College Algebra

Textbook Part II

Primary Authors:

Benjamin Atchison & Mark Morabito

This textbook is designed as the primary resource for instruction of a traditional College Algebra course at Framingham State University. Each section follows closely with its respective lesson(s) in the accompanying course pack, but offers more detailed explanations and additional worked out examples.

Although largely free of mathematical errors and "typos", students who identify any errors/typos in either the textbook or course pack are encouraged to report them to the instructor, and the reporting of any mathematical errors will be rewarded with small incentives in the form of additional course homework, quiz, or exam points.

The following chapters make up the first half of the course and cover the following content.

- Linear Equations and Inequalities
- Systems of Linear Equations
- Introduction to Functions
- Quadratic Equations and Inequalities

The following chapters make up the second half of the course and cover the following content.

- Advanced Function Concepts
- Polynomials
- Rational Functions

This text contains original content, as well as content adapted from each of the following open-source texts.

Contents

Measurable Outcomes

Below is a comprehensive list of the anticipated measurable outcomes and some essential prerequisite skills needed for successful completion of the College Algebra course. This list is based off of the course description and exit list topics of MATH 123 College Algebra at Framingham State University. Each outcome number aligns to its respective lesson in the accompanying course pack.

1 Solve general linear equations with variables on both sides of the equation.

2 Solve an equation that contains one or more absolute value(s).

3 Graph a linear equation by creating a table of values for x.
Identify the slope of a linear equation both graphically and algebraically.

4 Write the equation of a line in slope-intercept and point-slope form.

5 Write the equation of a line given a line parallel or perpendicular.

6 Solve, graph, and give interval notation for the solution to a linear inequality.
Create a sign diagram to identify those intervals where a linear expression is positive or negative.

7 Solve, graph, and give interval notation to the solution of a compound inequality.

8 Solve, graph, and give interval notation to the solution of an inequality containing absolute values.

9 Solve linear systems by graphing.

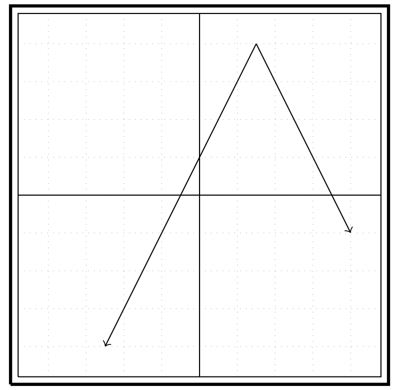
10 Solve linear systems by substitution.

11 Solve linear systems by addition and elimination.

12 Define a relation and a function; determine if a relation is a function.

13 Evaluate functions using appropriate notation.

14 Find the domain and range of a function from its graph.



Chapter 5

Advanced Function Concepts

Identifying Domain Algebraically (L30)

Objective: Identify the domain of a function that is described algebraically.

When trying to identify the domain of a function that has been described algebraically or whose graph is not known, we will often need to consider what is *not* permissible for the function, then exclude any values of x that will make the function undefined from the interval $(-\infty, \infty)$. What is left will be our domain. With virtually every algebraic function, this amounts to avoiding the following situations.

- Negatives under an even radical $\left(\sqrt{\ }\ ,\ \sqrt[4]{\ }\ ,\ \sqrt[6]{\ }\ ,\ \ldots\right)$
- Zero in a denominator

In the previous chapters, we dealt exclusively with linear equations. While equations of the form $y = mx + b$ represent y as a function of x, they are also included in a much larger family of functions known as *polynomials*. Polynomials are functions of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x + a_1 x + a_0,$$

where each of the coefficients a_i represent real numbers (with $a_n \neq 0$) and n represents a nonnegative integer. These functions include quadratics, which are of the form $y = ax^2 + bx + c$. Since polynomials contain no radicals or variables in a denominator, we can immediately conclude that their domain will always be all real numbers, or $(-\infty, \infty)$. We reiterate this with our first example.

Example 1. Find the domain of $f(x) = \frac{1}{3}x^2 - x$.

$f(x) = \dfrac{1}{3}x^2 - x$	No radicals or variables in a denominator
	No values of x need to be excluded
All real numbers or $(-\infty, \infty)$	Our solution

Our next example will be of a *rational function*, which is defined as a ratio of two polynomial functions. We will explore rational functions and their graphs in a later chapter. Since rational functions usually include expressions in a denominator, their domains will often require us to exclude one or more values of x.

Example 2. Find the domain of the function $f(x) = \dfrac{3x-1}{x^2+x-6}$.

$$
\begin{array}{rl}
f(x) = \dfrac{3x-1}{x^2+x-6} & \text{Cannot have zero in a denominator} \\[2ex]
x^2+x-6 \neq 0 & \text{Solve by factoring} \\
(x+3)(x-2) \neq 0 & \text{Set each factor not equal to zero} \\
x+3 \neq 0 \text{ and } x-2 \neq 0 & \text{Solve each inequality} \\
x \neq -3, 2 & \text{Our solution as an inequality} \\
(-\infty, -3) \cup (-3, 2) \cup (2, \infty) & \text{Our solution using interval notation}
\end{array}
$$

The notation in the previous example tells us that x can be any value except for -3 and 2. If x were to equal one of those two values, our expression in the denominator would reduce to zero and the function would consequently be undefined. Furthermore, although one can easily see that $x = \frac{1}{3}$ will make the numerator equal zero, since $x = \frac{1}{3}$ does not coincide with the two values obtained above (either -3 or 2), we should not exclude it from our domain.

This example further illustrates that whenever we are finding the domain of a rational function, we need not be concerned at all with the numerator, and instead must restrict our domain to exclude any value for x that would make the *denominator* equal to zero.

For our final two examples, we will introduce a square root in our function, first in the numerator and later in the denominator.

Example 3. Find the domain of $f(x) = \sqrt{-2x+3}$.

$$
\begin{array}{rl}
f(x) = \sqrt{-2x+3} & \text{Even radical; cannot have negative underneath} \\
-2x+3 \geq 0 & \text{Set greater than or equal to zero and solve} \\
-2x \geq -3 & \text{Remember to switch direction of inequality} \\[2ex]
x \leq \dfrac{3}{2} \text{ or } \left(-\infty, \dfrac{3}{2}\right] & \text{Our solution as an inequality or an interval}
\end{array}
$$

The notation in the above example states that our variable can be $\frac{3}{2}$ or any real number less than $\frac{3}{2}$. But any number greater than $\frac{3}{2}$ would make the function undefined.

Example 4. Find the domain of $m(x) = \dfrac{-x}{\sqrt{7x-3}}$.

The even radical tells us that we cannot have a negative value underneath. But also, the denominator cannot equal zero. This results in two inequalities.

$$7x-3 \geq 0 \text{ AND } 7x-3 \neq 0$$

Solving for x, we get the following.

$$x \geq \frac{3}{7} \quad \text{AND} \quad x \neq \frac{3}{7}$$

Our final solution is $x > \frac{3}{7}$, or $\left(\frac{3}{7}, \infty\right)$ as an interval. This represents the intersection of both inequalities above.

The previous two examples can be generalized as follows.

- In instances where the given function is a square root (or even radical), to find the domain we may set up and solve an inequality in which the entire expression underneath is set ≥ 0.
- In instances where the numerator of a given function is a polynomial and the denominator is a square root (or even radical), to find the domain we may set up and solve an inequality in which the expression underneath is set > 0 (strictly positive).

Since these two cases certainly do not handle every possible function than we may encounter, one should always be cautious when attempting to find the domain of any function.

Combining Functions

Function Arithmetic (L32)

Objective: Add, subtract, multiply, and divide functions.

In this section, we demonstrate how two (or more) functions can be combined to create new functions. This is accomplished using five common operations: the four basic arithmetic operations of addition, subtraction, multiplication and division, and a fifth operation that we will establish later in the section, known as a *composition*.

The notation for the four basic functions is as follows.

$$\begin{aligned}
\text{Addition} \quad & (f+g)(x) = f(x) + g(x) \\
\text{Subtraction} \quad & (f-g)(x) = f(x) - g(x) \\
\text{Multiplication} \quad & (f \cdot g)(x) = f(x)g(x) \\
\text{Division} \quad & \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \text{ where } g(x) \neq 0
\end{aligned}$$

As we will see in the next few examples, when applying the specified operations, one must be careful to completely simplify, by distributing and combining like terms where it is necessary. We will demonstrate this for each operation, highlighting the most critical steps in the process.

Example 5. Find $f+g$, where $f(x)=x^2-x-2$ and $g(x)=x+1$.

$(f+g)(x)$	Consider the problem
$f(x)+g(x)$	Rewrite as a sum of two functions
$(x^2-x-2)+(x+1)$	Substitute functions, inserting parentheses
$x^2-x-2+x+1$	Simplify; remove the parentheses
$x^2-x+x-2+1$	Combine like terms
$(f+g)(x)=x^2-1$	Our solution
$=(x-1)(x+1)$	Our solution in factored form

We include the factored form of $f+g$ in the previous example to reinforce the methods of factorization learned in an earlier chapter. Generally, either form (expanded or factored) would be considered acceptable.

Although the parentheses are not entirely necessary in our first example, we have included them nevertheless, to reinforce that each operation is applied to an *entire* function or expression. This will become more apparent in our next example (subtraction), when we will need to distribute a negative sign.

Example 6. Find $g-f$, where $f(x)=x^2-x-2$ and $g(x)=x+1$.

$(g-f)(x)$	Consider the problem
$g(x)-f(x)$	Rewrite as a difference of two functions
$(x+1)-(x^2-x-2)$	Substitute functions, inserting parentheses
$x+1-x^2+x+2$	Simplify; distribute the negative sign
$-x^2+x+x+1+2$	Combine like terms
$(g-f)(x)=-x^2+2x+3$	Our solution
$=-(x-3)(x+1)$	Our solution in factored form

Example 7. Find $(h\cdot k)(x)$, where $h(x)=3x^2-4x$ and $k(x)=x-2$.

$(h\cdot k)(x)$	Consider the problem
$h(x)\cdot k(x)$	Rewrite as a product of two functions
$(3x^2-4x)(x-2)$	Substitute functions, inserting parentheses
$3x^3-6x^2-4x^2+8x$	Expand by distributing
$3x^3-10x^2+8x$	Combine like terms
$(h\cdot k)(x)=3x^3-10x^2+8x$	Our solution
$=x(3x-4)(x-2)$	Our solution in factored form

Example 8. Find $\left(\dfrac{g}{f}\right)(x)$, where $f(x) = x^2 - x - 2$ and $g(x) = x + 1$.

$$\left(\frac{g}{f}\right)(x) \qquad \text{Consider the problem}$$

$$\frac{g(x)}{f(x)} \qquad \text{Rewrite as a quotient of two functions}$$

$$\frac{x+1}{x^2 - x - 2} \qquad \text{Substitute functions, parentheses unnecessary}$$

$$\frac{x+1}{(x+1)(x-2)} \qquad \text{Factor (if possible)}$$

$$x \neq -1 \quad \text{and} \quad x \neq 2 \qquad \text{Restrict denominator: } g(x) \neq 0$$

$$\frac{\cancel{x+1}}{\cancel{(x+1)}(x-2)} \qquad \text{Simplify: reduce } \frac{x+1}{x+1}$$

$$\left(\frac{g}{f}\right)(x) = \frac{1}{x-2}, \quad x \neq -1 \qquad \text{Our solution with added restriction}$$

The previous example presents us with a new precautionary measure that we must be careful not to overlook. This has to do with the simplification of g/f and the requirement that we include the necessary restriction of $x \neq -1$. Although the *domain* of the resulting quotient is still $x \neq -1, 2$, we have included $x \neq -1$ as part of our final answer, since the simplified expression allows us to easily determine that x cannot equal 2, but fails to carry through the additional restriction.

In general, whenever we simplify any function, we must be careful to insure that the domain of the resulting expression will be in agreement with the initial *unsimplified* expression. In the chapter on rational functions, we will see the graphical consequence that arises when the restriction $x \neq -1$ is overlooked.

Thus far, we have sought to create new functions by combining two functions f and g accordingly, keeping the variable x in place throughout. We could, however, just as easily evaluate the functions $f + g$, $f - g$, $f \cdot g$, and f/g at certain values of x. We do this in our next example.

Example 9. Find $(h \cdot k)(5)$, where $h(x) = 2x - 4$ and $k(x) = -3x + 1$.

$$h(x) = 2x - 4 \text{ and } k(x) = -3x + 1 \qquad \text{Evaluate each function at } 5$$

$$h(5) = 2(5) - 4 = 6 \qquad \text{Evaluate } h \text{ at } 5$$

$$k(5) = -3(5) + 1 = -14 \qquad \text{Evaluate } k \text{ at } 5$$

$$\begin{aligned}(h \cdot k)(5) &= (h(5)) \cdot (k(5)) && \text{Multiply the two results}\\ &= (6)(-14) \\ &= -84 && \text{Our solution}\end{aligned}$$

The clear advantage to this process is that the simplification can be substantially easier when the variable has been replaced with a constant. One major disadvantage, however, is that our end result represents only a single value, instead of an entire function. Particularly in situations where the resulting function is not demanded, students will likely find it more efficient to use this approach when evaluating $f+g$, $f-g$, $f \cdot g$ and f/g at a specified value.

Composite Functions (L33)

Objective: Construct, evaluate, and interpret composite functions.

In addition to the four basic arithmetic operations $(+, -, \cdot, \div)$, we will now discuss a fifth operation, known as a *composition* and denoted by $\circ$ (not to be confused with a product, $\cdot$). The result of a composition is called a *composite function* and is defined as follows.

$$\boldsymbol{(f \circ g)(x) = f(g(x))}$$

The notation $(f \circ g)(x)$ above should always be interpreted as "f of g of x". In this situation, we consider g to be the *inner* function, since it is being substituted into f for x. Consequently, we refer to f as the *outer* function.

Similarly, if we reversed the order of the two functions f and g, then the resulting composite function $(g \circ f)(x) = g(f(x))$ will have inner function f and outer function g, and should be interpreted as "g of f of x". As we will see, one should never assume that the two composite functions $f \circ g$ and $g \circ f$ will be equal.

The idea behind a composition, though relatively simple, can often pose a formidable challenge at first. We will begin by evaluating a composite function at a single value. This is accomplished by first evaluating the inner function at the specified value, and then substituting ("plugging in") the corresponding *output* into the outer function.

Example 10. Find $(f \circ g)(3)$, where $f(x) = x^2 - 2x + 1$ and $g(x) = x - 5$.

$$(f \circ g)(3) = f(g(3)) \qquad \text{Rewrite } f \circ g \text{ as inner and outer functions}$$

$$\begin{aligned} g(3) = (3) - 5 = -2 & && \text{Evaluate inner function at } x = 3 \\ & && \text{Use output of } -2 \text{ as input for } f \\ f(-2) = (-2)^2 - 2(-2) + 1 & && \text{Evaluate outer function at } x = -2 \\ = 4 + 4 + 1 & && \text{Simplify} \\ (f \circ g)(3) = 9 & && \text{Our solution}\end{aligned}$$

We can also identify a composite function in terms of the variable. In the next example, we will substitute the inner function into the outer function for every instance of the variable and then simplify. This approach is often referred to as the "inside-out" approach by some instructors.

Example 11. Find $(f \circ g)(x)$, where $f(x) = x^2 - x$ and $g(x) = x + 3$.

$$
\begin{array}{rl}
(f \circ g)(x) = f(g(x)) & \text{Rewrite } f \circ g \text{ as inner and outer functions} \\
 & \text{Our inner function is } g(x) = x + 3 \\
f(x+3) & \text{Replace each } x \text{ in } f \text{ with } (x+3) \\
 & \text{Make sure to include parentheses!} \\
(x+3)^2 - (x+3) & \text{Simplify; expand binomial} \\
(x^2 + 6x + 9) - (x+3) & \text{Distribute negative} \\
x^2 + 6x + 9 - x - 3 & \text{Combine like terms} \\
(f \circ g)(x) = x^2 + 5x + 6 & \text{Our solution} \\
= (x+3)(x+2) & \text{Our solution in factored form}
\end{array}
$$

It is important to reiterate that $(f \circ g)(x)$ usually will *not* equal $(g \circ f)(x)$ as the next example shows. Again, we will take the "inside-out" approach, where the inner function is now f and the outer function is g.

Example 12. Find $(g \circ f)(x)$, where $f(x) = x^2 - x$ and $g(x) = x + 3$.

$$
\begin{array}{rl}
(g \circ f)(x) = g(f(x)) & \text{Rewrite } g \circ f \text{ as inner and outer functions} \\
 & \text{Our inner function is } f(x) = x^2 - x \\
g(x^2 - x) & \text{Replace each } x \text{ in } f \text{ with } (x^2 - x) \\
(x^2 - x) + 3 & \text{Simplify; remove parentheses} \\
(g \circ f)(x) = x^2 - x + 3 & \text{Our solution}
\end{array}
$$

Notice that a simple calculation of the discriminant,

$$b^2 - 4ac = (-1)^2 - 4(1)(3) = -11 < 0,$$

tells us that the resulting composite function is irreducible (not factorable) over the real numbers.

Here is another example, for additional practice.

Example 13. Find $(m \circ n)(x)$, where $m(x) = 5x^2 - x + 1$ and $n(x) = x - 4$.

$$
\begin{array}{rl}
(m \circ n)(x) = m(n(x)) & \text{Rewrite } m \circ n \text{ as inner and outer functions} \\
 & \text{Our inner function is } n(x) = x - 4 \\
g(x-4) & \text{Replace each } x \text{ in } m \text{ with } (x-4) \\
 & \text{Make sure to include parentheses!} \\
5(x-4)^2 - (x-4) + 1 & \text{Simplify; expand binomial} \\
5(x^2 - 8x + 16) - (x-4) + 1 & \text{Distribute negative and the five} \\
5x^2 - 40x + 80 - x + 4 + 1 & \text{Combine like terms} \\
(m \circ n)(x) = 5x^2 - 41x + 85 & \text{Our solution}
\end{array}
$$

It is also possible to compose a function with itself, as the next example shows.

Example 14. Find $(g \circ g)(x)$, where $g(x) = x^2 - 2x$.

$(g \circ g)(x) = g(g(x))$	Rewrite $g \circ g$ as inner and outer functions
	Our inner function is $g(x) = x^2 - 2x$
$g(x^2 - 2x)$	Replace each x in g with $x^2 - 2x$
	Make sure to include parentheses!
$(x^2 - 2x)^2 - 2(x^2 - 2x)$	Simplify; expand binomial
$(x^4 - 4x^3 + 4x^2) - 2(x^2 - 2x)$	Distribute -2
$x^4 - 4x^3 + 4x^2 - 2x^2 + 4x$	Combine like terms
$(g \circ g)(x) = x^4 - 4x^3 + 2x^2 + 4x$	Our solution

We close this section by demonstrating the "outside-in" approach to finding a composite function $f \circ g$. The idea behind this approach is to *first* rewrite the outer function f by its given expression, replacing each instance of the variable with the general $g(x)$. To see that this will yield the same result as the "inside-out" approach, we will revisit example 11 above.

Example 15. Find $(f \circ g)(x)$, where $f(x) = x^2 - x$ and $g(x) = x + 3$.

$(f \circ g)(x) = f(g(x))$	Rewrite $f \circ g$ as inner and outer functions
	Our outer function is $f(x) = x^2 - x$
$[g(x)]^2 - [g(x)]$	Replace each x in f with $g(x)$
$(x+3)^2 - (x+3)$	Replace each $g(x)$ by $x + 3$
	Make sure to include parentheses!
$(x^2 + 6x + 9) - (x + 3)$	Simplify; expand binomial
$x^2 + 6x + 9 - x - 3$	Distribute negative
$x^2 + 5x + 6$	Combine like terms
$(f \circ g)(x) = x^2 + 5x + 6$	Our solution
$= (x+3)(x+2)$	Our solution in factored form

Inverse Functions

Definition and the Horizontal Line Test (L34)

Objective: Understand the definition of an inverse function and graphical implications. Determine whether a function is invertible.

In this section, we introduce the notion of an inverse function to a function f, and develop an understanding of the relationship (both algebraic and graphical) between a function f and its inverse.

One often considers the operations of addition and subtraction to be "opposites" of one another, and similarly for multiplication and division. The reason for this, naturally, is because each of these operations "undoes" the other. In mathematics, since the term "opposite" can take on different meanings, we instead consider addition and subtraction (or multiplication and division) to be *inverse operations* of one another. This notion of an inverse can be applied to entire functions, which we will now discuss.

We start by analyzing a very basic function which is reversible, a linear function. Consider the function $f(x) = 3x + 4$. Thinking of f as a process, we start with an input x and apply two steps, in order:

1. multiply by 3
2. add 4.

To reverse this process, we seek a function g which will undo each of these steps, by taking the output from f, $3x + 4$, and returning the original input x. If we think of the real-world reversible two-step process of first putting on socks then putting on shoes, to reverse the process, we first take off the shoes, and then we take off the socks. In much the same way, the function g should undo the last step of f first. That is, the function g should:

1. subtract 4, then
2. divide by 3.

Following this procedure, we get $g(x) = \dfrac{x-4}{3}$.

Now we can test our function to see if it conceptually agrees with our "feet, socks, and shoes" analogy. Just as in the first part of the process we began with our bare feet and ended up in shoes, the reverse process brings us back, in the end, to our bare feet. We can see if this holds for f and g by using what we already know about functions.

For example, if $x = 5$, then

$$f(5) = 3(5) + 4 = 15 + 4 = 19.$$

Substituting the output 19 from f as our new input for g, we get our original input for f.

$$g(19) = \frac{19-4}{3} = \frac{15}{3} = 5$$

To check that g does this for all x in the domain of f (not just a single value), we will need to find and simplify the composite function $(g \circ f)(x) = g(f(x))$.

$$g(f(x)) = g(3x+4) = \frac{(3x+\cancel{4}) - \cancel{4}}{3} = \frac{\cancel{3}x}{\cancel{3}} = x$$

If we carefully examine the arithmetic, as we simplify $g(f(x))$, we can actually see g "undoing" the addition of 4 first, followed by the multiplication by 3.

Not only does g "undo" f, but f also undoes g, which we can verify by once again looking at a composite function. This time we will find and simplify $(f \circ g)(x) = f(g(x))$.

$$f(g(x)) = f\left(\frac{x-4}{3}\right) = \cancel{3}\left(\frac{x-4}{\cancel{3}}\right) + 4 = (x - \cancel{4}) + \cancel{4} = x$$

In each composition, we began and ended with the variable x, which can be thought of as the bare feet in our analogy. Two functions f and g which are related in this manner are defined to be *inverse functions*, or simply *inverses*, of each other. More precisely, using the language of function composition, two functions f and g are said to be inverses if both:

- $g(f(x)) = x$ for all x in the domain of f, and
- $f(g(x)) = x$ for all x in the domain of g.

We say that a function f is *invertible* if an inverse function of f exists. If two functions g and f are inverses of each other, then we denote this by $g(x) = f^{-1}(x)$, and similarly $f(x) = g^{-1}(x)$. This notation can be a bit "gnarly" at first, since an inverse function f^{-1} of f must not be confused with the reciprocal function, $1/f$. The primary difference between these two functions is that a reciprocal function satisfies the property that

$$f(x) \cdot (1/f)(x) = 1,$$

whereas for inverses,

$$(f \circ f^{-1})(x) = x \quad \text{and} \quad (f^{-1} \circ f)(x) = x.$$

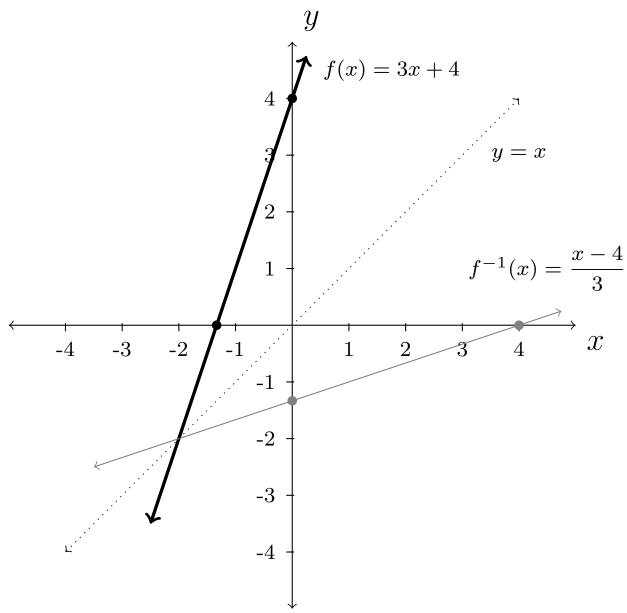
Using our function $f(x) = 3x + 4$, we can see this distinction.

- Original Function: $f(x) = 3x + 4$
- Inverse Function: $f^{-1}(x) = \dfrac{x-4}{3}$
- Reciprocal Function: $\left(\dfrac{1}{f}\right)(x) = \dfrac{1}{3x+4}$

Properties of Inverse Functions:
Let f and f^{-1} be inverse functions of one another.

- The range of f is the domain of f^{-1} and the domain of f is the range of f^{-1}.
- $f(a) = b$ if and only if $f^{-1}(b) = a$.
- The point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}.

As a direct consequence of the third property above, we will see that the graph of f^{-1} may be obtained by reflecting the graph of f about the line $y = x$. Again, we will use our example, by graphing the inverse functions $f(x) = 3x+4$ and $f^{-1}(x) = \dfrac{x-4}{3}$ on the same set of axes.



Again, from the third property, the figure above confirms that the $y-$intercept $(0, b)$ of the graph of f will be an $x-$intercept $(b, 0)$ of the graph of f^{-1}. Similarly, the $x-$intercept of the graph of f will be a $y-$intercept of the graph of f^{-1}.

Let us now turn our attention to the quadratic function $f(x) = x^2$. Is f invertible? If we consider the idea of "undoing" an operation, a likely candidate for the inverse of f is the function $g(x) = \sqrt{x}$. Checking the composition gives us

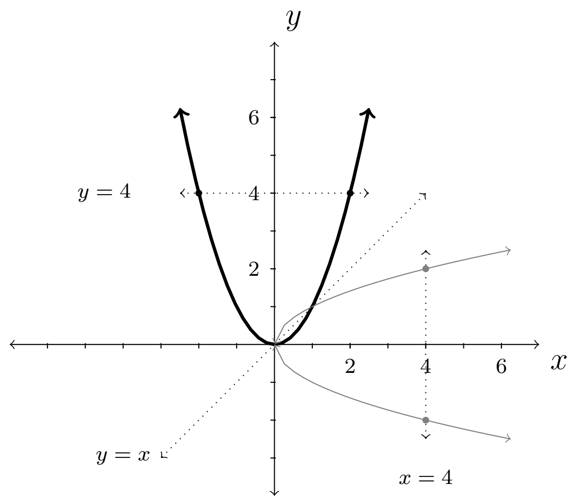
$$(g \circ f)(x) = g(f(x)) = \sqrt{x^2} = |x|,$$

which is not equal to x, for all real numbers in the domain of f, $(-\infty, \infty)$.

This subtle issue arises when we input a negative value for x into the composition above. For example, when $x = -2$, $f(-2) = (-2)^2 = 4$, but $g(4) = \sqrt{4} = 2$. Hence, g fails to return the original input $x = -2$ from its output of 4. What g does, however, is match the output 4 to a *different* input, namely $x = 2$, since $f(2)$ also equals 4.

Since both $f(-2)$ and $f(2)$ equal 4, it will be impossible to construct a function which inputs $x = 4$ and outputs *both* $x = 2$ and $x = -2$. This is due to the fact that, by definition, a function assigns a real number x with exactly one other real number.

Furthermore, we know that if and inverse f^{-1} of $f(x) = x^2$ exists, its graph can be obtained by reflecting the graph of x^2 about the line $y = x$.



In the above graph, we see that the vertical line $x = 4$ intersects the reflection of the parabola $y = x^2$ about the diagonal $y = x$ twice, which fails the Vertical Line Test, and as such, our proposed inverse cannot represent y as a function of x.

The vertical line $x = 4$ corresponds to the *horizontal line* $y = 4$ intersecting the graph of the parabola $y = x^2$. The fact that the horizontal line $y = 4$ intersects the graph of $y = x^2$ twice further confirms that two *different* inputs, namely $x = -2$ and $x = 2$, are paired with the *same* output, 4, which is the cause of all our trouble in attempting to find an inverse function to $f(x) = x^2$.

In general, in order for a function to be invertible, the function must have the property that any two inputs for x can never be paired with the same output, or else we will run into the same problem as with $f(x) = x^2$. We give this property a name.

> A function f is said to be *one-to-one* if f matches different inputs to different outputs. Equivalently, f is one-to-one if and only if whenever $f(c) = f(d)$, then $c = d$.

Graphically, we can identify one-to-one functions using the following test.

> **The Horizontal Line Test (HLT):**
> A function f is one-to-one if and only if no horizontal line intersects the graph of f more than once.

We say that the graph of a function *passes* the Horizontal Line Test if no horizontal line intersects the graph more than once; otherwise, we say the graph of the function *fails* the Horizontal Line Test.

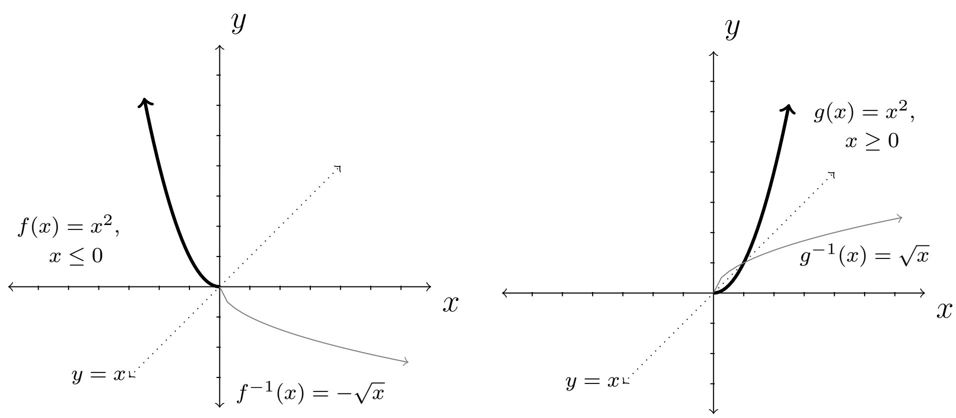
Lastly, we have argued that if f is invertible, then f must be one-to-one, since otherwise the reflection of the graph of $y = f(x)$ about the line $y = x$ will fail the Vertical Line Test. It turns out that being one-to-one is also enough to guarantee invertibility of a function f. To see this, we can think of f as the set of ordered pairs which constitute its graph. If switching the x- and y-coordinates of the points results in a function (i.e., passes the VLT), then f is invertible and we have found the graph of its inverse, f^{-1}. This is precisely what the Horizontal Line Test does for us: it checks to see whether or not a set of points describes x as a function of y.

We can now summarize our results.

Equivalent Conditions for Invertibility:
Suppose f is a function. The following statements are equivalent.

- f is invertible (f^{-1} exists).
- f is one-to-one.
- The graph of f passes the Horizontal Line Test.

In the case of $f(x) = x^2$, since the corresponding parabola fails the Horizontal Line Test, f is not invertible. If we were to restrict the domain of our function to either the left half ($x \leq 0$) or right half ($x \geq 0$) of the parabola, however, we could produce a function that passes the HLT and consequently has an inverse, as seen in the following two graphs.



In the next subsection, we will outline the process of determining whether or not a function is invertible, and if so, find its inverse function algebraically.

Finding Inverses Algebraically (L35)

Objective: Find the inverse of a given function.

Recall that a function f is one-to-one if and only if whenever $f(c) = f(d)$, then $c = d$. Using this definition, we will now test whether a given function is one-to-one and consequently invertible.

Example 16. Determine if the function $f(x) = \dfrac{1-2x}{5}$ is one-to-one.

Notice that f is a linear function with a nonzero slope. Hence, its graph passes the Horizontal Line Test. To confirm that f is one-to-one algebraically, we begin by assuming $f(c) = f(d)$ and attempt to deduce that $c = d$.

$$\begin{aligned} f(c) &= f(d) \\ \frac{1-2c}{5} &= \frac{1-2d}{5} \\ 1-2c &= 1-2d \\ -2c &= -2d \\ c &= d \checkmark \end{aligned}$$

Hence, f is one-to-one.

Example 17. Determine if the function $g(x) = \dfrac{2x}{1-x}$ is one-to-one.

The function g is known as a rational function, and will be formally discussed in a later chapter. To determine whether or not g is one-to-one, we must use an algebraic approach. Again, we begin with the assumption that $g(c) = g(d)$.

$$\begin{aligned} g(c) &= g(d) \\ \frac{2c}{1-c} &= \frac{2d}{1-d} \\ 2c(1-d) &= 2d(1-c) \\ 2c-2cd &= 2d-2dc \\ 2c &= 2d \\ c &= d \checkmark \end{aligned}$$

Hence, g is one-to-one.

Example 18. Determine if the function $h(x) = x^2 - 2x + 4$ is one-to-one.

Notice that h is a quadratic function, whose graph is a parabola, and consequently fails the Horizontal Line Test. This means that our function should not be one-to-one. We now verify this algebraically.

Let $h(c) = h(d)$. As we work our way through the problem, we encounter a nonlinear equation, which requires us to set the right-hand side equal to zero and factor accordingly.

$$\begin{array}{rcll} h(c) &=& h(d) & \\ c^2-2c+4 &=& d^2-2d+4 & \\ c^2-2c &=& d^2-2d & \\ c^2-d^2-2c+2d &=& 0 & \text{Factor by grouping} \\ (c+d)(c-d)-2(c-d) &=& 0 & \text{Difference of squares} \\ (c-d)((c+d)-2) &=& 0 & \\ c-d=0 &\text{or}& c+d-2=0 & \\ c=d &\text{or}& c=2-d & \end{array}$$

We get $c = d$ as one possibility, but we also get the possibility that $c = 2 - d$. This suggests that h will likely not be one-to-one.

Letting $d = 0$, we get $c = 0$ or $c = 2$. This implies that, $h(0) = 4$ and $h(2) = 4$, and we have produced two different inputs with the same output. Hence, h is not one-to-one, as anticipated.

Once we have established whether a function f is one-to-one, and consequently invertible, our next task is to identify f^{-1} precisely. In the previous part of this section, we noticed that switching each point, (x, y), of the graph of f produced a point (y, x) on the graph of f^{-1}. This is our motivation in the steps for finding an inverse algebraically, as we will be switching the x and y coordinates to do so.

Steps for finding the Inverse of a Function

1. Rewrite $f(x)$ as y.
2. Switch x and y.
3. Solve for y.
4. Rewrite y as $f^{-1}(x)$.

In the next few examples, we find the inverse of each function f, as well as confirm that the domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}. We also check each answer using function composition. We leave it as an exercise to the reader to graph each function (using a graphing utility where necessary), and verify that the two functions are reflections of each other about the line $y = x$.

Example 19. Find the inverse f^{-1} of the function $f(x) = \dfrac{1-2x}{5}$. Verify using compositions that f and f^{-1} are inverses, and that the domain and range of f equal the range and domain of f^{-1}, respectively.

We write $y = f(x)$ and proceed to switch x and y

$$\begin{aligned}
y &= f(x) \\
y &= \frac{1-2x}{5} \\
x &= \frac{1-2y}{5} && \text{Switch } x \text{ and } y \\
5x &= 1-2y && \text{Solve for } y \\
5x-1 &= -2y \\
\frac{5x-1}{-2} &= y \\
y &= -\frac{5}{2}x+\frac{1}{2}
\end{aligned}$$

We have $f^{-1}(x) = -\frac{5}{2}x + \frac{1}{2}$.

To verify this answer, we first check that $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f, which is all real numbers.

$$\begin{aligned}
\left(f^{-1} \circ f\right)(x) &= f^{-1}(f(x)) \\
&= -\frac{5}{2} f(x) + \frac{1}{2} \\
&= -\frac{5}{2}\left(\frac{1-2x}{5}\right) + \frac{1}{2} \\
&= -\frac{1}{2}(1-2x) + \frac{1}{2} \\
&= -\frac{1}{2} + x + \frac{1}{2} \\
&= x \checkmark
\end{aligned}$$

We now check that $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1} which is also all real numbers.

$$\begin{aligned}
\left(f \circ f^{-1}\right)(x) &= f(f^{-1}(x)) \\
&= \frac{1 - 2f^{-1}(x)}{5} \\
&= \frac{1 - 2\left(-\frac{5}{2}x + \frac{1}{2}\right)}{5} \\
&= \frac{1 + 5x - 1}{5} \\
&= \frac{5x}{5} \\
&= x \checkmark
\end{aligned}$$

Since both f and f^{-1} are linear functions with nonzero slopes, their domain and range is all real numbers, $(-\infty, \infty)$.

Example 20. Find the inverse g^{-1} of the function $g(x) = \dfrac{2x}{1-x}$. Verify using compositions that g and g^{-1} are inverses, and that the domain and range of g equal the range and domain of g^{-1}, respectively.

Notice that the domain of g is $(-\infty, 1) \cup (1, \infty)$. One can verify graphically, that the range of g is $(-\infty, -2) \cup (-2, \infty)$.

To find $g^{-1}(x)$, we start with $y = g(x)$.

$$\begin{aligned}
y &= g(x) \\
y &= \frac{2x}{1-x} \\
x &= \frac{2y}{1-y} && \text{Switch } x \text{ and } y \\
x(1-y) &= 2y && \text{Solve for } y\text{; clear denominator} \\
x - xy &= 2y && \text{Distribute } x \\
x &= xy + 2y && \text{Move } y \text{ terms to one side} \\
x &= y(x+2) && \text{Factor out } y \\
y &= \frac{x}{x+2} && \text{Divide by } x+2
\end{aligned}$$

We have $g^{-1}(x) = \dfrac{x}{x+2}$.

Notice that the domain of g^{-1} matches the range of g from earlier, $(-\infty, -2) \cup (-2, \infty)$. Again, we can use the graph of g^{-1} to verify that the range of g^{-1} also matches the domain of g, $(-\infty, 1) \cup (1, \infty)$.

To check that our inverse is correct, we first check that $\left(g^{-1} \circ g\right)(x) = x$.

$$\begin{aligned}
\left(g^{-1} \circ g\right)(x) &= g^{-1}(g(x)) \\
&= g^{-1}\left(\frac{2x}{1-x}\right)
\end{aligned}$$

$$\begin{aligned}
\left(g^{-1} \circ g\right)(x) &= \frac{\left(\dfrac{2x}{1-x}\right)}{\left(\dfrac{2x}{1-x}\right)+2} \\
&= \frac{\left(\dfrac{2x}{1-x}\right)}{\left(\dfrac{2x}{1-x}\right)+2} \cdot \frac{(1-x)}{(1-x)} \quad \text{Clear denominators} \\
&= \frac{2x}{2x+2(1-x)} \\
&= \frac{2x}{2x+2-2x} \\
&= \frac{2x}{2} \\
&= x \ \checkmark
\end{aligned}$$

Lastly, we check that $\left(g \circ g^{-1}\right)(x) = x$.

$$\begin{aligned}
\left(g \circ g^{-1}\right)(x) &= g\left(g^{-1}(x)\right) \\
&= g\left(\frac{x}{x+2}\right) \\
&= \frac{2\left(\dfrac{x}{x+2}\right)}{1-\left(\dfrac{x}{x+2}\right)} \\
&= \frac{2\left(\dfrac{x}{x+2}\right)}{1-\left(\dfrac{x}{x+2}\right)} \cdot \frac{(x+2)}{(x+2)} \quad \text{Clear denominators}
\end{aligned}$$

$$\begin{aligned}
&= \frac{2x}{(x+2)-x} \\
&= \frac{2x}{2} \\
&= x \checkmark
\end{aligned}$$

For our last two examples, we revisit the inverse relationship between quadratics and functions containing square roots.

Example 21. Find the inverse h^{-1} of the function $h(x) = 3\sqrt{x}+4$. Compare the domain and range of h with that of h^{-1}. Verify using compositions that h and h^{-1} are inverses.

Notice that the domain of h is $x \geq 0$, or $[0, \infty)$, and the range is $y \geq 4$, or $[4, \infty)$.

To find $h^{-1}(x)$, we start with $y = h(x)$.

$$\begin{aligned}
y &= h(x) \\
y &= 3\sqrt{x} + 4
\end{aligned}$$

$$\begin{aligned}
x &= 3\sqrt{y} + 4 && \text{Switch } x \text{ and } y \\
x - 4 &= 3\sqrt{y} && \text{Solve for } y \\
\frac{x-4}{3} &= \sqrt{y} \\
y &= \left(\frac{x-4}{3}\right)^2 && \text{Square both sides}
\end{aligned}$$

We have

$$h^{-1}(x) = \left(\frac{x-4}{3}\right)^2 = \frac{1}{9}(x-4)^2,$$

whose graph is a parabola, opening upwards with vertex $(4, 0)$.

Consequently, the range of h^{-1} is $y \geq 0$, or $[0, \infty)$, which coincides with the domain of h. In order for our functions to truly be inverses of one another, however, we must impose a restriction on the domain of h^{-1}, which would otherwise be all real numbers. Instead, we only take the right half of the graph of our parabola, which coincides with a domain of h^{-1} of $x \geq 4$, or $[4, \infty)$. This restriction guarantees that the domain of h^{-1} matches the range of h, and that the graph of h^{-1} passes the Horizontal Line Test, which is a requirement of invertibility. We leave it as an exercise to the reader to show that $(h \circ h^{-1})(x) = x$ and $(h^{-1} \circ h)(x) = x$.

For our last example, we begin with a quadratic function, whose domain has already been restricted, in order to guarantee the existence of an inverse.

Example 22. Find the inverse f^{-1} of the function $f(x) = -2x^2 - 20x - 30$, where $x \geq -5$. Verify using compositions that f and f^{-1} are inverses, and that the domain and range of f equal the range and domain of f^{-1}, respectively.

To find $f^{-1}(x)$, we start with $y = f(x)$.

$$\begin{aligned} y &= f(x) \\ y &= -2x^2 - 20x - 30 \\ x &= -2y^2 - 20y - 30 \quad \text{Switch } x \text{ and } y \end{aligned}$$

Any further attempt to solve for y, however, will lead us to a dead end. This is due in large part to the fact that we cannot combine the terms $-2y^2$ and $-20y$. Instead, we first convert the quadratic $f(x)$ to its vertex form.

$$h = \frac{-b}{2a} = \frac{-(-20)}{2(-2)} = \frac{20}{-4} = -5$$

$$k = f(h) = -2(-5)^2 - 20(-5) - 30 = -50 + 100 - 30 = 20$$

$$\text{Vertex Form: } f(x) = -2(x+5)^2 + 20, \text{ where } x \geq -5$$

We can now use our vertex form to find f^{-1}, as follows.

$$\begin{aligned} y &= f(x) \\ y &= -2(x+5)^2 + 20 \\ x &= -2(y+5)^2 + 20 && \text{Switch } x \text{ and } y \\ x - 20 &= -2(y+5)^2 && \text{Solve for } y \\ \frac{x-20}{-2} &= (y+5)^2 \\ \sqrt{\frac{x-20}{-2}} &= y + 5 && \text{Square root both sides} \\ \sqrt{\frac{x-20}{-2}} - 5 &= y \end{aligned}$$

So,

$$f^{-1}(x) = \sqrt{\frac{x-20}{-2}} - 5 = \sqrt{\frac{20-x}{2}} - 5.$$

Using our standard form for f, we see that the graph of f is the right half of a parabola (since we were given that $x \geq -5$), opening downward with vertex $(-5, 20)$. Thus we can conclude that the range of f is $y \leq 20$. Similarly, if we consider our answer for f^{-1}, we see that our inverse function has a domain of $20 - x \geq 0$, or $x \leq 20$, which agrees with the range of f. Furthermore, since a square root must always be nonnegative, we can conclude that the range of f^{-1} is $y \geq -5$, which agrees with the given domain restriction ($x \geq -5$) of f.

It is important to mention that in our steps for finding f^{-1}, we were required to introduce a square root into the equation. Although this would usually require us to include a $\pm$, our final answer only shows a positive square root. This is not by accident, but is in fact necessary, since including a $\pm$ will produce an expression whose graph fails the Vertical Line Test, and can therefore not be the correct inverse function of f. Furthermore, because we are given that the domain of f is $x \geq -5$, a decision must be made to only include a positive square root for f^{-1}, and disregard the case of a negative square root. If we were instead initially given that $x \leq -5$ for our quadratic f, our answer for f^{-1} would in fact require a negative square root. Interpreted graphically, such a change would correspond to the graph of f as the left half of our parabola ($x \leq -5$), instead of the right half ($x \geq -5$).

To conclude this section, we will check that $(f^{-1} \circ f)(x) = x$. We leave it as an exercise to the reader to confirm that $(f \circ f^{-1})(x) = x$. As when we found f^{-1}, in each case, it will again be beneficial to use the vertex form for f, rather than the standard form.

$$\begin{aligned}
\left(f^{-1} \circ f\right)(x) &= f^{-1}\left(f(x)\right) \\
&= f^{-1}\left(-2(x+5)^2+20\right) \\
&= \sqrt{\frac{20-(-2(x+5)^2+20)}{2}} - 5
\end{aligned}$$

$$\begin{aligned}
\left(f^{-1} \circ f\right)(x) &= \sqrt{\frac{20+2(x+5)^2-20}{2}} - 5 \\
&= \sqrt{\frac{2(x+5)^2}{2}} - 5 \\
&= \sqrt{(x+5)^2} - 5 \\
&= (x+5) - 5 \\
&= x \checkmark
\end{aligned}$$

Transformations

Introduction

In this section, we will continue to become more comfortable with general function notation and use it to establish a "database" of actions that may be applied to a particular function, each of which resulting in a predictable transformation of the graph of the original function. The three fundamental transformations which we will discuss are:

- translations, or "shifts"
- reflections
- scalings, or "stretches" and "shrinks"

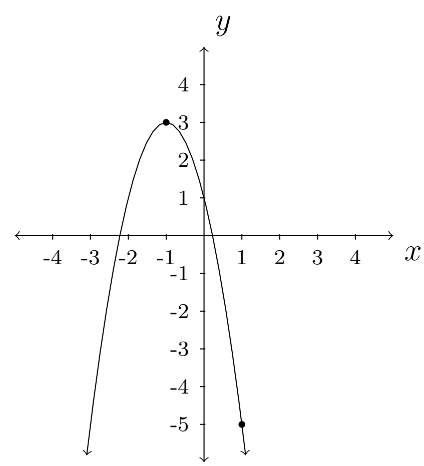
Each of these transformations will not only be identified by their name, but also by whether they have an effect on the original graph vertically or horizontally.

Eventually, once we have described, in detail, each action and its respective transformation, we will be able to consider transformations resulting from two or more actions on the given function. In fact, our first example, taken directly from the familiar chapter on quadratics (see page **??**) will demonstrate such a transformation.

Example 23. Sketch a complete graph of $g(x) = -2(x+1)^2 + 3$.

Recall that the vertex of the graph of g is at $(h, k) = (-1, 3)$. The negative leading coefficient ($a = -2$) reminds us that the graph opens (or points) downward. Although it is not necessary to find the intercepts in order to determine the general shape of the graph, we will include a table of points for the graph of g, which include both the $x-$ and $y-$intercepts, as well as a reference point at $(1, -5)$. We leave it as an exercise to the reader to verify that the values from the table are accurate. For the purposes of this example, we will only identify the vertex and reference point directly on our graph.

x	$g(x)$
-1	3
0	-1
1	-5
$-1+\sqrt{6}/2$	0
$-1-\sqrt{6}/2$	0



Although it should be relatively straightforward to deduce the graph of g using the methods from the previous chapter, if we were to instead consider the fundamental quadratic function $f(x) = x^2$ and compare it to g, we would actually notice *four* contributing factors which act on f and transform its graph to the graph of g shown above. We will identify each factor below, using a numbered list to help keep track of the changes. Later, we will see how rearranging the order of each of our actions can produce a different transformation of the original graph.

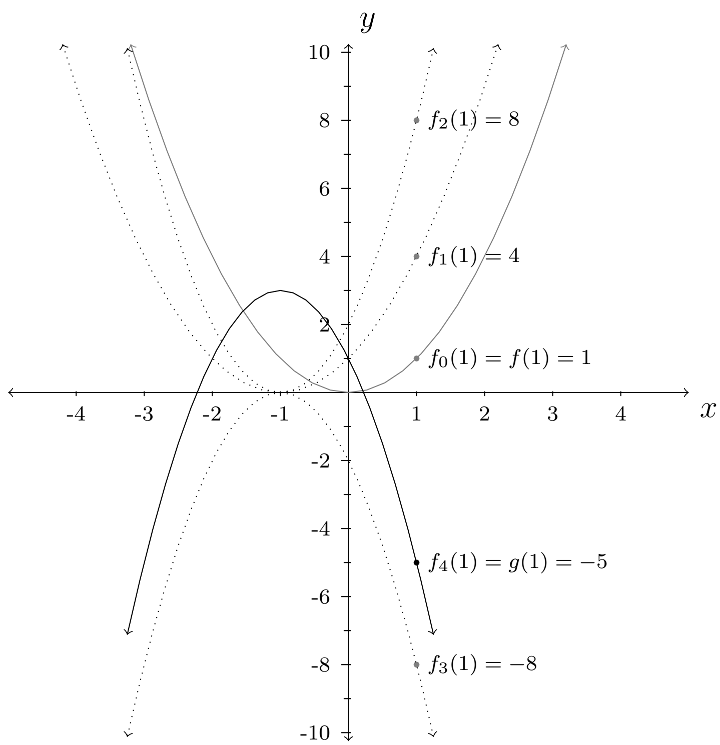
Original Function: $f(x) = x^2$ New Function: $g(x) = -2(x+1)^2 + 3$

Contributing factors, taken in order:

1. $+1$ inside parentheses results in a horizontal shift 1 unit left.
2. multiplier of 2 outside parentheses results in a vertical stretch by a factor of 2.
3. negative multiplier $(-)$ results in a reflection about the $x-$axis.
4. $+3$ outside of parentheses results in a vertical shift 3 units up.

By considering each individual action separately, we can actually determine a sequence of functions and corresponding graphical transformations that, taken as a whole, result in the graph of g. This sequence is detailed below, along with each of their graphs, which include a reference point when $x = 1$.

Number	Function			Resulting Action
0.	$f_0(x) = f(x)$	$=$	x^2	Original Function
1.	$f_1(x)$	$=$	$(x+1)^2$	Horizontal Shift
2.	$f_2(x)$	$=$	$2(x+1)^2$	Vertical Stretch
3.	$f_3(x)$	$=$	$-2(x+1)^2$	Reflection about $x-$axis
4.	$f_4(x) = g(x)$	$=$	$-2(x+1)^2 + 3$	Vertical Shift



Now that we have seen the result of a combination of multiple actions on a familiar function $f(x) = x^2$, we turn our attention to understanding the effects of each single action on the graph of an arbitrary function.

Translations (L36)

Objective: Graph or identify a function that is represented by either a vertical or horizontal translation of a known function.

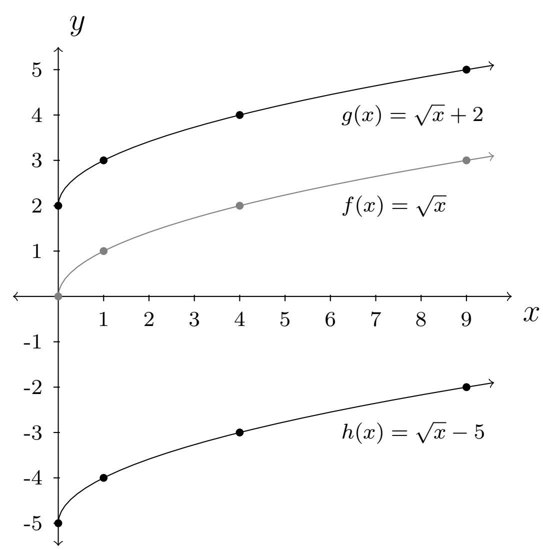
First, we consider the action of adding (or subtracting) a number to a function. In our earlier example, $f(x) = -2(x+1)^2 + 3$, this was done in two different places: once inside of the square (+1) and once outside of it (+3). And, as we have already seen, adding 1 inside of the square resulted in a horizontal shift of the graph, while adding 3 outside of the square resulted in a vertical shift. Such transformations are also referred to as *translations*. Our two cases demonstrate the possible effects that adding a number to a function can have on its graph. What remains to be seen is how the sign of the number can effect the graph.

We will begin by exploring vertical shifts of the function $f(x) = \sqrt{x}$.

Example 24. Graph the functions $g(x) = \sqrt{x} + 2$ and $h(x) = \sqrt{x} - 5$ and describe them as transformations of the graph of $f(x) = \sqrt{x}$.

First, observe that we can rewrite each function in terms of f as $g(x) = f(x) + 2$ and $h(x) = f(x) - 5$. It is also worth noting that the domain of all three functions is $[0, \infty)$. Next, we make a table of values to help sketch the graph of each function on the same set of axes.

x	$f(x)$	$g(x)$	$h(x)$
0	0	2	-5
1	1	3	-4
4	2	4	-3
9	3	5	-2



From our graph, we see that the graph of g represents a vertical shift of the graph of f up 2 units, while the graph of h represents a vertical shift of the graph of f down 5 units.

Our results are generalized as follows.

Vertical Shifts

Let f be a function and k a real number. Consider the function

$$g(x) = f(x) + k.$$

- If $k > 0$, then the graph of g represents a *vertical shift*, or translation, of the graph of f *up* k units.
- If $k < 0$, then the graph of g represents a *vertical shift*, or translation, of the graph of f *down* k units.

Next, we will explore horizontal shifts of the function $f(x) = \dfrac{1}{x}$.

