MATH 123 College Algebra
Course Pack

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Note: This course pack is required only for students whose *enrolled course instructor of record* is listed above.

This course pack contains lesson handouts for College Algebra. Each lesson follows closely with its respective section/subsection in the accompanying course textbook, available in Blackboard. Each lesson will be discussed during in-class lectures and/or labs, and consequently should be brought to class. All lessons are also be available in Blackboard.

IMPORTANT: Each lesson should be treated as a *secondary* resource to the course textbook, which offers more detailed explanations and additional worked out examples. In the event that you feel that a particular lesson does not provide enough insight or instruction regarding a particular topic, you should find that the accompanying textbook content will help to clarify things for more a complete understanding. **It is therefore expected that all students utilize the course textbook regularly,** alongside any course notes, lessons, and additional handouts.

Although largely free of mathematical errors and "typos", students who identify any errors/typos in either the lessons or textbook are encouraged to report them to the instructor, and the reporting of any mathematical errors will be rewarded with small incentives in the form of additional course homework, quiz, or exam points.

Lessons 1 through 30 make up the first half of the course and cover the following content.

- Linear Equations and Inequalities
- Systems of Linear Equations
- Introduction to Functions
- Quadratic Equations and Inequalities

Lessons 31 through 60 make up the second half of the course and cover the following content.

- Advanced Function Topics
- Polynomials
- Rational Functions

Measurable Outcomes

Below is a comprehensive list of the anticipated measurable outcomes and some essential prerequisite skills needed for successful completion of the College Algebra course. This list is based off of the course description and exit list topics of MATH 123 College Algebra at Framingham State University. Each outcome number aligns to its respective lesson.

1 Solve general linear equations with variables on both sides of the equation. Page 7

2 Solve an equation that contains one or more absolute value(s). Page 11

3 Graph a linear equation by creating a table of values for x.
Identify the slope of a linear equation both graphically and algebraically. Page 13

4 Write the equation of a line in slope-intercept and point-slope form. Page 17

5 Write the equation of a line given a line parallel or perpendicular. Page 21

6 Solve, graph, and give interval notation for the solution to a linear inequality.
Create a sign diagram to identify those intervals where a linear expression is positive or negative. Page 25

7 Solve, graph, and give interval notation to the solution of a compound inequality. Page 27

8 Solve, graph, and give interval notation to the solution of an inequality containing absolute values. Page 31

9 Solve linear systems by graphing. Page 35

10 Solve linear systems by substitution. Page 37

11 Solve linear systems by addition and elimination. Page 41

12 Define a relation and a function; determine if a relation is a function. Page 45

13 Evaluate functions using appropriate notation. Page 49

14 Find the domain and range of a function from its graph. Page 51

Lesson 1: Solving Linear Equations

Objective: Solve general linear equations with variables on both sides of the equation.

Students will be able to:

- Solve and check the solutions to linear equations.

Prerequisite Knowledge:

- Adding, subtracting, and multiplying fractions.
- Finding a least common multiple (LCM).
- Applying the distributive property.
- Checking solutions to equations.

Lesson:
I - Motivating Example(s):

We wish to solve the equation

$$\frac{2}{3}x - 2 = \frac{3}{2}x + \frac{1}{6}.$$

To do so, we will "clear out" all denominators in the equation by multiplying each term in the equation by a least common multiple of the denominators (LCM). In this case, our LCM is 6.

$$\mathbf{6} \cdot \frac{2}{3}x - \mathbf{6} \cdot 2 = \mathbf{6} \cdot \frac{3}{2}x + \mathbf{6} \cdot \frac{1}{6}$$

Cancel and reduce each term to eliminate all fractions.

$$\begin{aligned}
\cancel{\mathbf{6}} \cdot \frac{2}{\cancel{3}}x - \mathbf{6} \cdot 2 &= \cancel{\mathbf{6}} \cdot \frac{3}{\cancel{2}}x + \cancel{\mathbf{6}} \cdot \frac{1}{\cancel{6}} \\
\mathbf{2} \cdot 2x - \mathbf{6} \cdot 2 &= \mathbf{3} \cdot 3x + \mathbf{1} \cdot 1 \\
4x - 12 &= 9x + 1
\end{aligned}$$

Combine like terms and solve the resulting two-step equation for x.

