College Algebra

Textbook Part I

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This textbook is designed as the primary resource for instruction of a traditional College Algebra course at Framingham State University. Each section follows closely with its respective lesson(s) in the accompanying course pack, but offers more detailed explanations and additional worked out examples.

Although largely free of mathematical errors and "typos", students who identify any errors/typos in either the textbook or course pack are encouraged to report them to the instructor, and the reporting of any mathematical errors will be rewarded with small incentives in the form of additional course homework, quiz, or exam points.

The following chapters make up the first half of the course and cover the following content.

- Linear Equations and Inequalities
- Systems of Linear Equations
- Introduction to Functions
- Quadratic Equations and Inequalities

The following chapters make up the second half of the course and cover the following content.

- Advanced Function Concepts
- Polynomials
- Rational Functions

This text contains original content, as well as content adapted from each of the following open-source texts.

Contents

Measurable Outcomes

Below is a comprehensive list of the anticipated measurable outcomes and some essential prerequisite skills needed for successful completion of the College Algebra course. This list is based off of the course description and exit list topics of MATH 123 College Algebra at Framingham State University. Each outcome number aligns to its respective lesson in the accompanying course pack.

1 Solve general linear equations with variables on both sides of the equation.

2 Solve an equation that contains one or more absolute value(s).

3 Graph a linear equation by creating a table of values for x.
Identify the slope of a linear equation both graphically and algebraically.

4 Write the equation of a line in slope-intercept and point-slope form.

5 Write the equation of a line given a line parallel or perpendicular.

6 Solve, graph, and give interval notation for the solution to a linear inequality.
Create a sign diagram to identify those intervals where a linear expression is positive or negative.

7 Solve, graph, and give interval notation to the solution of a compound inequality.

8 Solve, graph, and give interval notation to the solution of an inequality containing absolute values.

9 Solve linear systems by graphing.

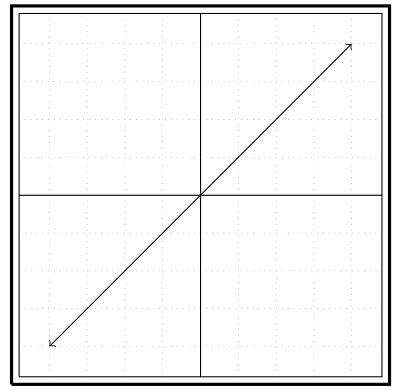
10 Solve linear systems by substitution.

11 Solve linear systems by addition and elimination.

12 Define a relation and a function; determine if a relation is a function.

13 Evaluate functions using appropriate notation.

14 Find the domain and range of a function from its graph.



Chapter 1

Linear Equations and Inequalities

Solving Linear Equations

One-Step Equations

Objective: Solve one-step linear equations by balancing using inverse operations.

Solving linear equations is an important and fundamental skill in algebra. In algebra, we are often presented with a problem where the answer is known, but part of the problem is missing. The missing part of the problem is what we seek to find. An example of such a problem is shown below.

Example 1.

$$4x + 16 = -4$$

Notice the above problem has a missing part, or unknown, that is marked by x. If we are given that the solution to this equation is $x = -5$, it could be plugged into the equation, replacing the x with -5. This is shown in Example 2.

Example 2.

$$\begin{aligned} 4(-5) + 16 &= -4 && \text{Multiply } 4(-5) \\ -20 + 16 &= -4 && \text{Add } -20 + 16 \\ -4 &= -4 && \text{True!} \end{aligned}$$

Now the equation comes out to a true statement! Notice also that if another number, for example, $x = 3$, was plugged in, we would not get a true statement as seen in Example 3.

Example 3.

$$\begin{aligned} 4(3) + 16 &= -4 && \text{Multiply } 4(3) \\ 12 + 16 &= -4 && \text{Add } 12 + 16 \\ 28 &\neq -4 && \text{False!} \end{aligned}$$

Due to the fact that this is not a true statement, this demonstrates that $x = 3$ is not the solution. However, depending on the complexity of the problem, this "guess and check" method is not very efficient. Thus, we take a more algebraic approach to solving equations. Here we will focus on what are called "one-step equations" or equations that only require one step to solve. While these equations often seem very fundamental, it is important to master the pattern for solving these problems so we can solve more complex problems.

Addition Problems

To solve equations, the general rule is to do the opposite, as demonstrated in the following example.

Example 4.

$$\begin{array}{rl} x + 7 = -5 & \text{The 7 is added to the } x \\ \underline{\mathbf{-7 \quad -7}} & \text{Subtract 7 from both sides to get rid of it} \\ x = -12 & \text{Our solution} \end{array}$$

It is important for the reader to recognize the benefit of checking an answer by plugging it back into the given equation, as we did with examples 2 and 3 above. This is a step that often gets overlooked by many individuals who may be eager to attempt the next problem. As is the case with most textbooks, we will often omit this step from this point forward, with the understanding that it will usually be an exercise that is left to the reader to verify the validity of each answer.

The same process is used in each of the following examples.

$4 + x = 8$	$7 = x + 9$	$5 = 8 + x$
$\underline{\mathbf{-4 \qquad -4}}$	$\underline{\mathbf{-9 \quad -9}}$	$\underline{\mathbf{-8 \quad -8}}$
$x = 4$	$-2 = x$	$-3 = x$

Table 1.1: Addition Examples

Subtraction Problems

In a subtraction problem, we get rid of negative numbers by adding them to both sides of the equation, as demonstrated in the following example.

Example 5.

$$\begin{array}{rl} x - 5 = 4 & \text{The 5 is negative, or subtracted from } x \\ \underline{\mathbf{+5 \quad +5}} & \text{Add 5 to both sides} \\ x = 9 & \text{Our solution} \end{array}$$

The same process is used in each of the following examples. Notice that each time we are getting rid of a negative number by adding.

In every example, we introduce the opposite operation of what is shown, in order to solve the given equation. This notion of opposites is more commonly referred to as an *inverse* operation. The inverse operation of addition is subtraction, and vice versa. Similarly, the inverse operation of multiplication is division, and vice versa, which we will see momentarily.

$-6 + x = -2$	$-10 = x - 7$	$5 = -8 + x$
$\underline{\mathbf{+6} \quad \mathbf{+\,6}}$	$\underline{\mathbf{+7} \quad \mathbf{+\,7}}$	$\underline{\mathbf{+8} \quad \mathbf{+\,8}}$
$x = 4$	$-3 = x$	$13 = x$

Table 1.2: Subtraction Examples

Multiplication Problems

With a multiplication problem, we get rid of the number by dividing on both sides, as demonstrated in the following examples.

Example 6.

$$\begin{array}{rl} 4x = 20 & \text{Variable is multiplied by } 4 \\ \overline{\mathbf{4}} \quad \overline{\mathbf{4}} & \text{Divide both sides by } 4 \\ x = 5 & \text{Our solution} \end{array}$$

With multiplication problems it is very important that care is taken with signs. If x is multiplied by a negative then we will divide by a negative. This is shown in example 7.

Example 7.

$$\begin{array}{rl} -5x = 30 & \text{Variable is multiplied by } -5 \\ \overline{\mathbf{-5}} \quad \overline{\mathbf{-5}} & \text{Divide both sides by } -5 \\ x = -6 & \text{Our solution} \end{array}$$

The same process is used in each of the following examples. Notice how negative and positive numbers are handled as each problem is solved.

$8x = -24$	$-4x = -20$	$42 = 7x$
$\overline{\mathbf{8}} \quad \overline{\mathbf{8}}$	$\overline{\mathbf{-4}} \quad \overline{\mathbf{-4}}$	$\overline{\mathbf{7}} \quad \overline{\mathbf{7}}$
$x = -3$	$x = 5$	$6 = x$

Table 1.3: Multiplication Examples

Division Problems

In division problems, we get rid of the denominator by multiplying on both sides, since multiplication is the opposite, or *inverse*, operation of division. This is demonstrated in the examples shown below.

Example 8.

$$\begin{array}{rl} \dfrac{x}{5} = -3 & \text{Variable is divided by 5} \\ (\mathbf{5})\dfrac{x}{5} = -3(\mathbf{5}) & \text{Multiply both sides by 5} \\ x = -15 & \text{Our solution} \end{array}$$

$\frac{x}{-7} = -2$	$\frac{x}{8} = 5$	$\frac{x}{-4} = 9$
$(\mathbf{-7})\frac{x}{-7} = -2(\mathbf{-7})$	$(\mathbf{8})\frac{x}{8} = 5(\mathbf{8})$	$(\mathbf{-4})\frac{x}{-4} = 9(\mathbf{-4})$
$x = 14$	$x = 40$	$x = -36$

Table 1.4: Division Examples

The process described above is fundamental to solving equations. Once this process is mastered, the problems we will see have several more steps. These problems may seem more complex, but the process and patterns used will remain the same.

Two-Step Equations

Objective: Solve two-step equations by balancing and using inverse operations.

After mastering the technique for solving one-step equations, we are ready to consider two-step equations. As we solve two-step equations, the important thing to remember is that everything works backwards! When working with one-step equations, we learned that in order to clear a "plus five" in the equation, we would subtract five from both sides. We learned that to clear "divided by seven" we multiply by seven on both sides. The same pattern applies to the order of operations. When solving for our variable x, we use order of operations backwards as well. This means we will add or subtract first, then multiply or divide second (then exponents, and finally any parentheses or grouping symbols, but that's another lesson).

Example 9.

$$4x - 20 = -8$$

We have two numbers on the same side as the x. We need to move the 4 and the 20 to the other side. We know to move the 4 we need to divide, and to move the 20 we will add 20 to both sides. If order of operations is done backwards, we will add or subtract first. Therefore we will add 20 to both sides first. Once we are done with that, we will divide both sides by 4. The steps are shown below.

$$\begin{array}{rl} 4x - 20 = -8 & \text{Start by focusing on the subtract 20} \\ \underline{\mathbf{+20} \quad \mathbf{+20}} & \text{Add 20 to both sides} \\ 4x = 12 & \text{Now we focus on the 4 multiplied by } x \\ \overline{\mathbf{4}} \quad \overline{\mathbf{4}} & \text{Divide both sides by 4} \\ x = 3 & \text{Our solution} \end{array}$$

Notice in our next example when we replace the x with 3 we get a true statement.

$$\begin{aligned} 4(3)-20&=-8 && \text{Multiply } 4(3)\\ 12-20&=-8 && \text{Subtract } 12-20\\ -8&=-8 && \text{True!} \end{aligned}$$

The same process is used to solve any two-step equation. Add or subtract first, then multiply or divide.

Example 10.

$$\begin{aligned} 5x+7&=7 && \text{Start by focusing on the plus } 7\\ \underline{\mathbf{-7}\quad}&\underline{\;\mathbf{-\,7}} && \text{Subtract } 7 \text{ from both sides}\\ 5x&=0 && \text{Now focus on the multiplication by } 5\\ \overline{\mathbf{5}}\quad&\;\overline{\mathbf{5}} && \text{Divide both sides by } 5\\ x&=0 && \text{Our solution} \end{aligned}$$

Notice the seven subtracted out completely! Many students get stuck on this point, do not forget that we have a number for "nothing left", and that number is zero. With this in mind the process is almost identical to our first example.

A common error students make with two-step equations is with negative signs. Remember the sign always stays with the number. Consider the following example.

Example 11.

$$\begin{aligned} 4-2x&=10 && \text{Start by focusing on the positive } 4\\ \underline{\mathbf{-4}\qquad}&\underline{\;\mathbf{-\,4}} && \text{Subtract } 4 \text{ from both sides}\\ -2x&=6 && \text{Negative (subtraction) stays on the } 2x\\ \overline{\mathbf{-2}}\;&\;\overline{\mathbf{-2}} && \text{Divide by } -2\\ x&=-3 && \text{Our solution} \end{aligned}$$

The same is true even if there is no apparent coefficient in front of the variable. The coefficient is 1 or -1 in this case. Consider the next example.

Example 12.

$$\begin{aligned} 8-x&=2 && \text{Start by focusing on the positive } 8\\ \underline{\mathbf{-8}\qquad}&\underline{\;\mathbf{-\,8}} && \text{Subtract } 8 \text{ from both sides}\\ -x&=-6 && \text{Negative(subtraction) stays on the } x\\ -1x&=-6 && \text{Remember, no number in front of variable means } 1\\ \overline{\mathbf{-1}}\;&\;\overline{\mathbf{-1}} && \text{Divide both sides by } -1\\ x&=6 && \text{Our solution} \end{aligned}$$

$$\begin{array}{l} -3x+7=-8 \\ \underline{\quad\ \mathbf{-7}\quad \mathbf{-7}} \\ \frac{-3x}{\mathbf{-3}}=\frac{-15}{\mathbf{-3}} \\ x=5 \end{array}$$

$$\begin{array}{l} -2+9x=7 \\ \underline{\mathbf{+2}\qquad\ \ \mathbf{+2}} \\ \frac{9x}{\mathbf{9}}=\frac{9}{\mathbf{9}} \\ x=1 \end{array}$$

$$\begin{array}{l} 8=2x+10 \\ \underline{\mathbf{-10}\qquad \mathbf{-10}} \\ \frac{-2}{\mathbf{2}}=\frac{2x}{\mathbf{2}} \\ -1=x \end{array}$$

$$\begin{array}{l} 7-5x=17 \\ \underline{\mathbf{-7}\qquad\quad\ \mathbf{-7}} \\ \frac{-5x}{\mathbf{-5}}=\frac{10}{\mathbf{-5}} \\ x=-2 \end{array}$$

$$\begin{array}{l} -5-3x=-5 \\ \underline{\mathbf{+5}\qquad\quad \mathbf{+5}} \\ \frac{-3x}{\mathbf{-3}}=\frac{0}{\mathbf{-3}} \\ x=0 \end{array}$$

$$\begin{array}{l} -3=\frac{x}{5}-4 \\ \underline{\mathbf{+4}\qquad\ \ \mathbf{+4}} \\ \\ \mathbf{(5)}(1)=\frac{x}{5}\mathbf{(5)} \\ 5=x \end{array}$$

Table 1.5: Two-Step Equation Examples

Solving two-step equations is a very important skill to master, as we study algebra. The first step is to add or subtract, the second is to multiply or divide. This pattern is seen in each of our examples thus far.

As problems in algebra become more complex the process covered here will remain the same. In fact, as we solve problems like those in the next example, each one of them will have several steps to solve, but the last two steps will resemble solving a two-step equation. This is why it is very important to master two-step equations now!

Example 13.

$$3x^2+4-x=6 \qquad\qquad \frac{1}{x-8}+\frac{1}{x}=\frac{1}{3}$$

$$\sqrt{5x-5}+1=x \qquad\qquad \log_5(2x-4)=1$$

General Equations

Objective: Solve general linear equations with variables on both sides.

Often as we are solving linear equations we will need to do some work to set them up into a form we are familiar with solving. This section will focus on manipulating an equation we are asked to solve in such a way that we can use our pattern for solving two-step equations to ultimately arrive at the solution.

One such issue that needs to be addressed is parentheses. Often the parentheses can get in the way of solving an otherwise easy problem. As you might expect we can get rid of the unwanted parentheses by using the distributive property. This is shown in the following example. Notice the first step is distributing, then it is solved like any other two-step equation.

Example 14.

$$
\begin{array}{rl}
4(2x-6)=16 & \text{Distribute 4 through parentheses} \\
8x-24=16 & \text{Focus on the subtraction first} \\
\underline{\mathbf{+24\ \ +24}} & \text{Add 24 to both sides} \\
8x=40 & \text{Now focus on the multiply by 8} \\
\overline{\mathbf{8}}\quad\ \overline{\mathbf{8}} & \text{Divide both sides by 8} \\
x=5 & \text{Our solution}
\end{array}
$$

Often after we distribute there will be some like terms on one side of the equation. Example 15 shows distributing to clear the parentheses and then combining like terms next. Notice we only combine like terms on the same side of the equation. Once we have done this, our next example solves just like any other two-step equation.

Example 15.

$$
\begin{array}{rl}
3(2x-4)+9=15 & \text{Distribute the 3 through the parentheses} \\
6x-12+9=15 & \text{Combine like terms, } -12+9 \\
6x-3=15 & \text{Focus on the subtraction first} \\
\underline{\mathbf{+3\ \ +3}} & \text{Add 3 to both sides} \\
6x=18 & \text{Now focus on multiply by 6} \\
\overline{\mathbf{6}}\quad\ \overline{\mathbf{6}} & \text{Divide both sides by 6} \\
x=3 & \text{Our solution}
\end{array}
$$

A second type of problem that becomes a two-step equation after a bit of work is one where we see the variable on both sides. This is shown in the following example.

Example 16.

$$4x-6=2x+10$$

Notice here the x is on both the left and right sides of the equation. This can make it difficult to decide which side to work with. We fix this by moving one of the terms with x to the other side, much like we moved a constant term. It doesn't matter which term gets moved, $4x$ or $2x$, however, it would be the author's suggestion to move the smaller term (to avoid negative coefficients). For this reason we begin this problem by clearing the positive $2x$ by subtracting $2x$ from both sides.

$$
\begin{array}{rl}
4x-6=2x+10 & \text{Notice the variable on both sides} \\
\underline{\boldsymbol{-2x}\qquad\ \boldsymbol{-\,2x}}\quad\ \ & \text{Subtract } 2x \text{ from both sides} \\
2x-6=10 & \text{Focus on the subtraction first} \\
\underline{\mathbf{+6\ \ +6}} & \text{Add 6 to both sides} \\
2x=16 & \text{Focus on the multiplication by 2} \\
\overline{\mathbf{2}}\quad\ \overline{\mathbf{2}} & \text{Divide both sides by 2} \\
x=8 & \text{Our solution}
\end{array}
$$

The previous example shows the check on this solution. Here the solution is plugged into the x on both the left and right sides before simplifying.

Example 17.

$$\begin{array}{rl}
4(8) - 6 = 2(8) + 10 & \text{Multiply } 4(8) \text{ and } 2(8) \text{ first} \\
32 - 6 = 16 + 10 & \text{Add and Subtract} \\
26 = 26 & \text{True!}
\end{array}$$

The next example illustrates the same process with negative coefficients. Notice first the smaller term with the variable is moved to the other side, this time by adding because the coefficient is negative.

Example 18.

$$\begin{array}{rl}
-3x + 9 = 6x - 27 & \text{Notice the variable on both sides, } -3x \text{ is smaller} \\
\underline{\boldsymbol{+3x \qquad + 3x}} & \text{Add } 3x \text{ to both sides} \\
9 = 9x - 27 & \text{Focus on the subtraction by 27} \\
\underline{\mathbf{+27 \qquad + 27}} & \text{Add 27 to both sides} \\
36 = 9x & \text{Focus on the multiplication by 9} \\
\overline{\mathbf{9}} \quad \overline{\mathbf{9}} & \text{Divide both sides by 9} \\
4 = x & \text{Our solution}
\end{array}$$

Linear equations can become particularly interesting when the two processes are combined. In the following problems we have parentheses and the variable on both sides. Notice in each of the following examples we distribute, then combine like terms, then move the variable to one side of the equation.

Example 19.

$$\begin{array}{rl}
2(x-5) + 3x = x + 18 & \text{Distribute the 2 through parentheses} \\
2x - 10 + 3x = x + 18 & \text{Combine like terms } 2x + 3x \\
5x - 10 = x + 18 & \text{Notice the variable is on both sides} \\
\underline{\boldsymbol{-x \qquad - x}} & \text{Subtract } x \text{ from both sides} \\
4x - 10 = 18 & \text{Focus on the subtraction of 10} \\
\underline{\mathbf{+10 \;\; + 10}} & \text{Add 10 to both sides} \\
4x = 28 & \text{Focus on multiplication by 4} \\
\overline{\mathbf{4}} \quad \overline{\mathbf{4}} & \text{Divide both sides by 4} \\
x = 7 & \text{Our solution}
\end{array}$$

Sometimes we may have to distribute more than once to clear several parentheses. Remember to combine like terms after you distribute!

Example 20.

$$
\begin{array}{rl}
3(4x-5)-4(2x+1)=5 & \text{Distribute 3 and } -4 \text{ through parentheses} \\
12x-15-8x-4=5 & \text{Combine like terms } 12x-8x \text{ and } -15-4 \\
4x-19=5 & \text{Focus on subtraction of 19} \\
\underline{\mathbf{+19 \quad +19}} & \text{Add 19 to both sides} \\
4x=24 & \text{Focus on multiplication by 4} \\
\overline{\mathbf{4}} \quad \overline{\mathbf{4}} & \text{Divide both sides by 4} \\
x=6 & \text{Our solution}
\end{array}
$$

This leads to a 5-step process to solve any linear equation. While all five steps aren't always needed, this can serve as a guide to solving equations.

1. Distribute through any parentheses.
2. Combine like terms on each side of the equation.
3. Get the variables on one side by adding or subtracting
4. Solve the remaining 2-step equation (add or subtract then multiply or divide)
5. Check your answer by plugging it back in for x to find a true statement. If your resulting statement is false, repeat the procedure, beginning with the first step.

The order of these steps is very important.

We can see each of the above five steps worked through our next example.

Example 21.

$$
\begin{array}{rl}
4(2x-6)+9=3(x-7)+8x & \text{Distribute 4 and 3 through parentheses} \\
8x-24+9=3x-21+8x & \text{Combine like terms } -24+9 \text{ and } 3x+8x \\
8x-15=11x-21 & \text{Notice the variable is on both sides} \\
\underline{\boldsymbol{-8x \qquad -8x}} & \text{Subtract } 8x \text{ from both sides} \\
-15=3x-21 & \text{Focus on subtraction of 21} \\
\underline{\mathbf{+21 \qquad +21}} & \text{Add 21 to both sides} \\
6=3x & \text{Focus on multiplication by 3} \\
\overline{\mathbf{3}} \quad \overline{\mathbf{3}} & \text{Divide both sides by 3} \\
2=x & \text{Our solution}
\end{array}
$$

Check:

$$
\begin{array}{rl}
4[2(2)-6]+9=3[(2)-7]+8(2) & \text{Plug 2 in for each } x\text{. Multiply inside parentheses} \\
4[4-6]+9=3[-5]+8(2) & \text{Finish parentheses on left, multiply on right} \\
4[-2]+9=-15+8(2) & \text{Finish multiplication on both sides} \\
-8+9=-15+16 & \text{Add} \\
1=1 & \text{True!}
\end{array}
$$

When we check our solution of $x = 2$ we found a true statement, $1 = 1$. Therefore, we know our solution $x = 2$ is the correct solution for the problem.

There are two special cases that can come up as we are solving these linear equations. The first is illustrated in the next two examples. Notice we start by distributing and moving the variables all to the same side.

Example 22.

$$\begin{array}{rl}
3(2x-5) = 6x-15 & \text{Distribute 3 through parentheses} \\
6x-15 = 6x-15 & \text{Notice the variable on both sides} \\
\underline{\boldsymbol{-6x \qquad -6x}} \quad & \text{Subtract } 6x \text{ from both sides} \\
-15 = -15 & \text{Variable is gone! True!}
\end{array}$$

Here the variable subtracted out completely! We are left with a true statement, $-15 = -15$. If the variables subtract out completely and we are left with a true statement, this indicates that the equation is always true, no matter what x is. Thus, for our solution we say "all real numbers" or $\mathbb{R}$.

It is worth mentioning that in both the previous and following examples, we are still *solving* a given equation for all possible values of x. When the variable is eliminated entirely, this can sometimes be confused with *checking* a solution.

Example 23.

$$\begin{array}{rl}
2(3x-5)-4x = 2x+7 & \text{Distribute 2 through parentheses} \\
6x-10-4x = 2x+7 & \text{Combine like terms } 6x-4x \\
2x-10 = 2x+7 & \text{Notice the variable is on both sides} \\
\underline{\boldsymbol{-2x \qquad -2x}} \quad & \text{Subtract } 2x \text{ from both sides} \\
-10 \neq 7 & \text{Variable is gone! False!}
\end{array}$$

Again, the variable subtracted out completely! However, this time we are left with a false statement, this indicates that the equation is never true, no matter what x is. Thus, for our solution we say "no solutions" or $\varnothing$.

Equations Containing Fractions (L1)

Objective: Solve linear equations with rational coefficients by multiplying by the least common multiple of the denominators to clear the fractions.

Often when solving linear equations we will need to work with an equation with fraction coefficients. We can solve these problems as we have in the past. This is demonstrated in our next example.

Example 24.

$$\frac{3}{4}x - \frac{7}{2} = \frac{5}{6} \qquad \text{Focus on subtraction}$$

$$\underline{+\mathbf{\frac{7}{2}} \quad +\mathbf{\frac{7}{2}}} \qquad \text{Add } \frac{7}{2} \text{ to both sides}$$

Notice we will need to get a common denominator to add $\frac{5}{6} + \frac{7}{2}$. We have a common denominator of 6. So we build up the denominator, $\frac{7}{2}\left(\frac{3}{3}\right) = \frac{21}{6}$, and we can now add the fractions:

$$\frac{3}{4}x - \frac{21}{6} = \frac{5}{6} \qquad \text{Same problem, with common denominator 6}$$

$$\underline{+\mathbf{\frac{21}{6}} \quad +\mathbf{\frac{21}{6}}} \qquad \text{Add } \frac{21}{6} \text{ to both sides}$$

$$\frac{3}{4}x = \frac{26}{6} \qquad \text{Reduce } \frac{26}{6} \text{ to } \frac{13}{3}$$

$$\frac{3}{4}x = \frac{13}{3} \qquad \text{Focus on multiplication by } \frac{3}{4}$$

We can get rid of $\frac{3}{4}$ by dividing both sides by $\frac{3}{4}$.

Dividing by a fraction is the same as multiplying by the reciprocal, so we will multiply both sides by $\frac{4}{3}$.

$$\left(\mathbf{\frac{4}{3}}\right)\frac{3}{4}x = \frac{13}{3}\left(\mathbf{\frac{4}{3}}\right) \qquad \text{Multiply by reciprocal}$$

$$x = \frac{52}{9} \qquad \text{Our solution}$$

While this process does help us arrive at the correct solution, the fractions can make the process quite difficult. This is why we have an alternate method for dealing with fractions - clearing fractions. Clearing fractions is nice as it gets rid of the fractions for the majority of the problem. We can easily clear the fractions by finding the least common multiple (LCM) of the denominators and multiplying each term by the LCM. This is shown in the next example, the same problem as our first example, but this time we will solve by clearing fractions.

Example 25.

$$\begin{array}{rl}
\dfrac{3}{4}x-\dfrac{7}{2}=\dfrac{5}{6} & \text{LCM} = 12, \text{multiply each term by } 12\\
\dfrac{(\mathbf{12})3}{4}x-\dfrac{(\mathbf{12})7}{2}=\dfrac{(\mathbf{12})5}{6} & \text{Reduce each 12 with denominators}\\
(\mathbf{3})3x-(\mathbf{6})7=(\mathbf{2})5 & \text{Multiply out each term}\\
9x-42=10 & \text{Focus on subtraction by 42}\\
\underline{\mathbf{+42}\quad\mathbf{+42}} & \text{Add 42 to both sides}\\
9x=52 & \text{Focus on multiplication by 9}\\
\overline{\mathbf{9}}\quad\overline{\mathbf{9}} & \text{Divide both sides by 9}\\
x=\dfrac{52}{9} & \text{Our solution}
\end{array}$$

The next example illustrates this as well. Notice the 2 isn't a fraction in the original equation, but to solve it we put the 2 over 1 to make it a fraction.

Example 26.

$$\begin{array}{rl}
\dfrac{2}{3}x-2=\dfrac{3}{2}x+\dfrac{1}{6} & \text{LCM} = 6, \text{multiply each term by } 6\\
\dfrac{(\mathbf{6})2}{3}x-\dfrac{(\mathbf{6})2}{1}=\dfrac{(\mathbf{6})3}{2}x+\dfrac{(\mathbf{6})1}{6} & \text{Reduce 6 with each denominator}\\
(\mathbf{2})2x-(\mathbf{6})2=(\mathbf{3})3x+(\mathbf{1})1 & \text{Multiply out each term}
\end{array}$$

$$\begin{array}{rl}
4x-12=9x+1 & \text{Notice variable on both sides}\\
\underline{\boldsymbol{-4x}\qquad\boldsymbol{-4x}} & \text{Subtract } 4x \text{ from both sides}\\
-12=5x+1 & \text{Focus on addition of 1}\\
\underline{\mathbf{-1}\qquad\mathbf{-1}} & \text{Subtract 1 from both sides}\\
-13=5x & \text{Focus on multiplication of 5}\\
\overline{\mathbf{5}}\quad\overline{\mathbf{5}} & \text{Divide both sides by 5}\\
-\dfrac{13}{5}=x & \text{Our solution}
\end{array}$$

We can use this same process if there are parenthesis in the problem. We will first distribute the coefficient in front of the parenthesis, then clear the fractions. This is seen in the following example.

Example 27.

$$
\begin{array}{rl}
\frac{3}{2}\left(\frac{5}{9}x+\frac{4}{27}\right)=3 & \text{Distribute } \frac{3}{2} \text{ through parenthesis, reducing if possible} \\
\frac{5}{6}x+\frac{2}{9}=3 & \text{LCM} = 18\text{, multiply each term by } 18 \\
\frac{\mathbf{(18)}5}{6}x+\frac{\mathbf{(18)}2}{9}=\mathbf{(18)}3 & \text{Reduce 18 with each denominator} \\
\mathbf{(3)}5x+\mathbf{(2)}2=\mathbf{(18)}3 & \text{Multiply out each term} \\
15x+4=54 & \text{Focus on addition of 4} \\
\underline{\mathbf{-4\ \ -4}} & \text{Subtract 4 from both sides} \\
15x=50 & \text{Focus on multiplication by 15} \\
\overline{\mathbf{15}}\quad\overline{\mathbf{15}} & \text{Divide both sides by 15, reduce on right side} \\
x=\frac{10}{3} & \text{Our solution}
\end{array}
$$

While the problem can take many different forms, the pattern to clear the fraction is the same, after distributing through any parentheses we multiply each term by the LCM and reduce. This will give us a problem with no fractions that is much easier to solve. The following example again illustrates this process.

Example 28.

$$
\begin{array}{rl}
\frac{3}{4}x-\frac{1}{2}=\frac{1}{3}(\frac{3}{4}x+6)-\frac{7}{2} & \text{Distribute } \frac{1}{3}\text{, reduce if possible} \\
\frac{3}{4}x-\frac{1}{2}=\frac{1}{4}x+2-\frac{7}{2} & \text{LCM} = 4\text{, multiply each term by } 4 \\
\frac{\mathbf{(4)}3}{4}x-\frac{\mathbf{(4)}1}{2}=\frac{\mathbf{(4)}1}{4}x+\frac{\mathbf{(4)}2}{1}-\frac{\mathbf{(4)}7}{2} & \text{Reduce 4 with each denominator} \\
\mathbf{(1)}3x-\mathbf{(2)}1=\mathbf{(1)}1x+\mathbf{(4)}2-\mathbf{(2)}7 & \text{Multiply out each term} \\
3x-2=x+8-14 & \text{Combine like terms } 8-14
\end{array}
$$

$$
\begin{array}{rl}
3x-2=x-6 & \text{Notice variable on both sides} \\
\underline{\boldsymbol{-x}\qquad\boldsymbol{-x}} & \text{Subtract } x \text{ from both sides} \\
2x-2=-6 & \text{Focus on subtraction by 2} \\
\underline{\mathbf{+2\ \ +2}} & \text{Add 2 to both sides} \\
2x=-4 & \text{Focus on multiplication by 2} \\
\overline{\mathbf{2}}\qquad\overline{\mathbf{2}} & \text{Divide both sides by 2} \\
x=-2 & \text{Our solution}
\end{array}
$$

Linear Equations Containing an Absolute Value (L2)

Objective: Solve linear equations containing an absolute value.

When solving equations with absolute value we can end up with more than one possible answer. This is because what is in the absolute value can be either negative or positive and we must account for both possibilities when solving equations. This is illustrated in the following example.

Example 29.

$$\begin{array}{rl} |x| = 7 & \text{Absolute value can be positive or negative} \\ x = 7 \ \text{ or } \ x = -7 & \text{Our solution} \end{array}$$

Notice that we have considered two possibilities, both the positive and negative. Either way, the absolute value of our number will be positive 7.

When we have absolute values in our problem it is important to first isolate the absolute value, then remove the absolute value by considering both the positive and negative solutions. Notice in the next two examples, all the numbers outside of the absolute value are moved to the other side first before we remove the absolute value bars and consider both positive and negative solutions.

Example 30.

$$\begin{array}{rl} 5 + |x| = 8 & \text{Notice absolute value is not alone} \\ \underline{\mathbf{-5} \qquad \mathbf{-5}} & \text{Subtract 5 from both sides} \\ |x| = 3 & \text{Absolute value can be positive or negative} \\ x = 3 \ \text{ or } \ x = -3 & \text{Our solution} \end{array}$$

Example 31.

$$\begin{array}{rl} -4|x| = -20 & \text{Notice absolute value is not alone} \\ \overline{\mathbf{-4}} \qquad \overline{\mathbf{-4}} & \text{Divide both sides by } -4 \\ |x| = 5 & \text{Absolute value can be positive or negative} \\ x = 5 \ \text{ or } \ x = -5 & \text{Our solution} \end{array}$$

Notice we never combine what is inside the absolute value with what is outside the absolute value. This is very important as it will often change the final result to an incorrect solution. The next example requires two steps to isolate the absolute value. The idea is the same as a two-step equation, add or subtract, then multiply or divide.

Example 32.

$$\begin{array}{rl}
5|x| - 4 = 26 & \text{Notice the absolute value is not alone} \\
\underline{+\mathbf{4} \ \ + \mathbf{4}} & \text{Add 4 to both sides} \\
5|x| = 30 & \text{Absolute value still not alone} \\
\overline{\mathbf{5}} \qquad \overline{\mathbf{5}} & \text{Divide both sides by 5} \\
|x| = 6 & \text{Absolute value can be positive or negative} \\
x = 6 \ \text{ or } \ x = -6 & \text{Our solution}
\end{array}$$

Again we see the same process, get the absolute value alone first, then consider the positive and negative solutions. Often the absolute value will have more than just a variable in it. In this case we will have to solve the resulting equations when we consider the positive and negative possibilities. This is shown in the next example.

Example 33.

$$\begin{array}{rl}
|2x - 1| = 7 & \text{Absolute value can be positive or negative} \\
2x - 1 = 7 \ \text{ or } \ 2x - 1 = -7 & \text{Two equations to solve}
\end{array}$$

Now notice we have two equations to solve, each equation will give us a different solution. Both equations solve like any other two-step equation.

$$\begin{array}{lcl}
2x - 1 = 7 & & 2x - 1 = -7 \\
\quad \underline{+\mathbf{1} \ \ +\mathbf{1}} & & \quad \underline{+\mathbf{1} \ \ \ +\mathbf{1}} \\
2x = 8 & \text{or} & 2x = -6 \\
\overline{\mathbf{2}} \quad \ \overline{\mathbf{2}} & & \overline{\mathbf{2}} \qquad \overline{\mathbf{2}} \\
x = 4 & & x = -3
\end{array}$$

Thus, from our previous example we have two solutions, $x = 4$ or $x = -3$.

Again, it is important to remember that the absolute value must be alone first before we consider the positive and negative possibilities. This is illustrated below.

Example 34.

$$2 - 4|2x + 3| = -18$$

To get the absolute value alone we first need to get rid of the 2 by subtracting, then divide by -4. Notice we cannot combine the 2 and -4 because they are not like terms, the -4 has the absolute value connected to it. Also notice we do not distribute the -4 into the absolute value. This is because the numbers outside cannot be combined with the numbers inside the absolute value. Thus we get the absolute value alone in the following way:

$$\begin{array}{rl}
2 - 4|2x + 3| = -18 & \text{Notice absolute value is not alone} \\
\underline{-\mathbf{2} \qquad\qquad\qquad - \mathbf{2}} & \text{Subtract 2 from both sides} \\
-4|2x + 3| = -20 & \text{Absolute value still not alone} \\
\overline{-\mathbf{4}} \qquad\quad \overline{-\mathbf{4}} & \text{Divide both sides by } -4 \\
|2x + 3| = 5 & \text{Absolute value can be positive or negative} \\
2x + 3 = 5 \ \text{ or } \ 2x + 3 = -5 & \text{Two equations to solve}
\end{array}$$

Now we just solve these two remaining equations to find our solutions.

$$\begin{array}{lcl}
\begin{aligned}2x+3&=5\\ \underline{\mathbf{-3}\quad}&\underline{\mathbf{-\ 3}}\\ 2x&=2\\ \overline{\mathbf{2}}\quad&\ \overline{\mathbf{2}}\\ x&=1\end{aligned} & \text{or} & \begin{aligned}2x+3&=-5\\ \underline{\mathbf{-3}\quad}&\underline{\mathbf{-\ 3}}\\ 2x&=-8\\ \overline{\mathbf{2}}\quad&\ \overline{\mathbf{2}}\\ x&=-4\end{aligned}
\end{array}$$

We now have our two solutions, $x = 1$ and $x = -4$.

As we are solving absolute value equations it is important to be aware of special cases. Remember the result of an absolute value must always be positive. Notice what happens in the next example.

Example 35.

$$\begin{array}{rl}
7+|2x-5|=4 & \text{Notice absolute value is not alone}\\
\underline{\mathbf{-7}\qquad\qquad\quad\mathbf{-\ 7}} & \text{Subtract 7 from both sides}\\
|2x-5|=-3 & \text{Result of absolute value is negative!}
\end{array}$$

Notice the absolute value equals a negative number! This is impossible with an absolute value. When this occurs we say there is "no solution" or $\varnothing$.

One other type of absolute value problem is when two absolute values are equal to each other. We still will consider both the positive and negative result, the difference here will be that we will have to distribute a negative into the second absolute value for the negative possibility.

Example 36.

$$\begin{array}{rl}
|2x-7|=|4x+6| & \text{Absolute value can be}\\
 & \quad\text{positive or negative}\\
2x-7=4x+6 & \text{Make second part of}\\
\text{or}\quad 2x-7=-(4x+6) & \quad\text{second equation negative}
\end{array}$$

Notice the first equation is the positive possibility and has no significant difference other than the missing absolute value bars. The second equation considers the negative possibility. For this reason we have a negative in front of the expression which will be distributed through the equation on the first step of solving. So we solve both these equations as follows:

$$\begin{array}{lcl}
\begin{aligned}2x-7&=4x+6\\ \underline{\boldsymbol{-2x}\quad}&\underline{\boldsymbol{-\ 2x}}\\ -7&=2x+6\\ \underline{\mathbf{-6}\quad}&\underline{\mathbf{-\ 6}}\\ -13&=2x\\ \overline{\mathbf{2}}\quad&\ \overline{\mathbf{2}}\\ -\tfrac{13}{2}&=x\end{aligned} & \text{or} & \begin{aligned}2x-7&=-(4x+6)\\ 2x-7&=-4x-6\\ \underline{\boldsymbol{+4x}\quad}&\underline{\boldsymbol{+\ 4x}}\\ 6x-7&=-6\\ \underline{\mathbf{+7}\quad}&\underline{\mathbf{+\ 7}}\\ 6x&=1\\ \overline{\mathbf{6}}\quad&\ \overline{\mathbf{6}}\\ x&=\tfrac{1}{6}\end{aligned}
\end{array}$$

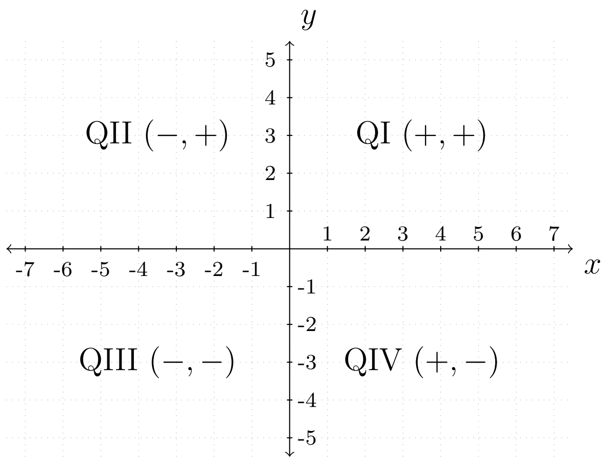
This gives us our two solutions, $x = -\frac{13}{2}$ or $x = \frac{1}{6}$.

Graphing Linear Equations

The Cartesian Plane

Objective: Locate and graph points using xy-coordinates

Often, to get an idea of the behavior of an equation we will make a picture that represents the solutions to the equation. Before we spend much time on making a visual representation of an equation, we first have to understand the basics of graphing. A *graph* is a set of points in the $xy-$plane, also known as the Cartesian plane. In most cases, a graph can simply be thought of as a "picture" of the points. We will see shortly that the graph of a linear equation is a visualization of the solutions to the equation. The following is an example of the Cartesian or $xy-$coordinate plane.



The plane is divided into four *quadrants*, or sections, by a horizontal number line (x-axis) and a vertical number line (y-axis).

Where the two lines, or axes, meet in the center is called the origin. This center origin is where $x = 0$ and $y = 0$.

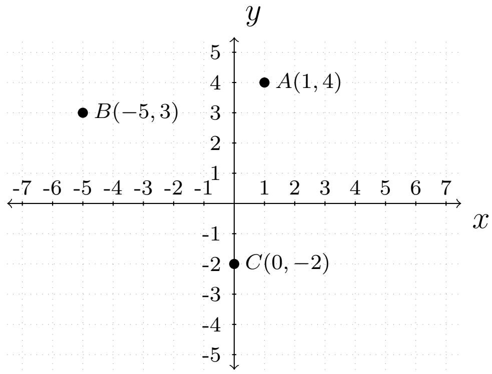
The quadrants are numbered using the roman numerals I, II, III, and IV, beginning with the top-right quadrant (where both x and y are positive) and moving counter-clockwise.

As we move to the right the numbers count up from zero, representing $x = 1, 2, 3 \ldots$. To the left the numbers count down from zero, representing $x = -1, -2, -3, \ldots$. Similarly, as we move up the numbers count up from zero, $y = 1, 2, 3, \ldots$, and as we move down count down from zero, $y = -1, -2, -3 \ldots$.

We can put dots on the graph which we will call points. Each point has an "address" that defines its location. The first number will be the value on the $x-$axis or horizontal number line. This is the distance the point moves left/right from the origin. The second number will represent the value on the $y-$axis or vertical number line. This is the distance the point moves up/down from the origin. The points are given as an ordered pair (x, y).

The following example finds the address or coordinate pair for each of several points on the coordinate plane.

Example 37. Give the coordinates of each point.



Tracing from the origin, point A is right 1, up 4. This becomes $A(1,4)$.

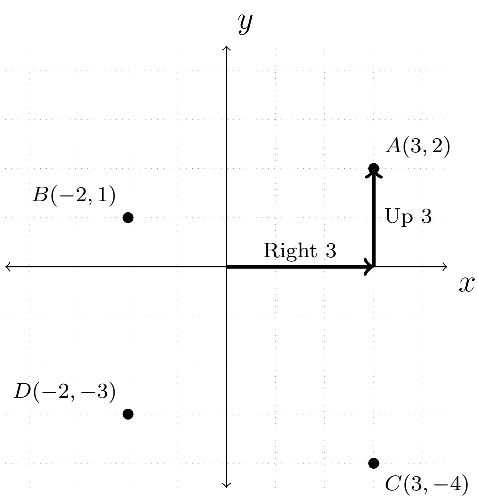
Point B is left 5, up 3. Left is backwards or negative so we have $B(-5,3)$.

Point C is straight down 2 units. There is no left or right. This means we go right zero so the point is $C(0,-2)$.

Our solution is $A(1,4), B(-5,3), C(0,-2)$.

Just as we can give the coordinates for a set of points, we can take a set of points and plot them on the plane.

Example 38. Graph the set of points:
$\{A(3,2), B(-2,1), C(3,-4), D(-2,-3), E(-3,0), F(0,2), G(0,0)\}$

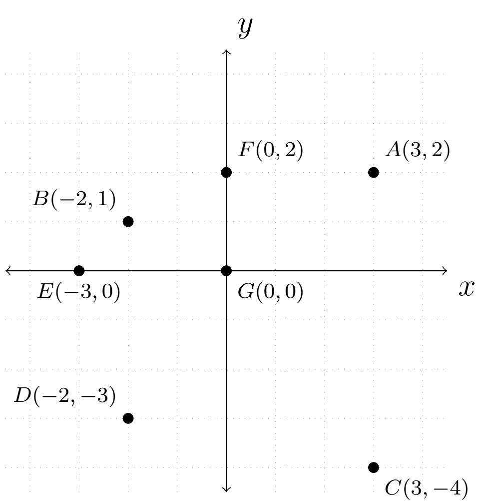


The first point, A is at $(3,2)$ this means $x = 3$ (right 3) and $y = 2$ (up 2). Following these instructions, starting from the origin, we get our point. This is also illustrated on the graph.

The second point, $B(-2,1)$, is left 2 (negative moves backwards), up 1.

The third point, $C(3,-4)$ is right 3, down 4 (negative moves backwards).

The fourth point, $D(-2,-3)$ is left 2, down 3 (both negative, both move backwards).



The last three points have zeros in them. We still treat these points just like the other points. If there is a zero there is just no movement.

First is $E(-3,0)$. This is left 3, and up zero, right on the $x-$axis.

Then is $F(0,2)$. This is right zero, and up two, right on the $y-$axis.

Finally is $G(0,0)$. This point has no movement, and thus is right on the origin.

Graphing Lines (L3)

Objective: Graph lines using xy-coordinates.

The main purpose of graphs is not to plot random points, but rather to give a picture of the solutions to an equation. We may have an equation such as $y = 2x - 3$. We may be interested in what type of solution are possible in this equation. We can visualize the solution by making a graph of possible x and y combinations that make this equation a true statement. We will have to start by finding possible x and y combinations. We will do this using a table of values.

Example 39.

Graph $y = 2x - 3$ We make a table of values

x	y
-1	
0	
1	

We will test three values for x. Any three can be used

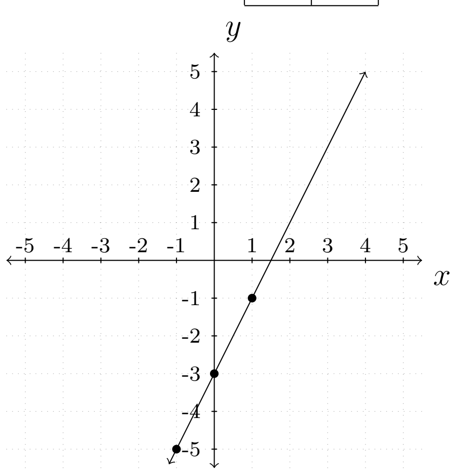
x	y
-1	-5
0	-3
1	-1

Evaluate each by replacing x with the given value

$x = -1$ $\quad y = 2(-1) - 3 = -2 - 3 = -5$

$x = 0$ $\quad y = 2(0) - 3 = 0 - 3 = -3$

$x = 1$ $\quad y = 2(1) - 3 = 2 - 3 = -1$



$(-1, -5), (0, -3)$, and $(1, -1)$

These become the points from our equation which we will plot on our graph.

Once the point are on the graph, connect the dots to make a line.

The graph is our solution.

What this line tells us is that any point on the line will work in the equation $y = 2x - 3$. For example, notice the graph also goes through the point $(2, 1)$. If we use $x = 2$, we should get $y = 1$. Sure enough, $y = 2(2) - 3 = 4 - 3 = 1$, just as the graph suggests. Thus we have the line is a picture of all the solutions for $y = 2x - 3$. We can use this table of values method to draw a graph of any linear equation.

Example 40.

Graph $2x - 3y = 6$	We will use a table of values

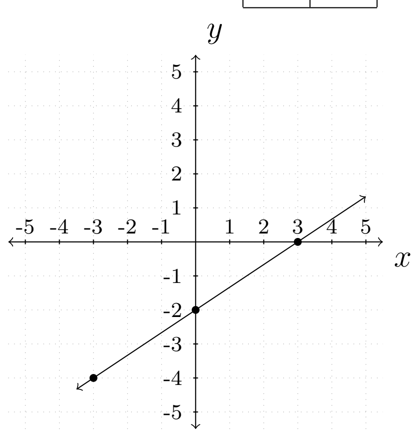
x	y
-3	
0	
3	

We will test three values for x. Any three can be used .

$2(-3) - 3y = 6$	Substitute each value in for x and solve for y
$-6 - 3y = 6$	Start with $x = -3$, multiply first
$\underline{\mathbf{+6} \quad\quad \mathbf{+\,6}}$	Add 6 to both sides
$-3y = 12$	Divide both sides by -3
$\overline{\mathbf{-3}} \quad \overline{\mathbf{-3}}$	
$y = -4$	solution for y when $x = -3$, add this to table
$2(0) - 3y = 6$	Next $x = 0$
$-3y = 6$	Multiplying clears the constant term
$\overline{\mathbf{-3}} \quad \overline{\mathbf{-3}}$	Divide each side by -3
$y = -2$	solution for y when $x = 0$, add this to table
$2(3) - 3y = 6$	Next $x = 3$
$6 - 3y = 6$	Multiply
$\underline{\mathbf{-6} \quad\quad \mathbf{-\,6}}$	Subtract 9 from both sides
$-3y = 0$	Divide each side by -3
$\overline{\mathbf{-3}} \quad \overline{\mathbf{-3}}$	
$y = 0$	solution for y when $x = -3$, add this to table

x	y
-3	-4
0	-2
3	0

Our completed table



The coordinate points from our table are then $(-3, -4), (0, -2)$, and $(3, 0)$

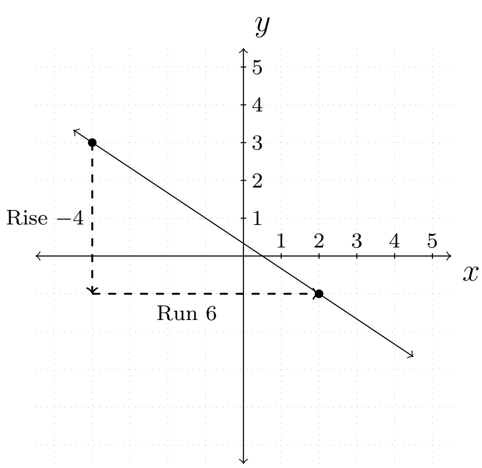
After we plot these points, we connect them to form our graph.

The Slope of a Line

Objective: Find the slope of a line given a graph or two points.

As we graph lines, we will want to be able to identify different properties of the lines we graph. One of the most important properties of a line is its slope. *Slope* is a measure of steepness. A line with a large slope, such as 25, is very steep or increases quickly. A line with a small slope, such as $\frac{1}{10}$ is very flat or increases gradually. We will also use slope to describe the direction of the line. A line that goes up from left to right will have a positive slope and a line that goes down from left to right will have a negative slope.

As we measure steepness we are interested in how fast the line rises compared to how far the line runs. For this reason we will describe slope as the fraction $\frac{\text{rise}}{\text{run}}$. Rise would be a vertical change, or a change in the y-values. Run would be a horizontal change, or a change in the x-values. So another way to describe slope would be the fraction $\frac{\text{change in } y}{\text{change in } x}$. It turns out that if we have a graph we can draw vertical and horizontal lines from one point to another to make what is called a slope triangle. The sides of the slope triangle give us our slope. Using this idea, we find the corresponding slopes for each of the lines that follow.

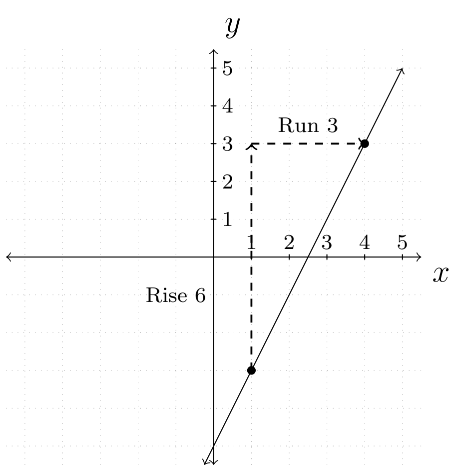


To find the slope of this line we will consider the rise, or vertical change and the run or horizontal change.

Drawing these lines in creates a triangle that we can use to count from one point to the next:

the graph goes down 4, right 6. This is a rise of -4 and a run 6.

As a fraction, we have, $\frac{-4}{6}$, or $-\frac{2}{3}$ when reduced, which is our slope.



To find the slope of this line, the rise is up 6, the run is right 3.

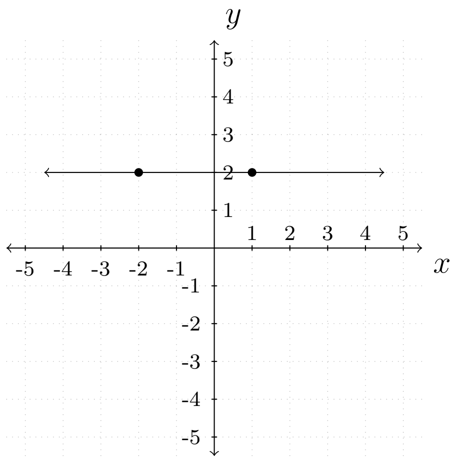
Our slope is then written as a fraction:

$\frac{\text{rise}}{\text{run}} = \frac{6}{3}$.

This fraction reduces to 2.

A slope of 2 is our solution.

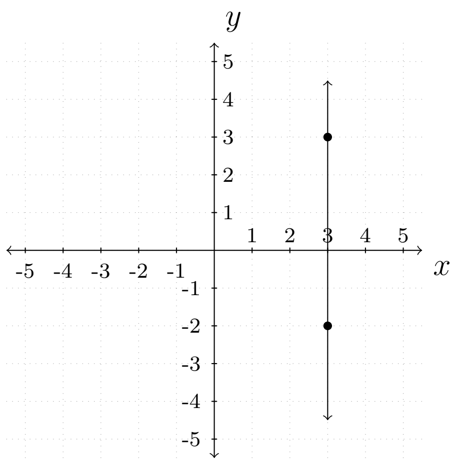
There are two special lines that have unique slopes that we need to be aware of. They are illustrated in the following examples.



In this graph there is no rise, but the run is 3 units.

This slope becomes $\frac{0}{3} = 0$.

This, and all *horizontal* lines have a slope of zero.



This line has a rise of 5, but no run.

The slope becomes $\frac{5}{0}$ = undefined, or $\varnothing$.

This, and all *vertical* lines have an undefined slope.

As you can see there is a big difference between having a zero slope and having no slope or undefined slope. Remember, slope is a measure of steepness. The first slope is not steep at all, in fact it is flat. Therefore it has a zero slope. The second slope can't get any steeper. It is so steep that there is no number large enough to express how steep it is. This is an undefined slope.

We can find the slope of a line through two points without seeing the points on a graph. We can do this using a slope formula. If the rise is the change in y values, we can calculate this by subtracting the y values of a point. Similarly, if run is a change in the x values, we can calculate this by subtracting the x values of a point. In this way we get the following equation for slope.

$$\text{The slope of a line through } (x_1, y_1) \text{ and } (x_2, y_2) \text{ is } \frac{y_2 - y_1}{x_2 - x_1}.$$

When mathematicians began working with slope, it was called the modular slope. For this reason we often represent the slope with the variable m. Now we have the following for slope.

$$\text{Slope} = m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

As we subtract the y values and the x values when calculating slope it is important we subtract them in the same order. This process is shown in the following examples.

Example 41.

$$\begin{array}{rl}
\text{Find the slope between} (-4,3) \text{ and } (2,-9) & \text{Identify } x_1, y_1, x_2, y_2 \\
(x_1, y_1) \text{ and } (x_2, y_2) & \text{Use slope formula, } m = \dfrac{y_2 - y_1}{x_2 - x_1} \\
m = \dfrac{-9-3}{2-(-4)} & \text{Simplify} \\
m = \dfrac{-12}{6} & \text{Reduce} \\
m = -2 & \text{Our solution}
\end{array}$$

Example 42.

$$\begin{array}{rl}
\text{Find the slope between} (4,6) \text{ and } (2,-1) & \text{Identify } x_1, y_1, x_2, y_2 \\
(x_1, y_1) \text{ and } (x_2, y_2) & \text{Use slope formula, } m = \dfrac{y_2 - y_1}{x_2 - x_1} \\
m = \dfrac{-1-6}{2-4} & \text{Simplify} \\
m = \dfrac{-7}{-2} & \text{Reduce, dividing by } -1 \\
m = \dfrac{7}{2} & \text{Our solution}
\end{array}$$

We may come up against a problem that has a zero slope (horizontal line) or no slope (vertical line) just as with using the graphs.