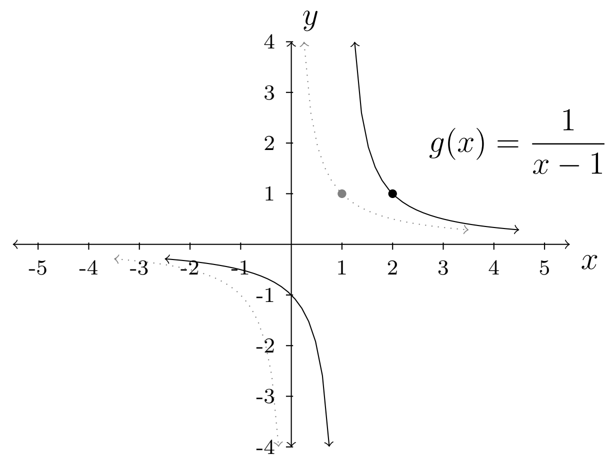
Example 25. Graph the functions $g(x) = \dfrac{1}{x-1}$ and $h(x) = \dfrac{1}{x+2}$ and describe them as transformations of the graph of $f(x) = \dfrac{1}{x}$.

First, observe that we can rewrite each function in terms of f as $g(x) = f(x-1)$ and $h(x) = f(x+2)$. Also notice that the domains of each function exclude a different value for x.

Function	Domain
$f(x) = \dfrac{1}{x}$	$x \neq 0$
$g(x) = \dfrac{1}{x-1}$	$x \neq 1$
$h(x) = \dfrac{1}{x+2}$	$x \neq -2$

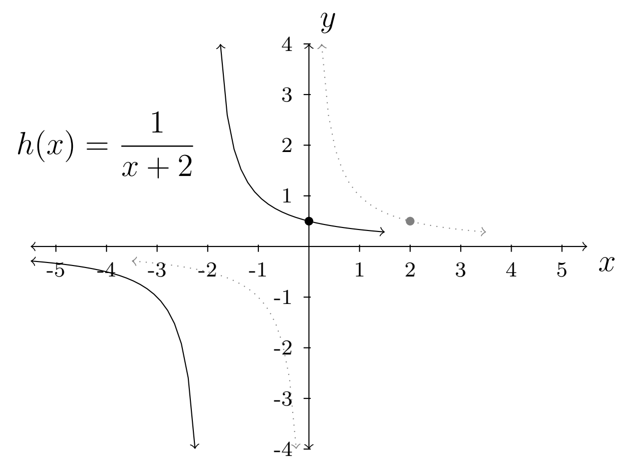
Again, we construct a table of values to help graph each function. In this example, it will be easier to compare each graph to our original graph one at a time. In each figure, the graph of f appears using a dotted curve. A set of two reference points, having the same $y-$coordinate has also been included in each graph.

x	$f(x)$	$g(x)$
-3	$-\frac{1}{3}$	$-\frac{1}{4}$
-2	$-\frac{1}{2}$	$-\frac{1}{3}$
-1	-1	$-\frac{1}{2}$
0	DNE	-1
1	1	DNE
2	$\frac{1}{2}$	1
3	$\frac{1}{3}$	$\frac{1}{2}$



From our graph, we see that the graph of g represents a horizontal shift of the graph of f to the right 1 unit.

x	$f(x)$	$h(x)$
-3	$-\frac{1}{3}$	-1
-2	$-\frac{1}{2}$	DNE
-1	-1	1
0	DNE	$\frac{1}{2}$
1	1	$\frac{1}{3}$
2	$\frac{1}{2}$	$\frac{1}{4}$
3	$\frac{1}{3}$	$\frac{1}{5}$



Similarly, we see that the graph of h represents a horizontal shift of the graph of f to the left 2 units.

It is worth noting that *adding* 2 from x in the case of h above resulted in a horizontal shift of the graph of f to the *left*. Often, this goes against what we would typically expect from adding a positive constant to x, since the left-half of the $x-$axis is considered the negative half (when $x < 0$).

Instead, if we consider plugging values into the expression $x + 2$, then evaluating $h(x) = f(x + 2)$ at $x = -2$ (two units to the *left* of zero) will yield the same $y-$coordinate as evaluating $f(x)$ at $x = 0$. We have also seen this notion at work when identifying the vertex of a parabola using the standard form of a quadratic function. For example, the graph of $h(x) = (x + 2)^2$ has a vertex at $(-2, 0)$, which is two units to the left of the vertex $(0, 0)$ associated with the graph of $f(x) = x^2$.

A similar observation is worth mentioning when we consider the resulting graph from *subtracting* a positive constant from x, as in the case of g above. In this case, the resulting transformation is a horizontal shift to the *right*.

Again, we can generalize our findings.

Horizontal Shifts
Let f be a function and h a real number. Consider the function

$$g(x) = f(x - h).$$

- If $h > 0$, then the graph of g represents a *horizontal shift*, or translation, of the graph of f *right* h units.
- If $h < 0$, then the graph of g represents a *horizontal shift*, or translation, of the graph of f *left* h units.

Before moving on to our next type of transformation, it is important to point out the nature of the associated transformation of the graph of a function $f(x)$, when adding (or subtracting) a constant either inside or outside of the function. Specifically, a change to the original function occurring outside, such as $f(x)+4$, results in a *vertical* change of the graph of the original function, whereas a change occurring inside, such as $f(x + 4)$, results in a *horizontal* change of the original graph. This will be a recurring theme, as we explore each of our remaining transformation types, and will be helpful as we encounter more advanced functions.

Reflections (L37)

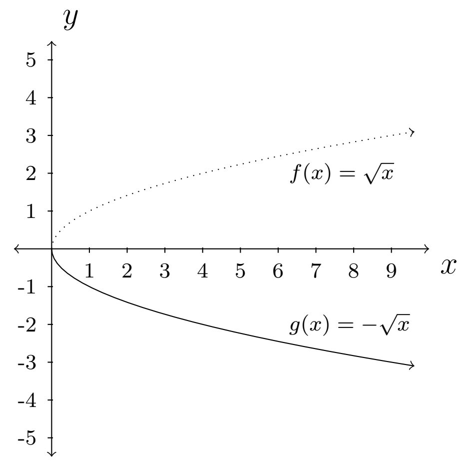
Objective: Graph or identify a function that is represented by either a vertical or horizontal reflection of a known function about the $y-$axis or $x-$axis, respectively.

Next, we consider the action of multiplication by -1. Given a function $f(x)$, we will consider the functions $-f(x)$ and $f(-x)$, whose graphs will represent reflections of the graph of f about either the $x-$axis or $y-$axis. Following along the same theme as in the previous subsection, one can initially guess that the graph of $-f(x)$ will represent a vertical reflection of the graph of f about the $x-$axis, since the negative sign occurs outside of the original function. This guess should also make sense to us, since multiplication of $f(x)$ by a negative sign would change the $y-$coordinate of any point $(x, y) = (x, f(x))$ on the graph of f, from either positive to negative or negative to positive.

Example 26. Graph the function $g(x) = -\sqrt{x}$ and describe it as a transformation of the graph of $f(x) = \sqrt{x}$.

Notice that since $g(x) = -f(x)$, the domain of g will be the same as that of f, $[0, \infty)$, whereas the range of g will be $(-\infty, 0]$. Next, we make a table of values to help sketch the graph of each function on the same set of axes.

x	$f(x)$	$g(x)$
0	0	0
1	1	-1
4	2	-2
9	3	-3

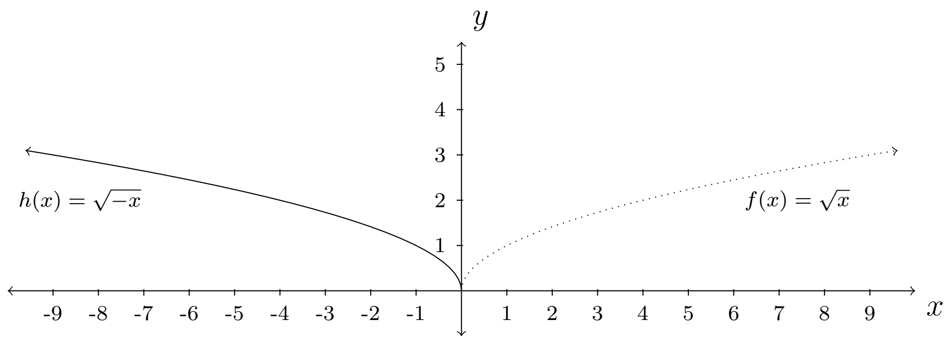


From our graph, we easily see that the graph of g represents a vertical reflection of the graph of f about the $x-$axis, as expected.

Example 27. Graph the function $h(x) = \sqrt{-x}$ and describe it as a transformation of the graph of $f(x) = \sqrt{x}$.

Notice that since $h(x) = f(-x)$, the range of h will be the same as that of f, $[0, \infty)$, whereas the domain of h will be $(-\infty, 0]$. Next, we make a table of values to help sketch the graph of each function on the same set of axes.

x	$f(x)$	$h(x)$
-9	DNE	-3
-4	DNE	-2
-1	DNE	-1
0	0	0
1	1	DNE
4	2	DNE
9	3	DNE



From our graph, we easily see that the graph of h represents a horizontal reflection of the graph of f about the $y-$axis, as expected.

Our two examples above are generalized as follows.

Reflections
Let f be a function.

- The graph of $g(x) = -f(x)$ represents a *reflection*, of the graph of f *about the x-axis* (a vertical change).
- The graph of $g(x) = f(-x)$ represents a *reflection*, of the graph of f *about the y-axis* (a horizontal change).

Scalings (L38)

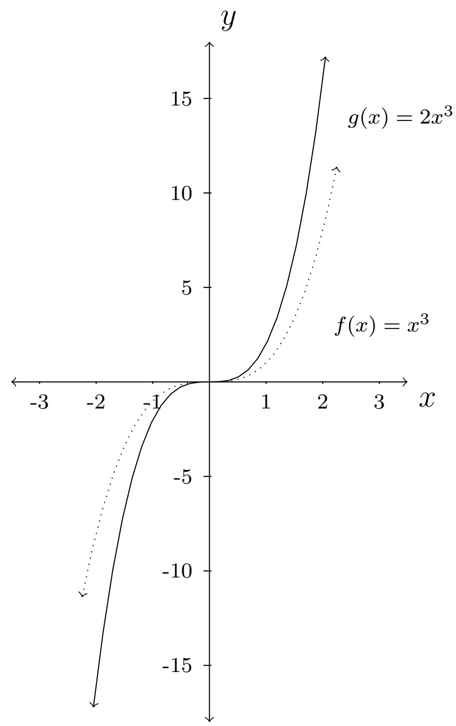
Objective: Graph or identify a function that is represented by either a vertical or horizontal scaling of a known function.

In this last portion of the transformations section, we focus our attention on scalings of the graph of a function f, also known as "stretches" or "shrinks". Again, we will look to classify both vertical and horizontal scalings, and will treat each case separately, beginning with vertical scalings.

Example 28. Graph the function $g(x) = 2x^3$ and describe it as a transformation of the graph of $f(x) = x^3$.

As we have seen in both of the previous subsections, we can anticipate a vertical effect on the graph of f, from the multiplication by 2 *outside*, or after, the cubing of x. In this case, we have that $g(x) = 2f(x)$, which will result in a doubling of every $y-$coordinate from our original graph. Our table and graphs below confirm this.

x	$f(x)$	$g(x)$
-3	-27	-54
-2	-8	-16
-1	-1	-2
0	0	0
1	1	2
2	8	16
3	27	54



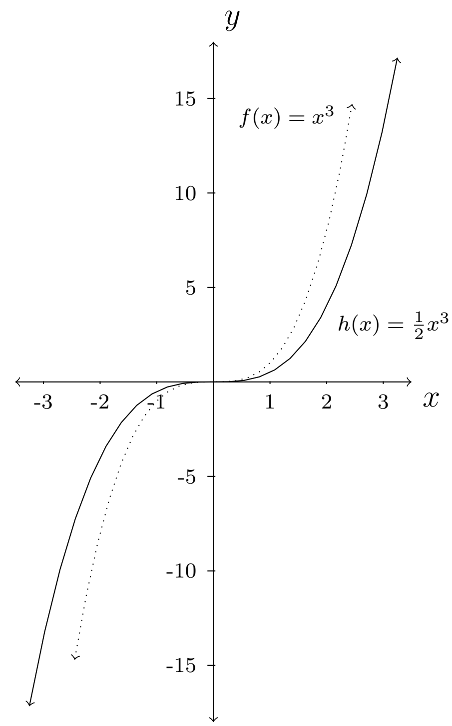
It is worth mentioning here that for the purposes of easily displaying our graphs, we have taken the liberty of adjusting the $y-$axis to easily fit the page. This adjustment should not affect our ability to identify the resulting transformation of the graph of f in any way.

As anticipated, our graph of g represents a vertical stretch of the graph of f. In this case, for a given value x, since every $y-$coordinate for the graph of g equals *twice* the value of the corresponding $y-$coordinate on the graph of f, we say that the resulting transformation is a *vertical stretch* of the graph of f *by a factor of* 2.

Example 29. Graph the function $h(x) = \frac{1}{2}x^3$ and describe it as a transformation of the graph of $f(x) = x^3$.

Again, we can anticipate a vertical effect on the graph of f, from the multiplication by $\frac{1}{2}$ *outside*, or after, the cubing of x. In this case, we have that $h(x) = \frac{1}{2}f(x)$, which will result in a halving of every $y-$coordinate from our original graph. Both the table and graph that follow confirm this.

x	$f(x)$	$h(x)$
-3	-27	$-\frac{27}{2}$
-2	-8	-4
-1	-1	$-\frac{1}{2}$
0	0	0
1	1	$\frac{1}{2}$
2	8	4
3	27	$\frac{27}{2}$



As with the previous example, for the purposes of easily displaying our graphs, we have taken the liberty of adjusting the $y-$axis to easily fit the page.

Now, our graph of h represents a vertical shrink of the graph of f. In this case, for a given value x, since every $y-$coordinate for the graph of h equals *half* the value of the corresponding $y-$coordinate on the graph of f, we say that the resulting transformation is a *vertical shrink* of the graph of f *by a factor of* 2.

Notice that despite the change from the last example (stretch to shrink), we still keep the same *factor* of 2. This is because the use of the term "shrink" tells us to *divide* by 2, as opposed to multiplying when the term "stretch" is used. Instead, if we were to mistakenly claim that the transformation for h represented a vertical shrink by a factor of $\frac{1}{2}$, this would actually mean that each $y-$coordinate for the graph of f should be divided by $\frac{1}{2}$, or doubled, which does not match the correct transformation for h.

In each of the previous two examples, we witnessed a vertical stretch when $f(x)$ was multiplied by 2 and a vertical shrink when $f(x)$ was multiplied by $\frac{1}{2}$. The fundamental difference in these two cases depends on the multiplier, and whether it is greater than or less than one. We now summarize each of these cases for vertical scalings.

Vertical Scalings
Let f be a function and a a positive real number. Consider the function

$$g(x) = af(x).$$

- If $a > 1$ the graph of g represents a *vertical stretch*, or expansion, of the graph of f *by a factor of* a.
- If $a < 1$ the graph of g represents a *vertical shrink*, or compression, of the graph of f *by a factor of* $1/a$.

For our last set of examples, we will analyze horizontal scalings.

Example 30. Graph the function $g(x) = (2x)^2$ and describe it as a transformation of the graph of $f(x) = x^2$.

Since $g(x) = f(2x)$, and the action occurs inside of the square, we will anticipate a horizontal effect on the graph of f. It is worth mentioning that the domain and range of g equal the domain and range of f. Again, we make a table to assist in graphing g.

x	$f(x)$	$g(x)$
0	0	0
1	1	4
$\frac{3}{2}$	$\frac{9}{4}$	9
2	4	16
3	9	36
4	16	64

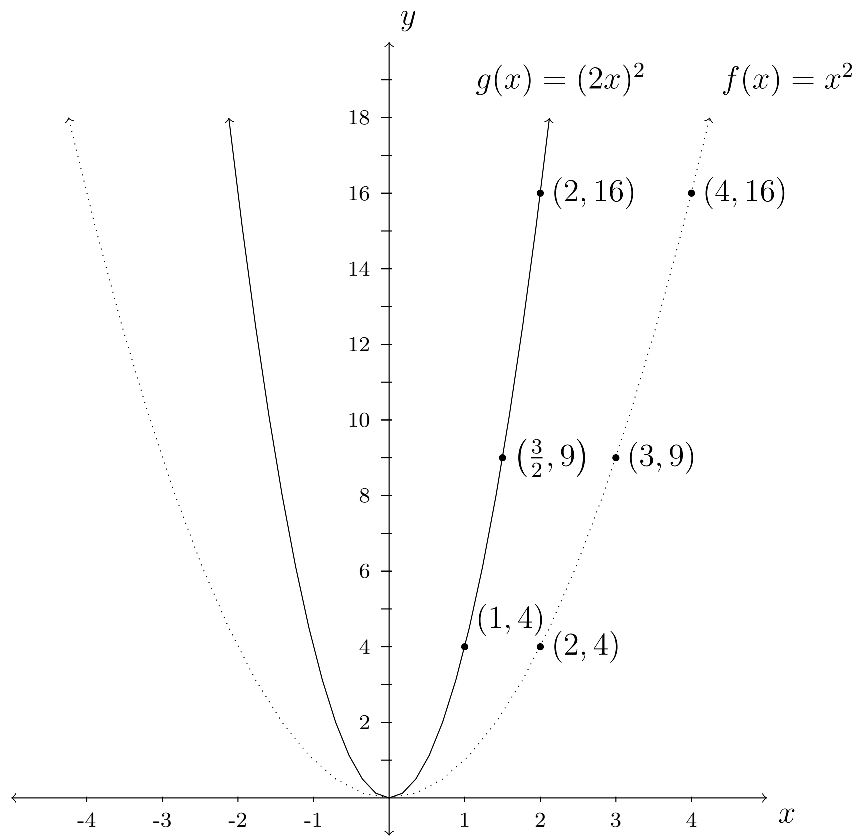
Notice that:

$g(1) = f(2) = 4$,

$g\left(\frac{3}{2}\right) = f(3) = 9$, and

$g(2) = f(4) = 16$.

In this example, we see that the points (x, y) for the graph of f will now correspond to the points $(x/2, y)$ for the graph of g. This results in a horizontal shrink (or compression) of the graph of f by a factor of 2, as shown in the graph below. The point at the origin $(0, 0)$, also the vertex, remains unchanged, since $0/2$ still equals 0. Notice that, as in the previous set of examples, we have adjusted the scaling of the $y-$axis, to easily display the graphs on the page. Again, this should not prevent us from correctly identifying the transformation.



Example 31. Graph the function $h(x) = \left(\dfrac{x}{2}\right)^2$ and describe it as a transformation of the graph of $f(x) = x^2$.

Since $h(x) = f\left(\frac{x}{2}\right)$, we will anticipate a horizontal effect on the graph of f. Again, we make a table to assist in graphing h.

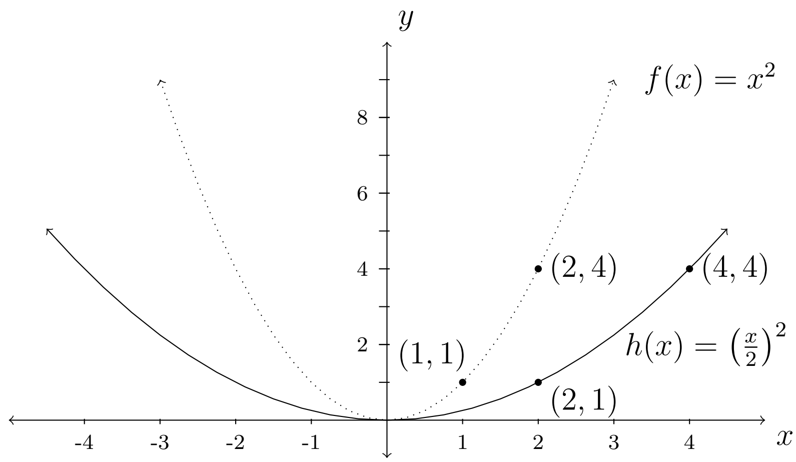
x	$f(x)$	$h(x)$
0	0	0
1	1	$\frac{1}{4}$
2	4	1
3	9	$\frac{9}{4}$
4	16	4

Notice that:

$h(2) = f(1) = 1$, and

$h(4) = f(2) = 4$.

In this example, we see that the points (x, y) for the graph of f will now correspond to the points $(2x, y)$ for the graph of h. This results in a horizontal stretch (or expansion) of the graph of f by a factor of 2, as shown in the graph below. As in the previous example, the origin $(0, 0)$ remains unchanged.



We are now ready to summarize the cases for horizontal scalings.

Horizontal Scalings

Let f be a function and b a positive real number. Consider the function

$$g(x) = f(bx).$$

- If $b > 1$ the graph of g represents a *horizontal shrink*, or compression, of the graph of f *by a factor of* b.
- If $b < 1$ the graph of g represents a *horizontal stretch*, or expansion, of the graph of f *by a factor of* $1/b$.

Transformations Summary (L39)

Objective: Graph or identify a function that is represented by a sequence of transformations of a known function.

Now that we have discussed each of the basic transformations of the graph of a function f, we are ready to consider combining two or more transformations, as demonstrated with our very first example of this section, $f(x) = -2(x+1)^2+3$. It will be critical that we keep track of the order of our actions on the function f, in order to correctly determine the resulting transformation of its graph. To assist in this, we now present the following theorem.

Transformations Summary
Let f be a function. Consider the function

$$g(x) = af(bx + h) + k,$$

where $a \neq 0$ and $b \neq 0$. Then, the graph of g may be obtained from the graph of f by following the sequence of transformations below.

1. **Horizontal Shift**:
   Shift the graph of f by h units to the the left if $h > 0$, or right if $h < 0$.
2. **Horizontal Scale/Horizontal Reflection**:
   Scale the graph from (1.) horizontally, as a shrink by a factor of $|b|$ if $|b| > 1$, or a stretch by a factor of $|1/b|$ if $0 < |b| < 1$.

   If $b < 0$, reflect the graph about the y-axis.
3. **Vertical Scale/Vertical Reflection**:
   Scale the graph from (2.) vertically, as a stretch by a factor of $|a|$ if $|a| > 1$, or a shrink by a factor of $|1/a|$ if $0 < |a| < 1$.

   If $a < 0$, reflect the graph about the x-axis.
4. **Vertical Shift**:
   Shift the graph from (3.) by k units up if $k > 0$, or down if $k < 0$.

In our first example, recall that the order of transformations was as follows.

$$f(x) = -2(x + 1)^2 + 3$$

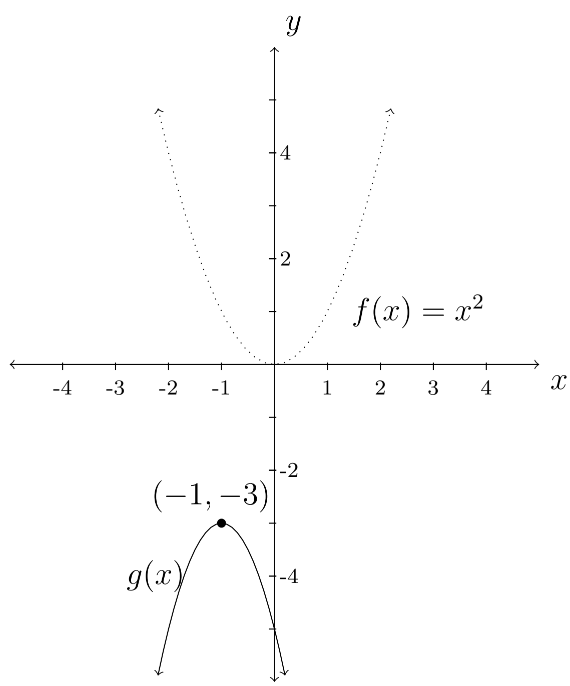
Contributing factors, taken in order:

1. $+1$ inside parentheses results in a horizontal shift 1 unit left.
2. multiplier of 2 outside parentheses results in a vertical stretch by a factor of 2.
3. negative multiplier $(-)$ results in a reflection about the $x-$axis.
4. $+3$ outside of parentheses results in a vertical shift 3 units up.

For our next example, we will rearrange the order of transformations from the previous example and utilize the graph of f to determine the graph of g. We can then work backwards, drawing upon our knowledge of quadratic equations to identify the corresponding function from our graph.

Example 32. Graph the resulting transformation of $f(x) = x^2$ from the sequence of transformations described below. Use the graph to determine the corresponding function $g(x)$. Using the Transformations Summary Theorem, compare your answer with the original sequence.

1. Vertical shift 3 units up
2. Vertical stretch by a factor of 2
3. Horizontal shift 1 unit left
4. Vertical reflection about the $x-$axis



Our resulting function is $g(x) = -2(x+1)^2 - 3$. Using the Transformations Summary Theorem, we see that our original sequence of transformations in this case also corresponds to the following sequence.

1. Horizontal shift 1 unit left
2. Vertical stretch by a factor of 2
3. Vertical reflection about the $x-$axis
4. Vertical shift 3 units down

We conclude this section with one final example.

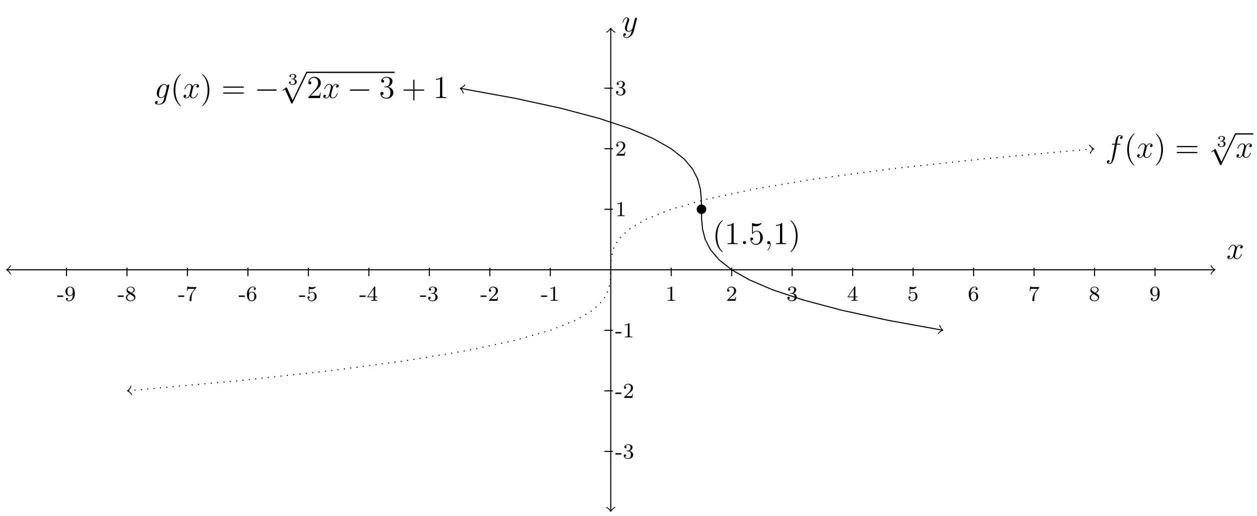
Example 33. Use the Transformations Summary Theorem to sketch a graph of the function below, as a transformation of $f(x) = \sqrt[3]{x}$.

$$g(x) = -\sqrt[3]{2x-3} + 1$$

Using the theorem, we can break down our graph as the following sequence of transformations of the graph of f. Since it will be difficult to keep track of each change in the graph, we use the point $(0,0)$ as a reference, keeping track of how it is affected by each change.

1. Horizontal shift 3 units to the right. $\Rightarrow$ $(3,0)$
2. Horizontal shrink by a factor of 2. $\Rightarrow$ $(1.5,0)$
3. Vertical reflection about the $x-$axis. $\Rightarrow$ $(1.5,0)$
4. Vertical shift 1 unit up. $\Rightarrow$ $(1.5,1)$

Our graph of g is shown below.



This last example demonstrates the significant challenge that comes from having to interpret the graph of a function that results from a sequence of two or more transformations. Although we have only kept track of how the function g affected one reference point in our example, by focusing our attention on just a few points of reference, we can come to a better understanding of the general shape of the graph of g, and how it relates to the graph of the fundamental function f.

Piecewise-Defined and Absolute Value Functions

Piecewise-Defined Functions (L40)

Objective: Define, evaluate, solve, and graph piecewise-defined functions

A *piecewise-defined* (or simply, a *piecewise*) function is a function that is defined in pieces. More precisely, a piecewise-defined function is a function that is presented using one or more expressions, each defined over non-intersecting intervals. An example of a piecewise-defined function is shown below.

$$f(x) \;=\; \begin{cases} 2x-1 & \text{if } \; x>0 \\ x^2-1 & \text{if } \; x\leq 0 \end{cases}$$

To evaluate a piecewise-defined function at a particular value of the variable, we must first compare our value to the various intervals (or domains) applied to each piece, and then substitute our value into the piece that coincides with the correct domain. For example, since $x = 1$ is greater than zero, we would use the expression $2x - 1$ to evaluate $f(1)$,

$$f(1) = 2(1) - 1 = 2 - 1 = 1.$$

Similarly, since $x = -1$ is less than zero, we would use the expression $x^2 - 1$ to evaluate $f(-1)$,

$$f(-1) = (-1)^2 - 1 = 1 - 1 = 0.$$

Below is a table of points obtained from the piecewise-defined function f above.

Example 34.

$$f(x) \;=\; \begin{cases} 2x-1 & \text{if } \; x>0 \\ x^2-1 & \text{if } \; x\leq 0 \end{cases}$$

x	$f(x)$
2	$2(2) - 1 = 3$
1	$2(1) - 1 = 1$
0	$(0)^2 - 1 = -1$
-1	$(-1)^2 - 1 = 0$
-2	$(-2)^2 - 1 = 3$

We have included an extra line between the values of $x = 0$ and $x = 1$ in the table above, in order to emphasize the changeover from one piece of our function $(2x - 1)$ to another $(x^2 - 1)$.

The value of $x = 0$ is very important, since it is an endpoint for the two domains of our function, $(0, \infty)$ and $(-\infty, 0]$.

A common misconception among students is to evaluate $f(0)$ at both $2x - 1$ and $x^2 - 1$ because it seems to "straddle" both individual domains. And although the values for both pieces are equal at $x = 0$,

$$2(0) - 1 = -1 = 0^2 - 1$$

this will often not be the case. Regardless, we must be careful to *always* associate $x = 0$ with $x^2 - 1$, since it is contained in our second piece's domain $(0 \leq 0)$ and not in our first. Our next example demonstrates what can happen with a piecewise function, if one mishandles such values of x.

Example 35.

$$g(x) = \begin{cases} 2x+1 & \text{if } x > 0 \\ x^2 - 1 & \text{if } x \leq 0 \end{cases}$$

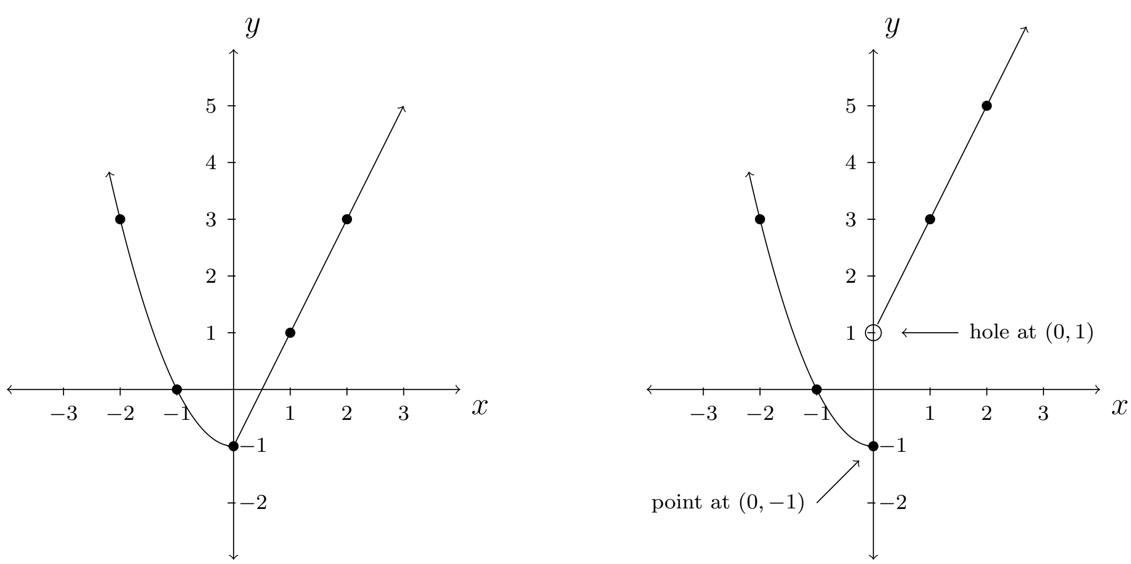
x	$g(x)$
2	$2(2)+1=5$
1	$2(1)+1=3$
0	$(0)^2-1=-1$
-1	$(-1)^2-1=0$
-2	$(-2)^2-1=3$

In this example, we see that both pieces for $g(x)$ do not "match up", since the values we obtain for both pieces at $x = 0$ do not agree:

$$g(0) = 0^2 - 1 = -1, \text{ but } 2(0) + 1 = 1.$$

Remember that when evaluating any function at a value of x in its domain, we should always only ever get a *single value* for $g(x)$, since this is how we defined a function earlier in the chapter. Furthermore, if we were to associate two values ($g(0) = \pm 1$) to $x = 0$, our graph would consequently contain points at $(0, -1)$ and $(0, 1)$, and therefore fail the Vertical Line Test.

When we consider the graphs of both f and g, since both pieces of f seem to "match up" at $x = 0$, we will see that the graph of f will be one *continuous* graph, with no breaks or separations appearing. On the other hand, since both pieces of g do not "match up" at $x = 0$, we will see that the graph of g will contain a break at $x = 0$, known as a *discontinuity* in the graph. The formal definition of a *continuous function* is one that is usually reserved for a follow-up course to Algebra (either Precalculus or Calculus). Both graphs are shown below.



Notice that in order for us to have a *complete* sketch of the graph of g, we have evaluated *both* pieces of g at $x = 0$, so that we can properly identify the *point* at the end of the quadratic piece x^2-1 and the *hole* at the end of the linear piece, $2x+1$. In general, whenever faced with the task of graphing a piecewise-defined function, one should always make sure to identify exactly where each piece of the graph starts and stops, even if a location corresponds to a hole, i.e., a coordinate pair that is *not* actually a point on the graph.

We can also observe, both from how our functions are defined (algebraically) and from their graphs that the domain of both f and g is all real numbers, or $(-\infty, \infty)$. To identify the range of each function, we can project each of our graphs onto the $y-$axis. In doing so, we obtain a range of $[-1, \infty)$ for both f and g. Notice that although both functions produce distinctly different graphs, their range is coincidentally the same, since the quadratic piece $x^2 - 1$ begins at the same minimum value ($y = -1$) for each graph.

As we have already discussed evaluating piecewise-defined functions at a value of x, we will now address the issue of solving an equation that involves a piecewise function for all possible values of x. We will do this, once again, using our functions f and g from before.

For some constant k, to find all x such that $f(x) = k$, we will use the strategy outlined below, which will be the same for any piecewise-defined function.

- Set each separate piece equal to k and solve for x.
- Compare your answers for x to the domain applied to each piece. Only keep those solutions that coincide with the specified domain.

We illustrate this approach by finding all possible zeros (or roots) of both f and g.

Example 36. Find the set of all zeros of

$$f(x) \;=\; \begin{cases} 2x-1 & \text{if } x > 0 \\ x^2-1 & \text{if } x \le 0 \end{cases}$$

.

$$\begin{aligned}
f(x) = 0 & \qquad \text{Apply to each piece separately} \\
\\
2x - 1 = 0,\ x > 0 & \qquad \text{First piece; solve for } x \\
x = \frac{1}{2},\ x > 0 & \qquad \text{One solution; coincides with domain} \\
\\
x^2 - 1 = 0\ , \quad x \le 0 & \qquad \text{Second piece; solve for } x \\
(x-1)(x+1) = 0,\ x \le 0 & \qquad \text{Solve by factoring} \\
x = \pm 1,\ x \le 0 & \qquad \text{Two potential solutions} \\
x = -1,\ x \le 0 & \qquad \text{Exclude } x = 1;\text{does not coincide with domain} \\
\\
f(x) = 0 \text{ when } x = -1,\ \frac{1}{2} & \qquad \text{Our answer}
\end{aligned}$$

Example 37. Find the set of all zeros of

$$g(x) = \begin{cases} 2x+1 & \text{if } x > 0 \\ x^2 - 1 & \text{if } x \le 0 \end{cases}$$

.

$$\begin{aligned}
g(x) &= 0 && \text{Apply to each piece separately} \\
2x + 1 = 0,\ x &> 0 && \text{First piece; solve for } x \\
x = -\frac{1}{2},\ x &> 0 && \text{Invalid solution; does not coincide with domain} \\
x^2 - 1 = 0\ ,\quad x &\le 0 && \text{Second piece; solve for } x \\
(x-1)(x+1) = 0,\ x &\le 0 && \text{Solve by factoring} \\
x = \pm 1,\ x &\le 0 && \text{Two potential solutions} \\
x = -1,\ x &\le 0 && \text{Exclude } x = 1; \text{does not coincide with domain} \\
g(x) = 0 \text{ when } x &= -1 && \text{Our answer}
\end{aligned}$$

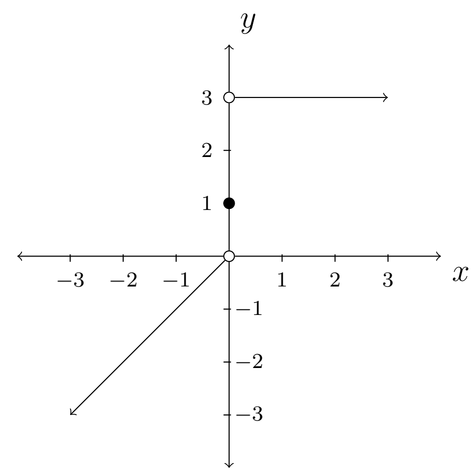
Each of the previous examples can also be confirmed by the graphs that we obtained earlier.

For our next example, we will graph a piecewise function that consists of three pieces.

Example 38.

$$h(x) = \begin{cases} 3 & \text{if } x > 0 \\ 1 & \text{if } x = 0 \\ x & \text{if } x < 0 \end{cases}$$

x	$h(x)$
2	3
1	3
0	1
-1	-1
-2	-2



The graph of h

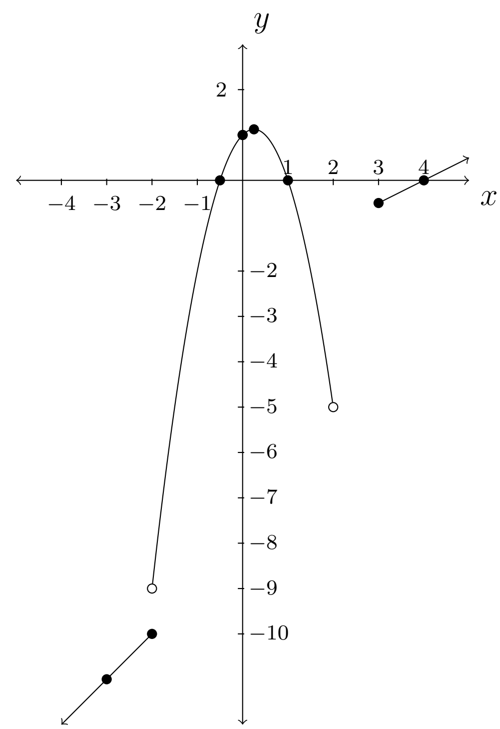
Here, we see that our graph consists of three pieces, one of which is a single point at $(0, 1)$. We can also once again determine both algebraically and graphically that our domain is $(-\infty, \infty)$. Using our graph, we obtain a range of $(-\infty, 0) \cup \{1\} \cup \{3\}$. Our complete graph also contains holes at $(0, 3)$ and $(0, 0)$.

We can easily identify all three of the coordinate pairs associated with $x = 0$ (two holes and one point) by evaluating all three pieces at $x = 0$. To reinforce this concept, we will present another example of a piecewise function that consists of three pieces.

Example 39.

$$f(x) = \begin{cases} \frac{x}{2} - 2 & \text{if } x \geq 3 \\ -2x^2 + x + 1 & \text{if } -2 < x < 2 \\ x - 8 & \text{if } x \leq -2 \end{cases}$$

x	$f(x)$
4	0
3	$-\frac{1}{2}$
1	0
$\frac{1}{4}$	$\frac{9}{8}$
0	1
$-\frac{1}{2}$	0
-1	-2
-2	-10
-3	-11



The graph of f

In the previous example, we see that there is a "gap" in our domain between the $x-$coordinates of 2 and 3. Hence, our domain is $(-\infty, 2) \cup [3, \infty)$. From our graph, we see that our range also contains a gap between the $y-$coordinates of -10 and -9. Hence, our range is $(-\infty, -10] \cup (-9, \infty)$. In our example we have have also identified several other essential coordinate pairs that should be included in our graph. We will now list each pair below, as well as the piece that is used to obtain it. We include the function f, once again, for reinforcement.

$$f(x) = \begin{cases} \frac{x}{2} - 2 & \text{if } x \geq 3 \\ -2x^2 + x + 1 & \text{if } -2 < x < 2 \\ x - 8 & \text{if } x \leq -2 \end{cases}$$

- A $y-$intercept at $(0, 1)$ from our second piece
- An $x-$intercept at $(4, 0)$ from our first piece
- Two $x-$intercepts at $(1, 0)$ and $\left(-\frac{1}{2}, 0\right)$ from our second piece
- A vertex at $\left(\frac{1}{4}, \frac{9}{8}\right)$ from our second piece
- An endpoint at $\left(3, -\frac{1}{2}\right)$ from our first piece
- An endpoint at $(-2, -10)$ from our third piece
- Two holes at $(-2, -9)$ and $(2, -5)$ from our second piece.

Lastly, we have included the point at $(-3, -11)$, to help identify the slope of the third piece of our graph.

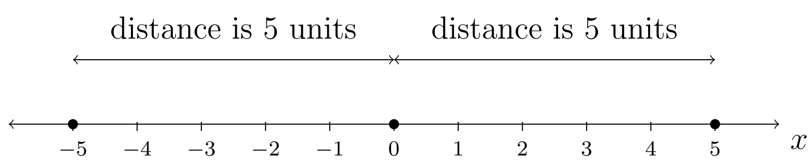
Although this example may first appear to be quite complicated, when considered on the level of each individual piece, we see that our training in the chapters leading up to this section has adequately prepared us to handle these, as well as more challenging piecewise-defined functions that we will eventually encounter.

Functions Containing an Absolute Value (L41)

Graphing Functions Containing an Absolute Value

Objective: Graph a variety of functions that contain an absolute value

There are a few ways to describe what is meant by the absolute value $|x|$ of a real number x. A common description is that $|x|$ represents the distance from the number x to 0 on the real number line. So, for example, $|5| = 5$ and $|-5| = 5$, since each is 5 units away from 0 on the real number line.



Another way to define an absolute value is by the equation $|x| = \sqrt{x^2}$. Using this definition, we have

$$|5| = \sqrt{(5)^2} = \sqrt{25} = 5 \qquad \text{and} \qquad |-5| = \sqrt{(-5)^2} = \sqrt{25} = 5.$$

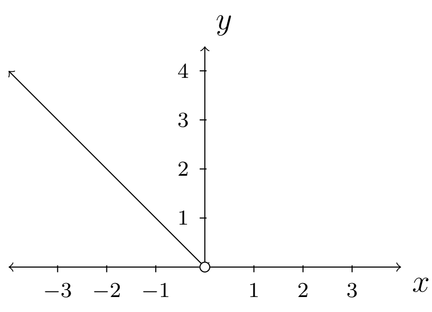
The long and short of both of these descriptions is that $|x|$ takes negative real numbers and assigns them to their positive counterparts, while it leaves positive real numbers (and zero) alone. This last description is the one we shall adopt, and is summarized in the following definition.

> The **absolute value** of a real number x, denoted $|x|$, is given by
>
> $$|x| = \begin{cases} -x, & \text{if} \quad x < 0 \\ x, & \text{if} \quad x \geq 0 \end{cases}$$

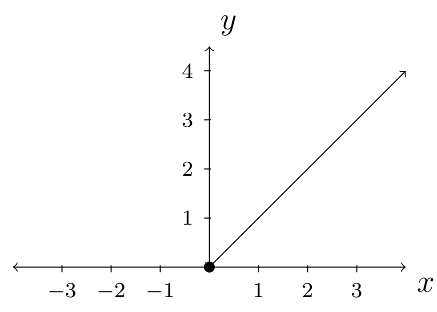
Notice that we have defined $|x|$ using a piecewise-defined function. To check that this definition agrees with what we previously understood to be the absolute value of x, observe that since $5 \geq 0$, to find $|5|$ we use the rule $|x| = x$, so $|5| = 5$. Similarly, since $-5 < 0$, we use the rule $|x| = -x$, so that $|-5| = -(-5) = 5$.

We will now graph some functions that contain an absolute value. Our strategy is to use our knowledge of the absolute value coupled with what we now know about graphing linear functions and piecewise-defined functions.

Example 40. Sketch a complete graph of $f(x) = |x|$.

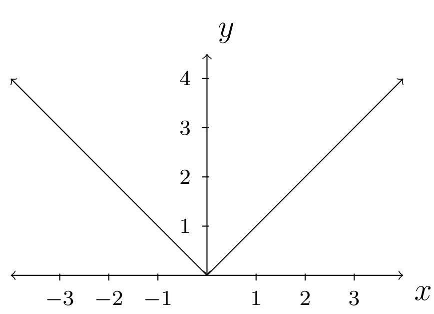


$f(x) = |x|, \; x < 0$



$f(x) = |x|, \; x \geq 0$

Notice that we have a hole at $(0, 0)$ in the graph when $x < 0$. As we have seen before, this is due to the fact that the points on $y = -x$ approach $(0, 0)$ as the x-values approach 0. Since x is required to be strictly less than zero on this interval, we include a hole at the origin. Notice, however, that when $x \geq 0$, we get to include the point at $(0, 0)$, which effectively fills in the hole from our first piece. Our final graph is shown below.

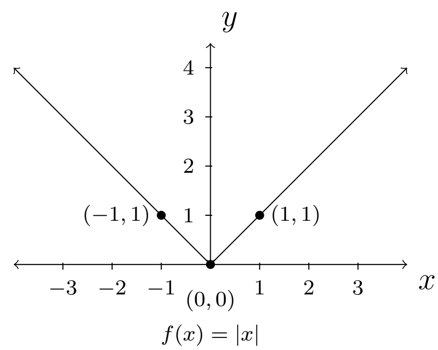


$f(x) = |x|$

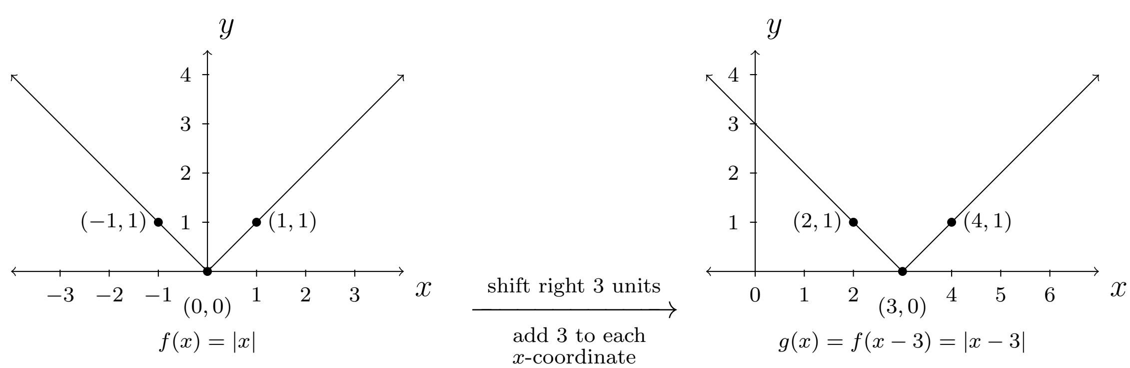
By projecting our graph onto the $x-$axis, we see that the domain of $f(x) = |x|$ is $(-\infty, \infty)$, as expected. Projecting onto the $y-$axis gives us our range of $[0, \infty)$. Our function is also increasing over the interval $[0, \infty)$ and decreasing over the interval $(-\infty, 0]$. We can also say that the graph of f has an absolute minimum at $y = 0$, since this coordinate coincides with the (absolute) lowest point on the graph, which occurs at the origin. From our graph, we can further conclude that there is no absolute maximum value of f, since the y values on the graph extend infinitely upwards.

Example 41. Use the graph of $f(x) = |x|$ to graph the function $g(x) = |x - 3|$.

We begin by graphing $f(x) = |x|$ and labeling three reference points: $(-1, 1)$, $(0, 0)$ and $(1, 1)$.

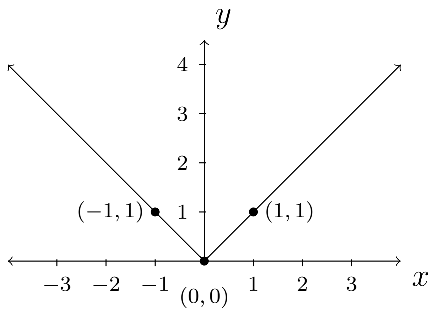


Since $g(x) = |x - 3| = f(x - 3)$, we will add 3 to each of the $x-$coordinates of the points on the graph of $y = f(x)$ to obtain the graph of $y = g(x)$. This shifts the graph of $y = f(x)$ to the *right* by 3 units and moves the points $(-1, 1)$ to $(2, 1)$, $(0, 0)$ to $(3, 0)$ and $(1, 1)$ to $(4, 1)$. Connecting these points in the classic '$\vee$' fashion produces the graph of $y = g(x)$.



Example 42. Use the graph of $f(x) = |x|$ to graph the function $h(x) = |x| - 3$.

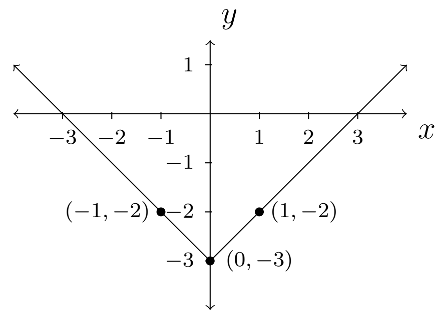
Since $h(x) = |x| - 3 = f(x) - 3$, we will subtract 3 from each of the $y-$coordinates of the points on the graph of $y = f(x)$ to obtain the graph of $y = h(x)$. This shifts the graph of $y = f(x)$ *down* by 3 units and moves the points $(-1, 1)$ to $(-1, -2)$, $(0, 0)$ to $(0, -3)$ and $(1, 1)$ to $(1, -2)$. Connecting these points with the '$\vee$' shape produces our graph of $y = h(x)$.



$f(x) = |x|$

shift down 3 units

subtract 3 from each y-coordinate



$h(x) = f(x) - 3 = |x| - 3$

Example 43. Use the graph of $f(x) = |x|$ to graph the function $k(x) = 4 - 2|3x + 1|$.

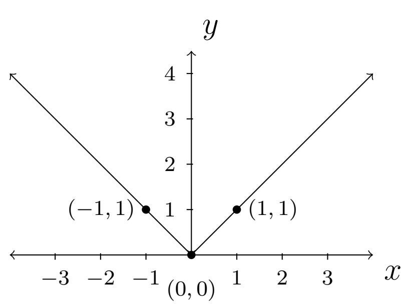
Notice that

$$\begin{aligned} k(x) &= 4 - 2|3x + 1| \\ &= 4 - 2f(3x + 1) \\ &= -2f(3x + 1) + 4. \end{aligned}$$

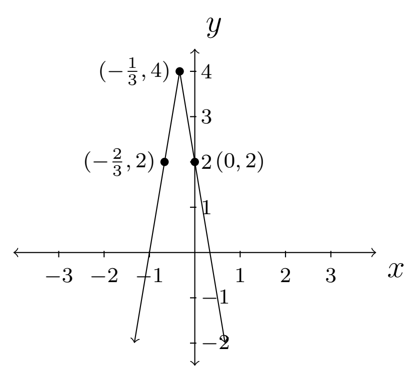
First, we will determine the corresponding transformations resulting from inside of the absolute value. Instead of $|x|$, we have $|3x + 1|$, so we must first subtract 1 from each of the $x-$coordinates of points on the graph of $y = f(x)$, then divide each of those new values by 3. This corresponds to a horizontal shift left by 1 unit followed by a horizontal shrink by a factor of 3. These transformations move the points $(-1, 1)$ to $\left(-\frac{2}{3}, 1\right)$, $(0, 0)$ to $\left(-\frac{1}{3}, 0\right)$ and $(1, 1)$ to $(0, 1)$.

Next, we will determine the corresponding transformations resulting from what appears outside of the absolute value. We must first multiply each $y-$coordinate of our new points by -2 and then *add* 4. Geometrically, this corresponds to a vertical *stretch* by a factor of 2, a reflection across the $x-$axis and finally, a vertical shift *up* by 4 units.

The resulting transformations move the points $\left(-\frac{2}{3}, 1\right)$ to $\left(-\frac{2}{3}, 2\right)$, $\left(-\frac{1}{3}, 0\right)$ to $\left(-\frac{1}{3}, 4\right)$ and $(0, 1)$ to $(0, 2)$. Connecting our final points with the usual '$\vee$' shape produces the graph of $y = k(x)$, shown below.



$f(x) = |x|$



$k(x) = -2f(3x + 1) + 4$

$= -2|3x + 1| + 4$

Absolute Value as a Piecewise Function (L42)

Objective: Interpret a function containing an absolute value as a piecewise-defined function.

