ is odd.

Suppose $f(x) = ax^n$ where $a \neq 0$ is a real number and $n \geq 1$ is an odd integer. The end behavior of the graph of $y = f(x)$ matches one of the following.

- For $a > 0$, as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$
- For $a < 0$, as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to -\infty$

Graphically:

As an immediate consequence of what we have just established, we can identify the end behavior of any polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0,$$

where $a_n \neq 0$, with the end behavior of the related power function $g(x) = a_n x^n$. The following example demonstrates this connection.

Example 5.39. Determine the end behavior of $f(x) = 4x^3 - x + 5$.

To determine the end behavior of f, it will help us to write it in the following form.

$$f(x) = 4x^3\left(1 - \frac{1}{4x^2} + \frac{5}{4x^3}\right)$$

Since we are only concerned with large positive and negative values of x ($x \to \pm\infty$), by using this form for f, we can see that as x becomes unbounded (in either direction), the terms $\dfrac{1}{4x^2}$ and $\dfrac{5}{4x^3}$ become smaller and smaller (approach 0). The table below confirms this.

x	$\frac{1}{4x^2}$	$\frac{5}{4x^3}$
-1000	0.00000025	-0.00000000125
-100	0.000025	-0.00000125
-10	0.0025	-0.00125
10	0.0025	0.00125
100	0.000025	0.00000125
1000	0.00000025	0.00000000125

In other words, as $x \to \pm\infty$, $f(x) \approx 4x^3\,(1 - 0 + 0) = 4x^3$, which is the leading term of f.

Our final answer is summarized below.

$$\text{As } x \to -\infty,\ f(x) \to -\infty.$$
$$\text{As } x \to \infty,\ f(x) \to \infty.$$

Below is another example to further cement the effect of the leading coefficient and degree of a polynomial on the end behavior of the graph of a polynomial.

Example 5.40. Determine the end behavior of $g(x) = -5x^4 + 2x^3 - 2x^2 + 4x - 1$.

As in the previous example, we can rewrite $g(x)$ as follows.

$$g(x) = -5x^4\left(1 - \frac{2}{5x} + \frac{2}{5x^2} - \frac{4}{5x^3} + \frac{1}{5x^4}\right)$$

Again, as x becomes unbounded (in either direction), the terms $-\dfrac{2}{5x}$, $\dfrac{2}{5x^2}$, $-\dfrac{4}{5x^3}$ and $\dfrac{1}{5x^4}$ will tend closer and closer to 0, so we need only focus on the leading term. Thus considering the power function $g(x) = -5x^4$. Since g has an even degree and a negative leading coefficient, the end behavior for the graph of f is as follows:

$$\text{As } x \to -\infty,\ f(x) \to -\infty.$$
$$\text{As } x \to \infty,\ f(x) \to -\infty.$$

In general, as in each of the previous examples, when determining the end behavior of the graph of a polynomial, if we factor out the leading term, the polynomial will always become:

$$f(x) = a_n x^n \left(1 + \frac{a_{n-1}}{a_n x} + \ldots + \frac{a_2}{a_n x^{n-2}} + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n}\right).$$

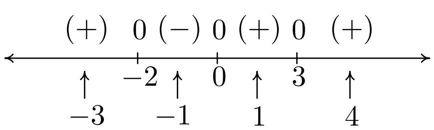
So, as $x \to \pm\infty$, any term with an x in the denominator will become smaller and smaller (approaches 0), and we have $f(x) \approx a_n x^n$.

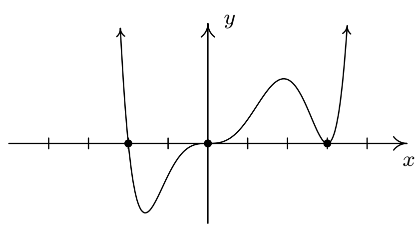
Geometrically, if we graph $y = f(x)$ using a graphing calculator, and continue to 'zoom out', the graph of it and its leading term will become indistinguishable. Next, we will address the *local behavior* of a polynomial function, which unlike its end behavior, will distinguish a polynomial f from its leading term.