Example 43.

$$\begin{array}{rl}
\text{Find the slope between } (-4,-1) \text{ and } (-4,-5) & \text{Identify } x_1, y_1, x_2, y_2 \\
(x_1, y_1) \text{ and } (x_2, y_2) & \text{Use slope formula, } m = \dfrac{y_2 - y_1}{x_2 - x_1} \\
m = \dfrac{-5-(-1)}{-4-(-4)} & \text{Simplify} \\
m = \dfrac{-4}{0} & \text{Can't divide by zero} \\
\text{Slope } m \text{ is undefined} & \text{Our solution}
\end{array}$$

Example 44.

$$
\begin{array}{rl}
\text{Find the slope between } (3,1) \text{ and } (-2,1) & \text{Identify } x_1, y_1, x_2, y_2 \\
(x_1, y_1) \text{ and } (x_2, y_2) & \text{Use slope formula, } m = \dfrac{y_2 - y_1}{x_2 - x_1} \\
m = \dfrac{1-1}{-2-3} & \text{Simplify} \\
m = \dfrac{0}{-5} & \text{Reduce} \\
m = 0 & \text{Our solution}
\end{array}
$$

Again, there is a big difference between no slope and a zero slope. Zero is an integer and it has a value, the slope of a flat horizontal line. No slope has no value, it is undefined, the slope of a vertical line.

Using the slope formula we can also find missing points if we know what the slope is. This is shown in the following two examples.

Example 45. Find the value of y between the points (2, y) and (5, - 1) with slope -3.

$$
\begin{array}{rl}
m = \dfrac{y_2 - y_1}{x_2 - x_1} & \text{We will plug values into the slope formula} \\
-3 = \dfrac{-1-y}{5-2} & \text{Simplify} \\
-3 = \dfrac{-1-y}{3} & \text{Multiply both sides by 3} \\
-3(\mathbf{3}) = \dfrac{-1-y}{3}(\mathbf{3}) & \text{Simplify} \\
-9 = -1 - y & \text{Add 1 to both sides} \\
\underline{+\mathbf{1} \quad +\mathbf{1}} & \\
-8 = -y & \text{Divide both sides by } -1 \\
\overline{-\mathbf{1}} \quad \overline{-\mathbf{1}} & \\
8 = y & \text{Our solution}
\end{array}
$$

Example 46. Find the value of x between the points (- 3, 2) and (x, 6) with slope $\frac{2}{5}$.

$$
\begin{array}{rl}
m = \dfrac{y_2 - y_1}{x_2 - x_1} & \text{We will plug values into slope formula} \\
\dfrac{2}{5} = \dfrac{6-2}{x-(-3)} & \text{Simplify} \\
\dfrac{2}{5} = \dfrac{4}{x+3} & \text{Multiply both sides by } (x+3) \\
\dfrac{2}{5}(x+3) = 4 & \text{Multiply by 5 to clear fraction}
\end{array}
$$

$$
\begin{aligned}
(\mathbf{5})\frac{2}{5}(x+3) &= 4(\mathbf{5}) && \text{Simplify}\\
2(x+3) &= 20 && \text{Distribute}\\
2x+6 &= 20 &&\\
\underline{\mathbf{-6} \quad} & \underline{\;\; \mathbf{-\,6}} && \text{Subtract 6 from both sides}\\
2x &= 14 && \text{Divide each side by 2}\\
\overline{\mathbf{2}} \quad & \quad \overline{\mathbf{2}} &&\\
x &= 7 && \text{Our solution}
\end{aligned}
$$

The Two Forms of a Linear Equation (L4)

Slope-Intercept Form

Objective: Find the equation of a line with a known slope and y-intercept.

When graphing a line we found one method we could use is to make a table of values. However, if we can identify some properties of the line, we may be able to make a graph much quicker and easier. One such method is finding the slope and the y-intercept of the equation. The slope can be represented by m and the y-intercept, where it crosses the axis and $x = 0$, can be represented by $(0, b)$ where b is the value where the graph crosses the vertical y-axis. Any other point on the line can be represented by (x, y). Using this information we will look at the slope formula and solve the formula for y.

Example 47.

$$
\begin{aligned}
m,\ (0,b),\ (x,y) & && \text{Use the slope formula}\\
\frac{y-b}{x-0} &= m && \text{Simplify}\\
\frac{y-b}{x} &= m && \text{Multiply both sides by } x\\
y-b &= mx && \text{Add } b \text{ to both sides}\\
\underline{\boldsymbol{+b} \quad} & \underline{\;\; \boldsymbol{+\,b}} &&\\
y &= mx+b && \text{Our solution}
\end{aligned}
$$

This equation, $y = mx + b$ can be thought of as the equation of any line that has a slope of m and a y-intercept of b. This formula is known as the slope-intercept formula or equation.

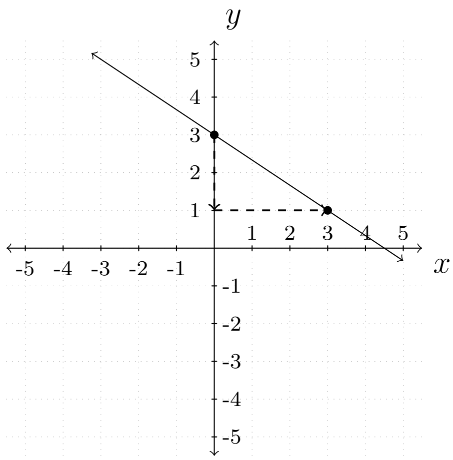
$$\boxed{\text{Slope-intercept equation: } y = mx + b}$$

If we know the slope and the y-intercept we can easily find the equation that represents the line.

Example 48.

$$
\begin{array}{rl}
\text{Slope} = \dfrac{3}{4},\quad y-\text{intercept} = -3 & \text{Use the slope} - \text{intercept equation} \\
y = mx + b & m \text{ is the slope,}\quad b \text{ is the } y - \text{intercept} \\
y = \dfrac{3}{4}x - 3 & \text{Our solution}
\end{array}
$$

We can also find the equation by looking at a graph and finding the slope and y-intercept.



Identify the point where the graph crosses the y-axis (0,3).

This means the y-intercept is 3.

Identify one other point and draw a slope triangle to find the slope.

The slope is $m = -\frac{2}{3}$.

Slope-intercept form: $y = mx + b$

Our Equation: $y = -\frac{2}{3}x + 3$

We can also move the opposite direction, using the equation identify the slope and y-intercept and graph the equation from this information. However, it will be important for the equation to first be in slope intercept form. If it is not, we will have to solve it for y so we can identify the slope and the y-intercept.

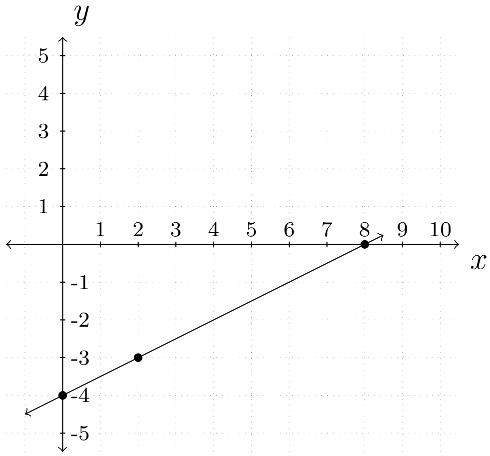
Example 49. Write the equation $2x - 4y = 6$ in slope-intercept form.

$$
\begin{array}{rl}
2x - 4y = 6 & \text{Solve for } y \\
\underline{\mathbf{-2x \qquad - 2x}} & \text{Subtract } 2x \text{ from both sides} \\
-4y = -2x + 6 & \text{Put } x \text{ term first} \\
\overline{\mathbf{-4}} \quad \overline{\mathbf{-4}} \quad \overline{\mathbf{-4}} & \text{Divide each term by } -4 \\
y = \dfrac{1}{2}x - \dfrac{3}{2} & \text{Our solution}
\end{array}
$$

Once we have an equation in slope-intercept form we can graph it by first plotting the y-intercept, then using the slope, finding a second point and connecting the dots.

Example 50. Graph $y = \dfrac{1}{2}x - 4$.

$$
\begin{array}{rl}
y = mx + b & \text{Slope} - \text{intercept equation} \\
m = \dfrac{1}{2},\ b = -4 & \text{Identify the slope, } m, \text{ and the } y - \text{intercept, } b
\end{array}
$$



Start with a point at the y-intercept of -4, $(0, -4)$.

Then use the slope $\frac{\text{rise}}{\text{run}}$ to find the next point, $(2, -3)$.

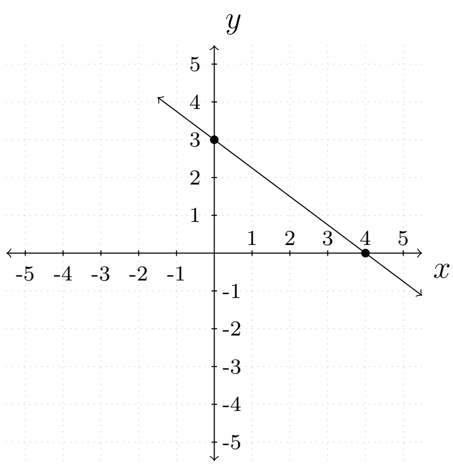
Once we have both points, connect the dots to get our graph.

Here, we have also identified the $x-$intercept $(8, 0)$, by setting $y = 0$ and solving for x:

$\frac{1}{2}x - 4 = 0$ when $x = 8$.

Example 51. Graph $3x + 4y = 12$.

$$\begin{array}{rl}
3x + 4y = 12 & \text{Not in slope-intercept form} \\
\underline{\boldsymbol{-3x} \qquad \boldsymbol{-3x}} & \text{Subtract } 3x \text{ from both sides} \\
4y = -3x + 12 & \text{Put the } x \text{ term first} \\
\overline{\mathbf{4}} \quad \overline{\mathbf{4}} \quad \overline{\mathbf{4}} & \text{Divide each term by } 4 \\
y = -\dfrac{3}{4}x + 3 & \text{Now in slope} - \text{intercept form} \\
m = -\dfrac{3}{4},\ b = 3 & \text{Identify } m \text{ and } b
\end{array}$$



Start with a point at the $y-$intercept, $(0, 3)$.

Then use the slope $\frac{\text{rise}}{\text{run}}$. Since the slope is negative, the graph will decrease from left to right. So we will drop 3 units and run 4 units to the right to find the next point.

Notice that our next point is also the $x-$intercept, $(4, 0)$.

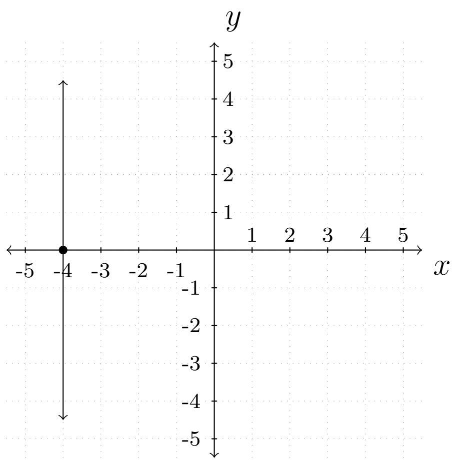
Once we have both points, connect the dots to get our graph.

We want to be very careful not to confuse using slope to find the next point with use a coordinate such as $(4, -2)$ to find an individual point. Coordinates such as $(4, -2)$ start from the origin and move horizontally first, and vertically second. Slope starts from a point on the line that could be anywhere on the graph. The numerator is the vertical change and the denominator is the horizontal change.

Lines with zero slope or no slope can make a problem seem very different. Such lines are either horizontal ($m = 0$) or vertical (m is undefined).

A horizontal line will have a slope of zero which when multiplied by x gives zero. So the equation simply becomes $y = 0x + b$ or just $y = b$. Remember that in this case, b also refers to where the line crosses the $y-$axis.

If we have no slope ($m = \varnothing$), our line is vertical, and the corresponding equation cannot be written in slope-intercept form. In this case, there is no y in our equation, and we simply write $x = a$, where a refers to the $x-$coordinate for the point where the line crosses the $x-$axis.



In this graph, because we have a vertical line (m is undefined), we do not use the slope-intercept form of a linear equation.

Rather, we set x equal to the $x-$coordinate of the $x-$intercept.

Our corresponding equation is $x = -4$.

Point-Slope Form

Objective: Find the equation of a line with a known slope and passing through a given point.

The slope-intercept form has the advantage of being simple to remember and use, however, it has one major disadvantage: we must know the y-intercept in order to use it! Generally we do not know the y-intercept, we only know one or more points (that are not the y-intercept). In these cases we can't use the slope intercept equation, so we will use a different, more flexible formula. If we let the slope of an equation be m, and a specific point on the line be (x_1, y_1), and any other point on the line be (x, y). We can use the slope formula to make a second equation.

Example 52.

$$\begin{aligned} m,\ (x_1, y_1),\ (x, y) & \qquad \text{Recall slope formula} \\ \frac{y_2 - y_1}{x_2 - x_1} = m & \qquad \text{Plug in values} \\ \frac{y - y_1}{x - x_1} = m & \qquad \text{Multiply both sides by} (x - x_1) \\ y - y_1 = m(x - x_1) & \qquad \text{Our equation} \end{aligned}$$

If we know the slope, m of an equation and any point on the line (x_1, y_1) we can easily plug these values into the equation above which will be called the point-slope formula or equation.

Point-slope equation: $y - y_1 = m(x - x_1)$

Example 53. Write the equation of the line through the point $(3, -4)$ with a slope of $\frac{3}{5}$.

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Plug values into point} - \text{slope formula} \\ y - (-4) &= \frac{3}{5}(x - 3) && \text{Simplify signs} \\ y + 4 &= \frac{3}{5}(x - 3) && \text{Our solution} \end{aligned}$$

Often, we will prefer final answers be written in slope-intercept form. If the directions ask for the answer in slope-intercept form we will simply distribute the slope, then solve for y.

Example 54.

Write the equation of the line through the point $(-6, 2)$ with a slope of $-\frac{2}{3}$ in slope-intercept form.

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Plug values into point} - \text{slope formula} \\ y - 2 &= -\frac{2}{3}(x - (-6)) && \text{Simplify signs} \\ y - 2 &= -\frac{2}{3}(x + 6) && \text{Distribute slope} \\ y - 2 &= -\frac{2}{3}x - 4 && \text{Solve for } y \text{ by adding 2 to both sides} \\ \underline{\mathbf{+2}} & \underline{\quad\quad\quad \mathbf{+2}} && \\ y &= -\frac{2}{3}x - 2 && \text{Our solution} \end{aligned}$$

An important thing to observe about the point slope formula is that the operation between the x's and y's is subtraction. This means when you simplify the signs you will have the opposite of the numbers in the point. We need to be very careful with signs as we use the point-slope formula.

In order to find the equation of a line we will always need to know the slope. If we don't know the slope to begin with we will have to do some work to find it first before we can get an equation.

Example 55. Find the equation of the line through the points $(-2, 5)$ and $(4, -3)$.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{First we must find the slope} \\ m = \frac{-3 - 5}{4 - (-2)} &= \frac{-8}{6} = -\frac{4}{3} && \text{Plug values in slope formula and evaluate} \\ y - y_1 &= m(x - x_1) && \text{Use point} - \text{slope formula,} \\ & && \quad \text{plugging in slope and either point} \end{aligned}$$

$$y-5=-\frac{4}{3}(x-(-2)) \qquad \text{Simplify signs}$$
$$y-5=-\frac{4}{3}(x+2) \qquad \text{Our solution}$$

Example 56.

Find the equation of the line through the points $(-3,4)$ and $(-1,-2)$ in slope-intercept form.

$$\begin{aligned}
m &= \frac{y_2-y_1}{x_2-x_1} && \text{First we must find the slope}\\
m = \frac{-2-4}{-1-(-3)} &= \frac{-6}{2}=-3 && \text{Plug values in slope formula and evaluate}\\
y-y_1 &= m(x-x_1) && \text{Use point} - \text{slope formula,}\\
&&& \quad \text{plugging in slope and either point}\\
y-4 &= -3(x-(-3)) && \text{Simplify signs}\\
y-4 &= -3(x+3) && \text{Distribute slope}\\
y-4 &= -3x-9 && \text{Solve for } y\\
\underline{\mathbf{+4}} & \underline{\qquad\quad \mathbf{+\,4}} && \text{Add 4 to both sides}\\
y &= -3x-5 && \text{Our solution}
\end{aligned}$$

Example 57.

Find the equation of the line through the points $(6,-2)$ and $(-4,1)$ in slope-intercept form.

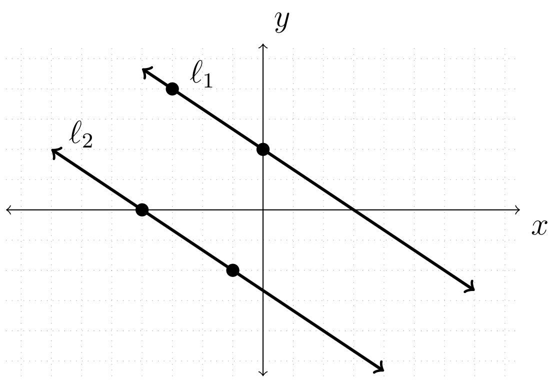
$$\begin{aligned}
m &= \frac{y_2-y_1}{x_2-x_1} && \text{First we must find the slope}\\
m = \frac{1-(-2)}{-4-6} &= \frac{3}{-10}=-\frac{3}{10} && \text{Plug values into slope formula and evaluate}
\end{aligned}$$

$$\begin{aligned}
y-y_1 &= m(x-x_1) && \text{Use point} - \text{slope formula,}\\
&&& \quad \text{plugging in slope and either point}\\
y-(-2) &= -\frac{3}{10}(x-6) && \text{Simplify signs}\\
y+2 &= -\frac{3}{10}(x-6) && \text{Distribute slope}\\
y+2 &= -\frac{3}{10}x+\frac{9}{5} && \text{Solve for } y\text{, by subtracting 2 from both sides}\\
\underline{\mathbf{-2}} & \underline{\qquad\qquad \mathbf{-\,\frac{10}{5}}} && \text{Use } \frac{10}{5} \text{ on right so we have a common denominator}\\
y &= -\frac{3}{10}x-\frac{1}{5} && \text{Our solution}
\end{aligned}$$

Parallel and Perpendicular Lines (L5)

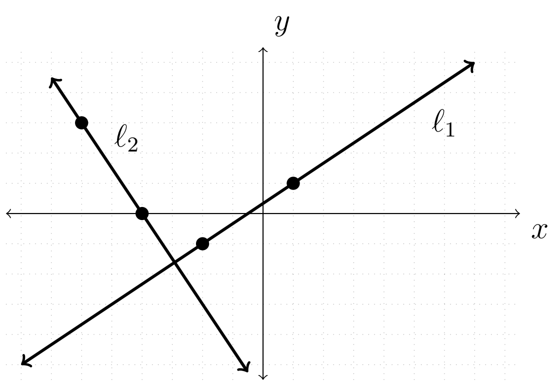
Objective: Identify the equation of a line that is either parallel or perpendicular to a given line.

There is an interesting connection between the slopes of lines that are parallel, as well as the slopes of lines that are perpendicular (meet at a right angle). This is shown in the following example.



This graph shows two parallel lines.

The slope (rise over run) of each line is "down 2, right 3," or $m_1 = m_2 = -\frac{2}{3}$.



This graph shows two perpendicular lines.

The slope (rise over run) of the more gradual line is "up 2, right 3," or $m_1 = \frac{2}{3}$.

The slope of the steeper line is "down 3, right 2," or $m_2 = -\frac{3}{2}$.

As the first graph above illustrates, parallel lines have the same slope.

On the other hand, perpendicular lines are said to have slopes that are *negative reciprocals* of one another. More precisely, if two lines with slopes m_1 and m_2 are known to be perpendicular, then $m_2 = -\dfrac{1}{m_1}$ (and so, $m_1 m_2 = -1$).

We can use these properties to make conclusions about parallel and perpendicular lines.

Example 58. Find the slope of a line parallel to $5y - 2x = 7$.

$5y - 2x = 7$	To find the slope we will put equation in slope $-$ intercept form
$\underline{\quad \mathbf{+2x} \quad \mathbf{+\,2x}}$	Add $2x$ to both sides
$5y = 2x + 7$	Put x term first
$\overline{\mathbf{5}} \quad \overline{\mathbf{5}} \quad \overline{\mathbf{5}}$	Divide each term by 5

$$
\begin{array}{rl}
y = \frac{2}{5}x + \frac{7}{5} & \text{The slope is the coefficient of } x \\
m = \frac{2}{5} & \text{Slope of given line} \\
 & \text{Parallel lines have the same slope} \\
m = \frac{2}{5} & \text{Our solution}
\end{array}
$$

Example 59. Find the slope of a line perpendicular to $3x - 4y = 2$.

$$
\begin{array}{rl}
3x - 4y = 2 & \text{To find slope we will put equation in slope} - \text{intercept form} \\
\underline{\boldsymbol{-3x} \qquad \boldsymbol{-3x}} & \text{Subtract } 3x \text{ from both sides} \\
-4y = -3x + 2 & \text{Put } x \text{ term first} \\
\overline{\boldsymbol{-4}} \quad \overline{\boldsymbol{-4}} \quad \overline{\boldsymbol{-4}} & \text{Divide each term by } -4 \\
y = \frac{3}{4}x - \frac{1}{2} & \text{The slope is the coefficient of } x \\
m = \frac{3}{4} & \text{Slope of given line} \\
 & \text{Perpendicular lines have negative reciprocal slopes} \\
m = -\frac{4}{3} & \text{Our solution}
\end{array}
$$

Once we have a slope, it is possible to find the complete equation of the desired line, if we know one point on it.

Example 60. Find the equation of a line through $(4, -5)$ and parallel to $2x - 3y = 6$.

$$
\begin{array}{rl}
2x - 3y = 6 & \text{We first need slope of parallel line} \\
\underline{\boldsymbol{-2x} \qquad \boldsymbol{-2x}} & \text{Subtract } 2x \text{ from each side} \\
-3y = -2x + 6 & \text{Put } x \text{ term first} \\
\overline{\boldsymbol{-3}} \quad \overline{\boldsymbol{-3}} \quad \overline{\boldsymbol{-3}} & \text{Divide each term by } -3 \\
y = \frac{2}{3}x - 2 & \text{Identify the slope, the coefficient of } x \\
m = \frac{2}{3} & \text{Parallel lines have the same slope} \\
m = \frac{2}{3} & \text{We will use this slope and our point} (4, -5) \\
y - y_1 = m(x - x_1) & \text{Plug this information into point} - \text{slope formula} \\
y - (-5) = \frac{2}{3}(x - 4) & \text{Simplify signs} \\
y + 5 = \frac{2}{3}(x - 4) & \text{Our solution}
\end{array}
$$

Example 61. Find the equation of the line through $(6,-9)$ perpendicular to $y=-\frac{3}{5}x+4$ in slope-intercept form.

$$\begin{aligned}
y &= -\frac{3}{5}x+4 && \text{Identify the slope, coefficient of } x\\
m &= -\frac{3}{5} && \text{Perpendicular lines have negative reciprocal slopes}\\
m &= \frac{5}{3} && \text{We will use this slope and our point}(6,-9)\\
y-y_1 &= m(x-x_1) && \text{Plug this information into point} - \text{slope formula}\\
y-(-9) &= \frac{5}{3}(x-6) && \text{Simplify signs}\\
y+9 &= \frac{5}{3}(x-6) && \text{Distribute slope}
\end{aligned}$$

$$\begin{array}{rl}
y+9=\frac{5}{3}x-10 & \text{Solve for } y\\
\underline{\mathbf{-9}\qquad\quad \mathbf{-\ 9}} & \text{Subtract 9 from both sides}\\
y=\frac{5}{3}x-19 & \text{Our solution}
\end{array}$$

Zero slopes and undefined slopes may seem like opposites (one is a horizontal line, one is a vertical line). Because a horizontal line is perpendicular to a vertical line we can say that an undefined slope and a zero slope are actually perpendicular slopes!

Example 62. Find the equation of the line through (3, 4) perpendicular to $x=-2$.

$$\begin{array}{rl}
x=-2 & \text{This equation has an undefined slope, a vertical line}\\
\text{Undefined slope} & \text{Perpendicular line then would have a zero slope}\\
m=0 & \text{Use this and our point}(3,4)\\
y-y_1=m(x-x_1) & \text{Plug this information into point} - \text{slope formula}\\
y-4=0(x-3) & \text{Distribute slope}\\
y-4=0 & \text{Solve for } y\\
\underline{\mathbf{+4}\ \ \mathbf{+\ 4}} & \text{Add 4 to each side}\\
y=4 & \text{Our solution}
\end{array}$$

Being aware that to be perpendicular to a vertical line means we have a horizontal line through a y value of 4, thus we could have jumped from this point right to the solution, $y=4$.

Linear Inequalities and Sign Diagrams (L6)

Objective: Solve, graph, and give interval notation for the solution to a linear inequality. Create a sign diagram to identify those intervals where a linear expression is positive or negative.

Linear Inequalities

When given a linear equation such as $x+2=5$, one can solve to obtain *one* solution ($x=3$). Although the method for solving an inequality is, in general, very similar to that for solving an equation, we will see that the solution to a inequality will usually include an entire range of values.

In order to solve any inequality, we must first understand the accompanying notation and respective terminology.

Symbol	Meaning
$<$	less than
$>$	greater than
$\leq$	less than or equal to
$\geq$	greater than or equal to
$\neq$	not equal to

For a more in-depth treatment of set notation (graphical, interval, or inequality notation) including unions and intersections, a review of the following open resource is strongly recommended: *Logic and Set Notation*

Notice that the "equals" symbol $=$ is not listed in the table above, as we will be working with *inequalities*, rather than equations. It is also worth mentioning that there are several alternate ways of describing the same symbol. For example, the phrases "at most" or "no more than" can easily be interchanged with "less than or equal to", and similarly for "at least", "no less than", and "greater than or equal to". Because of this, one needs to use a bit of caution, when faced with any problem that is presented verbally.

$$2 < 5, \qquad 1 > -1, \qquad 5 \leq 10, \qquad 3 \leq 3,$$
$$7 \geq -2, \qquad 4 \geq 4, \qquad -1 \neq 1$$

The examples above, though true, do not contain a variable. We now will work with inequalities containing one (or more) variable(s). Following the previous sections of this chapter, we will first concern ourselves with linear inequalities, followed by compound inequalities and inequalities that contain an absolute value. The solution to an inequality is the set of all real numbers that make the inequality true.

Example 63. Solve the linear inequality $x + 2 < 5$.

$$\begin{array}{rll} x + 2 &< 5 & \\ \underline{\mathbf{-2}} & \underline{\mathbf{-2}} & \text{Subtract 2 from both sides} \\ x &< 3 & \text{Our solution} \end{array}$$

Notice that we solve the previous inequality using the same method that one would use to solve the equation $x + 2 = 5$. Some differences will be seen later.

When describing the solution to a given inequality, it will often be useful to graph the solution on a number line and shade the section(s) of the number line that coincide with the solution set. The number line below illustrates our previous example.

Note that an open (unshaded) circle is often used in place of the parenthesis above. In each case, this notation denotes an *exclusion* of the value $x = 3$, since it is *not* a valid solution to the given inequality. Alternatively, a closed (shaded) circle or bracket would be used to denote *inclusion* of the boundary value, in the event that it *is* a valid solution.

It is also a good idea to test a few values in order to check our work.

Check:

Test Location	Test Value	Unsimplified	Simplified	Result
Shaded region	$x = 1$	$1 + 2 < 5$	$3 < 5$	True
Boundary value	$x = 3$	$3 + 2 < 5$	$5 < 5$	False
Unshaded region	$x = 5$	$5 + 2 < 5$	$7 < 5$	False

A common misconception that many students have with an inequality such as $x < 3$ and is worth mentioning has to do with the values between $x = 2$ and $x = 3$. Although we have seen that x cannot equal 3 in the given inequality, this does not mean that the solution set has a largest value at $x = 2$ (the largest *integer* solution to the inequality). In fact, there are infinitely many *real-number* solutions to the inequality between the integers 2 and 3. For example, 2.5, 2.7, 2.9, 2.99, 2.999, and 2.9999999999999999 are all valid solutions to $x < 3$. Because of this, one could say that the inequality is *bounded above by* $x = 3$, but there is no *largest* solution that satisfies it.

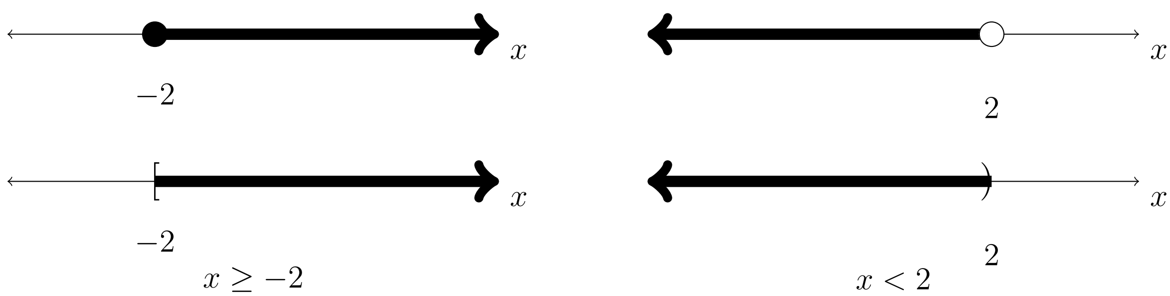
There are four primary ways of presenting the solution to an inequality:

1. In words (verbally): "All real numbers x greater than or equal to 4."
2. Using inequality (and set-builder) notation: $\{x|x \geq 4\}$.

3. Using interval notation: $[4, \infty)$.
4. Using real-number line notation (graphically):

In many of our examples, it will be acceptable to exclude the set-builder notation $\{x| \quad \}$ altogether, and instead simply present the inequality $x \geq 4$. Still, it is important that students recognize the meaning behind the notation ("The set of all real numbers x such that...").

Recall that for interval notation we use brackets [or] to denote *inclusion* of a boundary value, and parentheses (and) to denote *exclusion*. This notation can therefore be interchanged with a closed circle (inclusion) or an open circle (exclusion), when graphing a given solution set on the real-number line. As a convention, from this point forward we will adopt brackets and parentheses instead of closed and open circles for graphical representations of solution sets, since it presents a nice connection between interval and real-number line notation. Both notations, however, are generally accepted. An example is shown below.



Next, we will solve and present the solution to a linear equality using all four presentation methods.

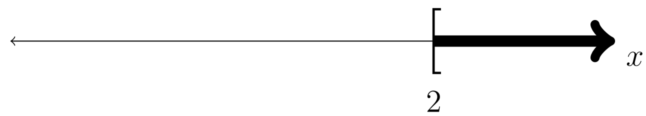
Example 64. Solve the linear inequality $4x - 3 \geq 5$.

$$\begin{array}{rll}
4x - 3 \geq 5 & & \\
\underline{+\mathbf{3}} \quad \underline{+\mathbf{3}} & & \text{Add 3 to both sides} \\
4x \geq 8 & & \\
\overline{\mathbf{4}} \quad \overline{\mathbf{4}} & & \text{Divide both sides by 4} \\
x \geq 2 & & \text{Our solution}
\end{array}$$

Our solution can be expressed as follows.

1. Verbally: "The set of all values of x that are greater than or equal to (at least) 2".

2. Inequality: $\{x|x \geq 2\}$
3. Interval: $[2, \infty)$
4. Real-number Line (Graphically):



Note: A closed (shaded) circle at $x = 2$ is also acceptable in place of a bracket.

Check:

Test Location	Test Value	Unsimplified	Simplified	Result
Shaded region	$x = 3$	$4(3) - 3 \geq 5$	$9 \geq 5$	True
Boundary value	$x = 2$	$4(2) - 3 \geq 5$	$5 \geq 5$	True
Unshaded region	$x = 0$	$4(0) - 3 \geq 5$	$-3 \geq 5$	False

Next, we would like to closely examine the impact that each of the four main operations ($+$, $-$, $\times$, $\div$) has on a given inequality. This will shed more light on one of the fundamental differences between solving an equation and solving an inequality. To demonstrate this, we will repeatedly use an obvious true statement, $4 < 10$.

Original Inequality: $4 < 10$

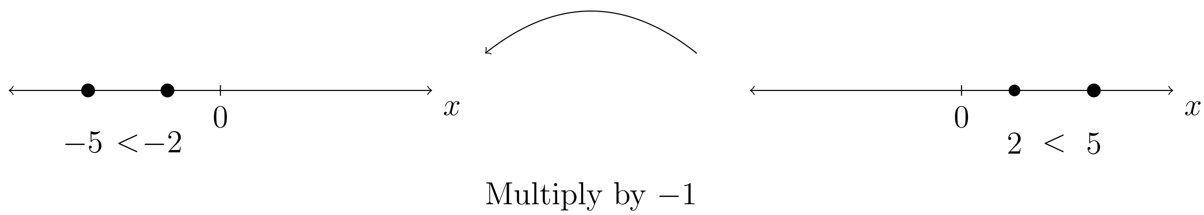
Action	Resulting Inequality	Outcome
Add 5	$9 < 15$	True
Subtract 5	$-1 < 5$	True
Add -3	$1 < 7$	True
Subtract -3	$7 < 13$	True

Note that since addition and subtraction are closely related, we see that the original inequality is also preserved when negative values are either added or subtracted. In other words, adding (or subtracting) -3 will also preserve the validity of the inequality. It is also worth noting that the action of adding -3 is analogous with that of subtracting 3, so there are no surprises. Later on, we will use the term *inverse* to describe the relationship between these two operations.

Original Inequality: $4 < 10$

Action	Resulting Inequality	Outcome
Multiply by 3	$12 < 30$	True
Divide by 2	$2 < 5$	True
Multiply by -3	$-12 < -30$	**False**
Divide by -2	$-2 < -5$	**False**

Here, we see that multiplication, and consequently division, by a negative value forces us to change the direction of the inequality ($-2 < -5$ changes to $-2 > -5$) in order to preserve its validity. This is best illustrated by the following diagram.



Note that as with addition and subtraction, the *inverse* relationship between the operations of multiplication and division is again at work, since for example, division by -2 is analogous to multiplication by $-1/2$.

We conclude our treatment of linear inequalities with a more complicated example. All our solution steps will be identical to those for solving a linear equation, with the only exception being those steps related to multiplication or division by a negative number.

Example 65. Solve the linear inequality $-1 - 2(x-3) \leq 5x - 9$.

$$\begin{array}{rcll}
-1 - 2(x-3) & \leq & 5x - 9 & \\
-1 - 2x + 6 & \leq & 5x - 9 & \text{Distribute } -2 \\
5 - 2x & \leq & 5x - 9 & \text{Combine like terms} \\
\underline{\mathbf{-5}} & & \underline{\mathbf{-5}} & \text{Subtract 5 from both sides} \\
-2x & \leq & 5x - 14 & \\
\underline{\boldsymbol{-5x}} & & \underline{\boldsymbol{-5x}} & \text{Subtract } 5x \text{ from both sides} \\
-7x & \leq & -14 & \\
\overline{\mathbf{-7}} & & \overline{\mathbf{-7}} & \text{Divide both sides by } -7 \\
x & \geq & 2 & \text{Our solution}
\end{array}$$

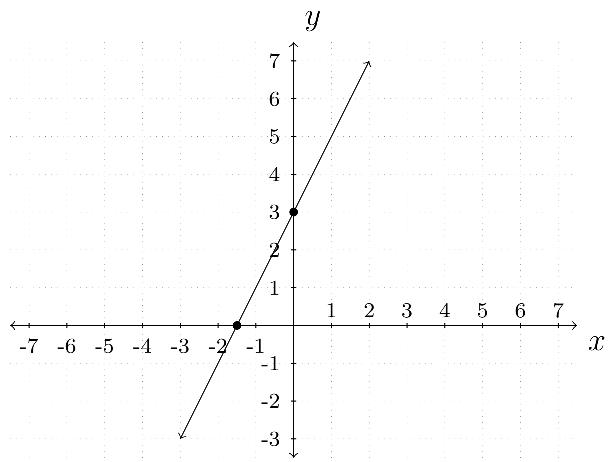
We leave it as an exercise to the reader to check that our solution is correct.

Introduction to Sign Diagrams

In a later chapter we will define a *function*, providing several examples of y as a function of x, and discuss in detail the processes associated with graphing certain families of functions. As both linear and quadratic functions present the most basic examples of polynomials, we will take this opportunity to introduce a tool, called a *sign diagram* (or chart), that will be incredibly useful for graphing these and more complicated functions. For the sake of the mathematics, it should be noted that the usefulness of the sign diagram for graphing is a direct consequence of the *continuity* of a function and the *Intermediate Value Theorem* (IVT). These concepts will be studied more closely in subsequent courses (e.g. Calculus)

Example 66. Graph the linear equation $y = 2x + 3$.

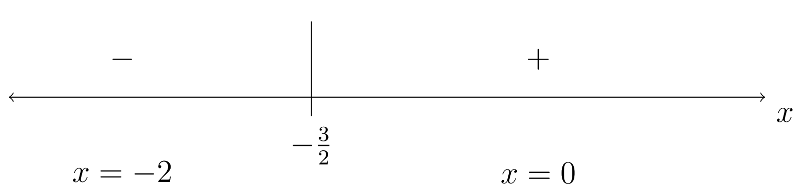
Our graph will have a y-intercept at the point $(0, 3)$. By setting $y = 0$, we obtain an x-intercept at the point $(-3/2, 0)$. We then obtain the following graph by plotting these two intercepts and connecting them.



When graphing any equation, it will be of particular interest to identify any x-intercepts on the graph. Though this will sometimes prove a daunting and even impossible task, as we have seen, it is relatively straightforward when faced with a linear equation. Recall that all lines which are not horizontal will have exactly one x-intercept. Horizontal lines will either have no x-intercepts or, in the case of the horizontal line $y = 0$, will have infinitely many x-intercepts. Once we know the x-intercept of the graph of our linear equation, we can easily determine the sign ($+$ or $-$) of the y-coordinate for every point to the left or right of our x-intercept. Since all lines are by their nature straight, this amounts to testing our equation, by plugging in a single *test value* for each interval on either side of our x-intercept.

In the case of our example, though we are free to choose any real-numbered test values we would like, we will make the more common selections of $x = -2$ and $x = 0$. Note that $x = -1$ would have been a perfectly fine value instead of $x = 0$, but it is often easier to plug $x = 0$ into a function than any other value. After plugging each test value into the equation, we determine the sign of the y-coordinate associated with $x = -2$ is negative $(-)$, since $2(-2) + 3 < 0$, and the sign of the y-coordinate associated with $x = 0$ is positive $(+)$, since $2(0) + 3 > 0$. Note that here we are *not* concerned with the actual values of the y-coordinates, just their respective signs. This point will be reiterated as we encounter more complicated mathematical expressions. The results of our calculations are presented on the real-number line shown below.

Example 67. Sign Diagram for $y = 2x + 3$.



Note that if constructed correctly, our sign diagram should be consistent with the graph of $y = 2x + 3$. Specifically, a plus $(+)$ corresponds to those points on the graph that sit *above* the x-axis, and a minus $(-)$ corresponds to those points that sit *below* the x-axis.

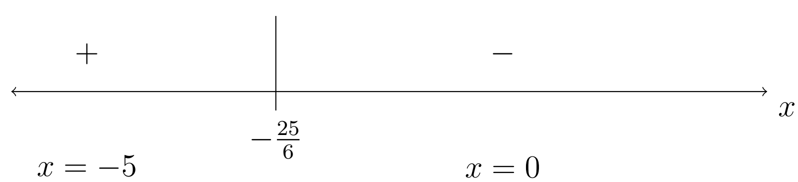
We now will summarize the steps for constructing a sign diagram for a linear equation (or function) with a nonzero slope.

1. If not provided, put the equation in slope-intercept form.
2. Determine the x-intercept of the graph of the equation. Mark this value (call it x_0) on a real-number line by placing a symbol $|$ directly above it that divides the line into two intervals, $(-\infty, x_0)$ and (x_0, ∞).
3. Identify a test value for each interval. Write your test values below their respective test intervals.
4. Determine the sign (either positive or negative) of the y-coordinate for each test value. Mark this on the real-number line by placing either a $+$ or $-$ above the interval.

Example 68. Construct a sign diagram for the linear equation $y = -12x - 50$.

By setting $y = 0$, we get $x = -\frac{50}{12} = -\frac{25}{6} = -4.1\overline{6}$. For test values, we will use $x = -5$ and $x = 0$.

Test Value	Resulting y-coordinate	Sign
$x = -5$	$-12(-5) - 50 = 60 - 50 > 0$	$+$
$x = 0$	$-12(0) - 50 = 0 - 50 < 0$	$-$



Note that in the instance of a horizontal line $m = 0$, our sign diagram will only require us to test a single value for the entire interval $(-\infty, \infty)$. It therefore suffices to just identify the sign of the y-intercept for the graph of our equation. Lastly, if the y-intercept is zero, then our sign diagram will have no test intervals to check, since all points on our graph will be of the form $(x, 0)$.

It is worth mentioning that here we have only sought to "set the table" for the construction of sign diagrams, using linear equations as a very basic introduction. Once we are exposed to more complicated equations and functions, we will see how the construction of a sign diagram will become more involved. In short, more complicated examples will include more x-intercepts, which will result in more test intervals to check. The process, however, will essentially remain the same as we have outlined, and the resulting sign diagram will be critical in understanding the graph of a function and solving any related inequalities.

Compound and Absolute Value Inequalities

Compound Inequalities (L7)

Objective: Solve, graph and give interval notation to the solution of compound inequalities and inequalities containing absolute values.

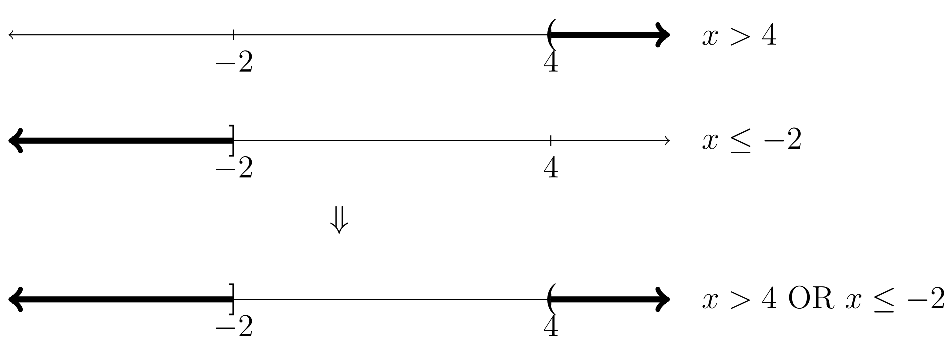
Several inequalities can be combined together to form what are called compound inequalities. There are three types of compound inequalities which we will investigate in this section.

The first type of a compound inequality is an OR inequality. For this type of inequality we want a true statement from either one inequality OR the other inequality OR both. When we are graphing these type of inequalities we will graph each individual inequality above the number line, then combine them together on the number line for our graph.

When we provide interval notation for our solution, if there are two different intervals to the graph we will put a $\cup$ between the two intervals. The $\cup$ symbol represents a *union* of the two intervals in our final answer.

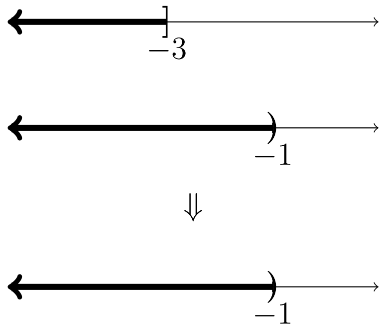
Example 69. Solve each inequality, graph the solution, and provide the interval notation of your solution.

$$\begin{array}{rcrl}
2x-5>3 & \text{or} & 4-x\geq 6 & \text{Solve each inequality}\\
\underline{\mathbf{+5}\ \ \mathbf{+5}} & & \underline{\mathbf{-4}\qquad \mathbf{-4}} & \text{Add or subtract first}\\
2x>8 & \text{or} & -x\geq 2 & \text{Divide}\\
\overline{\mathbf{2}}\quad\ \overline{\mathbf{2}} & & \overline{\mathbf{-1}}\ \ \overline{\mathbf{-1}} & \text{Dividing by negative flips sign}\\
x>4 & \text{or} & x\leq -2 & \text{Graph the inequalities separately,}\\
 & & & \quad\text{then combine}
\end{array}$$



Our answer is $(-\infty,-2]\cup(4,\infty)$.

There are several different results that could result from an OR statement. The graphs could be pointing different directions, as in the graph above. The graphs could also be pointing in the same direction, as in the graph below on the left. Lastly, the graphs could be pointing in opposite directions, but overlapping, as in the graph below on the right. Notice how interval notation works for each of these cases.

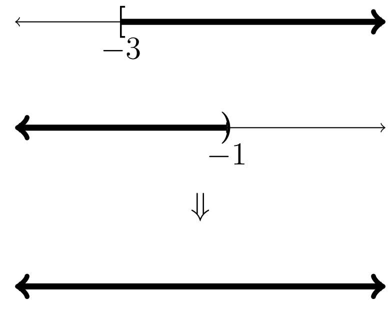


$$x \leq -3 \text{ or } x < -1$$

$$(-\infty, -3] \cup (-\infty, -1) = (-\infty, -1)$$

As the graphs overlap, we take the largest graph for our solution.

Interval notation: $(-\infty, -1)$



$$x \geq -3 \text{ or } x < -1$$

$$[-3, \infty) \cup (-\infty, -1) = (-\infty, \infty)$$

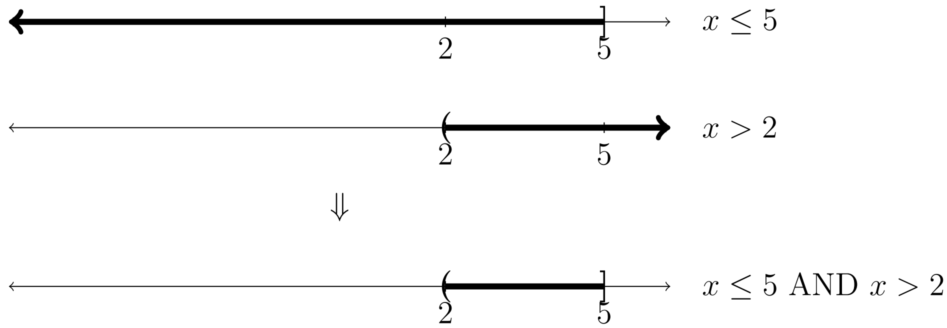
When the graphs are combined they cover the entire number line.

Interval notation: $(-\infty, \infty)$ or $\mathbb{R}$

The second type of compound inequality is an AND inequality. These inequalities require *both* statements to be true. If one is false, they both are false. When we graph these inequalities we can follow a similar process. First, graph both inequalities above the number line. This time, however, we will only consider where they overlap on the number line for our final graph. When our solution is given in interval notation it will be expressed in a manner very similar to single inequalities. The symbol that can be used for simplifying an AND inequality is known as an *intersection*, denoted by a $\cap$. When simplifying, the $\cap$ should not be needed or expressed in our final answer.

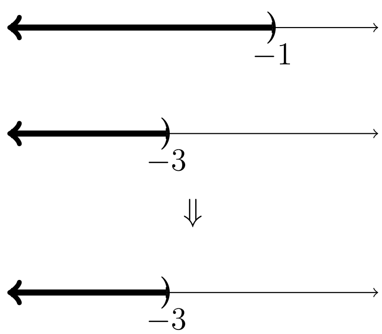
Example 70. Solve each inequality, graph the solution, and provide the interval notation of your solution.

$$\begin{array}{rcll}
2x + 8 \geq 5x - 7 & \text{and} & 5x - 3 > 3x + 1 & \text{Move variables to one side} \\
\underline{\boldsymbol{-2x \qquad - 2x}} & & \underline{\boldsymbol{-3x \qquad - 3x}} & \\
8 \geq 3x - 7 & \text{and} & 2x - 3 > 1 & \text{Add 7 or 3 to both sides} \\
\underline{\boldsymbol{+7 \qquad + 7}} & & \underline{\boldsymbol{+3 \quad + 3}} & \\
15 \geq 3x & \text{and} & 2x > 4 & \text{Divide} \\
\overline{\mathbf{3}} \quad \overline{\mathbf{3}} & & \overline{\mathbf{2}} \quad \overline{\mathbf{2}} & \\
5 \geq x & \text{and} & x > 2 & \text{Graph the inequalities} \\
 & & & \quad \text{separately, then combine}
\end{array}$$



Our answer is $(-\infty, 5] \cap (2, \infty) = (2, 5]$.

Again, as we graph AND inequalities, only the overlapping parts of the individual graphs makes it to the final number line. There are three different types of possibilities we could encounter when analyzing the overlap of an AND inequality. The first is shown in the above example; both intervals have some overlap, but point in opposite directions. The second occurs when the arrows both point in the same direction, as shown below on the left. The third occurs when the arrows point in opposite directions, but do not overlap, as shown below on the right. Notice how interval notation is expressed in each case.

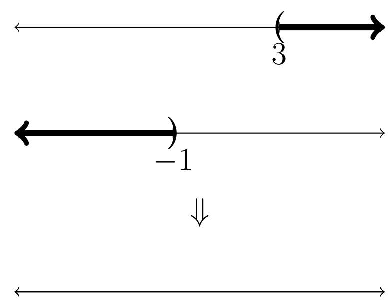


$x < -1$ and $x < -3$

$(-\infty, -1) \cap (-\infty, -3) = (-\infty, -3)$

In this graph, the overlap is only the smaller graph ($x < -2$), so this is what makes it to the final number line.

Interval notation: $(-\infty, -2)$



$x > 3$ and $x < -1$

$(3, \infty) \cap (-\infty, -1) = \varnothing$

In this graph there is no overlap of the parts. Because there is no overlap, no values make it to the final number line.

Interval notation: No solution or $\varnothing$

The third type of compound inequality is a special type of AND inequality, and occurs when our variable (or expression containing the variable) is between two numbers. We can write this as a single mathematical sentence with three parts, such as $5 < x \leq 8$, to show x is between 5 and 8 (or equal to 8). This type of inequality is often referred to as a *double inequality*, since it contains two inequalities. When solving these types of inequalities, as there are three parts to work with, in order to stay balanced we will do the same thing to all three parts (rather than just two sides), and eventually isolate the variable in the middle.

The resulting graph will contain all values between the numbers on either side of the double inequality, with appropriate brackets or parentheses on the ends.

Example 71. Solve each inequality, graph the solution, and provide the interval notation of your solution.

$$\begin{array}{rl}
-6 \leq -4x + 2 < 2 & \text{Subtract 2 from all three parts} \\
\underline{\mathbf{-2} \qquad \mathbf{-2} \quad \mathbf{-2}} & \\
-8 \leq -4x < 0 & \text{Divide all three parts by } -4 \\
\overline{\mathbf{-4}} \quad \overline{\mathbf{-4}} \quad \overline{\mathbf{-4}} & \text{Dividing by a negative flips the symbols} \\
2 \geq x > 0 & \text{Flip entire statement so values get larger left to right} \\
0 < x \leq 2 & \text{Graph } x \text{ between 0 and 2}
\end{array}$$

$0 < x \leq 2$

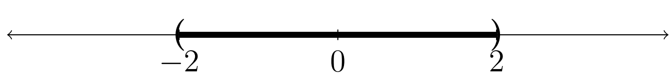
Our answer is $(0, 2]$.

Inequalities Containing Absolute Values (L8)

When an inequality contains an absolute value we will look to isolate the absolute value and eventually remove it, in order to graph the solution and express it using interval notation. The way that we treat the absolute value during this process depends on the direction of the inequality symbol.

Consider $|x| < 2$.

We define the absolute value as the distance from zero. Another way to read this inequality would be *the distance that the variable x is from zero is less than 2.* So on a number line we will shade all values of x that are less than 2 units away from zero. Alternatively stated, we will shade all values of x that are within 2 units of zero.



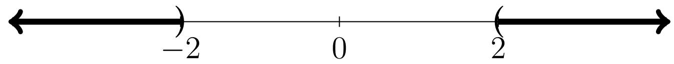
$-2 < x < 2$

Our solution set is all x in the interval $(-2, 2)$.

This graph looks just like the graphs of the double (compound) inequalities from the previous subsection! When an isolated absolute value is *less than* (or $\leq$) a number we will remove the absolute value by changing the problem to a double inequality, with the negative value on the left and the positive value on the right. So $|x| < 2$ becomes $-2 < x < 2$, as the graph above illustrates.

Consider $|x| > 2$.

Similarly, another way to read this inequality would be *the distance that x is from zero is greater than 2.* So on the number line we will shade all values of x that are more than 2 units away from zero.



$x < -2$ OR $x > 2$

Our solution set is all x in the union $(-\infty, -2) \cup (2, \infty)$.

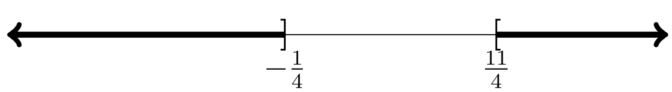
This graph looks just like the graphs of the OR compound inequalities from the previous subsection! When an isolated absolute value is *greater than* (or $\geq$) a number we will remove the absolute value by changing the problem to an OR inequality: the first inequality will look just like the original inequality, but with no absolute value; the second inequality will reverse the direction of the original inequality symbol, and changing the value to a negative. So $|x| > 2$ becomes $x > 2$ or $x < -2$, as the graph above illustrates.

For all absolute value inequalities we can also express our answers in interval notation, which is done the same way as for standard compound inequalities.

We can solve absolute value inequalities much like we solved absolute value equations. Our first step will be to isolate the absolute value. Next we will remove the absolute value by either making a double inequality if the absolute value is less than a number, or making an OR inequality if the absolute value is greater than a number. Then we will solve these inequalities. Remember, if we multiply or divide by a negative number during the process, the inequality symbol(s) must switch directions!

Example 72. Solve, graph, and provide interval notation for the following inequality.

$$\begin{array}{rcrl}
 & |4x-5| \geq 6 & & \text{Absolute value is greater, use OR} \\
4x-5 \geq 6 & \text{OR} & 4x-5 \leq -6 & \text{Solve} \\
\underline{\mathbf{+5 \quad +5}} & & \underline{\mathbf{+5 \quad +5}} & \text{Add } 5 \text{ to both sides} \\
4x \geq 11 & \text{OR} & 4x \leq -1 & \\
\overline{\mathbf{4}} \quad \overline{\mathbf{4}} & & \overline{\mathbf{4}} \quad \overline{\mathbf{4}} & \text{Divide both sides by } 4 \\
x \geq \dfrac{11}{4} & \text{OR} & x \leq -\dfrac{1}{4} & \text{Graph}
\end{array}$$

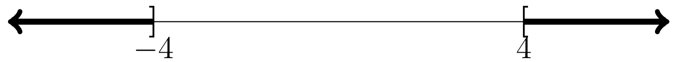


$x \leq -\frac{1}{4}$ OR $x \geq \frac{11}{4}$

Our solution set is all x in the union $(-\infty, -\frac{1}{4}] \cup [\frac{11}{4}, \infty)$.

Example 73. Solve, graph, and provide interval notation for the following inequality.

$$\begin{array}{rl}
-4-3|x| \leq -16 & \\
\underline{\mathbf{+4} \qquad\qquad \mathbf{+\,4}} & \text{Add } 4 \text{ to both sides} \\
-3|x| \leq -12 & \text{Divide both sides by } -3 \\
\overline{\mathbf{-3}} \quad \overline{\mathbf{-3}} & \text{Dividing by a negative switches the inequality} \\
|x| \geq 4 & \text{Absolute value is greater, use OR} \\
x \geq 4 \quad \text{OR} \quad x \leq -4 & \text{Graph}
\end{array}$$



$x \leq -4$ OR $x \geq 4$

Our solution set is all x in the union $(-\infty, -4] \cup [4, \infty)$.

In the previous example, we cannot combine -4 and -3 because there are no like terms, the -3 is being multiplied by an absolute value. So we must first clear the -4 by adding 4, then divide both sides by by -3. The next example is similar.

Example 74. Solve, graph, and provide interval notation for the solution.

$$\begin{array}{rl}
9 - 2|4x+1| > 3 & \\
\underline{\mathbf{-9} \qquad\qquad\quad \mathbf{-9}} & \text{Subtract } 9 \text{ from both sides} \\
-2|4x+1| > -6 & \text{Divide both sides by } -2 \\
\overline{\mathbf{-2}} \qquad \overline{\mathbf{-2}} & \text{Dividing by negative switches the inequality} \\
|4x+1| < 3 & \text{Absolute value is less, use double inequality} \\
-3 < 4x + 1 < 3 & \text{Solve} \\
\underline{\mathbf{-1}} \qquad \underline{\mathbf{-1}} \quad \underline{\mathbf{-1}} & \text{Subtract } 1 \text{ from all three parts} \\
-4 < 4x < 2 & \text{Divide all three parts by } 4 \\
\overline{\mathbf{4}} \quad \overline{\mathbf{4}} \quad \overline{\mathbf{4}} & \\
-1 < x < \dfrac{1}{2} & \text{Graph}
\end{array}$$

$-1 < x < \frac{1}{2}$

Our solution set is all x in the interval $(-1, \frac{1}{2})$.