By definition, we know that

$$|x| = \begin{cases} -x, & \text{if } \; x < 0 \\ x, & \text{if } \; x \geq 0 \end{cases}$$

If $m \neq 0$ and b is a real number, we may generalize the definition above as follows.

$$\begin{aligned} |mx+b| &= \begin{cases} -(mx+b), & \text{if } \; mx+b < 0 \\ mx+b\;, & \text{if } \; mx+b \geq 0 \end{cases} \\ &= \begin{cases} -mx-b, & \text{if } \; mx+b < 0 \\ mx+b, & \text{if } \; mx+b \geq 0 \end{cases} \end{aligned}$$

Notice that since we have never specified whether m is positive or negative above, it would not be wise to attempt to simplify either inequality in our new definition. Once we are given a value for m, as in our next example, we will be able to simplify our piecewise representation completely.

Example 44. Express $g(x) = |x-3|$ as a piecewise-defined function.

$$g(x) = |x-3| = \begin{cases} -(x-3), & \text{if } \; x-3 < 0 \\ (x-3), & \text{if } \; (x-3) \geq 0 \end{cases}$$

Simplifying, we get

$$g(x) = \begin{cases} -x+3, & \text{if } \; x < 3 \\ x-3, & \text{if } \; x \geq 3 \end{cases}$$

Our piecewise answer above should begin to make sense, when one considers the graph of g as a horizontal shift of $y = |x|$ to the right by 3 units.

Example 45. Express $h(x) = |x| - 3$ as a piecewise-defined function.

Since the variable within the absolute value remains unchanged, the domains for each piece in our resulting function will not change. Instead, we need only subtract 3 from each piece of our answer. Thus, we get the following representation.

$$h(x) = \begin{cases} -x-3, & \text{if } \; x < 0 \\ x-3, & \text{if } \; x \geq 0 \end{cases}$$

Similarly, this answer again seems reasonable, as the graph of $h(x) = |x| - 3$ represents a vertical shift of $y = |x|$ down by 3 units.

Example 46. Express $k(x) = 4 - 2|3x+1|$ as a piecewise-defined function and identify any $x-$ and $y-$intercepts on its graph. Determine the domain and range of $k(x)$.

We set $k(x) = 0$ to find any zeros: $4 - 2|3x+1| = 0$.

Isolating the absolute value gives us $|3x+1| = 2$, so either

$$3x+1 = 2 \qquad \text{or} \qquad 3x+1 = -2.$$

This results in $x = \frac{1}{3}$ or $x = -1$, so our $x-$intercepts are $\left(\frac{1}{3}, 0\right)$ and $(-1, 0)$.

For our $y-$intercept, substituting $x = 0$ into $k(x)$ gives us

$$y = k(0) = 4 - 2|3(0)+1| = 2.$$

So our y-intercept is at $(0, 2)$. Rewriting the expression for k as a piecewise function gives us the following.

$$\begin{aligned} k(x) &= \begin{cases} 4 - 2[-(3x+1)], & \text{if } 3x+1 < 0 \\ 4 - 2(3x+1), & \text{if } 3x+1 \geq 0 \end{cases} \\ &= \begin{cases} 4 + 6x + 2, & \text{if } 3x < -1 \\ 4 - 6x - 2, & \text{if } 3x \geq -1 \end{cases} \\ &= \begin{cases} 6x + 6, & \text{if } x < -\frac{1}{3} \\ -6x + 2, & \text{if } x \geq -\frac{1}{3} \end{cases} \end{aligned}$$

Either algebraically, or using the graph of k from page 243, we see that the domain of k is $(-\infty, \infty)$ while the range is $(-\infty, 4]$.

Practice Problems

Identifying Domain Algebraically

Find the domain of each of the following functions. Express your answers using interval notation.

1. $f(x) = |3x-2|$
2. $g(x) = (3x-2)^2$
3. $h(x) = \dfrac{1}{3x-2}$
4. $k(x) = \sqrt{3x-2}$
5. $k(x) = \sqrt[3]{3x-2}$
6. $\ell(x) = \sqrt[4]{2-3x}$
7. $m(x) = \dfrac{x-2}{\sqrt{3x-2}}$
8. $n(x) = \dfrac{\sqrt{3x-2}}{x-2}$

Find the domain of each of the following functions. Express your answers using interval notation.

9. $g(x) - 4x^2$
10. $f(x) = x^4 - 13x^3 + 56x^2 - 19$
11. $g(x) = x^2 - 4$
12. $k(x) = \dfrac{x}{x-8}$
13. $h(x) = \dfrac{x-5}{x+4}$

14. $h(x) = \dfrac{x-2}{x+1}$
15. $k(x) = \dfrac{x-2}{x-2}$
16. $k(x) = \dfrac{3x}{x^2+x-2}$
17. $g(x) = \dfrac{2x}{x^2-9}$
18. $f(x) = \dfrac{2x}{x^2+9}$
19. $h(x) = \dfrac{x+4}{x^2-36}$
20. $f(x) = \sqrt{3-x}$
21. $g(x) = \sqrt{2x+5}$
22. $f(x) = 5\sqrt{x-1}$
23. $h(x) = 9x\sqrt{x+3}$
24. $k(x) = \dfrac{\sqrt{7-x}}{x^2+1}$
25. $f(x) = \sqrt{6x-2}$
26. $g(x) = \dfrac{6}{\sqrt{6x-2}}$
27. $k(x) = \dfrac{4}{\sqrt{x-3}}$
28. $g(x) = \dfrac{x}{\sqrt{x-8}}$
29. $h(x) = \sqrt[3]{6x-2}$
30. $k(x) = \dfrac{6}{4-\sqrt{6x-2}}$
31. $f(x) = \dfrac{\sqrt{6x-2}}{x^2-36}$
32. $g(x) = \dfrac{\sqrt[3]{6x-2}}{x^2+36}$
33. $h(x) = \sqrt{x-7}+\sqrt{9-x}$
34. $h(t) = \dfrac{\sqrt{t}-8}{5-t}$
35. $f(r) = \dfrac{\sqrt{r}}{r-8}$
36. $k(v) = \dfrac{1}{4-\dfrac{1}{v^2}}$
37. $f(y) = \sqrt[3]{\dfrac{y}{y-8}}$
38. $k(w) = \dfrac{w-8}{5-\sqrt{w}}$

Combining Functions

Function Arithmetic

In each of the following exercises, use the pair of functions f and g to find the following values if they exist.

- $(f+g)(2)$
- $(f-g)(-1)$
- $(g-f)(1)$
- $(fg)\left(\frac{1}{2}\right)$
- $\left(\frac{f}{g}\right)(0)$
- $\left(\frac{g}{f}\right)(-2)$

1. $f(x) = 3x+1 \quad g(x) = 4-x$
2. $f(x) = x^2 \quad g(x) = -2x+1$
3. $f(x) = x^2-x \quad g(x) = 12-x^2$
4. $f(x) = 2x^3 \quad g(x) = -x^2-2x-3$
5. $f(x) = \sqrt{x+3} \quad g(x) = 2x-1$
6. $f(x) = \sqrt{4-x} \quad g(x) = \sqrt{x+2}$

7. $f(x) = 2x \qquad g(x) = \dfrac{1}{2x+1}$
8. $f(x) = x^2 \qquad g(x) = \dfrac{3}{2x-3}$
9. $f(x) = x^2 \qquad g(x) = \dfrac{1}{x^2}$
10. $f(x) = x^2 + 1 \qquad g(x) = \dfrac{1}{x^2+1}$

In each of the following exercises, use the pair of functions f and g to find the domain of the indicated function then find and simplify an expression for it.

- $(f+g)(x)$
- $(f-g)(x)$
- $(fg)(x)$
- $\left(\frac{f}{g}\right)(x)$

11. $f(x) = 2x + 1 \qquad g(x) = x - 2$
12. $f(x) = 1 - 4x \qquad g(x) = 2x - 1$
13. $f(x) = x^2 \qquad g(x) = 3x - 1$
14. $f(x) = x^2 - x \qquad g(x) = 7x$
15. $f(x) = x^2 - 4 \qquad g(x) = 3x + 6$
16. $f(x) = -x^2 + x + 6 \quad g(x) = x^2 - 9$
17. $f(x) = \dfrac{x}{2} \qquad g(x) = \dfrac{2}{x}$
18. $f(x) = x - 1 \qquad g(x) = \dfrac{1}{x-1}$
19. $f(x) = x \qquad g(x) = \sqrt{x+1}$
20. $f(x) = g(x) = \sqrt{x-5}$

For each of the following exercises, let f be the function defined by

$$f = \{(-3,4),(-2,2),(-1,0),(0,1),(1,3),(2,4),(3,-1)\}$$

and let g be the function defined

$$g = \{(-3,-2),(-2,0),(-1,-4),(0,0),(1,-3),(2,1),(3,2)\}.$$

Use f and g to compute each of the indicated values if they exist.

21. $(f+g)(-3)$
22. $(f-g)(2)$
23. $(fg)(-1)$
24. $(g+f)(1)$
25. $(g-f)(3)$
26. $(gf)(-3)$
27. $\left(\frac{f}{g}\right)(-2)$
28. $\left(\frac{f}{g}\right)(-1)$
29. $\left(\frac{f}{g}\right)(2)$
30. $\left(\frac{g}{f}\right)(-1)$
31. $\left(\frac{g}{f}\right)(3)$
32. $\left(\frac{g}{f}\right)(-3)$

Composite Functions

In each of the following exercises, use the given pair of functions to find the following values if they exist.

- $(g \circ f)(0)$
- $(f \circ g)(-1)$
- $(f \circ f)(2)$
- $(g \circ f)(-3)$
- $(f \circ g)\left(\frac{1}{2}\right)$
- $(f \circ f)(-2)$

1. $f(x) = x^2$, $g(x) = 2x + 1$
2. $f(x) = 4 - x$, $g(x) = 1 - x^2$
3. $f(x) = 4 - 3x$, $g(x) = |x|$
4. $f(x) = |x - 1|$, $g(x) = x^2 - 5$
5. $f(x) = 4x + 5$, $g(x) = \sqrt{x}$
6. $f(x) = \sqrt{3 - x}$, $g(x) = x^2 + 1$
7. $f(x) = \dfrac{3}{1 - x}$, $g(x) = \dfrac{4x}{x^2 + 1}$
8. $f(x) = \dfrac{x}{x + 5}$, $g(x) = \dfrac{2}{7 - x^2}$

In each of the following exercises, use the given pair of functions to find and simplify expressions for the following functions and state the domain of each using interval notation.

- $(g \circ f)(x)$
- $(f \circ g)(x)$
- $(f \circ f)(x)$

9. $f(x) = 2x + 3$, $g(x) = x^2 - 9$
10. $f(x) = x^2 - x + 1$, $g(x) = 3x - 5$
11. $f(x) = x^2 - 4$, $g(x) = |x|$
12. $f(x) = 3x - 5$, $g(x) = \sqrt{x}$
13. $f(x) = |x + 1|$, $g(x) = \sqrt{x}$
14. $f(x) = 3 - x^2$, $g(x) = \sqrt{x + 1}$
15. $f(x) = |x|$, $g(x) = \sqrt{4 - x}$
16. $f(x) = x^2 - x - 1$, $g(x) = \sqrt{x - 5}$
17. $f(x) = 3x - 1$, $g(x) = \dfrac{1}{x + 3}$
18. $f(x) = \dfrac{3x}{x - 1}$, $g(x) = \dfrac{x}{x - 3}$
19. $f(x) = \dfrac{x}{2x + 1}$, $g(x) = \dfrac{2x + 1}{x}$
20. $f(x) = \dfrac{2x}{x^2 - 4}$, $g(x) = \sqrt{1 - x}$

In each of the following exercises, use $f(x) = -2x$, $g(x) = \sqrt{x}$ and $h(x) = |x|$ to find and simplify expressions for the following functions and state the domain of each using interval notation.

21. $(h \circ g \circ f)(x)$
22. $(h \circ f \circ g)(x)$
23. $(g \circ f \circ h)(x)$
24. $(g \circ h \circ f)(x)$
25. $(f \circ h \circ g)(x)$
26. $(f \circ g \circ h)(x)$

In each of the following exercises, write the given function as a composition of two or more non-identity functions. (There are several correct answers, so check your answer using function composition.)

27. $p(x) = (2x+3)^3$
28. $P(x) = (x^2 - x + 1)^5$
29. $h(x) = \sqrt{2x-1}$
30. $H(x) = |7 - 3x|$
31. $r(x) = \dfrac{2}{5x+1}$
32. $R(x) = \dfrac{7}{x^2-1}$
33. $q(x) = \dfrac{|x|+1}{|x|-1}$
34. $Q(x) = \dfrac{2x^3+1}{x^3-1}$
35. $v(x) = \dfrac{2x+1}{3-4x}$
36. $w(x) = \dfrac{x^2}{x^4+1}$
37. Let $g(x) = -x$, $h(x) = x+2$, $j(x) = 3x$ and $k(x) = x - 4$. In what order must these functions be composed with $f(x) = \sqrt{x}$ to create $F(x) = 3\sqrt{-x+2} - 4$?
38. What linear functions could be used to transform $f(x) = x^3$ into $F(x) = -\frac{1}{2}(2x-7)^3 + 1$? What is the proper order of composition?

For each of the following exercises, let f be the function defined by

$$f = \{(-3,4), (-2,2), (-1,0), (0,1), (1,3), (2,4), (3,-1)\}$$

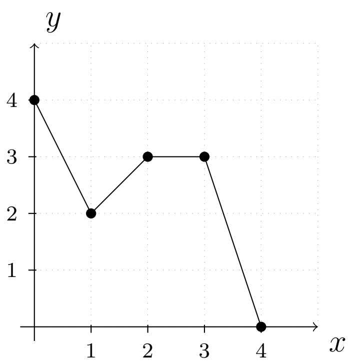
and let g be the function defined

$$g = \{(-3,-2), (-2,0), (-1,-4), (0,0), (1,-3), (2,1), (3,2)\}.$$

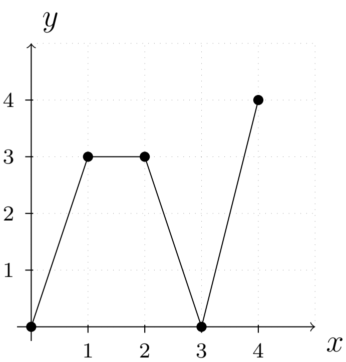
Use f and g to compute each of the indicated values if they exist.

39. $(f \circ g)(3)$
40. $f(g(-1))$
41. $(f \circ f)(0)$
42. $(f \circ g)(-3)$
43. $(g \circ f)(3)$
44. $g(f(-3))$
45. $(g \circ g)(-2)$
46. $(g \circ f)(-2)$
47. $g(f(g(0)))$

In each of the following exercises, use the graphs of $y = f(x)$ and $y = g(x)$ below to find the function value.



$y = f(x)$



$y = g(x)$

48. $(g \circ f)(1)$
49. $(f \circ g)(3)$
50. $(g \circ f)(2)$
51. $(f \circ g)(0)$
52. $(f \circ f)(1)$
53. $(g \circ g)(1)$

Inverse Functions

In each of the following exercises, show that the given function is one-to-one and find its inverse. Check your answers algebraically and graphically. Verify that the range of f is the domain of f^{-1} and vice-versa.

1. $f(x) = 6x - 2$
2. $f(x) = 42 - x$
3. $f(x) = \dfrac{x-2}{3} + 4$
4. $f(x) = 1 - \dfrac{4+3x}{5}$
5. $f(x) = \sqrt{3x-1} + 5$
6. $f(x) = 2 - \sqrt{x-5}$
7. $f(x) = 3\sqrt{x-1} - 4$
8. $f(x) = 1 - 2\sqrt{2x+5}$
9. $f(x) = \sqrt[5]{3x-1}$
10. $f(x) = 3 - \sqrt[3]{x-2}$
11. $f(x) = x^2 - 10x,\ x \geq 5$
12. $f(x) = 3(x+4)^2 - 5,\ x \leq -4$
13. $f(x) = x^2 - 6x + 5,\ x \leq 3$
14. $f(x) = 4x^2 + 4x + 1,\ x < -1$
15. $f(x) = \dfrac{3}{4-x}$
16. $f(x) = \dfrac{x}{1-3x}$
17. $f(x) = \dfrac{2x-1}{3x+4}$
18. $f(x) = \dfrac{4x+2}{3x-6}$

19. $f(x) = \dfrac{-3x-2}{x+3}$

20. $f(x) = \dfrac{x-2}{2x-1}$

Find the inverses of each of the following functions.

21. $f(x) = ax + b,\ a \neq 0$

22. $f(x) = a\sqrt{x-h} + k,\ a \neq 0, x \geq h$

23. $f(x) = ax^2 + bx + c$ where $a \neq 0$, $x \geq -\dfrac{b}{2a}$.

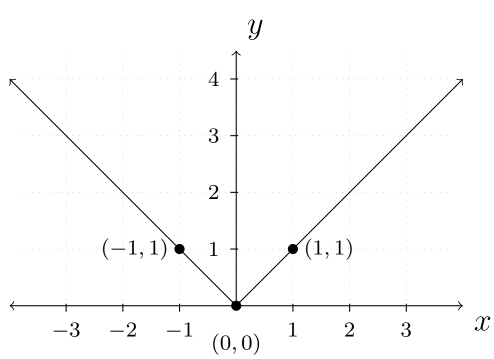
24. $f(x) = \dfrac{ax+b}{cx+d}$ where c and d are not both zero.

Transformations

Suppose $(2, -3)$ is on the graph of $y = f(x)$. In each of the following exercises, use the given point to find a point on the graph of the given transformed function.

1. $g(x) = f(x) + 3$
2. $g(x) = f(x+3)$
3. $g(x) = f(x) - 1$
4. $g(x) = f(x-1)$
5. $g(x) = 3f(x)$
6. $g(x) = f(3x)$
7. $g(x) = -f(x)$
8. $g(x) = f(-x)$
9. $g(x) = f(x-3) + 1$
10. $g(x) = 2f(x+1)$
11. $g(x) = 10 - f(x)$
12. $g(x) = 3f(2x) - 1$
13. $g(x) = \frac{1}{2}f(4-x)$
14. $g(x) = 5f(2x) + 3$
15. $g(x) = 2f(1-x) - 1$
16. $g(x) = \dfrac{f(3x)-1}{2}$
17. $g(x) = \dfrac{4 - f(3x-1)}{7}$

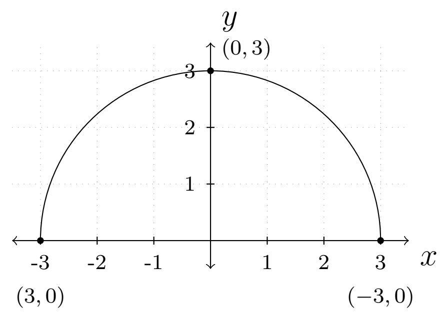
The complete graph of $f(x) = |x|$ is given below. In each of the following exercises, use it to graph the given transformed function.



$f(x) = |x|$

18. $g(x) = f(x) + 1$
19. $g(x) = f(x) - 2$
20. $g(x) = f(x + 1)$
21. $g(x) = f(x - 2)$
22. $g(x) = 2f(x)$
23. $g(x) = f(2x)$
24. $g(x) = 2 - f(x)$
25. $g(x) = f(2 - x)$
26. $g(x) = 2 - f(2 - x)$
27. Some of the answers to the previous nine exercises should be equal. Which ones are equal? What properties of the graph of $y = f(x)$ contribute to this?

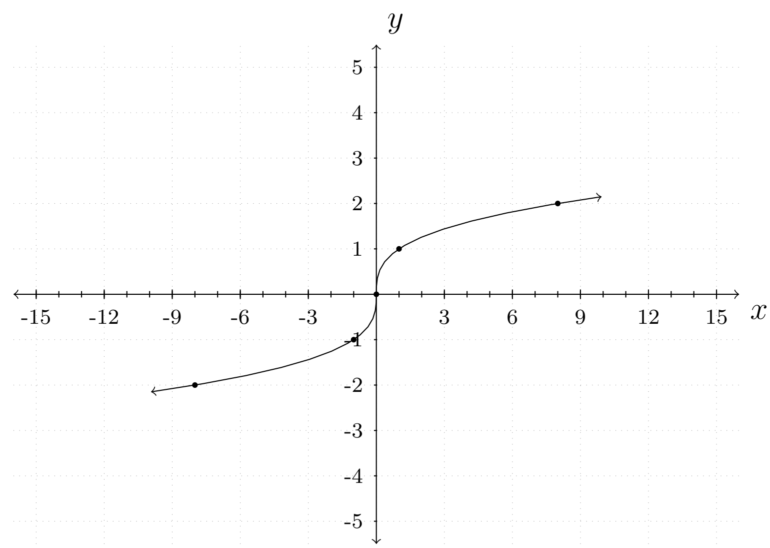
The complete graph of $f(x) = \sqrt{9 - x^2}$ is given below. Use the graph of f to graph the each of the given transformations.



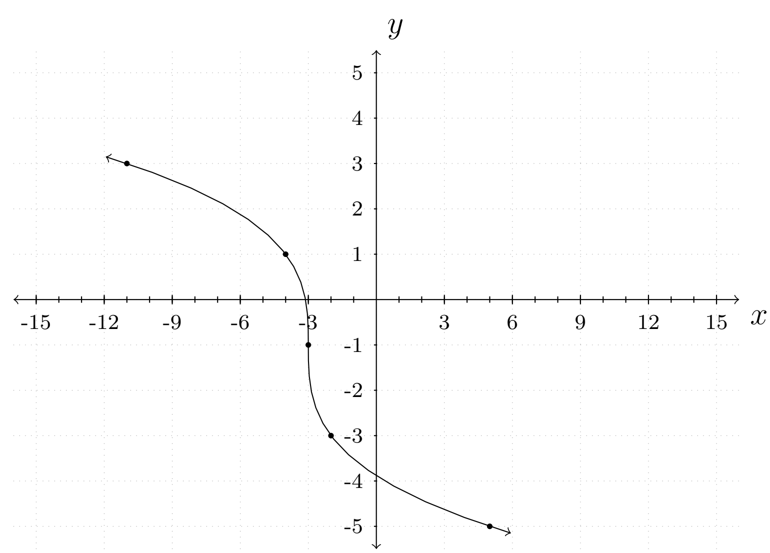
$f(x) = \sqrt{9 - x^2}$

28. $g(x) = f(x) + 3$
29. $h(x) = f(x) - \frac{1}{2}$
30. $j(x) = f\left(x - \frac{2}{3}\right)$
31. $a(x) = f(x + 4)$
32. $b(x) = f(x + 1) - 1$
33. $c(x) = \frac{3}{5}f(x)$
34. $d(x) = -2f(x)$
35. $k(x) = f\left(\frac{2}{3}x\right)$
36. $m(x) = -\frac{1}{4}f(3x)$
37. $n(x) = 4f(x - 3) - 6$
38. $p(x) = 4 + f(1 - 2x)$

39. The graphs of $y = f(x) = \sqrt[3]{x}$ and $y = g(x)$ are shown below. Find a formula for g based on transformations of the graph of f. Check your answer by confirming that the points shown on the graph of g satisfy the equation $y = g(x)$.



$f(x) = \sqrt[3]{x}$



$g(x)$

40. A function f is said to be *even* if $f(x) = f(-x)$. The graph of an even function will be symmetric about the $y-$axis, since $f(-x)$ represents a reflection of the graph of f about the $y-$axis. Determine both algebraically (using compositions) and graphically (using transformations) whether each of the following fundamental functions is even.

    (a) $g(x) = x^2$

    (b) $k(x) = \sqrt{x}$

    (c) $\ell(x) = |x|$

    (d) $m(x) = x^3$

    (e) $n(x) = \sqrt[3]{x}$

    (f) $p(x) = \dfrac{1}{x}$

    (g) $q(x) = \sqrt{9 - x^2}$

41. A function f is said to be *odd* if $-f(x) = f(-x)$. Since $f(-x)$ represents a reflection of the graph of f about the $y-$axis and $-f(x)$ represents a reflection of the graph of f about the $x-$axis, whenever these two reflections produce the same graph, the corresponding function will be odd. In this case, the graph of an odd function is said to be *symmetric about the origin.* Determine both algebraically (using compositions) and graphically (using transformations) whether each of the following fundamental functions is odd.

    (a) $g(x) = x^2$

    (b) $k(x) = \sqrt{x}$

    (c) $\ell(x) = |x|$

    (d) $m(x) = x^3$

    (e) $n(x) = \sqrt[3]{x}$

    (f) $p(x) = \dfrac{1}{x}$

    (g) $q(x) = \sqrt{9 - x^2}$

Let $f(x) = \sqrt{x}$. Find a formula for a function g whose graph is obtained from f from the given sequence of transformations.

42. (1) shift right 2 units; (2) shift down 3 units
43. (1) shift down 3 units; (2) shift right 2 units
44. (1) reflect across the x-axis; (2) shift up 1 unit
45. (1) shift up 1 unit; (2) reflect across the x-axis
46. (1) shift left 1 unit; (2) reflect across the y-axis; (3) shift up 2 units
47. (1) reflect across the y-axis; (2) shift left 1 unit; (3) shift up 2 units
48. (1) shift left 3 units; (2) vertical stretch by a factor of 2; (3) shift down 4 units
49. (1) shift left 3 units; (2) shift down 4 units; (3) vertical stretch by a factor of 2
50. (1) shift right 3 units; (2) horizontal shrink by a factor of 2; (3) shift up 1 unit
51. (1) horizontal shrink by a factor of 2; (2) shift right 3 units; (3) shift up 1 unit

Piecewise-Defined and Absolute Value Functions

Piecewise-Defined Functions

1. Let $f(x) = \begin{cases} x+5 & \text{if} \quad x \le -3 \\ \sqrt{9-x^2} & \text{if} \quad -3 < x \le 3 \\ -x+5 & \text{if} \quad x > 3 \end{cases}$

   Compute the following function values.

   (a) $f(-4)$ (b) $f(-3)$ (c) $f(3)$

   (d) $f(3.1)$ (e) $f(-3.01)$ (f) $f(2)$

2. Let $f(x) = \begin{cases} x^2 & \text{if} \quad x \le -1 \\ \sqrt{1-x^2} & \text{if} \quad -1 < x \le 1 \\ x & \text{if} \quad x > 1 \end{cases}$

   Compute the following function values.

   (a) $f(4)$ (b) $f(-3)$ (c) $f(1)$

   (d) $f(0)$ (e) $f(-1)$ (f) $f(-0.99)$

In each of the following exercises, find all possible x such that $f(x) = 0$. Then sketch the graph of the given piecewise-defined function. Use your graph to identify the domain and range of each function.

3. $f(x) = \begin{cases} 4-x & \text{if} \quad x \le 3 \\ 2 & \text{if} \quad x > 3 \end{cases}$

4. $f(x) = \begin{cases} x^2 & \text{if} \quad x \le 0 \\ 2x & \text{if} \quad x > 0 \end{cases}$

5. $f(x) = \begin{cases} -3 & \text{if} \quad x < 0 \\ 2x-3 & \text{if} \quad 0 \le x \le 3 \\ 3 & \text{if} \quad x > 3 \end{cases}$

6. $f(x) = \begin{cases} x^2-4 & \text{if} \quad x \le -2 \\ 4-x^2 & \text{if} \quad -2 < x < 2 \\ x^2-4 & \text{if} \quad x \ge 2 \end{cases}$

7. $f(x) = \begin{cases} -2x-4 & \text{if} \quad x < 0 \\ 3x & \text{if} \quad x \ge 0 \end{cases}$

8. $f(x) = \begin{cases} x^2 & \text{if} \quad x \le -2 \\ 3-x & \text{if} \quad -2 < x < 2 \\ 4 & \text{if} \quad x \ge 2 \end{cases}$

9. $f(x) = \begin{cases} \dfrac{1}{x} & \text{if} \quad -6 < x < -1 \\ x & \text{if} \quad -1 < x < 1 \\ \sqrt{x} & \text{if} \quad 1 < x < 9 \end{cases}$

Functions Containing an Absolute Value

In each of the following exercises, find the zeros of each function and the x- and y-intercepts of each graph, if any exist. Then graph the given absolute value function and express it as a piecewise-defined function. Use the graph to determine the domain and range of each function.

1. $f(x) = |x + 4|$
2. $f(x) = |x| + 4$
3. $f(x) = |4x|$
4. $f(x) = -3|x|$
5. $f(x) = |2x - 5|$
6. $f(x) = | - 2x + 5|$
7. $f(x) = 2|x - \frac{5}{2}|$
8. $f(x) = \dfrac{1}{3}|2x - 1|$
9. $f(x) = 3|x + 4| - 4$

Selected Answers

Identifying Domain Algebraically

1. $(-\infty, \infty)$
3. $(-\infty, \frac{2}{3}) \cup (\frac{2}{3}, \infty)$
5. $(-\infty, \infty)$
7. $(\frac{2}{3}, \infty)$
9. $(-\infty, \infty)$
11. $(-\infty, \infty)$
13. $(-\infty, -4) \cup (-4, \infty)$
15. $(-\infty, 2) \cup (2, \infty)$
17. $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
19. $(-\infty, -6) \cup (-6, 6) \cup (6, \infty)$
21. $[-\frac{5}{2}, \infty)$
23. $[-3, \infty)$
25. $[3, \infty)$
27. $(3, \infty)$
29. $(-\infty, \infty)$
31. $(-\infty, -6) \cup (-6, 6) \cup (6, \infty)$
33. $[7, 9]$
35. $(-\infty, 8) \cup (8, \infty)$
37. $(-\infty, \infty)$

Combining Functions

Function Arithmetic

1. $f(x) = 3x + 1, \quad g(x) = 4 - x$
   - $(f + g)(2) = 9$
   - $(f - g)(-1) = -7$
   - $(g - f)(1) = -1$
   - $(fg)(\frac{1}{2}) = \frac{35}{4}$
   - $\left(\frac{f}{g}\right)(0) = \frac{1}{4}$
   - $\left(\frac{g}{f}\right)(-2) = -\frac{6}{5}$

3. $f(x) = x^2 - x, \quad g(x) = 12 - x^2$
   - $(f+g)(2) = 10$
   - $(f-g)(-1) = -9$
   - $(g-f)(1) = 11$
   - $(fg)(\frac{1}{2}) = -\frac{47}{16}$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = \frac{4}{3}$

5. $f(x) = \sqrt{x+3}, \quad g(x) = 2x - 1$
   - $(f+g)(2) = 3 + \sqrt{5}$
   - $(f-g)(-1) = 3 + \sqrt{2}$
   - $(g-f)(1) = -1$
   - $(fg)(\frac{1}{2}) = 0$
   - $\left(\frac{f}{g}\right)(0) = -\sqrt{3}$
   - $\left(\frac{g}{f}\right)(-2) = -5$

7. $f(x) = 2x, \quad g(x) = \dfrac{1}{2x+1}$
   - $(f+g)(2) = \frac{21}{5}$
   - $(f-g)(-1) = -1$
   - $(g-f)(1) = -\frac{5}{3}$
   - $(fg)(\frac{1}{2}) = \frac{1}{2}$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = \frac{1}{12}$

9. $f(x) = x^2, \quad g(x) = \dfrac{1}{x^2}$
   - $(f+g)(2) = \frac{17}{4}$
   - $(f-g)(-1) = 0$
   - $(g-f)(1) = 0$
   - $(fg)(\frac{1}{2}) = 1$
   - $\left(\frac{f}{g}\right)(0) =$DNE
   - $\left(\frac{g}{f}\right)(-2) = \frac{1}{16}$

11. $f(x) = 2x + 1, \quad g(x) = x - 2$
   - $(f+g)(x) = 3x - 1,$ all reals
   - $(f-g)(x) = x + 3,$ all reals
   - $(fg)(x) = 2x^2 - 3x - 2,$ all reals
   - $\left(\frac{f}{g}\right)(x) = \frac{2x+1}{x-2}, \ x \neq 2$

13. $f(x) = x^2, \quad g(x) = 3x - 1$
   - $(f+g)(x) = x^2 + 3x - 1,$ all reals
   - $(f-g)(x) = x^2 - 3x + 1,$ all reals
   - $(fg)(x) = 3x^3 - x^2,$ all reals
   - $\left(\frac{f}{g}\right)(x) = \frac{x^2}{3x-1}, \ x \neq \frac{1}{3}$

15. $f(x) = x^2 - 4, \quad g(x) = 3x + 6$
   - $(f+g)(x) = x^2 + 3x + 2,$ all reals
   - $(f-g)(x) = x^2 - 3x - 10,$ all reals
   - $(fg)(x) = 3x^3 + 6x^2 - 12x - 24,$ all reals
   - $\left(\frac{f}{g}\right)(x) = \frac{x^2-4}{3x+6}, \ x \neq -2$

17. $f(x) = \dfrac{x}{2}, \quad g(x) = \dfrac{2}{x}$

- $(f+g)(x) = \frac{x^2+4}{2x},\ x \neq 0$
- $(f-g)(x) = \frac{x^2-4}{2x},\ x \neq 0$
- $(fg)(x) = 1,\ x \neq 0$
- $\left(\frac{f}{g}\right)(x) = \frac{x^2}{4},\ x \neq 0$

19. $f(x) = x, \quad g(x) = \sqrt{x+1}$

- $(f+g)(x) = x + \sqrt{x+1},\ x \geq -1$
- $(f-g)(x) = x - \sqrt{x+1},\ x \geq -1$
- $(fg)(x) = x\sqrt{x+1},\ x \geq -1$
- $\left(\frac{f}{g}\right)(x) = \frac{x}{\sqrt{x+1}},\ x > -1$

21. 2
23. 0
25. 3
27. DNE
29. 4
31. -2

Composite Functions

1. $f(x) = x^2, \quad g(x) = 2x + 1$

- $(g \circ f)(0) = 1$
- $(f \circ g)(-1) = 1$
- $(f \circ f)(2) = 16$
- $(g \circ f)(-3) = 19$
- $(f \circ g)(\frac{1}{2}) = 4$
- $(f \circ f)(-2) = 16$

3. $f(x) = 4 - 3x, \quad g(x) = |x|$

- $(g \circ f)(0) = 4$
- $(f \circ g)(-1) = 1$
- $(f \circ f)(2) = 10$
- $(g \circ f)(-3) = 13$
- $(f \circ g)(\frac{1}{2}) = \frac{5}{2}$
- $(f \circ f)(-2) = -26$

5. $f(x) = 4x + 5, \quad g(x) = \sqrt{x}$

- $(g \circ f)(0) = \sqrt{5}$
- $(f \circ g)(-1) =$DNE
- $(f \circ f)(2) = 57$
- $(g \circ f)(-3) =$DNE
- $(f \circ g)(\frac{1}{2}) = 4\sqrt{\frac{1}{2}} + 5$
- $(f \circ f)(-2) = -7$

7. $f(x) = \dfrac{3}{1-x}, \quad g(x) = \dfrac{4x}{x^2+1}$

- $(g \circ f)(0) = \frac{6}{5}$
- $(f \circ g)(-1) = 1$
- $(f \circ f)(2) = \frac{3}{4}$
- $(g \circ f)(-3) = \frac{48}{25}$
- $(f \circ g)(\frac{1}{2}) = -5$
- $(f \circ f)(-2) =$DNE

9. $f(x) = 2x + 3, \quad g(x) = x^2 - 9$
   - $(g \circ f)(x) = 4x^2 + 12x$
   - $(f \circ g)(x) = 2x^2 - 15$
   - $(f \circ f)(x) = 4x = 9$

11. $f(x) = x^2 - 4, \quad g(x) = |x|$
   - $(g \circ f)(x) = |x^2 - 4|$
   - $(f \circ g)(x) = x^2 - 4$
   - $(f \circ f)(x) = x^4 - 8x^2 + 12$

13. $f(x) = |x + 1|, \quad g(x) = \sqrt{x}$
   - $(g \circ f)(x) = \sqrt{|x + 1|}$
   - $(f \circ g)(x) = |\sqrt{x} + 1|$
   - $(f \circ f)(x) = |x + 1| + 1$

15. $f(x) = |x|, \quad g(x) = \sqrt{4 - x}$
   - $(g \circ f)(x) = \sqrt{4 - |x|}$
   - $(f \circ g)(x) = \sqrt{4 - x}$
   - $(f \circ f)(x) = |x|$

17. $f(x) = 3x - 1, \quad g(x) = \dfrac{1}{x + 3}$
   - $(g \circ f)(x) = \frac{1}{3x+2}$
   - $(f \circ g)(x) = \frac{x}{x+3}$
   - $(f \circ f)(x) = 9x - 4$

19. $f(x) = \dfrac{x}{2x + 1}, \quad g(x) = \dfrac{2x + 1}{x}$
   - $(g \circ f)(x) = \frac{4x+1}{x}$
   - $(f \circ g)(x) = \frac{2x+1}{5x+2}$
   - $(f \circ f)(x) = \frac{x}{4x+1}$

21. $h(g(f(x))) = |\sqrt{-2x}|$
23. $g(f(h(x))) = \sqrt{-2|x|}$
25. $f(h(g(x))) = -2|\sqrt{x}|$

27. $f(x) = x^3, \quad g(x) = 2x + 3$
29. $f(x) = \sqrt{x}, \quad g(x) = 2x - 1$
31. $f(x) = \dfrac{2}{x}, \quad g(x) = 5x + 1$
33. $f(x) = \dfrac{x + 1}{x - 1}, \quad g(x) = |x|$
35. $f(x) = \dfrac{x + 1}{3 - 2x}, \quad g(x) = 2x$
37. $k \circ j \circ f \circ h \circ g$

39. 4
41. 3
43. -4
45. 0
47. -3
49. 4
51. 4
53. 0

Inverse Functions

1. $f^{-1}(x) = \dfrac{x+2}{6}$
3. $f^{-1}(x) = 3x - 10$
5. $f^{-1}(x) = \frac{1}{3}(x-5)^2 + \frac{1}{3}$, $x \geq 5$
7. $f^{-1}(x) = \frac{1}{9}(x+4)^2 + 1$, $x \geq -4$
9. $f^{-1}(x) = \frac{1}{3}x^5 + \frac{1}{3}$
11. $f^{-1}(x) = 5 + \sqrt{x+25}$
13. $f^{-1}(x) = 3 - \sqrt{x+4}$
15. $f^{-1}(x) = \dfrac{4x-3}{x}$
17. $f^{-1}(x) = \dfrac{4x+1}{2-3x}$
19. $f^{-1}(x) = \dfrac{-3x-2}{x+3}$
21. $f^{-1}(x) = \dfrac{x-b}{a}$
23. $f^{-1}(x) = \dfrac{-b+\sqrt{b^2-4a(c-x)}}{2a}$

Transformations

1. $(2, 0)$
3. $(2, -4)$
5. $(2, -9)$
7. $(2, 3)$
9. $(5, -2)$
11. $(2, 13)$
13. $(2, -\frac{3}{2})$
15. $(-1, -7)$
17. $(1, 1)$

Each answer below describes the resulting transformation of the graph of $f(x) = |x|$.

19. Shift down 2 units
21. Shift right 2 units
23. Vertical stretch (or horizontal shrink) by a factor of 2
25. Shift right 2 units
27. Exercises 22. and 23. agree; 21. and 25. agree. $|kx| = |k| \cdot |x|$, where $k \in \mathbb{R}$

Each answer below describes the resulting transformation of the graph of $f(x) = \sqrt{9-x^2}$.

29. Shift down 1/2 units
31. Shift left 4 units
33. Vertical shrink by a factor of 5/3
35. Horizontal stretch by a factor of 3/2
37. Shift right 3 units, vertical stretch by a factor of 4, shift down 6 units

39. $g(x) = -2\sqrt[3]{x+3} - 1$
41. (d), (e), and (f)
43. $g(x) = \sqrt{x-2} - 3$
45. $g(x) = -\sqrt{x} - 1$
47. $g(x) = \sqrt{-x-1} + 2$
49. $g(x) = 2\sqrt{x+3} - 8$
51. $g(x) = \sqrt{2x-6} + 1$

Piecewise-Defined and Absolute Value Functions

Piecewise-Defined Functions

1. $f(x) = \begin{cases} x+5 & \text{if } x \leq -3 \\ \sqrt{9-x^2} & \text{if } -3 < x \leq 3 \\ -x+5 & \text{if } x > 3 \end{cases}$

   (a) $f(-4) = 1$ (b) $f(-3) = 2$ (c) $f(3) = 0$

   (d) $f(3.1) = 1.9$ (e) $f(-3.01) = 1.99$ (f) $f(2) = \sqrt{5}$

3. D: $(-\infty, \infty)$; R: $[1, \infty)$; No zeros
5. D: $(-\infty, \infty)$; R: $[-3, 3]$; $x = 3/2$
7. D: $(-\infty, \infty)$; R: $(-4, \infty)$; $x = -2, 0$
9. D: $(-6, -1) \cup (-1, 1) \cup (1, 9)$; R: $(-1, 1) \cup (1, 3)$; $x = 0$

Functions Containing an Absolute Value

1. No zeros; $y-$int at $(0, 4)$; D: $(-\infty, \infty)$; R: $[4, \infty)$

$$f(x) = \begin{cases} x+4 & \text{if } x \geq 0 \\ -x+4 & \text{if } x < 0 \end{cases}$$

3. Zero at $x = 0$; $y-$int at $(0, 0)$; D: $(-\infty, \infty)$; R: $[0, \infty)$

$$f(x) = \begin{cases} 4x & \text{if } x \geq 0 \\ -4x & \text{if } x < 0 \end{cases}$$

5. Zero at $x = \frac{5}{2}$; $y-$int at $(0, 5)$; D: $(-\infty, \infty)$; R: $[0, \infty)$

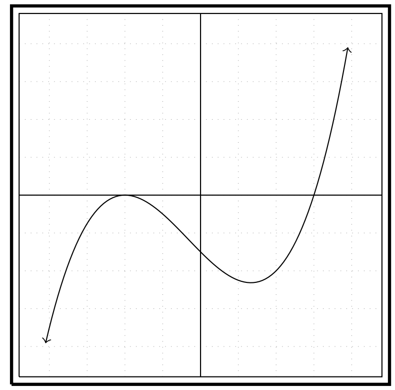
$$f(x) = \begin{cases} 2x-5 & \text{if } x \geq \frac{5}{2} \\ -2x+5 & \text{if } x < \frac{5}{2} \end{cases}$$

7. Zero at $x = \frac{5}{2}$; $y-$int at $(0, 5)$; D: $(-\infty, \infty)$; R: $[0, \infty)$

$$f(x) = \begin{cases} 2x-5 & \text{if } x \geq \frac{5}{2} \\ -2x+5 & \text{if } x < \frac{5}{2} \end{cases}$$

9. Zeros at $x = -\frac{16}{3}, \ -\frac{8}{3}$; $y-$int at $(0, 8)$; D: $(-\infty, \infty)$; R: $[-4, \infty)$

$$f(x) = \begin{cases} 3x+8 & \text{if } x \geq -4 \\ -3x-16 & \text{if } x < -4 \end{cases}$$



Chapter 6

Polynomials

Introduction and Terminology (L43)

Objective: Identify key features of and classify a polynomial by degree and number of nonzero terms.

A *polynomial* in terms of a variable x is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0,$$

where each *coefficient*, a_i, is a real number ($a_n \neq 0$) and the exponent, or *degree* of the polynomial, n, is a nonnegative integer.

Examples of polynomials include: $f(x) = x^2 + 5$, $f(x) = x$ and $f(x) = -3x^7 + 4x^3 - 5x$. Before classifying polynomials, we will take a moment to establish some key terminology. For our general polynomial above, the:

degree	is	n
coefficients	are	$a_n, a_{n-1}, \ldots, a_1, a_0$
leading coefficient	is	a_n
leading term	is	$a_n x^n$
constant term	is	$a_0 x^0 = a_0$.

A concrete example will help to clarify each of these terms.

Example 47. Identify the degree, leading coefficient, leading term and constant term for the polynomial

$$f(x) = -19x^5 + 4x^4 - 6x + 21.$$

The degree of this polynomial is $n = 5$, since five is the greatest exponent.

The leading term, which is the term that contains the greatest exponent (degree), is $a_n x^n = -19x^5$.

The leading coefficient is the real number being multiplied by x^n in the leading term, namely $a_n = -19$.

The constant term is $a_0 = 21$, which also represents the $y-$intercept for the graph of the given polynomial, just as it did in the chapter on quadratics.

The complete set of coefficients for the given polynomial is

$$\{a_5 = -19,\ a_4 = 4,\ a_3 = 0,\ a_2 = 0,\ a_1 = -6,\ a_0 = 21\}.$$

It is important to point out the fact that the previous example contains no *cubic* or *quadratic* terms, since the respective coefficients are both zero. This example demonstrates that not every polynomial will contain a nonzero coefficient for every term. As another example, the *power function* $f(x) = x^{10}$ is also characterized as a polynomial having degree $n = 10$, leading coefficient $a_{10} = 1$, and trailing coefficients $a_i = 0$ for $i = 9, 8, \ldots, 1, 0$.

Before we can identify and begin to classify a polynomial, we may need to simplify the given expression for x, by distributing and combining all like terms. The general form of a polynomial should be reminiscent of the standard form of a quadratic, with possibly more terms. Hence the name “polynomial”, meaning “many terms”.

The following example shows how to identify a polynomial after the necessary simplification has taken place.

Example 48. Identify the degree, leading coefficient, leading term and constant term for the given polynomial function.

$$\begin{aligned} f(x) &= 3(x+1)(x-1) + 4x^3 + 2x + 3 \\ &= 3(x^2 - 1) + 4x^3 + 2x + 3 \\ &= 3x^2 - 3 + 4x^3 + 2x + 3 \\ &= 4x^3 + 3x^2 + 2x \end{aligned}$$

Upon simplifying, we see that f has degree $n = 3$, since three is the greatest exponent.

The leading term is $4x^3$ with a leading coefficient of $a_n = 4$.

Since no constant term is shown, $a_0 = 0$ is our constant term.

Now that we can identify the essential components of a polynomial, we will categorize polynomials based upon their degree, as well as the number of terms, after all necessary simplification.

Types of Polynomials

Degree	Type	Example
0	Constant	-1
1	Linear	$2x+\sqrt{5}$
2	Quadratic	$5x^2-32x+2$
3	Cubic	$(-1/2)x^3$
4	Quartic	$-3x^4+2x^2+3x+1$
5	Quintic	$-2x^5$
6 or more	n^{th} Degree	$-2x^7+52x^6+12$

One point of note in the table above is the appearance of both rational and irrational coefficients $\left(-1/2 \text{ and } \sqrt{5}\right)$. The appearance of such coefficients is permissible in polynomials, since our coefficients a_i are only required to be real numbers. A coefficient containing the imaginary number $i=\sqrt{-1}$, on the other hand, is not permitted.

Polynomial Characterizations by Number of Nonzero Terms

Number of Terms	Name	Example
1	Monomial	$4x^5$
2	Binomial	$2x^3+1$
3	Trinomial	$-23x^{18}+4x^2+3x$
4	Tetranomial	$-23x^{18}+4x^2+3x+1$
5 or more	Polynomial	$-2x^4+x^3+15x^2-41x+12$

Example 49. Describe the type and characterization (number of terms) of the polynomial function shown below.

$$f(x)=-19x^5+4x^4-6x+21$$

Polynomials are typically named by their degree first and then their number of terms. The polynomial above is a *quintic tetranomial*; quintic because it is degree five and tetranomial because it contains four terms.

Example 50. Describe the type and characterization (number of terms) of the polynomial function shown below.

$$f(x)=x^3+x^2$$

The polynomial above is a *cubic binomial*, since it has degree three and contains two terms.

Example 51. Describe the type and characterization (number of terms) of the polynomial function shown below.

$$f(x)=21x^4+12x^2-3x^2-9x^2-22x^4$$

Upon simplifying, we see that the given polynomial reduces to $f(x)=-x^4$. As a result, our polynomial is a quartic (degree four) monomial (one term).

This section "sets the table" for the basic terminology that will be used throughout the chapter. In the next section, we will review some additional prerequisite factoring techniques which will be necessary for working with certain polynomials, and provide a brief summary of all factoring methods that have been discussed up to this point. Once we have finished our review of factoring, we will be ready to begin the natural (albeit lengthy) method of analyzing and graphing a polynomial function.

Sign Diagrams (L44)

Objective: Construct a sign diagram for a given polynomial expression.

If a polynomial function or expression is completely factored, it will be beneficial to us to construct a sign diagram for the polynomial, in order to answer questions about its graph and confirm any other findings. Therefore, we devote this section to the construction of a sign diagram for a factored polynomial. Note that expanded polynomials first require us to find a complete factorization prior to constructing a sign diagram. This will require us to first employ factoring techniques and possibly polynomial division, which we reserve for a later section.

Recall that the roots of a quadratic expression represent the dividers in its corresponding sign diagram. This carries over directly to a polynomial expression.

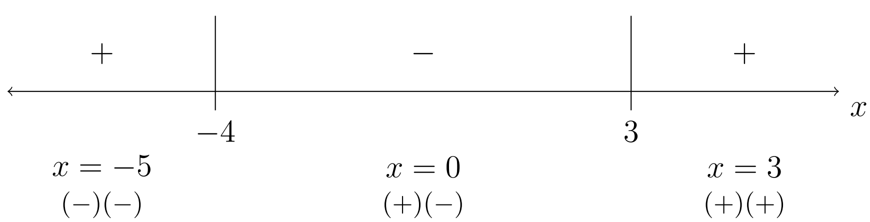
We begin with an example for quadratics.

Example 52. Construct a sign diagram for the polynomial function $f(x) = 2x^2 + 3x - 20$.

Although our first example is not factored, we can apply the *ac*-method to quickly factor our function.

$$\begin{aligned} f(x) &= 2x^2 + 3x - 20 \\ &= 2x^2 + 8x - 5x - 20 \\ &= 2x(x+4) - 5(x+4) \\ &= (x+4)(2x-5) \end{aligned}$$

This gives us two roots, $x = -4$ and $x = \frac{5}{2}$, which serve as the dividers in our accompanying diagram. For our three test values, we will use $x = -5, 0$, and 3.

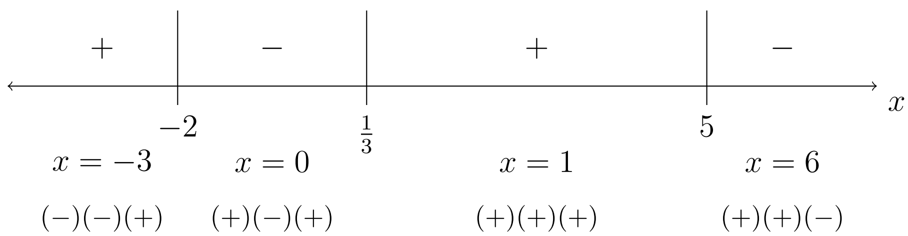


The previous example should be a familiar one, and one that we are comfortable with, since it ties in directly with the chapter on quadratics (degree-2 polynomials). For polynomials with a degree of $n \geq 3$, our diagram should look similar. The primary exceptions will be number of factors in our expression, and the number of dividers in our diagram. Again, we will focus primarily on polynomials which are already factored for our examples.

Example 53. Construct a sign diagram for the factored polynomial function

$$g(x) = (x+2)(3x-1)(5-x).$$

Our roots are $x = -2, \frac{1}{3}$, and 5. Consequently, the following diagram shows three dividers.

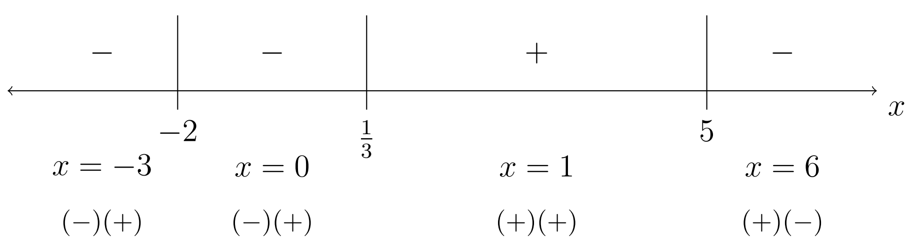


For our next example, we will make a slight change to the function g from the previous example, by including an extra factor of $x + 2$.

Example 54. Construct a sign diagram for the factored polynomial function

$$h(x) = (x+2)^2(3x-1)(5-x).$$

Since the roots of h equal those from g, our diagram will have the same dividers and test values.



In the previous diagram, we see that each of our sign calculations have excluded the $(x+2)^2$ factor, since it will always contribute a positive sign and therefore has no impact on the end result. For example, for the test value $x = -3$, we get

$$(-)^2(-)(+) = \cancel{(-)^2}(-)(+),$$

which reduces to a negative sign. This simplification in our sign calculation can be employed for any factor that appears in our function with an *even* exponent.

Additionally, our last two diagrams look almost identical, with the lone exception being the sign associated with our first interval, $(-\infty, -2)$. This should make some sense, however, since we only changed the factor associated with the root $x = -2$ from one example to the next. The reason behind the change in diagram will become more clear to us as we explore polynomials further.

For our last example, we will present both the sign diagram and the accompanying graph for the given polynomial. Although the techniques to graphing a polynomial have not yet been discussed, for any function it is often helpful to utilize a graphing utility such as Desmos, in order to better understand the makeup of the function and how its graph is related.