5.4.2 Zeros and Local Behavior

Objective: Determine how local behavior at each zero is determined by the multiplicity of its corresponding factor.

Consider $f(x) = x^3(x-3)^2(x+2)\left(x^2+1\right)$, whose sign diagram and graph are reproduced below for reference. Its end behavior is the same as that of its leading term x^8 based on our end behavior discussion in the first part of this section. This tells us that the graph of $y = f(x)$ starts and ends above the x-axis. In other words, $f(x)$ is $(+)$ as $x \to \pm\infty$, and as a result, we no longer need to evaluate f at the test values $x = -3$ and $x = 4$. Is there a way to eliminate the need to evaluate f at the other test values? What we would really need to know is how the function behaves near its zeros: does it cross through the x-axis at these points, as it does at $x = -2$ and $x = 0$, or does it simply touch and rebound like it does at $x = 3$? From the sign diagram, the graph of f will cross the x-axis whenever the signs on either side of the zero switch (like they do at $x = -2$ and $x = 0$); it will touch when the signs are the same on either side of the zero (as is the case with $x = 3$). What we need to determine is the reason behind whether or not the sign change occurs.





A sketch of $y = f(x)$

Fortunately, f was given to us in factored form: $f(x) = x^3(x-3)^2(x+2)$. When we attempt to determine the sign of $f(-4)$, we are attempting to find the sign of the number $(-4)^3(-7)^2(-2)$, which works out to be $(-)(+)(-)$ which is $(+)$. If we move to the other side of $x = -2$, and find the sign of $f(-1)$, we are determining the sign of $(-1)^3(-4)^2(+1)$, which is $(-)(+)(+)$ which gives us the $(-)$. Notice that signs of the first two factors in both expressions are the same in $f(-4)$ and $f(-1)$. The only factor which switches sign is the third factor, $(x+2)$, precisely the factor which gave us the zero $x = -2$. If we move to the other side of 0 and look closely at $f(1)$, we get the sign pattern $(+1)^3(-2)^2(+3)$ or $(+)(+)(+)$ and we note that, once again, going from $f(-1)$ to $f(1)$, the only factor which changed sign was the first factor, x^3, which corresponds to the zero $x = 0$. Finally, to find $f(4)$, we substitute to get $(+4)^3(+2)^2(+5)$ which is $(+)(+)(+)$ or $(+)$. The sign didn't change for the middle factor $(x-3)^2$. Even though this is the factor which corresponds to the zero $x = 3$, the fact that the quantity is *squared* kept the sign of the middle factor the same on either side of 3. If we look back at the exponents on the factors $(x+2)$ and x^3, we see that they both were both odd, so as we substitute values to the left and right of the corresponding zeros, the signs of the corresponding factors changed which resulted in the sign of the function value changing. This is the key to the behavior of the function near the zeros with odd degree. Alternatively, notice how the sign didn't change for the factor with an even exponent.

Definition. Suppose f is a polynomial function and m is a natural number. If $(x-c)^m$ is a factor of $f(x)$ but $(x-c)^{m+1}$ is not, then we say $x = c$ is a zero of **multiplicity** m.

Hence, rewriting $f(x) = x^3(x-3)^2(x+2)$ as $f(x) = (x-0)^3(x-3)^2(x-(-2))^1$, we see that $x = 0$ is a zero of multiplicity 3, $x = 3$ is a zero of multiplicity 2 and $x = -2$ is a zero of multiplicity 1.

Theorem 5.1. The Role of Multiplicity: Suppose f is a polynomial function and $x = c$ is a zero of multiplicity m.