In the previous example, we cannot distribute the -2 into the absolute value. In general, it is never recommended to distribute values inside or factor values outside of an absolute value. Our best way to solve is to first isolate the absolute value by clearing the values around it, then convert to the appropriate compound inequality (either a double inequality or an OR inequality) and solve.

It is important to remember that as we are solving these equations, an absolute value is always positive. If we end up with an absolute value that is less than a negative number, then we will have no solution because the absolute value will always be positive, and therefore greater than a negative. Similarly, if an absolute value is greater than a negative, this will always happen. Here our answer will be all real numbers. The next two examples demonstrate these special cases.

Example 75. Solve, graph, and provide interval notation for the solution.

$$\begin{array}{rl}
12+4|6x-1|<4 & \text{Subtract } 12 \text{ from both sides} \\
\underline{\mathbf{-12} \qquad\qquad \mathbf{-12}} & \\
4|6x-1|<-8 & \text{Divide both sides by } 4 \\
\overline{\mathbf{4}} \qquad \overline{\mathbf{4}} & \\
|6x-1|<-2 & \text{Absolute value cannot be less than a negative}
\end{array}$$

 x

Our solution set is *no solutions* or $\varnothing$.

Example 76. Solve, graph, and provide interval notation for the solution.

$$\begin{array}{rl}
5-6|x+7|\leq 17 & \\
\underline{\mathbf{-5} \qquad\qquad \mathbf{-5}} & \text{Subtract } 5 \text{ from both sides} \\
-6|x+7|\leq 12 & \text{Divide both sides by } -6 \\
\overline{\mathbf{-6}} \qquad \overline{\mathbf{-6}} & \text{Dividing by a negative flips the symbol} \\
|x+7|\geq -2 & \text{Absolute value is always greater than a negative}
\end{array}$$

 x

Our solution set is *all real numbers* or $(-\infty,\infty)$.

Practice Problems

Solving Linear Equations

One-Step Equations

Solve each equation.

1) $v+9=16$
2) $14=b+3$
3) $x-11=-16$
4) $-14=x-18$
5) $30=a+20$
6) $-1+k=5$
7) $x-7=-26$
8) $-13+p=-19$
9) $13=n-5$
10) $22=16+m$
11) $340=-17x$
12) $4r=-28$
13) $-9=\frac{n}{12}$
14) $\frac{5}{9}=\frac{b}{9}$
15) $20v=-160$
16) $-20x=-80$
17) $340=20n$
18) $\frac{1}{2}=\frac{a}{8}$
19) $16x=320$
20) $\frac{k}{13}=-16$
21) $-16+n=-13$
22) $21=x+5$
23) $p-8=-21$
24) $m-4=-13$
25) $180=12x$
26) $3n=24$
27) $20b=-200$
28) $-17=\frac{x}{12}$
29) $\frac{r}{14}=\frac{5}{14}$
30) $n+8=10$
31) $-7=a+4$
32) $v-16=-30$
33) $10=x-4$
34) $-15=x-16$
35) $13a=-143$
36) $-8k=120$
37) $\frac{p}{20}=-12$
38) $-15=\frac{x}{9}$
39) $9+m=-7$
40) $-19=\frac{n}{20}$

Two-Step Equations

Solve each equation.

1) $5 + \frac{n}{4} = 4$
2) $-2 = -2m + 12$
3) $102 = -7r + 4$
4) $27 = 21 - 3x$
5) $-8n + 3 = -77$
6) $-4 - b = 8$
7) $0 = -6v$
8) $-2 + \frac{x}{2} = 4$
9) $-8 = \frac{x}{5} - 6$
10) $-5 = \frac{a}{4} - 1$
11) $0 = -7 + \frac{k}{2}$
12) $-6 = 15 + 3p$
13) $-12 + 3x = 0$
14) $-5m + 2 = 27$
15) $24 = 2n - 8$
16) $-37 = 8 + 3x$
17) $2 = -12 + 2r$
18) $-8 + \frac{n}{12} = -7$
19) $\frac{b}{3} + 7 = 10$
20) $\frac{x}{1} - 8 = -8$
21) $152 = 8n + 64$
22) $-11 = -8 + \frac{v}{2}$
23) $-16 = 8a + 64$
24) $-2x - 3 = -29$
25) $56 + 8k = 64$
26) $-4 - 3n = -16$
27) $-2x + 4 = 22$
28) $67 = 5m - 8$
29) $-20 = 4p + 4$
30) $9 = 8 + \frac{x}{6}$
31) $-5 = 3 + \frac{n}{2}$
32) $\frac{m}{4} - 1 = -2$
33) $\frac{r}{8} - 6 = -5$
34) $-80 = 4x - 28$
35) $-40 = 4n - 32$
36) $33 = 3b + 3$
37) $87 = 3 - 7v$
38) $3x - 3 = -3$
39) $-x + 1 = -11$
40) $4 + \frac{a}{3} = 1$

General Linear Equations

Solve each equation.

1) $2 - (-3a - 8) = 1$
2) $2(-3n + 8) = -20$
3) $-5(-4 + 2v) = -50$
4) $2 - 8(-4 + 3x) = 34$
5) $66 = 6(6 + 5x)$
6) $32 = 2 - 5(-4n + 6)$
7) $0 = -8(p - 5)$
8) $-55 = 8 + 7(k - 5)$
9) $-2 + 2(8x - 7) = -16$
10) $-(3 - 5n) = 12$
11) $-21x + 12 = -6 - 3x$
12) $-3n - 27 = -27 - 3n$
13) $-1 - 7m = -8m + 7$
14) $56p - 48 = 6p + 2$
15) $1 - 12r = 29 - 8r$
16) $4 + 3x = -12x + 4$
17) $20 - 7b = -12b + 30$
18) $-16n + 12 = 39 - 7n$
19) $-32 - 24v = 34 - 2v$
20) $17 - 2x = 35 - 8x$

21) $-2 - 5(2 - 4m) = 33 + 5m$
22) $-25 - 7x = 6(2x - 1)$
23) $-4n + 11 = 2(1 - 8n) + 3n$
24) $-7(1 + b) = -5 - 5b$
25) $-6v - 29 = -4v - 5(v + 1)$
26) $-8(8r - 2) = 3r + 16$
27) $2(4x - 4) = -20 - 4x$
28) $-8n - 19 = -2(8n - 3) + 3n$
29) $-a - 5(8a - 1) = 39 - 7a$
30) $-4 + 4k = 4(8k - 8)$
31) $-57 = -(-p + 1) + 2(6 + 8p)$
32) $16 = -5(1 - 6x) + 3(6x + 7)$
33) $-2(m - 2) + 7(m - 8) = -67$
34) $7 = 4(n - 7) + 5(7n + 7)$
35) $50 = 8(7 + 7r) - (4r + 6)$
36) $-8(6 + 6x) + 4(-3 + 6x) = -12$
37) $-8(n - 7) + 3(3n - 3) = 41$
38) $-76 = 5(1 + 3b) + 3(3b - 3)$
39) $-61 = -5(5r - 4) + 4(3r - 4)$
40) $-6(x - 8) - 4(x - 2) = -4$
41) $-2(8n - 4) = 8(1 - n)$
42) $-4(1 + a) = 2a - 8(5 + 3a)$
43) $-3(-7v + 3) + 8v = 5v - 4(1 - 6v)$
44) $-6(x - 3) + 5 = -2 - 5(x - 5)$
45) $-7(x - 2) = -4 - 6(x - 1)$
46) $-(n + 8) + n = -8n + 2(4n - 4)$
47) $-6(8k + 4) = 8(6k + 3) + 12$
48) $-5(x + 7) = 4(-8x - 2)$
49) $-2(1 - 7p) = 8(p - 7)$
50) $8(-8n + 4) = 4(-7n + 8)$

Equations Containing Fractions

Solve each equation.

1) $\frac{3}{5}(1+p) = \frac{21}{20}$
2) $-\frac{1}{2} = \frac{3}{2}k + \frac{3}{2}$
3) $0 = -\frac{5}{4}(x - \frac{6}{5})$
4) $\frac{3}{2}n - \frac{8}{3} = -\frac{29}{12}$
5) $\frac{3}{4} - \frac{5}{4}m = \frac{113}{24}$
6) $\frac{11}{4} + \frac{3}{4}r = \frac{163}{32}$
7) $\frac{635}{72} = -\frac{5}{2}(-\frac{11}{4} + x)$
8) $-\frac{16}{9} = -\frac{4}{3}(\frac{5}{3} + n)$
9) $2b + \frac{9}{5} = -\frac{11}{5}$
10) $\frac{3}{2} - \frac{7}{4}v = -\frac{9}{8}$
11) $\frac{3}{2}(\frac{7}{3}n + 1) = \frac{3}{2}$
12) $\frac{41}{9} = \frac{5}{2}(x + \frac{2}{3}) - \frac{1}{3}x$
13) $-a - \frac{5}{4}(-\frac{8}{3}a + 1) = -\frac{19}{4}$
14) $\frac{1}{3}(-\frac{7}{4}k + 1) - \frac{10}{3}k = -\frac{13}{8}$
15) $\frac{55}{6} = -\frac{5}{2}(\frac{3}{2}p - \frac{5}{3})$
16) $-\frac{1}{2}(\frac{2}{3}x - \frac{3}{4}) - \frac{7}{2}x = -\frac{83}{24}$
17) $\frac{16}{9} = -\frac{4}{3}(-\frac{4}{3}n - \frac{4}{3})$
18) $\frac{2}{3}(m + \frac{9}{4}) - \frac{10}{3} = -\frac{53}{18}$
19) $-\frac{5}{8} = \frac{5}{4}(r - \frac{3}{2})$
20) $\frac{1}{12} = \frac{4}{3}x + \frac{5}{3}(x - \frac{7}{4})$
21) $-\frac{11}{3} + \frac{3}{2}b = \frac{5}{2}(b - \frac{5}{3})$
22) $\frac{7}{6} - \frac{4}{3}n = -\frac{3}{2}n + 2(n + \frac{3}{2})$
23) $-(-\frac{5}{2}x - \frac{3}{2}) = -\frac{3}{2} + x$
24) $-\frac{149}{16} - \frac{11}{3}r = -\frac{7}{4}r - \frac{5}{4}(-\frac{4}{3}r + 1)$
25) $\frac{45}{16} + \frac{3}{2}n = \frac{7}{4}n - \frac{19}{16}$
26) $-\frac{7}{2}(\frac{5}{3}a + \frac{1}{3}) = \frac{11}{4}a + \frac{25}{8}$
27) $\frac{3}{2}(v + \frac{3}{2}) = -\frac{7}{4}v - \frac{19}{6}$
28) $-\frac{8}{3} - \frac{1}{2}x = -\frac{4}{3}x - \frac{2}{3}(-\frac{13}{4}x + 1)$
29) $\frac{47}{9} + \frac{3}{2}x = \frac{5}{3}(\frac{5}{2}x + 1)$
30) $\frac{1}{3}n + \frac{29}{6} = 2(\frac{4}{3}n + \frac{2}{3})$

Equations Containing an Absolute Value

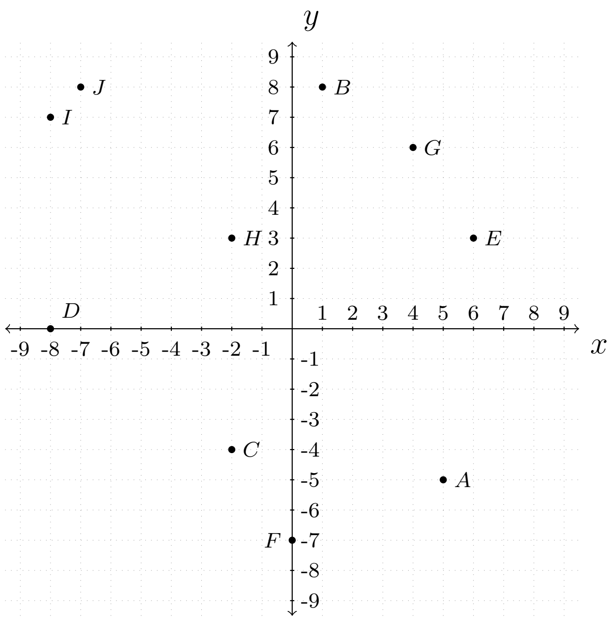
Solve each equation.

1) $|x| = 8$
2) $|n| = 7$
3) $|b| = 1$
4) $|x| = 2$
5) $|5 + 8a| = 53$
6) $|9n + 8| = 46$
7) $|3k + 8| = 2$
8) $|3 - x| = 6$
9) $|9 + 7x| = 30$
10) $|5n + 7| = 23$
11) $|8 + 6m| = 50$
12) $|9p + 6| = 3$
13) $|6 - 2x| = 24$
14) $|3n - 2| = 7$
15) $-7|-3 - 3r| = -21$
16) $|2 + 2b| + 1 = 3$
17) $7|-7x - 3| = 21$
18) $\frac{|-4-3n|}{4} = 2$
19) $\frac{|-4b-10|}{8} = 3$
20) $8|5p + 8| - 5 = 11$
21) $8|x + 7| - 3 = 5$
22) $3 - |6n + 7| = -40$
23) $5|3 + 7m| + 1 = 51$
24) $4|r + 7| + 3 = 59$
25) $3 + 5|8 - 2x| = 63$
26) $5 + 8|-10n - 2| = 101$
27) $|6b - 2| + 10 = 44$
28) $7|10v - 2| - 9 = 5$
29) $-7 + 8|-7x - 3| = 73$
30) $8|3 - 3n| - 5 = 91$

Graphing Linear Equations

The Cartesian Plane

1) **Find the coordinates of each point.**



2) **Graph each point on the $xy-$plane.**

$A(0,4)$	$F(-4,2)$
$B(0,3)$	$G(-3,0)$
$C(3,-2)$	$H(-3,4)$
$D(-2,-2)$	$I(1,0)$
$E(4,-2)$	$J(-5,5)$

Graphing Lines from Points

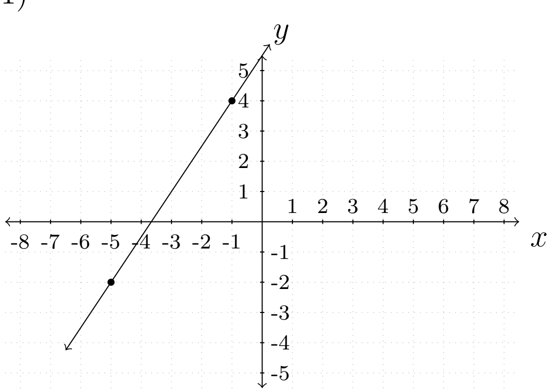
Sketch the graph of each line.

1) $y=-\frac{1}{4}x-3$
2) $y=x-1$
3) $y=-\frac{5}{4}x-4$
4) $y=-\frac{3}{5}x+1$
5) $y=-4x+2$
6) $y=\frac{5}{3}x+4$
7) $y=\frac{3}{2}x-5$
8) $y=-x-2$
9) $y=-\frac{4}{5}x-3$
10) $y=\frac{1}{2}x$
11) $x+5y=-15$
12) $8x-y=5$
13) $4x+y=5$
14) $3x+4y=16$
15) $2x-y=2$
16) $7x+3y=-12$
17) $x+y=-1$
18) $3x+4y=8$
19) $x-y=-3$
20) $9x-y=-4$

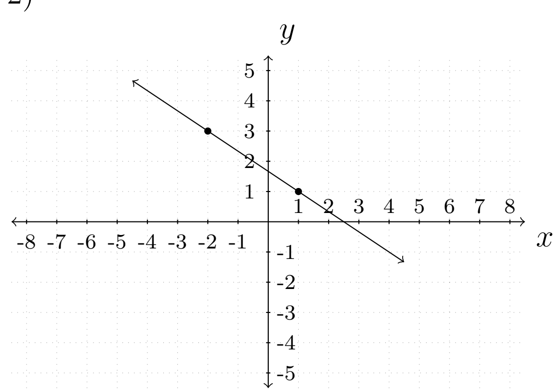
The Slope of a Line

Find the slope of each line.

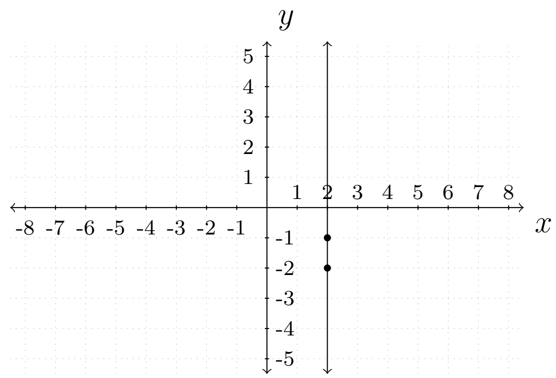
1)



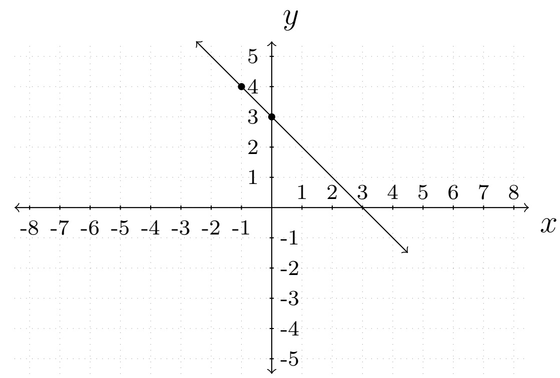
2)



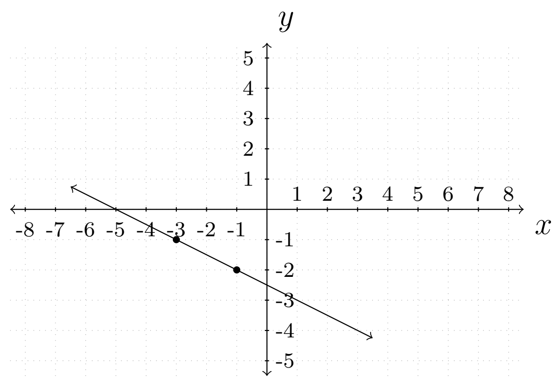
3)



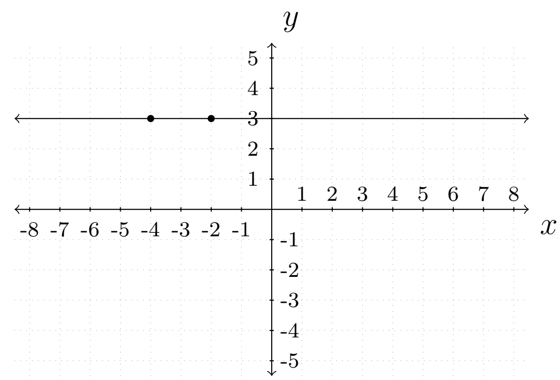
4)



5)



6)



Find the slope of the line through each pair of points.

7) $(-2,10),(-2,-15)$
8) $(1,2),(-6,-14)$
9) $(-15,10),(16,-7)$
10) $(13,-2),(7,7)$
11) $(10,18),(-11,-10)$
12) $(-3,6),(-20,13)$
13) $(-16,-14),(11,-14)$
14) $(13,15),(2,10)$
15) $(-4,14),(-16,8)$
16) $(9,-6),(-7,-7)$
17) $(12,-19),(6,14)$
18) $(-16,2),(15,-10)$
19) $(-5,-10),(-5,20)$
20) $(8,11),(-3,-13)$
21) $(-17,19),(10,-7)$
22) $(11,-2),(1,17)$
23) $(7,-14),(-8,-9)$
24) $(-18,-5),(14,-3)$
25) $(-5,7),(-18,14)$
26) $(19,15),(5,11)$

Find the value of x or y so that the line through the points has the given slope.

27) $(2,6)$ and $(x,2)$; slope : $\frac{4}{7}$
28) $(8,y)$ and $(-2,4)$; slope : $-\frac{1}{5}$
29) $(-3,-2)$ and $(x,6)$; slope : $-\frac{8}{5}$
30) $(-2,y)$ and $(2,4)$; slope : $\frac{1}{4}$
31) $(-8,y)$ and $(-1,1)$; slope : $\frac{6}{7}$
32) $(x,-1)$ and $(-4,6)$; slope : $-\frac{7}{10}$
33) $(x,-7)$ and $(-9,-9)$; slope : $\frac{2}{5}$
34) $(2,-5)$ and $(3,y)$; slope : 6
35) $(x,5)$ and $(8,0)$; slope : $-\frac{5}{6}$
36) $(6,2)$ and $(x,6)$; slope : $-\frac{4}{5}$

The Two Forms of a Linear Equation

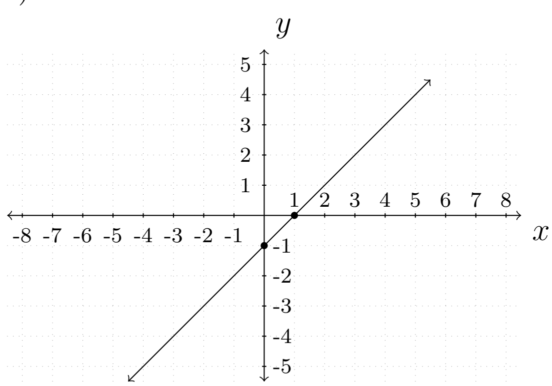
Slope-Intercept Form

Write the slope-intercept form of the equation of each line given the slope and the $y-$intercept.

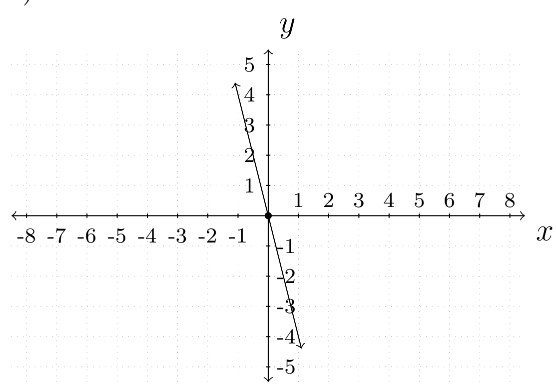
1) Slope = 2, y-intercept = 5
2) Slope = -6, y-intercept = 4
3) Slope = 1, y-intercept = -4
4) Slope = -1, y-intercept = -2
5) Slope = $-\frac{3}{4}$, y-intercept = -1
6) Slope = $-\frac{1}{4}$, y-intercept = 3
7) Slope = $\frac{1}{3}$, y-intercept = 1
8) Slope = $\frac{2}{5}$, y-intercept = 5

Write the slope-intercept form of the equation of each line.

1)



2)



Write each linear equation in slope-intercept form.

11) $x + 10y = -37$
12) $x - 10y = 3$
13) $2x + y = -1$
14) $6x - 11y = -70$
15) $7x - 3y = 24$
16) $4x + 7y = 28$
17) $x = -8$
18) $x - 7y = -42$
19) $y - 4 = -(x + 5)$
20) $y - 5 = \frac{5}{2}(x - 2)$
21) $y - 4 = 4(x - 1)$
22) $y - 3 = -\frac{2}{3}(x + 3)$
23) $y + 5 = -4(x - 2)$
24) $0 = x - 4$
25) $y + 1 = -\frac{1}{2}(x - 4)$
26) $y + 2 = \frac{6}{5}(x + 5)$

Sketch the graph of each line.

27) $y = \frac{1}{3}x + 4$
28) $y = -\frac{1}{5}x - 4$
29) $y = \frac{6}{5}x - 5$
30) $y = -\frac{3}{2}x - 1$
31) $y = \frac{3}{2}x$
32) $y = -\frac{3}{4}x + 1$
33) $x - y + 3 = 0$
34) $4x + 5 = 5y$
35) $-y - 4 + 3x = 0$
36) $-8 = 6x - 2y$
37) $-3y = -5x + 9$
38) $-3y = 3 - \frac{3}{2}x$

Point-Slope Form

Write the point-slope form of the equation of the line through the given point with the given slope.

1) through$(2, 3)$, slope = undefined
2) through$(1, 2)$, slope =undefined
3) through$(2, 2)$, slope $= \frac{1}{2}$
4) through$(2, 1)$, slope $= -\frac{1}{2}$
5) through$(-1, -5)$, slope =9
6) through$(2, -2)$, slope $= -2$
7) through$(-4, 1)$, slope $= \frac{3}{4}$
8) through$(4, -3)$, slope $= -2$
9) through$(0, -2)$, slope $= -3$
10) through$(-1, 1)$, slope $= 4$
11) through$(0, -5)$, slope $= -\frac{1}{4}$
12) through$(0, 2)$, slope $= -\frac{5}{4}$
13) through$(-5, -3)$, slope $= \frac{1}{5}$
14) through$(-1, -4)$, slope $= -\frac{2}{3}$
15) through$(-1, 4)$, slope $= -\frac{5}{4}$
16) through$(1, -4)$, slope $= -\frac{3}{2}$

Write the slope-intercept form of the equation of the line through the given point with the given slope.

17) through: $(-1, -5)$, slope $= 2$
18) through: $(2, -2)$, slope $= -2$
19) through: $(5, -1)$, slope $= -\frac{3}{5}$
20) through: $(-2, -2)$, slope $= -\frac{2}{3}$
21) through: $(-4, 1)$, slope $= \frac{1}{2}$
22) through: $(4, -3)$, slope $= -\frac{7}{4}$
23) through: $(4, -2)$, slope $= -\frac{3}{2}$
24) through: $(-2, 0)$, slope $= -\frac{5}{2}$
25) through: $(-5, -3)$, slope $= -\frac{2}{5}$
26) through: $(3, 3)$, slope $= \frac{7}{3}$
27) through: $(2, -2)$, slope $= 1$
28) through: $(-4, -3)$, slope =0
29) through:$(-3, 4)$, slope=undefined
30) through: $(-2, -5)$, slope =2
31) through: $(-4, 2)$, slope $= -\frac{1}{2}$
32) through: $(5, 3)$, slope $= \frac{6}{5}$

Write the point-slope form of the equation of the line through the given points.

33) through: $(-4, 3)$ and$(-3, 1)$
34) through: $(1, 3)$ and$(-3, 3)$
35) through: $(5, 1)$ and$(-3, 0)$
36) through: $(-4, 5)$ and$(4, 4)$
37) through: $(-4, -2)$ and$(0, 4)$
38) through: $(-4, 1)$ and$(4, 4)$
39) through: $(3, 5)$ and$(-5, 3)$
40) through: $(-1, -4)$ and$(-5, 0)$
41) through: $(3, -3)$ and$(-4, 5)$
42) through: $(-1, -5)$ and$(-5, -4)$

Write the slope-intercept form of the equation of the line through the given points.

43) through: $(-5, 1)$ and$(-1, -2)$
44) through: $(-5, -1)$ and$(5, -2)$
45) through: $(-5, 5)$ and$(2, -3)$
46) through: $(1, -1)$ and$(-5, -4)$
47) through: $(4, 1)$ and$(1, 4)$
48) through: $(0, 1)$ and$(-3, 0)$
49) through: $(0, 2)$ and$(5, -3)$
50) through: $(0, 2)$ and$(2, 4)$
51) through: $(0, 3)$ and$(-1, -1)$
52) through: $(-2, 0)$ and$(5, 3)$

Parallel and Perpendicular Lines

Find the slope of a line parallel to each given line.

1) $y = 2x + 4$
2) $y = -\frac{2}{3}x + 5$
3) $y = 4x - 5$
4) $y = -\frac{10}{3}x - 5$
5) $x - y = 4$
6) $6x - 5y = 20$
7) $7x + y = -2$
8) $3x + 4y = -8$

Find the slope of a line perpendicular to each given line.

9) $x = 3$
10) $y = -\frac{1}{2}x - 1$
11) $y = -\frac{1}{3}x$
12) $y = \frac{4}{5}x$
13) $x - 3y = -6$
14) $3x - y = -3$
15) $x + 2y = 8$
16) $8x - 3y = -9$

Write the point-slope form of the equation of the line described.

17) through : $(2, 5)$, parallel to $x = 0$
18) through: (5, 2), parallel to $y = \frac{7}{5}x + 4$
19) through : $(3, 4)$, parallel to $y = \frac{9}{2}x - 5$
20) through: $(1, -1)$, parallel to $y = -\frac{3}{4}x + 3$
21) through : $(2, 3)$, parallel to $y = \frac{7}{5}x + 4$
22) through : $(-1, 3)$, parallel to $y = -3x - 1$
23) through : $(4, 2)$, parallel to $x = 0$
24) through : $(1, 4)$, parallel to $y = \frac{7}{5}x + 2$
25) through: $(1,\ -5)$, perpendicular to $-x + y = 1$
26) through : $(1, -2)$, perpendicular to $-x + 2y = 2$
27) through : $(5, 2)$, perpendicular to $5x + y = -3$
28) through: (1, 3), perpendicular to $-x + y = 1$
29) through : $(4, 2)$, perpendicular to $-4x + y = 0$
30) through: $(-3, -5)$, perpendicular to $3x + 7y = 0$
31) through : $(2, -2)$ perpendicular to $3y - x = 0$
32) through: $(-2, 5)$. perpendicular to $y - 2x = 0$

Write the slope-intercept form of the equation of the line described.

33) through : $(4, -3)$, parallel to $y = -2x$
34) through : $(-5, 2)$, parallel to $y = \frac{3}{5}x$
35) through : $(-3, 1)$, parallel to $y = -\frac{4}{3}x - 1$
36) through : $(-4, 0)$, parallel to $y = -\frac{5}{4}x + 4$
37) through : $(-4, -1)$, parallel to $y = -\frac{1}{2}x + 1$
38) through : $(2, 3)$, parallel to $y = \frac{5}{2}x - 1$
39) through : $(-2, -1)$, parallel to $y = -\frac{1}{2}x - 2$
40) through : $(-5, -4)$, parallel to $y = \frac{3}{5}x - 2$
41) through : $(4, 3)$, perpendicular to $x + y = -1$
42) through : $(-3, -5)$, perpendicular to $x + 2y = -4$
43) through : $(5, 2)$, perpendicular to $x = 0$
44) through : $(5, -1)$, perpendicular to $-5x + 2y = 10$
45) through : $(-2, 5)$, perpendicular to $-x + y = -2$
46) through : $(2, -3)$, perpendicular to $-2x + 5y = -10$

47) through : $(4,-3)$, perpendicular to $-x+2y=-6$
48) through : $(-4,1)$, perpendicular to $4x+3y=-9$

Linear Inequalities and Sign Diagrams

Draw a graph for each inequality and provide interval notation.

1) $n>-5$
2) $n>4$
3) $-2\geq k$
4) $1\geq k$
5) $5\geq x$
6) $-5<x$

Solve each inequality, graph each solution, and provide interval notation.

7) $\frac{x}{11}\geq 10$
8) $-2\leq\frac{n}{13}$
9) $2+r<3$
10) $\frac{m}{5}\leq-\frac{6}{5}$
11) $8+\frac{n}{3}\geq 6$
12) $11>8+\frac{x}{2}$
13) $2>\frac{a-2}{5}$
14) $\frac{v-9}{-4}\leq 2$
15) $-47\geq 8-5x$
16) $\frac{6+x}{12}\leq -1$
17) $-2(3+k)<-44$
18) $-7n-10\geq 60$
19) $18<-2(-8+p)$
20) $5\geq\frac{x}{5}+1$
21) $24\geq-6(m-6)$

22) $-8(n-5)\geq 0$
23) $-r-5(r-6)<-18$
24) $-60\geq-4(-6x-3)$
25) $24+4b<4(1+6b)$
26) $-8(2-2n)\geq-16+n$
27) $-5v-5<-5(4v+1)$
28) $-36+6x>-8(x+2)+4x$
29) $4+2(a+5)<-2(-a-4)$
30) $3(n+3)+7(8-8n)<5n+5+2$
31) $-(k-2)>-k-20$
32) $-(4-5p)+3\geq-2(8-5p)$

Construct a sign diagram for each of following graphs/linear equations referenced below.

33)-38): Graphs (1) through (6) on page 62.

39)-50): Linear equations (27) through (38) on page 64.

Compound and Absolute Value Inequalities

Compound Inequalities

Solve each compound inequality, graph its solution, and provide interval notation.

1) $\frac{n}{3}\leq-3$ or $-5n\leq-10$
2) $6m\geq-24$ or $m-7<-12$
3) $x+7\geq 12$ or $9x<-45$
4) $10r>0$ or $r-5<-12$
5) $x-6<-13$ or $6x\leq-60$
6) $9+n<2$ or $5n>40$
7) $\frac{v}{8}>-1$ and $v-2<1$
8) $-9x<63$ and $\frac{x}{4}<1$
9) $-8+b<-3$ and $4b<20$
10) $-6n\leq 12$ and $\frac{n}{3}\leq 2$

11) $a + 10 \geq 3$ and $8a \leq 48$
12) $-6 + v \geq 0$ and $2v > 4$
13) $3 \leq 9 + x \leq 7$
14) $0 \geq \frac{x}{9} \geq -1$
15) $11 < 8 + k \leq 12$
16) $-11 \leq n - 9 \leq -5$
17) $-3 < x - 1 < 1$
18) $1 \leq \frac{p}{8} \leq 0$
19) $-4 < 8 - 3m \leq 11$
20) $3 + 7r > 59$ or $-6r - 3 > 33$
21) $-22 \leq 2n - 10 \leq -16$
22) $-6 - 8x \geq -6$ or $2 + 10x > 82$

23) $-5b + 10 \leq 30$ and $7b + 2 \leq -40$
24) $n + 10 \geq 15$ or $4n - 5 < -1$
25) $3x - 9 < 2x + 10$ and $5 + 7x \leq 10x - 10$
26) $4n + 8 < 3n - 6$ or $10n - 8 \geq 9 + 9n$
27) $-8 - 6v \leq 8 - 8v$ and $7v + 9 \leq 6 + 10v$
28) $5 - 2a \geq 2a + 1$ or $10a - 10 \geq 9a + 9$
29) $1 + 5k \leq 7k - 3$ or $k - 10 > 2k + 10$
30) $8 - 10r \leq 8 + 4r$ or $-6 + 8r < 2 + 8r$
31) $2x + 9 \geq 10x + 1$ and $3x - 2 < 7x + 2$
32) $-9m + 2 < -10 - 6m$ or $-m + 5 \geq 10 + 4m$

Inequalities Containing an Absolute Value

Solve each inequality, graph its solution, and provide interval notation.

1) $|x| < 3$
2) $|x| \leq 8$
3) $|2x| < 6$
4) $|x + 3| < 4$
5) $|x - 2| < 6$
6) $|x - 8| < 12$
7) $|x - 7| < 3$
8) $|x + 3| \leq 4$
9) $|3x - 2| < 9$
10) $|2x + 5| < 9$
11) $1 + 2|x - 1| \leq 9$
12) $10 - 3|x - 2| \geq 4$
13) $6 - |2x - 5| \geq 3$
14) $|x| > 5$
15) $|3x| > 5$
16) $|x - 4| > 5$
17) $|x - 3| \geq 3$
18) $|2x - 4| > 6$
19) $|3x - 5| \geq 3$
20) $3 - |2 - x| < 1$
21) $4 + 3|x - 1| \geq 10$
22) $3 - 2|3x - 1| \geq -7$
23) $3 - 2|x - 5| \leq -15$
24) $4 - 6|-6 - 3x| \leq -5$
25) $-2 - 3|4 - 2x| \geq -8$
26) $-3 - 2|4x - 5| \geq 1$
27) $4 - 5|-2x - 7| < -1$
28) $-2 + 3|5 - x| \leq 4$
29) $3 - 2|4x - 5| \geq 1$
30) $-2 - 3|-3x - 5| \geq -5$
31) $-5 - 2|3x - 6| < -8$
32) $6 - 3|1 - 4x| < -3$
33) $4 - 4|-2x + 6| > -4$
34) $-3 - 4|-2x - 5| \geq -7$
35) $|-10 + x| \geq 8$

Selected Answers

Solving Linear Equations

One-Step Equations

1) $v = 7$	9) $n = 18$	17) $n = 17$	25) $x = 15$	33) $x = 14$
5) $a = 10$	13) $n = -108$	21) $n = 3$	29) $r = 5$	37) $p = -240$

Two-Step Equations

1) $n = -4$	9) $x = -10$	17) $r = 7$	25) $k = 1$	33) $r = 8$
5) $n = 10$	13) $x = 4$	21) $n = 11$	29) $p = -6$	37) $v = -12$

General Linear Equations

1) $a = -3$	13) $m = 8$	25) $v = 8$	37) $n = -6$	49) $p = -9$
5) $x = 1$	17) $b = 2$	29) $a = -1$	41) $n = 0$	
9) $x = 0$	21) $m = 3$	33) $m = -3$	45) $x = 12$	

Equations Containing Fractions

1) $p = \frac{3}{4}$	9) $b = -2$	17) $n = 0$	25) $n = 16$
5) $m = -\frac{19}{6}$	13) $a = -\frac{3}{2}$	21) $b = \frac{1}{2}$	29) $x = \frac{4}{3}$

Equations Containing an Absolute Value

1) $x = \pm 8$	9) $x = -\frac{39}{7}, 3$	17) $x = 0, \frac{6}{7}$	25) $x = -2, 10$
5) $a = -\frac{29}{4}, 6$	13) $x = -9, 15$	21) $x = -8, -6$	29) $x = -\frac{13}{7}, 1$

Graphing Linear Equations

The Cartesian Plane

1) $\{A(5,-5),\ B(1,8),\ C(-2,-4),\ D(-8,0),\ E(6,3),$
$F(0,-7),\ G(4,6),\ H(-2,3),\ I(-8,7),\ J(-7,8)\}$

The Slope of a Line

1) $m = \frac{3}{2}$	9) $m = -\frac{17}{31}$	17) $m = -\frac{33}{6}$	25) $m = -\frac{7}{13}$	31) $y = -5$
5) $m = -\frac{1}{2}$	13) $m = 0$	21) $m = -\frac{26}{27}$	27) $x = -5$	35) $x = 2$

The Two Forms of a Linear Equation

Slope-Intercept Form

1) $y = 2x + 5$
3) $y = x - 4$
5) $y = -\frac{3}{4}x - 1$
7) $y = \frac{1}{3}x + 1$
9) $y = x - 1$
11) $y = -\frac{1}{10}x - \frac{37}{10}$
13) $y = -2x - 1$
15) $y = \frac{7}{3}x - 8$
17) $x = -8$ (m undefined)
19) $y = -x - 1$
21) $y = 4x$
23) $y = -4x + 3$
25) $y = -\frac{1}{2}x + 1$

Point-Slope Form

1) $x = 2$
5) $y + 5 = 9(x + 1)$
9) $y + 2 = -3(x - 0)$
13) $y + 3 = \frac{1}{5}(x + 5)$
17) $y = 2x - 3$
21) $y = \frac{1}{2}x + 3$
25) $y = -\frac{2}{5}x - 5$
29) $x = -3$
33) $y - 3 = -2(x + 4)$
37) $y + 2 = \frac{3}{2}(x + 4)$
41) $y + 3 = -\frac{8}{7}(x - 3)$
45) $y = -\frac{8}{7}x - \frac{5}{7}$
49) $y = -x + 2$

Parallel and Perpendicular Lines

1) $m = 2$
3) $m = 4$
5) $m = 1$
7) $m = -7$
9) $m = 0$
11) $m = 3$
13) $m = -3$
15) $m = 2$
17) $x = 2$

21) $y - 3 = \frac{7}{5}(x - 2)$
25) $y + 5 = -(x - 1)$
29) $y - 2 = -\frac{1}{4}(x - 4)$
33) $y = -2x + 5$
37) $y = -\frac{1}{2}x - 3$
41) $y = x - 1$
45) $y = -x + 3$

Linear Inequalities and Sign Diagrams

9) $(-\infty, 1)$
13) $(-\infty, 12)$
17) $(19, \infty)$
21) $[2, \infty)$
25) $(1, \infty)$
29) No Solution, $\varnothing$

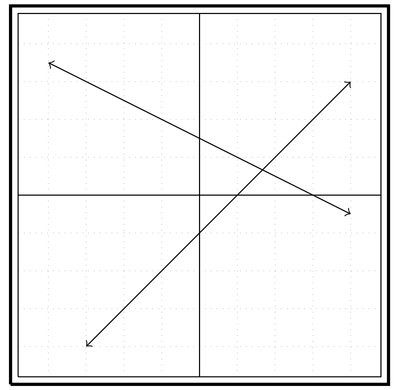
Compound and Absolute Value Inequalities

Compound Inequalities

1) $(-\infty, -9] \cup [2, \infty)$
5) $(-\infty, -7)$
9) $(-\infty, 5)$
13) $[-6, -2]$
17) $(-2, 2)$
21) $[-6, -3]$
25) $[5, 19)$
29) $(-\infty, -20) \cup [2, \infty)$

Inequalities Containing an Absolute Value

1) $(-3, 3)$
5) $(-4, 8)$
9) $(-\frac{7}{3}, \frac{11}{3})$
13) $[1, 4]$
15) $(-\infty, -\frac{5}{3}) \cup (\frac{5}{3}, \infty)$
17) $(-\infty, 0] \cup [6, \infty)$
19) $(-\infty, \frac{2}{3}] \cup [\frac{8}{3}, \infty)$
21) $(-\infty, -1] \cup [3, \infty)$
25) $[1, 3]$
29) $[1, \frac{3}{2}]$
33) $(2, 4)$



Chapter 2

Systems of Linear Equations

Introduction and Graphing (L9)

Objective: Solve a system of linear equations by graphing and identifying the point of intersection.

We have solved problems like $3x - 4 = 11$ by adding 4 to both sides and then dividing by 3 (solution is $x = 5$). We also have methods to solve equations with more than one variable in them. It turns out that to solve for more than one variable we will need the same number of equations as variables. For example, to solve for two variables such as x and y we will need two equations. When we have two (or more) equations we are working with, we call the set of equations a *system*. When solving a system of equations we are looking for a solution that satisfies each equation simultaneously. If our system consists of two equations in terms of x and y, this solution is usually described as an ordered pair (x, y). The following example demonstrates a solution for a system of two linear equations.

Example 77. Show (x, y)=(2,1) is the solution to the system

$$3x - y = 5 \qquad x + y = 3$$

$$\begin{aligned} (x, y) &= (2, 1) && \text{Identify } x \text{ and } y \text{ from the ordered pair} \\ x = 2,\ y &= 1 && \text{Plug these values into each equation} \\ \\ 3(2) - (1) &= 5 && \text{First equation} \\ 6 - 1 &= 5 && \text{Evaluate} \\ 5 &= 5 && \text{True} \\ \\ (2) + (1) &= 3 && \text{Second equation, evaluate} \\ 3 &= 3 && \text{True} \end{aligned}$$

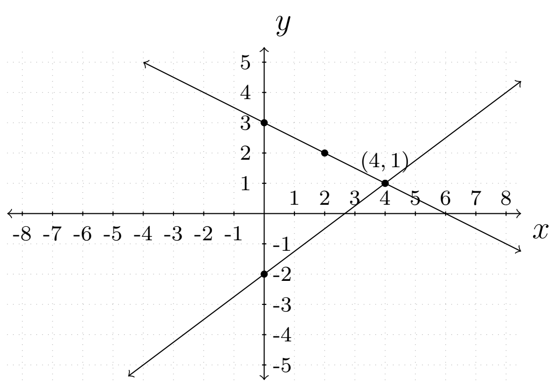
As we found a true statement for both equations we know $(2,1)$ is the solution to the system. It is in fact the only combination of numbers that works in both equations. In this section, we will attempt to identify a (simultaneous) solution to two equations, if such a solution exists. It stands to reason that if we use points to describe the solution, we can use graphs to find the solution.

If the graph of a line is a picture of all the solutions to its equation, we can graph two lines on the same coordinate plane to see the solutions of both equations. In particular, we are interested in finding all points that are a solution for both equations. This will be the point(s) where the two lines intersect! If we can find the intersection of the lines we have found the solution that works in both equations.

Example 78. Solve the following system of equations.

$$\begin{array}{rl} y = -\frac{1}{2}x + 3 & \\ y = \frac{3}{4}x - 2 & \end{array} \qquad \text{First identify slopes and } y - \text{intercepts}$$

$$\begin{array}{ll} \text{First Line}: & m = -\frac{1}{2}, \quad b = 3 \\ \text{Second Line}: & m = \frac{3}{4}, \quad b = -2 \end{array} \qquad \text{Next graph both lines on the same plane}$$



To graph each equation, we start at the y-intercept and use the slope to get the next point, then connect the dots.

Remember a line with a negative slope points downhill from left to right!

Find the intersection point, $(4,1)$. This is our solution.

Often our equations won't be in slope-intercept form and we will have to solve both equations for y first so we can identify the slope and $y-$intercept.

Example 79. Solve the following system of equations.

$$\begin{array}{r} 6x - 3y = -9 \\ 2x + 2y = -6 \end{array} \qquad \text{Solve each equation for } y$$

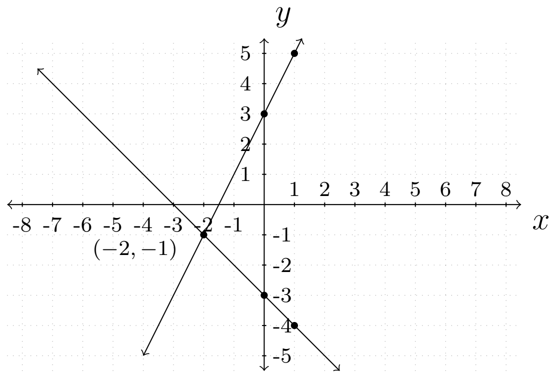
$$\begin{array}{rr} 6x - 3y = -9 & 2x + 2y = -6 \\ \underline{\mathbf{-6x \qquad - 6x}} & \underline{\mathbf{-2x \qquad - 2x}} \end{array} \qquad \text{Subtract } x \text{ terms}$$

$$\begin{array}{rr} \dfrac{-3y}{\mathbf{-3}} = \dfrac{-6x}{\mathbf{-3}} \dfrac{-9}{\mathbf{-3}} & \dfrac{2y}{\mathbf{2}} = \dfrac{-2x}{\mathbf{2}} \dfrac{-6}{\mathbf{2}} \end{array} \qquad \begin{array}{l}\text{Rearrange equations} \\ \text{Divide by coefficient of } y\end{array}$$

$y = 2x + 3$ $\quad y = -x - 3$ Identify slope and $y-$intercepts

First Line : $m = 2, \quad b = 3$
Second Line : $m = -1, \quad b = -3$

Next graph both lines on the same plane



To graph each equation, we start at the y-intercept and use the slope to get the next point, then connect the dots.

Remember a line with a negative slope decreases from left to right!

Using our slopes, we can find the intersection point, $(-2, -1)$. This is our solution.

As we are graphing our lines, it is possible to have one of two unexpected results. These are shown and discussed in the next two examples.

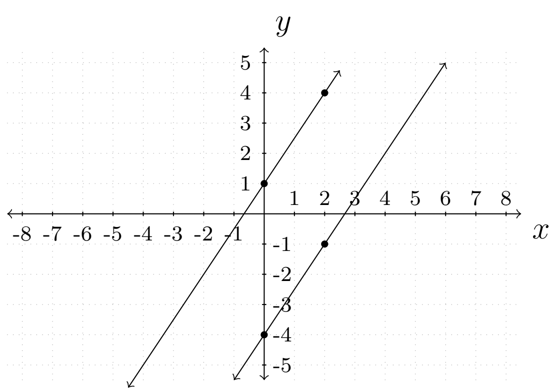
Example 80. Solve the following system of equations.

$$y = \frac{3}{2}x - 4 \qquad y = \frac{3}{2}x + 1$$

Identify the slope and $y-$intercept of each equation.

First Line : $m = \frac{3}{2}, \quad b = -4$
Second Line : $m = \frac{3}{2}, \quad b = 1$

Next graph both lines on the same plane



To graph each equation, we start at the y-intercept and use the slope to get the next point, then connect the dots.

The two lines do not intersect; they are parallel!

Since the lines do not intersect, we know that there is no point that will satisfy both equations.

There is no solution, or $\varnothing$.

Notice that we could also have recognized that both lines had the same slope. Remembering that parallel lines have the same slope one could conclude that there is no solution without having to graph the lines.

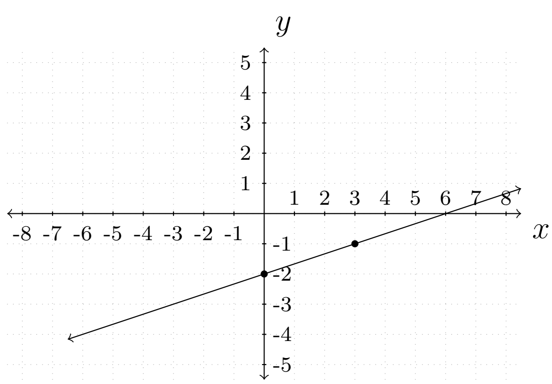
Example 81. Solve the following system of equations.

$$\begin{array}{l} 2x - 6y = 12 \\ 3x - 9y = 18 \end{array} \qquad \text{Solve each equation for } y$$

$$\begin{array}{rcl} 2x - 6y & = & 12 \\ \underline{\mathbf{-2x}} & & \underline{\mathbf{-\ 2x}} \end{array} \qquad \begin{array}{rcl} 3x - 9y & = & 18 \\ \underline{\mathbf{-3x}} & & \underline{\mathbf{-\ 3x}} \end{array} \qquad \text{Subtract } x \text{ terms}$$

$$\begin{array}{l} -6y = -2x + 12 \\ \overline{\mathbf{-6}} \quad \overline{\mathbf{-6}} \quad \overline{\mathbf{-6}} \\ y = \dfrac{1}{3}x - 2 \end{array} \qquad \begin{array}{l} -\ 9y = -3x + 18 \\ \overline{\mathbf{-9}} \quad \overline{\mathbf{-9}} \quad \overline{\mathbf{-9}} \\ y = \dfrac{1}{3}x - 2 \end{array} \qquad \begin{array}{l} \text{Put } x \text{ terms first} \\ \text{Divide by coefficient of } y \\ \text{Identify the slopes and } y - \text{intercepts} \end{array}$$

$$\begin{array}{ll} \text{First Line :} & m = \frac{1}{3}, \quad b = -2 \\ \text{Second Line :} & m = \frac{1}{3}, \quad b = -2 \end{array} \qquad \text{Next graph both lines on the same plane}$$



To graph each equation, we start at the y-intercept and use the slope to get the next point, then connect the dots.

Both equations are the same line!

As one line is directly on top of the other line, we can say that the lines intersect at every point on the line!

Here we say there are infinitely many solutions.

Notice that once we had both equations in slope-intercept form we could have recognized that the equations were the same. At this point one could state that there are infinitely many solutions without having to go through the work of graphing the equations.

The Substitution Method (L10)

Objective: Solve systems of equations using substitution.

Solving a system of equations by graphing has several limitations. First, it requires the graph to be perfectly drawn, if the lines are not straight we may arrive at the wrong answer. Second, graphing is not a great method to use if the answer is really large, over 100 for example, or a decimal, since a graph will not help us find an answer such as 3.2134. For these reasons we will rarely use graphs to solve a given system of equations. Instead, an algebraic approach will be used.

The first algebraic approach is called substitution. We will build the concepts of substitution through several examples, then end with a five-step process to solve problems using this method.

Example 82. Solve the following system of equations.

$$x = 5 \qquad\qquad y = 2x - 3$$

We already know x must equal 5, so we can substitute $x = 5$ into the other equation.

$$\begin{array}{ll} y = 2(\mathbf{5}) - 3 & \text{Evaluate: Multiply first} \\ y = 10 - 3 & \text{Next subtract} \\ y = 7 & \text{We now also have } y \\ (x, y) = (5, 7) & \text{Our solution} \end{array}$$

When we know what one variable equals we can plug that value (or expression) in for the variable in the other equation. It is very important that when we substitute, the substituted value goes in parentheses. The reason for this is shown in the next example.

Example 83. Solve the following system of equations.

$$2x - 3y = 7 \qquad\qquad y = 3x - 7$$

We begin by substituting $y = 3x - 7$ into the other equation.

$$\begin{array}{rl} 2x - 3(\mathbf{3x - 7}) = 7 & \text{Solve for } x\text{, distributing } -3 \text{ first} \\ 2x - 9x + 21 = 7 & \text{Combine like terms } 2x - 9x \\ -7x + 21 = 7 & \\ \underline{\mathbf{-21} \quad \mathbf{-21}} & \text{Subtract 21} \\ -7x = -14 & \\ \overline{\mathbf{-7}} \quad \overline{\mathbf{-7}} & \text{Divide by } -7 \\ x = 2 & \text{We now have our } x. \\ & \text{Substitute back in equation to find } y. \\ y = 3(\mathbf{2}) - 7 & \text{Evaluate: Multiply first} \\ y = 6 - 7 & \text{Next subtract} \\ y = -1 & \text{We now also have } y \\ (x, y) = (2, -1) & \text{Our solution} \end{array}$$

By using the entire expression $3x - 7$ to replace y in the other equation we were able to reduce the system to a single linear equation which we can easily solve for our first variable. However, the "lone" variable (a variable with a coefficient of 1) is not always alone on one side of the equation. If this happens we can isolate the lone variable by solving for it.

Example 84. Solve the following system of equations.

$$3x + 2y = 1 \qquad \boldsymbol{x} - 5y = 6$$

The lone variable is x. Isolate the lone variable by adding $5y$ to both sides.

$$\begin{array}{rl}
x - 5y = 6 & \\
\underline{\mathbf{+5}\boldsymbol{y} \quad \mathbf{+5}\boldsymbol{y}} & \\
x = 6 + 5y & \text{Substitute this into the untouched equation} \\
& \\
3(\mathbf{6 + 5}\boldsymbol{y}) + 2y = 1 & \text{Solve this equation, distributing 3 first} \\
18 + 15y + 2y = 1 & \text{Combine like terms } 15y + 2y \\
18 + 17y = 1 & \\
\underline{\mathbf{-18} \qquad \mathbf{-18}} & \text{Subtract 18 from both sides} \\
17y = -17 & \\
\overline{\mathbf{17}} \quad \overline{\mathbf{17}} & \text{Divide both sides by 17} \\
y = -1 & \text{We have our } y. \\
& \text{Substitute back in equation to find } x \\
x = 6 + 5(\mathbf{-1}) & \text{Evaluate: Multiply first, then subtract} \\
x = 1 & \text{We now also have } x \\
(x, y) = (1, -1) & \text{Our solution}
\end{array}$$

The process in the previous example is known as solving by substitution. This process is described and illustrated in the following table which lists the five steps to solving by substitution.

Problem	$4x - 2y = 2$ $2x + y = -5$
1. Find the lone variable.	Lone variable is y, in the second equation: $2x + \boldsymbol{y} = -5$
2. Solve for the lone variable.	Subtract $2x$ from both sides. $\boldsymbol{y = -5 - 2x}$
3. Substitute into the untouched equation.	$4x - 2(\boldsymbol{-5 - 2x}) = 2$
4. Solve.	$4x + 10 + 4x = 2$ $8x + 10 = 2$ $\underline{\mathbf{-10} \quad \mathbf{-10}}$ $8x = -8$ $\overline{\mathbf{8}} \quad \overline{\mathbf{8}}$ $\boldsymbol{x = -1}$
5. Plug into lone variable equation and evaluate.	$y = -5 - 2(\mathbf{-1})$ $y = -5 + 2$ $\boldsymbol{y = -3}$
Our solution	$(x, y) = (-1, -3)$

Sometimes we have several lone variables in a problem. In this case we will have the choice on which lone variable we wish to solve for, either will give the same final result.

Example 85. Solve the following system of equations.

$$x + y = 5 \qquad x - y = -1$$

Find the lone variable: either x or y in the first equation, or x in the second equation. We will choose x in the first equation.

$\boldsymbol{x} + y = 5$	Solve for the lone variable x
$\underline{\boldsymbol{-y} \quad \boldsymbol{-y}}$	Subtract y from both sides
$x = 5 - y$	Plug into the untouched equation, the second equation
$\boldsymbol{(5 - y)} - y = -1$	Combine like terms. Parentheses may be removed
$5 - 2y = -1$	
$\underline{\boldsymbol{-5} \qquad \boldsymbol{-5}}$	Subtract 5 from both sides
$-2y = -6$	
$\overline{\boldsymbol{-2}} \quad \overline{\boldsymbol{-2}}$	Divide both sides by -2
$y = 3$	We have our y!
$x = 5 - (3)$	Plug into lone variable equation and evaluate
$x = 2$	Now we have our x
$(x, y) = (2, 3)$	Our solution

Just as with graphing it is possible to have no solution $\varnothing$ (parallel lines) or infinite solutions (same line) with the substitution method. While we won't have a parallel line or the same line to look at and conclude if it is one or the other, the process takes an interesting turn as shown in the following example.

Example 86. Solve the following system of equations.

$$y + 4 = 3x \qquad 2y - 6x = -8$$

Find the lone variable: y in the first equation.