Example 55. Construct a sign diagram for the factored polynomial function

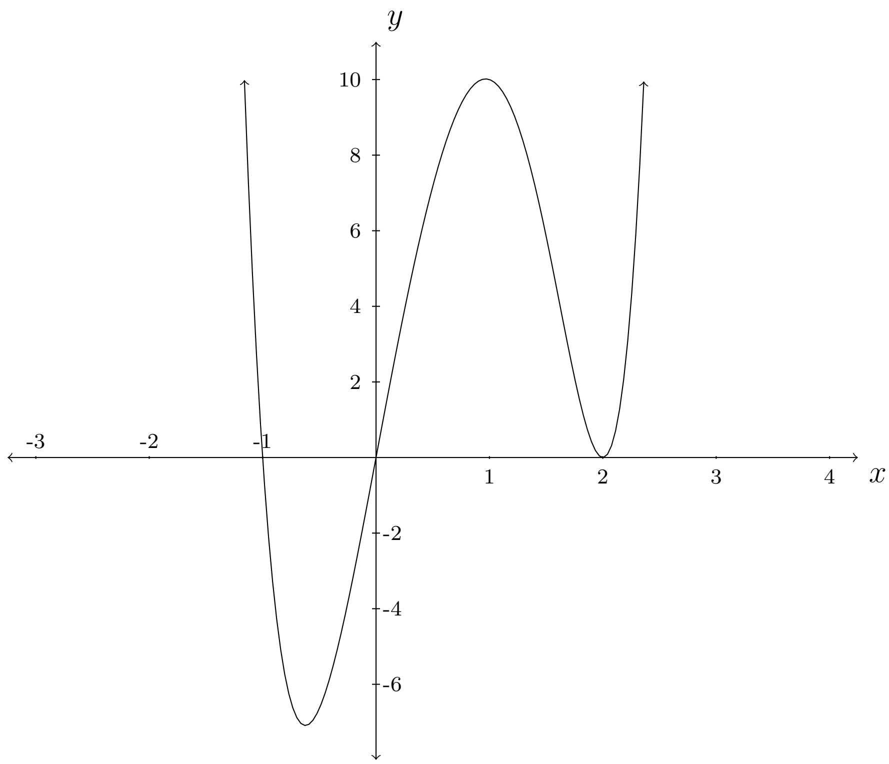
$$f(x) = x(x+1)(x-2)^2(x^2+4).$$

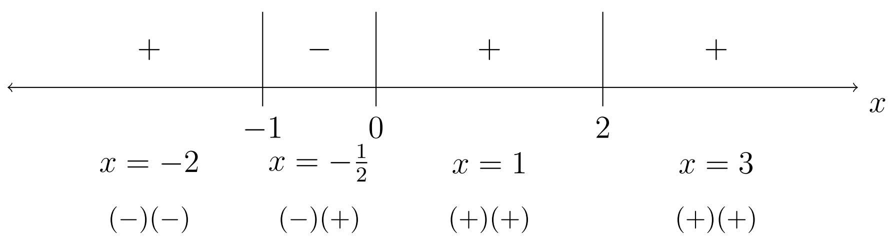
Use Desmos or a similar graphing utility to construct a graph of f.

Before we get started, it is important to spend some time discussing the factorization of f. Specifically, the factor of x will contribute a root of $x = 0$. This is the only instance in which our diagram requires a divider at $x = 0$.

Additionally, the factor of $x^2 + 4$ is often misinterpreted. By setting the expression equal to zero and solving for x, we see that the factor contributes two *imaginary* roots at $x = \pm 2i$. Furthermore, if we look more closely at this factor, we see that for any real number x, $x^2 + 4$ will always be positive. Hence, this factor will have no impact on our sign diagram calculations, and will be omitted. One should caution, however, that this factor does have an impact on the graph of f.

We can now conclude that the set of roots for f are $x = -1, 0$, and 2. The accompanying diagram and graph are shown below. As before, we have also omitted the factor of $(x-2)^2$, since the squared factor will not impact our signs.





By looking at the graph of our last example, one should begin to notice the relationship that the graph of a polynomial has with its precise makeup and, consequently, its sign diagram. In particular, close attention should be paid to the nature of the graph of f near its real roots. In the case of $x = -1$ and $x = 0$ in our last example, we see that the graph *crosses over* the $x-$axis. Alternatively, our graph *turns around* or "bounces off" at the root $x = 2$. This difference in the local behavior of the graph of f at its roots is not just a coincidence, but rather a consequence of the makeup of the function f, as we will see when we discuss the *multiplicity* of the root of a polynomial in a later section.

Factoring

Some Special Cases (L45)

Objective: Factor a general polynomial expression using one or more factorization methods.

When factoring polynomials there are a few special products that, if we can recognize them, can be easily broken down. The first is one we have seen before, when factoring some quadratics in which there is no linear term.

When expanding, we know that the product of a sum and difference of the same two terms results in a difference of two squares.

Difference of Two Squares: $a^2 - b^2 = (a+b)(a-b)$

Consequently, if faced with the difference of two squares, one can conclude that such an expression will always factor as a product of the sum and difference of their square roots. Our first four examples demonstrate this fact.

Example 56. Factor each of the given binomial expressions completely over the real numbers.

1. $x^2 - 16$
2. $9x^2 - 25y^2$
3. $x^2 - 24$
4. $2x^2 - 5$

1. In this first example, we see that $a = x$ and $b = 4$ for our difference of two squares.

$$\begin{aligned} x^2 - 16 &= (x)^2 - (4)^2 \\ &= (x+4)(x-4) \end{aligned}$$

2. Taking the square roots of $9x^2$ and $25y^2$ gives us $a = 3x$ and $b = 5y$ for our second expression.

$$\begin{aligned} 9x^2 - 25y^2 &= (3x)^2 - (5y)^2 \\ &= (3x+5y)(3x-5y) \end{aligned}$$

3. Our third expression poses a bit of a challenge, since it is the first which does not present us with the difference of two *perfect* squares. In this case, $a = x$, but $b = \sqrt{24} = \sqrt{4 \cdot 6} = 2\sqrt{6}$.

$$\begin{aligned} x^2 - 24 &= (x)^2 - \left(2\sqrt{6}\right)^2 \\ &= \left(x + 2\sqrt{6}\right)\left(x - 2\sqrt{6}\right) \end{aligned}$$

4. Similarly, our final expression presents us with two terms, neither of which are perfect squares. In this case, $a = \sqrt{2x^2} = \sqrt{2}x$ and $b = \sqrt{5}$.

$$\begin{aligned} 2x^2 - 5 &= \left(\sqrt{2}x\right)^2 - \left(\sqrt{5}\right)^2 \\ &= \left(\sqrt{2}x + \sqrt{5}\right)\left(\sqrt{2}x - \sqrt{5}\right) \end{aligned}$$

   Note that in this last case, we have $\sqrt{2}x$ (or $x\sqrt{2}$) in our factorization, and not $\sqrt{2x}$.

It is important to note that, unlike differences, a *sum* of squares will never factor over the real numbers. Such expressions only factor over the complex numbers. Hence, we say that they are *irreducible* over the reals. This can be seen in our next example, where we will attempt to employ the ac−method to factor.

Example 57. Factor the expression $x^2 + 36$ completely over the real numbers and over the complex numbers.

For the expression $x^2 + 36$, $ac = 36$ and $b = 0$, as we have no linear term. So we need to identify two integers, m and n, such that $m + n = 0$ and $m \cdot n = 36$. Our choices for $m \cdot n$ are $1 \cdot 36$, $2 \cdot 18$, $3 \cdot 12$, $4 \cdot 9$ and $6 \cdot 6$. But, since there are no combinations from these that will both multiply to 36 *and* add to 0, we conclude that the given expression is irreducible over the reals.

Notice that $x^2 + 36$ does, however, factor over the complex numbers.

$$\begin{aligned} x^2 + 36 &= x^2 - (-36) \\ &= x^2 - \left(\sqrt{-36}\right)^2 \\ &= x^2 - \left(\sqrt{36}\sqrt{-1}\right)^2 \\ &= x^2 - (6i)^2 \\ &= (x - 6i)(x + 6i) \end{aligned}$$

We can further make sense of this result by recalling the methods from the chapter on quadratics. Since the discriminant of $x^2 + 36$ is

$$\begin{aligned} b^2 - 4ac &= 0^2 - 4(1)(36) \\ &= -144 \\ &< 0, \end{aligned}$$

we know that the given expression has no real roots. Hence, any factorization must contain imaginary numbers. By setting the expression equal to zero and extracting square roots, we get $x = \pm 6i$, which further supports our factorization.

We present the general factorization for the sum of two squares over the complex numbers below.

$$\boxed{\text{Sum of Two Squares: } a^2 + b^2 = (a + bi)(a - bi)}$$

For graphing purposes, we will primarily be concerned with factorization over the real numbers.

In many cases, we can also recognize an entire expression as a perfect square (or a squared binomial).

$$\boxed{\text{Perfect Square: } a^2 + 2ab + b^2 = (a+b)^2}$$

While it might seem difficult to recognize a perfect square at first glance, by employing the $ac-$method, we can see that in the case where $m = n$, the resulting factorization will be a perfect square. In this case, we can factor by identifying the square roots of the first and last terms and using the sign from the middle term. This is demonstrated in the following example.

Example 58. Factor each of the given trinomial expressions completely over the real numbers.

1. $x^2 - 6x + 9$
2. $4x^2 + 20xy + 25y^2$

1. For our first expression, $a = 1,\ b = -6$, and $c = 9$. So we must find two integers m and n such that $m + n = -6$ and $mn = ac = 9$. In this case, the numbers we need are -3 and -3. Consequently, we will have a perfect square.

   Using the square roots of $a = 1$ and $c = 9$ and the negative sign from the linear term, our factorization is
   $$x^2 - 6x + 9 = (x-3)^2 .$$

2. For our second expression, $a = 4,\ b = 20$, and $c = 25$. So we are looking for an m and n such that $m + n = 20$ and $mn = ac = 100$. Quickly, we see that $m = n = 10$, and again, we have a perfect square.

   In this case, our factorization is
   $$4x^2 + 20xy + 25y^2 = (2x + 5y)^2 .$$

Another factoring shortcut involves sums and differences of cubes. Both sums and differences of cubes have very similar factorizations.

$$\boxed{\begin{array}{c}\text{Sum of Cubes: } a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right) \\ \text{Difference of Cubes: } a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)\end{array}}$$

As with all of the formulas in this section, we can verify those for a sum and difference of cubes by expanding the right-hand side. For example, for the difference of cubes,

$$\begin{aligned}(a-b)\left(a^2 + ab + b^2\right) &= a\left(a^2 + ab + b^2\right) - b\left(a^2 + ab + b^2\right) \\ &= a^3 + \cancel{a^2b} + \cancel{ab^2} - \cancel{a^2b} - \cancel{ab^2} - b^3 \\ &= a^3 - b^3\end{aligned}$$

Comparing the formulas one may notice that the only difference resides in the signs between the terms. One way to remember these two formulas is to think of "**SOAP**":

S	The first sign in our factorization is the **Same** sign as the given expression.
O	The second sign in our factorization is the **Opposite** sign as the given expression.
AP	The last sign in our factorization is **Always Positive**.

Example 59. Factor each of the given binomial expressions completely over the real numbers.

1. $m^3 - 27$
2. $125p^3 + 8r^3$

1. In our first expression, our desired cube roots for each term are $a = m$ and $b = 3$. Using the "Same, Opposite, Always Positive" acronym, we have the following factorization.

$$\begin{aligned} m^3 - 27 &= (m-3)\left((m)^2 + 3m + (3)^2\right) \\ &= (m-3)\left(m^2 + 3m + 9\right) \end{aligned}$$

2. In second expression, our desired cube roots for each term are

$$\begin{aligned} a &= \sqrt[3]{125p^3} \\ &= \sqrt[3]{125}\sqrt[3]{p^3} \\ &= 5p \end{aligned} \qquad \begin{aligned} b &= \sqrt[3]{8r^3} \\ &= \sqrt[3]{8}\sqrt[3]{r^3} \\ &= 2r \end{aligned}$$

Using the "Same, Opposite, Always Positive" acronym, we have the following factorization.

$$\begin{aligned} 125p^3 + 8r^3 &= (5p + 2r)\left((5p)^2 - (5p)(2r) + (2r)^2\right) \\ &= (5p + 2r)\left(25p^2 - 10pr + 4r^2\right) \end{aligned}$$

The second expression in our last example illustrates an important point. When we identify the first and last terms of the trinomial in our factorization, we must square each cube root in its entirety. In this case, both the coefficients and variables are squared, so that $(5p)^2$ becomes $25p^2$, and $(2r)^2$ becomes $4r^2$.

After factoring a sum or difference of cubes, it should be natural to attempt to factor the resulting trinomial expression (our second factor). As a general rule, however, this factor should always be irreducible over the reals, with the main exception being that of a GCF in the given expression that might have been initially overlooked.

Our last special case comes up frequently enough that we will devote the next subsection to it.

Quadratic Type (L46)

Objective: Recognize and factor a polynomial expression of quadratic type.

Recall that a quadratic expression in terms of a variable x is an expression of the form

$$ax^2 + bx + c.$$

If y is any algebraic expression, we say that the expression

$$ay^2 + by + c$$

is an expression of *quadratic type.*

In just about every case we will see, we will consider y as a power of x, $y = x^n$, so that our expression of quadratic type will appear as follows.

Quadratic Type:

$$ax^{2n} + bx^n + c \;=\; a\left[x^n\right]^2 + b\left[x^n\right] + c$$

If $y = x^3$, then the expression

$$ay^2 + by + c = ax^6 + bx^3 + c$$

would be an expression of quadratic type.

Similarly, if $y = x^4$, then the expression

$$ay^2 + by + c = ax^8 + bx^4 + c$$

would be an expression of quadratic type.

In each of these last two examples, notice the exponential pattern, where the middle term has an exponent that is half that of the leading term's. This will always be apparent, as long as the middle coefficient b is nonzero.

By viewing certain expressions as quadratic type, we can often apply more familiar methods, such as the $ac-$method, to obtain a complete factorization.

For example, if we let $y = x^2$, then the difference of fourth powers $x^4 - 16$ can be rewritten as a difference of squares, $y^2 - 4^2$, leading us to the complete factorization over the real numbers shown below.

$$\begin{aligned} x^4 - 16 &= \left(x^2\right)^2 - 4^2 \\ &= y^2 - 4^2, \quad y = x^2 \\ &= (y+4)(y-4) \\ &= \left(x^2+4\right)\left(x^2-4\right) \\ &= \left(x^2+4\right)(x+2)(x-2) \end{aligned}$$

Example 60. Factor the trinomial expression x^4+2x^2-24 completely over the real numbers.

Notice that the given trinomial exhibits quadratic type characteristics, since the degree of four is double the exponent appearing in the middle term. Consequently, we will let $y = x^2$ and rewrite the expression in terms of y.

$$y^2 + 2y - 24$$

Applying the $ac-$method, we see the following.

$$\begin{aligned} y^2 + 2y - 24 &= y^2 + 6y - 4y - 24 \\ &= y(y+6) - 4(y+6) \\ &= (y+6)(y-4) \end{aligned}$$

Substituting back for x, we have $(x^2+6)(x^2-4)$. The first factor is a sum of squares, which is irreducible over the reals. The second factor of $x^2 - 4$ is a difference of perfect squares, which we know is factorable as $(x+2)(x-2)$.

Our final factorization is

$$x^4 + 2x^2 - 24 = \left(x^2+6\right)(x+2)(x-2).$$

Example 61. Find all real roots of the polynomial expression $x^4 - 12x^2 + 27$.

In this example, we are not asked to factor the given expression, but instead to solve for when the expression equals zero. Still, we can start by finding a complete factorization, again substituting $y = x^2$ and employing the $ac-$method.

$$\begin{aligned} x^4 - 12x^2 + 27 &= y^2 - 12y + 27 \\ &= y^2 - 3y - 9y + 27 \\ &= y(y-3) - 9(y-3) \\ &= (y-3)(y-9) \\ &= \left(x^2-3\right)\left(x^2-9\right) \\ &= \left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)(x-3)(x+3) \end{aligned}$$

Here, we see that after using the $ac-$method and substituting back for x, we end up with two quadratic factors which can *both* be factored as a difference of squares. In the case of the first factor, $x^2 - 3$, our factorization requires a square root, since 3 is not a perfect square.

Setting each of the four factors equal to zero gives us our set of real roots, $\{\pm\sqrt{3}, \pm 3\}$.

In each of our last two examples, we have seen a degree-four polynomial having two and four real roots, respectively. We can also easily identify degree-four polynomials having no, one, or three real roots. The expression $x^4 + x^2 + 1$, for example, factors as $\left(x^2+1\right)^2$, which has only complex roots at $\pm i$. In general, a degree-n polynomial can have as few as zero and as many as n unique real roots. This is a fact which we will more formally state in a later section, once we have discussed the *multiplicity* of a root.

The following example should look familiar.

Example 62. Factor each of the following polynomial expressions completely over the real numbers.

1. $x^8 + 2x^4 - 24$
2. $x^6 + 2x^3 - 24$

Before we begin, notice that the coefficients for each of the given expressions match those in Example 60, with the only difference being the exponents appearing in each expression.

1. Despite the fact that the first expression has a higher degree, its factorization will be simpler than the second expression's. In this case, we will let $y = x^4$, and apply the $ac-$method as before.

$$\begin{aligned} x^8 + 2x^4 - 24 &= \left(x^4\right)^2 + 2\left(x^4\right) - 24 \\ &= y^2 + 2y - 24 \\ &= (y+6)(y-4) \\ &= \left(x^4+6\right)\left(x^4-4\right) \end{aligned}$$

   Though it might not be obvious, our first factor $x^4 + 6$ is in fact irreducible over the real numbers. One way we can realize this is to think of $x^4 + 6$ as a vertical shift of the graph of x^4 up six units. The resulting graph will lie entirely in the upper-half of the $xy-$plane, and therefore will not intersect the $x-$axis. Hence, the factor of $x^4 + 6$ will have no real roots, and consequently any factorization will involve the introduction of imaginary numbers. Alternatively, one might also notice that raising any real number to the fourth power and adding six will never produce an output of zero, leading us to again conclude that the expression has no real roots. Lastly, we could also recognize $x^4 + 6$ as a sum of squares, namely $(x^2)^2 + \left(\sqrt{6}\right)^2$, which we have already discussed as one expression type that is irreducible over the real numbers.

   On the other hand, we can view our second factor $x^4 - 4$ as a difference of two squares, and factor it as follows.

$$\begin{aligned} x^4 - 4 &= \left(x^2\right)^2 - 2^2 \\ &= \left(x^2+2\right)\left(x^2-2\right) \\ &= \left(x^2+2\right)\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right) \end{aligned}$$

   Our complete factorization over the reals is then

$$x^8 + 2x^4 - 24 = \left(x^4+6\right)\left(x^2+2\right)\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right).$$

2. In the case of the second expression, if we let $y = x^3$, we start out with the same two factors for y, which we rewrite as

$$\left(x^3+6\right)\left(x^3-4\right).$$

Although neither 6 nor 4 are perfect cubes, we can still break down each of the factors above by using the formulas for the sum and difference of cubes from earlier in the section. For our first factor, letting $a = x$ and $b = \sqrt[3]{6}$, we can write $x^3 + 6$ as

$$(a+b)\left(a^2 - ab + b^2\right) = \left(x + \sqrt[3]{6}\right)\left(x^2 - \sqrt[3]{6}x + \left(\sqrt[3]{6}\right)^2\right).$$

Similarly, for the second factor, if $a = x$ and $b = \sqrt[3]{4}$, we can write $x^3 - 4$ as

$$(a-b)\left(a^2 + ab + b^2\right) = \left(x - \sqrt[3]{4}\right)\left(x^2 + \sqrt[3]{4}x + \left(\sqrt[3]{4}\right)^2\right).$$

Our complete factorization over the reals is then

$$\begin{aligned} x^6 + 2x^3 - 24 &= \left(x^3 + 6\right)\left(x^3 - 4\right) \\ &= \left(x + \sqrt[3]{6}\right)\left(x^2 - \sqrt[3]{6}x + \left(\sqrt[3]{6}\right)^2\right)\left(x - \sqrt[3]{4}\right)\left(x^2 + \sqrt[3]{4}x + \left(\sqrt[3]{4}\right)^2\right). \end{aligned}$$

We end the subsection on quadratic type with one final example.

Example 63. Factor each polynomial expression completely over the reals and find its set of real roots.

1. $x^4 - 49$
2. $x^6 - 4x^3 - 5$

1. Setting $y = x^2$, we can quickly factor our first expression as a difference of squares, breaking down one of its factors in a similar manner.

$$\begin{aligned} x^4 - 49 &= \left(x^2\right)^2 - 49 \\ &= y^2 - (7)^2 \\ &= (y+7)(y-7) \\ &= \left(x^2 + 7\right)\left(x^2 - 7\right) \\ &= \left(x^2 + 7\right)\left(x + \sqrt{7}\right)\left(x - \sqrt{7}\right) \end{aligned}$$

From our two linear factors, we obtain $x = \pm 7$ as our two real roots.

2. For our second expression, setting $y = x^3$, we apply the $ac-$method.

$$\begin{aligned} x^6 - 4x^3 - 5 &= \left(x^3\right)^2 - 4\left(x^3\right) - 5 \\ &= y^2 - 4y - 5 \\ &= (y+1)(y-5) \\ &= \left(x^2 + 1\right)\left(x^2 - 5\right) \\ &= \left(x^2 + 1\right)\left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right) \end{aligned}$$

Our two linear factors give us $x = \pm\sqrt{5}$ as the real roots for our given expression.

Division

Polynomial (Long) Division (L47)

Objective: Apply polynomial division to a rational expression.

Up until this point, every polynomial expression that we have encountered has either already been provided in its factored form or is easily factorable using one or more of the many techniques that we have learned. We must be careful, however, to consider the very likely possibility that a given polynomial is not factorable using elementary methods (GCF, grouping, $ac-$method, etc.). In many cases, obtaining a complete factorization can prove almost impossible without the aid of mathematical computating software. Although, there is still one powerful tool that can help us to dissect certain factorable, yet formidable, polynomials. This tool is known as the *Rational Root Theorem*, and we will see its use in a later section.

In order to successfully employ the Rational Root Theorem, we first must understand polynomial division. As we will see, dividing polynomials is a process very similar to long division of whole numbers.

Before we begin with our first example, let's recall the terminology and format associated with division.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

Alternatively, multiplying both sides of the above equation by the divisor, we have the following.

$$\frac{\text{dividend}}{\cancel{\text{divisor}}} \cdot \cancel{\text{divisor}} = \text{quotient} \cdot \text{divisor} + \frac{\text{remainder}}{\cancel{\text{divisor}}} \cdot \cancel{\text{divisor}}$$
$$\text{dividend} = \text{quotient} \cdot \text{divisor} + \text{remainder}$$

We begin with dividing a polynomial by a monomial, which simply utilizes the distributive property. In the following example, we can think of the stated division as a distribution of the divisor (denominator) to each term in the dividend (numerator).

Example 64. Divide and simplify the following expressions.

1. $\dfrac{9x^5 + 6x^4 - 18x^3 - 24x^2}{3x^2}$
2. $\dfrac{8x^3 + 4x^2 - 2x + 6}{4x^2}$

1. By distributing the expression $\dfrac{1}{3x^2}$ to each of the four terms in the numerator, our first expression becomes

$$\frac{9x^5 + 6x^4 - 18x^3 - 24x^2}{3x^2} = \frac{9x^5}{3x^2} + \frac{6x^4}{3x^2} - \frac{18x^3}{3x^2} - \frac{24x^2}{3x^2}.$$

We can then reduce each individual quotient to produce the following expression.

$$3x^3 + 2x^2 - 6x - 8$$

In this case, our answer is $3x^3 + 2x^2 - 6x - 8$, and we can summarize our results as follows.

$$\frac{9x^5 + 6x^4 - 18x^3 - 24x^2}{3x^2} = 3x^3 + 2x^2 - 6x - 8$$

In this first example, our expression reduced completely, producing our quotient polynomial expression and a remainder of zero.

2. Again, we will begin with the second expression by splitting it up, or distributing the denominator to each of the three terms in the numerator. Reducing each individual quotient gives us our answer.

$$\begin{aligned}\frac{8x^3 + 4x^2 - 2x + 6}{4x^2} &= \frac{8x^3}{4x^2} + \frac{4x^2}{4x^2} - \frac{2x}{4x^2} + \frac{6}{4x^2} \\ &= 2x + 1 - \frac{1}{2x} + \frac{3}{2x^2}\end{aligned}$$

Unlike our first expression, here our expression does not reduce completely, i.e., it contains a nonzero remainder. Because of this, since our answer includes two rational (or fractional) expressions, one could also combine them to form a single rational expression.

$$\begin{aligned}\frac{8x^3 + 4x^2 - 2x + 6}{4x^2} &= 2x + 1 - \frac{1}{2x} \cdot \frac{x}{x} + \frac{3}{2x^2} \\ &= 2x + 1 - \frac{x}{2x^2} + \frac{3}{2x^2} \\ &= 2x + 1 + \frac{3 - x}{2x^2}\end{aligned}$$

Furthermore, if we wanted to identify the remainder in this example, we could multiply both sides of the equation by the divisor, $4x^2$.

$$\begin{aligned}\frac{8x^3 + 4x^2 - 2x + 6}{\cancel{4x^2}} \cdot \cancel{4x^2} &= (2x+1) \cdot 4x^2 + \left(\frac{3 - x}{2\cancel{x^2}}\right) \cdot 4\cancel{x^2} \\ \underbrace{8x^3 + 4x^2 - 2x + 6}_{\text{dividend}} &= \underbrace{(2x+1)}_{\text{quotient}}\, \underbrace{4x^2}_{\text{divisor}} + \underbrace{6 - 2x}_{\text{remainder}}\end{aligned}$$

Lastly, one final observation to point out from this particular answer is the initial reduction of the second term, $\frac{4x^2}{4x^2}$, which equals one (not zero), and therefore does not simply disappear from the expression altogether.

For division by polynomial expressions that contain more than a single term, long division is usually required. To illustrate the relationship between polynomial division and standard numerical long division, an example with whole numbers is provided in order to review the (general) steps that will also be used for polynomial long division.

Example 65. Divide 631 by 4.

$$
\begin{array}{r}
157\\
4\,\overline{)\,631}\\
\underline{400}\\
231\\
\underline{200}\\
31\\
\underline{28}\\
3
\end{array}
$$

Recalling the process associated with long division, we begin to construct our quotient (157) by comparing the divisor (4) with the largest placeholder of our dividend (631), then subtracting, and repeating this process until we have worked our way down to the ones digit. We know we have finished, once we are left with a remainder (3) that is less than our divisor.

Expressing our answer in the form

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}},$$

we have $\dfrac{631}{4} = 157 + \dfrac{3}{4}$. Or, in the alternate form

$$\text{dividend} = \text{quotient} \cdot \text{divisor} + \text{remainder},$$

we have $631 = 145 \cdot 4 + 3$.

The general process for division of polynomials follows closely with that for dividing integers. The only real difference is in the terminology that we use: *term* in place of *number/digit*, and *degree* instead of *value*.

General Steps for Polynomial (Long) Division

Let $D(x)$ and $d(x)$ represent two nonzero polynomial functions. The steps for simplifying the rational expression $\dfrac{D(x)}{d(x)}$ are as follows.

1. Divide the leading term of the dividend D by the leading term of the divisor d. Label the resulting term $a_n x^n$, and write it above the dividend. This will be the leading term of the quotient, $q(x)$.
2. Multiply $a_n x^n$ by the divisor, distribute, and simplify. Label this as $d_1(x)$ and write it directly below the dividend, D, making sure to align terms according to exponents.
3. Subtract the resulting terms from the dividend. Label the new expression D_1.
4. Repeat steps (1)-(3) for the divisor d and the new expression D_i until the degree of D_i is *less than* the degree of the divisor. Relabel the final new dividend as the remainder, $r(x)$. The entire polynomial expression appearing above the original dividend is the quotient, $q(x)$.

$$\begin{array}{rl} & \underline{\quad q(x) \quad} \\ d(x)\,\big) & D(x) \\ - & \underline{d_1(x)} \\ & \quad D_1(x) \\ - & \quad \underline{d_2(x)} \\ & \qquad \ddots \\ - & \qquad \underline{d_i(x)} \\ & \qquad D_i(x) = r(x) \end{array}$$

Step (3) above often tends to pose the greatest challenge for students. It is important to keep in mind that we are are always subtracting the top term from the bottom term, which is why we must change the signs of the term(s) on the bottom. In most cases, we will need to utilize the distributive property.

A basic example should clear up any confusion, and we begin by revisiting Example 64.

Example 66. Divide $9x^5 + 6x^4 - 18x^3 - 24x^2$ by $3x^2$. Simplify and express your answer in the form

$$\frac{D(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

$$3x^2\overline{\big)\; 9x^5 + 6x^4 - 18x^3 - 24x^2}$$

We set up our division process by first writing the dividend and the divisor in the appropriate locations.

$$\begin{array}{r|l} & \quad 3x^3 \\ \hline 3x^2 & 9x^5 + 6x^4 - 18x^3 - 24x^2 \end{array}$$

Next, we identify the leading term for our quotient, $3x^3$.

$$\begin{array}{rl} & \qquad\qquad 3x^3 \\ \cline{2-2} 3x^2 \big) & 9x^5 + 6x^4 - 18x^3 - 24x^2 \\ & \underline{-9x^5} \\ & \qquad 6x^4 \end{array}$$

Multiplying and subtracting produces our new expression, $D_1(x) = 6x^4$.

Notice that in this example we need only carry down the remaining terms from the dividend when our new expression contains terms which are alike to them. In this case, for example, while it is perfectly fine to carry down $-18x^3 - 24x^2$, it is not necessary until these terms play a role in the subtraction step.

$$\begin{array}{rl} & \qquad\qquad 3x^3 \; + 2x^2 \\ \cline{2-2} 3x^2 \big) & 9x^5 + 6x^4 - 18x^3 - 24x^2 \\ & \underline{-9x^5} \\ & \qquad 6x^4 \\ & \qquad \underline{-6x^4} \\ & \qquad\qquad -18x^3 \end{array}$$

Repeating our division steps gives us the second term in our quotient, $2x^2$.

Multiplying this term by the divisor, subtracting, and carrying down the next term in the dividend finishes up the second round of our steps for division. Our new expression is $D_2(x) = -18x^3$.

$$\begin{array}{rl} & \qquad\qquad 3x^3 \; + 2x^2 - 6x \\ \cline{2-2} 3x^2 \big) & 9x^5 + 6x^4 - 18x^3 - 24x^2 \\ & \underline{-9x^5} \\ & \qquad 6x^4 \\ & \qquad \underline{-6x^4} \\ & \qquad\qquad -18x^3 \\ & \qquad\qquad \underline{\;\;\; 18x^3} \\ & \qquad\qquad\qquad -24x^2 \end{array}$$

Again, we repeat our division steps to produce the third term in our quotient, $-6x$.

Multiplying and subtracting produces another new expression, $D_3(x) = -24x^2$. Since the degree of D_3 equals that of our divisor, $d(x) = 3x^2$, we will need to apply our steps for division one final time.

$$\begin{array}{rl} & \qquad\qquad 3x^3 \; + 2x^2 - 6x - 8 \\ \cline{2-2} 3x^2 \big) & 9x^5 + 6x^4 - 18x^3 - 24x^2 \\ & \underline{-9x^5} \\ & \qquad 6x^4 \\ & \qquad \underline{-6x^4} \\ & \qquad\qquad -18x^3 \\ & \qquad\qquad \underline{\;\;\; 18x^3} \\ & \qquad\qquad\qquad -24x^2 \\ & \qquad\qquad\qquad \underline{\;\;\; 24x^2} \\ & \qquad\qquad\qquad\qquad 0 \end{array}$$

After our fourth and final round of steps, our new expression produces a remainder of $r(x) = 0$. This should come as no real surprise, based upon our earlier calculations from Example 64. Many examples that we will encounter after this first one will not work out as nicely.

We express the results of our division in the required form as follows.

$$\frac{9x^5 + 6x^4 - 18x^3 - 24x^2}{3x^2} = 3x^3 + 2x^2 - 6x - 8 + \frac{0}{3x^2}$$

Because our divisor in this last example is a monomial, our steps do not require the distributive property. The next example includes a binomial divisor, which will emphasize the importance of the distributive property in the division process.

Example 67. Divide $3x^3 - 5x^2 - 32x + 7$ by $x - 4$. Simplify and express your answer in the form

$$\frac{D(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

$$
\begin{array}{rrrrr}
 & 3x^2 & +7x & -4 & \\
\cline{2-5}
x-4 \big) & 3x^3 & -5x^2 & -32x & +7 \\
 & -3x^3 & +12x^2 & & \\
\cline{2-3}
 & & 7x^2 & -32x & \\
 & & -7x^2 & +28x & \\
\cline{3-4}
 & & & -4x & +7 \\
 & & & 4x & -16 \\
\cline{4-5}
 & & & & -9
\end{array}
$$

For this second example, we have opted to show the entire division process all at once. Since our quotient,

$$3x^2 + 7x - 4,$$

is a trinomial, we must apply the steps for division three times.

In this second example, our remainder is the constant term $r(x) = -9$, which has one degree less than our linear divisor, $d(x) = x - 4$.

Our answer is $\dfrac{3x^3 - 5x^2 - 32x + 7}{x - 4} = 3x^2 + 7x - 4 + \dfrac{-9}{x - 4}$.

If we look at the first round of long division steps, we see a leading term in our quotient of $3x^2$. Multiplying by our divisor produces the following binomial expression.

$$3x^2(x - 4) = 3x^3 - 12x^2$$

Since we must subtract *each* term of this expression from our dividend, however, we see that the distributive property has already been applied. Specifically,

$$-(3x^3 - 12x^2) = -3x^3 + 12x^2.$$

What remains now is to simply add each term to the respective term in the dividend, which, if employed correctly, should *always* eliminate the leading term in each round of steps.

Failure to distribute a negative across *all* terms in this critical step will always lead to an incorrect quotient and remainder. This is the most common mistake in the polynomial division process, and is one that often takes a long amount of time to correct. Consequently, checking one's work is critical throughout the division process. We emphasize these points in this first example where distribution of a negative is required, in the hopes that it is not overlooked.

We continue with another example.

Example 68. Divide $6x^3 - 8x^2 + 10x + 103$ by $4 + 2x$. Simplify and express your answer in the form

$$\frac{D(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

Before we begin, we wish to point out the important prerequisite for polynomial division that all expressions be written in *descending power order*. In this case, we will start out by rewriting our divisor as $2x+4$.

$$\begin{array}{r|l} & 3x^2 - 10x \quad + 25 \\ \hline 2x+4 & 6x^3 \quad - 8x^2 + 10x + 103 \\ & -6x^3 - 12x^2 \\ \hline & -20x^2 + 10x \\ & 20x^2 + 40x \\ \hline & 50x + 103 \\ & -50x - 100 \\ \hline & 3 \end{array}$$

This example is similar to the previous one. Specifically, the divisor is a linear binomial, and the dividend has a degree of 3.

Consequently, the steps for polynomial division are employed three times, yielding a constant remainder.

Our answer should be expressed as follows.

$$\frac{6x^3 - 8x^2 + 10x + 103}{2x+4} = 3x^2 - 10x + 25 + \frac{3}{2x+4}$$

In each of the previous two examples the dividend contained only nonzero coefficients. In other words, no term was "skipped over" in the expression for $D(x)$. Our last example will address the importance of keeping track of polynomial *place holders*, in the event that a specific term carries with it a zero coefficient, and is therefore omitted from the original expression for the dividend, $D(x)$.

Our last example demonstrates the importance of these preliminary steps.

Example 69. Divide and simplify the given expression.

$$\frac{2x^4 + 42x - 4x^2}{x^2 + 3x} \qquad \text{Reorder dividend; need } x^3 \text{ term, add } 0x^3$$

$$x^2 + 3x \;\overline{)\; 2x^4 + \mathbf{0x^3} - 4x^2 + 42x} \qquad \text{Divide the leading terms: } \frac{2x^4}{x^2} = 2x^2$$

$$\begin{array}{rl} & \mathbf{2x^2} \\ x^2+3x & \overline{)\; 2x^4 + 0x^3 - 4x^2 + 42x} \\ & \underline{\mathbf{-2x^4 - 6x^3}} \\ & \quad \mathbf{-6x^3 - 4x^2} \end{array} \qquad \begin{array}{l} \text{Multiply this term by divisor: } 2x^2(x^2+3x) = 2x^4 + 6x^3 \\ \text{Subtract, changing terms} \\ \text{Bring down the next term, } -4x^2 \end{array}$$

$$\begin{array}{rl} & 2x^2\ \mathbf{-6x} \\ x^2+3x & \overline{)\; 2x^4 + 0x^3 - 4x^2 + 42x} \\ & \underline{-2x^4 - 6x^3} \\ & \quad -6x^3 - 4x^2 \\ & \quad \underline{\mathbf{+6x^3 + 18x^2}} \\ & \qquad \mathbf{14x^2 + 42x} \end{array} \qquad \begin{array}{l} \text{Repeat, divide new leading term by } x^2: \frac{-6x^3}{x^2} = -6x \\ \\ \text{Multiply this term by divisor: } -6x(x^2+3x) = -6x^3 - 18x^2 \\ \text{Subtract, changing signs} \\ \text{Bring down the next term, } 42x \end{array}$$

$$
\begin{array}{rl}
2x^2 - 6x\ \mathbf{+14} & \\
x^2+3x\ \overline{)\ 2x^4 + 0x^3 - 4x^2 + 42x} & \text{Repeat, divide new leading term by } x^2 : \dfrac{14x^2}{x^2} = 14 \\
\underline{-2x^4 - 6x^3} & \\
-6x^3 - 4x^2 & \\
\underline{+6x^3 + 18x^2} & \\
14x^2 + 42x & \text{Multiply this term by the divisor} : 14(x^2+3x) = 14x^2 + 42x \\
\underline{\boldsymbol{-14x^2 - 42x}} & \text{Subtract, changing signs} \\
0 & \text{Zero remainder} \\
& \\
2x^2 - 6x + 14 & \text{Our solution}
\end{array}
$$

So we have,

$$\frac{2x^4 - 4x^2 + 42x}{x^2 + 3x} = 2x^2 - 6x + 14$$

It is important to take a moment to check each problem, to verify that the exponents decrease incrementally and that none are skipped.

This final example also illustrates that, just as with classic numerical long division, sometimes our remainder will be zero.

Synthetic Division (L48)

Objective: Apply synthetic division to a rational expression.

Next, we will introduce a method of division that can be used to streamline the polynomial division process and is often preferred over the more traditional long division method. This method, known as *synthetic division*, although usually quicker than traditional polynomial division, this alternative method can only be implemented when the given divisor is *linear*. Specifically, we will require $d(x)$ to be of the form $x - c$.

For our first example, we will divide $x^3 + 4x^2 - 5x - 14$ by $x - 2$, which one can check will produce a quotient of $x^2 + 6x + 7$ and a remainder of zero using polynomial long division.

$$\frac{x^3 + 4x^2 - 5x - 14}{x - 2} = x^2 + 6x + 7$$

The method of synthetic division focuses primarily on the coefficients of both the divisor and dividend. We must still pay careful attention, however, to the powers of our exponents, which will serve as placeholders throughout the process. To start the process, we will write our coefficients in what we will refer to as a *synthetic division tableau* prior to dividing.

To divide $x^3 + 4x^2 - 5x - 14$ by $x - 2$, we first write 2 in the place of the divisor since 2 is zero of the factor $x - 2$ and we write the coefficients of $x^3 + 4x^2 - 5x - 14$ in for the dividend.

As our next step, we 'bring down' the first coefficient of the dividend. We will then multiply and add repeatedly.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & & & \\ \hline & & & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & & & \\ \hline & 1 & & & \end{array}$$

Next, take the 2 from the divisor and multiply by the 1 that was brought down to get 2. Write this underneath the 4, then add to get 6.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & & \\ \hline & 1 & & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & & \\ \hline & 1 & 6 & & \end{array}$$

Now multiply the 2 from the divisor by the 6 to get 12, and add it to the -5 to get 7.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & 12 & \\ \hline & 1 & 6 & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & 12 & \\ \hline & 1 & 6 & 7 & \end{array}$$

Finally, multiply the 2 in the divisor by the 7 to get 14, and add it to the -14 to get 0.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \mathbf{0} \end{array}$$

The first three numbers in the last row of our tableau will be the coefficients of the desired quotient polynomial. Remember, we started with a third degree polynomial and divided by a first degree polynomial, so the quotient will be a second degree polynomial. Hence the quotient is $x^2 + 6x + 7$. The number in bold represents the remainder, which is zero in this case.

Due in large part to its speed, synthetic division is often a 'tool of choice' for dividing polynomials by divisors of the form $x - c$. It is important to reiterate that synthetic division will *only* work for these kinds of divisors (linear divisors with leading coefficient 1), and we will need to use polynomial long division for divisors having degree larger than 1.

Another observation worth mentioning is that when a polynomial (of degree at least 1) is divided by $x - c$, the result will be a quotient polynomial of exactly one less degree than the original polynomial. This is a direct result of the divisor being a linear expression.

For a more complete understanding of the relationship between long and synthetic division, students are encouraged to trace each step in synthetic division back to its corresponding step in long division.

We conclude this section with three examples using synthetic division. We will summarize each example using the form below.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

Example 70. Use synthetic division to perform the following polynomial division. Find the quotient and the remainder polynomials.

$$\frac{5x^3 - 2x^2 + 1}{x - 3}$$

When setting up the synthetic division tableau, we need to enter 0 for the coefficient of x in the dividend as a placeholder, just like in polynomial division.

Setting up and working through the tableau gives us the following result.

$$\begin{array}{r|rrrr} 3 & 5 & -2 & 0 & 1 \\ & & 15 & 39 & 117 \\ \hline & 5 & 13 & 39 & \mathbf{118} \end{array}$$

Since the dividend was a third degree polynomial, the quotient is a quadratic polynomial with coefficients 5, 13 and 39. Our quotient is then $q(x) = 5x^2 + 13x + 39$ and the remainder is $r(x) = 118$.

Putting this all together, we have the following equation.

$$\frac{5x^3 - 2x^2 + 1}{x - 3} = 5x^2 + 13x + 39 + \frac{118}{x - 3}$$

Example 71. Use synthetic division to perform the following polynomial division. Find the quotient and the remainder polynomials.

$$\frac{x^3 + 8}{x + 2}$$

For this division, since we have a factor of $x + 2$, we must use the zero of $x = -2$ to begin. Here, we will once again stress that it is critical to take the time in order to ensure we have set the synthetic division tableau up correctly at the onset of the problem. Failure to do so will result in an incorrect answer, as well as a considerable amount time spent re-doing the problem.

$$\begin{array}{r|rrrr} -2 & 1 & 0 & 0 & 8 \\ & & -2 & 4 & -8 \\ \hline & 1 & -2 & 4 & \mathbf{0} \end{array}$$

We then obtain a quotient of $q(x) = x^2 - 2x + 4$ and remainder of $r(x) = 0$. This gives us the following equation.

$$\frac{x^3 + 8}{x + 2} = x^2 - 2x + 4$$

This answer is a great reminder of the factoring rules for cubic polynomials that we outlined earlier in the chapter.

Example 72. Use synthetic division to perform the following polynomial division. Find the quotient and the remainder polynomials.

$$\frac{4-8x-12x^2}{2x-3}$$

To divide $4-8x-12x^2$ by $2x-3$, two things must be done. First, we write the dividend in descending powers of x as $-12x^2-8x+4$. Second, since synthetic division works only for factors of the form $x-c$, we factor $2x-3$ as $2\left(x-\frac{3}{2}\right)$. Our strategy is to first divide $-12x^2-8x+4$ by 2, to get $-6x^2-4x+2$. Next, we divide by $x-\frac{3}{2}$. The tableau becomes

$$\begin{array}{c|ccc} \frac{3}{2} & -6 & -4 & 2 \\ & & -9 & -\frac{39}{2} \\ \hline & -6 & -13 & -\mathbf{\frac{35}{2}} \end{array}$$

From this, we get a quotient of $q(x)=-6x-13$ and a remainder of $r(x)=-\frac{35}{2}$. This gives us the following equation.

$$\frac{-6x^2-4x+2}{x-\frac{3}{2}} = -6x-13+\frac{-\frac{35}{2}}{x-\frac{3}{2}}$$

Multiplying both sides by of our equation by $\frac{2}{2}$ and distributing gives us our desired answer.

$$\frac{-12x^2-8x+4}{2x-3} = -6x-13+\frac{-35}{2x-3}$$

Note that we could also multiply both sides of our last equation by $2x-3$ to obtain the following equation.

$$-12x^2-8x+4=(2x-3)(-6x-13)-35$$

While both of the forms above are certainly equivalent, the previous one may remind us of the familiar classic division algorithm for integers, shown below.

$$\text{dividend} = (\text{divisor})\cdot(\text{quotient}) + \text{remainder}$$

The first form, however, will be particularly useful when we graph more complicated rational functions in the next chapter.

End Behavior (L49)

Objective: Identify and describe the end behavior of the graph of a polynomial function.

The *end behavior* of any function refers to what happens near the extreme ends of its graph. We also often refer to these as the "tails" of the graph. The ends of the graph of a function correspond to points having large positive or negative $x-$coordinates. Because of this, we can associate the expressions

$$x \to \infty \qquad \text{and} \qquad x \to -\infty$$

to the end behavior of a function. For example, the sentence

$$\text{As } x \to \infty, \;\; f(x) \to \infty.$$

describes a function for which the right-hand side of its graph, i.e. when $x \to \infty$, points upward. Alternatively, the sentence

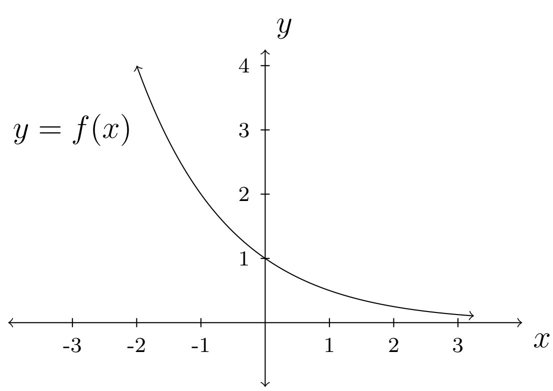
$$\text{As } x \to \infty, \;\; f(x) \to -\infty.$$

describes a function for which the right-hand side of its graph points downward.

In each of the above mathematical statements, we are identifying both a horizontal direction and a vertical direction:

1. the independent variable x getting large (either positively or negatively),
2. and the effect this has on the values of $f(x)$.

Example 73. Describe the end behavior of the function f whose graph is shown below.



Although this graph is one that we typically see in a precalculus setting (known as an exponential function), we can still discuss its end behavior. In this case, as the values of x increase, we see that the points on the graph approach the $x-$axis. This translates to the following statement.

$$\text{As } x \to \infty, \;\; f(x) \to 0.$$

On the other hand, as the values of x tend towards $-\infty$, we see that the $y-$coordinates for their respective points continue to increase. Hence, we can say the following.

$$\text{As } x \to -\infty, \;\; f(x) \to \infty.$$

Prior to this chapter, we have not had much need to discuss end behavior at great length, since most of the functions which we have been exposed to have been relatively easily diagnosed and graphed. A quadratic function, $f(x) = ax^2 + bx + c$, for example, will either open up or down, depending on the sign of the leading coefficient, a. As we begin to graph polynomials, however, we will see our graphs take more than a few turns, which will require us to have a better understanding about the nature of their tails.

For each algebraic function, the corresponding graph will describe two such statements: one for the left-hand side of the graph ($x \to -\infty$) and one for the right-hand side of the graph ($x \to \infty$). In the case of polynomials, there are only four cases for these two statements, summarized as follows.

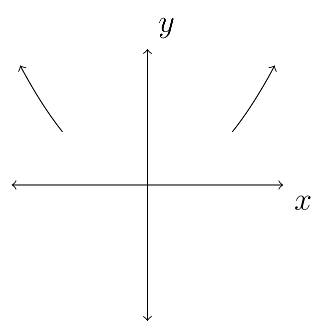
Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$$

be a polynomial function with degree n and nonzero leading coefficient a_n.

The end behavior of f is described by one of the following four cases.

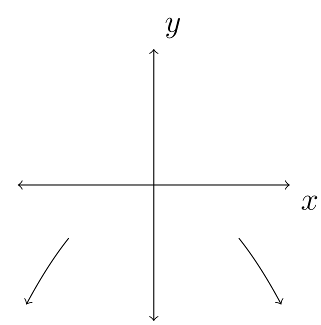
I. n even, $a_n > 0$



As $x \to \infty,\ f(x) \to \infty$

As $x \to -\infty,\ f(x) \to \infty$

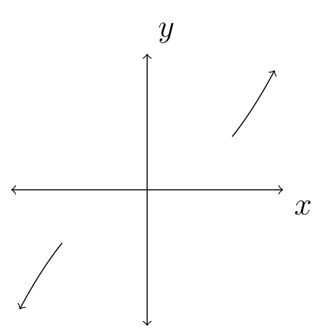
II. n even, $a_n < 0$



As $x \to \infty,\ f(x) \to -\infty$

As $x \to -\infty,\ f(x) \to -\infty$

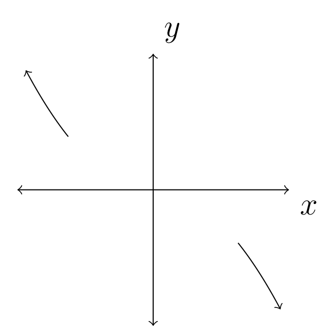
III. n odd, $a_n > 0$



As $x \to \infty,\ f(x) \to \infty$

As $x \to -\infty,\ f(x) \to -\infty$

IV. n odd, $a_n < 0$



As $x \to \infty,\ f(x) \to -\infty$

As $x \to -\infty,\ f(x) \to \infty$

An important initial observation of the previous figure is that the cases for the end behavior of a polynomial only depend on its leading term, $a_n x^n$. More specifically, the end behavior of a polynomial depends only on the parity of its degree n (even or odd) and the sign of its leading coefficient a_n (positive or negative). Additionally, we can see that cases I and II also include all quadratic functions (when $n = 2$).

Identifying the end behavior for an expanded polynomial, is much more straightforward than for a factored polynomial, as we will see in our next example.

Example 74. Determine the end behavior of each of the following functions.

1. $f(x) = 1 - 3x^4$
2. $g(x) = -2x^3 + 10000x^2 + 1000$
3. $h(x) = x(2x - 1)(x - 5)^2$
4. $k(x) = -2(1-3x)^2(x+1)(x-1)(x^2+1)$

1. The polynomial $f(x) = 1 - 3x^4$ is in expanded form, though not written in descending-power order. We can easily re-write f as $f(x) = -3x^4 + 1$. In this case, the degree $n = 4$ is even, and the leading coefficient $a_n = -3$ is negative. Hence, we are in case II:

   As $x \to -\infty$, $f(x) \to -\infty$. As $x \to \infty$, $f(x) \to -\infty$.

2. The polynomial g is also in expanded form, with odd degree $n = 3$ and negative leading coefficient, $a_n = -2$. The "large" quadratic and constant terms will not affect the end behavior of g, and so we are in case IV:

   As $x \to -\infty$, $g(x) \to \infty$. As $x \to \infty$, $g(x) \to -\infty$.

3. The polynomial h is written in factored form, which is helpful for identifying roots/$x-$intercepts, but not necessarily for describing the tails of the graph of h. Although we could, with enough time, expand h completely to describe the function's end behavior, this will quickly prove to be an inefficient strategy. Recall, however, that for end behavior we need only focus on finding the leading term, $a_n x^n$. We do this by identifying any parts of h that will contribute to the leading term. In this case, we identify in boldface font all contributing components to the leading term of h below.

   $$h(x) = \mathbf{x}(\mathbf{2x} - 1)(\mathbf{x} - 5)^{\mathbf{2}}$$

   So, when we expand, the leading term of h will be

   $$a_n x^n = x(2x)(x)^2 = 2x^4.$$

   We can now see that h has even degree $n = 4$, and positive leading coefficient, $a_n = 2$. Hence, we are in case I:

   As $x \to -\infty$, $h(x) \to \infty$. As $x \to \infty$, $h(x) \to \infty$.

4. Similarly, the polynomial k is written in factored form, and will require us to find the leading term, $a_n x^n$. Again, we identify all contributing components to the leading term of k in boldface font below.

$$k(x) = \mathbf{-2}(1\mathbf{-3x})^{\mathbf{2}}(\mathbf{x}+1)(\mathbf{x}-1)(\mathbf{x^2}+1)$$

So, when we expand, the leading term of k will be

$$\begin{aligned} a_n x^n &= -2(-3x)^2(x)(x)(x^2) \\ &= -2(9x^2)(x^4) \\ &= -18x^6. \end{aligned}$$

Through careful analysis, we see that k has an even degree, $n = 6$, and a negative leading coefficient, $a_n = -18$. Hence, we are in case II:

As $x \to -\infty$, $k(x) \to -\infty$. As $x \to \infty$, $k(x) \to -\infty$.

In the previous example, we witnessed a new technique to quickly identify the end behavior of a polynomial that is given in factored form. The idea behind this technique is to only focus on the contributing components to a polynomial's leading term, $a_n x^n$, ignoring all others. This essentially boils down to focusing on three things:

- any constant multiplier,
- the leading term of each factor,
- and the power associated with each factor.

In general, if we suppose that a polynomial f has the factorization

$$f(x) = c \cdot (\text{Factor 1})^{k_1} \cdot (\text{Factor 2})^{k_2} \cdot \ldots \cdot (\text{Factor m})^{k_m},$$

then the leading term for f will equal

$$a_n x^n = c \cdot (\text{Leading Term 1})^{k_1} \cdot (\text{Leading Term 2})^{k_2} \cdot \ldots \cdot (\text{Leading Term m})^{k_m}.$$

Note that "Leading Term 1" refers to the leading term of Factor 1, and so on for the other factors.

This approach is similar to one that we have likely seen for identifying the constant term for a factored polynomial. To identify the constant term, a_0 of f, we would have

$$a_n x^n = c \cdot (\text{Constant Term 1})^{k_1} \cdot (\text{Constant Term 2})^{k_2} \cdot \ldots \cdot (\text{Constant Term m})^{k_m}.$$

Example 75. Find the leading and constant terms for the given function, and use them to identify the end behavior and $y-$intercept of its graph.

$$f(x) = 3(-2x+1)^2(x-2)^2(x-5)$$

First, we boldface the contributors for the leading term.

$$f(x) = \mathbf{3}(\mathbf{-2x}+1)^{\mathbf{2}}(\mathbf{x}-2)^{\mathbf{2}}(\mathbf{x}-5)$$

This gives us the following.

$$\begin{aligned} a_n x^n &= 3(-2x)^2(x)^2(x) \\ &= 3(4x^2)x^3 \\ &= 12x^5 \end{aligned}$$

Next, we boldface the contributors for the constant term.

$$f(x) = \mathbf{3}(-2x{+}\mathbf{1})^{\mathbf{2}}(x{-}\mathbf{2})^{\mathbf{2}}(x{-}\mathbf{5})$$

This gives us the following.

$$\begin{aligned} a_0 &= 3(1)^2(-2)^2(-5) \\ &= 3(1)(4)(-5) \\ &= -60 \end{aligned}$$

Hence, we have that

$$f(x) = 12x^5 + \ldots + (-60),$$

with middle terms unknown.