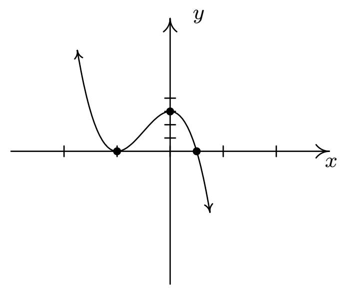
- If m is even, the graph of $y = f(x)$ touches and rebounds from the x-axis at $(c, 0)$, leaving the y-values to maintain the same sign on either side of the given zero.
- If m is odd, the graph of $y = f(x)$ crosses through the x-axis at $(c, 0)$, leaving the y-values to change signs on either side of the zero.

Our last example shows how end behavior and multiplicity allow us to sketch a decent graph without needing to create a sign diagram.

Example 5.41.

Sketch the graph of $f(x) = -3(2x-1)(x+1)^2$ using end behavior and the multiplicity of its zeros.

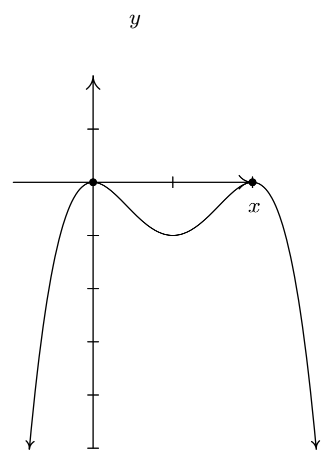
Solution. The end behavior of the graph of f will match that of its leading term. To find the leading term, we multiply by the leading terms of each factor to get $(-3)(2x)(x)^2 = -6x^3$. This tells us that the graph will start above the x-axis, in Quadrant II, and finish below the x-axis, in Quadrant IV. Next, we find the zeros of f. Fortunately for us, f is factored. Setting each factor equal to zero gives is $x = \frac{1}{2}$ and $x = -1$ as zeros. To find the multiplicity of $x = \frac{1}{2}$ we note that it corresponds to the factor $(2x - 1)$ and that exponent, and thus the multiplicity of this factor is 1. Since 1 is an odd number, we know that the graph of f will cross through the x-axis at $\left(\frac{1}{2}, 0\right)$. Since the zero $x = -1$ corresponds to the factor $(x + 1)^2 = (x - (-1))^2$, we find its multiplicity to be 2 which is an even number. As such, the graph of f will touch and rebound from the x-axis at $(-1, 0)$. Though we're not asked to, we can find the y-intercept by finding $f(0) = -3(2(0) - 1)(0 + 1)^2 = 3$. Thus, $(0, 3)$ is an additional point on the graph. Putting this together gives us the graph below.



Example 5.42.

Sketch the graph of $f(x) = x^2(x-2)^2$ using end behavior and the multiplicity of its zeros.

Solution. The end behavior of the graph of f will match that of its leading term. To find the leading term, we multiply by the leading terms of each factor to get $(x^2)(x^2) = x^4$. This tells us that the graph will start above the x-axis, in Quadrant II, and finish above the x-axis, in Quadrant I. Conversationally, we might say that the graph S̈tarts up(̈from the left) and ënds up(̈to the right). Next, we find the zeros of f since it is in factored form. Setting each factor equal to zero gives us zeros of $x = 0$ with multiplicity two, and $x = 2$ with multiplicity 2 as well. Since 2 is an even number, we know that the graph of f will touch the x-axis and rebound without changing sign at both $x = 0$ and $x = 2$. In addition, plugging in zero for x will yield a y-intercept of zero so $(0, 0)$ is an additional point on the graph. Putting this together gives us the following graph:



y
x

5.5 Polynomial Inequalities and Sign Diagrams

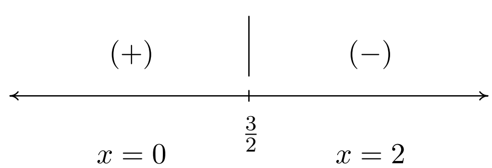
Objective: Solve and give interval notation for the solution to a polynomial inequality. Create a sign diagram to identify those intervals where a polynomial expression is positive or negative.