$\boldsymbol{y} + 4 = 3x$	Solve for the lone variable y
$\underline{\boldsymbol{-4} \quad \boldsymbol{-4}}$	Subtract 4 from both sides
$y = 3x - 4$	Plug into second equation
$2\boldsymbol{(3x - 4)} - 6x = -8$	Solve, distribute through parentheses
$6x - 8 - 6x = -8$	Combine like terms $6x - 6x$
$-8 = -8$	Variables are gone!

Since we are left with a true statement ($-8 = -8$), we conclude that there are infinitely many solutions.

Because we had a true statement, and no variables, we know that anything that works in the first equation, will also work in the second equation. However, we do not always end up with a true statement.

Example 87. Solve the following system of equations.

$$6x - 3y = -9 \qquad -2x + y = 5$$

Find the lone variable: y in the second equation.

$$\begin{array}{rl}
-2x + \boldsymbol{y} = 5 & \text{Solve for the lone variable} \\
\underline{+\boldsymbol{2x} \qquad + \boldsymbol{2x}} & \text{Add } 2x \text{ to both sides} \\
y = 5 + 2x & \text{Plug into untouched equation} \\
6x - 3(\boldsymbol{5 + 2x}) = -9 & \text{Solve, distribute through parentheses} \\
6x - 15 - 6x = -9 & \text{Combine like terms } 6x - 6x \\
-15 \neq -9 & \text{Variables are gone!}
\end{array}$$

Since we are left with a false statement ($-15 \neq -9$) and no variables, we know that nothing will work in both equations and we may conclude that there are no solutions, or $\varnothing$.

One more question needs to be considered: what if there is no lone variable? If there is no lone variable substitution can still work, we will just have to select one variable to solve for, and introduce fractions.

Example 88. Solve the following system of equations.

$$5x - 6y = -14 \qquad -2x + 4y = 12$$

There is no lone variable, so we will solve the first equation for x.

$$\begin{array}{rl}
5\boldsymbol{x} - 6y = -14 & \text{Solve for our variable } x \\
\underline{+\boldsymbol{6y} \quad + \boldsymbol{6y}} & \text{Add } 6y \text{ to both sides} \\
5x = -14 + 6y & \\
\overline{\boldsymbol{5}} \qquad \overline{\boldsymbol{5}} \quad \overline{\boldsymbol{5}} & \text{Divide each term by } 5 \\
x = \dfrac{-14}{5} + \dfrac{6y}{5} & \text{Plug into untouched equation} \\
-2\left(\dfrac{-14}{5} + \dfrac{6y}{5}\right) + 4y = 12 & \text{Solve, distribute through parentheses} \\
\dfrac{28}{5} - \dfrac{12y}{5} + 4y = 12 & \text{Clear fractions by multiplying by } 5 \\
\dfrac{28(\boldsymbol{5})}{5} - \dfrac{12y(\boldsymbol{5})}{5} + 4y(\boldsymbol{5}) = 12(\boldsymbol{5}) & \text{Reduce fractions and multiply} \\
28 - 12y + 20y = 60 & \text{Combine like terms } -12y + 20y \\
28 + 8y = 60 & \\
\underline{-\boldsymbol{28} \qquad - \boldsymbol{28}} & \text{Subtract } 28 \text{ from both sides} \\
8y = 32 & \\
\overline{\boldsymbol{8}} \quad \overline{\boldsymbol{8}} & \text{Divide both sides by } 8 \\
y = 4 & \text{We have our } y
\end{array}$$

$$\begin{array}{ll}
x = -\dfrac{14}{5} + \dfrac{6(4)}{5} & \text{Plug into lone variable equation, multiply} \\
x = -\dfrac{14}{5} + \dfrac{24}{5} & \text{Add fractions} \\
x = \dfrac{10}{5} & \text{Reduce fraction} \\
x = 2 & \text{Now we have our } x \\
(x, y) = (2, 4) & \text{Our solution}
\end{array}$$

Using the fractions does make the problem a bit trickier. This is why we have yet another method for solving systems of equations that will be discussed in the next section.

The Addition Elimination Method (L11)

Objective: Solve systems of equations using the addition/elimination method.

When solving systems we have found that graphing is very limited when solving equations. We then considered a second method known as substitution. This is probably the most used idea in solving systems in various areas of algebra. However, substitution can get ugly if we don't have a lone variable. This leads us to our second method for solving systems of equations. This method is known as either Elimination or Addition. We will set up the process in the following examples, then define the five step process we can use to solve by elimination.

Example 89. Solve the following system of equations.

$$x - 4y = 8 \qquad 5x + 4y = -24$$

$$\begin{array}{rl}
3x - 4y = 8 & \text{Notice opposite signs in front of } y \\
\underline{+ \quad 5x + 4y = -24} & \text{Add columns to eliminate } y \\
8x \qquad = -16 & \text{Solve for } x \\
\overline{\mathbf{8}} \qquad\quad \overline{\mathbf{8}} & \text{Divide by } 8 \\
x = -2 & \text{We have our } x! \\
5(-2) + 4y = -24 & \text{Plug into either original equation} \\
-10 + 4y = -24 & \text{Simplify} \\
\underline{+\mathbf{10} \qquad\quad + \mathbf{10}} & \text{Add } 10 \text{ to both sides} \\
4y = -14 & \\
\overline{\mathbf{4}} \qquad \overline{\mathbf{4}} & \text{Divide by } 4 \\
y = -\dfrac{7}{2} & \text{Now we have our } y! \\
(x, y) = \left(-2, -\dfrac{7}{2}\right) & \text{Our solution}
\end{array}$$

In the previous example one variable had opposites in front of it, $-4y$ and $4y$. Adding these together eliminated the y completely. This allowed us to solve for the x. This is the idea behind the addition method. However, generally we won't have opposites in front of one of the variables. In this case we will manipulate the equations to get the opposites we want by multiplying one or both equations (on both sides!). This is shown in the next example.

Example 90. Solve the following system of equations.

$$-6x + 5y = 22 \qquad 2x + 3y = 2$$

Notice that we can obtain "opposite" coefficients (one positive and one negative) in front of x by multiplying both sides of the second equation by 3.

$3(2x + 3y) = (2)3$	Distribute to get new second equation
$6x + 9y = \ 6$	New second equation
$\underline{-6x + 5y = 22}$	Add equations to eliminate x
$14y = 28$	
$\overline{\mathbf{14}} \quad \overline{\mathbf{14}}$	Divide both sides by 14
$y = 2$	We have our y!
$2x + 3(2) = 2$	Plug into one of the original equations
$2x + 6 = 2$	Simplify
$\underline{\mathbf{-6} \ \ \mathbf{-6}}$	Subtract 6 from both sides
$2x = -4$	
$\overline{\mathbf{2}} \quad \overline{\mathbf{2}}$	Divide both sides by 2
$x = -2$	We also have our x!
$(x, y) = (-2, 2)$	Our solution

When we looked at the x terms, $-6x$ and $2x$ we decided to multiply the $2x$ by 3 to get the opposites we were looking for. What we are looking for with our opposites is the least common multiple (LCM) of the coefficients. We also could have solved the above problem by looking at the terms with y, $5y$ and $3y$. The LCM of 3 and 5 is 15. So we would want to multiply both equations, the $5y$ by 3, and the $3y$ by-5 to get opposites, $15y$ and $-15y$. This illustrates an important point: for some problems we will have to multiply both equations by a constant (on both sides) to get the opposites we are looking for.

Example 91. Solve the following system of equations.

$$3x + 6y = -9 \qquad 2x + 9y = -26$$

Here, we can obtain opposite coefficients in front of y by finding the least common multiple (LCM) of 6 and 9, which is 18. We will therefore multiply both sides of both equations by the appropriate values to get $18y$ and $-18y$.

$$\begin{array}{rl} 3(3x+6y) = (-9)3 & \text{Multiply the first equation by 3} \\ 9x+18y = -27 & \end{array}$$

$$\begin{array}{rl} -2(2x+9y) = (-26)(-2) & \text{Multiply the second equation by } -2 \\ -4x-18y = 52 & \end{array}$$

$$\begin{array}{rl} 9x+18y = -27 & \text{Add two new equations together} \\ \underline{-4x-18y = 52} & \quad \text{to eliminate } y \\ 5x = 25 & \\ \overline{\mathbf{5}} \quad \overline{\mathbf{5}} & \text{Divide both sides by } 5 \\ x = 5 & \text{We have our solution for } x \end{array}$$

$$\begin{array}{rl} 3(5)+6y = -9 & \text{Plug into either original equation} \\ 15+6y = -9 & \text{Simplify} \\ \underline{\mathbf{-15} \qquad \mathbf{-15}} & \text{Subtract } 15 \text{ from both sides} \\ 6y = -24 & \\ \overline{\mathbf{6}} \quad \overline{\mathbf{6}} & \text{Divide both sides by } 6 \\ y = -4 & \text{Now we have our solution for } y \\ (x,y) = (5,-4) & \text{Our solution} \end{array}$$

As we get started, it is important for each problem that all variables and constants are aligned before we begin multiplying and adding equations. This is illustrated in the next example which includes the five steps we will go through to solve a problem using elimination.

Problem	$2x - 5y = -13$ $-3y + 4 = -5x$
1. Line up the variables and constants.	Rearrange the second equation $2x - 5y = -13$ $5x - 3y = -4$
2. Multiply to get opposites (use LCM).	First Equation : multiply by -5 $-5(2x - 5y) = (-13)(-5)$ $-10x + 25y = 65$ Second Equation : multiply by 2 $2(5x - 3y) = (-4)2$ $10x - 6y = -8$
3. Add equations to eliminate a variable.	$-10x + 25y = 65$ $\underline{10x - 6y = -8}$ $19y = 57$
4. Solve.	$\frac{19y}{\mathbf{19}} = \frac{57}{\mathbf{19}}$ $y = 3$
5. Plug back into either of the given equations and solve.	$2x - 5(3) = -13$ $2x - 15 = -13$ $\underline{\mathbf{+15} \quad \mathbf{+15}}$ $\frac{2x}{\mathbf{2}} = \frac{2}{\mathbf{2}}$ $x = 1$
Solution	$(x, y) = (1, 3)$

Just as with graphing and substitution, it is possible to have no solution or infinitely many solutions with elimination. If the variables all disappear from our problem, a true statement will always indicate infinitely many solutions and a false statement will always indicate no solutions.

Example 92. Solve the following system of equations.

$$2x - 5y = 3 \qquad -6x + 15y = -9$$

In order to obtain opposite coefficients in front of x, multiply the first equation by 3.

$$\begin{array}{rl} 3(2x - 5y) = (3)3 & \\ 6x - 15y = 9 & \text{Distribute} \end{array}$$

$$\begin{array}{rl} 6x - 15y = \ \ 9 & \\ \underline{-6x + 15y = -9} & \text{Add equations together} \\ 0 = 0 & \text{True statement} \end{array}$$

Since we are left with a true statement, we conclude that there are infinitely many solutions.

Example 93. Solve the following system of equations.

$$4x - 6y = 8 \qquad 6x - 9y = 15$$

Here, we will seek to obtain opposite coefficients for x. This means we must find the LCM of 4 and 6, which is 12. We will multiply both sides of both equations by the appropriate values in order to get $12x$ and $-12x$.

$$\begin{aligned} 3(4x - 6y) &= (8)3 && \text{Multiply first equation by } 3 \\ 12x - 18y &= 24 \end{aligned}$$

$$\begin{aligned} -2(6x - 9y) &= (15)(-2) && \text{Multiply second equation by } -2 \\ -12x + 18y &= -30 \end{aligned}$$

$$\begin{array}{rl} 12x - 18y = 24 & \\ \underline{-12x + 18y = -30} & \text{Add both new equations together} \\ 0 \neq -6 & \text{False statement} \end{array}$$

Since we are left with a false statement, we conclude that there are no solutions, or $\varnothing$.

We have now covered three different methods that can be used to solve a system of two equations with two variables: graphing, substitution, and addition/elimination. While all three can be used to solve any system, graphing works well for small integer solutions. Substitution works best when we have a lone variable, and addition/elimination works best when the other two methods fail. As each method has its own strengths, it is important that students become familiar with all three methods.

Three Variables

Objective: Solve systems of equations with three variables.

Recall that the graph of an equation containing two variables is a (two-dimensional) line. If we increase the number of variables in an equation to three, then the resulting graph will be a three-dimensional plane. This particular section deals with solving a system of equations containing three variables. Whereas the solution for a system of *two* equations is the set of points where their respective *lines* intersect, the solution for a system of *three* equations will be the set of points where all three respective *planes* intersect. Although we do not intend to undertake the arduous task of graphing even a single equation containing three variables in this setting, the visual is sometimes helpful in justifying a particular outcome, and is often critical to understanding in more advanced mathematics courses such as multivariate calculus and linear algebra.

The method for solving a system of equations with three (or more) variables is very similar to that for solving a system with two variables. When we had two variables we reduced the system down to one equation with one variable (by either substitution or addition/elimination).

With three variables we will reduce the system down to one equation with two variables (usually by addition/elimination), which we can then solve by either substitution or addition/elimination.

To reduce from three variables down to two it is very important to keep the work organized by lining up the variables vertically and using enough space to carefully keep track of everything. We will use addition/elimination with two equations to eliminate one variable. This new equation we will call (A). Then we will use a different pair of equations and use addition/elimination to eliminate the *same* variable. This second new equation we will call (B). Once we have done this we will have a system of two equations, (A) and (B), with the same two variables that we can solve using either method, substitution or elimination, depending on the context of the problem. This is demonstrated in the following examples.

Example 94. Solve the following system of equations.

$$\begin{aligned} 3x + 2y - z &= -1 \\ -2x - 2y + 3z &= 5 \\ 5x + 2y - z &= 3 \end{aligned}$$

Our strategy will be to first eliminate y using two different pairs of equations from those provided.

$$\begin{array}{rl} 3x + 2y - z = -1 & \text{Using the first two equations,} \\ \underline{-2x - 2y + 3z = \ \ 5} & \text{Add} \\ x + 2z = \ \ 4 & \text{Call this equation (A)} \end{array}$$

$$\begin{array}{rl} -2x - 2y + 3z = 5 & \text{Using the second two equations} \\ \underline{5x + 2y - z = 3} & \text{Add} \\ 3x + 2z = 8 & \text{Call this equation (B)} \end{array}$$

$$\begin{array}{rl} x + 2z = 4 & \text{Equation (A)} \\ 3x + 2z = 8 & \text{Equation (B)} \end{array}$$

$$\begin{array}{ll} -1(x + 2z) = (4)(-1) & \text{Multiply equation (A) by } -1 \\ -x - 2z = -4 & \text{Simplify} \end{array}$$

$$\begin{array}{rl} -x - 2z = -4 & \\ \underline{3x + 2z = \ \ 8} & \text{Add the two equations} \\ 2x = 4 & \\ \overline{\mathbf{2}} \quad \overline{\mathbf{2}} & \text{Divide by } 2 \\ x = 2 & \text{We now have } x! \\ & \text{Plug } x \text{ into either (A) or (B)} \end{array}$$

$(2) + 2z = 4$	We will use (A)
$\underline{\mathbf{-2} \qquad \mathbf{-2}}$	Subtract 2
$2z = 2$	
$\overline{\mathbf{2}} \quad \overline{\mathbf{2}}$	Divide by 2
$z = 1$	We now have z!
	Plug x and z into any of the original equations
$3(2) + 2y - (1) = -1$	We will use the first equation
	Simplify; reduce and combine constant terms
$2y + 5 = -1$	Solve for y
$\underline{\mathbf{-5} \quad \mathbf{-5}}$	Subtract 5
$2y = -6$	
$\overline{\mathbf{2}} \qquad \overline{\mathbf{2}}$	Divide by 2
$y = -3$	We now have y!
$(x, y, z) = (2, -3, 1)$	Our solution

As we are solving for $x, y,$ and z we will have an ordered triplet (x, y, z) instead of just the ordered pair (x, y). In the previous problem, y was easily eliminated using the addition method. Sometimes, however, we may have to do a bit of work to eliminate a variable. Just as with the addition of two equations, we may have to multiply the equations by a constant on both sides in order to get the opposites we want and eliminate the variable. As we do this, remember that it is important to eliminate the *same variable each time*, using two *different* pairs of equations.

Example 95. Solve the following system of equations.

$$\begin{aligned} 4x - 3y + 2z &= -29 \\ 6x + 2y - z &= -16 \\ -8x - y + 3z &= 23 \end{aligned}$$

Notice that no variable will easily eliminate. Although we are free to choose any variable to eliminate, we will choose x here. Remember, we will be eliminating x *twice*, using two different equations each time.

$4x - 3y + 2z = -29$	Begin with the first two equations
$6x + 2y - z = -16$	The LCM of 4 and 6 is 12

We will multiply both sides of the first equation by 3 to obtain $12x$. Similarly, we will multiply both sides of the second equation by -2 to obtain $-12x$.

$$\begin{aligned} 3(4x-3y+2z) &= (-29)3 && \text{Multiply the first equation by } 3\\ 12x-9y+6z &= -87 \end{aligned}$$

$$\begin{aligned} -2(6x+2y-z) &= (-16)(-2) && \text{Multiply the second equation by } -2\\ -12x-4y+2z &= 32 \end{aligned}$$

$$\begin{array}{rl} 12x-9y+6z = -87 & \\ \underline{-12x-4y+2z = \ \ 32} & \text{Add these two equations together}\\ -13y+8z = -55 & \text{Call this equation (A)} \end{array}$$

Next, we will use a different pair of equations.

$$\begin{array}{rl} 6x+2y-z = -16 & \text{Now use the second pair of equations}\\ -8x-y+3z = \ \ 23 & \text{The LCM of 6 and } -8 \text{ is } 24 \end{array}$$

Now, we will multiply both sides of the first equation by 4 to obtain $24x$, and both sides of the second equation by 3 to obtain $-24x$.

$$\begin{aligned} 4(6x+2y-z) &= (-16)4 && \text{Multiply the first equation by } 4\\ 24x+8y-4z &= -64 \end{aligned}$$

$$\begin{aligned} 3(-8x-y+3z) &= (23)3 && \text{Multiply the second equation by } 3\\ -24x-3y+9z &= 69 \end{aligned}$$

$$\begin{array}{rl} 24x+8y-4z = -64 & \\ \underline{-24x-3y+9z = \ \ 69} & \text{Add these two equations together}\\ 5y+5z = \ \ 5 & \text{Call this equation (B)} \end{array}$$

Now, using equations (A) and (B), we will solve the given system.

$$\begin{array}{rl} -13y+8z = -55 & \text{Equation (A)}\\ 5y+5z = \ \ 5 & \text{Equation (B)} \end{array}$$

$$\begin{array}{rl} 5y+5z = 5 & \text{Solve equation (B) for } z\\ \underline{\boldsymbol{-5y} \qquad \boldsymbol{-5y}} & \text{Subtract } 5y\\ 5z = 5-5y & \\ \overline{\mathbf{5}} \quad \overline{\mathbf{5}} \quad \overline{\mathbf{5}} & \text{Divide both sides by } 5\\ z = 1-y & \text{Equation for } z \end{array}$$

Next, substitute z into equation (A).

$$\begin{array}{rl} -13y+8\boldsymbol{(1-y)} = -55 & \text{Simplify}\\ -13y+8-8y = -55 & \text{Distribute} \end{array}$$

$$\begin{array}{rcl} -21y+8=-55 & & \text{Combine like terms} \\ \underline{\mathbf{-8}\quad\;\; \mathbf{-\ 8}} & & \text{Subtract } 8 \\ -21y=-63 & & \\ \overline{\mathbf{-21}}\quad\; \overline{\mathbf{-21}} & & \text{Divide by } -21 \\ y=3 & & \text{We have our } y! \end{array}$$

Now plug y into the equation for z.

$$\begin{array}{ll} z=1-(3) & \text{Evaluate} \\ z=-2 & \text{We have } z! \end{array}$$

Now, we can find x from one of our original equations. We will use the first equation.

$$\begin{array}{rl} 4x-3(3)+2(-2)=-29 & \text{Simplify} \\ 4x-13=-29 & \text{Combine like terms} \\ \underline{\mathbf{+13}\quad \mathbf{+\ 13}} & \text{Add } 13 \\ 4x=-16 & \\ \overline{\mathbf{4}}\qquad\; \overline{\mathbf{4}} & \text{Divide by } 4 \\ x=-4 & \text{We have our } x! \\ (x,y,z)=(-4,3,-2) & \text{Our solution} \end{array}$$

Just as with two variables and two equations, we can have special cases come up with three variables and three equations. Specifically, it is possible to encounter a system of equations that has infinitely many solutions, or none at all. The way we handle such systems is identical to that for a system containing only two equations/variables.

Example 96. Solve the following system of equations.

$$\begin{array}{r} 5x-4y+3z=-4 \\ -10x+8y-6z=\ 8 \\ 15x-12y+9z=-12 \end{array}$$

Again, we will choose to eliminate x.

$$\begin{array}{rl} 5x-4y+3z=-4 & \text{Begin with the first two equations} \\ -10x+8y-6z=\ 8 & \text{The LCM of 5 and } -10 \text{ is } 10 \end{array}$$

We will multiply both sides of the first equation by 2 to obtain $10x$. Since the second equation contains $-10x$, we do not need to multiply it by a constant.

$$\begin{array}{ll} 2(5x-4y+3z)=-4(2) & \text{Multiply the first equation by } 2 \\ 10x-8y+6z=-8 & \end{array}$$

$$\begin{array}{rl} 10x-8y+6z=-8 & \\ \underline{-10x+8y-6=\ 8} & \text{Add the two equations} \\ 0=0 & \text{A true statement} \end{array}$$

Since we are left with a true statement, we conclude that there are infinitely many solutions to the first two equations.

Remember, that our usual procedure requires us to eliminate a variable (x in this case) *twice*, using two different equations each time. Even though we have concluded that there are infinitely many simultaneous solutions to the first two equations, we still must consider two different equations. In this particular example, we will obtain the same outcome by choosing *any* two equations, and it is left as an exercise for the reader to show this.

Hint: What do you notice about the set of coefficients for each equation, in relation to each of the other two equations? Do you think our results are related to this?

Once we have eliminated the same variable *twice* and drawn the same conclusions as above, we can conclude that there are infinitely many simultaneous solutions (x, y, z) to *all three* equations, i.e., the entire system.

There are, in fact, cases where two equations will share infinitely many solutions, but the entire system of equations might *fail* to have any simultaneous solutions. This is why it is critical that we not rush to an incorrect conclusion. These more subtle cases will usually be treated in detail in a multivariate calculus or a linear algebra course.

Our last example demonstrates the only time when it is permissible to eliminate a variable from two equations in our system once.

Example 97. Solve the following system of equations.

$$\begin{aligned} 3x - 4y + z &= 2 \\ -9x + 12y - 3z &= -5 \\ 4x - 2y - z &= 3 \end{aligned}$$

Here, it will be slightly easier to eliminate z.

$$\begin{aligned} 3x - 4y + z &= 2 && \text{Begin with the first two equations} \\ -9x + 12y - 3z &= -5 && \text{The LCM of 1 and } -3 \text{ is } 3 \end{aligned}$$

We will multiply both sides of the first equation by 3 to obtain $3z$. Since the second equation contains $-3z$, we do not need to multiply it by a constant.

$$\begin{aligned} 3(3x - 4y + z) &= (2)3 && \text{Multiply the first equation by } 3 \\ 9x - 12y + 3z &= 6 \end{aligned}$$

$$\begin{array}{rl} 9x - 12y + 3z = 6 & \\ \underline{-9x + 12y - 3z = -5} & \text{Add the two equations} \\ 0 \neq 1 & \text{A false statement} \end{array}$$

Since we are left with a false statement, we conclude that there are no solutions to the given system.

Again, our usual procedure requires us to eliminate a variable (z in this case) *twice*, using two different equations each time. In this particular case, however, we need only eliminate the variable once. Since we obtained a false statement, which implies that there can be no solution to the first *two* equations in the system, it will be impossible to obtain a simultaneous solution to *all three* equations.

Equations with three (or more) variables are no more difficult to attempt to solve than those containing two variables, if we are careful to keep our information organized and eliminate the same variable twice, each time using two different pairs of equations. As with many problems, it is possible to solve each system several different ways. We can use different pairs of equations or eliminate variables in different orders. But as long as our information is organized and our algebra is correct, we should always arrive at the same conclusion.

Matrices

Objective: Represent a system of linear equations as an augmented matrix. Solve a system of linear equations using matrix row reduction.

In this section, we will solve systems of linear equations using matrices and row operations. The first step will be to represent a system as an augmented matrix, as in the following example.

Example 98.

System

$$\begin{aligned} x - 2y + z &= 7 \\ 3x - 5y + z &= 14 \\ 2x - 2y - z &= 3 \end{aligned}$$

Augmented Matrix

$$\left[\begin{array}{rrr|l} 1 & -2 & 1 & 7 \\ 3 & -5 & 1 & 14 \\ 2 & -2 & -1 & 3 \end{array}\right]$$

In our example the entries in the first three columns of the matrix are given by the coefficients of each of the variables in their corresponding equations; the first column contains the coefficients of x, the second column contains the coefficients of y, and the third the coefficients of z. The last column of the matrix will always contain the constant term from each equation, and is separated from the coefficient columns by a vertical line. Each row of the matrix should also match its respective equation in the ordered system.

The following row operations may be used to reduce an augmented matrix.

1. Interchange two rows.
2. Multiply all entries of a row by a nonzero constant.
3. Add one row to another row.

Furthermore, multiple row operations may be used in combination, as our first example will demonstrate.

Initially, our goal will be to transform (or reduce) the given augmented matrix using the row operations specified above into a matrix in *triangular form*. A matrix obtained from our original matrix that is in triangular form will have a solution that equals the solution for our original matrix, but which will be easier to identify.

We will now use the specified row reduction operations to transform our given matrix to a matrix in triangular form.

Example 99.

Original Matrix
$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 7 \\ 3 & -5 & 1 & 14 \\ 2 & -2 & -1 & 3 \end{array}\right]$$

Multiply Row 1 by -3 and
add to Row 2 (replacing Row 2)
Symbolic: R2+(-3)R1 $\Rightarrow$ R2
$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 7 \\ 0 & 1 & -2 & -7 \\ 2 & -2 & -1 & 3 \end{array}\right]$$

Multiply Row 1 by -2 and
add to Row 3
Symbolic: R3+(-2)R1 $\Rightarrow$ R3
$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 7 \\ 0 & 1 & -2 & -7 \\ 0 & 2 & -3 & -11 \end{array}\right]$$

Multiply Row 2 by -2 and
add to Row 3
Symbolic: R3+(-2)R2 $\Rightarrow$ R3
$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 7 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

The new matrix is now in triangular form, with resulting system of equations listed below.

$$\begin{aligned} x - 2y + z &= 7 \\ y - 2z &= -7 \\ z &= 3 \end{aligned}$$

At this point, we can easily solve the new system by first substituting $z = 3$ into the second equation to find y, and then substituting both known values for z and y into the first equation to find x. This results in the following solution, which the reader can easily verify.

$$(x, y, z) = (2, -1, 3)$$

It is also worth mentioning that, just as with every problem we have encountered, it is straightforward to check whether a particular answer is correct. In our previous example, this will amount to plugging $(x, y, z) = (2, -1, 3)$ into each equation and simplifying. Although this can be a tedious process, it is important to do every so often, in order to ensure accuracy. In the previous example, we see below that the answer checks out.

$$\begin{array}{rcccccc} x - 2y + z = 7 & = & 2 - 2(-1) + 3 & = & 2 + 2 + 3 & = & 7 \\ 3x - 5y + z = 14 & = & 3(2) - 5(-1) + 3 & = & 6 + 5 + 3 & = & 14 \\ 2x - 2y - z = 3 & = & 2(2) - 2(-1) - 3 & = & 4 + 2 - 3 & = & 3 \end{array}$$

The last matrix obtained in the previous example is said to be in *row echelon form.* A matrix is in row echelon form if the following conditions are satisfied.

1. Any row consisting entirely of zeros (if any exist) is listed at the bottom of the matrix.
2. The first coefficient entry of any nonzero row (i.e., a row that does not consist entirely of zeros) is 1. We will call such an entry a "leading one".
3. The leading ones indent. In other words, the column number for the leading ones increases from left to right as the row numbers increase from top to bottom.

In fact, if we continue to apply the permissible row operations to the row echelon form of a matrix, we can obtain a matrix in which all the columns that contain a leading one will have zeros elsewhere. This particular type of matrix is known as the *reduced row echelon form* of a matrix.

Continuing with our previous example, we will obtain the reduced row echelon form for our original augmented matrix.

Example 100.

Row Echelon Form
$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 7 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Multiply Row 2 by 2 and
add to Row 1
Symbolic: R1+(2)R2 $\Rightarrow$ R1
$$\left[\begin{array}{rrr|r} 1 & 0 & -3 & -7 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Multiply Row 3 by 2 and
add to Row 2
Symbolic: R2+(2)R3 $\Rightarrow$ R2
$$\left[\begin{array}{rrr|r} 1 & 0 & -3 & -7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Multiply Row 3 by 3 and
add to Row 1
Symbolic: R1+(3)R3 $\Rightarrow$ R1
$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Our resulting equations are shown below.

$$\begin{aligned} x &= 2 \\ y &= -1 \\ z &= 3 \end{aligned}$$

Consequently, no additional work is needed to obtain our solution!

This example helps demonstrate the benefit to solving a given system of equations by row reducing its corresponding augmented matrix. And, although the row echelon form was certainly helpful in completing our task, by continuing our row reduction to obtain the *reduced* row echelon form of the matrix we completely eliminated the requirement to directly solve any equations.

This is because the applied row operations have done the work of solving the equations for us. In fact, throughout our reduction process, it would not be difficult for us to "translate" each step into an application of the addition/elimination procedure learned earlier in the chapter. So, although row reducing an augmented matrix may appear somewhat as 'mathematical magic', it is nothing more than a prescribed arithmetic manipulation of coefficients and constants to achieve a solution to a system of equations.

We continue with our next example.

Example 101.

System	Augmented Matrix
$\begin{aligned} x+y+z&=3\\ 2x+y+4z&=8\\ x+2y-z&=1 \end{aligned}$	$\left[\begin{array}{ccc\|c} 1&1&1&3\\ 2&1&4&8\\ 1&2&-1&1 \end{array}\right]$
Multiply Row 1 by -2 and add to Row 2 Symbolic: R2+(-2)R1 $\Rightarrow$ R2	$\left[\begin{array}{ccc\|c} 1&1&1&3\\ 0&-1&2&2\\ 1&2&-1&1 \end{array}\right]$
Multiply Row 1 by -1 and add to Row 3 Symbolic: R3+(-1)R1 $\Rightarrow$ R3	$\left[\begin{array}{ccc\|c} 1&1&1&3\\ 0&-1&2&2\\ 0&1&-2&-2 \end{array}\right]$
Add Row 2 to Row 3 Symbolic: R3+R2 $\Rightarrow$ R3	$\left[\begin{array}{ccc\|c} 1&1&1&3\\ 0&-1&2&2\\ 0&0&0&0 \end{array}\right]$
Add Row 2 to Row 1 Symbolic: R1+R2 $\Rightarrow$ R1	$\left[\begin{array}{ccc\|c} 1&0&3&5\\ 0&-1&2&2\\ 0&0&0&0 \end{array}\right]$
Multiply Row 2 by -1 Symbolic: (-1)R2 $\Rightarrow$ R2	$\left[\begin{array}{ccc\|c} 1&0&3&5\\ 0&1&-2&-2\\ 0&0&0&0 \end{array}\right]$

Our last matrix is in reduced row echelon form, since the row containing all zeros occurs at the bottom and the two columns that contain leading ones also contain zeros elsewhere. The resulting system of equations is shown below.

$$\begin{aligned} x + 3z &= \ \ 5 \\ y - 2z &= -2 \\ 0 &= \ \ 0 \end{aligned}$$

The last equation in our system ($0 = 0$) above can be interpreted to mean that the variable z in this example is an *independent variable*. In other words, we are free to choose any real number for z (since $0 = 0$ is a true statement). On the other hand, the variables x and y in this case are *dependent variables*, since they depend on the choice of z. Specifically, solving for both x and y, we get $x = 5 - 3z$ and $y = -2 + 2z$. Since we are free to choose any value for z, we may conclude that there are infinitely many solutions to the given system of equations. Moreover, a solution to the given system must be of the following form.

$$(x, y, z) = (5 - 3z, -2 + 2z, z)$$

Furthermore, we may once again check that our solution makes sense by plugging it back into the original system.

$$\begin{aligned} x + y + z &= (5 - 3z) + (-2 + 2z) + z \\ &= 5 - 3z - 2 + 2z + z \\ &= (5 - 2) + (-3z + 2z + z) \\ &= 3 \end{aligned}$$

$$\begin{aligned} 2x + y + 4z &= 2(5 - 3z) + (-2 + 2z) + 4z \\ &= 10 - 6z - 2 + 2z + 4z \\ &= (10 - 2) + (-6z + 2z + 4z) \\ &= 8 \end{aligned}$$

$$\begin{aligned} x + 2y - z &= (5 - 3z) + 2(-2 + 2z) - z \\ &= 5 - 3z - 4 + 4z - z \\ &= (5 - 4) + (-3z + 4z - z) \\ &= 1 \end{aligned}$$

For out last example, we will work with a system of equations that will have no solution.

Example 102.

System	Augmented Matrix
$x + y + 3z = 2$ $3x + 4y + 10z = 5$ $x + 2y + 4z = 3$	$\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 2 \\ 3 & 4 & 10 & 5 \\ 1 & 2 & 4 & 3 \end{array}\right]$
Multiply Row 1 by -3 and add to Row 2 Symbolic: R2+(-3)R1 $\Rightarrow$ R2	$\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & 4 & 3 \end{array}\right]$

Multiply Row 1 by -1 and
add to Row 3
Symbolic: R3+(-1)R1 $\Rightarrow$ R3

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

Multiply Row 2 by -1 and
add to Row 3
Symbolic: R3+(-1)R2 $\Rightarrow$ R3

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

The resulting matrix is in row echelon form, but not reduced row echelon form. Notice that the last row of our matrix has corresponding equation $0 = 2$, which is false. Since our row reduction has resulted in a false statement, we conclude that the given system of equations has no solution. Therefore, we have no need to continue row reducing in order to obtain the reduced row echelon form.

We have now seen three examples of how matrices can be used to solve a system of equations containing three variables: one example with a single solution, one with infinitely many solutions, and one with no solution. Naturally, we can apply this approach to simpler systems, containing just two variables/equations, as well as to more complicated systems.

Practice Problems

Graphing

Solve each system by graphing.

1. $\begin{cases} y = -x + 1 \\ y = -5x - 3 \end{cases}$
2. $\begin{cases} y = -\frac{3}{4}x + 1 \\ y = -\frac{3}{4}x + 2 \end{cases}$
3. $\begin{cases} y = \frac{5}{3}x + 4 \\ y = -\frac{2}{3}x - 3 \end{cases}$
4. $\begin{cases} x - y = 4 \\ 2x + y = -1 \end{cases}$
5. $\begin{cases} 2x + y = 2 \\ x - y = 4 \end{cases}$
6. $\begin{cases} 9y + 6x = 36 \\ 3y - 6x = -12 \end{cases}$
7. $\begin{cases} 3 + y = -x \\ -4 - 6x = -y \end{cases}$
8. $\begin{cases} y = -\frac{5}{4}x - 2 \\ y = -\frac{1}{4}x + 2 \end{cases}$
9. $\begin{cases} y = 2x + 2 \\ y = -x - 4 \end{cases}$
10. $\begin{cases} y = \frac{1}{2}x + 4 \\ y = \frac{1}{2}x + 1 \end{cases}$
11. $\begin{cases} 6x + y = -3 \\ x + y = 2 \end{cases}$
12. $\begin{cases} x + 2y = 6 \\ 5x - 4y = 16 \end{cases}$
13. $\begin{cases} -2y + x = 4 \\ 2 = -x + \frac{1}{2}y \end{cases}$
14. $\begin{cases} 16 = -x - 4y \\ -2x = -4 - 4y \end{cases}$
15. $\begin{cases} y = -3 \\ y = -x - 4 \end{cases}$
16. $\begin{cases} y = \frac{1}{3}x + 2 \\ y = -\frac{5}{3}x - 4 \end{cases}$
17. $\begin{cases} x + 3y = -9 \\ 5x + 3y = 3 \end{cases}$
18. $\begin{cases} 2x + 3y = -6 \\ 2x + y = 2 \end{cases}$
19. $\begin{cases} 2x + y = -2 \\ x + 3y = 9 \end{cases}$
20. $\begin{cases} 2x - y = -1 \\ 3 = -2x - y \end{cases}$
21. $\begin{cases} -y + 7x = 4 \\ -y + 7x = 3 \end{cases}$
22. $\begin{cases} y = -x - 2 \\ y = \frac{2}{3}x + 3 \end{cases}$
23. $\begin{cases} y = 2x - 4 \\ y = \frac{1}{2}x + 2 \end{cases}$
24. $\begin{cases} x + 4y = -12 \\ 2x + y = 4 \end{cases}$
25. $\begin{cases} 3x + 2y = 2 \\ 3x + 2y = -6 \end{cases}$
26. $\begin{cases} x - y = 3 \\ 5x + 2y = 8 \end{cases}$
27. $\begin{cases} -2y = -4 - x \\ -2y = -5x + 4 \end{cases}$
28. $\begin{cases} -4 + y = x \\ x + 2 = -y \end{cases}$

Substitution

Solve each system by substitution.

1. $\begin{cases} y = -3x \\ y = 6x - 9 \end{cases}$

2. $\begin{cases} y = 6x + 4 \\ y = -3x - 5 \end{cases}$

3. $\begin{cases} y = 2x - 3 \\ y = -2x + 9 \end{cases}$

4. $\begin{cases} y = -6 \\ 3x - 6y = 30 \end{cases}$

5. $\begin{cases} -2x + 2y = 18 \\ y = 7x + 15 \end{cases}$

6. $\begin{cases} 7x - 2y = -7 \\ y = 7 \end{cases}$

7. $\begin{cases} -2x - y = -5 \\ x - 8y = -23 \end{cases}$

8. $\begin{cases} 3x + y = 9 \\ 2x + 8y = -16 \end{cases}$

9. $\begin{cases} x + 5y = 15 \\ -3x + 2y = 6 \end{cases}$

10. $\begin{cases} y = x + 5 \\ y = -2x - 4 \end{cases}$

11. $\begin{cases} y = 3x + 13 \\ y = -2x - 22 \end{cases}$

12. $\begin{cases} y = 7x - 24 \\ y = -3x + 16 \end{cases}$

13. $\begin{cases} 6x - 4y = -8 \\ y = -6x + 2 \end{cases}$

14. $\begin{cases} y = x + 4 \\ 3x - 4y = -19 \end{cases}$

15. $\begin{cases} x - 2y = -13 \\ 4x + 2y = 18 \end{cases}$

16. $\begin{cases} 6x + 4y = 16 \\ -2x + y = -3 \end{cases}$

17. $\begin{cases} -5x - 5y = -20 \\ -2x + y = 7 \end{cases}$

18. $\begin{cases} 2x + 3y = -10 \\ 7x + y = 3 \end{cases}$

19. $\begin{cases} y = -2x - 9 \\ y = 2x - 1 \end{cases}$

20. $\begin{cases} y = 3x + 2 \\ y = -3x + 8 \end{cases}$

21. $\begin{cases} y = 6x - 6 \\ -3x - 3y = -24 \end{cases}$

22. $\begin{cases} y = -5 \\ 3x + 4y = -17 \end{cases}$

23. $\begin{cases} y = -8x + 19 \\ -x + 6y = 16 \end{cases}$

24. $\begin{cases} x - 5y = 7 \\ 2x + 7y = -20 \end{cases}$

25. $\begin{cases} -6x + y = 20 \\ -3x - 3y = -18 \end{cases}$

26. $\begin{cases} 2x + y = 2 \\ 3x + 7y = 14 \end{cases}$

27. $\begin{cases} -2x + 4y = -16 \\ y = -2 \end{cases}$

28. $\begin{cases} y = -6x + 3 \\ y = 6x + 3 \end{cases}$

29. $\begin{cases} y = -2x - 9 \\ y = -5x - 21 \end{cases}$

30. $\begin{cases} -x + 3y = 12 \\ y = 6x + 21 \end{cases}$

31. $\begin{cases} 7x + 2y = -7 \\ y = 5x + 5 \end{cases}$

32. $\begin{cases} y = -2x + 8 \\ -7x - 6y = -8 \end{cases}$

33. $\begin{cases} 3x - 4y = 15 \\ 7x + y = 4 \end{cases}$

34. $\begin{cases} 7x + 5y = -13 \\ x - 4y = -16 \end{cases}$

35. $\begin{cases} 2x + y = -7 \\ 5x + 3y = -21 \end{cases}$

36. $\begin{cases} -2x + 2y = -22 \\ -5x - 7y = -19 \end{cases}$

Addition Elimination

Solve each system by elimination.

1. $\begin{cases} 4x + 2y = 0 \\ -4x - 9y = -28 \end{cases}$

2. $\begin{cases} -6x + 9y = 3 \\ 6x - 9y = -9 \end{cases}$

3. $\begin{cases} -x - 5y = 28 \\ -x + 4y = -17 \end{cases}$

4. $\begin{cases} 10x + 6y = 24 \\ -6x + y = 4 \end{cases}$

5. $\begin{cases} -7x + 4y = -4 \\ 10x - 8y = -8 \end{cases}$

6. $\begin{cases} -7x - 3y = 12 \\ -6x - 5y = 20 \end{cases}$

7. $\begin{cases} 9x + 6y = -21 \\ -10x - 9y = 28 \end{cases}$

8. $\begin{cases} -8x - 8y = -8 \\ 10x + 9y = 1 \end{cases}$

9. $\begin{cases} 0 = 9x + 5y \\ y = \frac{2}{7}x \end{cases}$

10. $\begin{cases} -7x + y = -10 \\ -9x - y = -22 \end{cases}$

11. $\begin{cases} 5x - 5y = -15 \\ 5x - 5y = -15 \end{cases}$

12. $\begin{cases} -10x - 5y = 0 \\ 10x + 10y = 30 \end{cases}$

13. $\begin{cases} x + 3y = -1 \\ 10x + 6y = -10 \end{cases}$

14. $\begin{cases} -6x + 4y = 4 \\ -3x - y = 26 \end{cases}$

15. $\begin{cases} -5x + 4y = 4 \\ -7x - 10y = -10 \end{cases}$

16. $\begin{cases} -4x - 5y = 12 \\ -10x + 6y = 30 \end{cases}$

17. $\begin{cases} -7x + 10y = 13 \\ 4x + 9y = 22 \end{cases}$

18. $\begin{cases} -6 - 42y = -12x \\ x - \frac{7}{2}y = \frac{1}{2} \end{cases}$

19. $\begin{cases} -9x + 5y = -22 \\ 9x - 5y = 13 \end{cases}$

20. $\begin{cases} 4x - 6y = -10 \\ 4x - 6y = -14 \end{cases}$

21. $\begin{cases} 2x - y = 5 \\ 5x + 2y = -28 \end{cases}$

22. $\begin{cases} 2x + 4y = 24 \\ 4x - 12y = 8 \end{cases}$

23. $\begin{cases} 5x + 10y = 20 \\ -6x - 5y = -3 \end{cases}$

24. $\begin{cases} 9x - 2y = -18 \\ 5x - 7y = -10 \end{cases}$

25. $\begin{cases} -7x + 5y = -8 \\ -3x - 3y = 12 \end{cases}$

26. $\begin{cases} 9y = 7 - x \\ -18y + 4x = -26 \end{cases}$

27. $\begin{cases} -x - 2y = -7 \\ x + 2y = 7 \end{cases}$

28. $\begin{cases} -3x + 3y = -12 \\ -3x + 9y = -24 \end{cases}$

29. $\begin{cases} -5x + 6y = -17 \\ x - 2y = 5 \end{cases}$

30. $\begin{cases} -6x + 4y = 12 \\ 12x + 6y = 18 \end{cases}$

31. $\begin{cases} -9x - 5y = -19 \\ 3x - 7y = -11 \end{cases}$

32. $\begin{cases} 3x + 7y = -8 \\ 4x + 6y = -4 \end{cases}$

33. $\begin{cases} 8x + 7y = -24 \\ 6x + 3y = -18 \end{cases}$

34. $\begin{cases} 21 = -9x + 12y \\ \frac{4}{3}y + \frac{7}{3}x = -1 \end{cases}$

Three Variables

Solve each of the following systems of equation.

1. $\begin{cases} 2x + y = z \\ 4x + z = 4y \\ y = x + 1 \end{cases}$

2. $\begin{cases} 3x + 2y = z + 2 \\ y = 1 - 2x \\ 3z = -2y \end{cases}$

3. $\begin{cases} x + y - z = 0 \\ x - y - z = 0 \\ x + y + 2z = 0 \end{cases}$

4. $\begin{cases} x + y - z = 0 \\ x + 2y - 4z = 0 \\ 2x + y + z = 0 \end{cases}$

5. $\begin{cases} m + 6n + 3p = 8 \\ 3m + 4n = -3 \\ 5m + 7n = 1 \end{cases}$

6. $\begin{cases} 2x + 3y = z - 1 \\ 3x = 8z - 1 \\ 5y + 7z = -1 \end{cases}$

7. $\begin{cases} x + 2y - z = 4 \\ 4x - 3y + z = 8 \\ 5x - y = 12 \end{cases}$

8. $\begin{cases} 4x + 12y + 16z = 0 \\ 3x + 4y + 5z = 0 \\ x + 8y + 11z = 0 \end{cases}$

9. $\begin{cases} 2x + y - 3z = 0 \\ x - 4y + z = 0 \\ 4x + 16y + 4z = 0 \end{cases}$

10. $\begin{cases} a - 2b + c = 5 \\ 2a + b - c = -1 \\ 3a + 3b - 2c = -4 \end{cases}$

11. $\begin{cases} x + y + z = 2 \\ 6x - 4y + 5z = 31 \\ 5x + 2y + 2z = 13 \end{cases}$

12. $\begin{cases} x + y + z = 6 \\ 2x - y - z = -3 \\ x - 2y + 3z = 6 \end{cases}$

13. $\begin{cases} p + q + r = 1 \\ p + 2q + 3r = 4 \\ 4p + 5q + 6r = 7 \end{cases}$

14. $\begin{cases} x - y + 2z = 0 \\ x - 2y + 3z = -1 \\ 2x - 2y + z = -3 \end{cases}$

15. $\begin{cases} x + 6y + 3z = 4 \\ 2x + y + 2z = 3 \\ 3x - 2y + z = 0 \end{cases}$

16. $\begin{cases} -2x + y - 3z = 1 \\ x - 4y + z = 6 \\ 4x + 16y + 4z = 24 \end{cases}$

17. $\begin{cases} x - 2y + 3z = 4 \\ 2x - y + z = -1 \\ 4x + y + z = 1 \end{cases}$

18. $\begin{cases} 4x - 7y + 3z = 1 \\ 3x + y - 2z = 4 \\ 4x - 7y + 3z = 6 \end{cases}$

19. $\begin{cases} 3x + y - z = 11 \\ x + 3y = z + 13 \\ x + y - 3z = 11 \end{cases}$

20. $\begin{cases} x - y + 2z = -3 \\ x + 2y + 3z = 4 \\ 2x + y + z = -3 \end{cases}$

21. $\begin{cases} 4x + 12y + 16z = 4 \\ 3x + 4y + 5z = 3 \\ x + 8y + 11z = 1 \end{cases}$

22. $\begin{cases} 3x + 2y + 2z = 3 \\ x + 2y - z = 5 \\ 2x - 4y + z = 0 \end{cases}$

23. $\begin{cases} x + 2y - 3z = 9 \\ 2x - y + 2z = -8 \\ 3x - y - 4z = 3 \end{cases}$

24. $\begin{cases} 4x - 3y + 2z = 40 \\ 5x + 9y - 7z = 47 \\ 9x + 8y - 3z = 97 \end{cases}$

25. $\begin{cases} 3x + y - z = 10 \\ 8x - y - 6z = -3 \\ 5x - 2y - 5z = 1 \end{cases}$

26. $\begin{cases} 3x + 3y - 2z = 13 \\ 6x + 2y - 5z = 13 \\ 5x - 2y - 5z = -1 \end{cases}$

27. $\begin{cases} 2x - 3y + 5z = 1 \\ 3x + 2y - z = 4 \\ 4x + 7y - 7z = 7 \end{cases}$

28. $\begin{cases} 3x - 4y + 2z = 1 \\ 2x + 3y - 3z = -1 \\ x + 10y - 8z = 7 \end{cases}$

29. $\begin{cases} 2w - 2x - 2y + 2z = 10 \\ w + x + y + z = -5 \\ 3w + 2x + 2y + 4z = -11 \\ w + 3x - 2y + 2z = -6 \end{cases}$

30. $\begin{cases} w - 2x + 3y - z = 8 \\ w - x - y + z = 4 \\ w + x + y + z = 22 \\ w - x + y + z = 14 \end{cases}$

31. $\begin{cases} w + x + y + z = 2 \\ w + 2x + 2y + 4z = 1 \\ w - x + y + z = 6 \\ w - 3x - y + z = 2 \end{cases}$

32. $\begin{cases} w + x - y + z = 0 \\ -w + 2x + 2y + z = 5 \\ w - 3x - y + z = 4 \\ 2w - x - y + 3z = 7 \end{cases}$

Matrix Notation

Construct an augmented matrix for each of the systems of equations referenced below. Then row reduce your matrix to its row echelon form and determine if the given system has (1) no solution, (2) infinitely many solutions, or (3) exactly one solution. If one solution exists, determine the reduced row echelon form for your matrix and use it to find the solution to the given system.

1) - 5): Systems (1) through (5) on page 97.

6) - 10): Systems (11) through (15) on page 97.

11) - 20): Systems (1) through (10) on page 98.

21) - 24): Systems (29) through (32) on page 98.

Selected Answers

Graphing

1) $(-1, 2)$
3) $(-3, -1)$
5) $(2, -2)$
7) $(-1, -2)$
9) $(-2, -2)$
11) $(-1, 3)$
13) $(-4, -4)$
15) $(-1, -3)$
17) $(3, -4)$
19) $(-3, 4)$
21) No solution, $\varnothing$
23) $(4, 4)$
25) No solution, $\varnothing$
27) $(2, 3)$

Substitution

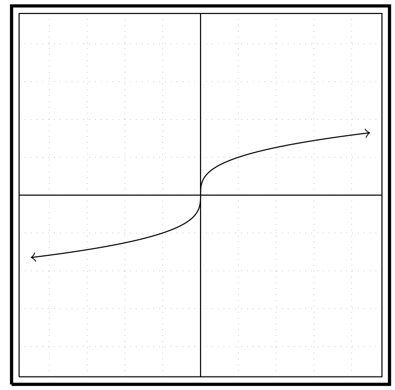
1) $(1, -3)$
3) $(3, 3)$
5) $(-1, 8)$
7) $(1, 3)$
9) $(0, 3)$
11) $(-7, -8)$
13) $(0, 2)$
15) $(1, 7)$
17) $(-1, 5)$
19) $(-2, -5)$
21) $(2, 6)$
23) $(2, 3)$
25) $(-2, 8)$
27) $(4, -2)$
29) $(-4, -1)$
31) $(-1, 0)$
33) $(1, -3)$
35) $(0, -7)$

Addition Elimination

1) $(-2, 4)$
3) $(-3, -5)$
5) $(4, 6)$
7) $(-1, -2)$
9) $(0, 0)$
11) Infinite
13) $(-1, 0)$
15) $(0, 1)$
17) $(1, 2)$
19) No solution, $\varnothing$
21) $(-2, -9)$
23) $(-2, 3)$
25) $(-1, -3)$
27) Infinite
29) $(1, -2)$
31) $(1, -2)$
33) $(-3, 0)$

Three Variables

3) $(0, 0, 0)$
9) $(0, 0, 0)$
10) $(1, -1, 2)$
15) $(-2, -1, 4)$
17) $(-1, 2, 3)$
24) $(10, 2, 3)$
28) No solution, $\varnothing$
29) $(1, -3, -2, -1)$



Chapter 3

Introduction to Functions

Notation and Basic Examples

Definitions and the Vertical Line Test (L12)

Objective: Identify functions and use correct notation to evaluate and solve functions for specific values.

A *relation* R is a set of points in the xy-plane. A relation in which each x-coordinate is paired with exactly one y-coordinate is said to describe y as a *function* of x. Relations which represent functions of x will often be denoted by f, or $f(x)$, rather than R. The set of all x-coordinates of the points in a function f is called the *domain* of f, and the set of all y-coordinates of the points in f is called the *range* of f.

Example 103. The following examples represent relations. Examples (5) and (6) also represent y as a *function* of x, $y = f(x)$, since each $x-$coordinate is paired with exactly one $y-$coordinate.

1. $\{(1,1),(2,-3),(2,0),(0,3),(-2,1/2)\}$
2. $\{(x,y) \mid x > 3 \text{ and } y \leq 2\}$
3. $x^2 + y^2 = 9$
4. $x = y^2$
5. $y = x^2$
6. $y = 3 - 2x$

Alternatively, one can define a function as a rule that assigns to each element of one set (the domain) exactly one element of a second set (the range). This definition is essentially the same as that given above, but avoids the term "relation" entirely. In each definition, however, the critical phrase that cannot be overlooked is "exactly one". This means that the first four relations given above cannot represent y as a function of x, since, for example, the third relation contains the points $(0, 3)$ and $(0, -3)$. On the other hand, each of the last two relations above can be considered to represent y as a function of x. Furthermore, their graphs should also look familiar, since they represent a quadratic equation ($y = x^2$) and a linear equation ($y = -2x + 3$).

In each of the last two examples above, we refer to the variable x as the *independent variable*, since we are free to choose any real number for x. We consequently refer to y as the *dependent variable*, since its value depends on the choice of value for x. One can also more simply refer to x as the *input* of the function and y as the *output*. This terminology naturally lends itself to what is the standard function notation of $f(x)$, read as "f of x". In the following example, we will use the given function to complete a table of values for x and $f(x)$. Each pair $(x, f(x))$ corresponds to a point (x, y) on the graph of f.

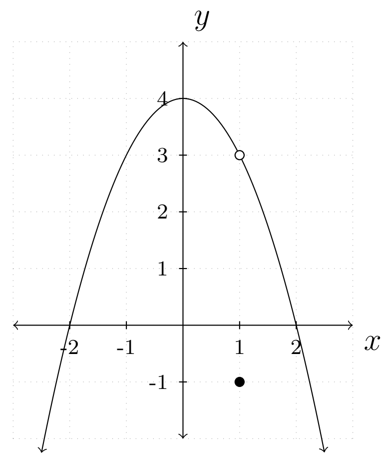
Example 104. $f(x) = x^2 - 4x - 5$

x	$f(x)$
-2	$(-2)^2 - 4(-2) - 5 = 7$
-1	$(-1)^2 - 4(-1) - 5 = 0$
0	$(0)^2 - 4(0) - 5 = -5$
1	$(1)^2 - 4(1) - 5 = -8$
2	$(2)^2 - 4(2) - 5 = -9$

To complete each row of the table, we simply substitute the specified value for x into the given equation and simplify. So, if we wanted to complete another row in the table, we could substitute $x = 3$ into the equation to obtain $f(3) = (3)^2 - 4(3) - 5 = -8$.

In the previous example, the y-coordinates for the relation $y = x^2 - 4x - 5$ are represented by $f(x)$, or more simply $y = f(x)$. It is important to note that the parentheses in function notation do not represent multiplication. This is a common misconception among students. Instead, one should consider the parentheses as an identifier, enclosing the value of x that the rule f is applied to. This will be especially important as we discuss composite functions later in the chapter.

In the following examples we will answer a variety of questions related to functions and their graphs. First, we will consider the case where we are presented with the graph of a particular function and asked to identify specific values of x or $f(x)$ from it.



The graph of f

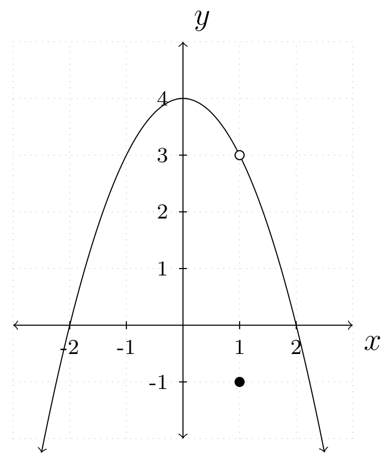
In our first scenario, we will be provided with an input x and asked to find the output $f(x)$. To find an output when given a specific input, locate the input value on the x-axis and follow the vertical line (above and below) the input value until it intersects, or "hits", the graph. The corresponding $y-$coordinate for the point of intersection will be the desired output, $y = f(x)$.