Since our degree, $n = 5$, is odd, and our leading coefficient, $a_n = 12$, is positive, we are in case III for end behavior.

As $x \to -\infty,\ f(x) \to -\infty.$ As $x \to \infty,\ f(x) \to +\infty.$

Our constant term also tells us that the graph of f has a $y-$intercept at $(0, -60)$.

It is natural to ask why the additional terms of a polynomial have no impact on its end behavior. To address this, let us consider factoring out the leading term from f, which will give us the following.

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0 \\ &= a_n x^n \left(1 + \frac{a_{n-1}}{a_n x} + \ldots + \frac{a_2}{a_n x^{n-2}} + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n}\right) \end{aligned}$$

If we use $g(x)$ to denote the expression in parentheses,

$$g(x) = 1 + \frac{a_{n-1}}{a_n x} + \ldots + \frac{a_2}{a_n x^{n-2}} + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n},$$

then

$$f(x) = a_n x^n \underbrace{\left(1 + \frac{a_{n-1}}{a_n x} + \ldots + \frac{a_2}{a_n x^{n-2}} + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n}\right)}_{g(x)}$$
$$= a_n x^n \cdot g(x).$$

But recall that the end behavior of a polynomial is determined when $x \to \pm\infty$. So, as x gets large (either positively or negatively), with the exception of the first term, all subsequent terms in the expression for g will approach zero.

$$\text{As } x \to \pm\infty, \;\; g(x) = 1 + \cancelto{0}{\frac{a_{n-1}}{a_n x}} + \ldots + \cancelto{0}{\frac{a_2}{a_n x^{n-2}}} + \cancelto{0}{\frac{a_1}{a_n x^{n-1}}} + \cancelto{0}{\frac{a_0}{a_n x^n}} \to 1.$$

Therefore, since $g(x)$ approaches 1, $f(x) = a_n x^n \cdot g(x)$ will approach its leading term, $a_n x^n$. Hence, we conclude that the end behavior of a polynomial f will coincide with the end behavior of its leading term.

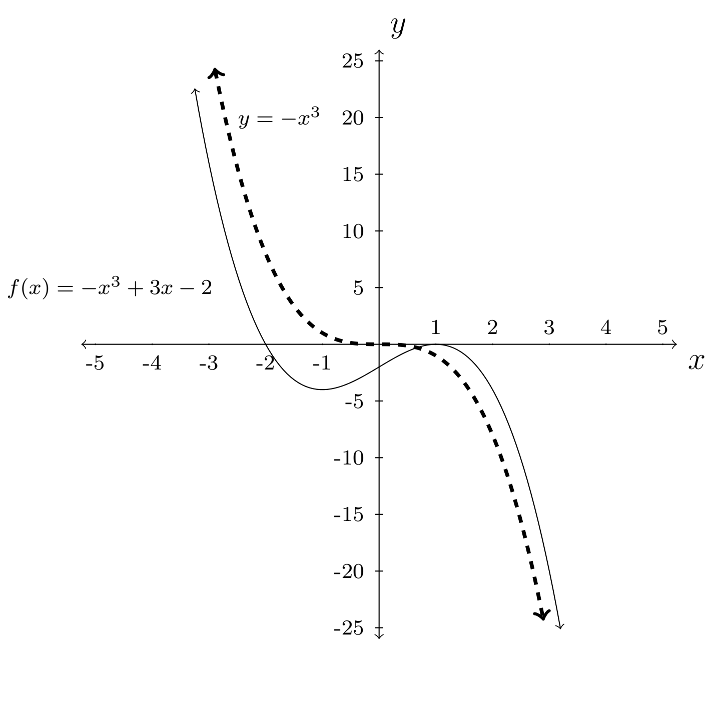
Furthermore, for any polynomial $f(x)$, if we were to graph the two curves $y = f(x)$ and $y = a_n x^n$ using Desmos or another graphing utility, and continue to 'zoom out', the two graphs would become virtually indistinguishable from one another. We demonstrate this in our next example.

Example 76. Determine the end behavior of the polynomial function below, and graph both the function and its leading term on a single set of axes.

$$f(x) = -x^3 + 3x - 2$$

The leading term of f is $-x^3$, with odd degree, $n = 3$, and negative leading coefficient, $a_n = -1$. Hence, we are in case IV:

As $x \to -\infty$, $f(x) \to \infty$. As $x \to \infty$, $f(x) \to -\infty$.



Local Behavior (L50)

Objective: Identify all real roots and their corresponding multiplicities for a polynomial function that is easily factorable.

In College Algebra and Precalculus, when we refer to the *local behavior* of a function f, we will be concerned with anything of interest in the interior of the graph of f, and not its end behavior. For polynomials, this is the $x-$ and $y-$intercepts of the graph. These points coincide with when $f(x) = 0$ for any $x-$intercepts, and when $x = 0$ in the case of the $y-$intercept. In Calculus, local behavior will also include points where the graph changes inflection or achieves a local maximum or minimum value.

Since we should be very familiar with finding a $y-$intercept at this point, we will start with a simple example.

Example 77. Find the $y-$intercept for each of the following polynomials.

1. $f(x) = 5x^3 - \frac{1}{2}x^2 + 6x - 18$
2. $g(x) = \frac{1}{2}(x-2)^2(x+5)(x-3)$

1. $f(0) = -18$. Hence, the graph of f has a $y-$intercept at $(0, -18)$.
2. In the case of g, we have to identify the constant term a_0 in the expanded form of the polynomial. Recalling Example 75 from our last section, we can easily obtain this value without the need to expand g in its entirety.

$$\begin{aligned} g(0) &= \frac{1}{2}(0-2)^2(0+5)(0-3) \\ &= \frac{1}{2}(-2)^2(5)(-3) \\ &= \frac{1}{2}(4)(-15) \\ &= 2(-15) \\ &= -30 \end{aligned}$$

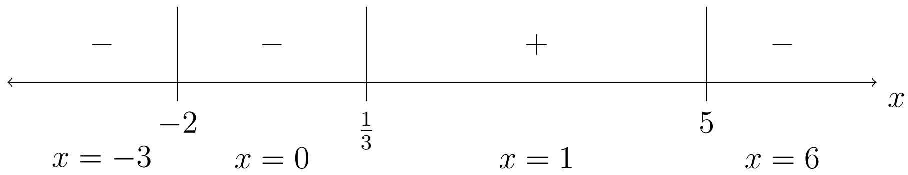
Hence, our $y-$intercept for the graph of g is $(0, -30)$.

Although it is certainly important to identify the $y-$intercept of any polynomial, the primary objective of this section will be finding the roots of a polynomial and classifying the respective $x-$intercepts of its graph. Since roots/$x-$intercepts coincide with when a function equals zero, $f(x) = 0$, this section will depend heavily on working with a polynomial that is either in factored form or for which a complete factorization is easily obtainable. In a subsequent section of this chapter, we will see more a more advanced technique for finding a complete factorization of a polynomial, using polynomial division and the Rational Root Theorem.

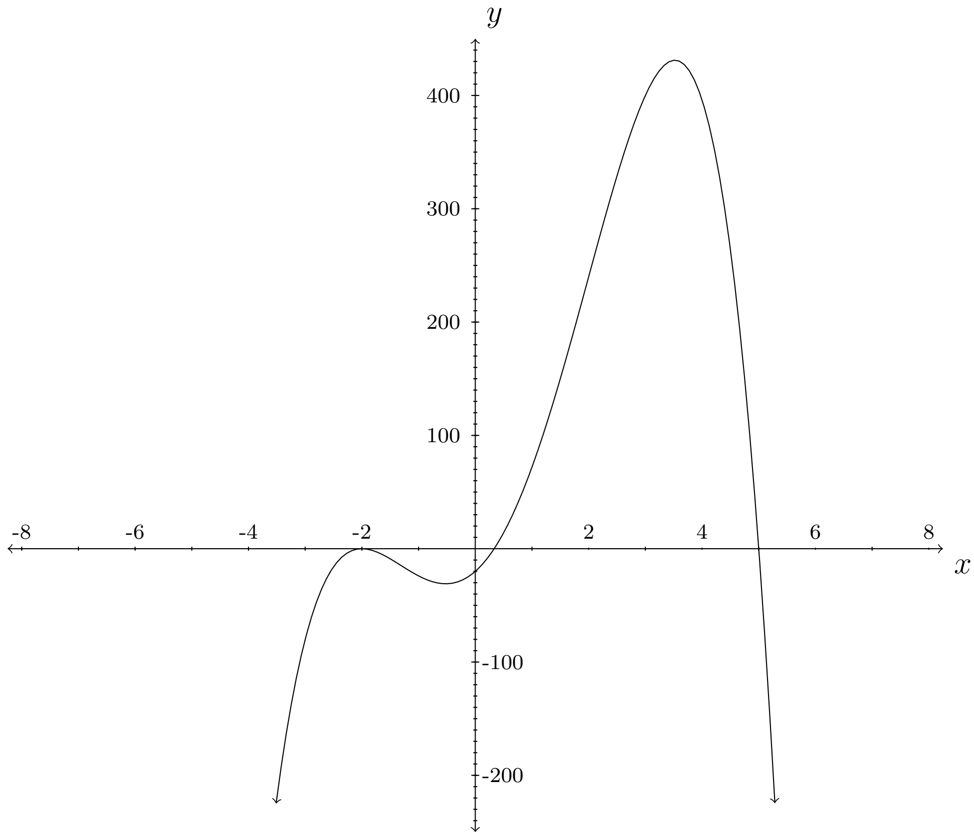
We begin our exploration of $x-$intercepts by revisiting a past example.

Example 78. Find all roots of the polynomial function $h(x) = (x+2)^2(3x-1)(5-x)$, and graph h using Desmos or another graphing utility. For each root, identify whether the graph of h crosses over or turns around at the corresponding $x-$intercept.

For this example, we will first recall the work done Example 54, where we identified the roots of h to be $x = -2, \frac{1}{3}$, and 5, as well as the following sign diagram.



Our graph of h is shown below.



Based upon our picture, we see that the graph of h crosses over the $x-$axis at $x = \frac{1}{3}$ and $x = 5$. The graph turns around at $x = -2$.

In our last example, we have included our sign diagram to point out a connection. Our diagram confirms the nature of each $x-$intercept without the need to graph h, since both sides of our *turnaround point* $x = -2$ show the *same* sign (either $+|+$ or $-|-$). Similarly, the signs *change* from either positive to negative ($+|-$) or negative to positive ($-|+$) for each of our *crossover points*.

In fact, this idea of turnaround and crossover points can be parsed down to one basic concept, known as the *multiplicity* of a root. We define the multiplicity of a root below, followed immediately by an example for clarification.

Suppose f is a polynomial function with real root $x = c$. For some positive integer k, if $(x-c)^k$ is a factor of f but $(x-c)^{k+1}$ is not, then we say $x = c$ is a root of f having associated multiplicity k.

Example 79. Determine the set of roots and corresponding multiplicities for the following functions.

1. $f(x) = x^6 - 2x^5 - 15x^4$
2. $g(x) = (x-6)^5(x+2)^2(x^2+1)$

1. Factoring f gives us the following.
$$\begin{aligned} f(x) &= x^6 - 2x^5 - 15x^4 \\ &= x^4(x^2 - 2x - 15) \\ &= x^4(x-5)(x+3) \end{aligned}$$
We then can easily see that f has a root at $x = 0$ with multiplicity four, and roots at $x = 5$ and $x = -3$, each with multiplicity one.
2. Since g is already factored, we see that $x = 6$ is a root having multiplicity five, and $x = -2$ is a root having multiplicity two. The factor of $x^2 + 1$ is meant to throw us off, since its roots are the imaginary numbers $\pm i$.

Another way of describing the multiplicity k of a root $x = c$ is that k represents the maximum number of factors of $(x-c)$ that divide the polynomial f (with a remainder of 0). That is,
$$f(x) = (x-c)^k \cdot q(x),$$
where $(x-c)$ is *not* a factor of the quotient $q(x)$.

If we apply this idea to g in our last example, we see that although $(x-6)^4$ divides our polynomial,
$$g(x) = (x-6)^4 \cdot \underbrace{(x-6)(x+2)^2(x^2+1)}_{q(x)},$$
the value of four does not represent the *maximum* number of factors of $(x-6)$ that divide g:
$$g(x) = (x-6)^5 \cdot \underbrace{(x+2)^2(x^2+1)}_{q(x)}.$$

At this point, we are ready to highlight the importance of multiplicities in graphing polynomials.

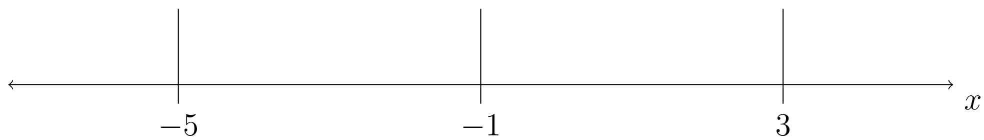
Let f be a polynomial function with a real root at $x = c$ having multiplicity k.

- If k is *even*, the corresponding $x-$intercept $(c, 0)$ is a *turnaround point.* In other words, the graph of f touches and rebounds from the x-axis at $(c, 0)$, leaving the $y-$values to maintain the same sign on either side of the root $x = c$.
- If k is *odd*, the corresponding $x-$intercept $(c, 0)$ is a *crossover point.* In other words, the graph of f crosses through the x-axis at $(c, 0)$, leaving the $y-$values to change signs on either side of the root $x = c$.

Combining this new notion about multiplicities of roots with all that we have already learned about polynomials will enable us to quickly identify all important aspects of a particular polynomial function, culminating in a sketch of its graph. We capitalize on this in our next example.

Example 80. Construct a sign diagram for the factored polynomial
$f(x) = -(x-3)^2(x+1)(x+5)^2$.

The dividers for our sign diagram come from the set of roots of f, namely $\{-5, -1, 3\}$.



Instead of assigning test values, however, we will use both multiplicities and end behavior to determine our various signs.

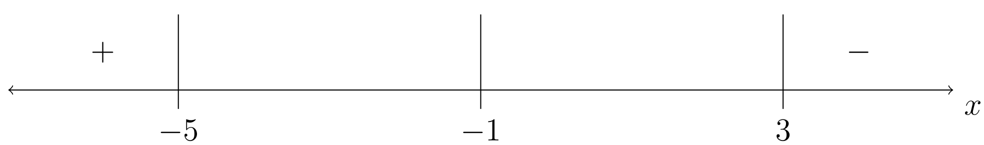
First, we identify the leading term of f.

$$a_n x^n = -(x)^2(x)(x)^2 = -x^5$$

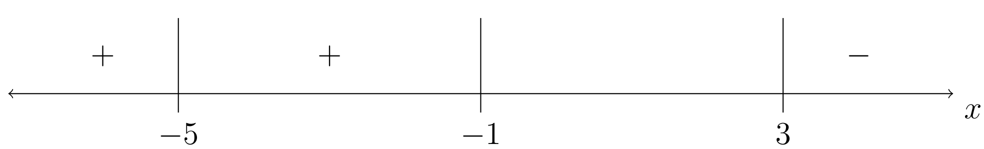
Since $a_n < 0$ and $n = 5$ is odd, our end behavior follows case IV:

$$\text{As } x \to -\infty,\ f(x) \to \infty. \qquad\qquad \text{As } x \to \infty,\ f(x) \to -\infty.$$

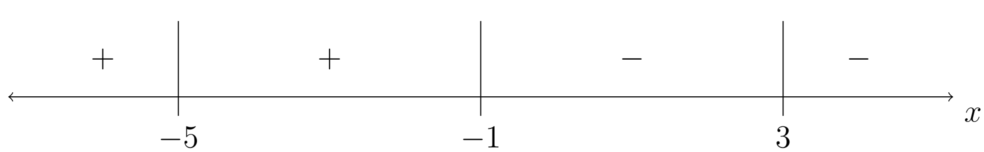
This tells us that our diagram will begin with a positive sign and end with a negative sign.



Furthermore, the multiplicity of the root $x = -5$ is two, which is even. So, our diagram must contain the same signs on either side of $x = -5$, namely two positive signs.



Applying this same idea to both of our remaining roots, we get the following diagram, and are done!



In fact, we can take this last example further, and easily sketch a graph of our polynomial. All that remains is to identify the $y-$intercept.

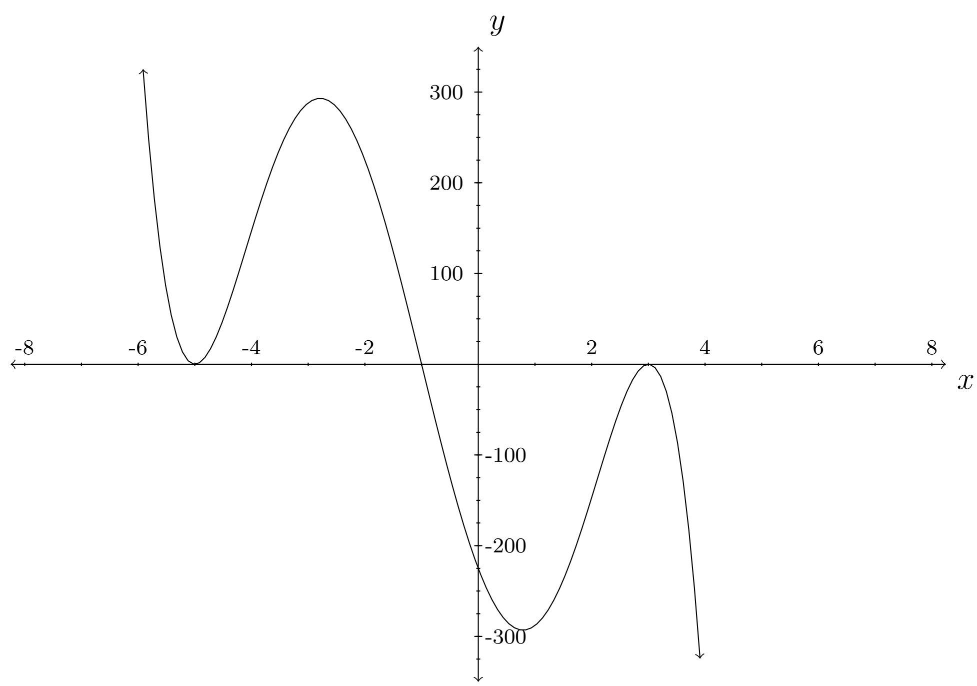
Example 81. Sketch a graph of the factored polynomial $f(x) = -(x-3)^2(x+1)(x+5)^2$, making sure to identify a clearly defined scale and any $x-$ and $y-$intercepts.

Since the roots, $x = 3$ and $x = -5$ have even multiplicities, their corresponding $x-$intercepts will be turnaround points. Our sign diagram confirms this, and further shows that the intercept at $x = -5$ will be a local minimum $(+|+)$, whereas the intercept at $x = 3$ will be a local maximum $(-|-)$. On the other hand, the root at $x = -1$ has an odd multiplicity, and the $x-$intercept at $x = -1$ will be a crossover point.

For a $y-$intercept, we evaluate the function at $x = 0$.

$$\begin{aligned} f(0) &= -(0-3)^2(0+1)(0+5)^2 \\ &= -(9)(1)(25) \\ &= -225 \end{aligned}$$

Since our $y-$intercept is a large negative value, we will have to shrink our scale for the $y-$axis accordingly.



This last example achieves what we have sought after since beginning the chapter. In it, we have identified the end behavior and all intercepts of a completely factored polynomial. We have further used both a sign diagram and a multiplicity argument in order to graph the given function.

What remains is to take a closer look at more challenging polynomials, for which a factorization may not be given or even easily identifiable.

The Rational Root Theorem (L51)

Objective: Apply the Rational Root Theorem to determine a set of possible rational roots for and a factorization of a given polynomial.

The Rational Root Theorem is used to identify a list of all possible rational roots for a given polynomial.

> **Rational Root Theorem:** Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial of degree n with $n \geq 1$, and a_0, a_1, $\ldots a_n$ are integers. If r is a rational root of f, then r is of the form $\pm\frac{p}{q}$, where p is a factor of the constant term a_0, and q is a factor of the leading coefficient a_n.

The Rational Root Theorem gives us a list of numbers to test as roots of a given polynomial using synthetic division, which is a nicer approach than simply guessing at possible roots. If none of the numbers in the list turn out to be roots, then either the polynomial has no real roots at all, or all of the real roots will be irrational numbers.

Example 82. Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. Use the Rational Root Theorem to list all of the possible rational roots of f.

To generate a complete list of rational roots, we need to take each of the factors of the constant term, $a_0 = -3$, and divide them by each of the factors of the leading coefficient $a_4 = 2$.

The factors of -3 are ± 1 and ± 3. Since the Rational Root Theorem tacks on a $\pm$ anyway, for the moment, we consider only the positive factors 1 and 3. The factors of 2 are 1 and 2, so the Rational Root Theorem gives the list $\left\{\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{3}{1}, \pm\frac{3}{2}\right\}$ or $\left\{\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 3\right\}$.

Additionally, we can evaluate f at each of the eight potential rational roots in our list, to see if any of them are indeed roots. Starting with ± 1, we see that

$$f(1) = 2 + 4 - 1 - 6 - 3 = -4 \neq 0 \qquad \text{and} \qquad f(-1) = 2 - 4 - 1 + 6 - 3 = 0.$$

Hence, we can conclude that $x = -1$ is a root of f and $x = 1$ is not. Using synthetic division, we can then divide f by the linear factor $x + 1$ as follows.

$$\begin{array}{r|rrrrr} -1 & 2 & 4 & -1 & -6 & -3 \\ & & -2 & -2 & 3 & 3 \\ \hline & 2 & 2 & -3 & -3 & \mathbf{0} \end{array}$$

We can then begin to factor f,

$$2x^4 + 4x^3 - x^2 - 6x - 3 = (x + 1)(2x^3 + 2x^2 - 3x - 3)$$

The resulting quotient polynomial is then factorable by grouping,

$$2x^3 + 2x^2 - 3x - 3 = (2x^2 - 3)(x + 1).$$

Factoring out a 2 from the expression $2x^2 - 3$, allows us to factor it as the difference of two squares,

$$\begin{aligned} 2x^2 - 3 &= 2\left(x^2 - \frac{3}{2}\right) \\ &= 2\left(x - \sqrt{\frac{3}{2}}\right)\left(x + \sqrt{\frac{3}{2}}\right) \\ &= 2\left(x - \frac{\sqrt{6}}{2}\right)\left(x + \frac{\sqrt{6}}{2}\right) \end{aligned}$$

So, a complete factorization for f would be

$$2x^4 + 4x^3 - x^2 - 6x - 3 = 2\left(x - \frac{\sqrt{6}}{2}\right)\left(x + \frac{\sqrt{6}}{2}\right)(x+1)^2,$$

and the set of real roots for f is $\left\{-1, \pm\frac{\sqrt{6}}{2}\right\}$.

Graphing Summary (L52)

Objective: Graph a polynomial function in its entirety.

At this point, we have addressed all key features of polynomials individually. This section pulls each of these aspects together, for a detailed analysis of a polynomial, culminating in a complete sketch of its graph. Along the way, we will need to address each of the following aspects for our polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$. It is important to note that there is no universally accepted order to this checklist.

- Find the $y-$intercept of the graph of f, $(0, f(0)) = (0, a_0)$.
- Use the degree n and leading coefficient a_n to determine the end behavior of the graph of f.
- Identify a complete factorization of f, and use it to find any $x-$intercepts of the graph of f. Using multiplicities, classify each $x-$intercept as a crossover or turnaround ("bounce") point.
- Using the $x-$intercepts, construct a sign diagram for f.

In each polynomial we encounter, we will carefully examine the function, making sure not to omit any of the checklist items above and to compare each item to those that precede it along the way for accuracy. Although the process will take some time, if we are thorough, our end result should be a complete, accurate sketch of the given polynomial.

Example 83. Sketch a complete graph of $f(x) = 14x^4 - 17x^3 - 6x^2 + 7x + 2$.

We will start with the $y-$intercept, which is $(0, 2)$.

Next, we see that f has even degree and positive leading coefficient. So, the tails of the graph of f both point upwards. In other words, as $x \to \pm\infty$, $f(x) \to \infty$.

Since f is degree-4, contains more than four terms, and is not of quadratic type, we will apply the Rational Root Theorem. In this case, our set of possible rational roots is

$$\left\{ \pm 1, \pm 2, \pm\frac{1}{2}, \pm\frac{1}{7}, \pm\frac{1}{14}, \pm\frac{2}{7} \right\}$$

Fortunately, we see that $f(1) = 14 - 17 - 6 + 7 + 2 = 0$. So, $x - 1$ is a factor of f. Dividing, we get:

$$\begin{array}{r} 14x^3 - 3x^2 - 9x - 2 \\ x-1 \overline{)\, 14x^4 - 17x^3 - 6x^2 + 7x + 2} \\ \underline{-14x^4 + 14x^3 \qquad\qquad\qquad\quad} \\ -3x^3 - 6x^2 \qquad\qquad \\ \underline{3x^3 - 3x^2 \qquad\qquad} \\ -9x^2 + 7x \quad\;\; \\ \underline{9x^2 - 9x \quad\;\;} \\ -2x + 2 \\ \underline{2x - 2} \\ 0 \end{array} \qquad \begin{array}{c|ccccc} 1 & 14 & -17 & -6 & 7 & 2 \\ & & 14 & -3 & -9 & -2 \\ \hline & 14 & -3 & -9 & -2 & \mathbf{0} \end{array}$$

So, $f(x) = (x - 1)(14x^3 - 3x^2 - 9x - 2)$. Applying the Rational Root Theorem a second time, we can see that $x = 1$ is also a root of the cubic factor of f, since $14 - 3 - 9 - 2 = 0$. Again, we can divide to factor f further.

$$\begin{array}{r} 14x^2 + 11x + 2 \\ x-1 \overline{)\, 14x^3 - 3x^2 - 9x - 2} \\ \underline{-14x^3 + 14x^2 \qquad\qquad} \\ 11x^2 - 9x \quad\;\; \\ \underline{-11x^2 + 11x \quad\;\;} \\ 2x - 2 \\ \underline{-2x + 2} \\ 0 \end{array} \qquad \begin{array}{c|cccc} 1 & 14 & -3 & -9 & -2 \\ & & 14 & 11 & 2 \\ \hline & 14 & 11 & 2 & \mathbf{0} \end{array}$$

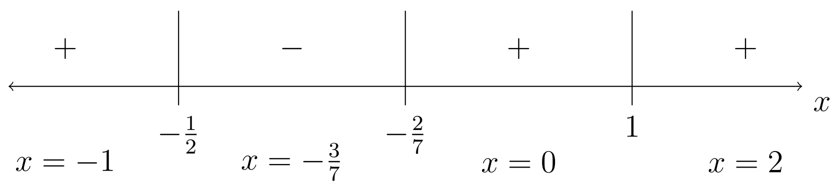
So, $f(x) = (x - 1)^2(14x^2 + 11x + 2)$. Factoring the remaining quadratic, we have

$$f(x) = (x - 1)^2(7x + 2)(2x + 1),$$

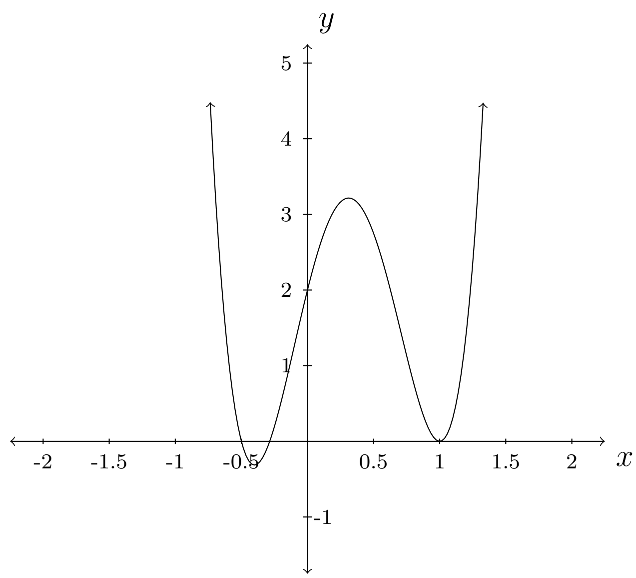
with accompanying set of roots $\{1, -\frac{1}{2}, -\frac{2}{7}\}$.

Using multiplicities, we conclude that the $x-$intercept $(1, 0)$ is a turnaround point, and the intercepts $\left(-\frac{1}{2}, 0\right)$ and $\left(-\frac{2}{7}, 0\right)$ are crossover points.

Though not necessary for graphing, a sign diagram confirms our end and local behavior findings.



Putting all of this information together results in the following graph.



Polynomial Inequalities (L53)

Objective: Solve a polynomial inequality by constructing a sign diagram.

Example 84. Solve the polynomial inequality

$$x^4 + 6x^2 - 15x \leq x^4 + 2x^3 - 7x^2.$$

Just as with quadratic inequalities, we begin by setting one side equal to zero. This gives us

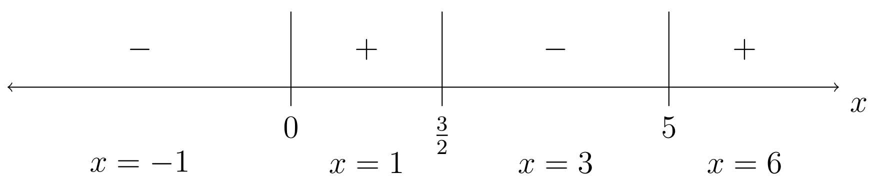
$$2x^3 - 13x^2 + 15x \geq 0.$$

In order to construct a sign diagram, we must find a factorization and identify the roots of the left-hand side of our inequality.

$$2x^3 - 13x^2 + 15x = 2x\left(x - \frac{3}{2}\right)(x - 5)$$

So the dividers in our diagram will be the roots $x = 0, \frac{3}{2}$, and 5. Below is a chart for testing the intervals in our sign diagram, as well as the end result.

Interval	Test Value	Signs	Result
$(-\infty, 0)$	$x = -1$	$(-)(-)(-)$	$-$
$(0, \frac{3}{2})$	$x = 1$	$(+)(-)(-)$	$+$
$(\frac{3}{2}, 5)$	$x = 3$	$(+)(+)(-)$	$-$
$(5, \infty)$	$x = 6$	$(+)(+)(+)$	$+$



So, using our diagram as an aide, we see that the solution to the inequality

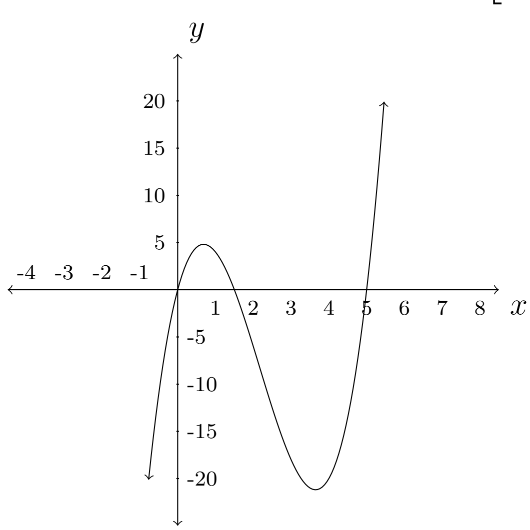
$$2x^3 - 13x^2 + 15x \geq 0,$$

as well as our original inequality

$$x^4 + 6x^2 - 15x \leq x^4 + 2x^3 - 7x^2,$$

will be

$$\left[0, \frac{3}{2}\right] \cup [5, \infty).$$



Since our given inequality was inclusive ($\leq$ or $\geq$), we include the corresponding endpoints in our answer.

We can verify that our answer is correct by comparing it to the graph of the function

$$f(x) = 2x^3 - 13x^2 + 15x,$$

which lies above (or on) the x−axis over the intervals in our answer.

Practice Problems

Introduction and Terminology

Identify the degree, set of coefficients, leading coefficient, leading term and constant term for each of the polynomials listed. Classify each polynomial by both degree and number of nonzero terms. If it is not already provided, write the polynomial in descending-power order.

1. $f(x) = -2x^3 - 1$
2. $f(x) = -2x^4 + 4x + 1$
3. $f(x) = 40 - x^3$
4. $f(x) = (x-1)^2$
5. $f(x) = 32x^5 + x^2 + x$
6. $f(x) = 4x^2 - 3x^4$
7. $f(x) = -2x^4 - 4x^2 - 6x - 8$
8. $f(x) = 5x + 3x^2 + x^3 + \sqrt{3}$
9. $f(x) = \frac{1}{2}x^4 - 5x^2 - \frac{1}{2}$
10. $f(x) = 12 - 6x + 3x^2 - 2x^3 - x^6$
11. $f(x) = -3x^4 - 12x^3 + x - 13$

Sign Diagrams

Construct a sign diagram for the factored polynomial functions below. Use Desmos to graph each function and check the accuracy of your diagram. Identify the interval(s) where the function is positive and where it is negative.

1. $f(x) = x^3(x-2)(x+2)$
2. $g(x) = (x^2+1)(1-x)$
3. $h(x) = x(x-3)^2(x+3)$
4. $k(x) = (3x-4)^3$
5. $\ell(x) = (x^2+2)(x^2+3)$
6. $m(x) = -2(x+7)^2(1-2x)^2$
7. $f(x) = (x^2-1)(x+4)$
8. $g(x) = (x^2-1)(x^2-16)$
9. $h(x) = -2x^3(3x-1)(2-x)$
10. $k(x) = (x^2-4x+1)(x+2)^2$

Factoring

Some Special Cases

Completely factor each of the following polynomial expressions.

1. $2x^2 - 11x + 15$
2. $5n^3 + 7n^2 - 6n$
3. $54u^3 - 16$
4. $54 - 128x^3$
5. $n^2 - n$
6. $2x^4 - 21x^2 - 11$
7. $24az - 18ah + 60yz - 45yh$
8. $5u^2 - 9uv + 4v^2$
9. $16x^2 + 48xy + 36y^2$
10. $-2x^3 + 128y^3$
11. $20uv - 60u^3 - 5xv + 15xu^2$
12. $2x^3 + 5x^2y + 3y^2x$

Quadratic Type

Completely factor each of the following polynomials over the real numbers and identify the set of all real roots.

1. $x^4 + 13x^2 + 40$
2. $x^4 - 5x^2 + 4$
3. $x^4 - 17x^2 + 16$
4. $x^4 - 3x^2 - 40$
5. $3x^4 - 32x^2 + 45$
6. $x^4 + x^2 - 12$
7. $x^4 - 3x^2 - 10$
8. $x^6 - 82x^3 + 81$
9. $8x^4 + 2x^2 - 3$
10. $2x^4 - 19x^2 + 9$

Division

Polynomial (Long) Division

Use polynomial long division to divide and simplify each of the given expressions. Express each answer in the form below.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

1. $\dfrac{20x^4 + x^3 + 2x^2}{4x^3}$
2. $\dfrac{5x^4 + 45x^3 + 4x^2}{9x}$
3. $\dfrac{20x^4 + x^3 + 40x^2}{10x}$
4. $\dfrac{3x^3 + 4x^2 + 2x}{8x}$
5. $\dfrac{12x^4 + 24x^3 + 3x^2}{6x}$
6. $\dfrac{5x^4 + 16x^3 + 16x^2}{4x}$
7. $\dfrac{10x^4 + 50x^3 + 2x^2}{10x^2}$
8. $\dfrac{3x^4 + 18x^3 + 27x^2}{9x^2}$
9. $\dfrac{x^2 - 2x - 71}{x + 8}$
10. $\dfrac{x^2 - 3x - 53}{x - 9}$
11. $\dfrac{x^2 + 13x + 32}{x + 5}$
12. $\dfrac{x^2 - 10x + 16}{x - 7}$
13. $\dfrac{x^2 - 2x - 89}{x - 10}$
14. $\dfrac{x^2 + 4x - 26}{x + 7}$
15. $\dfrac{x^2 - 4x - 38}{x - 8}$
16. $\dfrac{x^2 - 4}{x - 2}$
17. $\dfrac{x^3 + 15x^2 + 49x - 55}{x + 7}$
18. $\dfrac{x^3 - 26x - 41}{x + 4}$
19. $\dfrac{3x^3 + 9x^2 - 64x - 68}{x + 6}$
20. $\dfrac{9x^3 + 45x^2 + 27x - 5}{9x + 9}$
21. $\dfrac{x^3 - x^2 - 16x + 8}{x - 4}$
22. $\dfrac{x^2 - 10x + 22}{x - 4}$
23. $\dfrac{x^3 - 16x^2 + 71x - 56}{x - 8}$
24. $\dfrac{x^3 - 4x^2 - 6x + 4}{x - 1}$
25. $\dfrac{8x^3 - 66x^2 + 12x + 37}{x - 8}$
26. $\dfrac{3x^2 + 9x - 9}{3x - 3}$
27. $\dfrac{2x^2 - 5x - 8}{2x + 3}$
28. $\dfrac{3x^2 - 32}{3x - 9}$
29. $\dfrac{4x^2 - 23x - 38}{4x + 5}$
30. $\dfrac{2x^3 + 21x^2 + 25x}{2x + 3}$

31. $\dfrac{4x^3 - 21x^2 + 6x + 19}{4x + 3}$

32. $\dfrac{8x^3 - 57x^2 + 42}{8x + 7}$

33. $\dfrac{2x^3 + 12x^2 + 4x - 37}{2x + 6}$

34. $\dfrac{45x^2 + 56x + 19}{9x + 4}$

35. $\dfrac{10x^2 - 32x + 9}{10x - 2}$

36. $\dfrac{4x^2 - x - 1}{4x + 3}$

37. $\dfrac{27x^2 + 87x + 35}{3x + 8}$

38. $\dfrac{4x^2 - 33x + 28}{4x - 5}$

39. $\dfrac{48x^2 - 70x + 16}{6x - 2}$

40. $\dfrac{12x^3 + 12x^2 - 15x - 4}{2x + 3}$

41. $\dfrac{24x^3 - 38x^2 + 29x - 60}{4x - 7}$

Synthetic Division

Use synthetic division to divide and simplify each of the given expressions. Express each answer in the form below.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

1. $\dfrac{x^4 - 4x^3 + 2x^2 - x + 1}{x + 2}$
2. $\dfrac{x^4 - 2x^3 + 7x^2 - 6x + 3}{x - 2}$
3. $\dfrac{2x^4 - 2x^3 - 10x^2 + 1}{x + 2}$
4. $\dfrac{5x^4 - 2x^3 + 4x^2 - 5x}{x - 1}$
5. $\dfrac{-x^4 - x^3 + x^2 + x + 1}{x + 5}$
6. $\dfrac{x^4 - 3x^3 + 2x^2 - x + 1}{x - 4}$
7. $\dfrac{12x^4 - x^3 + x^2 - 3x + 1}{x + 2}$
8. $\dfrac{3x^4 + 3x^3 + 13x^2 - 4x + 14}{x + 1}$
9. $\dfrac{1x^4 - 3x^3 + 5x^2 - 14x + 2}{x - 2}$
10. $\dfrac{2x^4 - 2x + 1}{x + 3}$
11. $\dfrac{x^4 - 3x - 4}{x - 3}$
12. $\dfrac{x^4 - 4x^3 + 13x^2 - 5x + 7}{x - 4}$

Use synthetic division to divide and simplify each of the given expression from Exercises 9-41.

End Behavior

Determine the end behavior of each of the following functions. Write your answers as mathematical sentences. Graph each function on Desmos to check your answers.

1. $f(x) = -2x^3 + 4x + 1$
2. $g(x) = 32x^5 + x^2 + 15$
3. $h(x) = -3x^4 + 4x^2$
4. $k(x) = 15x^4 - 32x^2 - x - 14$
5. $\ell(x) = x^5 + 40$
6. $m(x) = 5x^5 + 3x^2 + x + 14$
7. $n(x) = 123x^4 - 7x^3 - 5x^2 - 3x + 1$
8. $p(x) = x^3 - 1$
9. $q(x) = -23x^6 + x^3 + x^2 + x + 1$

Identify the degree, leading coefficient, and constant term of each polynomial function below. Use the degree and leading coefficient to identify the end behavior of the graph of each function. Write your answers as mathematical sentences. Graph each function on Desmos to check your answers.

10. $f(x) = x^3(x-2)(x+2)$
11. $g(x) = (x^2+1)(1-x)$
12. $h(x) = x(x-3)^2(x+3)$
13. $k(x) = (3x-4)^3$
14. $\ell(x) = (x^2+2)(x^2+3)$
15. $m(x) = -2(x+7)^2(1-2x)^2$
16. $f(x) = (x^2-1)(x+4)$
17. $g(x) = (x^2-1)(x^2-16)$
18. $h(x) = -2x^3(3x-1)(2-x)$
19. $k(x) = (x^2-4x+1)(x+2)^2$

Local Behavior

Determine the set of roots and corresponding multiplicities for the following functions. In each case, classify the corresponding $x-$intercept as either a turnaround or crossover point. Use Desmos to check your answers.

1. $f(x) = x^3(x-2)(x+2)$
2. $g(x) = (x^2+1)(1-x)$
3. $h(x) = x(x-3)^2(x+3)$
4. $k(x) = (3x-4)^3$
5. $\ell(x) = (x^2+2)(x^2+3)$
6. $m(x) = -2(x+7)^2(1-2x)^2$
7. $f(x) = (x^2-1)(x+4)$
8. $g(x) = (x^2-1)(x^2-16)$
9. $h(x) = -2x^3(3x-1)(2-x)$
10. $k(x) = (x^2-4x+1)(x+2)^2$
11. $f(x) = \frac{1}{2}(x-2)^2(x+5)(x-3)$
12. $g(x) = (x+2)^2(3x-1)(5-x)$

The Rational Root Theorem

Use the Rational Root Theorem to identify a set of possible rational roots for each of the polynomial functions below. Evaluate the function at $x = 1$. If $x = 1$ is a real root, divide the polynomial by $x-1$ and factor the resulting quotient. If $x = 1$ is not a real root, evaluate the function at at least one of your remaining possible roots, in order to determine if they are actual roots of the polynomial. If successful, divide your polynomial by the respective factor and factor the remaining quotient. Use Desmos to help determine the actual set of real roots.

1. $f(x) = x^3 - 2x^2 - 5x + 6$
2. $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$
3. $f(x) = x^5 - x^4 - 37x^3 + 37x^2 + 36x - 36$
4. $f(x) = 3x^3 + 3x^2 - 11x - 10$

Use the Rational Root Theorem to identify a set of possible rational roots for each of the polynomial functions below. Evaluate the function at at least two of your possible roots, in order to determine if they are actual roots of the polynomial. If successful, divide your polynomial by the respective factor. Use Desmos to help determine the actual set of real roots.

5. $f(x) = x^4 - 2x^3 + 5x^2 - 8x + 4$
6. $f(x) = x^3 + 4x^2 - 11x + 6$
7. $f(x) = -2x^3 + 19x^2 - 49x + 20$
8. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
9. $f(x) = x^4 - 9x^2 - 4x + 12$
10. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
11. $f(x) = 6x^3 + 19x^2 - 6x - 40$

Graphing Summary

Factor each polynomial below, and sketch a complete graph of the function, making sure to have a clearly defined scale and label any intercepts. Use Desmos to compare your results.

1. $f(x) = -17x^3 + 5x^2 + 34x - 10$
2. $f(x) = x^4 - 9x^2 + 14$
3. $f(x) = 3x^4 - 14x^2 - 5$
4. $f(x) = 2x^4 - 7x^2 + 6$
5. $f(x) = x^5 - 2x^4 - x + 2$
6. $f(x) = 2x^5 + 3x^4 - 32x - 48$
7. $f(x) = x^6 - 6x^3 - 16$
8. $f(x) = 2x^6 - 7x^3 + 5$
9. $f(x) = -x^3 + 7x^2 - x + 7$
10. $f(x) = 3x^4 - 5x^3 - 12x^2$
11. $f(x) = 2x^3 - 5x^2 - x$
12. $f(x) = -x^4 - 2x^2 + 15$

Get a complete factorization of each polynomial below by first dividing the function by $x-1$. Then sketch a graph of the function, making sure to have a clearly defined scale and label any intercepts. Use Desmos to compare your results.

13. $f(x) = x^3 - 2x^2 - 5x + 6$
14. $f(x) = x^3 + 4x^2 - 11x + 6$
15. $f(x) = x^5 - x^4 - 37x^3 + 37x^2 + 36x - 36$
16. $f(x) = x^4 - 2x^3 + 5x^2 - 8x + 4$

Use the Rational Root Theorem and polynomial division to get a complete factorization of each polynomial function below. Then sketch a graph of the function, making sure to have a clearly defined scale and label any intercepts. Use Desmos to compare your results.

17. $f(x) = x^4 - 9x^2 - 4x + 12$
18. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
19. $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$
20. $f(x) = 3x^3 + 3x^2 - 11x - 10$
21. $f(x) = 6x^3 + 19x^2 - 34x - 40$
22. $f(x) = -2x^3 + 19x^2 - 49x + 20$
23. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
24. $f(x) = x^4 + 4x^3 - x - 4$
25. $f(x) = 2x^3 - 5x^2 - 52x + 60$
26. $f(x) = -x^3 - x^2 + 39x + 45$
27. $f(x) = -2x^4 + 7x^3 + 17x^2 - 28x - 36$
28. $f(x) = x^7 - 5x^6 - 24x^5 + 120x^4 - 25x^3 + 125x^2$

Polynomial Inequalities

Solve each polynomial inequality below, expressing your answers using interval notation. Use Desmos to help confirm that each answer is correct.

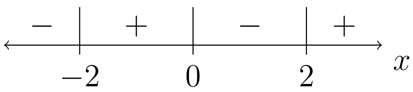
1. $x^4 + x^2 \geq 6$
2. $x^4 - 9x^2 \leq 4x - 12$
3. $4x^3 \geq 3x + 1$
4. $x^4 \leq 16 + 4x - x^3$
5. $3x^2 + 2x < x^4$
6. $\dfrac{x^3 + 2x^2}{2} < x + 2$
7. $\dfrac{x^3 + 20x}{8} \geq x^2 + 2$
8. $19x^2 + 20 > 2x^3 + 49x$
9. $x^3 < 4x^2$
10. $x^3 - 7x^2 \leq 12x - 84$
11. $(x - 1)^2 \geq 4$
12. $2x^4 > 5x^2 + 3$

Selected Answers

Introduction and Terminology

1. $n = 3, \quad a_n = -2, \quad a_n x^n = -2x^3, \quad a_0 = -1, \quad \{-2, 0, 0, -1\}$
3. $n = 3, \quad a_n = -1, \quad a_n x^n = -1x^3, \quad a_0 = 40, \quad \{-1, 0, 0, 40\}$
5. $n = 5, \quad a_n = 32, \quad a_n x^n = 32x^5, \quad a_0 = 0, \quad \{32, 0, 0, 1, 1, 0\}$
7. $n = 4, \quad a_n = -2, \quad a_n x^n = -2x^4, \quad a_0 = -8, \quad \{-2, 0, -4, -6, -8\}$
9. $n = 4, \quad a_n = \frac{1}{2}, \quad a_n x^n = \frac{1}{2}x^4, \quad a_0 = -\frac{1}{2}, \quad \{\frac{1}{2}, 0, -5, 0, -\frac{1}{2}\}$
11. $n = 4, \quad a_n = -3, \quad a_n x^n = -3x^4, \quad a_0 = -13, \quad \{-3, -12, 0, 1, -13\}$

Sign Diagrams

1.


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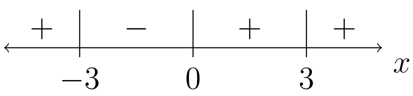

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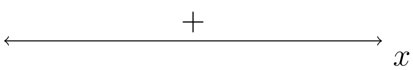
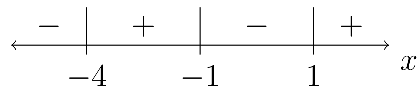
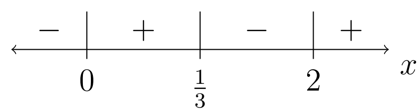


7.


9.