Recall that in Chapter 1 we were faced with having to solve linear inequalities such as $-2x + 3 \geq 0$. For the most part, this process was just as straight-forward as solving a linear equation, with the only exception being the careful consideration of division (or multiplication) by a negative. This simply required a change in the direction of the inequality.

For example, we now know that $-2x+3 \geq 0$ when $x \leq \frac{3}{2}$. This solution could then be reinforced graphically through the introduction of a sign diagram, which describes the intervals in which a given expression (or function) is positive or negative. An example for $f(x) = -2x + 3$ is shown below.

Case	Test Value	Unsimplified	Result
i	$x = 0$	$(-2(0) + 3)$	$(+)$
ii	$x = 2$	$(-2(2) + 3)$	$(-)$

Our end result can be summarized in the following *sign diagram.*



Later, in Chapter 3 we discovered that solving a quadratic inequality such as $-2x^2 + 13x - 15 < 0$ required a more careful treatment of the expression. In this case, since $-2x^2 + 13x - 15 = (-2x + 3)(x - 5)$, we recognized that the inequality $-2x^2 + 13x - 15 < 0$ holds whenever:

1. $-2x + 3 < 0$ and $x - 5 > 0$

OR

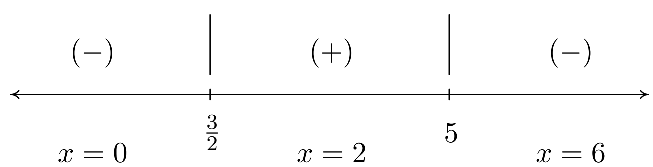
2. $-2x + 3 > 0$ and $x - 5 < 0.$

In other words, the inequality $-2x^2 + 13x - 15 < 0$ will hold for all values of x that, when plugged into the expression, yield opposite signs for each factor, since $(-) \cdot (+) = (-)$ and vice versa.

Similarly, if both factors of the original expression yield the same sign for a particular value of x, either $(+) \cdot (+)$ or $(-) \cdot (-)$, then we know that $-2x^2 + 13x - 15 > 0$ for x.

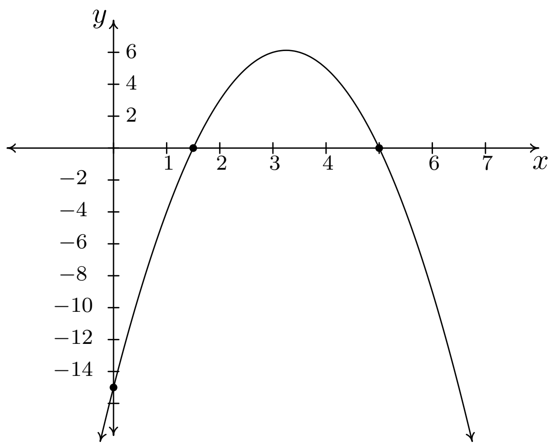
The resulting sign diagram for this case is shown below.

Case	Test Value	Unsimplified	Simplified	Result
i	$x = 0$	$(-2(0) + 3)(0 - 5)$	$(+)\cdot(-)$	$(-)$
ii	$x = 2$	$(-2(2) + 3)(2 - 5)$	$(-)\cdot(-)$	$(+)$
iii	$x = 6$	$(-2(6) + 3)(6 - 5)$	$(-)\cdot(+)$	$(-)$



We conclude that $-2x^2 + 13x - 15 > 0$ for x on the interval $\left(\frac{3}{2}, 5\right)$.

Once again, our sign diagram for $f(x) = -2x^2 + 13x - 15$ presented us with a nice visual of this, which we were able to relate to the graph of f. Specifically, if $-2x^2 + 13x - 15 > 0$, then the graph of f will reside *above* the x-axis. On the other hand, if $-2x^2 + 13x - 15 < 0$, then the graph of f will reside *below* the x-axis. We include the graph in this case for confirmation that our diagram and solution to the inequality are correct.