Example 105. Use the graph of f provided to find the desired outputs.

$$\begin{aligned} f(2) &= \ ? && \text{What is } y \text{ when } x = 2? \\ f(2) &= 0 && \text{Our answer} \\ \\ f(0) &= \ ? && \text{What is } y \text{ when } x = 0? \\ f(0) &= 4 && \text{Our answer} \\ \\ f(1) &= \ ? && \text{What is } y \text{ when } x = 1? \\ f(1) &= -1 && \text{Our answer} \end{aligned}$$

It is important to point out that many students will misinterpret the last example and incorrectly conclude that $f(1) = 3$, since the open circle at $(1, 3)$ appears to coincide with the rest of the graph of f. An *open* circle, however, is used to identify a *break* in the graph of f, also known as a point of *discontinuity*. In fact, the given function is not defined at $(1, 3)$, but rather at the *solid* (or *closed*) point $(1, -1)$. Hence, we get a corresponding value of $y = -1$ for $f(x)$.

Next, we will be provided with an output y and asked to find all corresponding inputs x such that $f(x) = y$. To find all possible inputs, we will make a simple adjustment to the method used in the previous example. Now, we will locate the output value on the y-axis and follow the horizontal line (left and right) of the output value until it intersects, or "hits", the graph. All corresponding $x-$coordinates for the points of intersection will represent the set of all values of x such that $f(x)$ equals our given output y and should be included as part of our final answer.



The graph of f

Example 106. Use the graph above to find all possible inputs that correspond to the specified output.

Find x where $f(x) = 0$.	Which inputs for x have an output of $y = 0$?
$x = -2, 2$	Our answers
Find x where $f(x) = 3$.	Which inputs for x have an output of $y = 3$?
$x = -1$	Our answer; We should not include $x = 1$.

Similarly, if we were also asked to find all possible x such that $f(x) = -1$, then we would end up with three values, since there are three points that intersect the horizontal line $y = -1$, namely $x \approx -2.2$, $x = 1$, and $x \approx 2.2$.

There are four major representations of functions: verbal (in words), numerical (using a table), symbolic (with an algebraic expression), and visual (with a graph). In many cases, we will be asked to identify one representation of y as a function of x when given a different representation. The next two examples demonstrate this.

Example 107. Provide the symbolic form for each of the following verbal descriptions of a function.

1. Add 2 to a value and then take the square root of the resulting value.

   Our answer $\qquad f(x) = \sqrt{2 + x}$

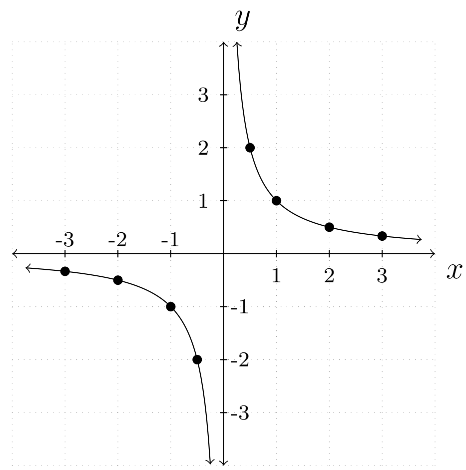
2. Take the square root of a value and then add 2 to the resulting value.

   Our answer $\qquad g(x) = \sqrt{x} + 2$

Note: It is often beneficial to rewrite $g(x)$ in the previous example as $g(x) = 2 + \sqrt{x}$, so as not to accidentally extend the radical to include the $+2$.

Example 108. Provide a graphical representation for the function given by the following table of values.

x	$f(x)$
-3	-1/3
-2	-1/2
-1	-1
-1/2	-2
1/2	2
1	1
2	1/2
3	1/3

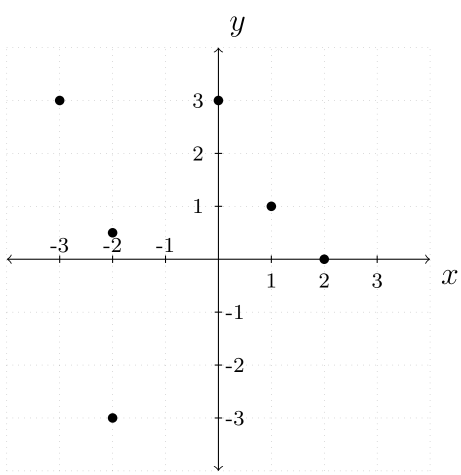


Since there are often advantages to working with either symbolic or graphical representations of functions, we will focus our attention on working with these two representations. One major test that is used to determine whether or not a graph of a relation represents y as a function of x is known as the Vertical Line Test. We will now state the Vertical Line Test as a mathematical theorem and then demonstrate its use.

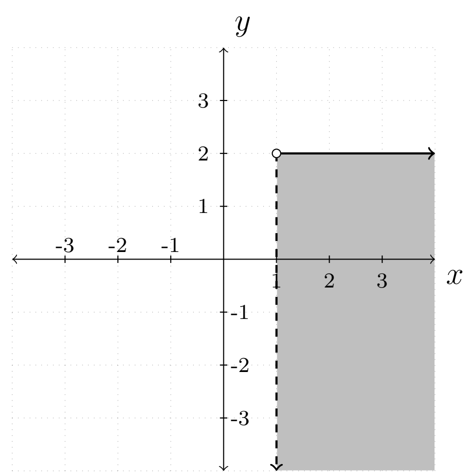
Vertical Line Test: A set of points in the xy-plane represents y as a function of x if and only if no two points lie on the same vertical line.

Alternatively stated, if a graph is known to represent y as a function of x, then there can be no vertical line that intersects the graph in more than one point. Conversely, if a known graph has the property that no vertical line intersects it in more than one point, then the given graph represents y as a function of x.

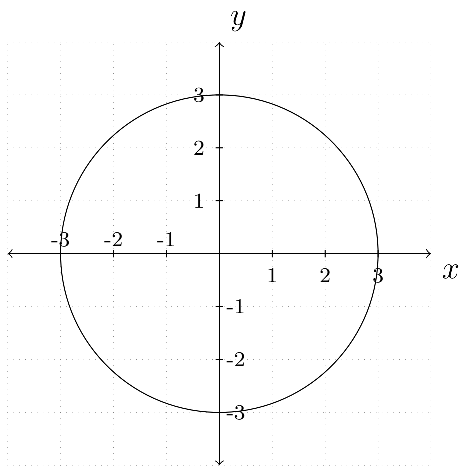
Example 109. Use the Vertical Line Test to determine whether or not each of the following graphs represent y as a function of x.



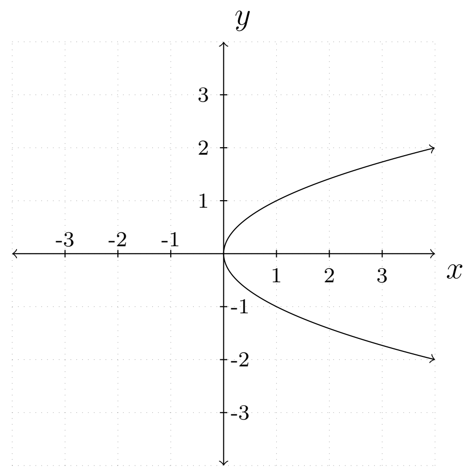
$\{(1,1),(-3,3),(-2,-3),(2,0),(0,3),(-2,\frac{1}{2})\}$



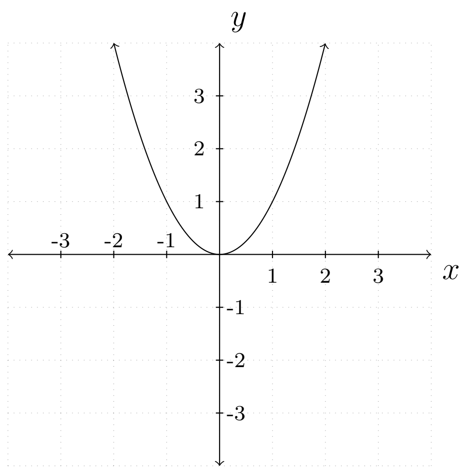
$\{(x,y) \mid x > 1, y \leq 2\}$



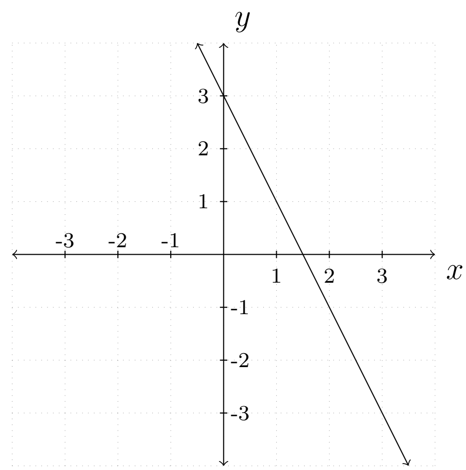
$x^2 + y^2 = 9$



$x = y^2$



$y = x^2$



$y = 3 - 2x$

In each example, we utilize the Vertical Line Test by "slicing" through each graph with several vertical lines, located at various values along the x-axis. Consequently, we can see that each of the first four examples at the beginning of the section *do not* represent y as a function of x, since in each case there exists at least one vertical line that intersects the graph in two (or possibly more) points. The last two examples *do* represent y as a function of x, since no such vertical line exists. As a result, we say that the first four examples *fail* the Vertical Line Test, and the last two examples *pass* the Vertical Line Test.

When we are presented with an equation, instead of a graph, we can still determine whether or not the equation represents y as a function of x by solving the equation for y and carefully considering the result. For example, if we consider the equation $x = y^2$, which corresponds to the fourth graph in our previous example (a sideways parabola), we know from the graph that $x = y^2$ cannot represent y as a function of x, since it fails the VLT.

If, instead of looking at the graph of $x = y^2$, we were to solve the equation for y, we would get $y = \pm\sqrt{x}$. The existence of the $\pm$ in this equation is the cause of our failure to have y

as a function of x, since, for example, $y = \pm 2$ when $x = 4$. This means that the points $(4, 2)$ and $(4, -2)$ will both be on our graph, which we cannot have for y to be a function of x. We conclude this section with two examples that demonstrate this algebraic analysis of an equation, in order to determine whether y represents a function of x.

Example 110. Determine whether the following equation represents y as a function of x.

$$x^2 + y^2 = 9$$

Solve the equation for y.

$$\begin{array}{rll}
x^2 + y^2 &= 9 & \text{Solve for } y \\
\underline{-\cancel{x^2}} \qquad & \underline{-x^2} & \text{Subtract } x^2 \\
y^2 &= 9 - x^2 & \\
\sqrt{y^2} &= \pm\sqrt{9 - x^2} & \text{Introduce a square root} \\
& & \quad \text{include a } \pm \text{ on right side} \\
y &= \pm\sqrt{9 - x^2} & y \text{ is not a function of } x
\end{array}$$

Due to the $\pm$, we can conclude that the equation does *not* represent y as a function of x.

Example 111. Determine whether the following equation represents y as a function of x.

$$2x + \frac{1}{y - 3} = 0$$

Solve the equation for y.

$$\begin{array}{rll}
\cancel{2x} + \dfrac{1}{y-3} &= 0 & \text{Solve for } y \\
\underline{-\cancel{2x}} \qquad\qquad & \underline{-2x} & \text{Subtract } 2x \\
\dfrac{1}{y-3} &= -2x & \\
\cancel{(y-3)} \cdot \dfrac{1}{\cancel{y-3}} &= (-2x) \cdot (y-3) & \text{Multiply by } y - 3 \\
1 &= (-2x)(y-3) & \\
\dfrac{1}{-2x} &= \dfrac{\cancel{(-2x)}(y-3)}{\cancel{-2x}} & \text{Divide by } -2x \\
-\dfrac{1}{2x} &= y - 3 & \\
\underline{+3} \qquad & \underline{+\cancel{3}} & \text{Add } 3 \\
y &= 3 - \dfrac{1}{2x} & y \text{ as a function of } x
\end{array}$$

We can conclude that y represents a function of x.

Evaluating (L13) and Solving (L31) Functions

Objective: Evaluate functions using appropriate notation.

Another function-related skill we will want to quickly master is evaluating functions at certain values of the independent variable (usually x). This is accomplished by substituting the specified value into the function for x and simplifying the resulting expression to find $f(x)$. This idea of "plugging in" values of x to find $f(x)$ is demonstrated in the following examples.

Example 112. Find $f(-2)$, where $f(x) = 3x^2 - 4x$.

$$\begin{aligned} f(x) &= 3x^2 - 4x && \text{Evaluate; Substitute } -2 \text{ for each } x \\ f(-2) &= 3(-2)^2 - 4(-2) && \text{Simplify using order of operations; exponent first} \\ f(-2) &= 3(4) - 4(-2) && \text{Multiply} \\ f(-2) &= 12 + 8 && \text{Add} \\ f(-2) &= 20 && \text{Our solution} \end{aligned}$$

Example 113. Find $h(4)$, where $h(x) = 3^{2x-6}$.

$$\begin{aligned} h(x) &= 3^{2x-6} && \text{Evaluate; Substitute 4 for } x \\ h(4) &= 3^{2(4)-6} && \text{Simplify exponent, multiplying first} \\ h(4) &= 3^{8-6} && \text{Subtract in exponent} \\ h(4) &= 3^2 && \text{Evaluate exponent} \\ h(4) &= 9 && \text{Our solution} \end{aligned}$$

Example 114. Find $k(-7)$, where $k(a) = 2|a+4|$.

$$\begin{aligned} k(a) &= 2|a+4| && \text{Evaluate; Substitute } -7 \text{ for } a \\ k(-7) &= 2|-7+4| && \text{Simplify, add inside absolute value} \\ k(-7) &= 2|-3| && \text{Evaluate absolute value} \\ k(-7) &= 2(3) && \text{Multiply} \\ k(-7) &= 6 && \text{Our solution} \end{aligned}$$

As the previous examples show, a function can take many different forms, but the method to evaluate the function is always the same: replace each instance of the variable with the specified value and simplify.

We can also substitute entire expressions into functions using this same process. This idea is known as a *composition* of two functions or expressions, and will be formally outlined in a later section. We present the following two examples as a preview of this concept.

Example 115. Find $g(3x)$, where $g(x) = x^4 + 1$.

$$\begin{aligned} g(x) &= x^4 + 1 && \text{Replace } x \text{ in the function with } (3x) \\ g(3x) &= (3x)^4 + 1 && \text{Simplify exponent} \\ g(3x) &= 81x^4 + 1 && \text{Our solution} \end{aligned}$$

Example 116. Find $p(t+1)$, where $p(t) = t^2 - t$.

$$\begin{array}{rl}
p(t) = t^2 - t & \text{Replace each } t \text{ in } p(t) \text{ with } (t+1) \\
p(t+1) = (t+1)^2 - (t+1) & \text{Simplify; square binomial} \\
p(t+1) = t^2 + 2t + 1 - (t+1) & \text{Distribute negative sign} \\
p(t+1) = t^2 + 2t + 1 - t - 1 & \text{Combine like terms} \\
p(t+1) = t^2 + t & \text{Our solution} \\
p(t+1) = t(t+1) & \text{Our solution in factored form}
\end{array}$$

As is the case with each of the previous examples, it is important to keep in mind that each expression (or function) will often use the same variable. Hence, it is critical that we recognize that each variable must be replaced by whatever expression appears in parentheses.

So far, all of the previous examples have shown how to find an output when given a specific input. Next, we will demonstrate how one can also algebraically find which input(s) x yield a required output $f(x)$. This is often referred to as *solving* a function (for a specific output).

Example 117. Given $f(x) = x^2 + 3x + 5$, find all x such that $f(x) = 5$.

$$\begin{array}{rl}
f(x) = x^2 + 3x + 5 & \text{Substitute 5 in for } f(x) \\
5 = x^2 + 3x + 5 & \text{Solve for } x \text{ by factoring} \\
0 = x^2 + 3x & \text{Set equal to 0} \\
0 = x(x+3) & \text{Factor} \\
x = 0 \text{ or } x = -3 & \text{Our solutions}
\end{array}$$

The above answer can be verified by checking. When we input $x = 0$ into the function, we simplify to find that $f(0) = 5$. Similarly, we see that when $x = -3$, $f(-3) = 5$.

Example 118. Given $h(x) = 4x - 1$, find all x such that $h(x) = -3$.

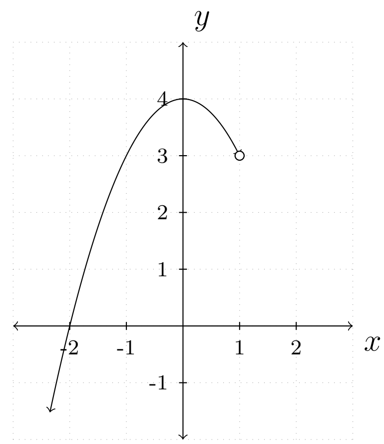
$$\begin{array}{rl}
h(x) = 4x - 1 & \text{Substitute } -3 \text{ for } h(x) \\
-3 = 4x - 1 & \text{Solve for } x \\
-2 = 4x & \text{Divide} \\
x = -\dfrac{1}{2} & \text{Our solution}
\end{array}$$

It is important that we become comfortable with function notation and how to use it, as we begin to transition to more advanced algebraic concepts.

Identifying Domain and Range Graphically (L14)

Objective: Identify the domain and range of a function that is described graphically. In this section, we will first discuss how one can identify the domain and range of a function using its graph. Later, we will explore finding the domain of a function using algebraic methods. As finding the range of a function using algebraic methods can often prove quite challenging, we will postpone this topic for another time.

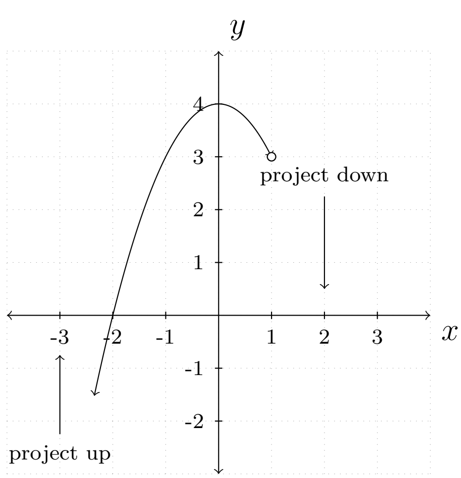
Example 119. Find the domain and range of the function f whose graph is given below.

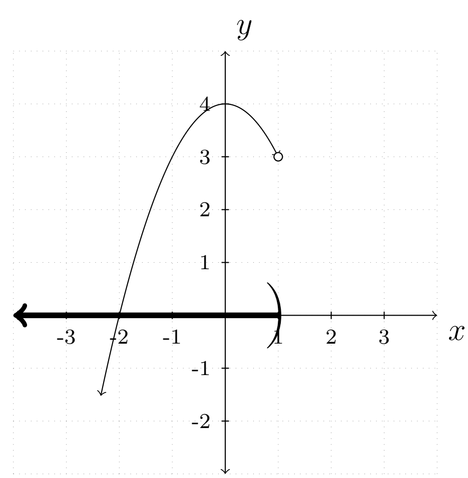


The graph of f

To determine the domain and range of f, we need to determine which x and y-values respectively occur as coordinates of points on the given graph.

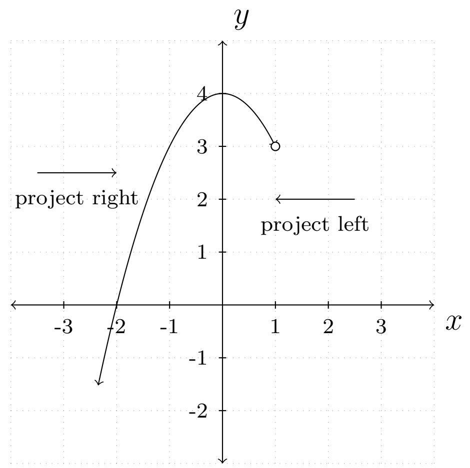
To find the domain, it will be helpful to imagine collapsing the curve onto the x-axis and determining the portion of the x-axis that gets covered. This is often described as *projecting* the curve onto the x-axis. Before we project, we need to pay attention to two subtle notations on the graph: the arrowhead on the lower left corner of the graph indicates that the graph continues to curve downwards to the left forever; and the open circle at $(1, 3)$ indicates that the point $(1, 3)$ is *not* on the graph, but all the points on the curve leading up to $(1, 3)$ are on the graph.

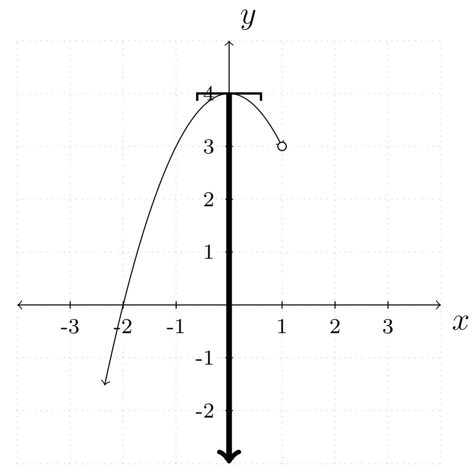




We see from the figure that if we project the graph of f to the x-axis, we get all real numbers less than 1. Using interval notation, we write the domain of f as $(-\infty, 1)$.

To determine the range of f, we use a similar method, projecting the curve onto the y-axis as follows.





Note that even though there is an open circle at $(1, 3)$, we still include the y value of 3 in our range, since the point $(-1, 3)$ is on the graph of f. We also include $y = 4$ in our answer, since the point $(0, 4)$ is also on our graph. Consequently, the range of f is all real numbers less than or equal to 4, or $(-\infty, 4]$.

Fundamental Functions (L15)

Objective: Graph and identify the domain, range, and intercepts of any of the ten fundamental functions.

In this section, we have listed ten fundamental function types which will be referenced throughout the rest of the text, as well as one example of each. Each type of function represents a "building block" for understanding the concepts of a traditional algebra course.

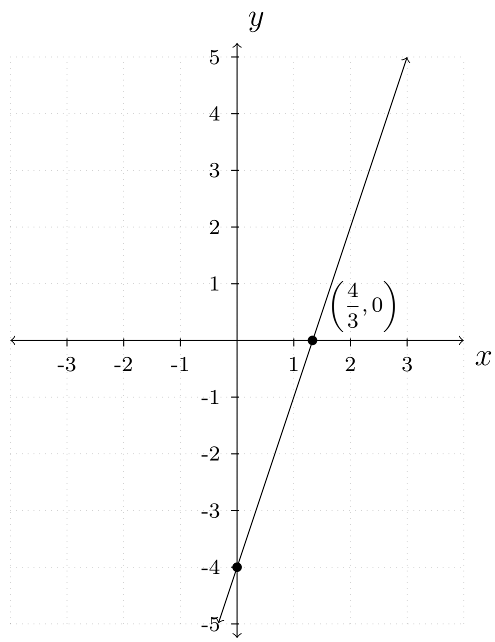
Students should be able to both identify and sketch a graph of each function, as well as identify its intercepts, domain (both graphically and algebraically), and range (graphically). Each representative form in the table below includes some element of generalization to reinforce understanding.

Function Type	Representative Form	Example
Linear	$mx+b$	$f(x)=3x-4$
Quadratic	ax^2+bx+c	$g(x)=x^2$
Square Root	$\sqrt{x-h}$	$k(x)=\sqrt{x}$
Absolute Value	$\|x-h\|$	$\ell(x)=\|x\|$
Cubic	$(x-h)^3$	$m(x)=x^3$
Cube Root	$\sqrt[3]{x-h}$	$n(x)=\sqrt[3]{x}$
Reciprocal (Rational)	$\dfrac{1}{x-h}$	$p(x)=\dfrac{1}{x}$
Semicircular	$\sqrt{r^2-x^2},\ r>0$	$q(x)=\sqrt{9-x^2}$
Exponential*	$a^x,\ a>0, a\neq 1$	$r(x)=2^x$
Logarithmic*	$\log_a(x),\ a>0, a\neq 1$	$s(x)=\log_2(x)$

*We have included Exponential and Logarithmic functions for a more complete list. These functions are more formally treated in a Precalculus setting.

Function Type: **Linear** $(m \neq 0)$
Example: $f(x) = 3x - 4$

x	$f(x)$
-3	-13
-2	-10
-1	-7
0	-4
1	-1
$\frac{4}{3}$	0
2	2
3	5



Graph of $f(x) = 3x - 4$

$y-$intercept: $(0, -4)$

$x-$intercept(s): $\left(\frac{4}{3}, 0\right)$

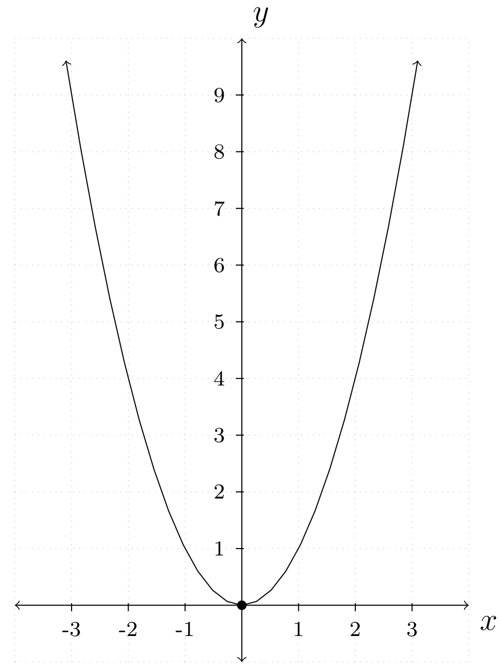
Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

Notes: If $m = 0$, then the corresponding graph of $f(x) = b$ is a horizontal line. The domain of f is still $(-\infty, \infty)$, but the range consists of a single value, $\{b\}$.

Function Type: **Quadratic**
Example: $g(x) = x^2$

x	$g(x)$
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9



Graph of $g(x) = x^2$

$y-$intercept: $(0, 0)$

$x-$intercept(s): $(0, 0)$

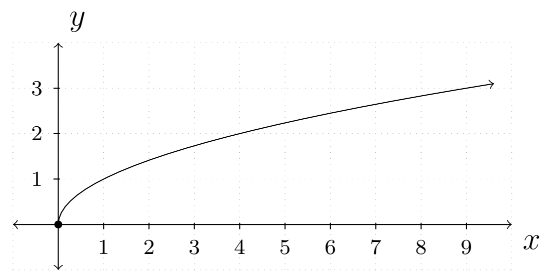
Domain: $(-\infty, \infty)$

Range: $[0, \infty)$, or $y \geq 0$

Notes: The domain of any quadratic function is $(-\infty, \infty)$. If $g(x) = a(x-h)^2 + k$, is a quadratic function in vertex form, then if $a > 0$, the corresponding parabola will be concave *up*, and the range of g will be $[k, \infty)$. If $a < 0$, then the corresponding parabola will be concave *down*, and the range of g will be $(-\infty, k]$. Quadratics will be covered extensively in the next chapter.

Function Type: **Square Root**
Example: $k(x) = \sqrt{x}$

x	$k(x)$
-1	undefined
0	0
1	1
2	$\sqrt{2} \approx 1.41$
3	$\sqrt{3} \approx 1.73$
4	2
9	3



Graph of $k(x) = \sqrt{x}$

$y-$intercept: $(0, 0)$

$x-$intercept(s): $(0, 0)$

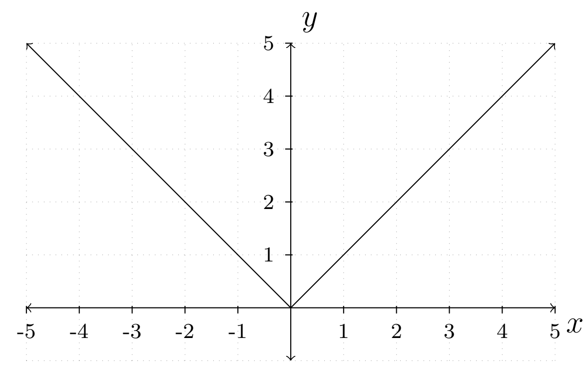
Domain: $[0, \infty)$, or $x \geq 0$

Range: $[0, \infty)$, or $y \geq 0$

Notes: The domain of a square root function of the form $k(x) = \sqrt{x - h}$ will be $x > h$. The range will be the same as in the example, $[0, \infty)$. The $x-$intercept will be $(h, 0)$.

Function Type: **Absolute Value**
Example: $\ell(x) = |x|$

x	$\ell(x)$
-3	3
-2	2
-1	1
0	0
1	1
2	2
3	3



Graph of $\ell(x) = |x|$

$y-$intercept: $(0, 0)$

$x-$intercept(s): $(0, 0)$

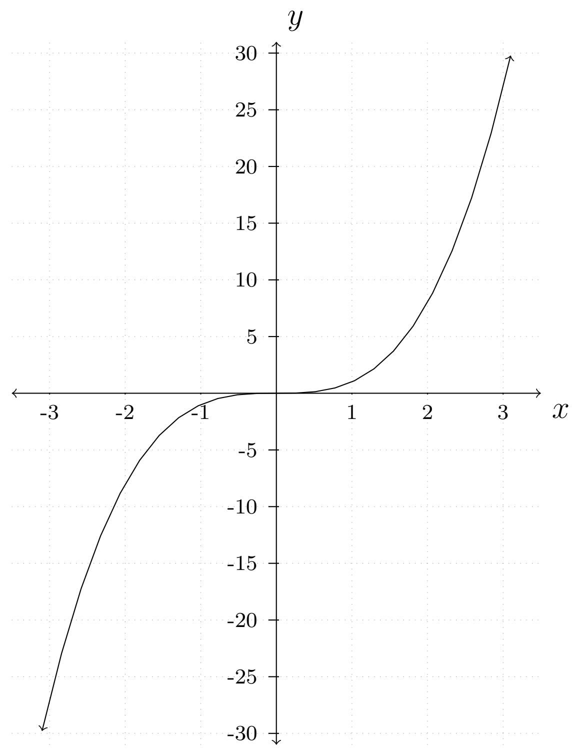
Domain: $(-\infty, \infty)$

Range: $[0, \infty)$, or $y \geq 0$

Notes: The domain and range of an absolute value function of the form $\ell(x) = |x - h|$ will remain the same as above. The $x-$intercept will be $(h, 0)$.

Function Type: **Cubic**
Example: $m(x) = x^3$

x	$m(x)$
-3	-27
-2	-8
-1	-1
0	0
1	1
2	8
3	27



Graph of $m(x) = x^3$

$y-$intercept: $(0, 0)$

$x-$intercept(s): $(0, 0)$

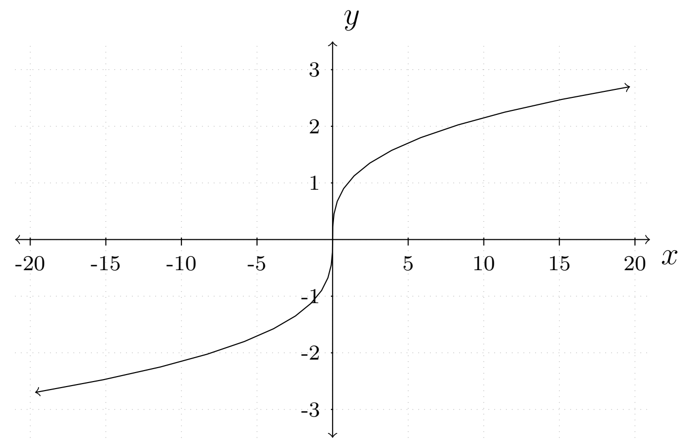
Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

Notes: The domain and range of a cubic function of the form $m(x) = (x - h)^3$ will remain the same as above. The $x-$intercept will be $(h, 0)$. The $y-$intercept will be $(0, -h^3)$.

Function Type: **Cube Root**
Example: $n(x) = \sqrt[3]{x}$

x	$n(x)$
-27	-3
-8	-2
-1	-1
0	0
1	1
8	2
27	3



Graph of $n(x) = \sqrt[3]{x}$

$y-$intercept: $(0, 0)$

$x-$intercept(s): $(0, 0)$

Domain: $(-\infty, \infty)$

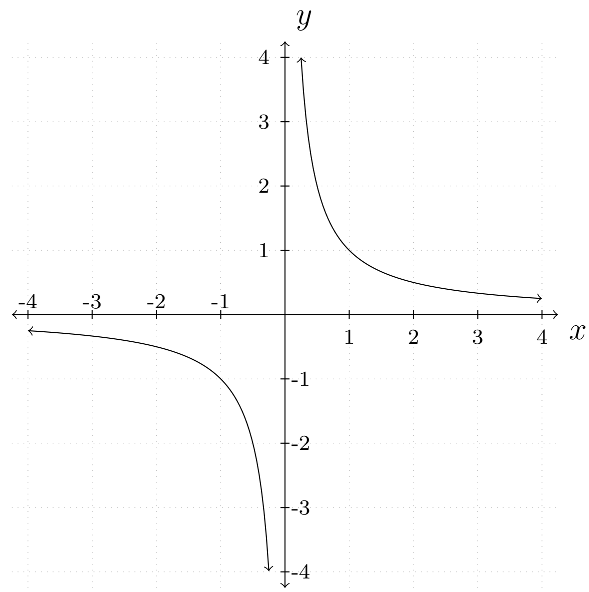
Range: $(-\infty, \infty)$

Notes: The domain and range of a cube root function of the form $n(x) = \sqrt[3]{x-h}$ will remain the same as above. The $x-$intercept will be $(h, 0)$. The $y-$intercept will be $(0, -\sqrt[3]{h})$.

Function Type: **Reciprocal (Rational)**

Example: $p(x) = \dfrac{1}{x}$

x	$p(x)$
-3	$-\dfrac{1}{3}$
-2	$-\dfrac{1}{2}$
-1	-1
0	undefined
1	1
2	$\dfrac{1}{2}$
3	$\dfrac{1}{3}$



Graph of $p(x) = \dfrac{1}{x}$

$y-$intercept: None

$x-$intercept(s): None

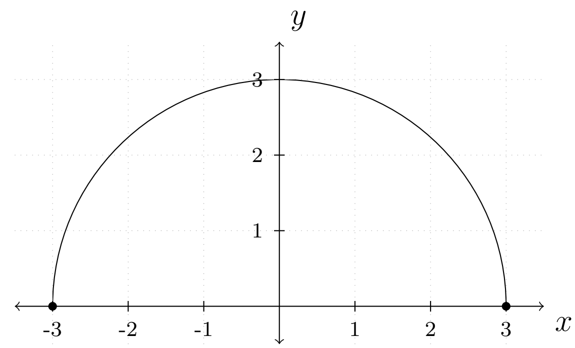
Domain: $(-\infty, 0) \cup (0, \infty)$, or $x \neq 0$

Range: $(-\infty, 0) \cup (0, \infty)$, or $y \neq 0$

Notes: The reciprocal function $\dfrac{1}{x}$ gets its name since each $y-$coordinate is the reciprocal of its corresponding $x-$coordinate, and vice versa. Although the more general representative function $\dfrac{1}{x-h}$ does not uphold this reciprocal property, we can still categorize both the reciprocal form and the more general form as specific types of *rational* functions. The domain of a function of the form $p(x) = \dfrac{1}{x-h}$ will be $(-\infty, h) \cup (h, \infty)$, or $x \neq h$. The range, however, will remain the same as the reciprocal function, $\dfrac{1}{x}$. The graph of $p(x) = \dfrac{1}{x-h}$ will have no $x-$intercept. The $y-$intercept will be at $\left(0, -\dfrac{1}{h}\right)$.

Function Type: **Semicircular**
Example: $q(x) = \sqrt{9 - x^2}$

x	$q(x)$
-3	0
-2	$\sqrt{5}$
-1	$\sqrt{8}$
0	0
1	$\sqrt{8}$
2	$\sqrt{5}$
9	0



Graph of $q(x) = \sqrt{9 - x^2}$

$y-$intercept: $(0, 3)$

$x-$intercept(s): $(-3, 0)$ and $(3, 0)$

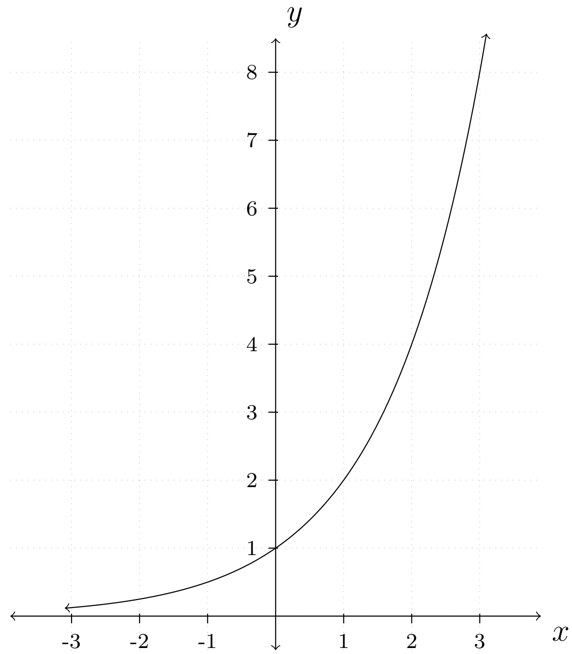
Domain: $[-3, 3]$, or $-3 \leq x \leq 3$

Range: $[0, 3]$, or $0 \leq y \leq 3$

Notes: The domain of a semicircular function of the form $q(x) = \sqrt{r^2 - x^2}$ will be $[-r, r]$, or $-r \leq x \leq r$. The range will be $[0, r]$, or $0 \leq y \leq r$. The graph of $q(x) = \sqrt{r^2 - x^2}$ will have $x-$intercepts at $(\pm r, 0)$ and a $y-$intercept at $(0, r)$.

Function Type: **Exponential**
Example: $r(x) = 2^x$

x	$r(x)$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8



Graph of $r(x) = 2^x$

$y-$intercept: $(0, 1)$

$x-$intercept(s): None

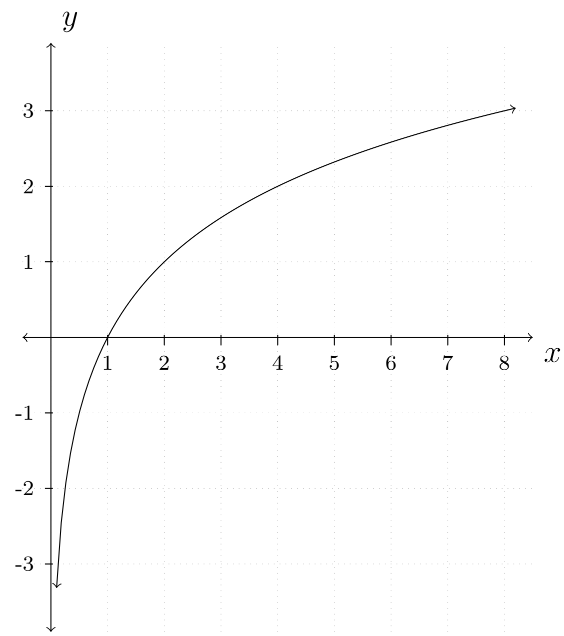
Domain: $(-\infty, \infty)$

Range: $(0, \infty)$, or $y > 0$

Notes: The domain, range, $x-$ and $y-$ intercepts of an exponential function of the form $r(x) = a^x$, where a is positive $(a \neq 1)$ will all be the same as above.

Function Type: **Logarithmic**
Example: $s(x) = \log_2 x$

x	$s(x)$
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3



Graph of $s(x) = \log_2 x$

$y-$intercept: None

$x-$intercept(s): $(1, 0)$

Domain: $(0, \infty)$, or $x > 0$

Range: $(-\infty, \infty)$

Notes: The domain, range, $x-$ and $y-$ intercepts of a logarithmic function of the form $s(x) = \log_a x$, where a is positive ($a \neq 1$) will all be the same as above.

Practice Problems

Notation and Basic Examples

Determine whether or not each relation represents y as a function of x.

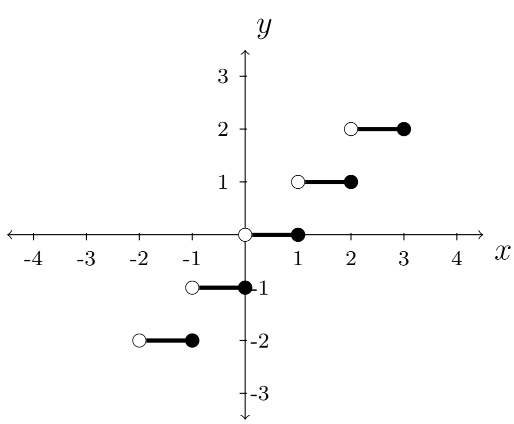
1. $\{(-3,9),\ (-2,4),\ (-1,1),\ (0,0),\ (1,1),\ (2,4),\ (3,9)\}$
2. $\{(-3,0),(1,6),(2,-3),(4,2),(-5,6),(4,-9),(6,2)\}$
3. $\{(-3,0),(-7,6),(5,5),(6,4),(4,9),(3,0)\}$
4. $\{(1,2),(4,4),(9,6),(16,8),(25,10),(36,12),\ldots\}$
5. $\{(x,y)\,|\,x$ is an odd integer, and y is an even integer$\}$
6. $\{(x,1)\,|\,x$ is an irrational number$\}$
7. $\{(1,0),\ (2,1),\ (4,2),\ (8,3),\ (16,4),\ (32,5),\ \ldots\}$
8. $\{\ldots,\ (-3,9),\ (-2,4),\ (-1,1),\ (0,0),\ (1,1),\ (2,4),\ (3,9),\ \ldots\}$
9. $\{(-2,y)\,|\,-3<y<4\}$
10. $\{(x,3)\,|\,-2\le x<4\}$

Determine if the following relations represent y as a function of x by making a table of values and graphing. Explain your reasoning. Use Desmos to confirm your results.

11. $x=y^3$
12. $y=x$
13. $xy=1$
14. $y=(x-3)^2$
15. $x=(y-3)^2$
16. $y<2x-5$

Determine whether each of the following relations represents y as a function of x.

17.



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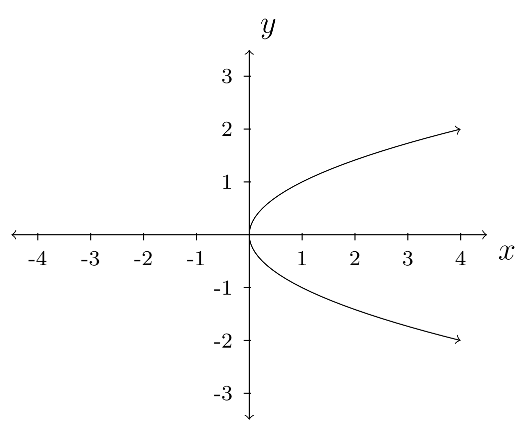

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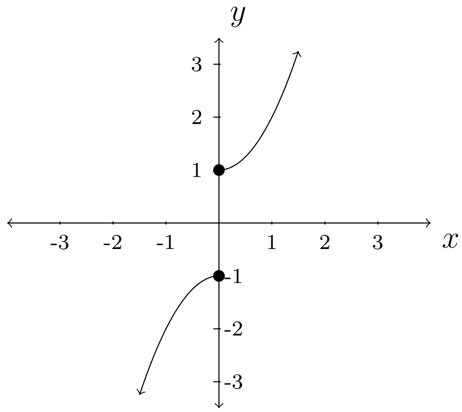

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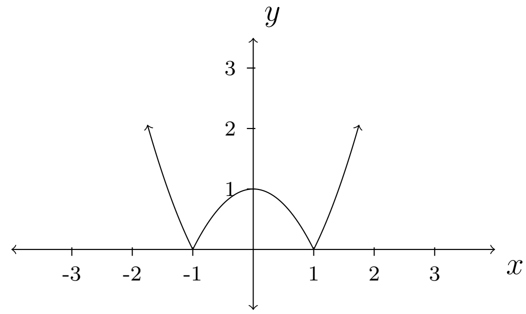


21.

x	y
3	-3
2	-2
1	-1
0	0
1	1
2	2
3	3

22.

x	y
-3	3
-2	2
-1	1
0	0
1	1
2	2
3	3

23.

x	y
-3	0
-2	0
-1	0
0	0
1	0
2	0
3	0

24.

x	y
3	8
2	4
1	2
0	1
-1	$1/2$
-2	$1/4$
-3	$1/8$

Determine whether each of the following equations represents y as a function of x.

25. $y = x^3 - x$

26. $y = \sqrt{x-2}$

27. $3x + 2y = 6$

28. $x^2 - y^2 = 1$

29. $y = \dfrac{x}{x^2 - 9}$

30. $x = -6$

31. $x = y^2 + 4$

32. $y = x^2 + 4$

33. $x^2 + y^2 = 4$

For each of the following statements, find an expression for $f(x)$.

34. f is a function that takes a real number x and performs the following three steps in the order given: (1) multiply by 2; (2) add 3; (3) divide by 4.
35. f is a function that takes a real number x and performs the following three steps in the order given: (1) add 3; (2) multiply by 2; (3) divide by 4.
36. f is a function that takes a real number x and performs the following three steps in the order given: (1) divide by 4; (2) add 3; (3) multiply by 2.
37. f is a function that takes a real number x and performs the following three steps in the order given: (1) multiply by 2; (2) add 3; (3) take the square root.
38. f is a function that takes a real number x and performs the following three steps in the order given: (1) add 3; (2) multiply by 2; (3) take the square root.

39. f is a function that takes a real number x and performs the following three steps in the order given: (1) add 3; (2) take the square root; (3) multiply by 2.
40. f is a function that takes a real number x and performs the following three steps in the order given: (1) take the square root; (2) subtract 13; (3) make the quantity the denominator of a fraction with numerator 4.
41. f is a function that takes a real number x and performs the following three steps in the order given: (1) subtract 13; (2) take the square root; (3) make the quantity the denominator of a fraction with numerator 4.
42. f is a function that takes a real number x and performs the following three steps in the order given: (1) take the square root; (2) make the quantity the denominator of a fraction with numerator 4; (3) subtract 13.
43. f is a function that takes a real number x and performs the following three steps in the order given: (1) make the quantity the denominator of a fraction with numerator 4; (2) take the square root; (3) subtract 13.

For each exercise, use the given function f to find and simplify each of the **nine** related values/expressions listed below.

- $f(1)$
- $f(-3)$
- $f\left(\frac{3}{2}\right)$
- $f(4x)$
- $4f(x)$
- $f(-x)$
- $f(x-4)$
- $f(x)-4$
- $f\left(x^2\right)$

44. $f(x) = 2x + 1$
45. $f(x) = 3 - 4x$
46. $f(x) = 2 - x^2$
47. $f(x) = x^2 - 3x + 2$
48. $f(x) = \sqrt{x-1}$
49. $f(x) = \dfrac{x}{x-1}$
50. $f(x) = 6$
51. $f(x) = 0$

For each exercise, use the given function f to find and simplify each of the **nine** related values/expressions listed below.

- $f(2)$
- $f(-2)$
- $f(2a)$
- $2f(a)$
- $f(a+2)$
- $f(a)+f(2)$
- $f\left(\frac{2}{a}\right)$
- $\frac{f(a)}{2}$
- $f(a+h)$

52. $f(x) = 2x - 5$
53. $f(x) = 5 - 2x$
54. $f(x) = 2x^2 - 1$
55. $f(x) = 3x^2 + 3x - 2$
56. $f(x) = \sqrt{2x+1}$
57. $f(x) = 1$

58. $f(x) = \dfrac{x}{2}$

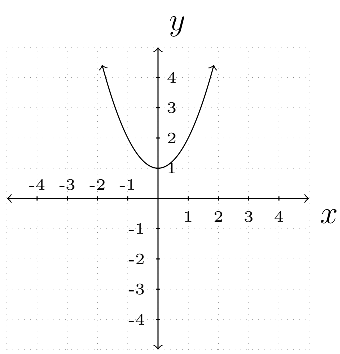
59. $f(x) = \dfrac{2}{x}$

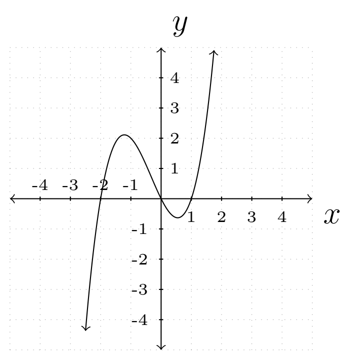
In each of the following exercises, use the given function f to find $f(0)$ and solve $f(x) = 0$

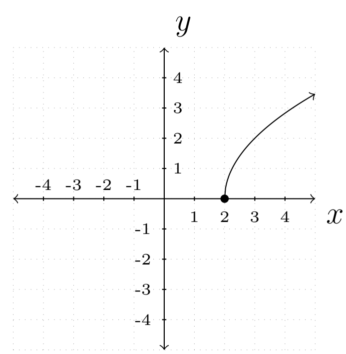
60. $f(x) = 2x - 1$
61. $f(x) = 3 - \frac{2}{5}x$
62. $f(x) = 2x^2 - 6$
63. $f(x) = x^2 - x - 12$
64. $f(x) = \sqrt{x+4}$
65. $f(x) = \sqrt{1-2x}$
66. $f(x) = \dfrac{3}{4-x}$
67. $f(x) = \dfrac{3x^2 - 12x}{4 - x^2}$

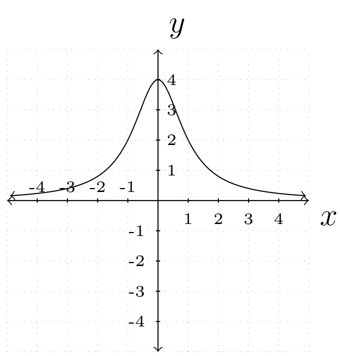
Identifying Domain and Range Graphically

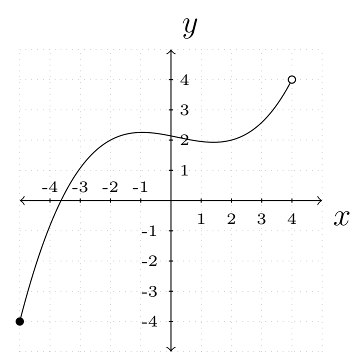
For each of the following graphs, identify the corresponding domain and range. Express your answers using interval notation.

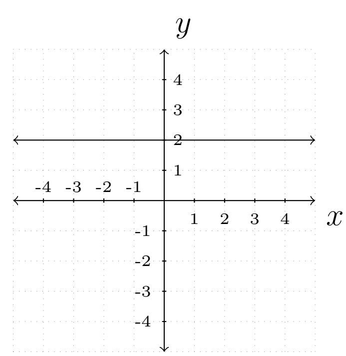
1. 

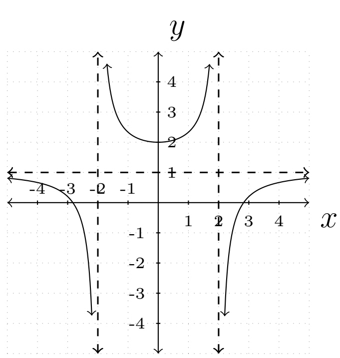
2. 

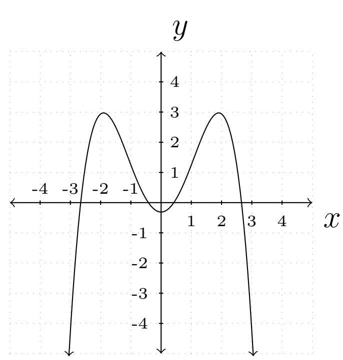
3. 

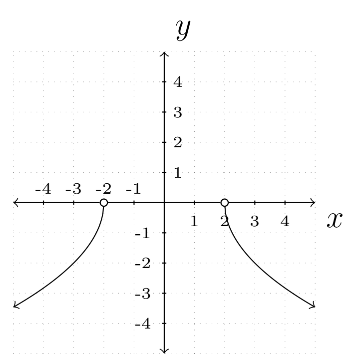
4. 

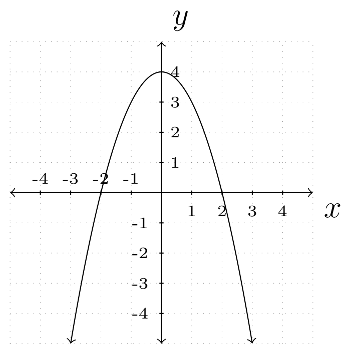
5. 

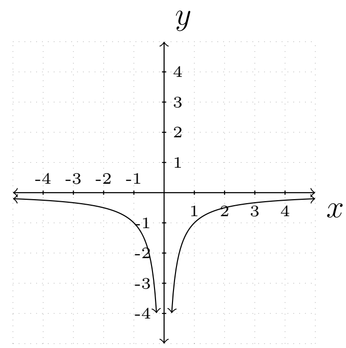
6. 

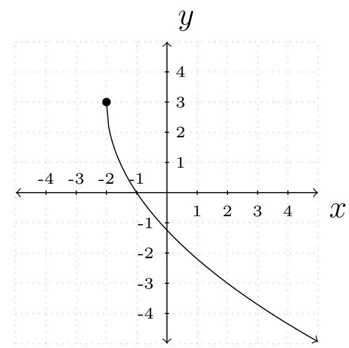
7. 

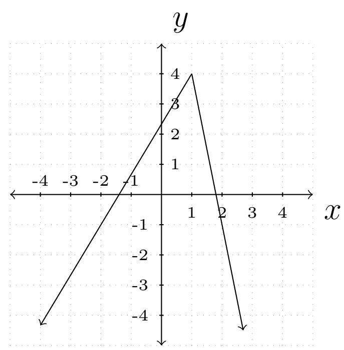
8. 

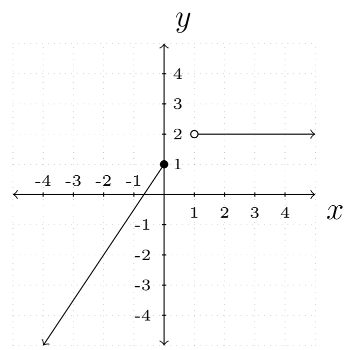
9. 

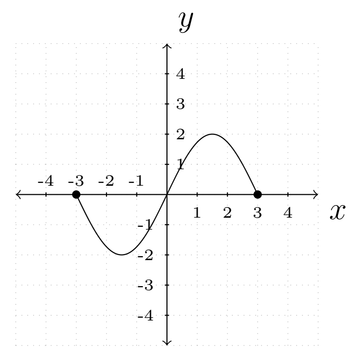
10. 

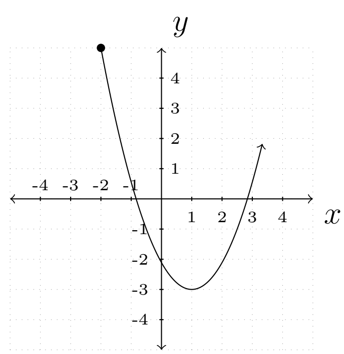
11. 

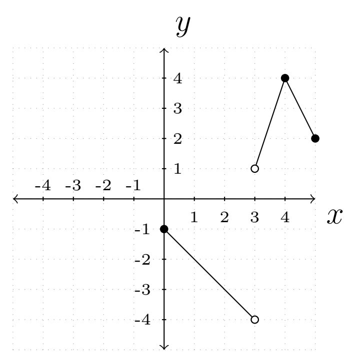
12. 

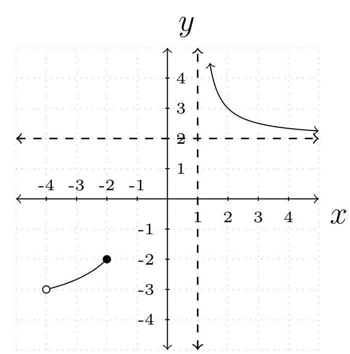
13. 

14. 

15. 

16. 

17. 

18. 