Factoring

Some Special Cases

1. $(2x-5)(x-3)$
3. $2(3x-2)(9x^2+6x+4)$
5. $n(n-1)$
7. $3(2a+15y)(4z-3h)$
9. $4(2x+3y)^2$
11. $-5(4u-x)(3u^2-v)$

Quadratic Type

1. $(x^2+8)(x^2+5)$
3. $(x-1)(x+1)(x-4)(x+4)$
5. $(3x^2-5)(x^2-9) = 3(x-\frac{\sqrt{15}}{3})(x-\frac{\sqrt{15}}{3})(x-3)(x+3)$
7. $(x^2-5)(x^2+2) = (x-\sqrt{5})(x+\sqrt{5})(x^2+2)$
9. $(2x^2-\frac{1}{2})(4x^2+3) = 2(x-\frac{\sqrt{2}}{2})(x+\frac{\sqrt{2}}{2})(4x^2+3)$

Division

Polynomial (Long) Division

1.

$$\begin{array}{r|l} & 5x + \frac{1}{4} \\ \hline 4x^3 & 20x^4 + x^3 + 2x^2 \\ & \underline{-\,20x^4} \\ & \qquad x^3 \\ & \qquad \underline{-\,x^3} \\ & \qquad\qquad 2x^2 \end{array}$$

3.
$$
\begin{array}{r}
2x^3 + \frac{1}{10}x^2 + 4x \\
10x\,\big)\overline{\,20x^4 + x^3 + 40x^2} \\
\underline{-20x^4} \\
x^3 \\
\underline{-x^3} \\
40x^2 \\
\underline{-40x^2} \\
0
\end{array}
$$

5.
$$
\begin{array}{r}
2x^3 + 4x^2 + \frac{1}{2}x \\
6x\,\big)\overline{\,12x^4 + 24x^3 + 3x^2} \\
\underline{-12x^4} \\
24x^3 \\
\underline{-24x^3} \\
3x^2 \\
\underline{-3x^2} \\
0
\end{array}
$$

7.
$$
\begin{array}{r}
x^2 + 5x + \frac{1}{5} \\
10x^2\,\big)\overline{\,10x^4 + 50x^3 + 2x^2} \\
\underline{-10x^4} \\
50x^3 \\
\underline{-50x^3} \\
2x^2 \\
\underline{-2x^2} \\
0
\end{array}
$$

9.
$$
\begin{array}{r}
x - 10 \\
x+8\,\big)\overline{\,x^2 - 2x - 71} \\
\underline{-x^2 - 8x} \\
-10x - 71 \\
\underline{10x + 80} \\
9
\end{array}
$$

11.
$$
\begin{array}{r}
x + 8 \\
x+5\,\big)\overline{\,x^2 + 13x + 32} \\
\underline{-x^2 - 5x} \\
8x + 32 \\
\underline{-8x - 40} \\
-8
\end{array}
$$

13.
$$
\begin{array}{r}
x + 8 \\
x-10\,\big)\overline{\,x^2 - 2x - 89} \\
\underline{-x^2 + 10x} \\
8x - 89 \\
\underline{-8x + 80} \\
-9
\end{array}
$$

15.
$$
\begin{array}{r}
x + 4 \\
x-8\,\big)\overline{\,x^2 - 4x - 38} \\
\underline{-x^2 + 8x} \\
4x - 38 \\
\underline{-4x + 32} \\
-6
\end{array}
$$

17.
$$
\begin{array}{r}
x^2 + 8x - 7 \\
x+7\,\big)\overline{\,x^3 + 15x^2 + 49x - 55} \\
\underline{-x^3 - 7x^2} \\
8x^2 + 49x \\
\underline{-8x^2 - 56x} \\
-7x - 55 \\
\underline{7x + 49} \\
-6
\end{array}
$$

19.
$$
\begin{array}{r}
3x^2 - 9x - 10 \\
x+6\,\big)\overline{\,3x^3 + 9x^2 - 64x - 68} \\
\underline{-3x^3 - 18x^2} \\
-9x^2 - 64x \\
\underline{9x^2 + 54x} \\
-10x - 68 \\
\underline{10x + 60} \\
-8
\end{array}
$$

21.
$$
\begin{array}{r}
x^2 + 3x - 4 \\
x-4\,\big)\overline{\,x^3 - x^2 - 16x + 8} \\
\underline{-x^3 + 4x^2} \\
3x^2 - 16x \\
\underline{-3x^2 + 12x} \\
-4x + 8 \\
\underline{4x - 16} \\
-8
\end{array}
$$

23.

$$
\begin{array}{r@{\,}rrrr}
 & & x^2 & -8x & +7 \\ \cline{2-5}
x-8) & x^3 & -16x^2 & +71x & -56 \\
 & -x^3 & +8x^2 & & \\ \cline{2-3}
 & & -8x^2 & +71x & \\
 & & 8x^2 & -64x & \\ \cline{3-4}
 & & & 7x & -56 \\
 & & & -7x & +56 \\ \cline{4-5}
 & & & & 0
\end{array}
$$

25.

$$
\begin{array}{r@{\,}rrrr}
 & & 8x^2 & -2x & -4 \\ \cline{2-5}
x-8) & 8x^3 & -66x^2 & +12x & +37 \\
 & -8x^3 & +64x^2 & & \\ \cline{2-3}
 & & -2x^2 & +12x & \\
 & & 2x^2 & -16x & \\ \cline{3-4}
 & & & -4x & +37 \\
 & & & 4x & -32 \\ \cline{4-5}
 & & & & 5
\end{array}
$$

27.

$$
\begin{array}{r@{\,}rrr}
 & & x & -4 \\ \cline{2-4}
2x+3) & 2x^2 & -5x & -8 \\
 & -2x^2 & -3x & \\ \cline{2-3}
 & & -8x & -8 \\
 & & 8x & +12 \\ \cline{3-4}
 & & & 4
\end{array}
$$

29.

$$
\begin{array}{r@{\,}rrr}
 & & x & -7 \\ \cline{2-4}
4x+5) & 4x^2 & -23x & -38 \\
 & -4x^2 & -5x & \\ \cline{2-3}
 & & -28x & -38 \\
 & & 28x & +35 \\ \cline{3-4}
 & & & -3
\end{array}
$$

31.

$$
\begin{array}{r@{\,}rrrr}
 & & x^2 & -6x & +6 \\ \cline{2-5}
4x+3) & 4x^3 & -21x^2 & +6x & +19 \\
 & -4x^3 & -3x^2 & & \\ \cline{2-3}
 & & -24x^2 & +6x & \\
 & & 24x^2 & +18x & \\ \cline{3-4}
 & & & 24x & +19 \\
 & & & -24x & -18 \\ \cline{4-5}
 & & & & 1
\end{array}
$$

33.

$$
\begin{array}{r@{\,}rrrr}
 & & x^2 & +3x & -7 \\ \cline{2-5}
2x+6) & 2x^3 & +12x^2 & +4x & -37 \\
 & -2x^3 & -6x^2 & & \\ \cline{2-3}
 & & 6x^2 & +4x & \\
 & & -6x^2 & -18x & \\ \cline{3-4}
 & & & -14x & -37 \\
 & & & 14x & +42 \\ \cline{4-5}
 & & & & 5
\end{array}
$$

35.

$$
\begin{array}{r@{\,}rrr}
 & & x & -3 \\ \cline{2-4}
10x-2) & 10x^2 & -32x & +9 \\
 & -10x^2 & +2x & \\ \cline{2-3}
 & & -30x & +9 \\
 & & 30x & -6 \\ \cline{3-4}
 & & & 3
\end{array}
$$

37.

$$
\begin{array}{r@{\,}rrr}
 & & 9x & +5 \\ \cline{2-4}
3x+8) & 27x^2 & +87x & +35 \\
 & -27x^2 & -72x & \\ \cline{2-3}
 & & 15x & +35 \\
 & & -15x & -40 \\ \cline{3-4}
 & & & -5
\end{array}
$$

39.

$$
\begin{array}{r@{\,}rrr}
 & & 8x & -9 \\ \cline{2-4}
6x-2) & 48x^2 & -70x & +16 \\
 & -48x^2 & +16x & \\ \cline{2-3}
 & & -54x & +16 \\
 & & 54x & -18 \\ \cline{3-4}
 & & & -2
\end{array}
$$

41.

$$
\begin{array}{r@{\,}rrrr}
 & & 6x^2 & +x & +9 \\ \cline{2-5}
4x-7) & 24x^3 & -38x^2 & +29x & -60 \\
 & -24x^3 & +42x^2 & & \\ \cline{2-3}
 & & 4x^2 & +29x & \\
 & & -4x^2 & +7x & \\ \cline{3-4}
 & & & 36x & -60 \\
 & & & -36x & +63 \\ \cline{4-5}
 & & & & 3
\end{array}
$$

Synthetic Division

1. $$\begin{array}{r|rrrrr} -2 & 1 & -4 & 2 & -1 & 1 \\ & & -2 & 12 & -28 & 58 \\ \hline & 1 & -6 & 14 & -29 & \mathbf{59} \end{array}$$

3. $$\begin{array}{r|rrrrr} -2 & 2 & -2 & -10 & 0 & 1 \\ & & -4 & 12 & -4 & 8 \\ \hline & 2 & -6 & 2 & -4 & \mathbf{9} \end{array}$$

5. $$\begin{array}{r|rrrrr} -5 & -1 & -1 & 1 & 1 & 1 \\ & & 5 & -20 & 95 & -480 \\ \hline & -1 & 4 & -19 & 96 & -\mathbf{479} \end{array}$$

7. $$\begin{array}{r|rrrrr} -2 & 12 & -1 & 1 & -3 & 1 \\ & & -24 & 50 & -102 & 210 \\ \hline & 12 & -25 & 51 & -105 & \mathbf{211} \end{array}$$

9. $$\begin{array}{r|rrrrr} 2 & 1 & -3 & 5 & -14 & 2 \\ & & 2 & -2 & 6 & -16 \\ \hline & 1 & -1 & 3 & -8 & -\mathbf{14} \end{array}$$

11. $$\begin{array}{r|rrrrr} 3 & 1 & 0 & 0 & -3 & -4 \\ & & 3 & 9 & 27 & 72 \\ \hline & 1 & 3 & 9 & 24 & \mathbf{68} \end{array}$$

End Behavior

1. As $x \to -\infty,\ f(x) \to \infty$. As $x \to \infty,\ f(x) \to -\infty$.
3. As $x \to -\infty,\ h(x) \to -\infty$. As $x \to \infty,\ h(x) \to -\infty$.
5. As $x \to -\infty,\ \ell(x) \to -\infty$. As $x \to \infty,\ \ell(x) \to \infty$.
7. As $x \to -\infty,\ n(x) \to \infty$. As $x \to \infty,\ n(x) \to \infty$.
9. As $x \to -\infty,\ q(x) \to -\infty$. As $x \to \infty,\ q(x) \to -\infty$.
11. $a_n x^n = -1x^3$, $a_0 = 1$, As $x \to -\infty,\ f(x) \to \infty$. As $x \to \infty,\ f(x) \to -\infty$.
13. $a_n x^n = 27x^3$, $a_0 = -64$, As $x \to -\infty,\ k(x) \to -\infty$. As $x \to \infty,\ k(x) \to \infty$.
15. $a_n x^n = -8x^4$, $a_0 = -98$, As $x \to -\infty,\ m(x) \to -\infty$. As $x \to \infty,\ m(x) \to -\infty$.
17. $a_n x^n = 1x^4$, $a_0 = 16$, As $x \to -\infty,\ g(x) \to \infty$. As $x \to \infty,\ g(x) \to \infty$.
19. $a_n x^n = 1x^4$, $a_0 = 2$, As $x \to -\infty,\ k(x) \to \infty$. As $x \to \infty,\ k(x) \to \infty$.

Local Behavior

1. $x_1 = 0, k_1 = 3$, Crossover; $x_2 = 2, k_2 = 1$, Crossover; $x_3 = -2, k_3 = 1$, Crossover
3. $x_1 = 0, k_1 = 1$, Crossover; $x_2 = 3, k_2 = 2$, Turnaround; $x_3 = -3, k_3 = 1$, Crossover
5. No real roots
7. $x_1 = 1, k_1 = 1$, Crossover; $x_2 = -1, k_2 = 1$, Crossover; $x_3 = -4, k_3 = 1$, Crossover
9. $x_1 = 0, k_1 = 3$, Crossover; $x_2 = \frac{1}{3}, k_2 = 1$, Crossover; $x_3 = 2, k_3 = 1$, Crossover
11. $x_1 = 2, k_1 = 2$, Turnaround; $x_2 = -5, k_2 = 1$, Crossover; $x_3 = 3, k_3 = 1$, Crossover

The Rational Root Theorem

1. $f(x) = \big(x-1\big)\big(x+2\big)\big(x-3\big)$
   List of possible rational roots: $\{\pm 6, \pm 3, \pm 2, \pm 1\}$
3. $f(x) = \big(x-1\big)^2\big(x+1\big)\big(x-6\big)\big(x+6\big)$
   List of possible rational roots: $\{\pm 36, \pm 18, \pm 12, \pm 9, \pm 6, \pm 4, \pm 3, \pm 2, \pm 1\}$
5. $f(x) = \big(x^2+4\big)\big(x-1\big)^2$
   List of possible rational roots: $\{\pm 4, \pm 2, \pm 1\}$
7. $f(x) = \,-2\big(x-\frac{1}{2}\big)\big(x-4\big)\big(x-5\big)$
   List of possible rational roots: $\{\pm 20, \pm 10, \pm 5, \pm 4, \pm\frac{5}{2}, \pm 2, \pm 1, \pm\frac{1}{2}\}$
9. $f(x) = \big(x-1\big)\big(x+2\big)^2\big(x-3\big)$
   List of possible rational roots: $\{\pm 12, \pm 6, \pm 4, \pm 3, \pm 2, \pm 1\}$
11. $f(x) = 6\big(x+2\big)\big(x-\frac{4}{3}\big)\big(x+\frac{5}{2}\big)$
    List of possible rational roots:
    $\{\pm 40, \pm 20, \pm\frac{40}{3}, \pm 10, \pm 8, \pm\frac{20}{3}, \pm 5, \pm 4, \pm\frac{10}{3}, \pm\frac{8}{3}, \pm\frac{5}{2}, \pm 2, \pm\frac{5}{3}, \pm\frac{4}{3}, \pm 1, \pm\frac{5}{6}, \pm\frac{2}{3}, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}\}$

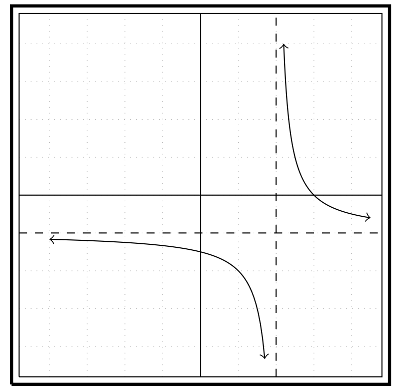
Graphing Summary

1. $f(x) = \,-17\big(x-\frac{5}{17}\big)\big(x+\sqrt{2}\big)\big(x-\sqrt{2}\big)$
3. $f(x) = (3x^2+1)\,\big(x+\sqrt{5}\big)\big(x-\sqrt{5}\big)$
5. $f(x) = \big(x^2+1\big)\big(x-1\big)\big(x+1\big)\big(x-2\big)$
7. $f(x) = (x^3-8)(x^3+2) = (x-2)(x^2+2x+4)(x+\sqrt[3]{2})(x^2-\sqrt[3]{2}x+\sqrt[3]{4})$
9. $f(x) = -(x^2+1)\,\big(x-7\big)$
11. $f(x) = 2x\left(x-\frac{5}{4}+\frac{\sqrt{33}}{4}\right)\left(x-\frac{5}{4}-\frac{\sqrt{33}}{4}\right)$
13. $f(x) = \big(x-1\big)\big(x+2\big)\big(x-3\big)$
15. $f(x) = \big(x-1\big)^2\big(x+1\big)\big(x-6\big)\big(x+6\big)$
17. $f(x) = \big(x-1\big)\big(x+2\big)^2\big(x-3\big)$

19. $f(x) = 2\left(x+1\right)\left(x-\frac{1}{2}\right)\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)$
21. $f(x) = 6\left(x+2\right)\left(x-\frac{4}{3}\right)\left(x+\frac{5}{2}\right)$
23. $f(x) = 36\left(x-\frac{1}{2}\right)^2\left(x+\frac{1}{3}\right)^2$
25. $f(x) = 2\left(x-6\right)\left(x+\frac{7}{4}+\frac{\sqrt{129}}{4}\right)\left(x+\frac{7}{4}-\frac{\sqrt{129}}{4}\right)$
27. $f(x) = -2\left(x+1\right)\left(x-2\right)\left(x+2\right)\left(x-\frac{9}{2}\right)$

Polynomial Inequalities

1. $(-\infty, \sqrt{2}] \cup [\sqrt{2}, \infty)$
3. $[1, \infty)$
5. $(-\infty, 0) \cup (2, \infty)$
7. $[4, \infty)$
9. $(-\infty, 4)$
11. $(-\infty, -1] \cup [3, \infty)$



Chapter 7

Rational Functions

Introduction and Terminology (L54)

Objective: Define and identify key features of rational functions

A *rational function* is a function that can be represented as a ratio (or fraction) of two polynomials p and q. The general form of a rational function f is

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0},$$

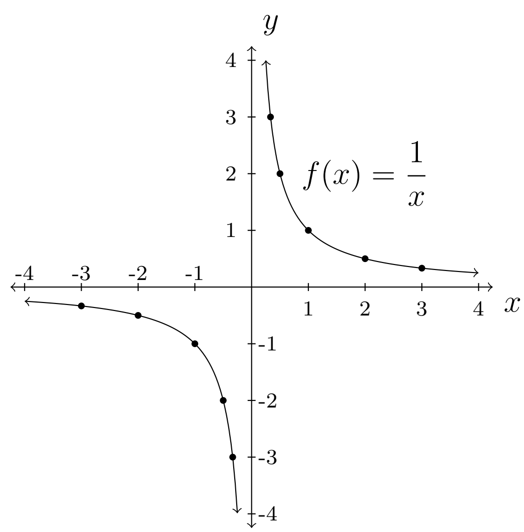
where each of the a_i and b_j are real numbers (for $i, j = 0, 1, 2, \ldots$), with a_n and $b_m \neq 0$, and both m and n are nonnegative integers.

We have already encountered at least one example of a rational function, namely $f(x) = \dfrac{1}{x}$, whose graph we should also be familiar with.

Example 85. $f(x) = \dfrac{1}{x}$

Domain: $x \neq 0$ or $(-\infty, 0) \cup (0, \infty)$

Range: $y \neq 0$ or $(-\infty, 0) \cup (0, \infty)$



As with any function, we can easily identify the $y-$intercept of the graph of f by evaluating the function at $x = 0$.

$$f(0) = \frac{p(0)}{q(0)} = \frac{\cancel{a_n \cdot 0^n} + \cancel{a_{n-1} \cdot 0^{n-1}} + \ldots + \cancel{a_1 \cdot 0} + a_0}{\cancel{b_m \cdot 0^m} + \cancel{b_{m-1} \cdot 0^{m-1}} + \ldots + \cancel{b_1 \cdot 0} + b_0} = \frac{a_0}{b_0}$$

Hence, the graph of f will have a $y-$intercept at the point $\left(0, \dfrac{a_0}{b_0}\right)$.

For rational functions that are already factored, finding the $y-$intercept just requires some simplification after substituting zero for x. The following example demonstrates this.

Example 86. $f(x) = \dfrac{x^3 - x^2 - 8x + 12}{-2x^2 + 14x - 12}$ $\qquad g(x) = \dfrac{(x-2)^2(x+3)}{2(x-6)(1-x)}$

The $y-$intercept of the graph of f is $(0, \frac{12}{-12}) = (0, -1)$.

To find the $y-$intercept of the graph of g, we evaluate $g(0)$ below.

$$g(0) = \frac{(0-2)^2(0+3)}{2(0-6)(1-0)} = \frac{(-2)^2(3)}{2(-6)(1)} = \frac{12}{-12} = -1$$

Hence, as was the case with f, the $y-$intercept of the graph of g is $(0, -1)$. In fact, it is left as an exercise to the reader to show that f and g are the same function.

To identify the domain of a rational function $f(x) = \frac{p(x)}{q(x)}$, we must eliminate all real numbers x which make the denominator equal to zero. In other words, the domain of f is the set of all x such that $q(x) \neq 0$. Identifying the domain of a rational function that is given in factored form is relatively straightforward, whereas rational functions that are given in expanded form must first be factored. Again, we provide an example for each case.

Example 87. $f(x) = \dfrac{3(x+4)(x-2)^2}{(x+3)^2(2x-3)}$ $\qquad g(x) = \dfrac{-x^2 - 4x + 45}{2x^3 - 5x^2 - 18x + 45}$

The domain of f is $x \neq -3, \frac{3}{2}$, or $(-\infty, -3) \cup (-3, \frac{3}{2}) \cup (\frac{3}{2}, \infty)$.

To find the domain of g, one must first use grouping and the $ac-$method to obtain the following factorization of g.

$$\begin{aligned} g(x) &= \frac{-x^2 - 4x + 45}{2x^3 - 5x^2 - 18x + 45} \\ &= \frac{-(x+9)(x-5)}{(x^2-9)(2x-5)} \\ &= \frac{-(x+9)(x-5)}{(x-3)(x+3)(2x-5)} \end{aligned}$$

We now can identify three zeros for the denominator of g that must be excluded from our domain.

The domain of g is $x \neq 3, -3, \frac{5}{2}$, or $(-\infty, -3) \cup (-3, \frac{5}{2}) \cup (\frac{5}{2}, 3) \cup (3, \infty)$.

Notice that in our previous example, the interval notation for each domain always contains one more interval than the number of values that are excluded from the domain. This will always be the case, since we can think of our excluded values as partitioning dividers of the $x-$axis, $(-\infty, \infty)$. In other words, three dividers partition the real number line into four intervals. The multiplicity of each excluded zero of the denominator $q(x)$ will also play a role in the nature of the graph of f, as we will see in a later section.

To find all possible $x-$intercepts for the graph of $f(x) = \frac{p(x)}{q(x)}$, we set the function equal to zero and solve for all possible x, keeping *only* those values that are also in our domain. Since $f(x)$ can only equal zero if its numerator is zero, this amounts to finding all roots of the polynomial p.

$$\begin{aligned} f(x) &= 0 \\ \frac{p(x)}{q(x)} &= 0 \\ \cancel{q(x)} \cdot \frac{p(x)}{\cancel{q(x)}} &= 0 \cdot q(x) \\ p(x) = 0,\ q(x) \neq 0 & \end{aligned}$$

Furthermore, referring to the results from the chapter on polynomials, we can again use the multiplicity of each zero of p to determine whether the corresponding $x-$intercept will represent a crossover or turnaround point, as in the following example.

Example 88. $f(x) = \dfrac{(x+3)^2(x-1)(x-4)}{(x-1)^2(x^2+2)}$

The numerator has three zeros ($x = -3, 1,$ and 4), but the corresponding graph only has two $x-$intercepts, since $x = 1$ is also a zero of the denominator, and therefore not in the domain of f. The $x-$intercept at $(-3, 0)$ is a turnaround point, since the multiplicity of the zero $x = -3$ is even. The $x-$intercept at $(4, 0)$ is a crossover point, since the multiplicity of the zero $x = 4$ is odd.

The graph of the reciprocal function $f(x) = \frac{1}{x}$ in our first example of this chapter also has two interesting characteristics, known as *asymptotes*. Asymptotes do not appear in the graph of a polynomial, but often show up when analyzing rational functions and more advanced functions such as exponentials and logarithms. An asymptote usually appears in the form of a line (horizontal, vertical, or slanted) that the graph of a function f approaches. The graph of $f(x) = \frac{1}{x}$ has a horizontal asymptote at $y = 0$, since the end behavior of the graph (as $x \to \pm\infty$) approaches zero, as well as a vertical asymptote at $x = 0$, since the local behavior of the graph near $x = 0$ (as $x \to 0$) tends towards $\pm\infty$.

Later on, we will outline the procedures for finding both horizontal and vertical asymptotes for a rational function, as well as the case where the graph of f has a slant (or oblique) asymptote. We will also see an example of the special case of a curvilinear asymptote, in which the graph of f approaches a nonlinear curve, as x approaches $\pm\infty$. Although all polynomials are, by definition, rational functions (with denominator $q(x) = 1$), the concept of an asymptote (horizontal, vertical, or otherwise) demonstrates a critical aspect that separates most rational functions from polynomials. We close this section with a few more examples of rational functions and their graphs. While each function shares a common domain ($x \neq 5$), the corresponding graphs exhibit some clear differences. Specifically, close attention should be paid to the existence, location, and nature of the graph of each function near the $y-$int and $x-$int(s), as well as the horizontal and vertical asymptotes.

Example 89.

$$f(x) = \frac{-2x+4}{x-5} = \frac{-2(x-2)}{x-5}$$

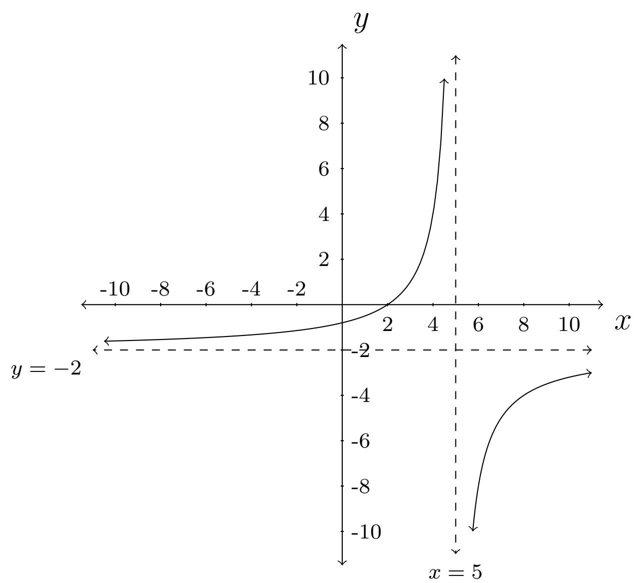
$y-$intercept at $(0, -\frac{4}{5})$

Domain: $x \neq 5$ or $(-\infty, 5) \cup (5, \infty)$

$x-$intercept at $(2, 0)$

Horizontal asymptote at $y = -2$

Vertical asymptote at $x = 5$



Example 90.

$$g(x) = \frac{x^2-4x+4}{x-5} = \frac{(x-2)^2}{x-5}$$

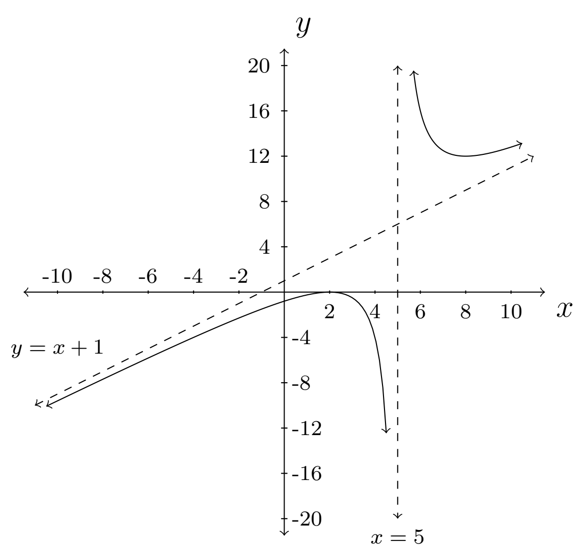
$y-$intercept at $(0, -\frac{4}{5})$

Domain: $x \neq 5$ or $(-\infty, 5) \cup (5, \infty)$

$x-$intercept at $(2, 0)$

Slant (oblique) asymptote at $y = x + 1$

Vertical asymptote at $x = 5$



Example 91.

$$h(x) = \frac{x^2+25}{x^2-10x+25} = \frac{x^2+25}{(x-5)^2}$$

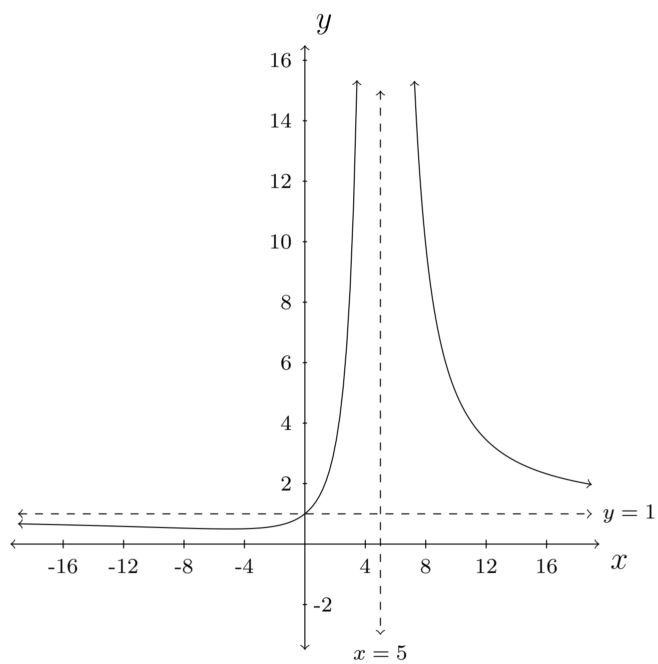
$y-$intercept at $(0, 1)$

Domain: $x \neq 5$ or $(-\infty, 5) \cup (5, \infty)$

No $x-$intercepts

Horizontal asymptote at $y = 1$

Vertical asymptote at $x = 5$



Example 92.

$$k(x) = \frac{x^3-5x^2}{10x-50} = \frac{x^2(x-5)}{10(x-5)}$$

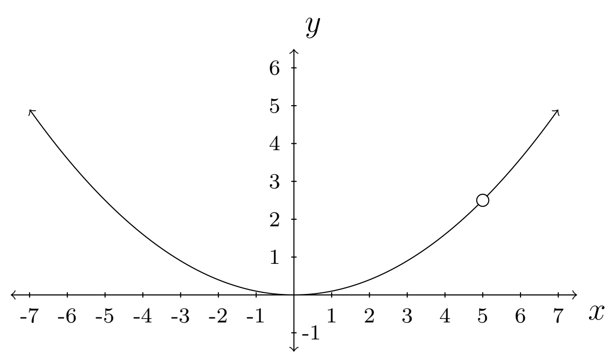
$y-$intercept at $(0, 0)$

Domain: $x \neq 5$ or $(-\infty, 5) \cup (5, \infty)$

$x-$intercept at $(0, 0)$

No horizontal or vertical asymptotes

Hole at $(5, \frac{5}{2})$



Sign Diagrams (L55)

Objective: Construct a sign diagram for a given rational function.

As with polynomial functions, throughout this chapter we will periodically reference the sign diagram of a rational function or expression, to both answer questions about particular functions and verify our work. As before, there is a reliance on factorization that is needed for construction of a sign diagram, since the roots of a given expression will be used as the dividers in the corresponding sign diagram.

We begin with an example for polynomial functions.

Example 93. Construct a sign diagram for the polynomial function $f(x) = x^3 - 5x^2 + 3x + 9$. Use the fact that $f(-1) = 0$.

Since f is a polynomial that has four terms, we could first try to factor f by grouping. But we quickly see that this method will fail to yield a factorization.

$$\begin{aligned} f(x) &= x^3 - 5x^2 + 3x + 9 \\ &= x^2(x-5) + 3(x+3) \end{aligned}$$

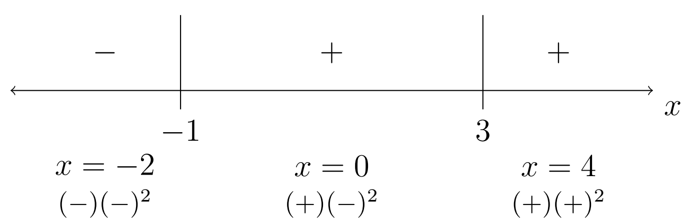
$$x - 5 \neq x - 3$$

Instead, if we use the fact that $f(-1) = 0$, we can employ polynomial division to factor f completely.

$$\begin{array}{r|l} & x^2 - 6x + 9 \\ \hline x+1 & x^3 - 5x^2 + 3x + 9 \\ & -x^3 - x^2 \\ \hline & \quad -6x^2 + 3x \\ & \quad 6x^2 + 6x \\ \hline & \qquad 9x + 9 \\ & \qquad -9x - 9 \\ \hline & \qquad\quad 0 \end{array}$$

$$\begin{aligned} f(x) &= x^3 - 5x^2 + 3x + 9 \\ &= (x+1)(x^2 - 6x + 9) \\ &= (x+1)(x-3)^2 \end{aligned}$$

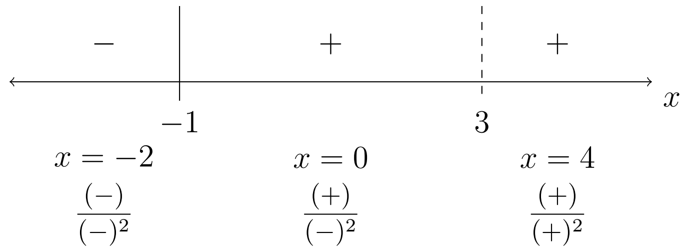
Roots of f: $x = -1, 3$



Since a rational function includes an expression in a denominator, the only additional consideration that we need to make to construct a sign diagram is to identify the roots of both the numerator *and* the denominator as dividers in our diagram. In other words, the dividers of our diagram for a rational function f will consist of all roots and all values not in the domain of f.

Example 94. Construct a sign diagram for the rational function $g(x) = \dfrac{x+1}{(x-3)^2}$.

Since $x = -1$ is a root of g, and $g(3)$ is undefined (the domain of g is $x \neq 3$), we will again place dividers at -1 and 3.



In Example 94, since $x = 3$ is not in the domain of g, we have used a dashed divider to signify this fundamental difference from Example 93. As we will see later, this results in different graphical implications. In other words, the graph of f behaves differently when $x = c$ is a root of f versus when it is excluded from the domain.

Example 95. Construct a sign diagram for the rational function $h(x) = \dfrac{x^3 - 5x^2 + 3x + 9}{x - 3}$.

Since the numerator of h matches f from Example 93, we can factor h as follows.

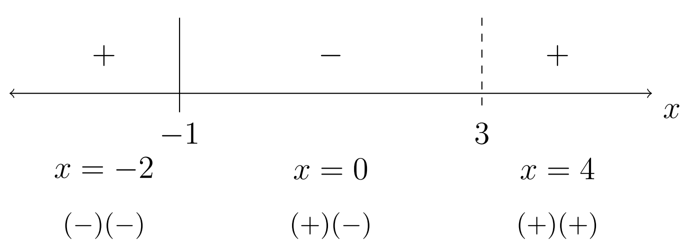
$$h(x) = \frac{(x+1)(x-3)^2}{x-3}$$

Although we might be tempted to cancel out the denominator of h completely, it still remains that $h(3)$ is undefined, i.e., the domain of h is $x \neq 3$. Hence, we will include a dashed divider in our sign diagram at $x = 3$.

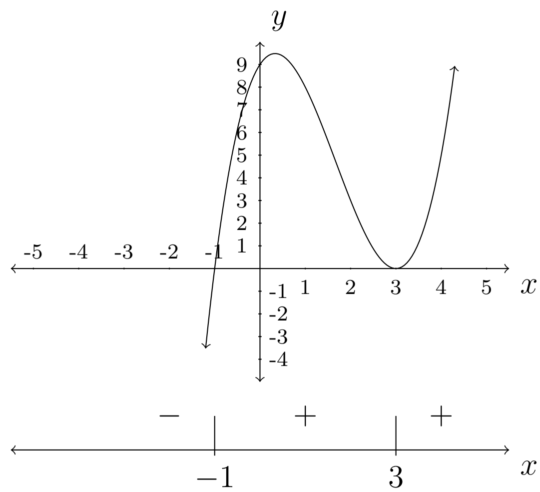
When determining the various signs for our diagram, however, we can consider working with the simplified expression

$$\frac{(x+1)(x-3)^{\cancel{2}}}{\cancel{x-3}} = (x+1)(x-3),$$

since none of our test values equal 3.

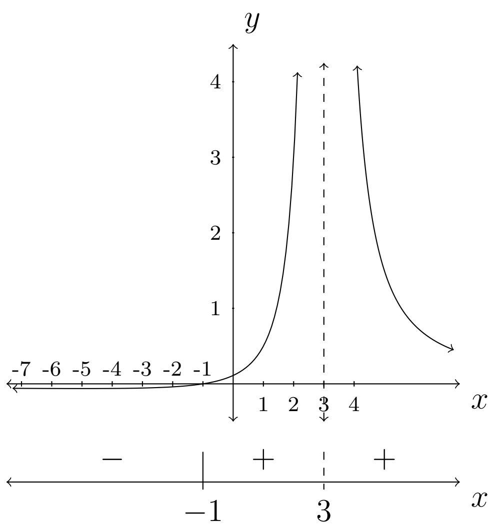


Although we are not yet ready to completely graph a rational function, let's look at the graphs of each of our last three examples, in order to see the differences at $x = 3$.

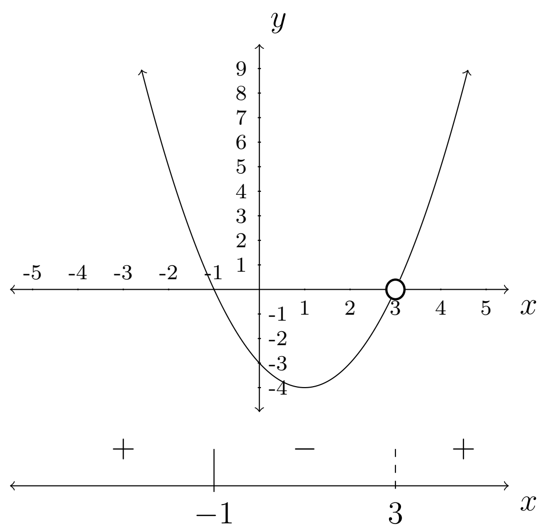


$f(x) = (x+1)(x-3)^2$

In the case of Example 93 above, there should be no real surprises, since we are given a cubic polynomial with positive leading coefficient. The multiplicity of $x = -1$ being odd produces a sign change in our diagram, which results in a "crossover point" at our corresponding $x-$intercept. Alternatively, the even multiplicity of $x = 3$ creates a "turnaround point" (or bounce off) at the corresponding $x-$intercept.



$$g(x) = \frac{x+1}{(x-3)^2}$$



$$h(x) = \frac{(x+1)(x-3)^2}{x-3}$$

In Examples 94 and 95 above, we see the same $x-$intercept at $x = -1$ as in Example 93. The fundamental difference between polynomials and rational functions centers around what happens at $x = 3$. Unlike our first graph, both rational functions exhibit a break in the graph at this value of x, since it is not in the domain of either function.

Still, the difference in the expressions for both g and h results in two distinctly different breaks in each of our graphs (a vertical asymptote in the first graph and a hole in the second graph). We will more closely examine these differences in a later section. For the purposes of this section, we simply wish to stress the importance of identifying the corresponding divider in each of our sign diagrams.

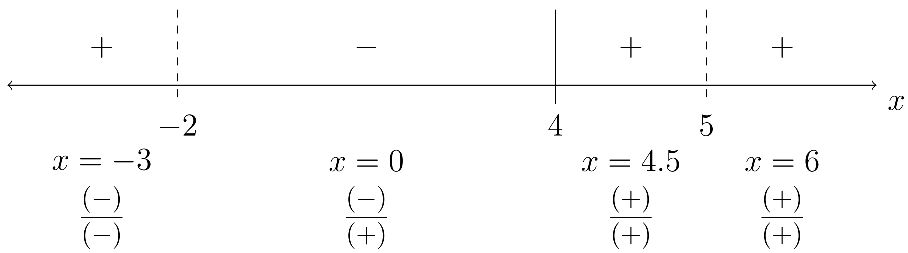
Example 96. Construct a sign diagram for the rational function $f(x) = \dfrac{x^2 - 9x + 20}{x^2 - 3x - 10}$.

A factorization of f gives us the following.

$$f(x) = \frac{(x-4)(x-5)}{(x+2)(x-5)}$$

Hence, we need dividers for $x = -2, 4$, and 5. Since our domain is $x \neq -2, 5$ we will use dashed dividers for these values, and a solid divider for the $x-$intercept at $x = 4$.

As in Example 95, we will use the simplified expression $\dfrac{x-4}{x+2}$ for each of our test values, since we are not testing $x = 5$.



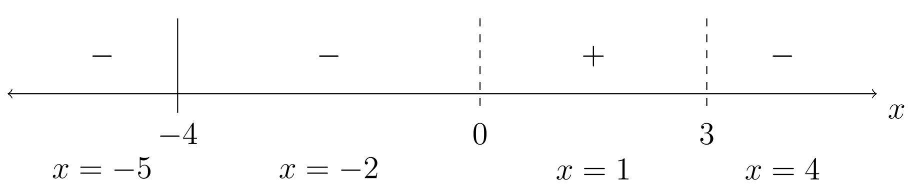
We leave it as an exercise to the reader to graph

$$f(x) = \frac{x^2 - 9x + 20}{x^2 - 3x - 10} = \frac{(x-4)(x-5)}{(x+2)(x-5)}$$

using Desmos, in order to see the differences near the two values excluded from our domain, $x = -2$ and $x = 5$.

We conclude this section with one final example

Example 97. Identify a rational function f having the following sign diagram.



We know that at each divider value c, our function must have a corresponding factor of $x - c$. Furthermore, solid dividers will see our factor in the numerator, whereas dashed dividers signify that c is not in our domain, and so our factor must lie in the denominator. This gives us the following candidate for f.

$$\frac{x+4}{x(x-3)}$$

But also, the lack of a sign change at $x = -4$ (negative to negative) tells us that the corresponding $x-$intercept will be a turnaround point. This further tells us that the multiplicity at $x = -4$ must be even. So we refine our candidate to the following.

$$\frac{(x+4)^2}{x(x-3)}$$

Since our signs change at $x = 0$ (negative to positive) and $x = 3$ (positive to negative), we will keep an odd multiplicity of 1 for each of their respective factors in the denominator.

A simple check of one of our test values will tell us whether our answer is correct.

When $x = -5$, we get $\dfrac{(-)^2}{(-)(-)} = +$, which is not what we want. This suggests that each of our signs will be the opposite of what we are looking for. By multiplying our candidate by a negative, we obtain our final answer.

$$f(x) = \frac{-(x+4)^2}{x(x-3)}$$

We leave it as an exercise for the reader to check that our function does indeed match the given diagram.

End Behavior

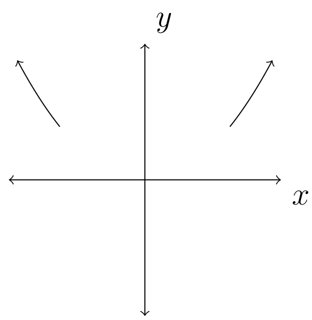
Horizontal Asymptotes (L56)

Objective: Identify a horizontal asymptote in the graph of a rational function.

Next, we will look at the end (or long run) behavior of the graph of a rational function f, as $x \to \pm\infty$. For clarity, we will first state the main result of this subsection.

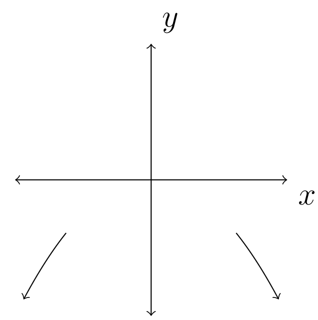
> Let $f(x) = \dfrac{p(x)}{q(x)}$ be a rational function with leading terms $a_n x^n$ and $b_m x^m$ of $p(x)$ and $q(x)$, respectively.
>
> - If $n = m$, the graph of f will have a horizontal asymptote at $y = \dfrac{a_n}{b_m}$.
> - If $n < m$, the graph of f will have a horizontal asymptote at $y = 0$.
> - If $n > m$, the graph of f will not have a horizontal asymptote.

Since any polynomial is, by definition, also a rational function, we will begin by including the possibilities that $f(x) \to \infty$ or $f(x) \to -\infty$ for either the left (as $x \to -\infty$) or right (as $x \to \infty$) end behavior of the graph of a rational function f.



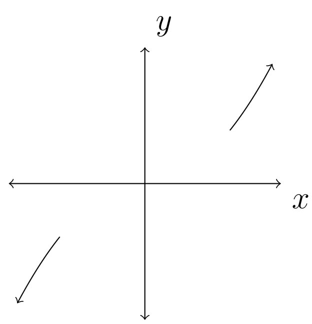
As $x \to \infty,\ f(x) \to \infty$

As $x \to -\infty,\ f(x) \to \infty$



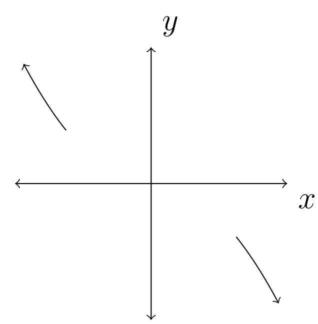
As $x \to \infty,\ f(x) \to -\infty$

As $x \to -\infty,\ f(x) \to -\infty$



As $x \to \infty,\ f(x) \to \infty$

As $x \to -\infty,\ f(x) \to -\infty$



As $x \to \infty,\ f(x) \to -\infty$

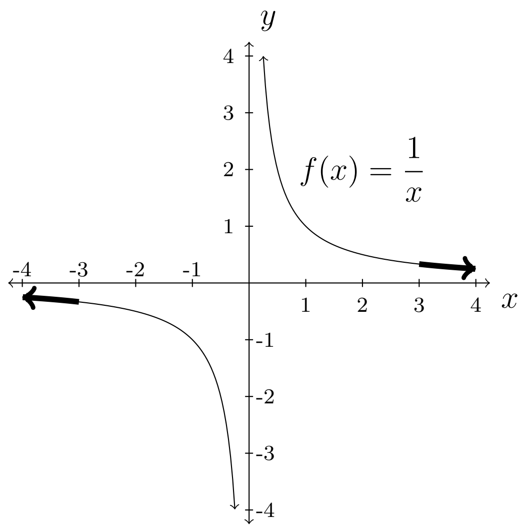
As $x \to -\infty,\ f(x) \to \infty$

Recall that we used two aspects of a polynomial to identify the end behavior of its graph:

1. the parity of the degree (even or odd), and
2. the sign of the leading coefficient (positive or negative).

As with polynomials, we will use the degree and leading coefficient of both the numerator and denominator of a rational function f, to identify the end behavior of its graph.

To start, let us again consider the graph of the reciprocal function $f(x) = \dfrac{1}{x}$.



This example presents us with the first instance in which a graph does not tend towards either ∞ or $-\infty$, but instead "levels off" as the values of x grow in either the positive (right) or negative (left) direction.

$$\text{As } x \to \infty,\ f(x) \to 0^+.$$

$$\text{As } x \to -\infty,\ f(x) \to 0^-.$$

Here, we use a $+$ or $-$ in the exponent to further describe how the tails of the graph approach 0, either from *above* $(+)$ or from *below* $(-)$. These identifiers can just as easily be omitted entirely, but provide a bit more insight into the graph of the function f. The tails of the graph are thickened for additional emphasis of this concept.

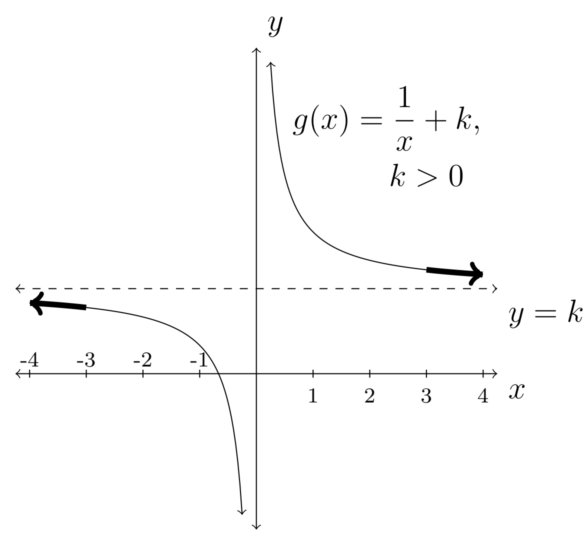
In fact, for any real number k, we can transform the graph above, by simply adding k to the function, to produce a new rational function whose graph levels off at k. The resulting graph represents a vertical shift of the graph of $\frac{1}{x}$ by k units. The shift is up when $k > 0$ and down when $k < 0$.

Example 98.

$$\begin{aligned} g(x) &= f(x) + k \\ &= \frac{1}{x} + k \cdot \frac{x}{x} \\ &= \frac{kx+1}{x} \end{aligned}$$

$$\text{As } x \to \infty,\ g(x) \to k^+.$$

$$\text{As } x \to -\infty,\ g(x) \to k^-.$$



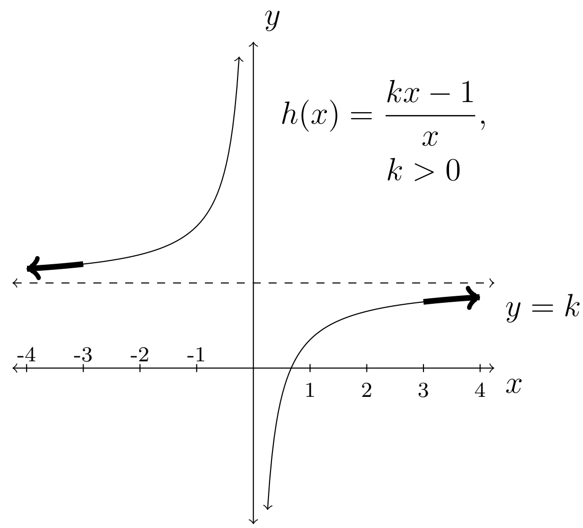
Furthermore, if we first replace x by $-x$ in f, and then add k, this will reflect the graph of f about the $y-$axis and shift it vertically by k units, producing a slightly different end behavior as in our previous two examples.

Example 99.

$$\begin{aligned} h(x) &= f(-x)+k \\ &= \frac{1}{-x}+k \\ &= -\frac{1}{x}+k\cdot\frac{x}{x} \\ &= \frac{kx-1}{x} \end{aligned}$$

As $x \to \infty,\ h(x) \to k^-$.

As $x \to -\infty,\ h(x) \to k^+$.



One of the most important takeaways from each of the above examples is that, unlike a polynomial, a rational function can possibly "level off" along a horizontal line $y = k$, for any real number k, as x approaches either ∞ or $-\infty$. In this case, we say that the corresponding graph has a *horizontal asymptote* at $y = k$.

Notice that in each of the previous two examples, when adding $k \neq 0$, we have increased the degree of the numerator (from 0 to 1), which matches the degree of the denominator.

$$g(x) = \frac{kx^1+1}{1x^1} \qquad\qquad h(x) = \frac{kx^1-1}{1x^1}$$

In fact, whenever the numerator and denominator of a rational function f have the *same* degree, the corresponding graph will have a horizontal asymptote along the line $y = \frac{a_n}{b_m}$, where a_n and b_m represent the leading coefficients of the numerator and denominator of f, respectively.

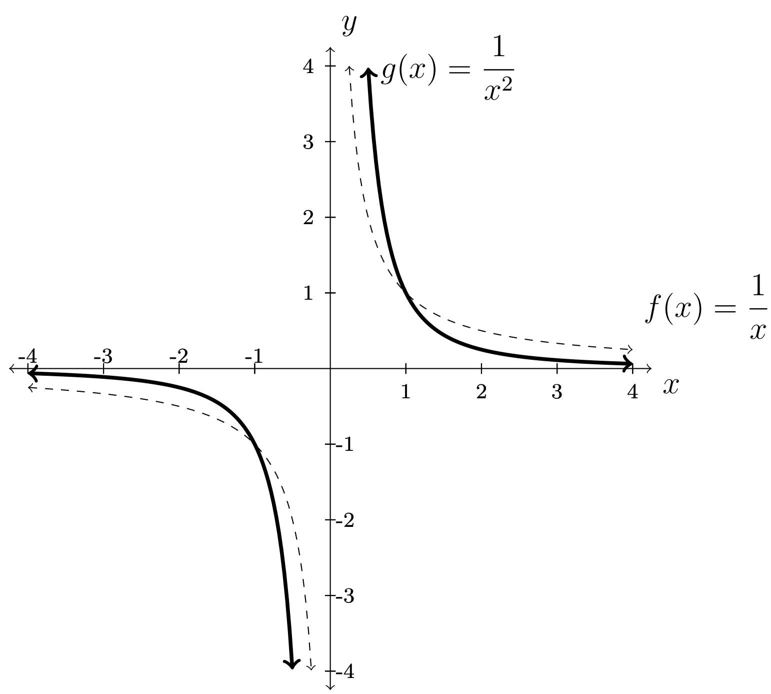
Stated more formally:

> Let $f(x) = \dfrac{p(x)}{q(x)}$, with leading terms $a_n x^n$ and $b_m x^m$ of $p(x)$ and $q(x)$, respectively. If $n = m$, the graph of f will have a horizontal asymptote at $y = \dfrac{a_n}{b_m}$. In other words, when the degrees of p and q are equal, as $x \to \pm\infty$, $f(x) \to \dfrac{a_n}{b_m}$.

We see this at work in Example 89, where the graph of $f(x) = \dfrac{-2x+4}{x-5}$ has a horizontal asymptote at $y = \frac{-2}{1} = -2$, and again in Example 91, where the graph of $h(x) = \dfrac{x^2+25}{x^2-10x+25}$ has a horizontal asymptote at $y = \frac{1}{1} = 1$. In each of these cases, we can now easily confirm that, for example, as $x \to \pm\infty$, $h(x) \to 1$. To determine the precise nature of the tails or ends of the graph ($h(x) \to 1^+$ versus $h(x) \to 1^-$), however, requires further analysis. This is aided by polynomial division and often a sign diagram, which we will see in a subsequent section.

We have now seen that the graph of a rational function will "level off" along a horizontal line $y = \dfrac{a_n}{b_m}$ when n and m are equal. What remains is to determine the end behavior when $n \neq m$. This gives us two additional cases to consider: 1) $n < m$ and 2) $n > m$.

In the case when $n < m$, we may again look to $f(x) = \dfrac{1}{x}$, which has a horizontal asymptote along the $x-$axis, i.e., the line $y = 0$. In fact, this will generally be the case when $n < m$, since we can think of the denominator, $q(x)$, as growing much faster than the numerator, $p(x)$, whenever x approaches either ∞ or $-\infty$. If we look at the difference in degrees of p and q, we can further see that the nature with which our graph approaches the $x-$axis changes. Specifically, When $m - n$ is larger (2,3,...), the graph will approach the $x-$axis more quickly. We will use $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x^2}$ to illustrate this point. Recall that both f and g have numerators which have a degree of zero.



As x gets large, the act of squaring causes the denominator to grow more quickly, which in turn, makes the values of $g(x) = \dfrac{1}{x^2}$ approach zero more quickly than the values of $f(x) = \dfrac{1}{x}$. Regardless, in both cases, the horizontal asymptote is the same:

> Let $f(x) = \dfrac{p(x)}{q(x)}$, with degrees n and m of $p(x)$ and $q(x)$, respectively. If $n < m$, the graph of f will have a horizontal asymptote at $y = 0$. In other words, as $x \to \pm\infty$, $f(x) \to 0$.

Revisiting Example 87, the graph of $g(x) = \dfrac{-x^2 - 4x + 45}{2x^3 - 5x^2 - 18x + 45}$ will again have a horizontal asymptote at $y = 0$, since $m > n$. In this case, we can further reason the nature of the graph of g (whether it approaches 0 above or below the $x-$axis) by looking at the leading terms for both the numerator and denominator of g.

$$g(x) = \frac{-x^2 - 4x + 45}{2x^3 - 5x^2 - 18x + 45}$$

Since the numerator of g has a negative leading coefficient and an even degree, as $x \to \infty$, the numerator of g will approach $-\infty$. The denominator, however, has a positive leading coefficient and an odd degree. Consequently, as $x \to \infty$, the denominator of g will approach ∞. But since $m > n$ and the numerator and denominator differ in sign, as $x \to \infty$, $g(x) \to 0^-$. All of this means that the graph of g will approach the $x-$axis from *below* on the right.

Alternatively, as $x \to -\infty$, we can see that the graph of g will approach the $x-$axis from *above* on the left, since both the numerator and denominator of g will approach $-\infty$.

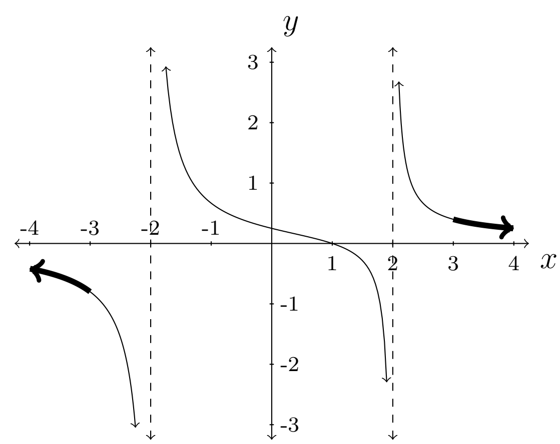
Example 100. In this example, we see the graph of

$$f(x) = \frac{x-1}{x^2-4}.$$

Since the degree of the denominator is greater than the degree of the numerator, we see that the graph of f (again) levels off along the $x-$axis, i.e., the line $y = 0$.

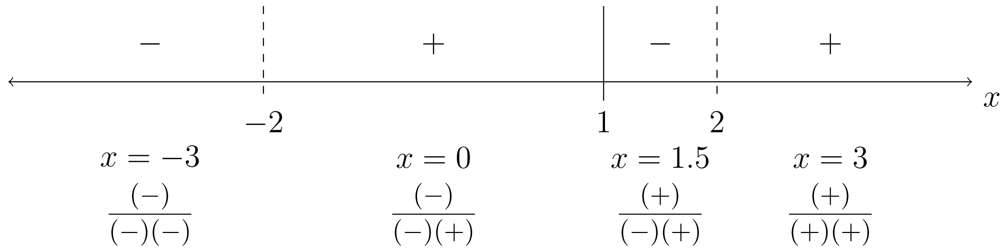
We can also readily determine whether the graph approaches the $x-$axis above or below without too much difficulty.

$$\text{As } x \to \infty,\ f(x) \to 0^+.$$
$$\text{As } x \to -\infty,\ f(x) \to 0^-.$$



In the case when $m > n$, a sign diagram can also verify whether $f(x) \to 0$ from above or below, and we have included one here. The signs on the ends of the diagram will correspond to the nature of the tails of the graph above.

$$f(x) = \frac{x-1}{x^2-4} = \frac{x-1}{(x-2)(x+2)}$$

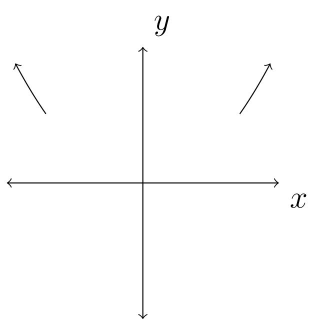


Our last case to consider for a rational function $f(x)$, is when the numerator has a greater degree than the denominator, $n > m$. This case includes all polynomial functions, which represent rational functions, in which the denominator equals the constant (degree-0) function $q(x) = 1$. In this case, as $x \to \pm\infty$, we can think of the numerator as growing faster than the denominator. The result is a graph that has no horizontal asymptote.