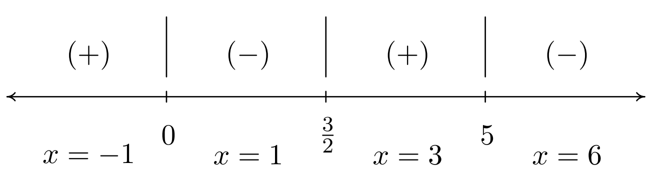


Notice that our treatment of a quadratic inequality proved significantly more challenging than solving a linear inequality due to the introduction of a second factor. This same treatment must be applied to a polynomial inequality, which may contain one (linear), two (quadratic), three (cubic), or even more factors. Working off of our previous two inequalities, we will consider the following cubic polynomial, which includes an additional factor of x.

$$\begin{aligned} f(x) &= -2x^3 + 13x^2 - 15x \\ &= x(-2x+3)(x-5) \end{aligned}$$

Here we will make our sign diagram, being careful to plug in any test values located between the zeros, just as in the previous examples.

Case	Test Value	Unsimplified	Simplified	Result
i	$x = -1$	(-1)$(-2(-1)+3)(-1-5)$	$(-)\cdot(+)\cdot(-)$	$(+)$
ii	$x = 1$	$(1)(-2(1)+3)(1-5)$	$(+)\cdot(+)\cdot(-)$	$(-)$
iii	$x = 3$	$(3)(-2(3)+3)(3-5)$	$(+)\cdot(-)\cdot(-)$	$(+)$
iv	$x = 6$	$(6)(-2(6)+3)(6-5)$	$(+)\cdot(-)\cdot(+)$	$(-)$



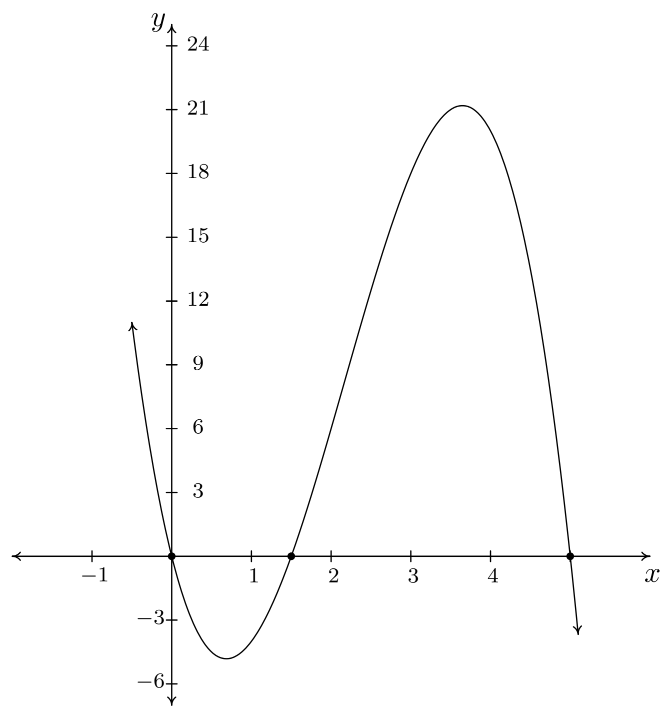
So, if we were asked to solve the polynomial inequality

$$-2x^3 + 13x^2 - 15x \leq 0,$$

we could conclude from our diagram that the solution is

$$\left[0, \frac{3}{2}\right] \cup [5, \infty).$$

Once again, we can verify that our answer is correct by comparing it to the graph of f, shown below.



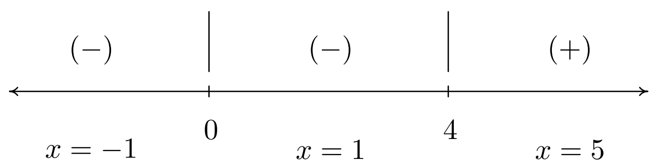
Now that we have arrived at sign diagrams of cubic functions, we will give one more example of solving a cubic inequality.