$$\begin{aligned}
4x - 12 &= 9x + 1 \\
-12 &= 5x + 1 \\
-13 &= 5x \\
x &= -\frac{13}{5}
\end{aligned}$$

Check your answer by plugging it back into the **original** equation and simplifying.

$$\begin{aligned}
\frac{2}{3} \cdot \left(-\frac{13}{5}\right) - 2 &= \frac{3}{2} \cdot \left(-\frac{13}{5}\right) + \frac{1}{6} \\
-\frac{26}{15} - 2 &= -\frac{39}{10} + \frac{1}{6}
\end{aligned}$$

Multiply through by the LCM and simplify.

$$\begin{aligned}
\mathbf{30}\cdot\left(-\frac{26}{15}\right)-\mathbf{30}\cdot 2 &= \mathbf{30}\cdot\left(-\frac{39}{10}\right)+\mathbf{30}\cdot\frac{1}{6}\\
\cancel{\mathbf{30}}\cdot\left(-\frac{26}{\cancel{15}}\right)-\mathbf{30}\cdot 2 &= \cancel{\mathbf{30}}\cdot\left(-\frac{39}{\cancel{10}}\right)+\cancel{\mathbf{30}}\cdot\frac{1}{\cancel{6}}\\
\mathbf{2}\cdot(-26)-\mathbf{30}\cdot 2 &= \mathbf{3}\cdot(-39)+\mathbf{5}\cdot 1\\
-52-60 &= -117+5\\
-112 &= -112\ \checkmark
\end{aligned}$$

Since the resulting equation is true, our solution is correct.

II - Demo/Discussion Problems:

Solve each equation. Check your answer.

1. $\dfrac{3}{4}x-\dfrac{7}{2}=\dfrac{5}{6}$
2. $\dfrac{3}{2}\left(\dfrac{5}{9}x+\dfrac{4}{27}\right)=3$
3. $\dfrac{3}{4}x-\dfrac{1}{2}=\dfrac{1}{3}\left(\dfrac{3}{4}x+6\right)-\dfrac{7}{2}$

III - Practice Problems:

Solve each equation.

1) $2-(-3a-8)=1$
2) $2(-3n+8)=-20$
3) $-5(-4+2v)=-50$
4) $2-8(-4+3x)=34$
5) $66=6(6+5x)$
6) $32=2-5(-4n+6)$
7) $0=-8(p-5)$
8) $-55=8+7(k-5)$
9) $-2+2(8x-7)=-16$
10) $-(3-5n)=12$
11) $-21x+12=-6-3x$
12) $-3n-27=-27-3n$
13) $-1-7m=-8m+7$
14) $56p-48=6p+2$
15) $1-12r=29-8r$
16) $4+3x=-12x+4$
17) $20-7b=-12b+30$
18) $-16n+12=39-7n$
19) $-32-24v=34-2v$
20) $17-2x=35-8x$
21) $-2-5(2-4m)=33+5m$
22) $-25-7x=6(2x-1)$
23) $-4n+11=2(1-8n)+3n$
24) $-7(1+b)=-5-5b$
25) $-6v-29=-4v-5(v+1)$
26) $-8(8r-2)=3r+16$
27) $2(4x-4)=-20-4x$
28) $-8n-19=-2(8n-3)+3n$
29) $-a-5(8a-1)=39-7a$
30) $-4+4k=4(8k-8)$
31) $-57=-(-p+1)+2(6+8p)$
32) $16=-5(1-6x)+3(6x+7)$
33) $-2(m-2)+7(m-8)=-67$
34) $7=4(n-7)+5(7n+7)$

35) $50 = 8(7 + 7r) - (4r + 6)$
36) $-8(6 + 6x) + 4(-3 + 6x) = -12$
37) $-8(n - 7) + 3(3n - 3) = 41$
38) $-76 = 5(1 + 3b) + 3(3b - 3)$
39) $-61 = -5(5r - 4) + 4(3r - 4)$
40) $-6(x - 8) - 4(x - 2) = -4$
41) $-2(8n - 4) = 8(1 - n)$
42) $-4(1 + a) = 2a - 8(5 + 3a)$
43) $-3(-7v + 3) + 8v = 5v - 4(1 - 6v)$
44) $-6(x - 3) + 5 = -2 - 5(x - 5)$
45) $-7(x - 2) = -4 - 6(x - 1)$
46) $-(n + 8) + n = -8n + 2(4n - 4)$
47) $-6(8k + 4) = 8(6k + 3) + 12$
48) $-5(x + 7) = 4(-8x - 2)$
49) $-2(1 - 7p) = 8(p - 7)$
50) $8(-8n + 4) = 4(-7n + 8)$

Solve each equation.