> Let $f(x) = \dfrac{p(x)}{q(x)}$, with degrees n and m of $p(x)$ and $q(x)$, respectively. If $n > m$, the graph of f will have no horizontal asymptote.

The possibilities for the tails of the graph of f ($f(x) \to \infty$ or $f(x) \to -\infty$) are, again, determined by the leading terms for both the numerator and denominator of f.

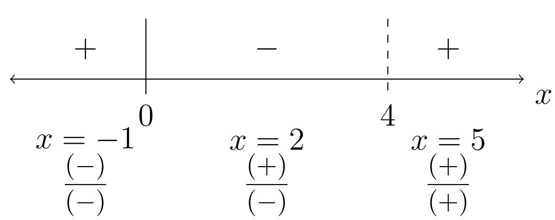
Example 101. Consider the function $f(x) = \dfrac{x^3}{x-4}$. The numerator has a degree of 3, and the denominator has a degree of 1. Hence, the graph of f has no horizontal asymptotes. As $x \to \infty$, we can further see that both the numerator and denominator will approach $+\infty$. Hence, we can conclude that $f(x) \to \infty$. Alternatively, as $x \to -\infty$, we see that both the numerator and denominator will approach $-\infty$, and so $f(x) \to \infty$. This tells us that the tails of f will both point upwards. As in the previous example, a sign diagram confirms this.



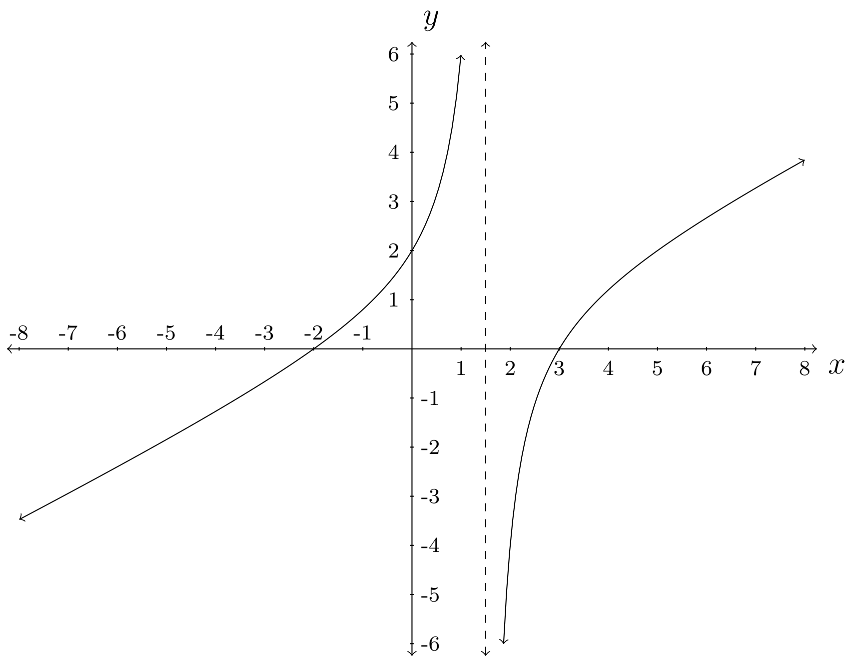
As $x \to \infty$, $f(x) \to \infty$

As $x \to -\infty$, $f(x) \to \infty$

$$f(x) = \frac{x^3}{x-4}$$



Example 102. Let $g(x) = \dfrac{x^2 - x - 6}{2x - 3}$. Again, the graph of g will not have a horizontal asymptote, since the degree of the numerator is greater than the degree of the denominator. We include the graph of g below.



Examples 90 and 92 present us with two additional rational functions whose graphs do not include a horizontal asymptote. In each case, as $x \to \infty$, the $y-$coordinate also approaches ∞. Still, the nature in which the $y-$coordinates grow as x grows is distinctly different for each function and its graph. The same can be said for our last two examples. This has to do with the difference between the degrees of the numerator and denominator, and we will address this next.

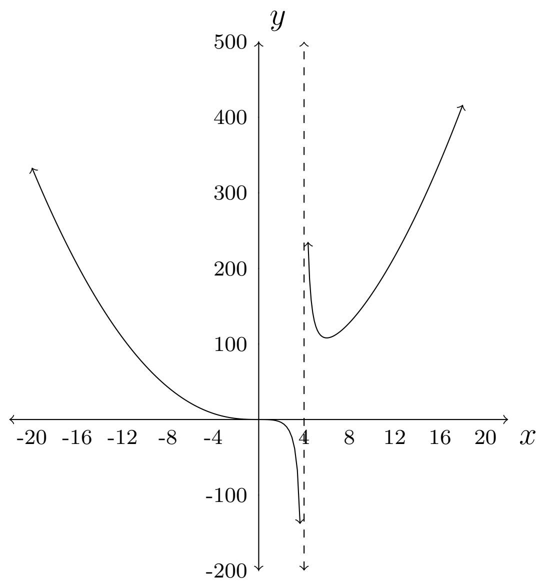
Slant Asymptotes (L57)

Objective: Identify a slant or curvilinear asymptote in the graph of a rational function.

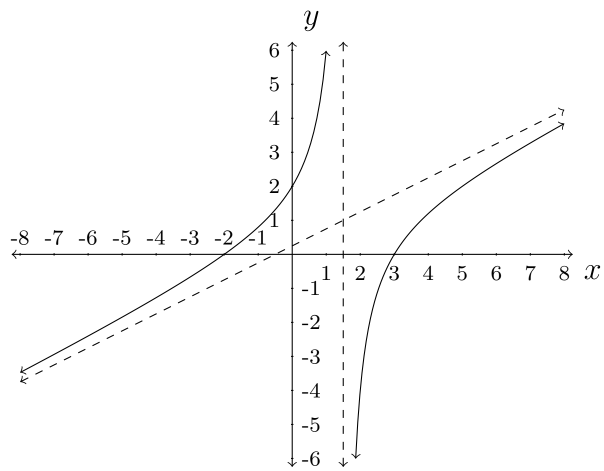
For a rational function

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0},$$

when $n > m$, we know that the graph of f will have no horizontal asymptotes. Depending upon the difference between n and m, however, there is more to discover about the nature of the graph of f, as $x \to \pm\infty$. For example, below are the graphs of Examples 101 and 102.



$$f(x) = \frac{x^3}{x-4}$$



$$g(x) = \frac{x^2 - x - 6}{2x - 3}$$

In the case of $g(x) = \dfrac{x^2 - x - 6}{2x - 3}$, we see that as $x \to \pm\infty$, the graph of g actually approaches a linear asymptote. Whereas horizontal asymptotes are horizontal lines, having a slope of zero, this new type of linear asymptote has a non-zero slope and is consequently *slanted*. Hence, we say that the graph of g contains a *slant* or *oblique asymptote*.

On the other hand, the graph of $f(x) = \dfrac{x^3}{x-4}$ does not appear to contain a slant asymptote. In fact, as $x \to \pm\infty$, the graph of f resembles a parabola. In cases such as these, we could say that the graph of f contains a *curvilinear asymptote*. In other words, the graph of f approaches some identifiable non-linear curve, as x approaches $\pm\infty$.

The central idea behind slant and curvilinear asymptotes is the same, and only requires an understanding of polynomial division. Nevertheless, we will focus almost entirely on slant asymptotes in this section, leaving the topic of curvilinear asymptotes for our last example.

So what causes the graph of a rational function to have such asymptotes? We ponder this question, keeping in mind that we have already imposed the requirement that the degree of the numerator be greater than the denominator, $n > m$. Simply stated, it is the difference between the degrees, $n - m$, which will determine whether the corresponding graph possesses a slant or curvilinear asymptote.

Let's look more closely at $g(x) = \dfrac{x^2 - x - 6}{2x - 3}$. In this case, the degree of the numerator is equal one more than the degree of the denominator, $n = m + 1$ (or $n - m = 1$). This is precisely the case in which the corresponding graph will always contain a slant asymptote!

Let $f(x) = \dfrac{p(x)}{q(x)}$, with degrees n and m of $p(x)$ and $q(x)$, respectively. If $n = m + 1$, the graph of f will have a slant (or oblique) asymptote. In other words, when $n = m + 1$, as $x \to \pm\infty$, the graph of f will approach some line $y = cx + d$, where $c \neq 0$.

Next comes the question of how to find the equation of our slant asymptote. The answer comes to us by employing polynomial division. Recall that when we divide two polynomials, we have the following result.

$$\frac{D(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)} \qquad \text{or} \qquad \frac{\text{Dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

Here, we use the labels d, D, q, r to represent the *divisor, Dividend, quotient,* and *remainder*, respectively. Relabeling in terms of our rational function, $f(x) = \dfrac{p(x)}{q(x)}$, we have the following.

$$\frac{p(x)}{q(x)} = h(x) + \frac{r(x)}{q(x)}$$

But in the case where the degree of p is one more than the degree of q, the resulting quotient function, $h(x)$, will be the desired linear equation for our slant asymptote.

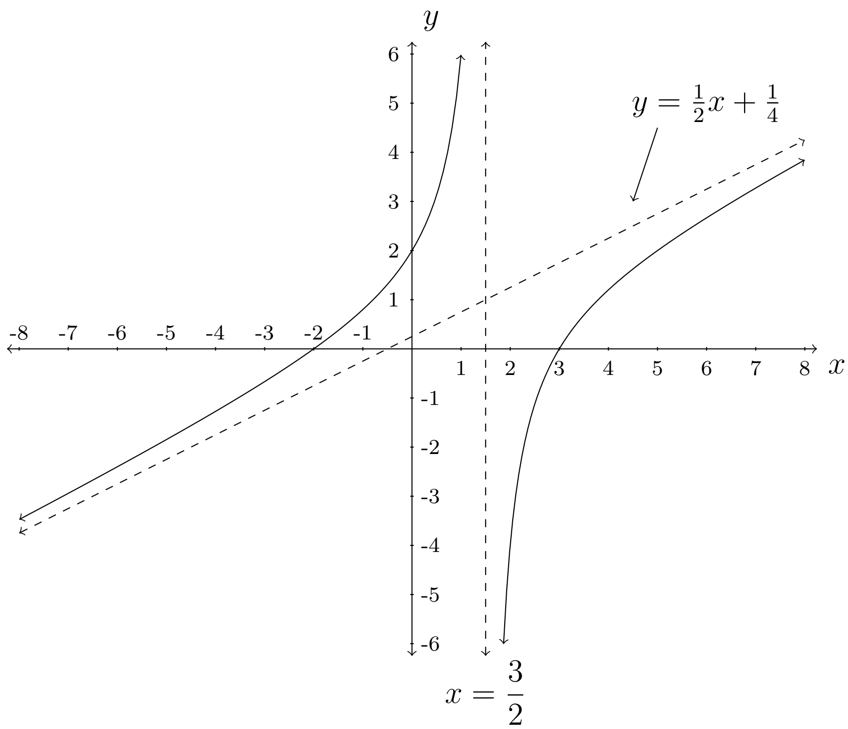
$$\frac{p(x)}{q(x)} \;=\; \underbrace{cx + d}_{\substack{\text{slant}\\ \text{asymptote}}} \;+\; \frac{r(x)}{q(x)}$$

Example 103. Find the equation of the slant asymptote in the graph of $g(x) = \dfrac{x^2 - x - 6}{2x - 3}$.

$$\begin{array}{r|rrr}
 & \tfrac{1}{2}x & +\tfrac{1}{4} & \\ \hline
2x-3 & x^2 & -x & -6 \\
 & -x^2 & +\tfrac{3}{2}x & \\ \hline
 & & \tfrac{1}{2}x & -6 \\
 & & -\tfrac{1}{2}x & +\tfrac{3}{4} \\ \hline
 & & & -\tfrac{21}{4}
\end{array}$$

Hence, $\dfrac{x^2 - x - 6}{2x - 3} = \frac{1}{2}x + \frac{1}{4} + \dfrac{-\frac{21}{4}}{2x - 3}$.

So, the graph of g has a slant asymptote at $y = \frac{1}{2}x + \frac{1}{4}$.



$$g(x) = \frac{x^2 - x - 6}{2x - 3} \;=\; \tfrac{1}{2}x + \tfrac{1}{4} + \frac{-\frac{21}{4}}{2x - 3}$$

It is worth noting in this last example, that the trailing expression $\dfrac{-\frac{21}{4}}{2x-3}$, involving the remainder in our polynomial division, can be extremely helpful in determine whether or not the graph of g sits above or below the slant asymptote, as $x \to \pm\infty$. In this case, when x is a large positive value, our trailing expression will be negative. Hence, as $x \to \infty$ the values of $g(x)$ on the right-side tail of our graph will lie slightly *below* our slant asymptote, since

$$\frac{x^2 - x - 6}{2x - 3} = \tfrac{1}{2}x + \tfrac{1}{4} + (\text{a small negative value}).$$

On the other hand, as $x \to -\infty$, the trailing expression will be positive, and so

$$\frac{x^2 - x - 6}{2x - 3} = \tfrac{1}{2}x + \tfrac{1}{4} + (\text{a small positive value}).$$

Hence, the left-side tail of the graph of g will lie slightly *above* our slant asymptote.

Example 104. Find the equation of the slant asymptote in the graph of $f(x) = \dfrac{-2x^3 + x^2 - 2x + 3}{x^2 + 1}$.

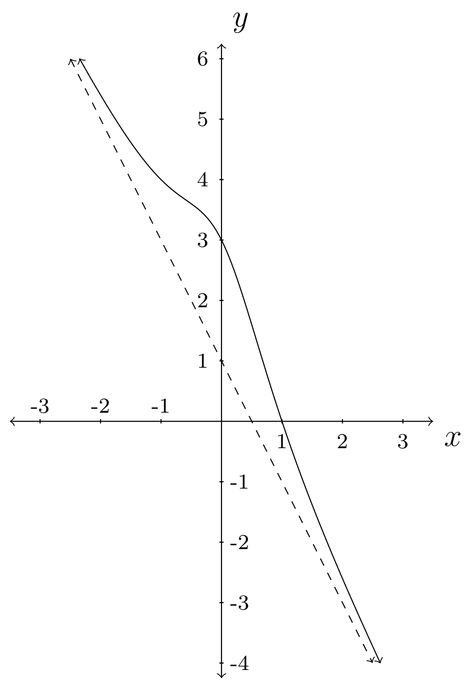
In this example, again, we know that the graph of f will include a slant asymptote, since the degree of the numerator is one more than the degree of the denominator, $3 = 2 + 1$.

$$
\begin{array}{r|l}
 & - 2x + 1 \\ \hline
x^2 + 1 & -2x^3 + x^2 - 2x + 3 \\
 & \underline{2x^3 + 2x} \\
 & x^2 + 3 \\
 & \underline{- x^2 - 1} \\
 & 2
\end{array}
$$

Hence, $\dfrac{-2x^3 + x^2 - 2x + 3}{x^2 + 1} = -2x + 1 + \dfrac{2}{x^2 + 1}$.

So, the graph of f has a slant asymptote at $y = -2x + 1$.

Notice that the trailing expression above, $\dfrac{2}{x^2 + 1}$ is *positive for all* x. Hence, as $x \to \pm\infty$, the graph of f will lie *above* the slant asymptote $y = -2x + 1$. We include the graph of f below for completeness.



$$f(x) = \frac{-2x^3 + x^2 - 2x + 3}{x^2 + 1} = -2x + 1 + \frac{2}{x^2 + 1}$$

Example 105. Construct a rational function $f(x) = \dfrac{p(x)}{q(x)}$ that has a domain of $x \neq 2$ and a slant asymptote along the line $y = x - 3$.

In this example, we can work backwards from what we just learned to construct the desired rational function f. We will start by filling in the necessary information to the right hand side of the expression below.

$$\frac{p(x)}{q(x)} \;=\; \underbrace{cx+d}_{\substack{\text{slant}\\\text{asymptote}}} \;+\; \frac{r(x)}{q(x)}$$

Since $x \neq 2$, we may use $q(x) = x - 2$ for our denominator. Similarly, we can replace $cx + d$ with our given asymptote, $x - 3$. Since there are no other restrictions for our function, we are free to choose any polynomial expression for $r(x)$. Since $q(x) = x - 2$, $r(x)$ will be a constant function. For this example, we will use the identity polynomial, $r(x) = 1$.

$$\frac{p(x)}{x-2} = x - 3 + \frac{1}{x-2}$$

All that remains is to obtain a common denominator on the right-hand side, in order to identify the numerator, $p(x)$.

$$\begin{aligned} f(x) &= \frac{x-3}{1} \cdot \frac{x-2}{x-2} + \frac{1}{x-2} \\ &= \frac{x^2-5x+6}{x-2} + \frac{1}{x-2} \\ &= \frac{x^2-5x+7}{x-2} \end{aligned}$$

Our desired polynomial is $f(x) = \dfrac{x^2-5x+7}{x-2}$.

Notice that our answer fits the criteria for a slant asymptote, since $n = m + 1$. Also, we can easily identify many other functions that satisfy this particular problem by changing the expression for $r(x)$ to another constant.

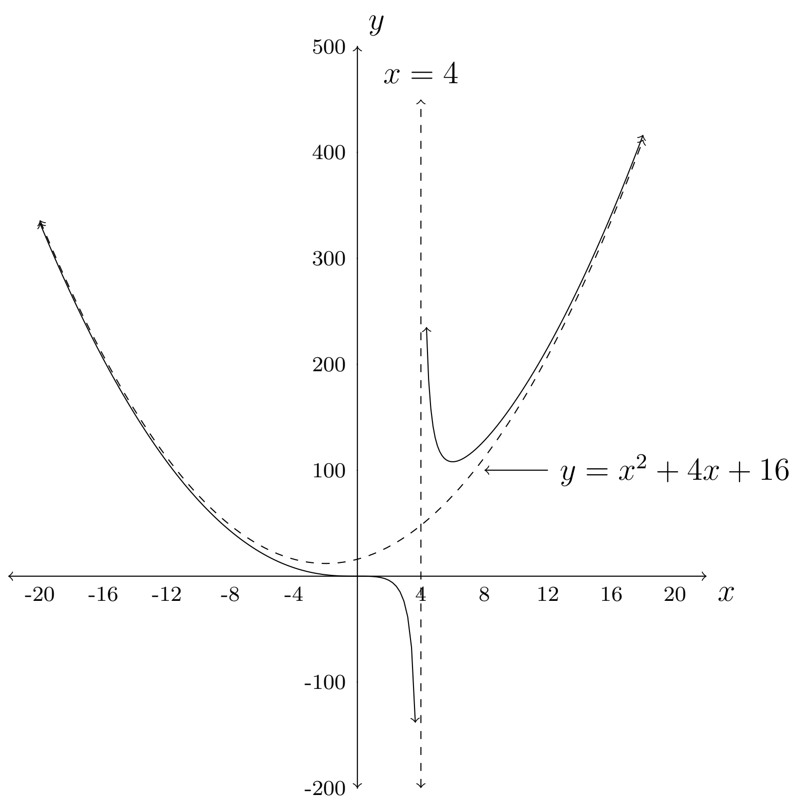
We leave it as an exercise to the reader to determine whether there might be other possibilities for our denominator $q(x)$.

Example 106. In the case of Example 101, we see that $f(x) = \dfrac{x^3}{x-4}$ does not satisfy our criteria for the existence of a slant asymptote, since $n - m \neq 1$. This should now make perfect sense, however, since polynomial division will not produce a linear quotient expression, but rather a *quadratic*, as shown below.

$$
\begin{array}{r|l}
 & x^2 + 4x + 16 \\
\hline
x-4 & x^3 \\
 & -x^3 + 4x^2 \\
\hline
 & 4x^2 \\
 & -4x^2 + 16x \\
\hline
 & 16x \\
 & -16x + 64 \\
\hline
 & 64
\end{array}
$$

$$
\begin{aligned}
f(x) &= \frac{x^3}{x-4} \\
&= \underbrace{x^2 + 4x + 16}_{\text{curvilinear asymptote}} + \frac{64}{x-4}
\end{aligned}
$$

Consequently, as $x \to \pm\infty$, we can indeed see that the graph of f approaches the *curvilinear asymptote* $y = x^2 + 4x + 16$.



$$f(x) = \frac{x^3}{x-4} = x^2 + 4x + 16 + \frac{64}{x-4}$$

In general, for a given rational function, $f(x) = \dfrac{p(x)}{q(x)}$, with degrees n and m of $p(x)$ and $q(x)$, respectively, it is the difference in degrees, $n - m$, that will dictate the nature of the associated end behavior asymptote for the graph of f. Specifically, if $n - m = 1$, the tails of the graph will resemble a linear graph (having non-zero slope), if $n - m = 2$, the tails will resemble a parabola, if $n - m = 3$, the tails will resemble a cubic graph, and so on. We close the section on end behavior with a table that summarizes the key takeaways.

Applying polynomial division:

$$f(x) = \frac{p(x)}{q(x)} = h(x) + \frac{r(x)}{q(x)}$$
$$n = \text{degree of } p$$
$$m = \text{degree of } q$$

$n - m$	End Behavior of the Graph of f
< 0	Horizontal Asymptote at $y = 0$*
0	Horizontal Asymptote at $y = a_n/b_m$*
1	Slant Asymptote at $y = h(x)$
> 1	Curvilinear Asymptote at $y = h(x)$

*Note that in this case, our asymptote will actually still equal $h(x)$.

Local Behavior

Recall that the domain of a rational function $f(x) = \dfrac{p(x)}{q(x)}$ is the set of all x such that $q(x) \neq 0$. We call those values not in the domain of f *discontinuities*, since they will correspond to a break in the graph of f. In this section we will explore what happens to the graph of a rational function near a given discontinuity.

This boils down to two cases:

1. vertical asymptotes, known as *infinite discontinuities,* and
2. holes, known as *removable discontinuities.*

In order to discuss either of the aforementioned cases, we will need to consider a "simplified expression" for a rational function f.

For example, in Example 95, we looked at the function $h(x) = \dfrac{(x+1)(x-3)^2}{x-3}$.

As long as x does not equal 3, we can think of $(x+1)(x-3)$ as a simplified expression for h, since the two expressions will always produce the same value for $x \neq 3$.

For example, when $x = 5$, we get

$$h(5) = \frac{(5+1)(5-3)^2}{5-3} = \frac{(5+1)(5-3)^{\cancel{2}}}{\cancel{5-3}} = (5+1)(5-3) = 6 \cdot 2 = 12$$

Similarly, if we look at Example 96, a simplified expression for

$$f(x) = \frac{(x-4)(x-5)}{(x+2)(x-5)}$$

would be $\dfrac{x-4}{x+2}$.

It is important to note that one should never assume that a simplified expression for a rational function f will equal f for *all* values of x. This is evident in each of our examples, since the domain of our simplified expression does not equal the domain of the given function. Nevertheless, the two expressions are closely related, and are needed in order to identify the various discontinuities in the graph of a given rational function.

Vertical Asymptotes (L58)

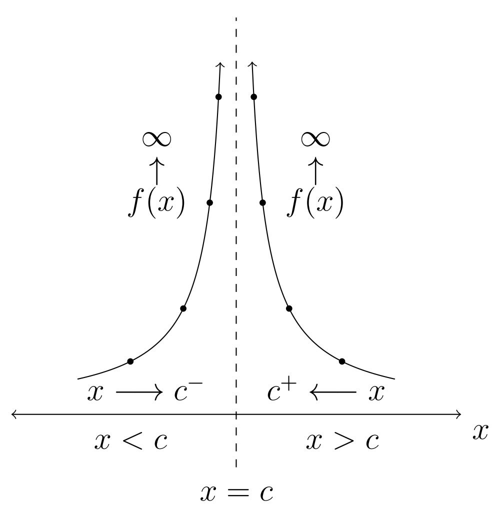
Objective: Identify one or more vertical asymptotes in the graph of a rational function.

The central idea around a vertical asymptote, say $x = c$, is that as x approaches the value of c, either from the left or the right, the values for the corresponding function $f(x)$ will approach either ∞ or $-\infty$.

Approaching from the right: As $x \to c^+,\ f(x) \to \pm\infty$.

Approaching from the left: As $x \to c^-,\ f(x) \to \pm\infty$.

We should be clear here, in that when we say x approaches c *from the right,* what is meant is that we are evaluating the function at values of x that are getting arbitrarily close go c, but are all *greater* than c, i.e., $x > c$. This is precisely why we can write $x \to c^+$ in the statement above. The + in the exponent signifies that $x > c$. The same can be said for when x approaches c from the left. The following graph further illustrates this point.



Our previous graph shows that as x approaches c from either direction, the values for $f(x)$ approach $+\infty$. If, instead, we reflected the right-hand side of the graph across the $x-$axis, we would say that as $x \to c^+$, $f(x) \to -\infty$, since the right-hand side would now point downwards.

Up until this point, we have seen several examples of graphs of rational functions that contain vertical asymptotes. We are now ready to formally state the condition for the existence of a vertical asymptote.

> Let $f(x)$ be a rational function and let $g(x)$ represent the simplified expression for f. If $x = c$ is not in the domain of *both* f and g, then the graph of f will have a vertical asymptote at $x = c$.

In the case where f cannot be simplified, any value not in the domain will correspond to a vertical asymptote in the graph. Examples 89 and 91 are a good place to start. We revisit each of them now.

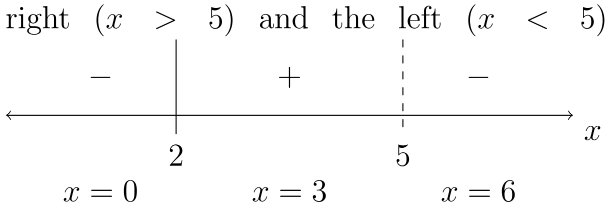
Example 107. $f(x) = \dfrac{-2x+4}{x-5} = \dfrac{-2(x-2)}{x-5}$

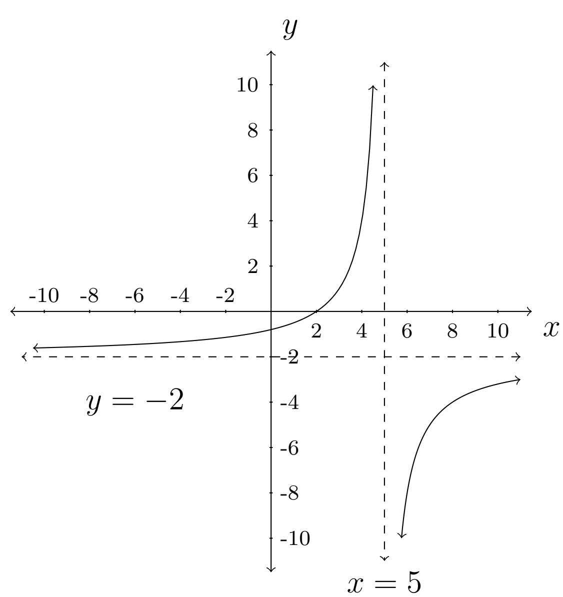
We quickly see that the expression for f is already simplified. In this case,

$$\text{As } x \to 5^+,\ f(x) \to -\infty.$$

$$\text{As } x \to 5^-,\ f(x) \to \infty.$$

A sign diagram for f will also help us to confirm whether $f(x)$ will approach ∞ or $-\infty$, as x approaches 5 from both the right $(x > 5)$ and the left $(x < 5)$.





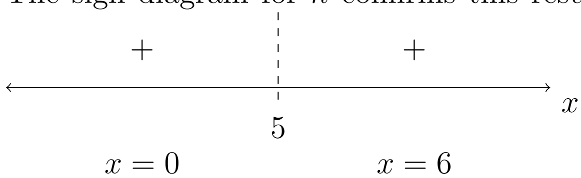
Example 108. $h(x) = \dfrac{x^2+25}{x^2-10x+25} = \dfrac{x^2+25}{(x-5)^2}$

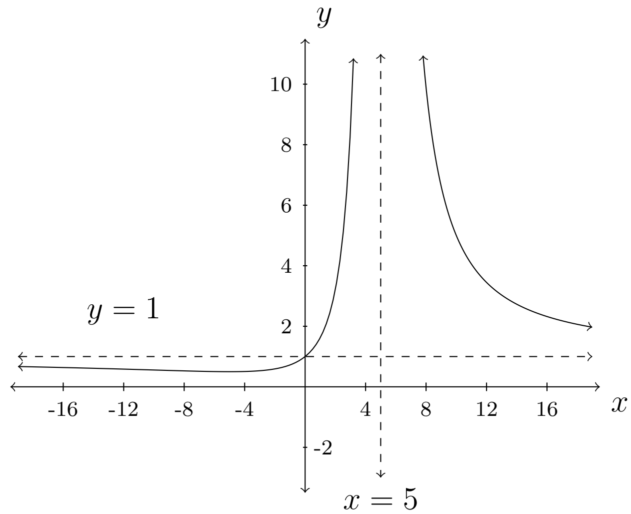
Again, the expression for h is already simplified. In this case,

$$\text{As } x \to 5^+,\ f(x) \to \infty.$$

$$\text{As } x \to 5^-,\ f(x) \to \infty.$$

The sign diagram for h confirms this result.





If, unlike the previous two examples, we were faced with a rational function whose simplified expression did *not* equal the original function, we would still have a vertical asymptote at $x = c$, as long as a factor of $(x - c)$ remained in the simplified expression. This is precisely the same as saying that $x = c$ is not in the domain of both the original function and the simplified expression. A good example of this is Example 96, which we revisit now.

Example 109. $f(x) = \dfrac{x^2-9x+20}{x^2-3x-10} = \dfrac{(x-4)(x-5)}{(x+2)(x-5)}$

Domain of f: $x \neq -2, 5$

Simplified Expression for f: $\dfrac{x-4}{x+2}$

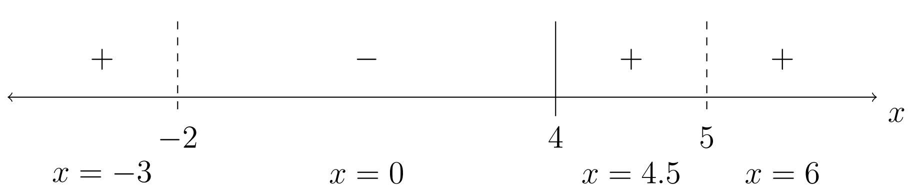
Domain of Simplified Expression: $x \neq -2$

It follows that the graph of f has a vertical asymptote at $x = -2$. Notice, however, that f does *not* have a vertical asymptote at $x = 5$, since $x = 5$ is in the domain of the simplified expression. We will revisit this example, yet again, in the very near future, to discuss the nature of the graph of f near $x = 5$.

The sign diagram for f (shown below) confirms the following statements.

$$\text{As } x \to -2^+,\ f(x) \to -\infty.$$

$$\text{As } x \to -2^-,\ f(x) \to \infty.$$



If we look more closely at each of our last three examples, we can begin to make sense of why a particular function approaches $+\infty$ or $-\infty$, as x approaches our particular discontinuity. The explanation it tied to the notion of *multiplicity* from the chapter on polynomial functions.

Recall that the multiplicity of a root c for a polynomial function $p(x)$ is the maximum number of times, k, that the factor of $x - c$ appears in the polynomial's complete factorization. For example, if we consider the polynomial

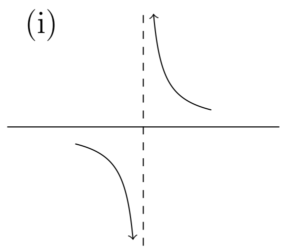
$$p(x) = 2x^3(x-4)^2(x-5),$$

the roots $x_1 = 0$ $x_2 = 4$, and $x_3 = 5$ would have respective multiplicities $k_1 = 3$, $k_2 = 2$, and $k_3 = 1$.

The multiplicity k of a root c helped us to determine whether the corresponding $x-$intercept was a *crossover* point (when k is odd) or a *turnaround* point (when k is even). It turns out that we can use this same idea for visualizing vertical asymptotes, when $x = c$ is a root of the denominator $q(x)$ of a rational function $f(x) = \dfrac{p(x)}{q(x)}$.

Let f be a rational function with vertical asymptote at $x = c$, and let k be the multiplicity of c in the denominator of the *simplified expression* for f.
If k is odd, then the graph of f near $x = c$ will resemble one of the following:

(i)

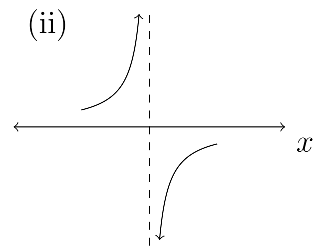


$x = c$

As $x \to c^+$, $f(x) \to \infty$

As $x \to c^-$, $f(x) \to -\infty$

(ii)



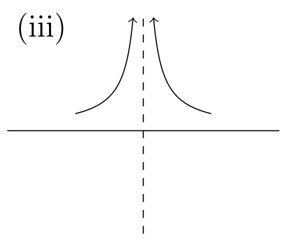
$x = c$

As $x \to c^+$, $f(x) \to -\infty$

As $x \to c^-$, $f(x) \to \infty$

If k is even, then the graph of f near $x = c$ will resemble one of the following:

(iii)

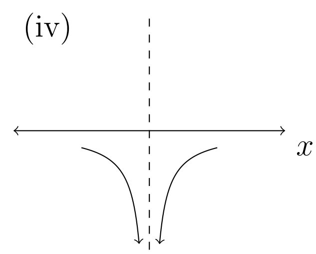


$x = c$

As $x \to c^+$, $f(x) \to \infty$

As $x \to c^-$, $f(x) \to \infty$

(iv)



$x = c$

As $x \to c^+$, $f(x) \to -\infty$

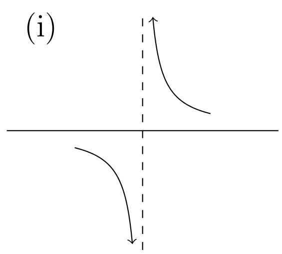
As $x \to c^-$, $f(x) \to -\infty$

We clearly see this idea at work for $f(x) = \dfrac{-2(x-2)}{x-5}$ (Example 107) and $h(x) = \dfrac{x^2+25}{(x-5)^2}$ (Example 108), whose graphs both show a vertical asymptote at $x = 5$. In the case of f, the multiplicity of $x = 5$ in our denominator ($k = 1$) is odd. Consequently, the two sides of our graph near $x = 5$ approach the vertical asymptote on opposite sides of the $x-$axis. Alternatively, the multiplicity of $x = 5$ in the denominator of h is even ($k = 2$). Consequently, the two sides of our graph near $x = 5$ approach the vertical asymptote on the same side of the $x-$axis.

Furthermore, for a given rational function with vertical asymptote at $x = c$, once we have identified the associated multiplicity k, we can use a sign diagram to determine precisely which case matches our graph: (i) or (ii) when k is odd, and (iii) or (iv) when k is even.

Example 110. We already know that $f(x) = \dfrac{(x-4)(x-5)}{(x+2)(x-5)}$ has a vertical asymptote at $x = -2$ from Example 109.

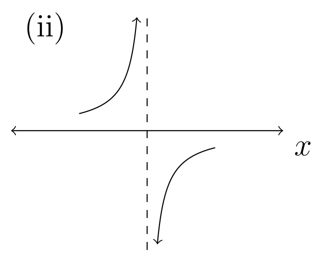
Since the multiplicity of $x = -2$ in the denominator of our simplified expression $\dfrac{x-4}{x+2}$ is odd ($k = 1$), we know that the nature of the graph near $x = -2$ will resemble one of the following cases.



$x = -2$

As $x \to -2^+$, $f(x) \to \infty$

As $x \to -2^-$, $f(x) \to -\infty$

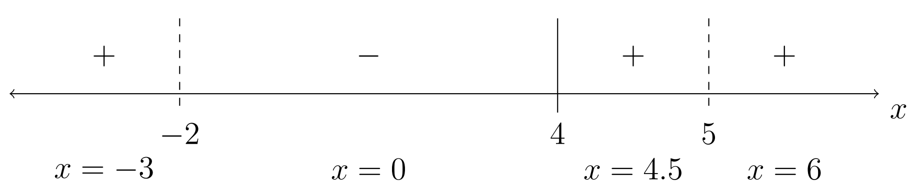


$x = -2$

As $x \to -2^+$, $f(x) \to -\infty$

As $x \to -2^-$, $f(x) \to \infty$

But recall that our sign diagram for f was as follows.



This tells us that our graph will point *upwards* to the left of $x = -2$ and *downwards* to the right of $x = -2$. Hence, case (ii) is the correct graph for our function near $x = -2$. And our corresponding statements match those from before.

$$\text{As } x \to -2^+,\ f(x) \to -\infty.$$

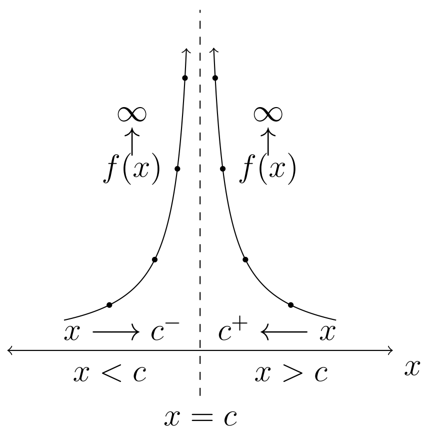
$$\text{As } x \to -2^-,\ f(x) \to \infty.$$

Holes (L59)

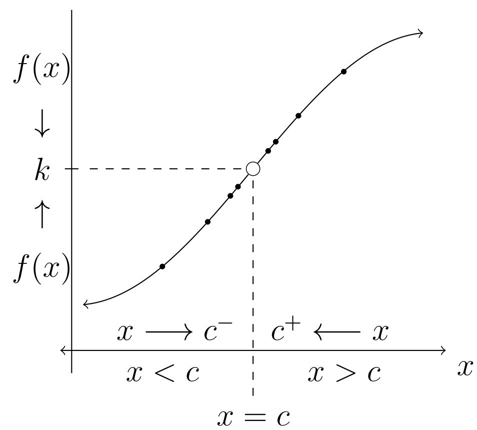
Objective: Identify the precise location of one or more holes in the graph of a rational function.

While Vertical Asymptotes correspond to infinite discontinuities, a hole corresponds to a *removable discontinuity*, since the removal of a single point along a continuous curve creates the hole.

Suppose that the rational function $f(x)$ has a discontinuity at $x = c$, i.e., c is not in the domain of f. If $x = c$ is a vertical asympote of the graph of f, we just saw that as $x \to c$, $f(x) \to \pm\infty$. If $x = c$ represents a hole in the graph of f, however, we will see that as $x \to c$, $f(x) \to k$, for some real number k. This is the fundamental difference between infinite and removable discontinuities.



Infinite Discontinuity



Removable Discontinuity

In the case of the graph of the left, recall that we have the following statements.

$$\text{As } x \to c^+,\ f(x) \to \infty. \qquad \text{As } x \to c^-,\ f(x) \to \infty.$$

Similarly, in the case of the graph on the right, we employ the same idea, using k^+ and k^- in order to identify whether or not the graph of f approaches k from *above* if $f(x) > k$ and *below* if $f(x) < k$.

$$\text{As } x \to c^+,\ f(x) \to k^+. \qquad \text{As } x \to c^-,\ f(x) \to k^-.$$

In virtually all cases, however, it will be sufficient enough to simply state that as $x \to c$, $f(x) \to k$, since further analysis will often prove difficult.

We now state the requirement for a hole, which, as with vertical asymptotes, depends on both the rational function f and its simplified expression.

Let $f(x)$ be a rational function and let $g(x)$ represent the simplified expression for f. If $x = c$ is not in the domain of f, but *is* in the domain of g, then the graph of f will have a hole at $(c, g(c))$.

For the existence of a vertical asymptote, the identified discontinuity had to be excluded from the domain of *both* f and its simplified expression. This is not the case for a hole, however, as the simplified expression g is, in fact, defined at $x = c$. Furthermore, the value $g(c)$ tells us the precise location of our hole along the $y-$axis.

Example 92 is our first example with a hole, and we revisit it now.

Example 111. Original Function:

$$k(x) = \frac{x^3 - 5x^2}{10x - 50} = \frac{x^2(x-5)}{10(x-5)}$$

Domain of k: $x \neq 5$ or $(-\infty, 5) \cup (5, \infty)$

The original function k is undefined at $x = 5$

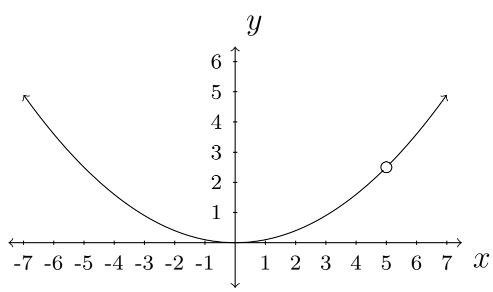
Simplified Expression:

$$g(x) = \frac{x^2\cancel{x-5}}{10\cancel{(x-5)}} = \frac{x^2}{10}$$

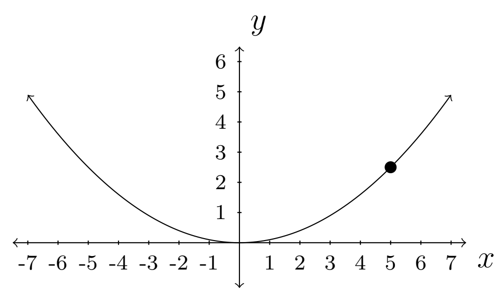
Domain of g: $(-\infty, \infty)$

$$g(5) = \frac{25}{10} = \frac{5}{2}$$

Conclusion: The graph of k has a hole at $(5, g(5)) = (5, \frac{5}{2})$.



Graph of $k(x) = \dfrac{x^3 - 5x^2}{10x - 50}$



Graph of $g(x) = \dfrac{x^2}{10}$

In this first example, we observe a somewhat obvious fact that we have neglected to state until now:

> The graph of a rational function f will always equal the graph of its simplified expression for any x *in the domain* of f.

In this case, the graphs of $k(x) = \dfrac{x^3 - 5x^2}{10x - 50}$ and the familiar quadratic function $g(x) = \dfrac{x^2}{10}$ only differ in their behavior at $x = 5$.

We now take one last look at a familiar example (109), and include its graph for completeness

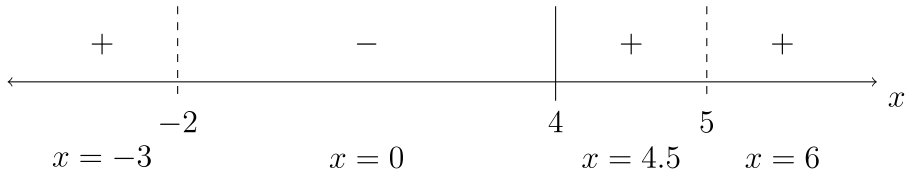
Example 112. $f(x) = \dfrac{x^2 - 9x + 20}{x^2 - 3x - 10} = \dfrac{(x-4)(x-5)}{(x+2)(x-5)}$

The simplified expression for f is $g(x) = \dfrac{x-4}{x+2}$.

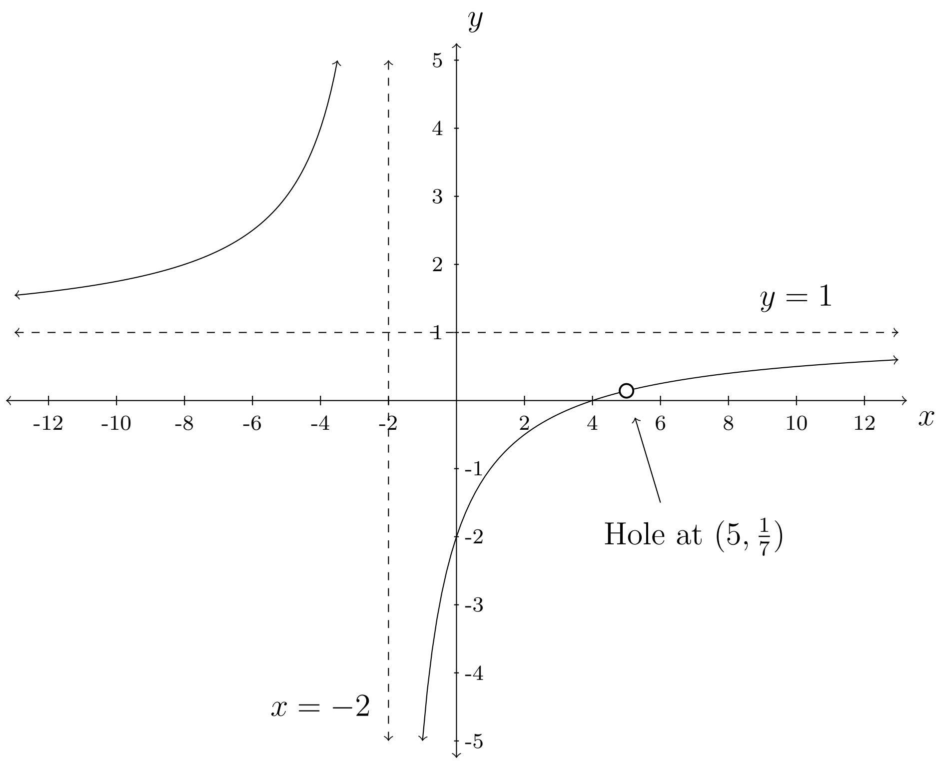
Notice that $x = 5$ is in the domain of g, with $g(5) = \frac{5-4}{5+2} = \frac{1}{7}$.

Since $x = 5$ is not in the domain of f, the graph of f will have a hole at $(5, \frac{1}{7})$.

Recall that our sign diagram for f is as follows.



We now are ready to make sense of the complete graph of f, presented below.



$$f(x) = \frac{x^2 - 9x + 20}{x^2 - 3x - 10} = \frac{(x-4)(x-5)}{(x+2)(x-5)} = \frac{x-4}{x+2}, \ x \neq 5$$

In each of our last two examples, the simplified expression for our given rational function has seen a complete elimination of the "offending factor" $(x-c)$ from both the numerator and the denominator. Based upon our criteria for identifying holes, this is certainly a requirement for the denominator. As we will see with our next example, however, it is possible that not all offending factors will completely disappear from the numerator. In this situation, our hole will simply reside on the $x-$axis, since $g(c)$ will equal zero in our simplified expression.

Example 113. Find the domain of $f(x) = \dfrac{4x^2}{x^3+3x^2-4x}$, and identify all discontinuities. In each case, determine whether the discontinuity is infinite (vertical asymptote) or removable (hole). Once again, we begin by obtaining a complete factorization of f in order to identify its domain and simplified expression.

$$\begin{aligned} f(x) &= \frac{4x^2}{x^3+3x^2-4x} \\ &= \frac{4x^2}{x(x^2+3x-4)} \\ &= \frac{4x^2}{x(x+4)(x-1)} \end{aligned}$$

The domain of f is $x \neq -4, 0, 1$ or

$$(-\infty, -4) \cup (-4, 0) \cup (0, 1).$$

The simplified expression for f is

$$g(x) = \frac{4x}{(x+4)(x-1)}.$$

Here, we see that the graph of f has three discontinuities, occurring at those values not in the domain, $x = -4, 0,$ and 1. Since the factors of $x+4$ and $x-1$ still appear in the denominator of the simplified expression, it follows that the discontinuities at $x = -4$ and $x = 1$ will be infinite, and the graph of f will have vertical asymptotes along these lines.

Our third discontinuity at $x = 0$ will be removable, since the simplified expression does not contain a factor of x in its denominator. Hence, the graph of f will have a hole at the point $(0, g(0)) = (0, 0)$.

We leave it as an exercise for the reader to verify our answer by graphing f using Desmos.

Graphing Summary (L60)

Objective: Graph a rational function in its entirety.

At this point, we have addressed all key features of rational functions individually. This section pulls each of these aspects together, for a detailed analysis of a rational function, culminating in a complete sketch of its graph. Along the way, we will need to address each of the following aspects for our rational function $f(x) = \dfrac{p(x)}{q(x)}$. It is important to note that there is no universally accepted order to this checklist.

- Find the $y-$intercept of the graph of f, $(0, f(0))$, if it exists.
- Use the degrees and leading coefficients of p and q to determine whether the graph of f has a horizontal asymptote. If the graph of f has a slant asymptote, use polynomial division to find where it is located.
- Identify a complete factorization of f, and use it to find the domain of the function. This is the set of all x, such that $q(x) \neq 0$.

- Find any $x-$intercepts of the graph of f. This is the set of all x in the domain of f, such that $p(x) = 0$. Using multiplicities, classify each $x-$intercept as a crossover or turnaround ("bounce") point.
- Find the simplified expression g for the given function f, and use it to identify any vertical asymptotes or holes in the graph of f. Use multiplicities to help visualize the nature of the graph of f near its vertical asymptotes. If f has a hole at $x = c$, use g to help plot the hole's precise location at $(c, g(c))$.
- Using both the $x-$intercepts and the discontinuities (those x not in the domain), construct a sign diagram for f.

In each example that follows, we will carefully examine the given function, making sure not to omit any of the checklist items above and to compare each item to those that precede it along the way for accuracy. Although the process will take some time, if we are thorough, our end result should be a complete, accurate sketch of the given rational function. We will start by revisiting our last example.

Example 114. Sketch a complete graph of the rational function below, making sure to have a clearly defined scale and label all key features of your graph (intercepts, asymptotes, and holes).

$$f(x) = \frac{4x^2}{x^3 + 3x^2 - 4x}$$

In this first example, we see that the graph of f will not have a $y-$intercept, since $f(0) = \frac{0}{0}$, which is undefined.

Since the degree of the numerator is less than the degree of the denominator, we conclude that the graph of f has a horizontal asymptote along the $x-$axis, $y = 0$.

Our graph also has no $x-$intercepts, since our numerator only equals zero when $x = 0$, which we know is not in our domain of f.

Furthermore, from the work in our last example, we know that f has a complete factorization of

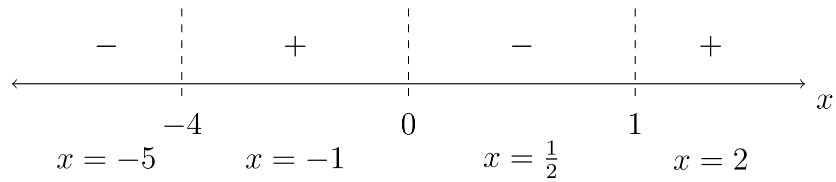
$$f(x) = \frac{4x^2}{x(x+4)(x-1)},$$

with corresponding domain $x \neq -4, 0, 1$, and simplified expression

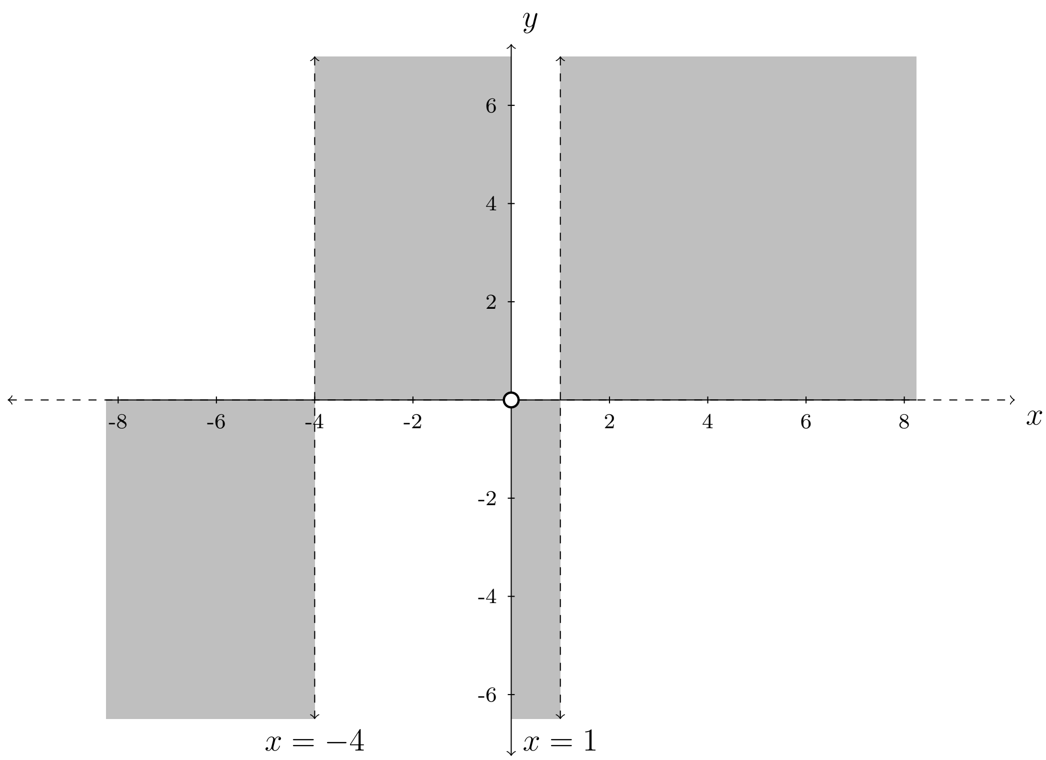
$$g(x) = \frac{4x}{(x+4)(x-1)}.$$

Consequently, the graph of f has vertical asymptotes at $x = -4$ and $x = 1$ and a hole at $(0, g(0)) = (0, 0)$.