Example 5.43. Find all x such that $x^3 < 4x^2$.

Before we begin solving the given inequality, we must *always* begin by setting one side equal to zero, leaving us with the inequality $x^3 - 4x^2 < 0$.

By factoring $x^3 - 4x^2$ as $x^2(x - 4)$, we can identify zeros of the expression on the left-hand side at $x = 0$ and $x = 4$. From here, we construct a sign diagram.

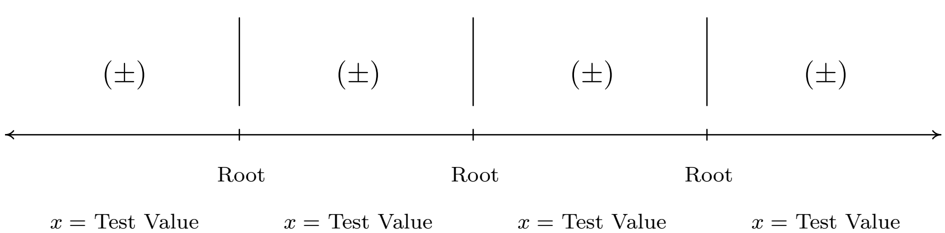
Case	Test Value	Unsimplified	Simplified	Result
i	$x = -1$	$(-1)^2$(-1-4)	$(+)\cdot(-)$	$(-)$
ii	$x = 1$	$(1)^2$(1-4)	$(+)\cdot(-)$	$(-)$
iii	$x = 5$	$(5)^2$(5-4)	$(+)\cdot(+)$	$(+)$



Finally, our answer includes the regions where the expression $x^3 - 4x^2$ is less than but not equal to zero, which we will represent as the union of two intervals: $(-\infty, 0) \cup (0, 4)$. Our careful steps have insured that this is also the solution for the original inequality of $x^3 < 4x^2$. We leave it as an exercise to the reader to sketch a graph of the function $f(x) = x^3 - 4x^2$, and confirm that our answer is correct.

At this point it is worth mentioning that one should never assume that the signs appearing in a sign diagram will always alternate from $(+)$ to $(-)$ or vice versa. This is a common misconception by students, and our previous example illustrates its fallacy. In fact, the appearance of a $(-)$ on either side of the root $x = 0$ in our previous example, is directly related to the *even multiplicity* of the root.

Additionally, the value at $x = 0$ is identified as a "critical point" in our previous sign diagram *only* because it appears as a *root* of $x^3 - 4x^2$. Although we are always interested in identifying the point that corresponds to $x = 0$ (a $y-$intercept) on a graph, we need only label $x = 0$ on a sign diagram if it is either a root or is being used as a test value for a particular interval. The following diagram has been included for reinforcement of this concept.



Whether we are faced with a linear function (degree 1) a quadratic function (degree 2), or a polynomial having a much larger degree (3,4,...), if the polynomial can be easily factored, one can always quickly construct a sign diagram to get a better idea of the behavior of the polynomial function and its corresponding graph. The method for this is the same as that which we have outlined in each of the previous examples. The difficulty only lies in what function or inequality we are presented with.

5.6 Practice Problems

5.6.1 Notation and Basic Examples

Objective: Identify the degree, leading coefficient, leading term, and constant term. Classify each polynomial by type (linear, quadratic, cubic, ...) and number of terms (monomial, binomial, trinomial, ...).