Selected Answers

Notation and Basic Examples

1) Function
3) Function
5) Not a function
7) Function
9) Not a function
11) Function
13) Function
15) Function
17) Function
19) Not a function
21) Not a function
23) Function
25) Function
27) Function
29) Function
31) Not a function
33) Not a function

35) $f(x)=\dfrac{x+3}{2}$

37) $f(x)=\sqrt{2x+3}$

39) $f(x)=2\sqrt{x+3}$

41) $f(x)=\dfrac{4}{\sqrt{x-13}}$

43) $f(x)=\sqrt{\frac{4}{x}-13}$

45) $f(x)=3-4x$

- $f(1)=-1$
- $f(-3)=15$
- $f\left(\frac{3}{2}\right)=-3$
- $f(4x)=3-16x$
- $4f(x)=12-16x$
- $f(-x)=3+4x$
- $f(x-4)=19-4x$
- $f(x)-4=-1-4x$
- $f\left(x^2\right)=3-4x^2$

47) $f(x)=x^2-3x+2$

- $f(1)=0$
- $f(-3)=20$
- $f\left(\frac{3}{2}\right)=-\frac{1}{4}$
- $f(4x)=16x^2-12x+2$
- $4f(x)=4x^2-12x+8$
- $f(-x)=x^2+3x+2$
- $f(x-4)=x^2-11x+30$
- $f(x)-4=x^2-3x-2$
- $f\left(x^2\right)=x^4-3x^2+2$

49) $f(x)=\dfrac{x}{x-1}$

- $f(1)=$undefined
- $f(-3)=\frac{3}{4}$
- $f\left(\frac{3}{2}\right)=3$
- $f(4x)=\dfrac{4x}{4x-1}$
- $4f(x)=\dfrac{4x}{x-1}$
- $f(-x)=\dfrac{x}{x+1}$
- $f(x-4)=\dfrac{x-4}{x-5}$
- $f(x)-4=\dfrac{-3x+4}{x-1}$
- $f\left(x^2\right)=\dfrac{x^2}{x^2-1}$

51) $f(x)=0$

- $f(1)=0$
- $f(-3)=0$
- $f\left(\frac{3}{2}\right)=0$
- $f(4x)=0$
- $4f(x)=0$
- $f(-x)=0$
- $f(x-4)=0$
- $f(x)-4=-4$
- $f\left(x^2\right)=0$

53) $f(x)=5-2x$

- $f(2)=1$
- $f(-2)=9$
- $f(2a)=5-4a$
- $2f(a)=10-4a$
- $f(a+2)=1-2a$
- $f(a)+f(2)=6-2a$

- $f\left(\dfrac{2}{a}\right) = \dfrac{5a-4}{a}$
- $\dfrac{f(a)}{2} = \dfrac{5}{2} - a$
- $f(a+h) = 5 - 2a - 2h$

55) $f(x) = 3x^2 + 3x - 2$

- $f(2) = 16$
- $f(-2) = 4$
- $f(2a) = 12a^2 + 6a - 2$
- $2f(a) = 6a^2 + 6a - 4$
- $f(a+2) = 3a^2+15a+16$
- $f(a)+f(2) = 3a^2+3a+14$
- $f\left(\dfrac{2}{a}\right) = \dfrac{-2a^2+6a+12}{a^2}$
- $\dfrac{f(a)}{2} = \dfrac{3}{2}a^2 + \dfrac{3}{2}a - 1$
- $f(a+h) = 3a^2 + 6ah + 3h^2 + 3a + 3h - 2$

57) $f(x) = 1$

- $f(2) = 1$
- $f(-2) = 1$
- $f(2a) = 1$
- $2f(a) = 2$
- $f(a+2) = 1$
- $f(a) + f(2) = 2$
- $f\left(\dfrac{2}{a}\right) = 1$
- $\dfrac{f(a)}{2} = \dfrac{1}{2}$
- $f(a+h) = 1$

59) $f(x) = \dfrac{2}{x}$

- $f(2) = 1$
- $f(-2) = -1$
- $f(2a) = \dfrac{1}{a}$
- $2f(a) = \dfrac{4}{a}$
- $f(a+2) = \dfrac{2}{a+2}$
- $f(a) + f(2) = \dfrac{2+a}{a}$
- $f\left(\dfrac{2}{a}\right) = a$
- $\dfrac{f(a)}{2} = \dfrac{1}{a}$
- $f(a+h) = \dfrac{2}{a+h}$

61) $f(0) = 3$; $f(x) = 0$ for $x = 15/2$

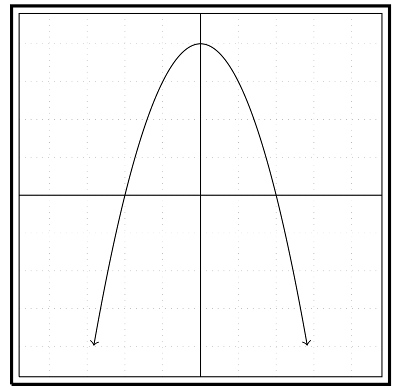
63) $f(0) = -12$; $f(x) = 0$ for $x = -3, 4$

65) $f(0) = 1$; $f(x) = 0$ for $x = 1/2$

67) $f(0) = 0$; $f(x) = 0$ for $x = 0, 4$

Domain and Range

Exercise	Domain	Range
1)	$(-\infty, \infty)$	$[1, \infty)$
3)	$[2, \infty)$	$[0, \infty)$
5)	$[-5, 4)$	$[-4, 4)$
7)	$(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$	$(-\infty, 1) \cup [2, \infty)$
9)	$(-\infty, -2) \cup (2, \infty)$	$(-\infty, 0)$
11)	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0)$
13)	$(-\infty, \infty)$	$(-\infty, 4]$
15)	$[-3, 3]$	$[-2, 2]$
17)	$[0, 3) \cup (3, 5]$	$(-4, -1] \cup (1, 4]$



Chapter 4

Quadratic Equations and Inequalities

Introduction (L16)

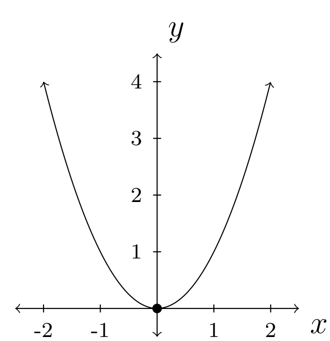
Objective: Recognize and classify a quadratic equation algebraically and graphically.

A quadratic equation is an equation of the form

$$y = ax^2 + bx + c,$$

where the *coefficients* of a,b, and c are real numbers and $a \neq 0$. This form is most commonly referred to as the *standard form* of a quadratic. We call a the *leading coefficient*, ax^2 the *leading term* (also known as the *quadratic term*), bx the *linear term* and c the *constant term* of the equation. The quadratic term ax^2, must have a nonzero coefficient in order for the equation to be a quadratic (otherwise y would be linear, in slope-intercept form). The most fundamental quadratic equation is $y = x^2$ and its graph, like all quadratics, is known as a *parabola*.

Example 120. $y = x^2$

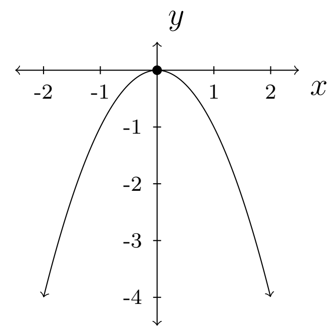


From the standard form, since $a > 0$, the graph opens upwards and is said to be *concave up*.

As a result, there is a minimum point, known as the *vertex*, located at the origin, $(0, 0)$.

Notice the symmetry over the y-axis.

Example 121. $y = -x^2$



Since $a = -1$, the graph opens downward or we say that it is *concave down*.

Every parabola with a negative leading coefficient ($a < 0$) will be concave down with a maximum value at its vertex.

The graph above has the same vertex as that in the previous example, but is a reflection of the previous graph about the y-axis. This flip of the graph is known as a transformation and will be discussed in the next chapter.

Aside from the shape and concavity, there is little else that the standard form immediately provides for graphing a quadratic. Additional aspects related to graphing quadratics will be covered a bit later in the chapter. Following this introduction, we will primarily focus on factoring quadratics from standard form. With all of the algebraic material that will follow, however, it will help to have a graphical sense of a quadratic equation.

An Introduction to the Vertex Form

Objective: Recognize and utilize the vertex form to graph a quadratic.

The most useful form for graphing a quadratic equation is the *vertex form*. A quadratic equation is said to be in vertex form if it is represented as

$$y = a(x - h)^2 + k,$$

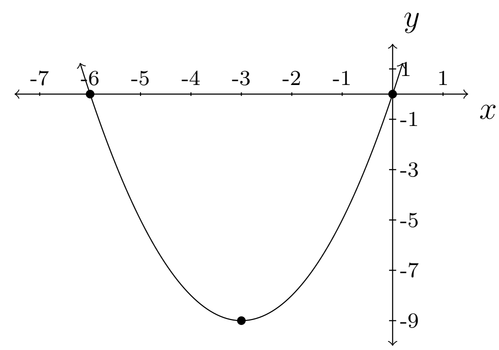
where h and k are real numbers.

It is important to note that the value a appearing above is also the leading coefficient from the standard form for a quadratic. Later, we will see the relationships between the coefficients a, b, and c in the standard form with h and k above.

When $a = 1$, the graph of a quadratic equation given in vertex form can be represented as a *shift*, or translation, of the original or "parent equation" $y = x^2$ presented earlier. The vertex form, unlike the standard form, allows us to immediately identify the vertex of the resulting parabola, which will be the point (h, k).

Next, we will see a few examples of quadratics in vertex form, the last of which is a bit surprising.

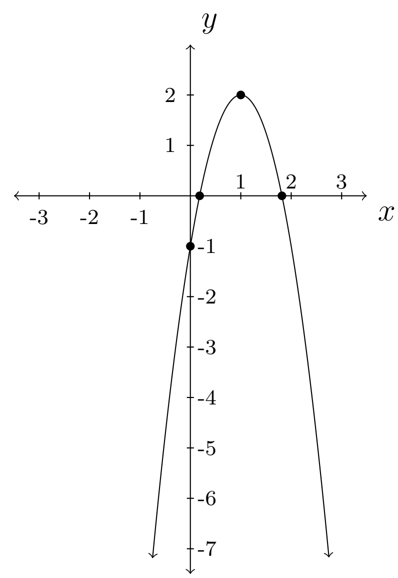
Example 122. $y = (x+3)^2 - 9$



The vertex is at $(-3,-9)$ and the graph can be realized as the graph of $y = x^2$ shifted left 3 units and down 9 units from the origin.

Since our graph is concave up there will be two x-intercepts as the function opens upward from below the x-axis.

Example 123. $y = -3(x-1)^2 + 2$



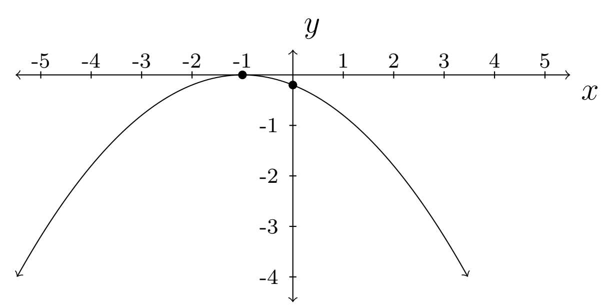
The vertex is at $(1,2)$ and represents a translation of the vertex for the graph of $y = x^2$ right 1 unit and up 2 units.

This graph is also concave down, since the leading coefficient $a = -3$ is less than zero.

Moreover, since $|a| > 1$, the shape of the graph is narrower than those which we have seen thus far.

Just like the previous example, this graph will have two x-intercepts as its vertex is above the x-axis and it opens downward.

Example 124. $y = -\frac{1}{5}(x+1)^2$



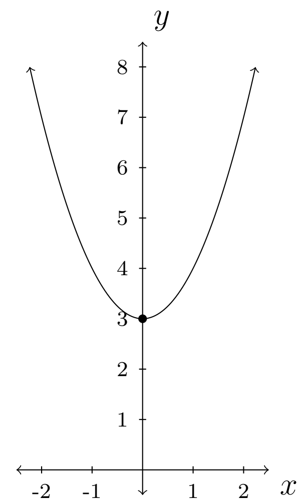
The vertex is at $(-1,0)$ and represents a translation of the vertex for the graph of $y = x^2$ left 1 unit.

There is no vertical shift, since there is no addition of a constant outside of the given expression.

Our graph is concave down and is much wider than any example we have seen thus far. This is on account of the fact that a is both negative and $|a| < 1$.

The following example shows an equation represented in both vertex and standard forms.

Example 125. $y = x^2 + 3$



The vertex is at $(0, 3)$ and our graph is a shift of the graph of $y = x^2$ up 3 units.

Since our graph is concave up with a vertex above the x-axis, there will be no real x-intercepts.

Notice that there is no horizontal shift because no number has been added or subtracted to x prior to it being squared.

This final example above may be recognized as a quadratic equation in standard form, where $b = 0$. Since there is no linear term, this quadratic is also in vertex form.

More generally, the graph of any equation of the form

$$y = ax^2 + c$$

has a y-intercept and vertex at $(0, c)$, since the resulting parabola represents a only a vertical shift of the graph of $y = x^2$ by c units and no horizontal shift.

Factoring Methods

Greatest Common Factors (L17)

Objective: Find the greatest common factor (GCF) and factor it out of an expression.

In order to discuss the factorization methods of this section, it will be necessary to introduce some of the terminology a bit early. In particular, in this section we will be working with *polynomial expressions*. While most of our work will be with polynomials containing a single variable, it will be helpful to see a few examples of polynomials that contain two (or more) variables.

Both linear and quadratic expressions of a variable x are basic examples of polynomials. A more general description of a polynomial in terms of the variable x is

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

where n is a nonnegative integer and $a_0, a_1, \ldots, a_{n-1}, a_n$ represent real coefficients ($a_n \neq 0$).

A basic interpretation of this description is a sum of n terms, each containing a real coefficient (possibly equal to 0), where the associated power of the variable is a positive integer (or possibly 0, in the case of the constant term $a_0 = a_0 x^0$).

The expression $8x^4 - 12x^3 + 32x$ would be an example of a polynomial, in which the power n (known as the *degree* of the polynomial) equals 4, and the coefficients are as follows.

$$a_4 = 8, \quad a_3 = -12, \quad a_2 = 0, \quad a_1 = 32, \quad a_0 = 0$$

If we inserted another variable(s) into each of the terms of our expression, we could create a polynomial expression in terms of two (or more) variables. An example of this would be

$$8x^4y - 12x^3y^2 + 32x.$$

While there is much more that we could say about this important concept of algebra, we will postpone a more in-depth treatment of polynomials until a later chapter, and move on to the topic of factorization.

Factoring a polynomial could be considered as the "opposite" action of multiplying (or expanding) polynomials together. In working with polynomial expressions, there are many benefits to identifying both its expanded and factored forms. Specifically, we will use factored polynomials to help us solve equations, learn behaviors of graphs, and understand more complicated rational expressions. Because so many concepts in algebra depend on being able to factor polynomials, it is critical that we establish strong factorization skills.

In this first part of the section, we will focus on factoring using the greatest common factor or GCF of a polynomial. When multiplying polynomials, we employ the distributive property, as demonstrated below.

$$4x^2(2x^2 - 3x + 8) = 8x^4 - 12x^3 + 32x^2$$

Here, we will work with the same expression, but with a backwards approach, starting with the expanded form and obtaining one that is partially (or completely) factored.

We will start with $8x^4 - 12x^3 + 32x^2$ and try and work backwards to reach $4x^2(2x^2 - 3x + 8)$.

To do this we have to be able to first identify what the GCF of a polynomial is. We will first introduce this concept by finding the GCF of a set of integers. To find a GCF of two or more integers, we must find the largest integer d that divides nicely into each of the given integers. Alternatively stated, d should be the largest factor of each of the integers in our set. This can often be determined with quick "mental math", as shown in the following example.

Example 126. Find the GCF of 15, 24, and 27.

$$\begin{array}{rl} \dfrac{15}{3} = 5, \quad \dfrac{24}{3} = 8, \quad \dfrac{27}{3} = 9 & \text{Each of the numbers can be divided by 3} \\ \text{GCF} = 3 & \text{Our solution} \end{array}$$

When there are variables in our problem we can first find the GCF of the numbers, then we can identify any variables that appear in every term and factor them out, taking the smallest exponent in each case. This is shown in the next example.

Example 127. Find the GCF of $24x^4y^2z$, $18x^2y^4$, and $12x^3yz^5$.

$$\frac{24}{6}=4,\quad \frac{18}{6}=3,\quad \frac{12}{6}=2 \qquad \text{Each number can be divided by 6}$$

$$x^2y \qquad x \text{ and } y \text{ appear in all three terms, taking the lowest exponent for each variable}$$

$$\text{GCF}=6x^2y \qquad \text{Our solution}$$

To factor out a GCF from a polynomial we first need to identify the GCF of all the terms, this is the part that goes in front of the parentheses, then we divide each term by the GCF in order to determine what should appear inside of the parentheses. This is demonstrated in the following examples.

Example 128. Find and factor out the GCF of the given polynomial expression.

$$4x^2-20x+16 \qquad \text{GCF is 4, divide each term by 4}$$

$$\frac{4x^2}{4}=x^2,\quad \frac{-20x}{4}=-5x,\quad \frac{16}{4}=4 \qquad \text{This is what is left inside the parentheses}$$

$$4(x^2-5x+4) \qquad \text{Our solution}$$

With factoring we can always check our solutions by expanding or multiplying out the answer. As in the example above, this usually will involve some form of the distributive property. Our end result upon checking should match the original expression.

Example 129. Find and factor out the GCF of the given polynomial expression.

$$25x^4-15x^3+20x^2 \qquad \text{GCF is } 5x^2\text{, divide each term by } 5x^2$$

$$\frac{25x^4}{5x^2}=5x^2,\quad \frac{-15x^3}{5x^2}=-3x,\quad \frac{20x^2}{5x^2}=4$$

This is what is left inside the parentheses.

$$5x^2(5x^2-3x+4) \qquad \text{Our solution}$$

Example 130. Find and factor out the GCF of the given polynomial expression.

$$3x^3y^2z+5x^4y^3z^5-4xy^4 \qquad \text{GCF is } xy^2\text{, divide each term by } xy^2$$

$$\frac{3x^3y^2z}{xy^2}=3x^2z,\quad \frac{5x^4y^3z^5}{xy^2}=5x^3yz^5,\quad \frac{-4xy^4}{xy^2}=-4y^2$$

This is what is left inside the parentheses.

$$xy^2(3x^2z + 5x^3yz^5 - 4y^2) \qquad \text{Our solution}$$

Example 131. Find and factor out the GCF of the given polynomial expression.

$$21x^3 + 14x^2 + 7x \qquad \text{GCF is } 7x, \text{divide each term by } 7x$$

$$\frac{21x^3}{7x} = 3x^2, \quad \frac{14x^2}{7x} = 2x, \quad \frac{7x}{7x} = 1$$

This is what is left inside the parentheses.

$$7x(3x^2 + 2x + 1) \qquad \text{Our solution}$$

It is important to note that in the previous example, the GCF of $7x$ was also one of the original terms. Dividing this term by the GCF left us with 1. A common mistake is to try to factor out the $7x$ and leave a value of zero. Factoring, however, will never make terms disappear completely. Any (nonzero) number or term that is divided by itself will always equal 1. Therefore, we must make certain to not forget to include a 1 in our solution.

Often the line showing the division is not written in the work of factoring the GCF, and we will simply identify the GCF and put it in front of the parentheses. This step is one that will eventually be understood, and can therefore be omitted once the skill has been mastered. The following two examples demonstrate this.

Example 132. Find and factor out the GCF of the given polynomial expression.

$$12x^5y^2 - 6x^4y^4 + 8x^3y^5$$

Notice, the GCF is $2x^3y^2$. Write $2x^3y^2$ in front of the parentheses and divide each term by it, writing the resulting terms inside the parentheses.

$$2x^3y^2(6x^2 - 3xy^2 + 4y^3) \qquad \text{Our solution}$$

Example 133. Find and factor out the GCF of the given polynomial expression.

$$18a^4b^3 - 27a^3b^3 + 9a^2b^3$$

Notice, the GCF is $9a^2b^3$. Write $9a^2b^3$ in front of the parentheses and divide each term by it, writing the resulting terms inside the parentheses.

$$9a^2b^3(2a^2 - 3a + 1) \qquad \text{Our solution}$$

Again, in the previous problem, when dividing $9a^2b^3$ by itself, the resulting term is 1, not zero. Be very careful that each term is accounted for in your final solution, and never forget that we can easily check our answers by expanding.

Factor by Grouping (L18)

Objective: Factor a tetranomial (four-term) expression by grouping.

The first thing we will always do when factoring is try to factor out a GCF. A GCF is often a *monomial* (a single term) like in the expression $5xy+10xz$. Here, the GCF is the monomial $5x$, so we would have $5x(y+2z)$ as our answer. However, a GCF does not have to be a monomial. It could, in fact, be a *binomial* and contain two terms. To see this, consider the following two examples.

Example 134. Find and factor out the GCF of the given expression.

$$\begin{array}{rl} 3ax-7bx & \text{Both terms have } x \text{ in common, factor it out} \\ x(3a-7b) & \text{Our solution} \end{array}$$

Now we will work with the same expression, replacing x with $(2a+5b)$.

Example 135. Find and factor out the GCF of the given expression.

$$\begin{array}{rl} 3a(2a+5b)-7b(2a+5b) & \text{Both terms have } (2a+5b) \text{ in common,} \\ & \quad\text{factor it out} \\ (2a+5b)(3a-7b) & \text{Our solution} \end{array}$$

In the same way that we factored out a GCF of x we can factor out a GCF which is a binomial, such as $(2a+5b)$ in the example above. This process can be extended to factoring expressions in which there is either no apparent GCF or there is more factoring that can be done after the GCF has been factored. At this point, we will introduce another useful factorization strategy, known as *grouping.* Grouping is typically employed when faced with an expression containing four terms.

Throughout this section, it is important to reinforce the fact that factoring is essentially expansion (multiplication) done in reverse. Therefore, we will first look at problem which requires us to multiply two expressions, and then try to reverse the process.

Example 136. Write the expanded form for the given expression.

$$\begin{array}{rl} (2a+3)(5b+2) & \text{Distribute } (2a+3) \text{ into second parentheses} \\ 5b(2a+3)+2(2a+3) & \text{Distribute each monomial} \\ 10ab+15b+4a+6 & \text{Our solution} \end{array}$$

Our solution above has four terms in it. We arrived at this solution by focusing on the two parts, $5b(2a+3)$ and $2(2a+3)$.

When attempting to factor by grouping, we will always divide an expression into two parts, or groups: group one will contain the first two terms of our expression and group two will contain the last two terms. Then we can identify and factor the GCF out of each group. In doing this, our hope is that what is left over in each group will be the same expression. If the resulting expressions match, we can then factor out this matching GCF from both of our designated groups, writing what is left in a new set of parentheses.

Although the description of this method can sound rather complicated, the next few examples will help to clear up any lingering questions. We will start by working through the last example in reverse, factoring instead of multiplying.

Example 137. Factor the given expression.

$10ab + 15b + 4a + 6$	Split expression into two groups
$\boxed{10ab + 15b \mid +4a + 6}$	GCF on left is $5b$, on the right is 2
$\boxed{5b(2a + 3) \mid +2(2a + 3)}$	$(2a + 3)$ appears twice! Factor out this GCF
$(2a + 3)(5b + 2)$	Our solution

The key for grouping to be successful is for the two binomials to match exactly, once the GCF has been factored out of both groups. If there is any difference between the two binomials, we either have to do some adjusting or we cannot factor by grouping. Consider the following example.

Example 138. Factor the given expression.

$6x^2 + 9xy - 14x - 21y$	Split expression into two groups
$\boxed{6x^2 + 9xy \mid -14x - 21y}$	GCF on left is $3x$, on right is 7
$\boxed{3x(2x + 3y) \mid +7(-2x - 3y)}$	The signs in the parentheses do not match!

When the signs on both terms do not match, we can easily make them match by factoring a negative out of the GCF on the right side. Instead of 7 we will use -7. This will change the signs inside the second set of parentheses.

$\boxed{3x(2x + 3y) \mid -7(2x + 3y)}$	$(2x + 3y)$ appears twice! Factor out this GCF
$(2x + 3y)(3x - 7)$	Our solution

It will often be easy to recognize if we will need to factor out a negative sign when grouping. Specifically, if the first term of the first binomial is positive, the first term of the second binomial will also need to be positive. Similarly, if the first term of the first binomial is negative, the first term of the second binomial will also need to be negative.

Example 139. Factor the given expression.

$5xy - 8x - 10y + 16$	Split the expression into two groups
$\boxed{5xy - 8x} \boxed{-10y + 16}$	GCF on left is x, on right we need to factor out a negative, we will use -2
$\boxed{x(5y - 8)} \boxed{-2(5y - 8)}$	$(5y - 8)$ appears twice! Factor out this GCF
$(5y - 8)(x - 2)$	Our solution

Occasionally, when factoring out a GCF from either group, it will appear as though there is nothing that can be factored out. In this case a GCF of either 1 or -1 should be used. Often this will be all that is required, in order to match up the two binomials.

Example 140. Factor the given expression.

$12ab - 14a - 6b + 7$	Split the expression into two groups
$\boxed{12ab - 14a} \boxed{-6b + 7}$	GCF on left is $2a$, on right use GCF of -1
$\boxed{2a(6b - 7)} \boxed{-1(6b - 7)}$	$(6b - 7)$ appears twice! Factor out this GCF
$(6b - 7)(2a - 1)$	Our solution

Example 141. Factor the given expression.

$6x^3 - 15x^2 + 2x - 5$	Split expression into two groups
$\boxed{6x^3 - 15x^2} \boxed{+2x - 5}$	GCF on left is $3x^2$, on right use GCF of 1
$\boxed{3x^2(2x - 5)} \boxed{+1(2x - 5)}$	$(2x - 5)$ appears twice! Factor out this GCF
$(2x - 5)(3x^2 + 1)$	Our solution

When grouping, the selection or assignment of terms for each group can also be an area of concern. In particular, if after factoring out the GCF from the preassigned groups, the binomials do not match *and* cannot be adjusted as in the previous examples, a change in the group assignments may be necessary. In the next example we will demonstrate this by eventually moving the second term to the end of the given expression, to see if grouping may still be used.

Example 142. Factor the given expression.

$$4a^2 - 21b^3 + 6ab - 14ab^2 \quad \text{Split the expression into two groups}$$

$$\boxed{4a^2 - 21b^3 \mid +6ab - 14ab^2} \quad \text{GCF on left is } 1, \text{ on right is } 2ab$$

$$\boxed{1(4a^2 - 21b^3) \mid +2ab(3 - 7b)} \quad \text{Binomials do not match! Move second term to end}$$

$$4a^2 + 6ab - 14ab^2 - 21b^3 \quad \text{Start over, split expression into two groups}$$

$$\boxed{4a^2 + 6ab \mid -14ab^2 - 21b^3} \quad \text{GCF on left is } 2a, \text{ on right is } -7b^2$$

$$\boxed{2a(2a + 3b) \mid -7b^2(2a + 3b)} \quad (2a + 3b) \text{ appears twice! Factor out this GCF}$$

$$(2a + 3b)(2a - 7b^2) \quad \text{Our solution}$$

When rearranging terms the expression might still appear to be out of order. Sometimes after factoring out the GCF the resulting binomials appear “backwards”. There are two scenarios where this can happen: one with addition and one with subtraction. In the first scenario, if the binomials are say $(a + b)$ and $(b + a)$, then we do not have to do any extra work. This is because addition is a *commutative* operation. This means that the sum of two terms is the same, regardless of their order. For example, $5 + 3 = 3 + 5 = 8$.

Example 143. Factor the given expression.

$$7 + y - 3xy - 21x \quad \text{Split the expression into two groups}$$

$$\boxed{7 + y \mid -3xy - 21x} \quad \text{GCF on left is } 1, \text{ on the right is } -3x$$

$$\boxed{1(7 + y) \mid -3x(y + 7)} \quad y + 7 \text{ and } 7 + y \text{ are equal, use either one}$$

$$(y + 7)(1 - 3x) \quad \text{Our solution}$$

In the second scenario, if the binomials contain subtraction, then we need to be a bit more careful. For example, if the binomials are $(a - b)$ and $(b - a)$, we will factor a negative sign out of either group (usually the second). Notice what happens when we factor out a -1 in the following example.

Example 144. Factor the given expression.

$$
\begin{array}{rl}
(b-a) & \text{Factor out a } -1 \\
-1(-b+a) & \text{Resulting binomial contains addition,} \\
 & \quad \text{we may switch the order} \\
-1(a-b) & \text{The order of the subtraction has been switched!}
\end{array}
$$

Generally we will not show all of the steps in the previous example when simplifying. Instead, we will simply factor out a negative sign and switch the order of the subtraction to make the resulting binomials. As with previous concepts, this omission should only be made by the student when the skill has been mastered. We conclude our discussion of grouping with one final example.

Example 145. Factor the given expression.

$$
\begin{array}{rl}
8xy-12y+15-10x & \text{Split the expression into two groups} \\
\\
\boxed{8xy-12y}\boxed{15-10x} & \text{GCF on left is } 4y, \text{ on right is } 5 \\
\\
\boxed{4y(2x-3)}\boxed{+5(3-2x)} & \text{Need to switch order,} \\
 & \quad \text{Factor negative sign out of second binomial} \\
\\
\boxed{4y(2x-3)}\boxed{-5(2x-3)} & (2x-3) \text{ appears twice! Factor out this GCF} \\
\\
(2x-3)(4y-5) & \text{Our solution}
\end{array}
$$

Trinomials with Leading Coefficient $a = 1$ (L19)

Objective: Factor a trinomial with a leading coefficient of one.

Factoring polynomial expressions that contain three terms, or *trinomials*, is the most essential factorization skill to algebra. Consequently, it is also the most important factorization skill to master. Again, since factoring is basically multiplication performed in reverse, we will start with a multiplication example and look at how we can reverse the process.

Example 146. Write the expanded form for the given expression.

$$
\begin{array}{rl}
(x+6)(x-4) & \text{Distribute } (x+6) \text{ through second parentheses} \\
x(x+6)-4(x+6) & \text{Distribute each monomial through parentheses} \\
x^2+6x-4x-24 & \text{Combine like terms} \\
x^2+2x-24 & \text{Our solution}
\end{array}
$$

Notice that if we reverse the last three steps of the previous example, the process looks like grouping. This is because it is grouping! In the second-to-last line, the GCF of the first two terms is x and the GCF of the last two terms is -4. In this manner, we will factor trinomials by writing them as a polynomial containing four terms, and then factor by grouping. This is demonstrated in the following example, which is the previous one done in reverse.

Example 147. Factor the given expression.

$$\begin{array}{rl} x^2+2x-24 & \text{split middle (linear) term into} +6x-4x \\ x^2+6x-4x-24 & \text{Grouping : GCF on left is } x, \text{ on right is } -4 \\ x(x+6)-4(x+6) & (x+6) \text{ appears twice, factor out this GCF} \\ (x+6)(x-4) & \text{Our solution} \end{array}$$

The trick to make these problems work resides in how we split the middle (or linear) term. Why did we choose $+6x-4x$ and not $+5x-3x$? The reason is because $6x-4x$ is the only combination that will allow grouping to work! So how do we know what is the one combination that we need? To find the correct way to split the middle term we will use what is called the *ac*-method. Later, we will discuss why it is called the *ac*-method.

The idea behind the *ac*-method is that we must find a pair of numbers that *multiply* to get the last (or constant) term in the expression and *add* to get the coefficient of the middle (or linear) term. In the previous example, we would want two numbers whose product is -24 and sum is $+2$. The only numbers that can do this are 6 and -4, since $6\cdot-4=-24$ and $6+(-4)=2$. This method is demonstrated in the next few examples.

Example 148. Factor the given expression.

$$\begin{array}{rl} x^2+9x+18 & \text{Need to multiply to } 18, \text{add to } 9 \\ x^2+6x+3x+18 & \text{Use } 6 \text{ and } 3, \text{ split the middle term} \\ x(x+6)+3(x+6) & \text{Factor by grouping} \\ (x+6)(x+3) & \text{Our solution} \end{array}$$

Example 149. Factor the given expression.

$$\begin{array}{rl} x^2-4x+3 & \text{Need to multiply to } 3, \text{add to } -4 \\ x^2-3x-x+3 & \text{Use } -3 \text{ and } -1, \text{ split the middle term} \\ x(x-3)-1(x-3) & \text{Factor by grouping} \\ (x-3)(x-1) & \text{Our solution} \end{array}$$

Example 150. Factor the given expression.

$$\begin{array}{rl} x^2-8x-20 & \text{Need to multiply to } -20, \text{add to } -8 \\ x^2-10x+2x-20 & \text{Use } -10 \text{ and } 2, \text{ split the middle term} \\ x(x-10)+2(x-10) & \text{Factor by grouping} \\ (x-10)(x+2) & \text{Our solution} \end{array}$$

Often when factoring we are faced with an expression containing two variables. These expressions are treated just like those containing only one variable. As in the next example, we will still use the coefficients to decide how to split the linear term.

Example 151. Factor the given expression.

$$\begin{array}{rl} a^2-9ab+14b^2 & \text{Need to multiply to } 14, \text{add to } -9 \\ a^2-7ab-2ab+14b^2 & \text{Use } -7 \text{ and } -2, \text{ split the middle term} \\ a(a-7b)-2b(a-7b) & \text{Factor by grouping} \\ (a-7b)(a-2b) & \text{Our solution} \end{array}$$

As the past few examples has shown, it is very important to be aware of negatives in finding the right pair of numbers used to split the linear term. Consider the following example, done *incorrectly*, ignoring negative signs.

Example 152. Factor the given expression.

$$\begin{array}{rl} x^2+5x-6 & \text{Need to multiply to } 6, \text{add to } 5 \\ x^2+2x+3x-6 & \text{Use } 2 \text{ and } 3, \text{ split the middle term} \\ x(x+2)+3(x-2) & \text{Factor by grouping} \\ ??? & \text{Binomials do not match!} \end{array}$$

Because we did not consider the negative sign with the constant term of -6 to find our pair of numbers, the binomials did not match and grouping was unsuccessful. Now we show factorization done correctly.

Example 153. Factor the given expression.

$$\begin{array}{rl} x^2+5x-6 & \text{Need to multiply to } -6, \text{add to } 5 \\ x^2+6x-x-6 & \text{Use } 6 \text{ and } -1, \text{ split the middle term} \\ x(x+6)-1(x+6) & \text{Factor by grouping} \\ (x+6)(x-1) & \text{Our solution} \end{array}$$

At this point, one might notice a shortcut for factoring such expressions. Once we identify the two numbers that are used to split the linear term, these will be the two numbers in each of our factors! In the previous example, the numbers used to split the linear term were 6 and -1, our factors turned out to be $(x+6)(x-1)$.

This shortcut will not always work out, as we will see momentarily. We can use it, however, when we have a leading coefficient of $a=1$ for our quadratic term ax^2, which has been the case for all of the trinomials we have factored thus far. This shortcut is employed in the next few examples.

Example 154. Factor the given expression.

$$\begin{array}{rl} x^2-7x-18 & \text{Need to multiply to } -18, \text{add to } -7 \\ & \text{Use } -9 \text{ and } 2, \text{write the factors} \\ (x-9)(x+2) & \text{Our solution} \end{array}$$

Example 155. Factor the given expression.

$$\begin{array}{rl} m^2 - mn - 30n^2 & \text{Need to multiply to } -30, \text{add to } -1 \\ & \text{Use } 5 \text{ and } -6, \text{write the factors} \\ & \text{Do not forget second variable!} \\ (m+5n)(m-6n) & \text{Our solution} \end{array}$$

It is also certainly possible to have a trinomial that does not factor using the *ac*-method. If there is no combination of numbers that multiplies and adds to the correct numbers, then we say that we cannot factor the polynomial "nicely", or easily. Later on in the chapter, we will learn of some other methods and terminology for factoring quadratic expressions of this type. The next example is of a quadratic expression that is not easily factorable.

Example 156. Factor the given expression.

$$\begin{array}{rl} x^2 + 2x + 6 & \text{Need to multiply to } 6, \text{add to } 2 \\ 1 \cdot 6 \text{ and } 2 \cdot 3 & \text{Only possibilities to multiply to } 6, \text{ none add to } 2 \\ \text{Not easily factorable} & \text{Our solution} \end{array}$$

Later, we will discover that the quadratic expression above cannot be factored over the real numbers. In other words, there exist no real numbers r and s such that

$$x^2 + 2x + 6 = (x - r)(x - s)$$

Such expressions are said to be *irreducible over the reals*, and any factorization will require us to use *complex* numbers. Complex numbers will be discussed later on in the chapter.

When factoring any expression, it is important to not forget about first identifying a GCF of all the given terms. If all the terms in an expression have a common factor we will want to first factor out the GCF before using any other method.

Example 157. Factor the given expression.

$$\begin{array}{rl} 3x^2 - 24x + 45 & \text{GCF of all terms is } 3, \text{ factor this out first} \\ 3(x^2 - 8x + 15) & \text{Need to multiply to } 15, \text{add to } -8 \\ & \text{Use } -5 \text{ and } -3, \text{ write the factors} \\ 3(x-5)(x-3) & \text{Our solution} \end{array}$$

Again it is important to comment on the shortcut of jumping right to the factors, this only works if the leading coefficient $a = 1$. In the example above, we applied the shortcut only *after* we factored out a GCF of 3. Next, we will look at how this process changes when $a \neq 1$.

Trinomials with Leading Coefficient $a \neq 1$ (L20)

Objective: Factor a trinomial with a leading coefficient of $a \neq 1$.

When factoring trinomials we used the ac-method to split the middle (or linear) term and then factor by grouping. The ac-method gets its name from the general trinomial expression, $ax^2 + bx + c$, where a, b, and c are the leading coefficient, linear coefficient, and constant term, respectively.

The ac-method is named as such because we will use the product $a \cdot c$ to help find out what two numbers we will need for grouping later on. Previously, we always found two numbers whose product was equal to c, since the leading coefficient a was 1 in our expression (so $ac = 1c = c$). Now we will be working with trinomials where $a \neq 1$, so we will need to identify two numbers that multiply to ac and add to b. Aside from this adjustment, the process will be the same as before.

Example 158. Factor the given expression.

$$\begin{aligned} 3x^2 + 11x + 6 &\quad \text{Multiply to } ac \text{ or } (3)(6) = 18, \text{ add to } 11 \\ 3x^2 + 9x + 2x + 6 &\quad \text{The numbers are 9 and 2, split the linear term} \\ 3x(x+3) + 2(x+3) &\quad \text{Factor by grouping} \\ (x+3)(3x+2) &\quad \text{Our solution} \end{aligned}$$

When $a = 1$, we were able to use a shortcut, using the numbers that split the linear term for our factors. The previous example illustrates an important point: the shortcut does not work when $a \neq 1$. Therefore, we must go through all the steps of grouping in order to factor the expression.

Example 159. Factor the given expression.

$$\begin{aligned} 8x^2 - 2x - 15 &\quad \text{Multiply to } ac \text{ or } (8)(-15) = -120, \text{ add to } -2 \\ 8x^2 - 12x + 10x - 15 &\quad \text{The numbers are } -12 \text{ and } 10, \text{ split the linear term} \\ 4x(2x-3) + 5(2x-3) &\quad \text{Factor by grouping} \\ (2x-3)(4x+5) &\quad \text{Our solution} \end{aligned}$$

Example 160. Factor the given expression.

$$\begin{aligned} 10x^2 - 27x + 5 &\quad \text{Multiply to } ac \text{ or } (10)(5) = 50, \text{ add to } -27 \\ 10x^2 - 25x - 2x + 5 &\quad \text{The numbers are } -25 \text{ and } -2, \text{ split the linear term} \\ 5x(2x-5) - 1(2x-5) &\quad \text{Factor by grouping} \\ (2x-5)(5x-1) &\quad \text{Our solution} \end{aligned}$$

The same process will work for trinomials containing two variables.

Example 161. Factor the given expression.

$$\begin{array}{rl} 4x^2 - xy - 5y^2 & \text{Multiply to } ac \text{ or } (4)(-5) = -20, \text{ add to } -1 \\ 4x^2 + 4xy - 5xy - 5y^2 & \text{The numbers are } 4 \text{ and } -5, \text{ split the middle term} \\ 4x(x+y) - 5y(x+y) & \text{Factor by grouping} \\ (x+y)(4x-5y) & \text{Our solution} \end{array}$$

As always, when factoring we will first look for a GCF before using any other method, including the ac-method. Factoring out the GCF first also has the added bonus of making the coefficients smaller, so other methods become easier.

Example 162. Factor the given expression.

$$\begin{array}{rl} 18x^3 + 33x^2 - 30x & \text{GCF is } 3x, \text{ factor this out first} \\ 3x(6x^2 + 11x - 10) & \text{Multiply to } ac \text{ or} (6)(-10) = -60, \text{ add to } 11 \\ 3x(6x^2 + 15x - 4x - 10) & \text{The numbers are } 15 \text{ and } -4, \text{ split the linear term} \\ 3x[3x(2x+5) - 2(2x+5)] & \text{Factor by grouping} \\ 3x(2x+5)(3x-2) & \text{Our solution} \end{array}$$

As was the case with trinomials when $a = 1$, not all trinomials can be factored easily. If there are no combinations that multiply and add correctly, then we can say the trinomial is not easily factorable. In such cases, the expression will require a new method of factorization, and may even be shown to be irreducible over the real numbers (the factorization will require complex numbers). We will encounter such expressions and learn how to properly handle them before the end of this chapter. We conclude this section with one such example.

Example 163. Factor the given expression.

$$\begin{array}{rl} 3x^2 + 2x - 7 & \text{Multiply to } ac \text{ or } (3)(-7) = -21, \text{ add to } 2 \\ -3(7) \text{ and } -7(3) & \text{Only two ways to multiply to } -21, \text{ neither adds to } 2 \\ \text{Not easily factorable} & \text{Our solution} \end{array}$$

It turns out that the previous example *is* factorable over the real numbers, but we will postpone this discovery until later.

Solving by Factoring (L21)

Objective: Solve polynomial equations by factoring and using the Zero Factor Property.

When solving linear equations such as $2x - 5 = 21$ we can solve for the variable directly by adding 5 and dividing by 2 to get 13. When working with quadratic equations (or higher degree polynomials), however, we cannot simply isolate the variable as we did with linear equations. One property that we can use to solve for the variable is known as the zero factor property.

Zero Factor Property : If $ab = 0$ then either $a = 0$ or $b = 0$.

The zero factor property tells us that if the product of two factors is zero, then one of the factors must be zero. We can use this property to help us solve factored polynomials as in the following example.

Example 164. Solve the given equation for all possible values of x.

$$\begin{array}{rl}
(2x-3)(5x+1)=0 & \text{One factor must be zero}\\
2x-3=0 \ \text{ or } \ 5x+1=0 & \text{Set each factor equal to zero}\\
\underline{\mathbf{+3}\quad\mathbf{+3}} \qquad \underline{\mathbf{-1}\quad\mathbf{-1}} & \text{Solve each equation}\\
2x=3 \quad \text{or} \quad 5x=-1 & \\
\overline{\mathbf{2}}\quad\overline{\mathbf{2}} \qquad\quad \overline{\mathbf{5}}\quad\overline{\mathbf{5}} & \\
x=\dfrac{3}{2} \ \text{ or } \ -\dfrac{1}{5} & \text{Our solution}
\end{array}$$

For the zero factor property to work we must have factors to set equal to zero. This means if an expression is not already factored, we must first factor it.

Example 165. Solve the given equation for all possible values of x.

$$\begin{array}{rl}
4x^2+x-3=0 & \text{Factor using the } ac\text{-method,}\\
 & \quad \text{multiply to } -12, \text{ add to } 1\\
4x^2-3x+4x-3=0 & \text{The numbers are } -3 \text{ and } 4, \text{ split the linear term}\\
x(4x-3)+1(4x-3)=0 & \text{Factor by grouping}
\end{array}$$

$$\begin{array}{rl}
(4x-3)(x+1)=0 & \text{One factor must be zero}\\
4x-3=0 \ \text{ or } \ x+1=0 & \text{Set each factor equal to zero}\\
\underline{\mathbf{+3}\quad\mathbf{+3}} \qquad \underline{\mathbf{-1}\quad\mathbf{-1}} & \text{Solve each equation}\\
4x=3 \quad \text{or} \quad x=-1 & \\
\overline{\mathbf{4}}\quad\overline{\mathbf{4}} & \\
x=\dfrac{3}{4} \ \text{ or } \ -1 & \text{Our solution}
\end{array}$$

Another important aspect of the zero factor property is that before we factor, our equation must equal zero. If it does not, we must move terms around so it does equal zero. Although it is not necessary, it will generally be easier to keep our leading term ax^2 positive.

Example 166. Solve the given equation for all possible values of x.

$$\begin{array}{rl}
x^2 = 8x - 15 & \text{Set equal to zero by moving terms to the left} \\
x^2 - 8x + 15 = 0 & \text{Factor using the } ac\text{-method,} \\
 & \quad \text{multiply to } 15, \text{ add to } -8 \\
(x-5)(x-3) = 0 & \text{The numbers are } -5 \text{ and } -3 \\
x - 5 = 0 \text{ or } x - 3 = 0 & \text{Set each factor equal to zero} \\
\underline{\mathbf{+5 \quad +5}} \qquad \underline{\mathbf{+3 \quad +3}} & \text{Solve each equation} \\
x = 5 \text{ or } 3 & \text{Our solution}
\end{array}$$

Example 167. Solve the given equation for all possible values of x.

$$\begin{array}{rl}
(x-7)(x+3) = -9 & \text{Not equal to zero, multiply first} \\
x^2 - 7x + 3x - 21 = -9 & \text{Combine like terms} \\
x^2 - 4x - 21 = -9 & \text{Move } -9 \text{ to other side so equation equals zero} \\
\underline{\mathbf{+9 \quad +9}} & \\
x^2 - 4x - 12 = 0 & \text{Factor using the } ac\text{-method,} \\
 & \quad \text{multiply to } -12, \text{ add to } -4 \\
(x-6)(x+2) = 0 & \text{The numbers are } 6 \text{ and } -2 \\
x - 6 = 0 \text{ or } x + 2 = 0 & \text{Set each factor equal to zero} \\
\underline{\mathbf{+6 \quad +6}} \qquad \underline{\mathbf{-2 \quad -2}} & \text{Solve each equation} \\
x = 6 \text{ or } -2 & \text{Our solution}
\end{array}$$

Example 168. Solve the given equation for all possible values of x.

$$\begin{array}{rl}
3x^2 + 4x - 5 = 7x^2 + 4x - 14 & \text{Set equal to zero by} \\
 & \quad \text{moving terms to the right} \\
0 = 4x^2 - 9 & \text{Factor using difference of squares} \\
0 = (2x+3)(2x-3) & \text{One factor must be zero} \\
2x + 3 = 0 \text{ or } 2x - 3 = 0 & \text{Set each factor equal to zero} \\
\underline{\mathbf{-3 \quad -3}} \qquad \underline{\mathbf{+3 \quad +3}} & \text{Solve each equation} \\
2x = -3 \quad \text{or} \quad 2x = 3 & \\
\overline{\mathbf{2}} \quad \overline{\mathbf{2}} \qquad \overline{\mathbf{2}} \quad \overline{\mathbf{2}} & \\
x = -\frac{3}{2} \text{ or } \frac{3}{2} & \text{Our solution}
\end{array}$$

Most quadratic equations will have two unique real solutions. It is possible, however, to have only one real solution as the next example illustrates.

Example 169. Solve the given equation for all possible values of x.

$$\begin{array}{rl}
4x^2 = 12x - 9 & \text{Set equal to zero by moving terms to left} \\
4x^2 - 12x + 9 = 0 & \text{Factor using the } ac\text{-method,} \\
 & \quad \text{multiply to 36, add to } -12 \\
4x^2 - 6x - 6x + 9 = 0 & \text{Use } -6 \text{ and } -6, \text{ split the linear term} \\
2x(2x-3) - 3(2x-3) = 0 & \text{Factor by grouping} \\
(2x-3)^2 = 0 & \text{A perfect square!} \\
2x - 3 = 0 & \text{Set this factor equal to zero} \\
\underline{\mathbf{+3 \quad +3}} & \text{Solve the equation} \\
2x = 3 & \\
\overline{\mathbf{2}} \quad \overline{\mathbf{2}} & \\
x = \dfrac{3}{2} & \text{Our solution}
\end{array}$$

As always, it will be important to factor out the GCF first if we have one. This GCF is also a factor, and therefore must also be set equal to zero using the zero factor property. The next example illustrates this.

Example 170. Solve the given equation for all possible values of x.

$$\begin{array}{rl}
4x^2 = 8x & \text{Set equal to zero by moving the terms to left} \\
 & \text{Be careful, } 4x^2 \text{ and } 8x \text{ are not like terms!} \\
4x^2 - 8x = 0 & \text{Factor out the GCF of } 4x \\
4x(x-2) = 0 & \text{One factor must be zero} \\
4x = 0 \ \text{ or } \ x - 2 = 0 & \text{Set each factor equal to zero} \\
\overline{\mathbf{4}} \quad \overline{\mathbf{4}} \qquad \underline{\mathbf{+2 \quad +2}} & \text{Solve each equation} \\
x = 0 \ \text{ or } \ 2 & \text{Our solution}
\end{array}$$

If our polynomial is not a quadratic, as in the next example, we may end up with more than two solutions.

Example 171. Solve the given equation for all possible values of x.

$$\begin{array}{rl}
2x^3 - 14x^2 + 24x = 0 & \text{Factor out the GCF of } 2x \\
2x(x^2 - 7x + 12) = 0 & \text{Factor with } ac\text{-method,} \\
 & \quad \text{multiply to 12, add to } -7 \\
2x(x-3)(x-4) = 0 & \text{The numbers are } -3 \text{ and } -4 \\
2x = 0 \ \text{ or } \ x - 3 = 0 \ \text{ or } \ x - 4 = 0 & \text{Set each factor equal to zero} \\
\overline{\mathbf{2}} \quad \overline{\mathbf{2}} \qquad \underline{\mathbf{+3 \quad +3}} \qquad \underline{\mathbf{+4 \quad +4}} & \text{Solve each equation} \\
x = 0 \ \text{ or } \ 3 \ \text{ or } \ 4 & \text{Our solution}
\end{array}$$

Example 172. Solve the given equation for all possible values of x.

$$\begin{array}{rl}
6x^2+21x-27=0 & \text{Factor out the GCF of 3} \\
3(2x^2+7x-9)=0 & \text{Factor with } ac\text{-method,} \\
 & \quad \text{multiply to } -18\text{, add to } 7 \\
3(2x^2+9x-2x-9)=0 & \text{The numbers are 9 and } -2 \\
3[x(2x+9)-1(2x+9)]=0 & \text{Factor by grouping} \\
3(2x+9)(x-1)=0 & \text{One factor must be zero}
\end{array}$$

$$\begin{array}{llll}
3=0 \text{ or } 2x+9=0 \text{ or } x-1=0 & & & \text{Set each factor equal to zero} \\
\mathbf{3\neq 0} \quad \underline{\mathbf{-9}\ \ \mathbf{-9}} \quad \underline{\mathbf{+1}\ \ \mathbf{+1}} & & & \text{Solve each equation} \\
\quad 2x=-9 \quad \text{or} \quad x=1 & & & \\
\quad \overline{\mathbf{2}} \quad\ \ \overline{\mathbf{2}} & & & \\
\quad x=-\dfrac{9}{2} \text{ or } 1 & & & \text{Our solution}
\end{array}$$

In the previous example, the GCF did not have a variable in it. When we set this factor equal to zero we got a false statement. No solutions come from this factor. We can only disregard setting the GCF factor equal to zero if it is a constant.

Just as not all polynomials can be easily factored, all equations cannot be easily solved by factoring. If an equation does not factor easily, we will have to solve it using another method. These other methods are saved for another section.

It is a common question to ask if it is permissible to get rid of the square on the variable x^2 by taking the square root of both sides of the equation. Although it is sometimes possible, there are a few properties of square roots that we have not covered yet, and thus it is more common to inadvertently break a rule of roots that we may not yet be aware of. Because of this, we will postpone a discussion of roots until we see how they can be employed properly to solve quadratic equations. For now, we will advise to *not* take the square root of both sides of an equation!

Square Roots and the Imaginary Number i

Square Roots (L22)

Objective: Simplify and evaluate expressions involving square roots.

Recall that we define a radical (or n^{th} root) as follows.

$$\sqrt[n]{a}=a^{1/n},$$

where a is a nonnegative real number and n a positive integer.

We refer to n as the *index* of the radical and a as the *radicand*. Square roots (when $n = 2$) are the most common type of radical used in mathematics. A square root "un-squares" a number. In other words, if $a^2 = b$, then $\sqrt[2]{b} = a$. This relationship between a square and a square root is similar to the relationship between multiplication and division, as well as the relationship between addition and subtraction. In each case, the two operations are said to be *inverse* operations of each other. The idea behind inverses and the notion of an inverse function is one that will be discussed in detail in a later chapter.

Note that although we have written the index of 2 for the square root of b in the previous paragraph, in general, the index of a square root is usually omitted ($\sqrt[2]{b} = \sqrt{b}$). Using numbers, since $5^2 = 25$ we say the square root of 25 is 5, and write $\sqrt{25} = 5$.

While a great deal more could be said about radicals and how they fit in with the properties of exponents, for now we will focus our attention on properly working with expressions that contain a square root.

The following example gives several square roots.

Example 173.

$\sqrt{0} = 0$	$\sqrt{121} = 11$
$\sqrt{1} = 1$	$\sqrt{625} = 25$
$\sqrt{4} = 2$	$\sqrt{-81} =$ Undefined

The final example of $\sqrt{-81}$ is currently considered to be undefined, since the square root of a negative number does not equal a real number. This is because if we square a positive or a negative number, the answer will be positive, not to mention that $0^2 = 0$. Thus we can only take square roots of nonnegative numbers (positive numbers or zero). In the second part of this section, we will define a method we can use to work with and evaluate negative square roots. For now we will simply say they are undefined.

Not all numbers have a "nice" (or *rational*) square root. For example, if we found $\sqrt{8}$ on our calculator, the answer would be 2.8284271247461900097... , and even this number is a rounded approximation of the square root. To be as accurate as possible, we will never use the calculator to find decimal approximations of square roots. Instead we will express roots in simplest radical form. We will do this using a property known as the product rule of radicals (in this case, square roots).

$$\textbf{Product Rule of Square Roots}: \sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$$

More generally,

$$\textbf{Product Rule of Radicals}: \sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$

We can use the product rule of square roots to simplify an expression such as $\sqrt{180} = \sqrt{36 \cdot 5}$ by splitting it into two roots, $\sqrt{36} \cdot \sqrt{5}$, and simplifying the first root, $6\sqrt{5}$. The trick in this process is being able to recognize that an expression like $\sqrt{180}$ may be rewritten as $\sqrt{36 \cdot 5}$, since $180 = 36 \cdot 5$. In the case of $\sqrt{8}$, we may write $\sqrt{8} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}$.

There are several ways of applying the product rule of square roots. The most common and, with a bit of practice, fastest method is to find perfect squares that divide nicely into the radicand (the number under the radical). This is demonstrated in the next example.

Example 174. Completely simplify the given radical.

$\sqrt{75}$	75 is divisible by 25, a perfect square
$\sqrt{25\cdot 3}$	Split into factors
$\sqrt{25}\cdot\sqrt{3}$	Product rule, take the square root of 25
$5\sqrt{3}$	Our solution

If there is a coefficient in front of the radical to begin with, the problem merely becomes a big multiplication problem, as seen in the next example.

Example 175. Completely simplify the given radical.

$5\sqrt{63}$	63 is divisible by 9, a perfect square
$5\sqrt{9\cdot 7}$	Split into factors
$5\sqrt{9}\cdot\sqrt{7}$	Product rule, take the square root of 9
$5\cdot 3\sqrt{7}$	Multiply coefficients
$15\sqrt{7}$	Our solution

As we simplify radicals using this method it is important to be sure our final answer can not be simplified further, as seen in the next example.

Example 176. Completely simplify the given radical.

$\sqrt{72}$	72 is divisible by 9, a perfect square
$\sqrt{9\cdot 8}$	Split into factors
$\sqrt{9}\cdot\sqrt{8}$	Product rule, take the square root of 9
$3\sqrt{8}$	But 8 is also divisible by a perfect square, 4
$3\sqrt{4\cdot 2}$	Split into factors
$3\sqrt{4}\cdot\sqrt{2}$	Product rule, take the square root of 4
$3\cdot 2\sqrt{2}$	Multiply
$6\sqrt{2}$	Our solution

The previous example could also have been done in fewer steps if we had noticed that $72 = 36\cdot 2$, but often it can take longer to discover the larger perfect square than to simplify in several steps.

Variables often are part of the radicand as well. When taking the square roots of one (or more) variable(s), we can divide the associated exponent of the variable by two, and write the new exponent outside of the root. For example, $\sqrt{x^{10}} = x^5$. This follows from a familiar property of exponents, shown below.

$$(x^m)^n = x^{mn}$$

Applying this to a square root, we have

$$\sqrt{x^m} = (x^m)^{1/2} = x^{m/2}.$$

So, $\sqrt{x^{10}} = x^{10/2} = x^5$. This makes sense, since

$$\begin{aligned}
\left(x^5\right)^2 &= x^5 \cdot x^5 \\
&= \underbrace{x \cdot x \cdot \ldots \cdot x}_{10 \text{ times}} \\
&= x^{10} \\
&= x^{5 \cdot 2}.
\end{aligned}$$

In summary, when squaring, we multiply the exponent by two. So, when taking a square root, we divide the exponent by two. The following example demonstrates this property.