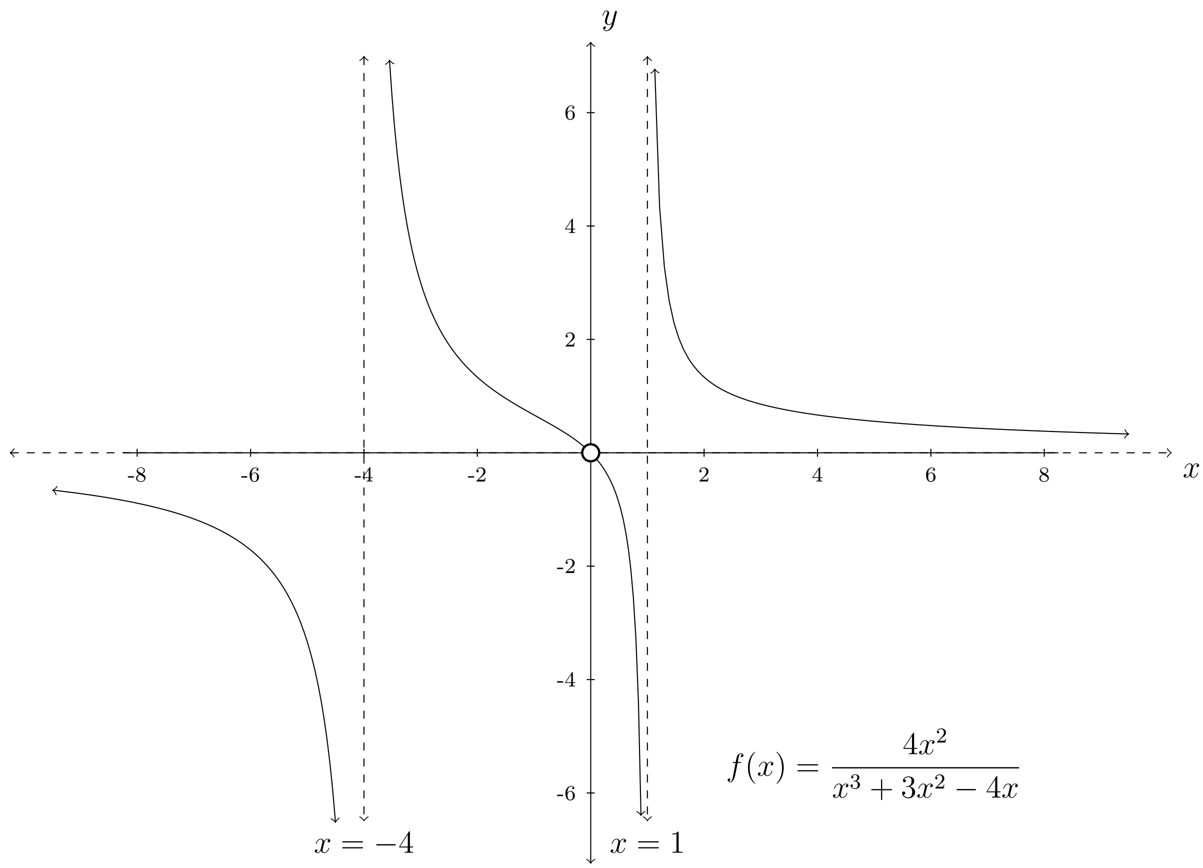
Since the multiplicities of both $x = -4$ and $x = 1$ in the denominator of f are both one (odd), we know that the graph of f will approach each vertical asymptote from opposite sides of the $x-$axis. The following sign diagram confirms this observation.



We are now ready to try our hand at graphing f, and begin our graph by defining a scale for both the $x-$ and $y-$axes, and identifying all intercepts, and asymptotes. This should always be our first step to successfully sketching a decent-looking graph. To emphasize this point, we first show an initial graph that identifies each of these features, and further shades those areas of the $xy-$plane that correspond to our sign diagram above.



We now carefully sketch the graph of f based upon our findings.



Example 115. Sketch a complete graph of the rational function below, making sure to have a clearly defined scale and label all key features of your graph (intercepts, asymptotes, and holes).

$$f(x) = \frac{x^2 - 6x + 9}{x^2 + x - 6}$$

Again, we start by evaluating f at $x = 0$ to identify the $y-$intercept. We get $(0, \frac{9}{-6}) = (0, -\frac{3}{2})$.

Since the degrees of the numerator and denominator are equal and the leading coefficients are also equal, we know that the graph of f will have a horizontal asymptote at the line $y = 1$.

A complete factorization of f gives us

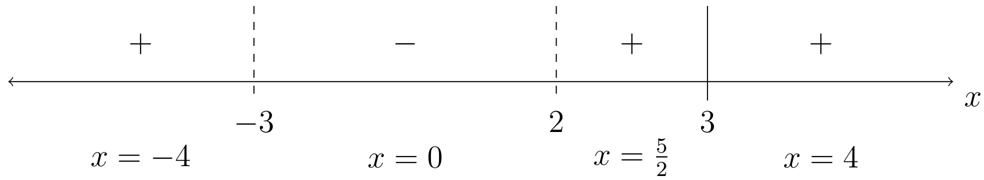
$$f(x) = \frac{(x-3)^2}{(x+3)(x-2)},$$

which is also our simplified expression.

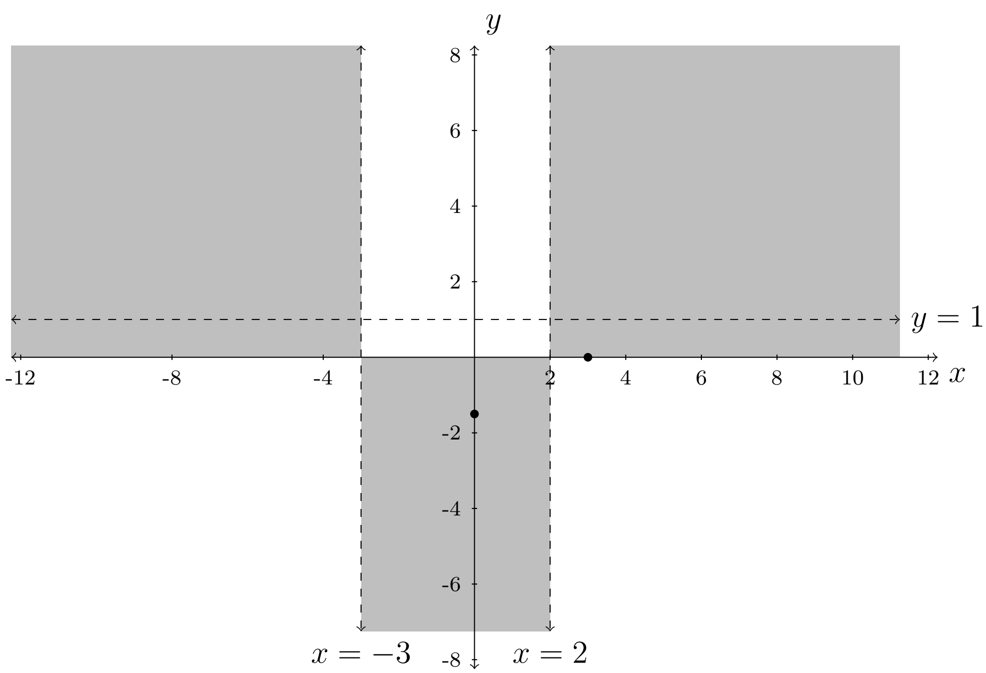
Using our factorization, we see that the graph of f will have an $x-$intercept at $(3, 0)$. This will be a turnaround point, due to the even multiplicity of the root $x = 3$ in the numerator.

Our domain for f is $x \neq -3, 2$, and the corresponding graph will have vertical asymptotes at $x = -3$ and $x = 2$.

Our sign diagram is as follows.



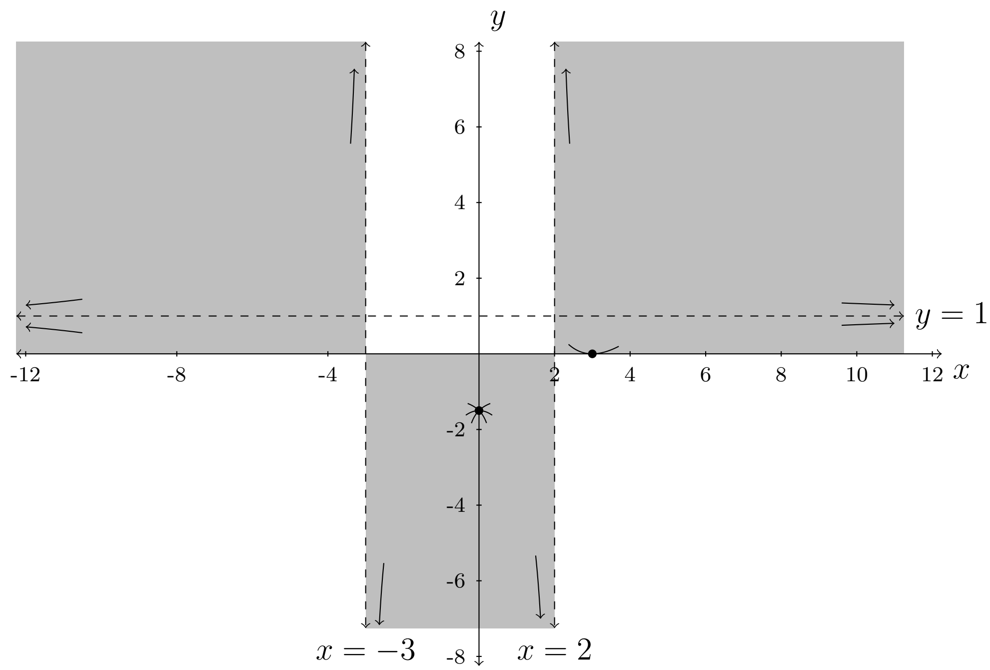
As with our last example, we will build up to the graph of f by first defining a scale, identifying our intercepts and asymptotes, and shading the corresponding areas that should include our graph.



In this particular example, without further algebraic analysis of our function, we will have to make a choice as to how the graph of f approaches the horizontal asymptote, $y = 1$. For example, does it approach from above or below? Does the graph ever cross the horizontal asymptote? In order to answer this last question, one could attempt to solve the equation $f(x) = 1$ for all possible x in order to see exactly where our graph crosses the line $y = 1$.

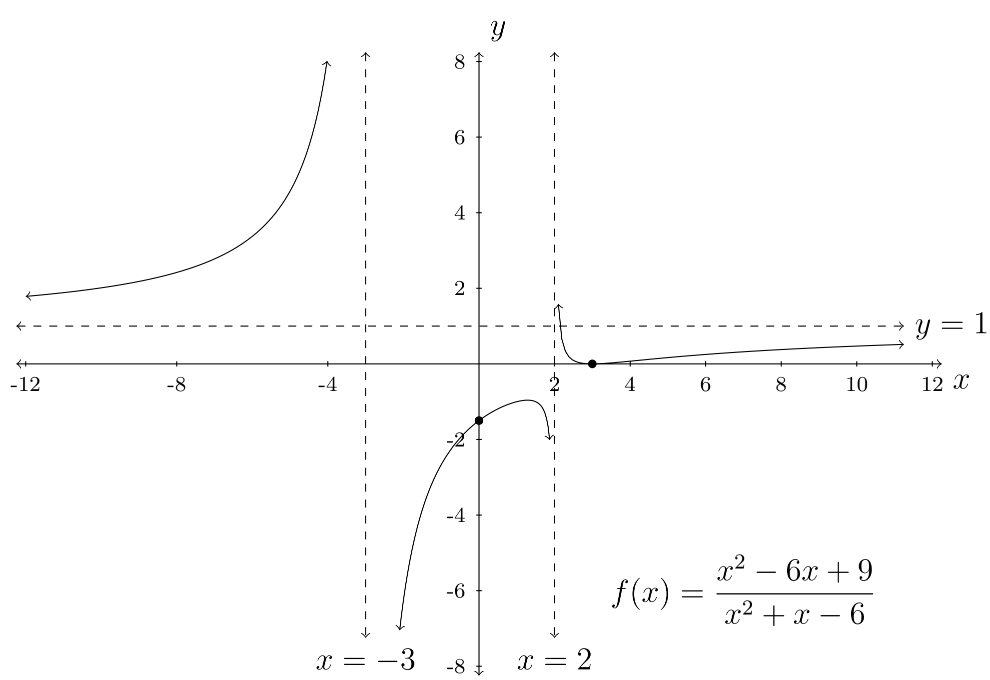
Similarly, we cannot know if our $y-$intercept is also a local maximum value for our graph or how exactly the graph intersects this point without using advanced methods that are typically covered in a calculus course.

On account of these subtleties, before we see the complete graph of f, we will take one more step, and begin to sketch the possibilities for our graph near the asymptotes and our intercepts.



At this point it is important to reinforce the fact that our graph is meant to be a *rough sketch* of the actual graph of f. So, we have to make some choices about how to connect everything up properly. A more complete sketch of f requires advanced techniques and concepts that one would likely see in precalculus or calculus. Nevertheless, as long as there is a logical, smooth connection to each aspect of our graph, we can rest assured that our analysis of the function is sufficient.

We can now conclude this example with the actual graph of f.



As a result, we learn that as $x \to \infty$, $f(x) \to 1^-$, and consequently, our graph does not cross over the horizontal asymptote to the right of our $x-$intercept. Furthermore, our $y-$intercept is not a local maximum for the graph of f, as is often the case. Lastly, as $x \to 2$, we see that the graph of f seems to hold more tightly to the vertical asymptote than when $x \to -3$.

Example 116. Sketch a complete graph of the rational function below, making sure to have a clearly defined scale and label all key features of your graph (intercepts, asymptotes, and holes).

$$f(x) = \frac{x^3 - 16x}{3x^2 - 6x - 72}$$

We will begin by obtaining a complete factorization of f and a simplified expression.

$$\begin{aligned} f(x) &= \frac{x(x^2-16)}{3(x^2-2x-24)} \\ &= \frac{x(x+4)(x-4)}{3(x+4)(x-6)} \end{aligned}$$

The simplified expression for f is

$$\begin{aligned} g(x) &= \frac{x(x-4)}{3(x-6)} \\ &= \frac{x^2-4x}{3x-18} \end{aligned}$$

We quickly see that the graph of f has both an $x-$ and $y-$intercept at the origin, since $f(0) = 0$.

Since the degree of the numerator is one greater than the degree of the denominator, we know that the graph of f will have a slant asymptote, and we can use polynomial division to find its precise location. It is worth noting, however, that we can apply polynomial division to *either* our original function or its simplified expression in order to find our slant asymptote, as demonstrated below.

Original Function:

$$\begin{array}{r|l} & \frac{1}{3}x + \frac{2}{3} \\ \hline 3x^2-6x-72 & x^3 \qquad - 16x \\ & -x^3 + 2x^2 + 24x \\ \hline & 2x^2 + 8x \\ & -2x^2 + 4x + 48 \\ \hline & 12x + 48 \end{array}$$

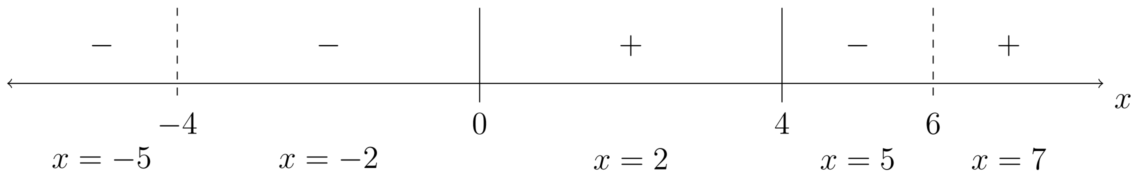
Simplified Expression:

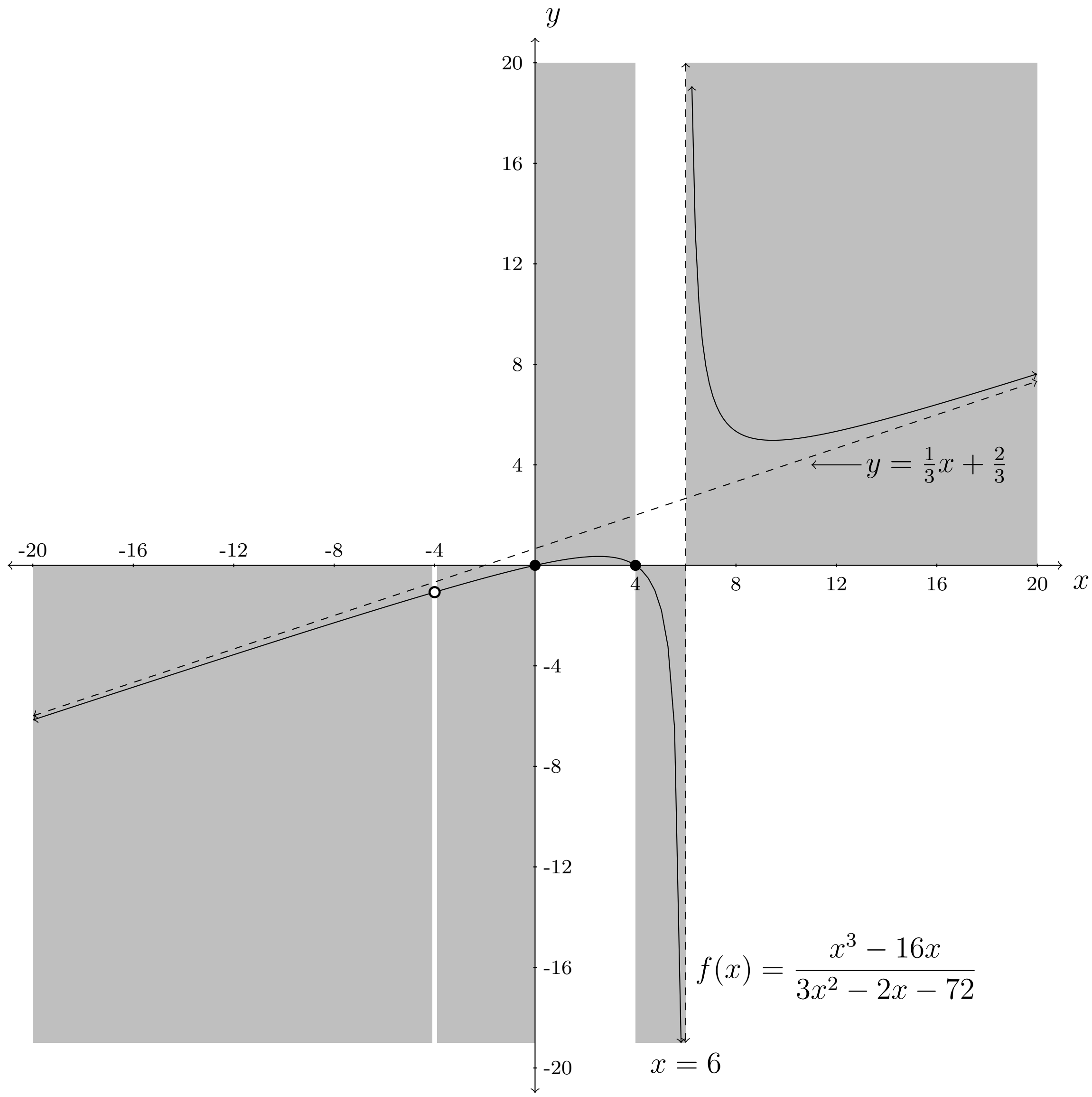
$$\begin{array}{r|l} & \frac{1}{3}x + \frac{2}{3} \\ \hline 3x-18 & x^2 - 4x \\ & -x^2 + 6x \\ \hline & 2x \\ & -2x + 12 \\ \hline & 12 \end{array}$$

Each calculation carries the same number of steps and yields the same equation for our slant asymptote, $y = \frac{1}{3}x + \frac{2}{3}$. Since dividing the simplified expression for our function should always be a simpler process, it makes sense to use this approach when we encounter similar problems.

Observe that the domain of f is $x \neq -4, 6$. Since the factor of $x + 4$ appears in both the numerator and denominator, the graph of f will contain a hole at $(-4, g(-4)) = (-4, -\frac{16}{15})$ and a vertical asymptote at $x = 6$. Our graph will also include crossover $x-$intercepts at $x = 0$ and $x = 4$, as their multiplicities in the numerator are both odd.

The sign diagram for f is shown below, and we conclude with the graph of f. Though not necessary, we have again included a shading of the regions of our graph that correspond to our sign diagram.

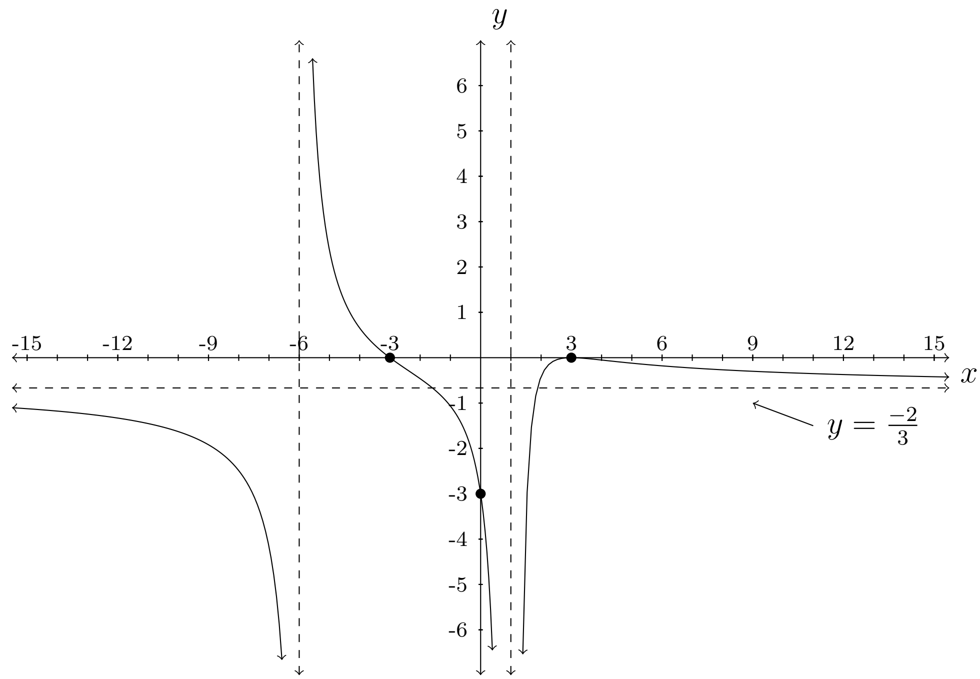




Our graph and, more importantly, its accompanying sign diagram are essential in determining when f is either positive or negative. For example, one could say that $f(x) > 0$ for all x in the set $(0, 4) \cup (6, \infty)$, and $f(x) < 0$ when x is in the set $(-\infty, -4) \cup (-4, 0) \cup (4, 6)$. We will revisit this idea at the beginning of our next section, when we are asked to solve a rational inequality, rather than graph a particular function.

While it is essential that we are able to analyze and graph a rational function, it is equally important that we can work backwards, and correctly interpret a graph to identify the key aspects of an otherwise unknown function. Our last example does just that.

Example 117. Find an explicit form for the rational function whose graph is shown below.



Although it may be easy to feel overwhelmed at what this problem is asking us to do, at this point we can rest assured that the skills outlined throughout the chapter will enable us to systematically analyze each aspect of the graph above, in order to construct the rational function whose graph matches our graph.

We begin by first making the following observations, in no particular order, along with the implications for our function $f(x)$.

$y-$intercept at $(0,-3)$	$\Longrightarrow$	$f(0)=-3$
Crossover $x-$intercept at $x=-3$	$\Longrightarrow$	$(x+3)^1$ appearing in numerator
Turnaround $x-$intercept at $x=3$	$\Longrightarrow$	$(x-3)^2$ appearing in numerator
Vertical asymptote at $x=-6$, with ends pointing in opposite directions	$\Longrightarrow$	$(x+6)^1$ appearing in denominator
Vertical asymptote at $x=1$, with ends pointing in same direction	$\Longrightarrow$	$(x-1)^2$ appearing in denominator
Horizontal asymptote at $y=\frac{-2}{3}$	$\Longrightarrow$	Numerator and denominator have same degree, n $\frac{a_n}{b_n}=\frac{-2}{3}$

Since many of our observations involve a *factor* of some kind, we will begin construction of f in its factored form. An expanded form can then be easily obtained by multiplying, if it is required.

We will start by focusing on the implications from our $x-$intercepts and those values not in our domain (corresponding to any vertical asymptotes and holes). Consequently, an initial candidate for f could be

$$f(x) = \frac{(x+3)(x-3)^2}{(x+6)(x-1)^2}.$$

At this point, we will need to check our candidate against all additional aspects of the graph outlined above, making adjustments as necessary. But, we soon see that

$$f(0) = \frac{(0+3)(0-3)^2}{(0+6)(0-1)^2} = \frac{27}{6} \neq -3,$$

as our graph shows. Additionally, though the degrees of both the numerator and denominator are equal (in this case, three), it is not hard to see that our numerator and denominator will both have a leading coefficient of 1, which will not produce the necessary horizontal asymptote at $y = \frac{-2}{3}$.

The solution to this problem is found by making an adjustment to our initial guess, by introducing a *scalar multiplier* k.

$$f(x) = \frac{k(x+3)(x-3)^2}{(x+6)(x-1)^2}$$

All that remains is to determine what k equals. Remember that we know it must be negative, in order to produce a horizontal asymptote at $y = -1$. In order to find the precise multiple that is needed, we must substitute our $y-$intercept into the equation above and solve for k.

$$\begin{aligned} f(0) &= \frac{k(0+3)(0-3)^2}{(0+6)(0-1)^2} \\ &= \frac{27k}{6} \\ &= \frac{9k}{2} \end{aligned}$$

But we know that $f(0) = -3$. So we are left with having to solve $\dfrac{9k}{2} = -3$.

In this case, we get $k = \dfrac{-2}{3}$. Our final answer for f is therefore

$$f(x) = \frac{-2(x+3)(x-3)^2}{3(x+6)(x-1)^2},$$

which further satisfies the requirement for our horizontal asymptote at $y = \frac{-2}{3}$.

Expanding our answer gives us $f(x) = \dfrac{-2x^3 + 6x^2 + 18x - 54}{3x^3 + 12x^2 - 33x + 18}$.

Rational Inequalities

Objective: Solve a rational inequality by constructing a sign diagram.

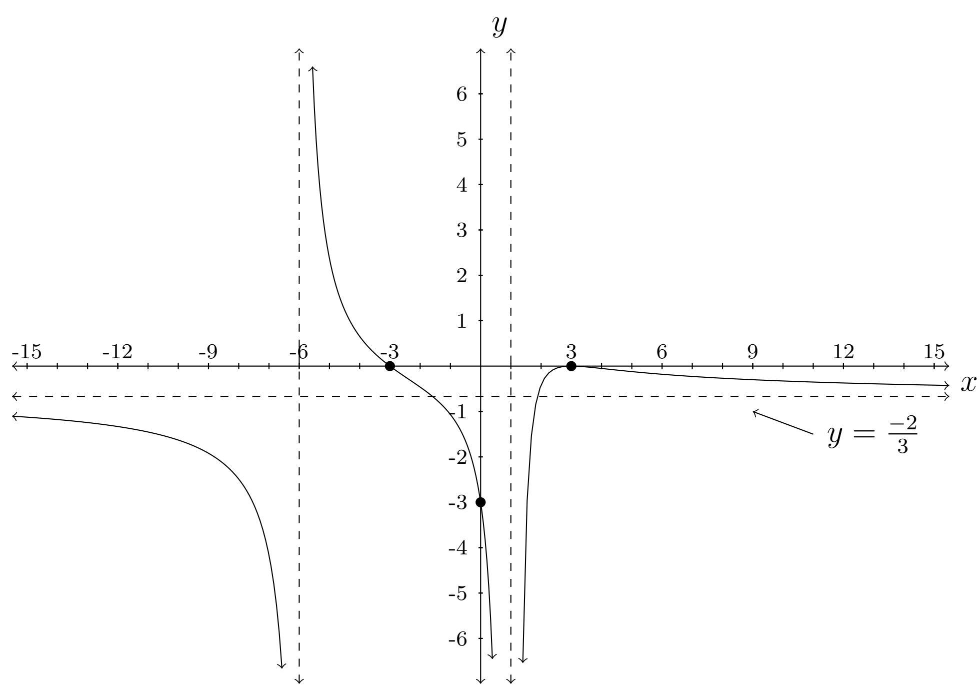
Identifying the solution of a rational inequality is one very practical application of the sign diagram and graph of a rational function. For example, if we are asked to identify when a function $f(x)$ is greater (or less) than zero, we know that this answer will correspond to those values of x such that the point $(x, f(x))$ is above (or below) the $x-$axis. We see this at work at the end of Example 116, where we concluded that the function

$$f(x) = \frac{x^3 - 16x}{3x^2 - 6x - 24} = \frac{x(x+4)(x-4)}{3(x+4)(x-6)}$$

is positive for all x in the set $(0,4) \cup (6,\infty)$ and negative for all x in the set $(-\infty,-4) \cup (-4,0) \cup (4,6)$.

Similarly, we can use the graph for Example 117 to determine when the function $f(x) > 0$. This example is a good starting place, since we were not initially given the expression for f, and had to find it.

Example 118. Determine when the given graph is positive, i.e., when $f(x) > 0$. Express your answer using interval notation.



To answer this question, we need only identify those values of x that correspond to points lying *above* the $x-$axis. Our answer is the single interval $(-6,-3)$.

Note that if we were asked to find when $f(x) \geq 0$ in the previous example, we would need to include the two $x-$intercepts at $x = \pm 3$. Hence, $f(x) \geq 0$ for all x in the set $(-6,-3] \cup \{3\}$.

Recall that since $x = 3$ is a single value, rather than an entire interval of values, we use braces to denote its inclusion in our answer.

Looking at our last example from another angle, let's suppose that we were starting out with a *function*, rather than a graph. We seek to answer the question of when $f(x) > 0$ without the benefit of this visual aide, and will do this using a sign diagram.

Example 119. Solve the inequality $f(x) > 0$ for the function

$$f(x) = \frac{-2x^3 + 6x^2 + 18x - 54}{3x^3 + 12x^2 - 33x + 18}$$

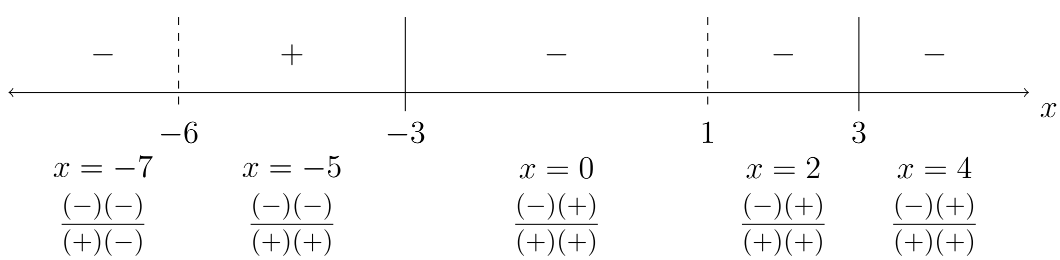
Recall that the given function is that obtained from our graph in the previous example, so we know that our answer should be $(-6, -3)$.

In every problem from here on out, we will need to find a factored form for f so that we can construct a sign diagram. Since this function is a carry-over from a previous example, we know its factorization,

$$f(x) = \frac{-2(x+3)(x-3)^2}{3(x+6)(x-1)^2}.$$

Had we not known this information, factoring f could take a considerable amount of work, since, for example, our denominator $3x^3 + 12x^2 - 33x + 18$ is not easily factorable.

To find the sign diagram for f, we need to identify our $x-$intercepts, as well as those x not in the domain. From our factorization, we see that this is the set $x = \{-6, -3, 1, 3\}$, with $x \pm 3$ being our intercepts. Our diagram is shown below.



One important observation in our diagram is in the calculation of each sign. For each test value, we have excluded the *squared* factors in the numerator and denominator, since both $(x-3)^2$ and $(x-1)^2$ will always contribute a positive sign and not affect the end result. For example, when $x = 0$, we get

$$\frac{(-)(+)(-)^2}{(+)(+)(-)^2},$$

which reduces to the result that we see above. Similarly, we could have excluded the $(+)$ that appears in the denominator of each test value's sign calculation, since the constant multiplier of 3 will have no impact on sign.

At this point we are essentially done, since the factorization and construction of our diagram has done the bulk of the work for us. Since we are asked to find all x such that $f(x) > 0$, we see that this equals to the union of all intervals that correspond to a + sign. This gives us our anticipated answer of $(-6,-3)$.

Recalling our discussion of the last example, if we wished to answer the follow-up question of when $f(x) \geq 0$, we would just need to include all boundary values in our diagram that correspond to $x-$intercepts (when $f(x) = 0$). From our diagram, this would be any value of x that has a *solid* divider, remembering that dashed dividers correspond to values not in our domain. In this case, the function $f(x) \geq 0$ for all x in the set $(-6,-3] \cup \{3\}$.

In each example where we are asked to find when a rational expression or function f is positive, negative, ≥ 0, or ≤ 0, we can take this same approach:

1. Identify a complete factorization of the expression.
2. Construct a sign diagram.
3. Find all intervals that correspond to the desired inequality.
4. In the case of $\geq$ or $\leq$, make sure to include any $x-$intercepts.

Example 120. Solve the inequality

$$\frac{4x^2 - 4x + 1}{x^3 - x^2 - 17x - 15} \leq 0.$$

As with our last example, we will require some additional information for factoring our denominator. In this case, we will include the fact that $x = -1$ is a root of the denominator, and apply polynomial division below. Note that if we did not know this information, we would need to use the Rational Root Theorem to factor the denominator completely.

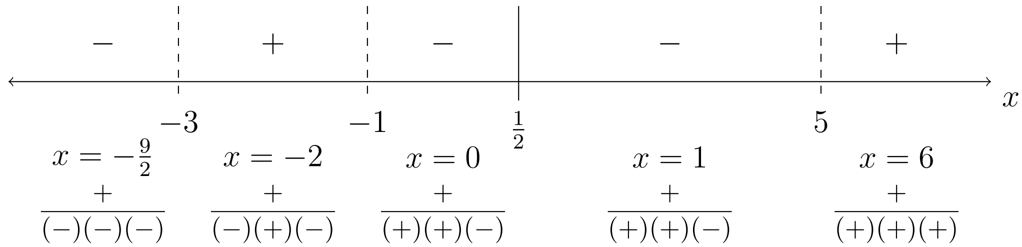
$$\begin{array}{r|rrrr}
 & & x^2 & -2x & -15 \\ \hline
x+1 & x^3 & -x^2 & -17x & -15 \\
 & -x^3 & -x^2 & & \\ \hline
 & & -2x^2 & -17x & \\
 & & 2x^2 & +2x & \\ \hline
 & & & -15x & -15 \\
 & & & 15x & +15 \\ \hline
 & & & & 0
\end{array}$$

Our denominator factors as follows.

$$\begin{aligned}
&x^3 - x^2 - 17x - 15 \\
&= (x+1)(x^2 - 2x - 15) \\
&= (x+1)(x+3)(x-5)
\end{aligned}$$

A complete factorization of our expression is $\dfrac{(2x-1)^2}{(x+1)(x+3)(x-5)}$, with critical values at $x = -3, -1, \frac{1}{2}$, and 5.

This corresponds to the following diagram.



Our answer will correspond to all "negative" intervals in the diagram above (intervals with a $-$). Since the inequality we are asked to solve is not strict (it includes when the expression equals zero), we can combine the two intervals with endpoints at $x = \frac{1}{2}$ into one.

$$\text{We conclude that } \frac{4x^2-4x+1}{x^3-x^2-17x-15} \leq 0 \text{ for all } x \text{ in the set } (-\infty,-3)\cup(-1,5).$$

Next, how might we handle solving an equation or inequality that does not compare an expression to zero, but some other number or another expression? For example,

$$\frac{x(x-3)}{x+2} < 4.$$

Since each of our previous answers required us to use a sign diagram in order to determine whether an expression was positive or negative, it should seem logical to set one side of the given equation or inequality equal to zero, and proceed as before. All that remains is to obtain a common denominator so that we have one rational expression on the non-zero side. Our next example demonstrates this.

Example 121. Solve the inequality $\dfrac{x(x-3)}{x+2} < 4$.

$$\frac{x(x-3)}{x+2} < 4$$

$$\frac{x(x-3)}{x+2} - 4 < \cancel{4} - \cancel{4}$$

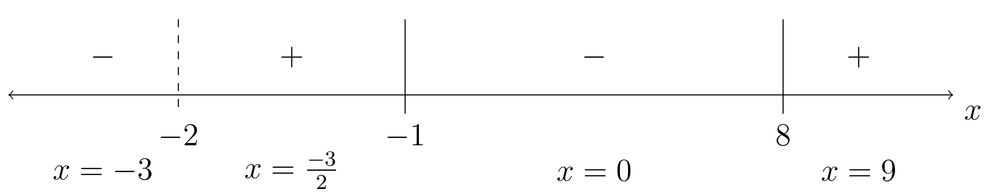
$$\frac{x(x-3)}{x+2} - 4\cdot\frac{x+2}{x+2} < 0$$

$$\frac{x^2-3x-4x-8}{x+2} < 0$$

$$\frac{x^2-7x-8}{x+2} < 0$$

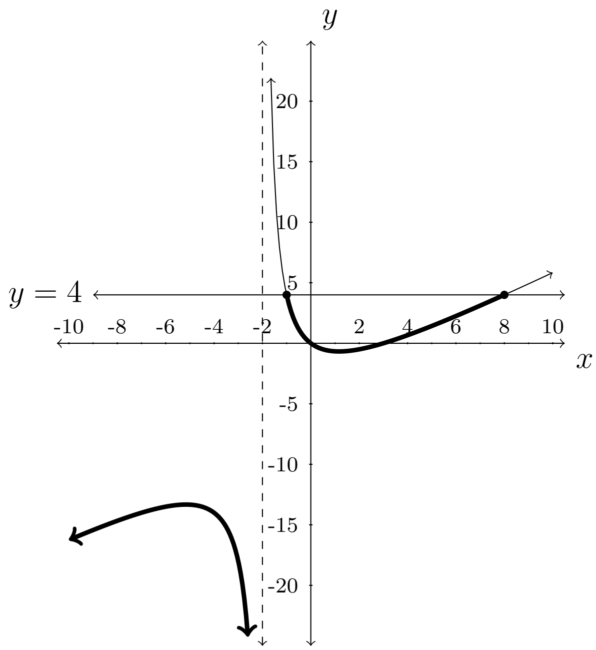
$$\frac{(x-8)(x+1)}{x+2} < 0$$

A sign diagram for the expression $\dfrac{(x-8)(x+1)}{x+2}$ is shown below.

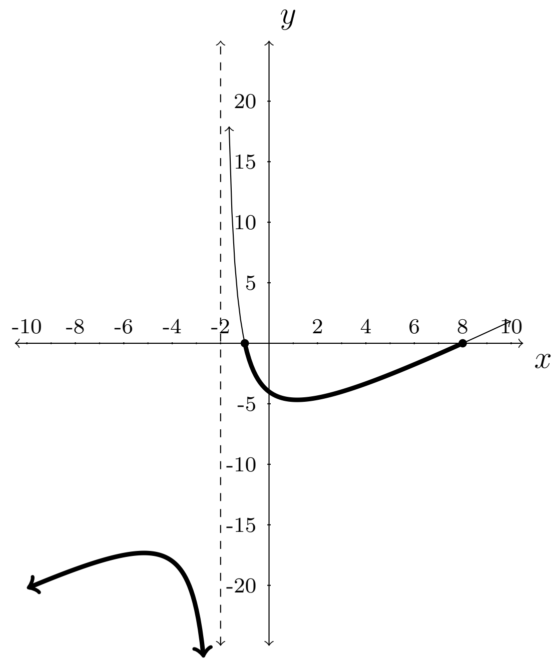


Thus, our inequality holds for all x in the set $(-\infty, -2) \cup (-1, 8)$.

Below, we show the graph of two functions, $f(x) = \dfrac{x(x-3)}{x+2}$ and $g(x) = \dfrac{(x-8)(x+1)}{x+2}$. In the case of the first graph, we see that our solution set coincides with those points that lie below the line $y = 4$, whereas in the case of the second graph, our solution set coincides with those points lying below the $x-$axis (the line $y = 0$). In both graphs, the points coinciding with our solution set appear in bold.



$$f(x) = \frac{x(x-3)}{x+2}$$



$$g(x) = \frac{(x-8)(x+1)}{x+2}$$

This same approach of setting one side of an inequality equal to zero and constructing a sign diagram should be taken with any rational inequality.

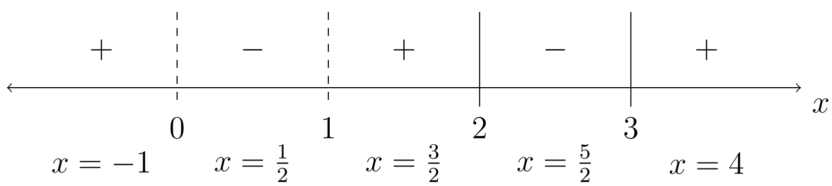
Our final example compares two basic rational expressions.

Example 122. Solve the following inequality

$$\frac{x-6}{x} \geq \frac{-2}{x-1}$$

$$\begin{aligned}
\frac{x-6}{x} &\geq \frac{-2}{x-1} \\
\frac{x-6}{x} + \frac{2}{x-1} &\geq \cancel{\frac{-2}{x-1}} + \cancel{\frac{2}{x-1}} \\
\frac{x-6}{x} + \frac{2}{x-1} &\geq 0 \\
\frac{(x-6)}{x} \cdot \frac{(x-1)}{(x-1)} + \frac{2}{(x-1)} \cdot \frac{x}{x} &\geq 0 \\
\frac{x^2-7x+6+2x}{x(x-1)} &\geq 0 \\
\frac{x^2-5x+6}{x(x-1)} &\geq 0 \\
\frac{(x-3)(x-2)}{x(x-1)} &\geq 0
\end{aligned}$$

This corresponds to the following sign diagram.



From our diagram, we see that $\frac{(x-3)(x-2)}{x(x-1)} \geq 0$, and consequently, $\frac{x-6}{x} \geq \frac{-2}{x-1}$ for all x in the set $(-\infty, 0) \cup (1, 2] \cup [3, \infty)$.

We leave it as an exercise to the reader to confirm our findings using Desmos.

Practice Problems

Perform the indicated tasks below for each of the following rational functions.

- Find the $y-$intercept of the corresponding graph.
- Find the domain of the function.
- Find all $x-$intercepts.
- Graph the function using Desmos, and identify the existence of any horizontal or vertical asymptotes.

1. $f(x) = \dfrac{x+4}{x+2}$
2. $f(x) = \dfrac{x-8}{x-2}$
3. $f(x) = \dfrac{2x-3}{x+3}$
4. $f(x) = \dfrac{2x}{x-1}$
5. $f(x) = \dfrac{x+5}{5-x}$
6. $f(x) = \dfrac{x+4}{2x-1}$
7. $f(x) = \dfrac{x^2}{x-2}$
8. $f(x) = \dfrac{3x+5}{3-x}$
9. $f(x) = \dfrac{x+1}{x^2+1}$

Perform the indicated tasks below for each of the following rational functions.

- Find the $y-$intercept of the corresponding graph.
- Find the domain of the function.
- Find all $x-$intercepts.
- Graph the function using Desmos, and identify the existence of any horizontal, vertical, or slant asymptotes.

10. $f(x) = \dfrac{(x-3)(x+4)}{x-6}$
11. $f(x) = \dfrac{(x-4)^2}{x+2}$
12. $f(x) = \dfrac{x-10}{(x+5)^2}$
13. $f(x) = \dfrac{(x+5)(3x+1)}{(x-1)^2}$
14. $f(x) = \dfrac{(x+3)(x-2)}{(x-4)(x+2)}$
15. $f(x) = \dfrac{2(x-1)^2(x+2)}{(2x-1)^2(x^2+1)}$

For each of exercises 16 through 19, there are many acceptable answers.

16. Find a rational function having a domain of $(-\infty, -3) \cup (-3, \infty)$ and whose graph has an $x-$intercept at $(6, 0)$.
17. Find a rational function whose graph has vertical asymptotes at $x = -2$ and $x = -1$, an $x-$intercept at $(6, 0)$, and a $y-$intercept at $(0, 3)$.
18. A student is asked to construct a rational function whose graph has a vertical asymptote at $x = \frac{1}{2}$, an $x-$intercept at $(2, 0)$, and a $y-$intercept at $(3, 0)$. The student's answer is shown below.
$$f(x) = \frac{x-2}{2x-1} + 3$$
Explain why the student's answer is incorrect, and make the necessary changes to find the desired function.
19. Find a rational function having a domain of $(-\infty, -\sqrt{3}) \cup (-\sqrt{3}, \sqrt{3}) \cup (\sqrt{3}, \infty)$, and whose graph has $x-$intercepts at $(-3, 0)$ and $(3, 0)$, and a $y-$intercept at $(0, 12)$.

Perform the indicated tasks below for each of the following rational functions.

- Find the $y-$intercept of the corresponding graph.
- Factor the function completely over the real numbers.
Note: Functions labeled $r(x)$ will require the Rational Root Theorem.

- Find the domain of the function.
- Find all $x-$intercepts.
- Graph the function using Desmos, and identify the existence of any horizontal, vertical, or slant asymptotes.

20. $f(x) = \dfrac{x^2 - 4}{x^2 - x - 12}$
21. $f(x) = \dfrac{2x^2 - 14x + 24}{x^2 - 6}$
22. $f(x) = \dfrac{6x^2 - 13x - 5}{x^3 - 3x^2 - 10x}$
23. $f(x) = \dfrac{x^2 + 10x + 25}{x^3 - 5x^2 + x - 5}$
24. $f(x) = \dfrac{x^2 - 2x - 2}{2x^2 - 7x - 4}$
25. $f(x) = \dfrac{x^4 - 20x^2 + 64}{x^3 + 8x^2 - x - 8}$
26. $r(x) = \dfrac{2x^3 + 15x^2 + 16x - 12}{x^2 + 6x + 9}$
27. $r(x) = \dfrac{x^2 - 3x - 10}{x^3 + 12x^2 + 3x + 10}$
28. $r(x) = \dfrac{16 - x^4}{x^4 + 16x^3 - 3x^2 - 46x + 32}$

"Take the problem further" – Choose any of exercises 1 through 9 above, and multiply either the numerator or denominator (or both) by the factor $(x + 1)$. Label your new function $g(x)$. Graph both f (the old function) and g on Desmos. Describe any similarities and differences between the two graphs.

Next, change the newly included factor from $(x + 1)$ to $(x + 1)^2$. Again, describe how this has impacted the new graph. We will look more closely at the impacts that such changes have on rational functions in the next few sections.

Selected Answers

$y-$intercept; Domain; $x-$intercept; Horizontal, Vertical, and Slant Asymptotes

1. $(0, 2)$; $(-\infty, -2) \cup (-2, \infty)$; $(-4, 0)$; HA at $y = 1$, VA at $x = -2$
2. $(0, 4)$; $(-\infty, 2) \cup (2, \infty)$; $(8, 0)$; HA at $y = 1$, VA at $x = 2$
3. $(0, -1)$; $(-\infty, -3) \cup (-3, \infty)$; $(\frac{3}{2}, 0)$; HA at $y = 2$, VA at $x = -3$
4. $(0, 0)$; $(-\infty, 1) \cup (1, \infty)$; $(0, 0)$; HA at $y = 2$, VA at $x = 1$
5. $(0, 1)$; $(-\infty, 5) \cup (5, \infty)$; $(-5, 0)$; HA at $y = -1$, VA at $x = 5$
6. $(0, -4)$; $(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$; $(-4, 0)$; HA at $y = \frac{1}{2}$, VA at $x = \frac{1}{2}$
7. $(0, 0)$; $(-\infty, 2) \cup (2, \infty)$; $(0, 0)$; HA at $y = 0$, VA at $x = 2$
8. $(0, \frac{5}{3})$; $(-\infty, 3) \cup (3, \infty)$; $(-\frac{5}{3}, 0)$; HA at $y = -3$, VA at $x = 3$
9. $(0, 1)$; $(-\infty, \infty)$; $(-1, 0)$; HA at $y = 0$, No VA

10. $(0, 2)$; $(-\infty, 6) \cup (6, \infty)$; $(-4, 0), (3, 0)$; No HA, VA at $x = 6$, SA at $y = x + 7$

11. $(0,8)$; $(-\infty,-2)\cup(-2,\infty)$; $(4,0)$; No HA, VA at $x=-2$, SA at $y=x-10$
12. $(0,-\frac{2}{5})$; $(-\infty,-5)\cup(-5,\infty)$; $(10,0)$; HA at $y=0$, VA at $x=-5$
13. $(0,5)$; $(-\infty,1)\cup(1,\infty)$; $(-5,0),(-\frac{1}{3},0)$; HA at $y=3$, VA at $x=1$
14. $(0,\frac{3}{4})$; $(-\infty,-2)\cup(-2,4)\cup(4,\infty)$; $(-3,0),(2,0)$; HA at $y=1$, VA at $x=-2$ and $x=4$
15. $(0,4)$; $(-\infty,\frac{1}{2})\cup(\frac{1}{2},\infty)$; $(-2,0),(1,0)$; HA at $y=0$, VA at $x=\frac{1}{2}$
16. $f(x)=\dfrac{x-6}{x+3}$
17. $f(x)=\dfrac{-(x-6)}{(x+2)(x+1)}$
18. The student's graph has a vertical asymptote at $x=\frac{1}{2}$, but adding three to $\dfrac{x-2}{2x+1}$ shifts all points up three units, so the desired $x-$intercept will no longer be at $(2,0)$, since
$$\frac{x-2}{2x-1}+3=\frac{x-2+3(2x-1)}{2x-1}=\frac{7x-5}{2x-1}.$$
Instead, the student needs to rescale the original function by the appropriate factor. In this case, $f(x)=\dfrac{3(x-2)}{2(2x-1)}$.
19. $f(x)=\dfrac{4(x^2-9)}{x^2-3}$
20. $f(x)=\dfrac{x^2-4}{x^2-x-12}=\dfrac{(x+2)(x-2)}{(x+3)(x-4)}$
    $y-$int at $(0,\frac{1}{3})$
    Domain: $(-\infty,-3)\cup(-3,4)\cup(4,\infty)$
    $x-$int(s) at $(-2,0),(2,0)$
    HA at $y=1$, VA at $x=-3$ and $x=4$
21. $f(x)=\dfrac{2x^2-14x+24}{x^2-6}=\dfrac{2(x-3)(x-4)}{(x+\sqrt{6})(x-\sqrt{6})}$
    $y-$int at $(0,-4)$
    Domain: $(-\infty,-\sqrt{6})\cup(-\sqrt{6},\sqrt{6})\cup(\sqrt{6},\infty)$
    $x-$int(s) at $(3,0),(4,0)$
    HA at $y=2$, VA at $x=-\sqrt{6}$ and $x=\sqrt{6}$
22. $f(x)=\dfrac{6x^2-13x-5}{x^3-3x^2-10x}=\dfrac{(3x+1)(2x-5)}{x(x+2)(x-5)}$
    No $y-$int
    Domain: $(-\infty,-2)\cup(-2,0)\cup(0,5)\cup(5,\infty)$
    $x-$int(s) at $(-\frac{1}{3},0),(\frac{5}{2},0)$
    HA at $y=0$, VA at $x=-2$, $x=0$, and $x=5$

23. $f(x) = \dfrac{x^2+10x+25}{x^3-5x^2+x-5} = \dfrac{(x+5)^2}{(x-5)(x^2+1)}$

$y-$int at $(0,-5)$
Domain: $(-\infty,5)\cup(5,\infty)$
$x-$int(s) at $(-5,0)$
HA at $y=0$, VA at $x=5$

24. $f(x) = \dfrac{x^2-2x-2}{2x^2-7x-4} = \dfrac{(x-(1-\sqrt{3}))(x-(1+\sqrt{3}))}{(2x+1)(x-4)}$

$y-$int at $(0,\frac{1}{2})$
Domain: $(-\infty,-\frac{1}{2})\cup(-\frac{1}{2},4)\cup(4,\infty)$
$x-$int(s) at $(1-\sqrt{3},0),(1+\sqrt{3},0)$
HA at $y=\frac{1}{2}$, VA at $x=-\frac{1}{2}$ and $x=4$

25. $f(x) = \dfrac{x^4-20x^2+64}{x^3+8x^2-x-8} = \dfrac{(x+2)(x-2)(x+4)(x-4)}{(x+8)(x+1)(x-1)}$

$y-$int at $(0,-8)$
Domain: $(-\infty,-8)\cup(-8,-1)\cup(-1,1)\cup(1,\infty)$
$x-$int(s) at $(-4,0),(-2,0),(2,0)$, and $(4,0)$
No HA, VA at $x=-8$, $x=-1$, and $x=1$, SA at $y=x-8$

26. $r(x) = \dfrac{2x^3+15x^2+16x-12}{x^2+6x+9} = \dfrac{(x+6)(x+2)(2x-1)}{(x+3)^2}$

$y-$int at $(0,-\frac{4}{3})$
Domain: $(-\infty,-3)\cup(-3,\infty)$
$x-$int(s) at $(-6,0),(-2,0)$, and $(\frac{1}{2},0)$
No HA, VA at $x=-3$, SA at $y=2x+3$

27. $r(x) = \dfrac{x^2-3x-10}{x^3+12x^2+3x+10} = \dfrac{(x+2)(x-5)}{(x+10)(x+1)^2}$

$y-$int at $(0,-1)$
Domain: $(-\infty,-10)\cup(-10,-1)\cup(-1,\infty)$
$x-$int(s) at $(-2,0),(5,0)$
HA at $y=0$, VA at $x=-10$ and $x=-1$

28. $r(x) = \dfrac{16-x^4}{x^4+16x^3-3x^2-46x+32} = \dfrac{-(x+2)(x-2)(x^2+4)}{(x+16)(x+2)(x-1)^2}$

$y-$int at $(0,\frac{1}{2})$
Domain: $(-\infty,-16)\cup(-16,-2)\cup(-2,1)\cup(1,\infty)$
$x-$int(s) at $(2,0)$ (Hole at $(-2,0)$)
HA at $y=-1$, VA at $x=-16$ and $x=1$