1) $y = -2x^3 + 4x + 1$
2) $y = 32x^5 + x^2 + 15$
3) $y = -3x^4 + 4x^2$
4) $y = 15x^4 - 32x^2 - x - 14$
5) $y = x^5 + 40$
6) $y = 5x^5 + 3x^2 + x + 14$
7) $y = 123x^4 - 7x^3 - 5x^2 - 3x + 1$
8) $y = -2x^3 - 1$
9) $y = -23x^6 + x^3 + x^2 + x + 1$
10) $y = -3x^4 - 15x^4 + x^4 - 27x^3 + x^2 - 13$

5.6.2 Factoring

Special Products

Objective: Classify each expression as one of the following and perform the necessary factorization: difference of squares, perfect squares, and/or sum and difference of cubes.

1) $x^4 - 81$
2) $x^4 - 1$
3) $2x^4 - 32$
4) $x^2 - 25$
5) $x^2 - 2x + 1$
6) $x^3 - 8$
7) $16x^2 + 24xy + 9y^2$
8) $x^3 + 8$
9) $25x^2 + 20xy + 4y^2$
10) $x^3 - 27$

Quadratic Type

Factor each quadratic type completely over the real numbers.

1) $x^4 + 13x^2 + 40$
2) $x^4 + x^2 - 12$
3) $x^4 - 5x^2 + 4$
4) $x^4 - 3x^2 - 10$
5) $x^4 - 17x^2 + 16$
6) $x^6 - 82x^3 + 81$
7) $x^4 + 13x^2 + 40$
8) $8x^4 + 2x^2 - 3$
9) $3x^4 - 32x^2 + 45$
10) $2x^4 - 19x^2 + 9$

Factoring Summary

Factor each expression completely, making certain to factor out a GCF whenever possible.

1) $24az - 18ah + 60yz - 45yh$
2) $2x^2 - 11x + 15$
3) $5u^2 - 9uv + 4v^2$
4) $16x^2 + 48xy + 36y^2$
5) $-2x^3 + 128y^3$
6) $20uv - 60u^3 - 5xv + 15xu^2$
7) $5n^3 + 7n^2 - 6n$
8) $2x^3 + 5x^2y + 3y^2x$
9) $54u^3 - 16$
10) $54 - 128x^3$
11) $n^2 - n$
12) $2x^4 - 21x^2 - 11$

5.6.3 Division

Polynomial Division

Use polynomial long division to divide and simplify the given expression. Express each answer in the form below.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