Example 177. Completely simplify the given radical.

$$\begin{array}{rl}
-5\sqrt{18x^4y^6z^{10}} & \text{18 is divisible by 9, a perfect square} \\
-5\sqrt{9 \cdot 2x^4y^6z^{10}} & \text{Split into factors} \\
-5\sqrt{9} \cdot \sqrt{2} \cdot \sqrt{x^4} \cdot \sqrt{y^6} \cdot \sqrt{z^{10}} & \text{Product rule applied to all parts} \\
-5 \cdot 3x^2y^3z^5\sqrt{2} & \text{Simplify roots, divide exponents by 2} \\
-15x^2y^3z^5\sqrt{2} & \text{Multiply coefficients, Our solution}
\end{array}$$

We can't always nicely divide the exponent on a variable by two, since sometimes we will have a positive remainder. If there is a positive remainder, this means the remainder is left inside the radical, and the whole number portion (or quotient) represents the exponent that should appear outside of the radical. The next example demonstrates this.

Example 178. Completely simplify the given radical.

$$\begin{array}{rl}
\sqrt{20x^5y^9z^6} & \text{20 is divisible by 4, a perfect square} \\
\sqrt{4 \cdot 5x^5y^9z^6} & \text{Split into factors} \\
\sqrt{4} \cdot \sqrt{5} \cdot \sqrt{x^5} \cdot \sqrt{y^9} \cdot \sqrt{z^6} & \text{Simplify, divide exponents by 2} \\
 & \quad \text{Remainder is left inside} \\
2x^2y^4z^3\sqrt{5xy} & \text{Our solution}
\end{array}$$

If we focus on the variable y in the previous example, when we divide the exponent 9 by 2, we get a quotient of 4 and a remainder of 1 ($9 = 2 \cdot 4 + 1$). Consequently, $\sqrt{y^9} = y^4\sqrt{y}$. This same idea also applies to x above, since the exponent 5 is odd and therefore will have a remainder of 1. Since the exponent for z is even, it is divisible by 2, and so the radical in our final answer does not contain z.

Introduction to Complex Numbers (L23)

Objective: Simplify expressions involving complex numbers.

In mathematics, when the current number system does not provide the tools to solve the problems the culture is working with, we tend to develop new ways for solving the problem. Throughout history, this has been the case with the need for a number that represents nothing (0), smaller than zero (negatives), between integers (fractions), and between fractions (irrational numbers). This is also the case for square roots of negative numbers. To work with the square root of a negative number, mathematicians have defined what we now know as imaginary and complex numbers.

$$\textbf{Imaginary Number } \boldsymbol{i}: \ \boldsymbol{i^2 = -1} \ \textbf{(thus } \boldsymbol{i = \sqrt{-1}}\textbf{)}$$

Examples of imaginary numbers include $3i, -6i, \frac{3}{5}i$ and $3i\sqrt{5}$. A *complex number* is one that contains both a real and imaginary part, such as $2 + 5i$.

$$\textbf{Complex Number: } \boldsymbol{a + bi}, \text{ where } \boldsymbol{a} \text{ and } \boldsymbol{b} \text{ are real numbers, } \boldsymbol{i = \sqrt{-1}}$$

With this definition, the square root of a negative number will no longer be considered undefined. We now will be able to perform basic operations with the square root of a negative number. First we will consider powers of imaginary numbers. We will do this by manipulating our definition of $i^2 = -1$. If we multiply both sides of the definition by i, the equation becomes $i^3 = -i$. Then if we multiply both sides of the equation again by i, the equation becomes $i^4 = -i^2 = -(-1) = 1$, or simply $i^4 = 1$. Multiplying again by i gives $i^5 = i$. One more time gives $i^6 = i^2 = -1$.

This pattern continues, and we can see a cycle forming. Specifically, as the exponents on i increase, our simplified value for i^n will cycle through the simplified values for i, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$. As there are 4 different possible answers in this cycle, if we divide the exponent n by 4 and consider the remainder, we can easily simplify any power of i by knowing the following four values:

Cyclic Property of Powers of i

$$\begin{aligned} i^0 &= 1 \\ i^1 &= i \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 = i^0 &= 1 \end{aligned}$$

Example 179. Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl} i^{35} & \text{Divide exponent by 4} \\ 35 = 4 \cdot 8 + 3 & \text{Use remainder as exponent for } i \\ i^3 & \text{Simplify} \\ -i & \text{Our solution} \end{array}$$

Example 180. Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl} i^{124} & \text{Divide exponent by 4} \\ 124 = 4 \cdot 31 + 0 & \text{Use remainder as exponent for } i \\ i^0 & \text{Simplify} \\ 1 & \text{Our solution} \end{array}$$

When performing the basic mathematical operations (addition, subtraction, multiplication, division) we may treat i just like any other variable. This means that when adding and subtracting complex numbers we may simply combine like terms.

Example 181. Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl} (2 + 5i) + (4 - 7i) & \text{Combine like terms, } \ 2 + 4 \ \text{ and } \ 5i - 7i \\ 6 - 2i & \text{Our solution} \end{array}$$

It is important to recognize what operation we are applying. A common mistake in the previous example is to view the parentheses and think that one must distribute. The previous example, however, requires addition. So we simply add (or combine) the like terms.

For problems involving subtraction the idea is the same, but we must first remember to distribute the negative to each term in the parentheses.

Example 182. Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl} (4 - 8i) - (3 - 5i) & \text{Distribute the negative} \\ 4 - 8i - 3 + 5i & \text{Combine like terms, } \ 4 - 3 \ \text{ and } \ -8i + 5i \\ 1 - 3i & \text{Our solution} \end{array}$$

Addition and subtraction may also appear in a single problem.

Example 183. Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl} (5i) - (3 + 8i) + (-4 + 7i) & \text{Distribute the negative} \\ 5i - 3 - 8i - 4 + 7i & \text{Combine like terms, } \ 5i - 8i + 7i \ \text{ and } \ -3 - 4 \\ -7 + 4i & \text{Our solution} \end{array}$$

Multiplying two (or more) complex numbers is similar to the multiplication of two binomials with one key exception. In each problem, we will want to simplify our final answer so that it does not contain any power of i greater than or equal to 2. This will always enable us to write our answer in the standard form of $a + bi$. We now show this in general below, remembering that $i^2 = -1$.

$$\begin{array}{rl}
(c+di)(g+hi) & \text{Expand} \\
cg+chi+dgi+dhi^2 & \text{Simplify, } \ i^2=-1 \\
cg+chi+dgi-dh & \text{Combine like terms} \\
(cg-dh)+(ch+dg)i & \text{Our solution, in standard form}
\end{array}$$

Here, $cg - dh$ represents the real part a and $ch + dg$ represents the imaginary part b of our resulting complex number $a + bi$.

Next we will see several examples to reinforce the concept. We will begin with the product of two imaginary numbers.

Example 184. Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl}
(3i)(7i) & \text{Multiply, } \ 3\cdot 7 \ \text{ and } \ i\cdot i \\
21i^2 & \text{Simplify, } \ i^2=-1 \\
21(-1) & \text{Multiply} \\
-21 & \text{Our solution}
\end{array}$$

Example 185. Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl}
5i(3i-7) & \text{Distribute} \\
15i^2-35i & \text{Simplify, } \ i^2=-1 \\
15(-1)-35i & \text{Multiply} \\
-15-35i & \text{Our solution}
\end{array}$$

Example 186. Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl}
(2-4i)(3+5i) & \text{Expand} \\
6+10i-12i-20i^2 & \text{Simplify, } \ i^2=-1 \\
6+10i-12i-20(-1) & \text{Multiply} \\
6+10i-12i+20 & \text{Combine like terms } \ 6+20 \ \text{ and } \ 10i-12i \\
26-2i & \text{Our solution}
\end{array}$$

Example 187. Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl}
(3i)(6i)(2-3i) & \text{Multiply first two monomials} \\
18i^2(2-3i) & \text{Simplify, } \ i^2=-1 \\
18(-1)(2-3i) & \text{Multiply} \\
-18(2-3i) & \text{Distribute} \\
-36+54i & \text{Our solution}
\end{array}$$

Notice that in the previous example we chose to simplify i^2 before distributing. This could also have been done *after* distributing $18i^2$ through $(2-3i)$. The resulting expression of $36i^2 - 54i^3$ will then simplify to match our solution above.

Recall that when squaring a binomial such as $(a-b)^2$, we must be careful to expand *completely*, and not forget the inner and outer terms of the product.

$$\begin{aligned}(a-b)^2 &= (a-b)(a-b)\\ &= a^2 - ab - ab + b^2\\ &= a^2 - 2ab + b^2\end{aligned}$$

The next example demonstrates this using complex numbers.

Example 188. Write the given expression as $a+bi$, where a and b are real numbers.

$(4-5i)^2$	Rewrite as a product of two binomials
$(4-5i)(4-5i)$	Expand
$16-20i-20i+25i^2$	Simplify, $i^2=-1$
$16-20i-20i-25$	Combine like terms
$-9-40i$	Our solution

When simplifying rational expressions (fractions) that contain imaginary or complex numbers in a denominator, we will employ the same strategy as that which is used for eliminating square roots from a denominator. This is a logical progression, since we defined i so that $i^2 = \sqrt{-1}$. We refer to this strategy as *rationalizing the denominator*, since the end result will be an expression in which the denominator is a rational number (it contains no radicals).

As we did with complex multiplication, we will first demonstrate the technique generally, followed by several examples.

$\dfrac{c+di}{g+hi}$	Multiply top and bottom by $g-hi$
$\dfrac{c+di}{g+hi}\cdot\left(\dfrac{g-hi}{g-hi}\right)$	Expand numerator and denominator
$\dfrac{cg-chi+dgi-dhi^2}{g^2-ghi+ghi-h^2i^2}$	Simplify, $i^2=-1$
$\dfrac{cg-chi+dgi+dh}{g^2-\cancel{ghi}+\cancel{ghi}+h^2}$	Combine like terms in top and bottom
$\dfrac{(cg+dh)+(dg-ch)i}{g^2+h^2}$	Rewrite as $a+bi$
$\left(\dfrac{cg+dh}{g^2+h^2}\right)+\left(\dfrac{dg-ch}{g^2+h^2}\right)i$	Our solution, in standard form

Here, $\dfrac{cg+dh}{g^2+h^2}$ represents the real part a and $\dfrac{dg-ch}{g^2+h^2}$ represents the imaginary part b of our resulting complex number $a+bi$. Remember that c, d, g and h all represent real numbers, so our denominator g^2+h^2 is also a real number.

As shown above, the expression that we will typically choose to rationalize with (in this case $g-hi$) is known as the *complex conjugate* to the original denominator $(g+hi)$. When multiplying two complex numbers that are conjugates to one another, the resulting product in our denominator (g^2+h^2) should have no imaginary part.

For our first example, we will start with a denominator which only contains an imaginary part, $0+bi$. In this case, although the complex conjugate would equal $0-bi$, we only need to multiply the numerator and denominator by i, since multiplying by $-bi$ would result in an eventual cancellation of $-b$ from the entire expression.

Example 189. Write the given expression as $a+bi$, where a and b are real numbers.

$\dfrac{7+3i}{-5i}$	A monomial in denominator, multiply by i
$\dfrac{7+3i}{-5i}\left(\dfrac{i}{i}\right)$	Distribute i in numerator
$\dfrac{7i+3i^2}{-5i^2}$	Simplify $i^2=-1$
$\dfrac{7i+3(-1)}{-5(-1)}$	Multiply
$\dfrac{7i-3}{5}$	Simplify, split up fraction
$\dfrac{7i}{5}-\dfrac{3}{5}$	Rewrite as $a+bi$
$-\dfrac{3}{5}+\dfrac{7}{5}i$	Our solution

As shown in the previous example, a solution for such problems can be written several different ways, for example $\dfrac{-3+7i}{5}$ or $-\dfrac{3}{5}+\dfrac{7}{5}i$. Although both answers are generally accepted, we will keep our final answers consistent with the definition of a complex number:

$$a+bi = (\text{Real part}) \ + \ (\text{Imaginary part})i.$$

Example 190. Write the given expression as $a + bi$, where a and b are real numbers.

$$\frac{2-6i}{4+8i} \quad \text{Binomial in denominator,}$$
$$\text{multiply by conjugate, } 4-8i$$
$$\frac{2-6i}{4+8i}\left(\frac{4-8i}{4-8i}\right) \quad \text{Expand the numerator,}$$
$$\text{denominator is a difference of two squares}$$
$$\frac{8-16i-24i+48i^2}{16-64i^2} \quad \text{Simplify } i^2=-1$$
$$\frac{8-16i-24i+48(-1)}{16-64(-1)} \quad \text{Multiply}$$
$$\frac{8-16i-24i-48}{16+64} \quad \text{Combine like terms}$$
$$\frac{-40-40i}{80} \quad \text{Reduce, factor out } 40 \text{ and divide}$$
$$\frac{-1-i}{2} \quad \text{Rewrite as } a+bi$$
$$-\frac{1}{2}-\frac{1}{2}i \quad \text{Our solution}$$

By rewriting $\sqrt{-1}$ as i, we can now simplify square roots with negatives underneath. We will use the product rule and simplify the negative as a factor of negative one. This is shown in the following examples.

Example 191. Write the given expression as $a + bi$, where a and b are real numbers.

$$\sqrt{-16} \quad \text{Consider the negative as a factor of } -1$$
$$\sqrt{-1\cdot 16} \quad \text{Take each root, square root of } -1 \text{ is } i$$
$$4i \quad \text{Our solution}$$

Example 192. Write the given expression as $a + bi$, where a and b are real numbers.

$$\sqrt{-24} \quad \text{Find perfect square factors . Factor out } -1$$
$$\sqrt{-1\cdot 4\cdot 6} \quad \text{Square root of } -1 \text{ is } i\text{, square root of 4 is 2}$$
$$2i\sqrt{6} \quad \text{Move } i \text{ over}$$
$$(2\sqrt{6})i \quad \text{Our solution}$$

When simplifying complex radicals, it is important that we take the -1 out of the radical (as an i) before we combine radicals.

Notice also that in the previous example our final answer is $(2\sqrt{6})i$ and not $2\sqrt{6i}$. Although the parentheses are not technically needed, they are included because there is a subtle mathematical difference between these two values, since having i *underneath* a square root ($\sqrt{6i}$) is not equivalent to having it *beside* the square root ($\sqrt{6}i$). This common mistake can be easily avoided by taking care not to extend the square root too far when writing our final answer. The parentheses are simply an added precaution. The same care must be made in order to distinguish an expression like $\sqrt{6}x$ from $\sqrt{6x}$.

Example 193. Write the given expression as $a + bi$, where a and b are real numbers.

$\sqrt{-6}\sqrt{-3}$	Simplify the negatives, bringing i out of radicals
$(i\sqrt{6})(i\sqrt{3})$	Multiply, $i^2 = -1$
$-\sqrt{18}$	Simplify the radical
$-\sqrt{9 \cdot 2}$	Take square root of 9
$-3\sqrt{2}$	Our solution

Lastly, when reducing fractions that involve i, as is often the case, we must take extra care to properly simplify and avoid any common mistakes. This is demonstrated in the following example.

Example 194. Write the given expression as $a + bi$, where a and b are real numbers.

$\dfrac{-15 - \sqrt{-200}}{20}$	We will simplify the radical first
$\sqrt{-200}$	Find perfect square factors . Factor out -1
$\sqrt{-1 \cdot 100 \cdot 2}$	Take square root of -1 and 100
$10i\sqrt{2}$	Put this back into original expression
$\dfrac{-15 - 10i\sqrt{2}}{20}$	Factor out 5 and divide
$\dfrac{-3 - 2i\sqrt{2}}{4}$	Simplify answer, split up fraction
$-\dfrac{3}{4} - \dfrac{2i\sqrt{2}}{4}$	Reduce, move i to side
$-\dfrac{3}{4} - \dfrac{\sqrt{2}}{2}i$	Our solution

By using $i = \sqrt{-1}$ we will be able to simplify expressions and solve problems that we could not before. In the next few sections, we will see how this will enable us to better understand quadratic equations and their graphs.

Vertex Form and Graphing (L24)

The Vertex Form

Objective: Express a quadratic equation in vertex form.

Recall the two forms of a quadratic equation, shown below. In both forms, assume $a \neq 0$.

Standard Form: $y = ax^2 + bx + c$, where a, b, and c are real numbers

Vertex Form: $y = a(x-h)^2 + k$, where a, h, and k are real numbers

Unlike the standard form, a quadratic equation written in vertex form allows for immediate recognition of the vertex (h, k), which will always coincide with either a maximum (if $a < 0$) or a minimum (if $a > 0$) on the accompanying graph, called a parabola. Additionally, using the vertex form, we can easily identify the *axis of symmetry* for the parabola, which is a vertical line $x = h$ that passes through the x-coordinate of the vertex and "splits" the graph into two identical halves.

When graphing parabolas, it will help to think of the axis of symmetry as a vertical line over which either half of the graph could be "folded", to produce the other half. This will allow us to reflect (by symmetry) any point on the parabola to the other side of the axis of symmetry, and identify another point on the graph. As a result, both points will have the same y-coordinate, and will be (horizontally) equidistant from the axis of symmetry. By reflecting points about the axis of symmetry, we can graph not just one, but two points on the graph, for every single value of x that we plug into the given equation.

Example 195. Consider $y = -2(x+1)^2 + 3$.

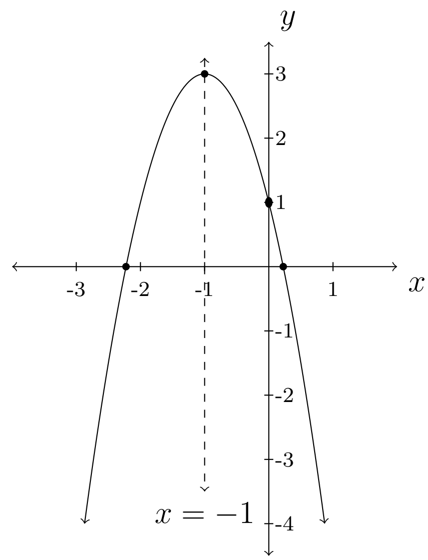
In this example we can see immediately that the vertex is at $(-1, 3)$. It is important that we not overlook the negative value for h. The axis of symmetry, passes through the x-coordinate for the vertex, $x = -1$.

Now to find more points on the parabola we can plug in $x = 0$. We can see that $y = -2(0+1)^2 + 3 = 1$, so $(0, 1)$ is a point on our parabola.

Since the point we just located sits one unit to the right of the axis of symmetry, we also know that the point $(-2, 1)$, sitting one unit to the left of the axis of symmetry will also be a point on our graph. We can always check this by plugging $x = -2$ into the equation and solving for y.

Similarly, we can plug in $x = 1$, a coordinate that is two units to the right of the axis of symmetry and get a y-coordinate of -5.

Thus an x-coordinate two units left of the axis, $x = -3$, will also yield a y-coordinate of -5. The accompanying graph shows our parabola, with the axis of symmetry appearing as a dashed vertical line at $x = -1$.



We began the discussion of vertex form in the introductory section of this chapter. It follows naturally to learn how to transform a quadratic equation that is given in standard form into one written in vertex form.

If $y = ax^2 + bx + c$ $(a \neq 0)$, we can identify the x-coordinate for the vertex (and consequently the equation for the axis of symmetry) using the following formula.

$$h = -\frac{b}{2a}$$

After identifying h, we can determine based upon the sign of the leading coefficient a whether the vertex will be a maximum (if a is negative, $a < 0$) or a minimum (if a is positive, $a > 0$). The equation for the vertical line $x = h$ will be our axis of symmetry.

Finally, we know that the y-coordinate for our vertex must occur somewhere on the axis of symmetry. This can easily be found by plugging $x = h$ back into the given equation for our quadratic, and simplifying to find the y-coordinate, which we will relabel as k.

Once we have h and k, we can use them, along with a, to write the vertex form for our quadratic,

$$y = a(x - h)^2 + k.$$

The following examples will clearly demonstrate this process.

Example 196. Identify the vertex and axis of symmetry for the parabola represented by the given quadratic equation.

$$\begin{aligned}
y = x^2 + 8x - 12 &\quad \text{Given an equation in standard form}\\
a = 1, \quad b = 8, \quad c = -12 &\quad \text{Identify } a, b, \text{ and } c\\
h = -\frac{b}{2a} = -\frac{8}{2(1)} = -4 &\quad \text{Identify } h\\
x = -4 &\quad \text{Use } h \text{ for axis of symmetry, a vertical line}\\
k = (-4)^2 + 8(-4) - 12 &\quad \text{Plug in } h \text{ to find } k\\
k = 16 - 32 - 12 = -28 &\quad \text{Simplify}\\
(-4, -28) &\quad \text{Write the vertex as an ordered pair } (h, k)
\end{aligned}$$

Example 197. Identify the vertex and axis of symmetry for the parabola represented by the given quadratic equation.

$$\begin{aligned}
y = -3x^2 + 6x - 1 &\quad \text{Given an equation in standard form}\\
a = -3, \quad b = 6, \quad c = -1 &\quad \text{Identify } a, b, \text{ and } c\\
h = -\frac{b}{2a} = -\frac{6}{2(-3)} = 1 &\quad \text{Identify } h\\
x = 1 &\quad \text{Use } h \text{ for axis of symmetry, a vertical line}\\
k = -3(1)^2 + 6(1) - 1 &\quad \text{Plug in } h \text{ to find } k\\
k = -3 + 6 - 1 = 2 &\quad \text{Simplify}\\
(1, 2) &\quad \text{Write the vertex as an ordered pair } (h, k)
\end{aligned}$$

Example 198. Identify the vertex and axis of symmetry for the parabola represented by the given quadratic equation.

$$\begin{aligned}
y = -x^2 - 12 &\quad \text{Given an equation in standard form}\\
a = -1, \quad b = 0, \quad c = -12 &\quad \text{Identify } a, b, \text{ and } c\\
h = -\frac{b}{2a} = -\frac{0}{2(-1)} = 0 &\quad \text{Identify } h\\
x = 0 &\quad \text{Use } h \text{ for axis of symmetry, a vertical line}\\
k = -(0)^2 - 12 &\quad \text{Plug in } h \text{ to find } k\\
k = 0 - 12 = -12 &\quad \text{Simplify}\\
(0, -12) &\quad \text{Write the vertex as an ordered pair } (h, k)
\end{aligned}$$

There is a more algebraic (and complicated) method of transforming a quadratic equation given in standard form into one that is in vertex form, known as *completing the square.* This method will be explained in detail towards the end of the chapter.

We will also see how the vertex form can be particularly useful when solving a quadratic equation, in order to identify the x-intercepts of the corresponding parabola. Solving a quadratic equation using the vertex form is know as the method of *extracting square roots*, and will be seen once we have had a thorough discussion of square roots, as well as complex numbers.

Graphing Quadratics

Objective: Graph equations in both standard and vertex forms.

Up until now, we have discussed the general shape of the graph of a quadratic equation (known as a *parabola*), but have only seen a few examples. Furthermore, most of our examples have only identified the vertex of the parabola, and perhaps an $x-$ or $y-$intercept of the graph. Although these examples have been able to show us the general shape of each graph (where it is centered, whether it opens up or down, whether it is narrow or wide), our steps for obtaining each graph have not followed a standard procedure. Here, we will define that procedure more precisely, and provide a few examples for reinforcement.

One way that we can always build a picture of the general shape of a graph is to make a table of values, as we will do in our first example.

Example 199. Sketch a graph of the quadratic equation $y = x^2 - 4x + 3$ by making a table of values and plotting points on the graph.

We will test five values to get an idea of the shape of the graph.

x	0	1	2	3	4
y					

$$
\begin{aligned}
y = (0)^2 + 4(0) + 3 = 0 - 0 + 3 &= 3 && \text{Plug in 0 for } x \text{ and evaluate.}\\
y = (1)^2 - 4(1) + 3 = 1 - 4 + 3 &= 0 && \text{Plug in 1 for } x \text{ and evaluate.}\\
y = (2)^2 - 4(2) + 3 = 4 - 8 + 3 &= -1 && \text{Plug in 2 for } x \text{ and evaluate.}\\
y = (3)^2 - 4(3) + 3 = 9 - 12 + 3 &= 0 && \text{Plug in 3 for } x \text{ and evaluate.}\\
y = (4)^2 - 4(4) + 3 = 16 - 16 + 3 &= 3 && \text{Plug in 4 for } x \text{ and evaluate.}
\end{aligned}
$$

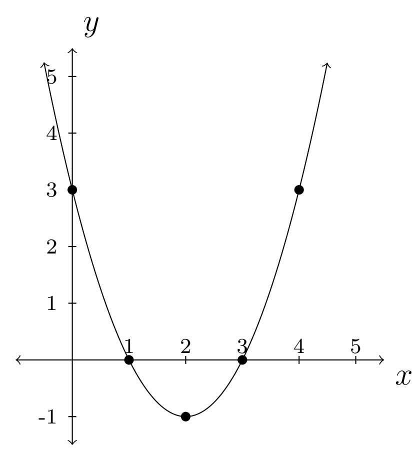
Our completed table is below.

x	0	1	2	3	4
y	3	0	-1	0	3

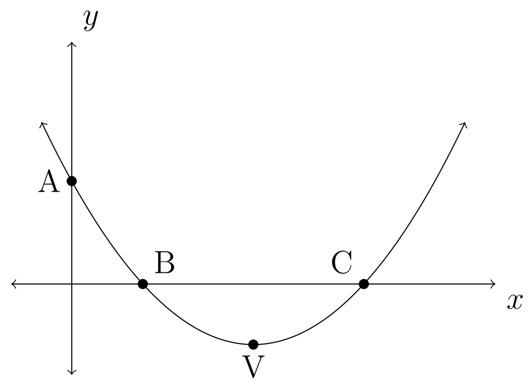
Plot the points on the $xy-$plane.

Plot the points $(0, 3), (1, 0), (2, -1), (3, 0)$, and $(4, 3)$.

Connect the dots with a smooth curve.



The above method to graph a parabola works for any equation, however, it can be very difficult to find a sufficient collection of points in order to identify the overall shape of the complete graph. For this reason, we will now formally identify several key points on the graph of a parabola, which will enable us to always determine a complete graph. These points are the $y-$intercept, $x-$intercepts, and the vertex (h, k).



Point A: y-intercept; where the graph crosses the vertical y-axis (when $x = 0$).

Points B and C: x-intercepts; where the graph crosses the horizontal x-axis (when $y = 0$)

Point V: vertex (h, k); The point of the minimum (or maximum) value, where the graph changes direction.

We will use the following method to find each of the key points on our parabola.

Steps for graphing a quadratic in standard form, $y = ax^2 + bx + c$.

1. Identify and plot the vertex: $h = -\dfrac{b}{2a}$. Plug h into the equation to find k. Resulting point is (h, k).
2. Identify and plot the y-intercept: Set $x = 0$ and solve. The y-intercept will correspond to the constant term c. Resulting point is $(0, c)$.
3. Identify and plot the x-intercept(s): Set $y = 0$ and solve for x. Depending on the expression, we will end up with zero, one or two $x-$intercepts.

Important: Up until now, we have only discussed how to solve a quadratic equation for x by factoring. If an expression is not easily factorable, we may not be able to identify the x-intercepts. Soon, we will learn of two additional methods for finding x-intercepts, which will prove especially useful, when an equation is not easily factorable.

After plotting these points we can connect them with a smooth curve to find a complete sketch of our parabola!

Example 200. Provide a complete sketch of the equation $y = x^2 + 4x + 3$.

$$
\begin{array}{rl}
y = x^2 + 4x + 3 & \text{Find the key points} \\
\\
h = -\dfrac{4}{2(1)} = -\dfrac{4}{2} = -2 & \text{To find the vertex, use } h = -\dfrac{b}{2a} \\
k = (-2)^2 + 4(-2) + 3 & \text{Plug } h \text{ into the equation to find } k \\
k = 4 - 8 + 3 & \text{Evaluate} \\
k = -1 & \text{The } y\text{-coordinate of the vertex} \\
(-2, -1) & \text{Vertex as a point} \\
\\
y = 3 & (0, c) \text{ is the } y\text{-intercept}
\end{array}
$$

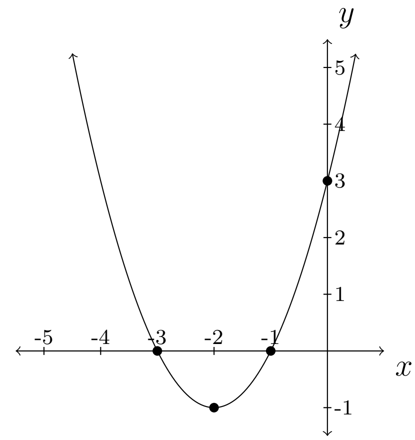
$$\begin{array}{rl}
0 = x^2 + 4x + 3 & \text{To find the } x\text{-intercept we solve the equation} \\
0 = (x+3)(x+1) & \text{Factor} \\
x + 3 = 0 \ \text{ and } \ x + 1 = 0 & \text{Set each factor equal to zero} \\
\underline{\mathbf{-3 \quad -3}} \qquad \underline{\mathbf{-1 \quad -1}} & \text{Solve each equation} \\
x = -3 \quad \text{and} \quad x = -1 & \text{Our } x\text{-intercepts}
\end{array}$$

Graph the y-intercept at $(0, 3)$,

the x-intercepts at $(-3, 0)$ and $(-1, 0)$,

and the vertex at $(-2, -1)$.

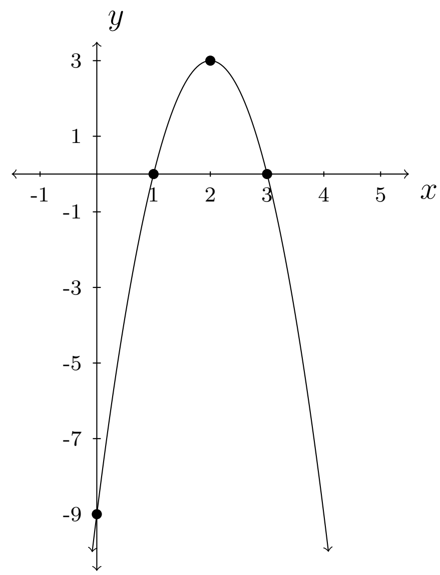
Connect the dots with a smooth curve in a 'U'-shape to get our parabola.



Remember that if $a > 0$, then our parabola will open upwards, as in the previous example. In our next example, $a < 0$, and the resulting parabola will open downwards.

Example 201. Provide a complete sketch of the equation $y = -3x^2 + 12x - 9$.

$$\begin{array}{rl}
y = -3x^2 + 12x - 9 & \text{Find key points} \\
 & \\
h = -\dfrac{12}{2(-3)} = -\dfrac{12}{-6} = 2 & \text{To find the vertex, use } h = -\dfrac{b}{2a} \\
k = -3(2)^2 + 12(2) - 9 & \text{Plug } h \text{ into the equation to find } k \\
k = -3(4) + 24 - 9 & \text{Evaluate} \\
k = 3 & \text{The } y\text{-coordinate of the vertex} \\
(2, 3) & \text{Vertex as a point} \\
 & \\
y = -9 & (0, c) \text{ is the } y\text{-intercept} \\
 & \\
0 = -3x^2 + 12x - 9 & \text{To find the } x\text{-intercept we solve the equation} \\
0 = -3(x^2 - 4x + 3) & \text{Factor out GCF first} \\
0 = -3(x-3)(x-1) & \text{Factor remaining trinomial} \\
x - 3 = 0 \ \text{ and } \ x - 1 = 0 & \text{Set each factor with a variable equal to zero} \\
\underline{\mathbf{+3 \quad +3}} \qquad \underline{\mathbf{+1 \quad +1}} & \text{Solve each equation} \\
x = 3 \quad \text{and} \quad x = 1 & \text{Our } x\text{-intercepts}
\end{array}$$



Graph the y-intercept at $(0, -9)$,

the x-intercepts at $(3, 0)$ and $(1, 0)$,

and the vertex at $(2, 3)$.

Connect the dots with a smooth curve in a 'U'-shape to get our parabola.

Remember that the graph of any quadratic is a parabola with the same 'U'-shape (opening up or down). If we plot our points and we cannot connect them in the correct 'U'-shape, then one or more of our points is likely to be incorrect. If this happens, a simple check of our calculations should identify where any mistakes were made! Each of our examples have involved quadratics that were easily factorable. Although we can still graph quadratics such as $y = x^2 - 3$ without actually identifying the $x-$intercepts, being able to identify them by solving $x^2 - 3 = 0$ and other more involved quadratic equations for x is a skill that we will eventually come to master.

The Method of Extracting Square Roots

Solve by Square Roots (L25)

Objective: Solve quadratic equations of the form $ax^2 + c = 0$ by introducing a square root.

Up until now, when attempting to solve an equation such as $x^2 - 4 = 0$, we have had no choice but to factor the expression on the left and set each factor equal to zero.

Example 202. Solve the given equation for all possible values of x.

$$
\begin{aligned}
x^2 - 4 &= 0 && \text{Factor, difference of two squares}\\
(x-2)(x+2) &= 0 && \text{Use Zero Factor Property}\\
x - 2 = 0 \text{ or } x + 2 &= 0 && \text{Solve}\\
x = 2 \quad \text{or} \quad &-2 && \text{Our solution}
\end{aligned}
$$

As an alternative method, in this subsection, we will look at solving the expression $ax^2+c = 0$ using an alternative method. Instead of attempting to factor the expression, we will introduce a square root, when solving for x. In each case, when faced with such an expression, our solution can be reached by applying the following three stops, in the specified order.

Solve $ax^2 + c = 0$.

Step	Equation
1. Subtract c.	$ax^2 = -c$
2. Divide by a.	$x^2 = -\dfrac{c}{a}$
3. Take a square root.	$x = \pm\sqrt{-\dfrac{c}{a}}$

Example 203. Solve the given equation for all possible values of x.

$$\begin{aligned} x^2 - 4 &= 0 && \text{Add } 4 \\ x^2 &= 4 && \text{Take a square root} \\ x &= \pm 2 && \text{Our solution} \end{aligned}$$

Recall that the values $x = 2$ and $x = -2$ are known as the *zeros* or *roots* of the equation $y = x^2 - 4$. Observe that the graphical interpretation of a zero is an $x-$intercept (when $y = 0$). In this case, the $x-$intercepts of the resulting parabola are at $(2, 0)$ and $(-2, 0)$.

Example 204. Solve the given equation for all possible values of x.

$$\begin{aligned} 5x^2 + 60 &= 0 && \text{Subtract } 60 \\ 5x^2 &= -60 && \text{Divide by } 5 \\ x^2 &= -12 && \text{Take a square root} \\ x &= \pm\sqrt{-12} && \text{Imaginary roots; Simplify} \\ x &= \pm 2\sqrt{3}i && \text{Our solution} \end{aligned}$$

In this example, we see that our two solutions, $x = 2\sqrt{3}i$ and $x = -2\sqrt{3}i$ are not real. Hence, the corresponding parabola for $y = 5x^2 + 60$ will have no $x-$intercepts.

In what follows, we will refer to the more general form of this method as *extracting square roots.*

Extracting Square Roots (L26)

Objective: Solve quadratic equations using the method of extracting square roots.

We will now introduce a new technique for identifying the zeros of a quadratic equation, known as the method of *extracting square roots.* The method of extracting square roots will only be employed once we have identified the vertex form for a given quadratic, $y = a(x-h)^2 + k$. The general steps for the method are shown below, and the requirement of the vertex form will be essential.

Example 205. Determine the zeros of the quadratic equation $y = ax^2 + bx + c$, where $a \neq 0$.

First obtain the vertex form: $h = -\dfrac{b}{2a}$, set $x = h$ to find k.

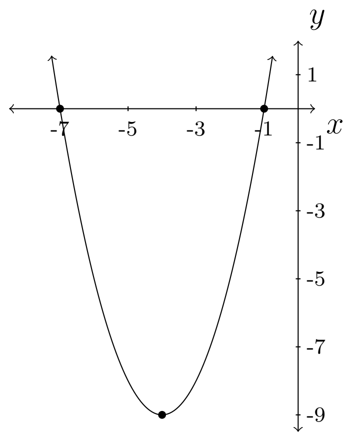
$$\begin{aligned}
a(x-h)^2 + k &= 0 && \text{Vertex form} \\
\underline{\boldsymbol{-k} \quad} & \underline{\boldsymbol{-k}} && \text{Subtract } k \text{ from both sides} \\
a(x-h)^2 &= -k \\
\overline{\boldsymbol{a}} \qquad & \overline{\boldsymbol{a}} && \text{Divide both sides by } a \\
(x-h)^2 &= -\frac{k}{a} \\
\sqrt{(x-h)^2} &= \pm\sqrt{-\frac{k}{a}} && \text{Take square root of both sides} \\
& && \quad \text{to extract radicand, } x-h \\
x - h &= \pm\sqrt{-\frac{k}{a}} \\
\underline{\boldsymbol{+h}} \qquad & \underline{\boldsymbol{+h}} && \text{Add } h \text{ to both sides} \\
x &= h \pm \sqrt{-\frac{k}{a}} && \text{Our solution}
\end{aligned}$$

In the previous example, there are two important points to consider. First is the introduction of the square root into the equation. This step is the reason behind the name of the method, and its success hinges upon the fact that the vertex form contains a single instance of the variable x. Unlike with the vertex form, if we were to introduce a square root directly to the equation $ax^2 + bx + c = 0$ (using the standard form), we would immediately reach a dead end, and be unable to simplify the resulting equation. This is primarily because we cannot combine the "unlike" terms ax^2 and bx, and we cannot split up sums (and differences) of terms underneath a square root.

Additionally, it is critical that we include a '$\pm$' on the right side of the equation once the square root has been introduced. The justification for this follows from the fact that there are always two values (one positive and one negative) that will equal the value underneath a square root (assuming that value is nonzero, since $\sqrt{0} = 0$). For example, $\sqrt{4} = \pm 2$ and $\sqrt{-9} = \pm 3i$.

We now present a few examples that demonstrate the method, as well as some of the possibilities for the number of zeros, and consequently, the number of $x-$intercepts of the corresponding graph.

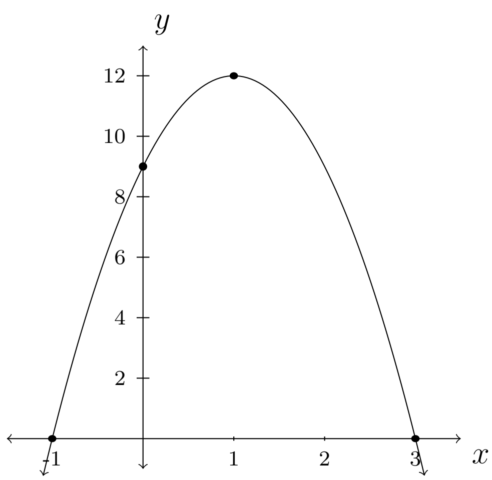
Example 206. Use the method of extracting square roots to find the zeros of the equation $y = (x+4)^2 - 9$.



$$
\begin{array}{rl}
0 = (x+4)^2 - 9 & \text{Set equal to zero and solve} \\
\underline{\mathbf{+9} \qquad\qquad \mathbf{+9}} & \text{Isolate the square} \\
9 = (x+4)^2 & \\
\pm\sqrt{9} = \sqrt{(x+4)^2} & \text{Square root both sides} \\
\pm 3 = x + 4 & \text{Solve for } x \\
\underline{\mathbf{-4}} \quad \underline{\mathbf{-4}} & \text{Subtract } 4 \\
x = \pm 3 - 4 & \text{Two solutions} \\
x = 3 - 4 \Rightarrow x = -1 & \text{One solution} \\
x = -3 - 4 \Rightarrow x = -7 & \text{The other solution}
\end{array}
$$

Our zeros are $x = -7$ and -1. The corresponding $x-$intercepts are at $(-7, 0)$ and $(-1, 0)$.

Example 207. Use the method of extracting square roots to find the zeros of the equation $y = -3(x-1)^2 + 12$.



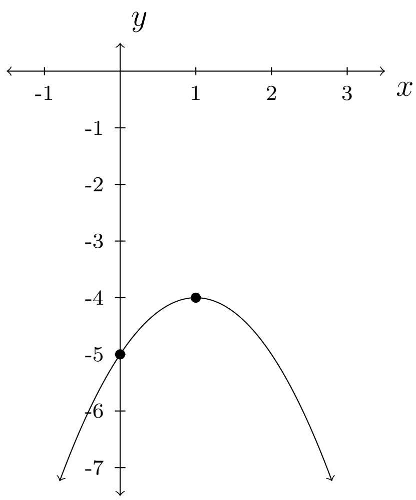
$$
\begin{array}{rl}
0 = -3(x-1)^2 + 12 & \text{Set equal to zero and solve} \\
\underline{\mathbf{-12} \qquad\qquad \mathbf{-12}} & \text{Subtract } 12 \\
-12 = -3(x-1)^2 & \text{Isolate the square,} \\
\overline{\mathbf{-3}} \qquad \overline{\mathbf{-3}} & \quad \text{divide both sides by } -3 \\
4 = (x-1)^2 & \\
\pm\sqrt{4} = \sqrt{(x-1)^2} & \text{Square root both sides} \\
\pm 2 = x - 1 & \text{Solve for } x \\
\underline{\mathbf{+1}} \quad \underline{\mathbf{+1}} & \text{Add } 1 \\
x = \pm 2 + 1 & \text{Two solutions} \\
x = 1 - 2 \Rightarrow x = -1 & \text{One solution} \\
x = 1 + 2 \Rightarrow x = 3 & \text{The other solution}
\end{array}
$$

Our two zeros are $x = -1$ and $x = 3$.

In some cases, the introduction of a square root results in an imaginary number. This scenario coincides with our corresponding parabola having no $x-$intercepts. In the previous example, if we were to change the sign of k from $+12$ to -12, the corresponding parabola would still open downwards, while having a vertex at $(1, -12)$, located below the $x-$axis. This will result in the appearance of a $\sqrt{-4} = 2i$, rather than a $\sqrt{4}$, in our solution. Consequently, there will be no real zeros for the equation and no $x-$intercepts on its graph.

We conclude this section with a final example, which will also result in no real zeros.

Example 208. Use the method of extracting square roots to find the zeros of the equation $y = -1(x-1)^2 - 4$.



$$
\begin{array}{rcll}
0 &=& -1(x-1)^2 - 4 & \text{Set equal to zero and solve} \\
\underline{\mathbf{+4}} & & \underline{\mathbf{+4}} & \text{Add } 4 \\
4 &=& -1(x-1)^2 & \text{Isolate the square,} \\
\overline{\mathbf{-1}} & & \overline{\mathbf{-1}} & \quad \text{divide both sides by } -1 \\
-4 &=& (x-1)^2 & \\
\pm\sqrt{-4} &=& \sqrt{(x-1)^2} & \text{Square root both sides} \\
\pm 2i &=& x-1 & \text{Solve for } x \\
\underline{\mathbf{+1}} & & \underline{\mathbf{+1}} & \text{Add } 1 \\
x &=& \pm 2i + 1 & \text{Two solutions} \\
x &=& 1 - 2i & \text{One solution} \\
x &=& 1 + 2i & \text{The other solution}
\end{array}
$$

Once again, the negative appearing under the square root results in two complex zeros (no real zeros). Graphically, the function never touches or crosses the $x-$axis.

Completing the Square

Objective: Solve quadratic equations by completing the square.

In this section, we will introduce a method for obtaining the vertex form of a quadratic function from the standard form, without having to rely on the vertex formula $h = -\dfrac{b}{2a}$. This method is known as *completing the square.* To complete the square, and convert a quadratic expression $ax^2 + bx + c$ from standard form to the vertex form $a(x-h)^2 + k$ (without our prior knowledge of the relationship between h, a and b), we will first start by considering the expression $ax^2 + bx$.

Observe that if a quadratic is of the form $x^2 + bx + c$, and *is* a perfect square, the constant term, c, can be found by the formula $\left(\frac{1}{2} \cdot b\right)^2$. This is shown in the following examples. In each example, we will find the number needed to complete the perfect square, and then factor it.

Example 209. Identify the constant term c that is needed to factor the given trinomial as a perfect square.

$$x^2+8x+c \qquad c=\left(\frac{1}{2}\cdot b\right)^2 \text{ and our } b=8$$

$$\left(\frac{1}{2}\cdot 8\right)^2=4^2=16 \qquad \text{The necessary constant term is } 16$$

$$x^2+8x+16 \qquad \text{Our desired trinomial; factor}$$

$$(x+4)^2 \qquad \text{Our solution}$$

Example 210. Identify the constant term c that is needed to factor the given trinomial as a perfect square.

$$x^2-7x+c \qquad c=\left(\frac{1}{2}\cdot b\right)^2 \text{ and our } b=7$$

$$\left(\frac{1}{2}\cdot 7\right)^2=\left(\frac{7}{2}\right)^2=\frac{49}{4} \qquad \text{The necessary constant term is } \frac{49}{4}$$

$$x^2-7x+\frac{49}{4} \qquad \text{Our desired trinomial; factor}$$

$$\left(x-\frac{7}{2}\right)^2 \qquad \text{Our solution}$$

Example 211. Identify the constant term c that is needed to factor the given trinomial as a perfect square.

$$x^2+\frac{5}{3}x+c \qquad c=\left(\frac{1}{2}\cdot b\right)^2 \text{ and our } b=\frac{5}{3}$$

$$\left(\frac{1}{2}\cdot\frac{5}{3}\right)^2=\left(\frac{5}{6}\right)^2=\frac{25}{36} \qquad \text{The necessary constant term is } \frac{25}{36}$$

$$x^2+\frac{5}{3}x+\frac{25}{36} \qquad \text{Our desired trinomial; factor}$$

$$\left(x+\frac{5}{6}\right)^2 \qquad \text{Our solution}$$

The process demonstrated in the previous examples may be used to obtain the vertex form of a quadratic. The following set of steps describes the process used to complete the square. Since all three of the previous examples contained a leading coefficient of $a=1$, an example where $a\neq 1$ has been included below to illustrate the special care that must be taken in this case.

Expression	$3x^2+18x-6$
1. Separate constant term from variables	$(3x^2+18x)-6$
2. Factor out a from each term in parentheses	$3(x^2+6x)-6$
3. Determine value to complete the square: $\left(\frac{1}{2}\cdot b\right)^2$	$\left(\frac{1}{2}\cdot 6\right)^2=3^2=9$
4. Add & subtract value to expression in parentheses	$3(x^2+6x+9-9)-6$
5. Separate subtracted value from other three terms, making sure to multiply by a	$3(x^2+6x+9)-3(9)-6$
6. Combine constant terms outside parentheses	$3(x^2+6x+9)^2-27-6$
7. Factor remaining trinomial	$3(x+3)^2-33$

Example 212. Use the method of completing the square to solve the given equation.

$$
\begin{aligned}
4x^2+40x+51&=0 && \text{Equation in standard form}\\
(4x^2+40x)+51&=0 && \text{Separate constant term}\\
4(x^2+10x)+51&=0 && \text{Factor out } a\\
\left(\frac{1}{2}\cdot 10\right)^2&=5^2=25 && \text{Complete the square : find } \left(\frac{1}{2}\cdot b\right)^2\\
4(x^2+10x+25-25)+51&=0 && \text{Add and subtract } 25 \text{ inside parentheses}\\
4(x^2+10x+25)-4(25)+51&=0 && \text{Separate trinomial}\\
4(x^2+10x+25)^2-100+51&=0 && \text{Simplify: combine constant terms,}\\
& && \quad\text{factor trinomial}\\
4(x+5)^2-49&=0 && \text{Solve by extracting square roots}\\
(x+5)^2&=\frac{49}{4} && \text{Isolate the square}\\
\sqrt{(x+5)^2}&=\pm\sqrt{\frac{49}{4}} && \text{Square root both sides}\\
x+5&=\pm\frac{7}{2} && \text{Subtract } 5 \text{ from both sides}\\
\underline{\mathbf{-5}}\quad&\quad\underline{\mathbf{-5}} && \\
x&=-5\pm\frac{7}{2} && \\
x&=-\frac{17}{2}\ \text{ or }\ -\frac{3}{2} && \text{Our solution}
\end{aligned}
$$

Example 213. Use the method of completing the square to solve the given equation.

$$
\begin{array}{rl}
x^2-3x-2=0 & \text{Equation in standard form} \\
(x^2-3x)-2=0 & \text{Separate constant term} \\
 & \text{Leading coefficient is } a=1 \\
\left(\frac{1}{2}\cdot -3\right)^2=\left(-\frac{3}{2}\right)^2=\frac{9}{4} & \text{Complete the square : find } \left(\frac{1}{2}\cdot b\right)^2 \\
\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)-2=0 & \text{Add and subtract } \frac{9}{4} \text{ inside parentheses} \\
\left(x^2-3x+\frac{9}{4}\right)-\frac{9}{4}-2=0 & \text{Separate trinomial} \\
\left(x^2-3x+\frac{9}{4}\right)-\frac{9}{4}-2=0 & \text{Simplify: combine constant terms,} \\
 & \quad\text{factor trinomial} \\
\left(x-\frac{3}{2}\right)^2-\frac{17}{4}=0 & \text{Solve by extracting square roots} \\
\left(x-\frac{3}{2}\right)^2=\frac{17}{4} & \text{Isolate the square} \\
\sqrt{\left(x-\frac{3}{2}\right)^2}=\pm\sqrt{\frac{17}{4}} & \text{Square root both sides} \\
x-\frac{3}{2}=\pm\frac{\sqrt{17}}{2} & \text{Reduce square root;} \\
\underline{+\frac{\mathbf{3}}{\mathbf{2}}} \qquad \underline{+\frac{\mathbf{3}}{\mathbf{2}}} & \text{Add } \frac{3}{2} \text{ to both sides} \\
x=\frac{3}{2}\pm\frac{\sqrt{17}}{2} & \\
x=\frac{3+\sqrt{17}}{2} \text{ or } \frac{3-\sqrt{17}}{2} & \text{Our solution}
\end{array}
$$

As the previous example shows, completing the square when $a = 1$ can be seen as slightly easier than when $a \neq 1$. Our last example demonstrates how we can more also handle the case when $a \neq 1$ early on in our solution, by simply dividing the equation by a.

Example 214. Use the method of completing the square to solve the given equation.

$$\begin{array}{rl}
3x^2 - 2x + 7 = 0 & \text{Equation in standard form} \\
\overline{\mathbf{3}} \quad \overline{\mathbf{3}} \quad \overline{\mathbf{3}} \quad \overline{\mathbf{3}} & \text{Divide both sides by 3} \\
x^2 - \frac{2}{3}x + \frac{7}{3} = 0 & \text{Resulting equation has } a = 1 \\
\left(\frac{1}{2} \cdot -\frac{2}{3}\right)^2 = \left(-\frac{1}{3}\right)^2 = \frac{1}{9} & \text{Complete the square: find } \left(\frac{1}{2} \cdot b\right)^2 \\
x^2 - \frac{2}{3}x + \frac{1}{9} - \frac{1}{9} + \frac{7}{3} = 0 & \text{Add and subtract } \frac{1}{9} \text{ to left side} \\
\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) - \frac{1}{9} + \frac{7}{3} = 0 & \text{Combine constant terms by} \\
& \quad \text{obtaining a common denominator} \\
-\frac{1}{9} + \frac{7}{3} = -\frac{1}{9} + \frac{21}{9} = \frac{20}{9} & \\
\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) + \frac{20}{9} = 0 & \text{Factor trinomial} \\
\left(x - \frac{1}{3}\right)^2 + \frac{20}{9} = 0 & \text{Solve by extracting square roots} \\
\left(x - \frac{1}{3}\right)^2 = -\frac{20}{9} & \text{Isolate the square} \\
\sqrt{\left(x - \frac{1}{3}\right)^2} = \pm\sqrt{\frac{-20}{9}} & \text{Square root both sides} \\
x - \frac{1}{3} = \frac{\pm 2i\sqrt{5}}{3} & \text{Simplify the radical} \\
\underline{+\frac{\mathbf{1}}{\mathbf{3}}} \qquad \underline{+\frac{\mathbf{1}}{\mathbf{3}}} & \text{Add } \frac{1}{3} \text{ to both sides} \\
x = \frac{1}{3} \pm \frac{2\sqrt{5}}{3}i & \text{Our solution}
\end{array}$$

As we mentioned earlier, completing the square is simply an alternative method to the vertex formula for converting a quadratic expression from standard form into vertex form. Still, as many of the previous examples have demonstrated, we will often need to work with fractions and be comfortable finding common denominators when solving quadratic equations using this method. Although this can be intimidating, with enough practice, one should be able to easily solve almost any quadratic equation by completing the square.

In the next section, we will present one final method for determining the zeros of a quadratic.

The Quadratic Formula (L28) and the Discriminant (L27)

Objective: Solve quadratic equations using the quadratic formula. Use the discriminant to determine the number of real solutions to a quadratic equation.

Recall that the general from of a quadratic equation is $y = ax^2 + bx + c$, where $a \neq 0$. We are now ready to solve the general equation $ax^2 + bx + c = 0$ for x by completing the square, which we show in the following example.

Example 215. Solve the equation $ax^2 + bx + c = 0$ for all values of x using the method of completing the square.

$$ax^2 + bx + c = 0 \quad \text{Divide each term by } a$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad \text{Separate constant term } \frac{c}{a}$$

$$\left(x^2 + \frac{b}{a}x\right) + \frac{c}{a} = 0 \quad \text{Complete the square}$$

$$\left(\frac{1}{2} \cdot \frac{b}{a}\right)^2 = \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} \quad \text{Add and subtract } \frac{b^2}{4a^2} \text{ inside parentheses}$$

$$\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + \frac{c}{a} = 0 \quad \text{Separate trinomial}$$

$$\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a^2} + \frac{c}{a} = 0 \quad \text{Simplify:}$$

$$-\frac{b^2}{4a^2} + \frac{c}{a}\left(\frac{4a}{4a}\right) = -\frac{b^2}{4a^2} + \frac{4ac}{4a^2} = -\frac{b^2 - 4ac}{4a^2} \quad \text{(1) Combine constant terms}$$

$$\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) = \left(x + \frac{b}{2a}\right)^2 \quad \text{(2) Factor trinomial}$$

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} = 0 \quad \text{Now solve by extracting square roots}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \quad \text{Isolate the square}$$

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \quad \text{Square root both sides}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \quad \text{Subtract } \frac{b}{2a} \text{ from both sides}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{Write as single fraction}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{Our solution}$$

This solution is a very important one to us. Since we solved a *general* equation by completing the square, we can now use this formula to solve any quadratic equation. Once we identify what $a, b,$ and c are, we can substitute those values into the equation $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and simplify in order to find our solution to the given quadratic. This formula is known as the *quadratic formula*. We call the expression underneath the square root, $b^2 - 4ac$, the *discriminant* of the quadratic equation $ax^2 + bx + c = 0$, and will see its importance later on in the section.

Quadratic Formula: The solutions to $ax^2 + bx + c = 0$ are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Discriminant: The discriminant of a quadratic equation $ax^2 + bx + c = 0$ is the expression

$$D = b^2 - 4ac.$$

We can use the quadratic formula to solve any quadratic, this is shown in the following examples.

Notice that we focus on calculating the discriminant first, and that it will have a major impact on the type of solutions that we receive.

Example 216. Solve the given equation for all values of x.

$$\begin{array}{rl}
x^2 + 3x + 2 = 0 & \text{Identify } a, b, \text{ and } c \\
a = 1,\ b = 3,\ c = 2 & \text{Use quadratic formula} \\
x = \dfrac{-3 \pm \sqrt{3^2 - 4(1)(2)}}{2(1)} & \text{Substitute } a, b, \text{ and } c \text{ without simplifying} \\
x = \dfrac{-3 \pm \sqrt{3^2 - 4(1)(2)}}{2(1)} & \\
x = \dfrac{-3 \pm \sqrt{9 - 8}}{2} & \text{Simplify} \\
x = \dfrac{-3 \pm \sqrt{1}}{2} & \text{Discriminant is } 1 \\
x = \dfrac{-3 \pm 1}{2} & \text{Evaluate } \pm;\ \text{write as two equations} \\
x = \dfrac{-3 + 1}{2} \text{ or } \dfrac{-3 - 1}{2} & \text{Simplify} \\
x = \dfrac{-2}{2} \text{ or } \dfrac{-4}{2} & \\
x = -1 \text{ or } -2 & \text{Our solutions}
\end{array}$$

Notice that the previous equation resulted in two real solutions. This is directly related to the discriminant being positive (in this case, 1). If the discriminant had been zero, then we would not have had anything underneath the square root, meaning that the plus or minus ($\pm$) would have had no effect on the rest of the procedure. Consequently, we would have only had one real solution. Furthermore, since the discriminant was a perfect square, we actually could have factored our quadratic from the start.

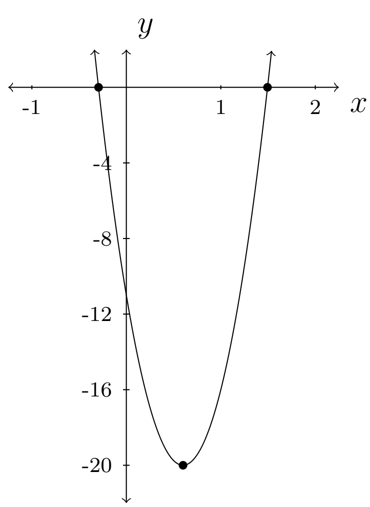
$$x^2 + 3x + 2 = (x + 1)(x + 2)$$

It is important to mention that when solving using the quadratic formula, we must remember to first set the given equation equal to zero and make sure the quadratic is in standard form.

Example 217. Solve the given equation for all values of x.

$$\begin{aligned}
25x^2 &= 30x + 11 && \text{First set equal to zero} \\
25x^2 - 30x - 11 &= 0 && \text{Identify } a, b, \text{ and } c \\
a = 25,\ b = -30,\ c &= -11 && \text{Use quadratic formula}
\end{aligned}$$

$$\begin{aligned}
x &= \frac{30 \pm \sqrt{(-30)^2 - 4(25)(-11)}}{2(25)} && \text{Substitute } a, b, \text{ and } c \text{ without simplifying} \\
x &= \frac{30 \pm \sqrt{(-30)^2 - 4(25)(-11)}}{2(25)} && \\
x &= \frac{30 \pm \sqrt{2000}}{50} && \text{Discriminant is } 2000 \\
x &= \frac{30 \pm 20\sqrt{5}}{50} && \text{Divide each term by 10} \\
x &= \frac{3 \pm 2\sqrt{5}}{5} && \text{Our solutions}
\end{aligned}$$

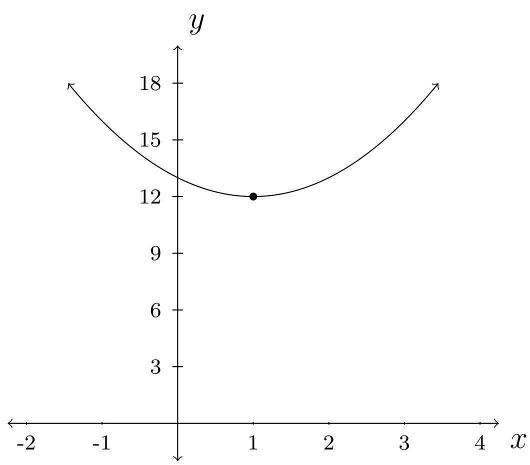


In each of the previous two examples the discriminant was positive, and consequently, there were two real solutions. Graphically, quadratics with a positive discriminant will intersect the x-axis at two distinct points.

The included graph shows the two real solutions to $25x^2 - 30x - 11 = 0$. This example demonstrates the importance of our efforts to relate an algebraic solution to a graphical representation, in order to help internalize the meaning behind the quadratic formula.

Example 218. Solve the given equation for all values of x.

$$
\begin{aligned}
3x^2 + 4x + 8 &= 2x^2 + 6x - 5 && \text{First set equation equal to zero} \\
x^2 - 2x + 13 &= 0 && \text{Identify } a, b, \text{ and } c \\
a = 1,\ b = -2,\ c &= 13, && \text{Use quadratic formula} \\
x &= \frac{2 \pm \sqrt{(-2)^2 - 4(1)(13)}}{2(1)} && \text{Substitute } a, b, \text{ and } c \text{ without simplifying} \\
x &= \frac{2 \pm \sqrt{4 - 52}}{2} && \text{Simplify} \\
x &= \frac{2 \pm \sqrt{-48}}{2} && \text{Discriminant is } -48 \\
x &= \frac{2 \pm 4i\sqrt{3}}{2} && \text{Simplify: reduce radical, divide by 2} \\
x &= 1 \pm 2i\sqrt{3} && \text{Our solutions}
\end{aligned}
$$



The previous example has two complex solutions that are not real. Consequently, we see that graphically our parabola has no $x-$intercepts. This results from the discriminant being negative, -48 in this case.

When using the quadratic formula, it is possible to *not* obtain two unique real (or complex) solutions. If the discriminant under the square root simplifies to zero, we can end up with only *one* real solution.

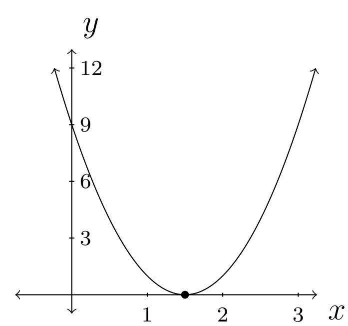
As it turns out, this single solution will coincide with the vertex of our parabola, (h, k). Recalling that $h = -\frac{b}{2a}$, we can verify that this result makes sense, when we consider that a discriminant of zero will eliminate the term $\pm\frac{\sqrt{b^2 - 4ac}}{2a}$ from our quadratic formula completely. What we are left with is precisely h.

Our next example will result in a single real solution, and will coincide to a parabola that touches the x-axis exactly once, at its vertex.

Example 219. Solve the given equation for all values of x.

$$
\begin{aligned}
4x^2 - 12x + 9 &= 0 && \text{Identify } a, b, \text{ and } c \\
a = 4,\ b = -12,\ c &= 9, && \text{Use quadratic formula} \\
x &= \frac{12 \pm \sqrt{(-12)^2 - 4(4)(9)}}{2(4)} && \text{Substitute } a, b, \text{ and } c \text{ without simplifying} \\
x &= \frac{12 \pm \sqrt{144 - 144}}{8} && \text{Simplify} \\
x &= \frac{12 \pm \sqrt{0}}{8} && \text{Discriminant is zero} \\
x &= \frac{12 \pm 0}{8} && \text{We get one real solution} \\
x &= \frac{12}{8} && \text{Reduce fraction} \\
x &= \frac{3}{2} && \text{Our solution}
\end{aligned}
$$

A graph of our resulting parabola confirms our previous result of a single zero, and consequently one x-intercept. In this case, the x-intercept should equal the vertex.



If a term is absent from our quadratic, we can still use the quadratic formula and simply use zero in place of the missing coefficient. The order of terms, however, is still important. If, for example, the linear term was absent, we would use $b = 0$. And, if the constant term is missing, we would use $c = 0$.

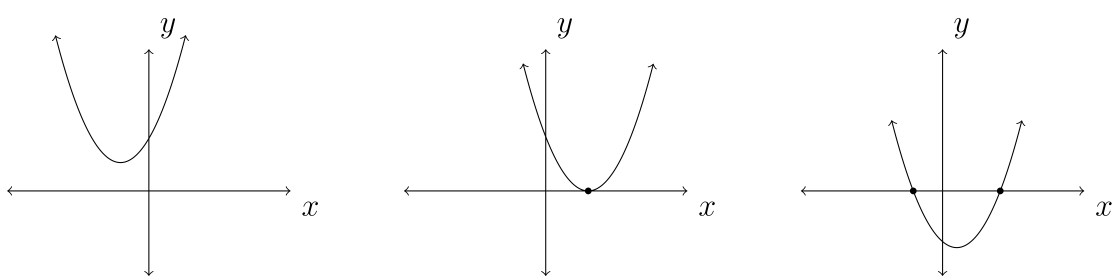
It is necessary that we take extra precautions when using the quadratic formula, since one false step can lead to a substantial amount of time lost. Taking the time to write the quadratic in standard form, set equal to zero, and identify the correct values for a,b, and c is crucial to the success of the quadratic formula.

Example 220. Solve the given equation for all values of x.

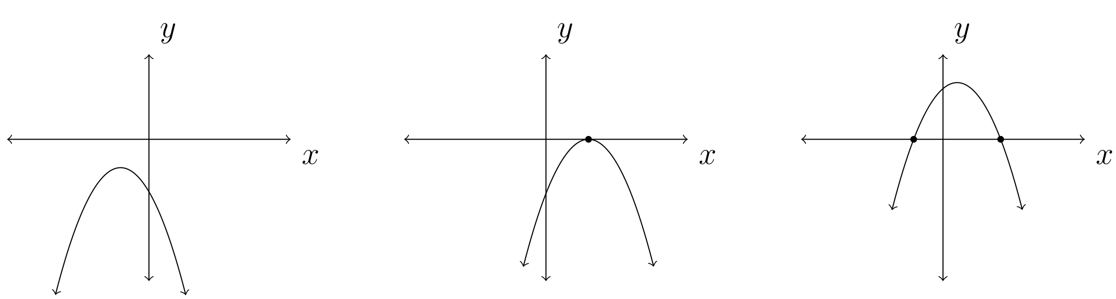
$$\begin{aligned}
3x^2+7&=0 && \text{Identify } a, b, \text{ and } c\\
a=3,\ b=0\text{ (missing term)},\ c&=7 && \text{Use quadratic formula}\\
x&=\frac{-0\pm\sqrt{0^2-4(3)(7)}}{2(3)} && \text{Substitute } a, b, \text{ and } c \text{ without simplifying}\\
x&=\frac{\pm\sqrt{-84}}{6} && \text{Simplify; discriminant is } -84\\
x&=\frac{\pm 2i\sqrt{21}}{6} && \text{Reduce radical and divide by } 2\\
x&=\frac{\pm i\sqrt{21}}{3} && \text{Our solutions}
\end{aligned}$$

We leave it as an exercise to the reader to graph the corresponding parabola and confirm that our solution is correct. Remember, the fact that we have two imaginary solutions means that our parabola should have no $x-$intercepts.

As we have seen in the previous examples, the discriminant determines the nature and quantity of the solutions of the quadratic formula. The following collection of graphs summarizes both the graphical and algebraic consequences for each type of discriminant (negative, zero, or positive).



Negative Discriminant	Zero Discriminant	Positive Discriminant
$b^2-4ac<0$	$b^2-4ac=0$	$b^2-4ac>0$
No Real Solutions	One Real Solution	Two Real Solutions



We have now outlined three different methods to use to solve a quadratic equation: factoring, extracting square roots, and using the quadratic formula. It is important to be familiar with all three methods, since each has its advantages. The following table suggests a procedure to help determine which method might be best to use for solving a given quadratic equation.

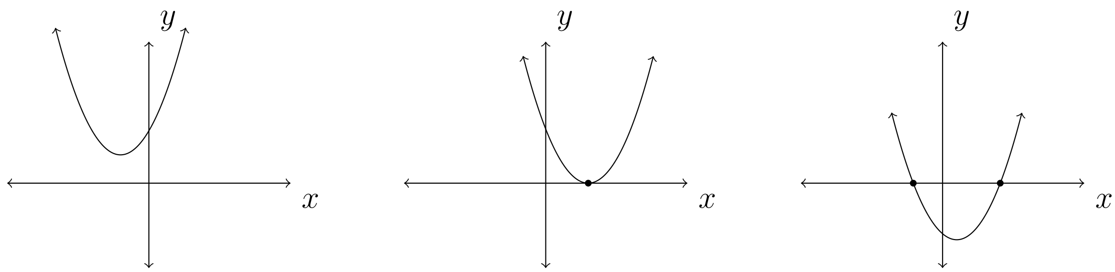
1. If we can easily factor, solve by factoring	$x^2-5x+6=0$ $(x-2)(x-3)=0$ $x=2$ or $x=3$
2. If $a=1$ and b is even, complete the square (or use the vertex formula) and extract square roots	$x^2+2x-4=0$ $\left(\frac{1}{2}\cdot 2\right)^2=1^2=1$ $(x^2+2x+1)-1-4=0$ $(x+1)^2-5=0$ $(x+1)^2=5$ $x+1=\pm\sqrt{5}$ $x=-1\pm\sqrt{5}$
3. As a last resort, apply the quadratic formula	$x^2-3x+4=0$ $x=\dfrac{3\pm\sqrt{(-3)^2-4(1)(4)}}{2(1)}$ $x=\dfrac{3\pm i\sqrt{7}}{2}$

The above table is merely a suggestion for approaching quadratic equations. Recall that completing the square and extracting square roots, as well as the quadratic formula may always be used to solve any quadratic, but often may not be the most efficient or "cleanest" method. Factoring can be very efficient but only works if the given equation can be easily factored.

Quadratic Inequalities and Sign Diagrams (L29)

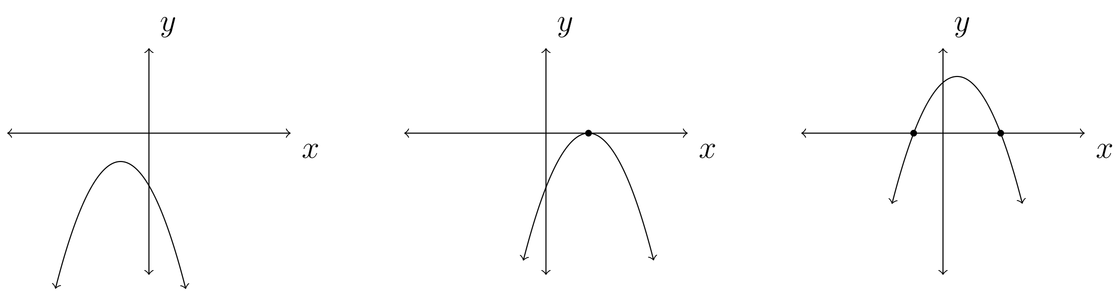
Objective: Solve and give interval notation for the solution to a quadratic inequality. Create a sign diagram to identify those intervals where a quadratic expression is positive or negative.

Recall that the *vertex form* for a quadratic equation is $y=a(x-h)^2+k$, where $a\neq 0$ and (h,k) represents the *vertex* of the corresponding graph, called a *parabola*. If $a>0$, then the parabola opens upward, and if $a<0$, then the parabola opens downward. With any quadratic equation, we have seen that there are three possibilities for the number of *zeros* or *roots* of the equation $(0,1,$ or $2)$. Assuming $a>0$, we illustrate these possibilities in the graphs below.



Notice also that each of these three graphs lie above the x-axis over different intervals. In the case of the parabola on the left, the entire graph lies above the x-axis, whereas the middle parabola lies above the x-axis everywhere *except* at its x-intercept (where $y = 0$). Even more interesting is the parabola on the right, which contains two *separate* intervals where its graph lies above the x-axis.

Considering the case where $a < 0$, we see three similar graphs as those appearing above, with the only major difference being the opening of each parabola downward instead of upward (when $a > 0$). When we consider again those intervals where each graph lies above the x-axis, each parabola below exhibits a different behavior than those where $a > 0$.



Now, each of the first two graphs have no points that lie above the x-axis, whereas the last graph, on the right, lies above the x-axis over the interval that is between its x-intercepts.

Each of these six graphs above exhibit all of the various possibilities for the *sign* of a quadratic expression $ax^2 + bx + c$, where $a \neq 0$. As was the case with linear equations in the previous chapter, we can determine the general shape of the graph of a quadratic equation (or function) through identification of its zeros and construction of a sign diagram. As a consequence, we will also see the care that must be taken when asked to solve a quadratic inequality.

Let us begin with what should be a familiar example, $y = x^2 - 1$, which we can recall has a factorization of $y = (x + 1)(x - 1)$.

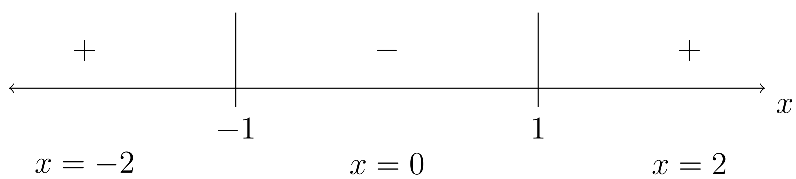
Example 221. Solve the quadratic inequality $x^2 - 1 < 0$.

As has often been the case, our first instinct is to add 1 to both sides of the given inequality, obtaining $x^2 < 1$. Our next guess is most likely to take a square root of both sides of the given inequality. Here, however, is where we encounter a common "pitfall", which begs the question: how does one handle radicals and inequalities?

The answer is that unlike with solving linear inequalities, one should not attempt to solve for the variable x, but rather set the given inequality equal to zero and attempt to *factor* the resulting expression on the other side. In doing this, we obtain $(x+1)(x-1) < 0$. Recalling that $x = \pm 1$ are zeros of the given expression, we can therefore rule them out of our solution. Next, we will *test* the expression on the left by plugging in three values for x: (i) $x < -1$, (ii) $-1 < x < 1$, and (iii) $x > 1$.

Case	Test Value	Unsimplified	Simplified	Result
i	$x = -2$	$(-2+1)(-2-1)$	$(-)\cdot(-)$	$(+)$
ii	$x = 0$	$(0+1)(0-1)$	$(+)\cdot(-)$	$(-)$
iii	$x = 2$	$(2+1)(2-1)$	$(+)\cdot(+)$	$(+)$

Our end result can be summarized in the following *sign diagram.*



From our sign diagram, we can conclude that $x^2 - 1 < 0$ when $-1 < x < 1$, or using interval notation, $(-1, 1)$.

Example 222. Solve the inequality $x^2 \geq 1$.

Here, we need only subtract -1 from both sides of the inequality, to obtain $x^2 - 1 \geq 0$. After factoring the left-hand side, We may then use the sign diagram from our previous example. Our solution set will be the *union* of two intervals, $(-\infty, -1] \cup [1, \infty)$.

Example 223. Solve the inequality $-(x-1)^2 + 9 \geq 0$.

Notice that the left-hand side of our inequality is in vertex form. So we will draw upon our knowledge of the graph of $y = -(x-1)^2 + 9$ later on to confirm our answer.

We start by expanding the left-hand side to obtain

$$-(x^2 - 2x + 1) + 9 \geq 0,$$

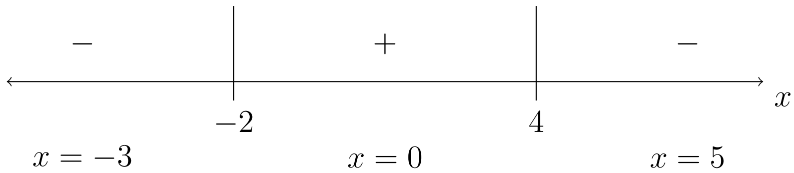
which reduces to

$$-x^2 + 2x + 8 \geq 0.$$

After factoring, we obtain

$$-(x+2)(x-4) \geq 0.$$

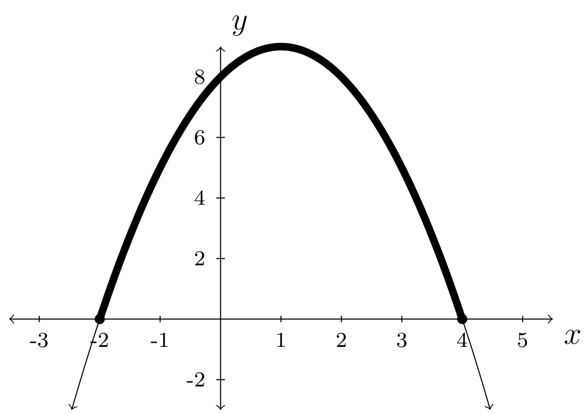
Since both $x = -2$ and $x = 4$ are zeros of the left-hand side, for our sign diagram, we will therefore test $x = -3$, $x = 0$, and $x = 5$. It is important to not overlook the negative sign that appears in front of our inequality when testing our values. Our results are shown in the following diagram.



From our sign diagram, we can determine that

$$-(x-1)^2 + 9 \geq 0 \text{ when } -2 \leq x \leq 4.$$

Again, the vertex form $y = -(x-1)^2 + 9$ confirms this, since the corresponding parabola will have a vertex of $(1, 9)$, which lies above the x-axis, and will open downward, as the leading coefficient $a = -1$ is negative. This implies that there will be two x-intercepts, which we found to be at the points $(-2, 0)$ and $(4, 0)$. Hence the graph will be nonnegative over an interval between (and including) the x-intercepts. To reinforce this, we provide the graph below, highlighting the portion that coincides with our desired interval.



In our next example, we will touch upon the notion of the *multiplicity* of a zero for a given equation/function, and how it affects the graph.

Example 224. Solve the inequality $x^2 + 4x > -4$.

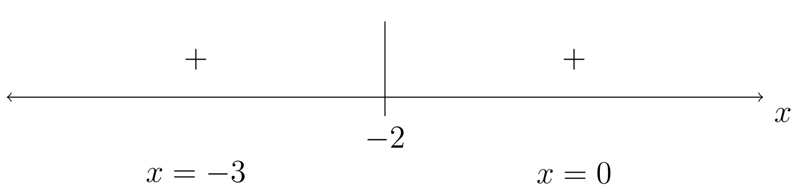
Setting the right-hand side to zero gives us

$$x^2 + 4x + 4 > 0.$$

Factoring, we then have

$$(x+2)^2 > 0.$$

Hence, we have only one zero for the left-hand side ($x = -2$), which means that there are only two intervals to test.



Our solution set may be represented as the inequality $x \neq -2$, or as the union of intervals $(-\infty, -2) \cup (-2, \infty)$.

Notice that $x = -2$ was a zero in each of the last two examples. In the first example, a change in sign occurred (negative to positive) as the values of x increased from one side of our zero to the other. In the second example, however, both the values below and above $x = -2$ yield positive signs.

This result has to do with the number of factors of $(x+2)$ appearing in our expression. This number is known as the *multiplicity* of the zero $x = -2$. Briefly stated, the *parity* of a zero's multiplicity (whether the number of factors is even or odd) will determine whether or not the sign of the given expression on either side of the zero remains the same or changes. This notion will be quite useful when graphing complicated functions, and will be revisited in the chapter on polynomial functions.

Example 225. Solve the inequality $x^2 + 4x < -4$.

Since we have only switched the direction of our inequality in the last example, we may conclude that the inequality has no solution set, represented by the empty set, $\varnothing$.

Up until this point, all of our examples have reduced to expressions that can easily be factored. As this is often not the case for quadratic expressions, we will now attempt to solve some more challenging inequalities.

Example 226. Solve the inequality $x^2 - x + 1 > 0$.

After brief inspection, we see that the expression on the left-hand side is not easily factorable. At this point, in order to determine if any real zeros exist for $x^2 - x + 1$, we have a few methods to choose from. We will use the quadratic formula, shown below.

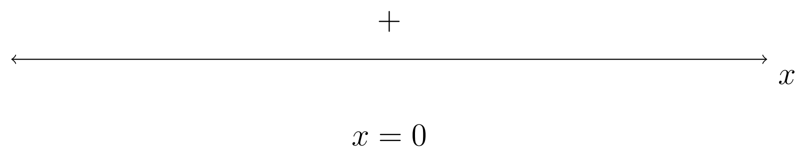
$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$x = \frac{1}{2} \pm \frac{\sqrt{-3}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

Since we are left with a negative under the square root, we conclude that the given expression has no real zeros. Hence, the corresponding parabola will have no x-intercepts. Note: A slightly quicker method would have been to simply calculate the discriminant, shown below.

$$D = (-1)^2 - 4(1)(1) = -3 < 0$$

As our leading coefficient $a = 1$ in the above expression is positive, we know that the corresponding parabola will open upward. Using this information, along with the fact that there are no x-intercepts, we may conclude that the entire parabola must lie above the x-axis. The corresponding sign diagram, with chosen test value of $x = 0$, is included for completeness.



Hence, our solution set for the inequality $x^2 - x + 1 > 0$ is all real numbers, $(-\infty, \infty)$.

Example 227. Solve the inequality $x^2 > 4x - 1$.

Setting the right-hand side to zero, we have $x^2 - 4x + 1 > 0$.

Although we could again resort to the quadratic formula, we will instead identify the vertex form of the expression on the left, shown below.

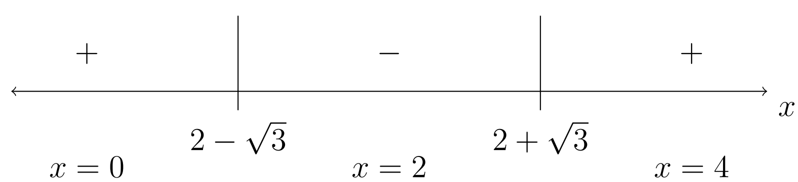
$$h = -\frac{-4}{2(1)} = 2 \qquad\qquad k = 2^2 - 4(2) + 1 = -3$$

$$x^2 - 4x + 1 = (x - 2)^2 - 3$$

So, setting $(x - 2)^2 - 3$ equal to zero and extracting square roots, we obtain two real zeros at $x = 2 \pm \sqrt{3}$. It then follows that

$$x^2 - 4x + 1 = \left(x - (2 - \sqrt{3})\right)\left(x - (2 + \sqrt{3})\right).$$

Since we have two real zeros, we will construct a sign diagram, using test values on either side of $2 - \sqrt{3} \approx 0.27$ and $2 + \sqrt{3} \approx 3.73$. Our results are shown below.



Note that since we already obtained the vertex of $(2, -3)$, we have chosen $x = 2$ as a test value for our middle interval.

From the above diagram, we conclude that $x^2 > 4x - 1$ precisely on the union of intervals $(-\infty, 2 - \sqrt{3}) \cup (2 + \sqrt{3}, \infty)$.

Practice Problems

Introduction

After simplifying, classify each equation as linear, quadratic, or neither. If the equation is a quadratic, then specify whether it is concave up or down.

1) $y = x^2 + 9$
2) $y = 5 - 2x + x^2$
3) $y = x + 6 - 3x$
4) $y = 5x + x^2 - 3x - 3x^2$
5) $y = -5x + 3 + 2x - 3x^2 + 6$
6) $y = 3x^2 + -x + x - 3x^2 + 6$
7) $y = (x - 1)(x + 2) + 3$
8) $y = (x - 5)(2x + 3) - 2(x - 3)$
9) $y = (x - 4)(x + 4) - (x + 1)^2$
10) $y = (2x - 4)(x - 1) - 2(x + 3)^2 + 3x^2$

An Introduction to the Vertex Form

Identify the vertex and concavity (concave up or down) of each quadratic.

11) $y = (x - 3)^2 + 4$
12) $y = (x - 2)^2 + 5$
13) $y = 6(x + 3)^2 + 4$
14) $y = -2(x - 3)^2 + 4$
15) $y = -2(x - 1)^2 - 7$
16) $y = -(x + 1)^2$
17) $y = 7x^2 + 4$
18) $y = -\frac{1}{23}(x - 8)^2 + 5$
19) $y = x^2 + 4$
20) $y = 5x^2 + 23$

Factoring Methods

Greatest Common Factors

Factor the common factor out of each expression.

1) $4 + 8b^2$
2) $x - 5$
3) $45x^2 - 25$
4) $-n - 2n^2$
5) $56 - 35p$
6) $50x - 80y$
7) $7ab - 35a^2b$
8) $27x^2y^5 - 72x^3y^2$
9) $-3a^2b + 6a^3b^2$
10) $8x^3y^2 + 4x^3$
11) $-5x^2 - 5x^3 - 15x^4$
12) $-32n^9 + 32n^6 + 40n^5$
13) $20x^4 - 30x + 30$
14) $21p^6 + 30p^2 + 27$
15) $28m^4 + 40m^3 + 8$
16) $-10x^4 + 20x^2 + 12x$
17) $30b^9 + 5ab - 15a^2$
18) $27y^7 + 12y^2x + 9y^2$
19) $-48a^2b^2 - 56a^3b - 56a^5b$
20) $30m^6 + 15mn^2 - 25$
21) $20x^8y^2z^2 + 15x^5y^2z + 35x^3y^3z$
22) $3p + 12q - 15q^2r^2$
23) $50x^2y + 10y^2 + 70xz^2$
24) $30y^4z^3x^5 + 50y^4z^5 - 10y^4z^3x$
25) $30qpr - 5qp + 5q$
26) $28b + 14b^2 + 35b^3 + 7b^5$
27) $-18n^5 + 3n^3 - 21n + 3$
28) $30a^8 + 6a^5 + 27a^3 + 21a^2$
29) $-40x^{11} - 20x^{12} + 50x^{13} - 50x^{14}$
30) $-24x^6 - 4x^4 + 12x^3 + 4x^2$
31) $-32mn^8 + 4m^6n + 12mn^4 + 16mn$
32) $-10y^7 + 6y^{10} - 4y^{10}x - 8y^8x$

Factor by Grouping

Factor each expression completely.

33) $40r^3 - 8r^2 - 25r + 5$
34) $35x^3 - 10x^2 - 56x + 16$
35) $3n^3 - 2n^2 - 9n + 6$
36) $14v^3 + 10v^2 - 7v - 5$
37) $15b^3 + 21b^2 - 35b - 49$
38) $6x^3 - 48x^2 + 5x - 40$
39) $3x^3 + 15x^2 + 2x + 10$
40) $28p^3 + 21p^2 + 20p + 15$
41) $35x^3 - 28x^2 - 20x + 16$
42) $7n^3 + 21n^2 - 5n - 15$
43) $7xy - 49x + 5y - 35$
44) $42r^3 - 49r^2 + 18r - 21$
45) $32xy + 40x^2 + 12y + 15x$
46) $15ab - 6a + 5b^3 - 2b^2$
47) $16xy - 56x + 2y - 7$
48) $3mn - 8m + 15n - 40$
49) $2xy - 8x^2 + 7y^3 - 28y^2x$
50) $5mn + 2m - 25n - 10$
51) $40xy + 35x - 8y^2 - 7y$
52) $8xy + 56x - y - 7$
53) $32uv - 20u + 24v - 15$
54) $4uv + 14u^2 + 12v + 42u$
55) $10xy + 30 + 25x + 12y$
56) $24xy + 25y^2 - 20x - 30y^3$
57) $3uv + 14u - 6u^2 - 7v$
58) $56ab + 14 - 49a - 16b$
59) $16xy - 3x - 6x^2 + 8y$

Trinomials with Leading Coefficient $a = 1$

Factor each expression completely.

60) $p^2 + 17p + 72$
61) $x^2 + x - 72$
62) $n^2 - 9n + 8$
63) $x^2 + x - 30$
64) $x^2 - 9x - 10$
65) $x^2 + 13x + 40$
66) $b^2 + 12b + 32$
67) $b^2 - 17b + 70$
68) $x^2 + 3x - 70$
69) $x^2 + 3x - 18$
70) $n^2 - 8n + 15$
71) $a^2 - 6a - 27$
72) $p^2 + 15p + 54$
73) $p^2 + 7p - 30$
74) $n^2 - 15n + 56$
75) $m^2 - 15mn + 50n^2$
76) $u^2 - 8uv + 15v^2$
77) $m^2 - 3mn - 40n^2$
78) $m^2 + 2mn - 8n^2$
79) $x^2 + 10xy + 16y^2$
80) $x^2 - 11xy + 18y^2$
81) $u^2 - 9uv + 14v^2$
82) $x^2 + xy - 12y^2$
83) $x^2 + 14xy + 45y^2$
84) $x^2 + 4xy - 12y^2$
85) $4x^2 + 52x + 168$
86) $5a^2 + 60a + 100$
87) $5n^2 - 45n + 40$
88) $6a^2 + 24a - 192$
89) $5v^2 + 20v - 25$
90) $6x^2 + 18xy + 12y^2$
91) $5m^2 + 30mn - 90n^2$
92) $6x^2 + 96xy + 378y^2$
93) $6m^2 - 36mn - 162n^2$

Trinomials with Leading Coefficient $a \neq 1$

Factor each expression completely.

94) $7x^2 - 48x + 36$
95) $7n^2 - 44n + 12$
96) $7b^2 + 15b + 2$
97) $7v^2 - 24v - 16$
98) $5a^2 - 13a - 28$
99) $5n^2 - 7n - 24$
100) $2x^2 - 5x + 2$
101) $3r^2 - 4r - 4$
102) $2x^2 + 19x + 35$
103) $7x^2 + 29x - 30$
104) $2b^2 - b - 3$
105) $5x^2 - 26x + 24$
106) $5x^2 + 13x + 6$
107) $3r^2 + 16r + 21$
108) $3x^2 - 17x + 20$
109) $3u^2 + 13uv - 10v^2$
110) $3x^2 + 17xy + 10y^2$
111) $7x^2 - 2xy - 5y^2$
112) $5x^2 + 28xy - 49y^2$
113) $5u^2 + 31uv - 28v^2$
114) $6x^2 - 39x - 21$
115) $10a^2 - 54a - 36$
116) $21x^2 - 87x - 90$
117) $21n^2 + 45n - 54$
118) $14x^2 - 60x + 16$
119) $4r^2 + r - 3$
120) $6x^2 + 29x + 20$

121) $6p^2 + 11p - 7$
122) $4x^2 - 17x + 4$
123) $4r^2 + 3r - 7$
124) $4x^2 + 9xy + 2y^2$
125) $4m^2 + 6mn + 6n^2$
126) $4m^2 - 9mn - 9n^2$
127) $4x^2 - 6xy + 30y^2$
128) $4x^2 + 13xy + 3y^2$
129) $18u^2 - 3uv - 36v^2$
130) $12x^2 + 62xy + 70y^2$
131) $16x^2 + 60xy + 36y^2$
132) $24x^2 - 52xy + 8y^2$
133) $12x^2 + 50xy + 28y^2$

Solving by Factoring

Set each of the following expressions equal to zero and solve for the given variable.

1) - 15): Expressions (60) through (74) on page 190.

16) - 30): Expressions (94) through (108) on page 190.

31) - 40): Expressions (114) through (123) on page 190.

Square Roots and the Imaginary Number i

Square Roots

Simplify each of the following square roots completely.

1) $\sqrt{245}$
2) $\sqrt{125}$
3) $\sqrt{36}$
4) $\sqrt{196}$
5) $\sqrt{12}$
6) $\sqrt{72}$
7) $3\sqrt{12}$
8) $5\sqrt{32}$
9) $6\sqrt{128}$
10) $7\sqrt{128}$
11) $-8\sqrt{392}$
12) $-7\sqrt{63}$
13) $\sqrt{192n}$
14) $\sqrt{343b}$
15) $\sqrt{196v^2}$
16) $\sqrt{100n^3}$
17) $\sqrt{252x^2}$
18) $\sqrt{200a^3}$
19) $-\sqrt{100k^4}$
20) $-4\sqrt{175p^4}$
21) $-7\sqrt{64x^4}$
22) $-2\sqrt{128n}$
23) $-5\sqrt{36m}$
24) $8\sqrt{112p^2}$
25) $\sqrt{45x^2y^2}$
26) $\sqrt{72a^3b^4}$
27) $\sqrt{16x^3y^3}$
28) $\sqrt{512a^4b^2}$
29) $\sqrt{320x^4y^4}$
30) $\sqrt{512m^4n^3}$
31) $6\sqrt{80xy^2}$
32) $8\sqrt{98mn}$
33) $5\sqrt{245x^2y^3}$
34) $2\sqrt{72x^2y^2}$
35) $-2\sqrt{180u^3v}$
36) $-5\sqrt{72x^3y^4}$
37) $-8\sqrt{180x^4y^2z^4}$
38) $6\sqrt{50a^4bc^2}$
39) $2\sqrt{80hj^4k}$
40) $-\sqrt{32xy^2z^3}$
41) $-4\sqrt{54mnp^2}$
42) $-8\sqrt{32m^2p^4q}$

Introduction to Complex Numbers

Rewrite each of the following complex numbers in the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$.

1) $3 - (-8 + 4i)$
2) $(3i) - (7i)$
3) $(7i) - (3 - 2i)$
4) $5 + (-6 - 6i)$
5) $(-6i) - (3 + 7i)$
6) $(-8i) - (7i) - (5 - 3i)$
7) $(3 - 3i) + (-7 - 8i)$
8) $(-4 - i) + (1 - 5i)$
9) $(i) - (2 + 3i) - 6$
10) $(5 - 4i) + (8 - 4i)$
11) $(6i)(-8i)$
12) $(3i)(-8i)$
13) $(-5i)(8i)$
14) $(8i)(-4i)$
15) $(-7i)^2$

16) $(-i)(7i)(4-3i)$
17) $(6+5i)^2$
18) $(8i)(-2i)(-2-8i)$
19) $(-7-4i)(-8+6i)$
20) $(3i)(-3i)(4-4i)$
21) $(-4+5i)(2-7i)$
22) $-8(4-8i)-2(-2-6i)$
23) $(-8-6i)(-4+2i)$
24) $(-6i)(3-2i)-(7i)(4i)$
25) $(1+5i)(2+i)$
26) $(-2+i)(3-5i)$
27) $\dfrac{-9+5i}{i}$
28) $\dfrac{-3+2i}{-3i}$
29) $\dfrac{-10-9i}{6i}$
30) $\dfrac{-4+2i}{3i}$
31) $\dfrac{-3-6i}{4i}$
32) $\dfrac{-5+9i}{9i}$
33) $\dfrac{10-i}{-i}$
34) $\dfrac{10}{5i}$
35) $\dfrac{4i}{-10+i}$
36) $\dfrac{9i}{1-5i}$
37) $\dfrac{8}{7-6i}$
38) $\dfrac{4}{4+6i}$
39) $\dfrac{7}{10-7i}$
40) $\dfrac{9}{-8-6i}$
41) $\dfrac{5i}{-6-i}$
42) $\dfrac{8i}{6-7i}$
43) $\sqrt{-81}$
44) $\sqrt{-45}$
45) $\sqrt{-10}\sqrt{-2}$
46) $\sqrt{-12}\sqrt{-2}$
47) $\frac{3+\sqrt{-27}}{6}$
48) $\frac{-4-\sqrt{-8}}{-4}$
49) $\frac{8-\sqrt{-16}}{4}$
50) $\frac{6+\sqrt{-32}}{4}$
51) i^{73}
52) i^{251}
53) i^{48}
54) i^{68}
55) i^{62}
56) i^{181}
57) i^{154}
58) i^{51}

Vertex Form and Graphing

The Vertex Form

Identify whether the quadratic is in vertex form, standard form, or both. If it is in vertex form, then identify the vertex (h,k).

1) $y=(x-12)^2+5$
2) $y=-3(x-3)^2+5$
3) $y=x^2+8$
4) $y=2(x-4)^2$
5) $y=-4(x-1)^2+2$
6) $y=-5(x-7)^2$
7) $y=x^2+3x+4$
8) $y=x^2-1$
9) $y=x^2-3$
10) $y=(x-1)^2-3$
11) $y=(x-1)^2$
12) $y=x^2$

Each quadratic equation below has been given in standard form. Rewrite each equation in vertex form.

13) $y=x^2+2x-1$
14) $y=-3x^2-12x-5$
15) $y=3x^2+12x-1$
16) $y=x^2+2x$
17) $y=x^2+6$
18) $y=-5x^2-40x$
19) $y=x^2+8x$
20) $y=x^2$
21) $y=x^2+4x-2$
22) $y=x^2+16x-2$
23) $y=4x^2+10x$

Graphing Quadratics

Find the vertex and intercepts of the following quadratics. Use this information to graph the resulting parabola.

1) $y = x^2 - 2x - 8$
2) $y = x^2 - 2x - 3$
3) $y = 2x^2 - 12x + 10$
4) $y = 2x^2 - 12x + 16$
5) $y = -2x^2 + 12x - 18$
6) $y = -2x^2 + 12x - 10$
7) $y = -3x^2 + 24x - 45$
8) $y = -3x^2 + 12x - 9$
9) $y = -x^2 + 4x + 5$
10) $y = -x^2 + 4x - 3$
11) $y = -x^2 + 6x - 5$
12) $y = -2x^2 + 16x - 30$
13) $y = -2x^2 + 16x - 24$
14) $y = 2x^2 + 4x - 6$
15) $y = 3x^2 + 12x + 9$
16) $y = 5x^2 + 30x + 45$
17) $y = 5x^2 - 40x + 75$
18) $y = 5x^2 + 20x + 15$
19) $y = -5x^2 - 60x - 175$
20) $y = -5x^2 + 20x - 15$

The Method of Extracting Square Roots

Find the x-intercepts of each quadratic by setting $y = 0$ and using the method of extracting square roots.

1) $y = (x - 12)^2 - 5$
2) $y = -3(x - 3)^2 + 30$
3) $y = x^2 - 16$
4) $y = 2(x - 4)^2 - 200$
5) $y = -4(x - 1)^2 + 20$
6) $y = -2(x - 7)^2 + 50$
7) $y = -4(x + 6)^2 + 8$
8) $y = x^2 - 4$
9) $y = (x - 4)^2 - 9$
10) $y = (x - 1)^2 - 25$
11) $y = (x + 2)^2 + 16$
12) $y = 9(x - 11)^2 - 81$

Completing the Square

Find the value that completes the square and then rewrite the given expression as a perfect square.

1) $x^2 - 30x +$ ______
2) $a^2 - 24a +$ ______
3) $m^2 - 36m +$ ______
4) $x^2 - 34x +$ ______
5) $x^2 - 15x +$ ______
6) $r^2 - \frac{1}{9}r +$ ______
7) $y^2 - y +$ ______
8) $p^2 - 17p +$ ______

Solve each equation by completing the square.

9) $x^2 - 16x + 55 = 0$
10) $n^2 - 8n - 12 = 0$
11) $v^2 - 8v + 45 = 0$
12) $b^2 + 2b + 43 = 0$
13) $6x^2 + 12x + 63 = 0$
14) $3x^2 - 6x + 47 = 0$
15) $5k^2 - 10k + 48 = 0$
16) $8a^2 + 16a - 1 = 0$
17) $x^2 + 10x - 57 = 4$
18) $p^2 - 16p - 52 = 0$
19) $n^2 - 16n + 67 = 4$
20) $m^2 - 8m - 3 = 6$
21) $2x^2 + 4x + 38 = -6$
22) $6r^2 + 12r - 24 = -6$
23) $8b^2 + 16b - 37 = 5$
24) $6n^2 - 12n - 14 = 4$
25) $x^2 = -10x - 29$
26) $v^2 = 14v + 36$

27) $n^2 = -21 + 10n$
28) $a^2 - 56 = -10a$
29) $3k^2 + 9 = 6k$
30) $5x^2 = -26 + 10x$
31) $2x^2 + 63 = 8x$
32) $5n^2 = -10n + 15$
33) $p^2 - 8p = -55$
34) $x^2 + 8x + 15 = 8$
35) $7n^2 - n + 7 = 7n + 6n^2$
36) $n^2 + 4n = 12$
37) $13b^2 + 15b + 44 = -5 + 7b^2 + 3b$
38) $-3r^2 + 12r + 49 = -6r^2$
39) $5x^2 + 5x = -31 - 5x$
40) $8n^2 + 16n = 64$
41) $v^2 + 5v + 28 = 0$
42) $b^2 + 7b - 33 = 0$
43) $7x^2 - 6x + 40 = 0$
44) $4x^2 + 4x + 25 = 0$
45) $k^2 - 7k + 50 = 3$
46) $a^2 - 5a + 25 = 3$
47) $5x^2 + 8x - 40 = 8$
48) $2p^2 - p + 56 = -8$
49) $m^2 = -15 + 9m$
50) $n^2 - n = -41$
51) $8r^2 + 10r = -55$
52) $3x^2 - 11x = -18$
53) $5n^2 - 8n + 60 = -3n + 6 + 4n^2$
54) $4b^2 - 15b + 56 = 3b^2$
55) $-2x^2 + 3x - 5 = -4x^2$
56) $10v^2 - 15v = 27 + 4v^2 - 6v$

The Quadratic Formula and the Discriminant

Use the discriminant in order to determine the number of real roots for each equation. If an equation is shown to have one (or two) real root(s), set $y = 0$ and use the quadratic formula to find them.

1) $y = x^2 + 2x - 1$
2) $y = -3x^2 - 12x - 5$
3) $y = 3x^2 + 12x - 1$
4) $y = x^2 + 2x$
5) $y = x^2 + 6$
6) $y = -5x^2 - 40x$
7) $y = x^2 + 8x$
8) $y = x^2$
9) $y = x^2 + 4x - 2$
10) $y = x^2 + 16x - 2$
11) $y = 4x^2 + 10x$

Solve each equation using the quadratic formula.

12) $4a^2 + 6 = 0$
13) $3k^2 + 2 = 0$
14) $2x^2 - 8x - 2 = 0$
15) $6n^2 - 1 = 0$
16) $2m^2 - 3 = 0$
17) $5p^2 + 2p + 6 = 0$
18) $3r^2 - 2r - 1 = 0$
19) $2x^2 - 2x - 15 = 0$
20) $4n^2 - 36 = 0$
21) $3b^2 + 6 = 0$
22) $v^2 - 4v - 5 = -8$
23) $2x^2 + 4x + 12 = 8$
24) $2a^2 + 3a + 14 = 6$
25) $6n^2 - 3n + 3 = -4$
26) $3k^2 + 3k - 4 = 7$
27) $4x^2 - 14 = -2$
28) $7x^2 + 3x - 16 = -2$
29) $4n^2 + 5n = 7$
30) $2p^2 + 6p - 16 = 4$
31) $m^2 + 4m - 48 = -3$
32) $3n^2 + 3n = -3$
33) $3b^2 - 3 = 8b$
34) $2x^2 = -7x + 49$
35) $3r^2 + 4 = -6r$
36) $5x^2 = 7x + 7$
37) $6a^2 = -5a + 13$
38) $8n^2 = -3n - 8$
39) $6v^2 = 4 + 6v$
40) $2x^2 + 5x = -3$
41) $x^2 = 8$
42) $4a^2 - 64 = 0$
43) $2k^2 + 6k - 16 = 2k$
44) $4p^2 + 5p - 36 = 3p^2$
45) $12x^2 + x + 7 = 5x^2 + 5x$
46) $-5n^2 - 3n - 52 = 2 - 7n^2$
47) $7m^2 - 6m + 6 = -m$
48) $7r^2 - 12 = -3r$
49) $3x^2 - 3 = x^2$
50) $2n^2 - 9 = 4$
51) $6b^2 = b^2 + 7 - b$

Quadratic Inequalities and Sign Diagrams

Construct a sign diagram for each of the following expressions/equations. Then using interval notation, describe the set of values for which the given expression is greater than or equal to zero.

1) - 5): Expressions (60) through (64) on page 190.

6) - 10): Expressions (94) through (98) on page 190.

11) - 15): Expressions (114) through (118) on page 190.

16) - 20): Equations (1) through (5) on page 194.

Selected Answers

Introduction

L=linear, Q=quadratic, N=neither, U=concave up, D=concave down

1) $y = x^2 + 9$, QU
3) $y = -2x + 6$, L
5) $y = -3x^2 - 3x + 9$, QD
7) $y = x^2 + x + 1$, QU
9) $y = -2x - 17$, L

An Introduction to the Vertex Form

U=concave up, D=concave down

11) $(3, 4)$, U
13) $(-3, 4)$, U
15) $(1, -7)$, D
17) $(0, 4)$, U
19) $(0, 4)$, U

Factoring Methods

Greatest Common Factors

1) $4(1 + 2b^2)$
3) $5(9x^2 - 5)$
5) $7(8 - 5p)$
7) $7ab(1 - 5a)$
9) $-3a^2b(1 - 2ab)$
11) $-5x^2(1 + x + 3x^2)$
13) $10(2x^4 - 3x + 3)$
15) $4(7m^4 + 10m^3 + 2)$
17) $5(6b^9 + ab - 3a^2)$
19) $-8a^2b(6b - 7a - 7a^3)$
21) $5x^3y^2z(4x^5z + 3x^2 + 7y)$
23) $10(5x^2y + y^2 + 7xz^2)$
25) $5q(6pr - p + 1)$
27) $-3(6n^5 - n^3 + 7n - 1)$
29) $-10x^{11}(4+2x-5x^2+5x^3)$
31) $-4mn(8n^7-m^5-3n^3-4)$

Factor by Grouping

33) $(8r^2 - 5)(5r - 1) = \left(2\sqrt{2}r - \sqrt{5}\right)\left(2\sqrt{2}r + \sqrt{5}\right)(5r - 1)$
35) $(n^2 - 3)(3n - 2) = \left(n - \sqrt{3}\right)\left(n + \sqrt{3}\right)(3n - 2)$

37) $(3b^2-7)(5b+7)=\left(\sqrt{3}b-\sqrt{7}\right)\left(\sqrt{3}b+\sqrt{7}\right)(5b+7)$
39) $(3x^2+2)(x+5)$
41) $(7x^2-4)(5x-4)=\left(\sqrt{7}x-2\right)\left(\sqrt{7}x+2\right)(5x-4)$

43) $(7x+5)(y-7)$
45) $(8x+3)(4y+5x)$
47) $(8x+1)(2y-7)$
49) $(2x+7y^2)(y-4x)$
51) $(5x-y)(8y+7)$
53) $(4u+3)(8v-5)$
55) $(5x+6)(2y+5)$
57) $(3u-7)(v-2u)$
59) $(8y-3x)(2x+1)$

Trinomials with Leading Coefficient $a=1$

60) $(p+9)(p+8)$
62) $(n-8)(n-1)$
64) $(x-10)(x+1)$
66) $(b+4)(b+8)$
68) $(x+10)(x-7)$
70) $(n-5)(n-3)$
72) $(p+9)(p+6)$
74) $(n-7)(n-8)$
76) $(u-5v)(u-3v)$
78) $(m+4n)(m-2n)$
80) $(x-9y)(x-2y)$
82) $(x+4y)(x-3y)$
84) $(x+6y)(x-2y)$
86) $5(a+2)(a+10)$
88) $6(a+8)(a-4)$
90) $6(x+2y)(x+y)$
92) $6(x+9y)(x+7y)$

Trinomials with Leading Coefficient $a\neq 1$

94) $(7x-6)(x-6)$
96) $(7b+1)(b+2)$
98) $(5a+7)(a-4)$
100) $(2x-1)(x-2)$
102) $(2x+5)(x+7)$
104) $(2b-3)(b+1)$
106) $(5x+3)(x+2)$
108) $(3x-5)(x-4)$
110) $(3x+2y)(x+5y)$
112) $(5x-7y)(x+7y)$
114) $3(2x+1)(x-7)$
116) $3(7x+6)(x-5)$
118) $3(7x-2)(x-4)$
120) $(6x+5)(x+4)$
122) $(4x-1)(x-4)$
124) $(4x+y)(x+2y)$
126) $(4m+3n)(m-3n)$
128) $(4x+y)(x+3y)$
130) $2(3x+5y)(2x+7y)$
132) $4(6x-y)(x-2y)$

Solving by Factoring

1) $p=-9,-8$
3) $n=1,8$
5) $x=-1,10$
7) $b=-4,-8$
9) $x=-10,7$
11) $n=3,5$
13) $p=-6,-9$
15) $n=7,8$
17) $n=2/7,6$
19) $v=-4/7,4$
21) $n=-8/5,3$
23) $r=-2/3,2$
25) $x=-5,6/7$
27) $x=6/5,4$
29) $r=-3,-7/3$
31) $x=-1/2,7$
33) $x=-6/7,5$
35) $x=2/7,4$
37) $x=-4,-5/6$
39) $x=1/4,4$

Square Roots and the Imaginary Number i

Square Roots

1) $7\sqrt{5}$
3) 6
5) $2\sqrt{3}$
7) $6\sqrt{3}$
9) $48\sqrt{2}$
11) $-112\sqrt{2}$
13) $8\sqrt{3n}$
15) $14v$
17) $6x\sqrt{7}$
19) $-10k^2$
21) $-56x^2$
23) $-30\sqrt{m}$

25) $3xy\sqrt{5}$
27) $4xy\sqrt{xy}$
29) $8x^2y^2\sqrt{5}$
31) $24y\sqrt{5x}$
33) $35xy\sqrt{5y}$
35) $-12u\sqrt{5uv}$
37) $-48x^2yz^2\sqrt{5}$
39) $8j^2\sqrt{5hk}$
41) $-12p\sqrt{6mn}$

Introduction to Complex Numbers

1) $11+4i$
3) $-3+9i$
5) $-3-13i$
7) $-4-11i$
9) $-8-2i$
11) 48
13) 40
15) -49
17) $11+60i$
19) $80-10i$
21) $27+38i$
23) $44+8i$
25) $-3+11i$
27) $5+9i$
29) $-\frac{3}{2}+\frac{5}{3}i$
31) $-\frac{3}{2}+\frac{3}{4}i$
33) $1+10i$
35) $\frac{4}{101}-\frac{40}{101}i$
37) $\frac{56}{85}+\frac{48}{85}i$
39) $\frac{70}{149}+\frac{49}{149}i$
41) $-\frac{5}{37}-\frac{30}{37}i$
43) $9i$
45) $2\sqrt{5}$
47) $\frac{1}{2}+\frac{\sqrt{3}}{2}i$
49) $2-i$
51) i
53) 1
55) -1
57) -1

Vertex Form and Graphing

The Vertex Form

V=vertex form, S=standard form, B=both

1) V, $(12,5)$
3) B, $(0,8)$
5) V, $(1,2)$
7) S
9) B, $(0,-3)$
11) V, $(1,0)$
13) $y=(x+1)^2-2$
15) $y=3(x+2)^2-13$
17) $y=x^2+6$
19) $y=(x+4)^2-16$
21) $y=(x+2)^2-6$
23) $y=4\left(x+\frac{5}{4}\right)^2-\frac{25}{4}$

Graphing Quadratics

No.) y-int, vertex, x-int(s)

1) $(0,-8)$, $(1,-9)$, $(-2,0),(4,0)$
3) $(0,10)$, $(3,-8)$, $(1,0)$, $(5,0)$
5) $(0,-18)$, $(3,0)$, $(3,0)$
7) $(0,-45)$, $(4,3)$, $(3,0),(5,0)$
9) $(0,5)$, $(2,9)$, $(-1,0),(5,0)$
11) $(0,-5)$, $(3,4)$, $(1,0),(5,0)$
13) $(0,-24)$, $(4,8)$, $(2,0),(6,0)$
15) $(0,9)$, $(-2,-3)$, $(-3,0),(-1,0)$
17) $(0,75)$, $(4,-5)$, $(3,0),(5,0)$
19) $(0,-175)$, $(-6,5)$, $(-7,0),(-5,0)$

The Method of Extracting Square Roots

1) $12\pm\sqrt{5}$
3) ± 4
5) $1\pm\sqrt{5}$
7) $-6\pm\sqrt{2}$
9) $1,7$
11) $-2\pm 4i$

Completing the Square

1) $x^2 - 30x + \underline{\ 225\ } = (x-15)^2$
3) $m^2 - 36m + \underline{\ 324\ } = (m-18)^2$
5) $x^2 - 15x + \underline{\ 225/4\ } = (x-15/2)^2$
7) $y^2 - y + \underline{\ 1/4\ } = (y-1/2)^2$

11) $v = 4 \pm \sqrt{29}i$
15) $k = 1 \pm \frac{\sqrt{215}}{5}i$
17) $x = -\frac{5}{2} \pm \sqrt{86}$
21) $x = -1 \pm \sqrt{21}i$
25) $x = -5 \pm 2i$
27) $n = 3,\ 7$
31) $x = 2 \pm \sqrt{29}i$
35) $n = 1,\ 7$
37) $b = -1 \pm \frac{\sqrt{258}}{6}i$
41) $v = -\frac{5}{2} \pm \frac{\sqrt{87}}{2}i$
45) $k = \frac{7}{2} \pm \frac{\sqrt{137}}{2}i$
47) $x = -4,\ \frac{12}{5}$
51) $r = -\frac{5}{8} \pm \frac{\sqrt{415}}{8}i$
55) $x = -\frac{5}{2},\ 1$

The Quadratic Formula and the Discriminant

1) Two real roots, $x = -1 \pm \sqrt{2}$
3) Two real roots, $x = -2 \pm \frac{\sqrt{39}}{3}$
5) No real roots
7) Two real roots, $x = 0,\ -8$
9) Two real roots, $x = -2 \pm \sqrt{6}$
11) Two real roots, $x = 0,\ -\frac{5}{2}$

13) $k = \pm\frac{\sqrt{6}}{3}i$
15) $n = \pm\frac{\sqrt{6}}{6}$
17) $p = -\frac{1}{5} \pm \frac{\sqrt{29}}{5}i$
21) $b = \pm\sqrt{2}i$
25) $n = \frac{1}{4} \pm \frac{\sqrt{159}}{12}i$
27) $x = \pm\sqrt{3}$
31) $m = 5,\ -9$
35) $r = -1 \pm \frac{\sqrt{3}}{3}i$
37) $a = -\frac{5}{12} \pm \frac{\sqrt{337}}{12}$
41) $x = \pm 2\sqrt{2}$
45) $x = \frac{2}{7} \pm \frac{3\sqrt{5}}{7}i$
47) $m = \frac{5}{14} \pm \frac{\sqrt{143}}{14}i$
49) $x = \pm\frac{\sqrt{6}}{2}$

Quadratic Inequalities and Sign Diagrams

1) $(-\infty, -9] \cup [-8, \infty)$
3) $(-\infty, 1] \cup [8, \infty)$
5) $(-\infty, -1] \cup [10, \infty)$
7) $\left(-\infty, \frac{2}{7}\right] \cup [6, \infty)$
9) $\left(-\infty, -\frac{4}{7}\right] \cup [4, \infty)$
11) $\left(-\infty, -\frac{1}{2}\right] \cup [7, \infty)$
13) $\left(-\infty, -\frac{6}{7}\right] \cup [5, \infty)$
15) $(-\infty, -3] \cup \left[\frac{6}{7}, \infty\right)$
17) $\left[-2 - \frac{\sqrt{21}}{3}, -2 + \frac{\sqrt{21}}{3}\right]$
19) $(-\infty, -2] \cup [0, \infty)$