1) $\frac{20x^4+x^3+2x^2}{4x^3}$

2) $\frac{5x^4+45x^3+4x^2}{9x}$

3) $\frac{20n^4+n^3+40n^2}{10n}$

4) $\frac{3k^3+4k^2+2k}{8k}$

5) $\frac{12x^4+24x^3+3x^2}{6x}$

6) $\frac{5p^4+16p^3+16p^2}{4p}$

7) $\frac{10n^4+50n^3+2n^2}{10n^2}$

8) $\frac{3m^4+18m^3+27m^2}{9m^2}$

9) $\frac{x^2-2x-71}{x+8}$

10) $\frac{r^2-3r-53}{r-9}$

11) $\frac{n^2+13n+32}{n+5}$

12) $\frac{b^2-10b+16}{b-7}$

13) $\frac{v^2-2v-89}{v-10}$

14) $\frac{x^2+4x-26}{x+7}$

15) $\frac{a^2-4a-38}{a-8}$

16) $\frac{x^2-10x+22}{x-4}$

17) $\frac{45p^2+56p+19}{9p+4}$

18) $\frac{48k^2-70k+16}{6k-2}$

19) $\frac{10x^2-32x+9}{10x-2}$

20) $\frac{n^2+7n+15}{n+4}$

21) $\frac{4r^2-r-1}{4r+3}$

22) $\frac{3m^2+9m-9}{3m-3}$

23) $\frac{n^2-4}{n-2}$

24) $\frac{2x^2-5x-8}{2x+3}$

25) $\frac{27b^2+87b+35}{3b+8}$

26) $\frac{3v^2-32}{3v-9}$

27) $\frac{4x^2-33x+28}{4x-5}$

28) $\frac{4n^2-23n-38}{4n+5}$

29) $\frac{a^3+15a^2+49a-55}{a+7}$

30) $\frac{8k^3-66k^2+12k+37}{k-8}$

31) $\frac{x^3-26x-41}{x+4}$

32) $\frac{x^3-16x^2+71x-56}{x-8}$

33) $\frac{3n^3+9n^2-64n-68}{n+6}$

34) $\frac{k^3-4k^2-6k+4}{k-1}$

35) $\frac{x^3-46x+22}{x+7}$

36) $\frac{2n^3+21n^2+25n}{2n+3}$

37) $\frac{9p^3+45p^2+27p-5}{9p+9}$

38) $\frac{8m^3-57m^2+42}{8m+7}$

39) $\frac{r^3-r^2-16r+8}{r-4}$

40) $\frac{2x^3+12x^2+4x-37}{2x+6}$

41) $\frac{12n^3+12n^2-15n-4}{2n+3}$

42) $\frac{24b^3-38b^2+29b-60}{4b-7}$

43) $\frac{4v^3-21v^2+6v+19}{4v+3}$

Synthetic Division

Use synthetic division to divide and simplify the given expression. Express each answer in the form below.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

1) $\frac{x^4-4x^3+2x^2-x+1}{x+2}$

2) $\frac{12x^4-x^3+x^2-3x+1}{x+2}$

3) $\frac{x^4-2x^3+7x^2-6x+3}{x-2}$

4) $\frac{3x^4+3x^3+13x^2-4x+14}{x+1}$

5) $\frac{2x^3-2x^3+10x^2+1}{x+2}$

6) $\frac{1x^4-3x^3+5x^2-14x+2}{x-2}$

7) $\frac{5x^3-2x^3+4x^2-5x}{x-5}$

8) $\frac{2x^4-2x+1}{x+3}$

9) $\frac{-x^4-x^3+x^2+x+1}{x+5}$

10) $\frac{x^4-3x-4}{x-3}$

11) $\frac{x^4-3x^3+2x^2-x+1}{x-4}$

12) $\frac{x^4-4x^3+13x^2-5x+7}{x-4}$

5.6.4 Graphing

End Behavior

Use the degree and leading coefficient of each polynomial function below to identify the end behavior of its graph.

1) $f(x) = -2x^3 + 4x + 1$
2) $g(x) = 32x^5 + x^2 + 15$
3) $h(x) = -3x^4 + 4x^2$
4) $k(x) = 15x^4 - 32x^2 - x - 14$
5) $\ell(x) = x^5 + 40$
6) $m(x) = 5x^5 + 3x^2 + x + 14$
7) $n(x) = 123x^4 - 7x^3 - 5x^2 - 3x + 1$
8) $p(x) = x^3 - 1$
9) $q(x) = -23x^6 + x^3 + x^2 + x + 1$

Identify the degree and leading coefficient of each polynomial function below. Use the degree and leading coefficient to identify the end behavior of the graph of each function.

10) $f(x) = x^3(x-2)(x+2)$
11) $g(x) = (x^2+1)(1-x)$
12) $h(x) = x(x-3)^2(x+3)$
13) $k(x) = (3x-4)(3-4x)$
14) $\ell(x) = (x^2+2)(x^2+3)$
15) $m(x) = -2(x+7)^2(1-2x)^2$
16) $f(x) = (x^2-1)(x+4)$
17) $g(x) = (x^2-1)(x^2-16)$
18) $h(x) = -2x^3(3x-1)(2-x)$
19) $k(x) = (x^2-4x+1)(x+2)^2$

Zeros and Local Behavior

Identify the set of real zeros and their respective multiplicities for each of functions (10)-(19) above. Provide a rough sketch of each function, making sure to identify any $x-$ and $y-$intercepts. Use a graphing utility to check your graph.

5.6.5 Polynomial Inequalities and Sign Diagrams

Construct a sign diagram for each of functions (10)-(19) on page **397****. Use your diagram to identify the intervals where each function is nonnegative ($y \geq 0$).**