51) $\frac{3}{5}(1 + p) = \frac{21}{20}$

52) $-\frac{1}{2} = \frac{3}{2}k + \frac{3}{2}$

53) $0 = -\frac{5}{4}(x - \frac{6}{5})$

54) $\frac{3}{2}n - \frac{8}{3} = -\frac{29}{12}$

55) $\frac{3}{4} - \frac{5}{4}m = \frac{113}{24}$

56) $\frac{11}{4} + \frac{3}{4}r = \frac{163}{32}$

57) $\frac{635}{72} = -\frac{5}{2}(-\frac{11}{4} + x)$

58) $-\frac{16}{9} = -\frac{4}{3}(\frac{5}{3} + n)$

59) $2b + \frac{9}{5} = -\frac{11}{5}$

60) $\frac{3}{2} - \frac{7}{4}v = -\frac{9}{8}$

61) $\frac{3}{2}(\frac{7}{3}n + 1) = \frac{3}{2}$

62) $\frac{41}{9} = \frac{5}{2}(x + \frac{2}{3}) - \frac{1}{3}x$

63) $-a - \frac{5}{4}(-\frac{8}{3}a + 1) = -\frac{19}{4}$

64) $\frac{1}{3}(-\frac{7}{4}k + 1) - \frac{10}{3}k = -\frac{13}{8}$

65) $\frac{55}{6} = -\frac{5}{2}(\frac{3}{2}p - \frac{5}{3})$

66)$-\frac{1}{2}(\frac{2}{3}x - \frac{3}{4}) - \frac{7}{2}x = -\frac{83}{24}$

67) $\frac{16}{9} = -\frac{4}{3}(-\frac{4}{3}n - \frac{4}{3})$

68) $\frac{2}{3}(m + \frac{9}{4}) - \frac{10}{3} = -\frac{53}{18}$

69) $-\frac{5}{8} = \frac{5}{4}(r - \frac{3}{2})$

70) $\frac{1}{12} = \frac{4}{3}x + \frac{5}{3}(x - \frac{7}{4})$

71) $-\frac{11}{3} + \frac{3}{2}b = \frac{5}{2}(b - \frac{5}{3})$

72) $\frac{7}{6} - \frac{4}{3}n = -\frac{3}{2}n + 2(n + \frac{3}{2})$

73) $-(-\frac{5}{2}x - \frac{3}{2}) = -\frac{3}{2} + x$

74) $-\frac{149}{16} - \frac{11}{3}r = -\frac{7}{4}r - \frac{5}{4}(-\frac{4}{3}r + 1)$

75) $\frac{45}{16} + \frac{3}{2}n = \frac{7}{4}n - \frac{19}{16}$

76) $-\frac{7}{2}(\frac{5}{3}a + \frac{1}{3}) = \frac{11}{4}a + \frac{25}{8}$

77) $\frac{3}{2}(v + \frac{3}{2}) = -\frac{7}{4}v - \frac{19}{6}$

78) $-\frac{8}{3} - \frac{1}{2}x = -\frac{4}{3}x - \frac{2}{3}(-\frac{13}{4}x + 1)$

79) $\frac{47}{9} + \frac{3}{2}x = \frac{5}{3}(\frac{5}{2}x + 1)$

80) $\frac{1}{3}n + \frac{29}{6} = 2(\frac{4}{3}n + \frac{2}{3})$

Lesson 2: Equations Containing Absolute Values

Objective: Solve an equation that contains one or more absolute value(s).

Students will be able to:

- Solve and check the solutions to an equation that contains an absolute value.

Prerequisite Knowledge:

- Evaluating absolute value expressions.
- Applying the distributive property.
- Checking solutions to equations.

Lesson:
I - Motivating Example(s):

$$\begin{aligned} |x| &= 7 && \text{Absolute value can be positive or negative} \\ x &= \pm 7 && \text{Our solution} \end{aligned}$$

Notice that we have considered two possibilities, both the positive and negative. Either way, the absolute value of our number will be positive 7. When we have absolute values in our problem it is important to first isolate the absolute value, then remove the absolute value by considering both the positive and negative solutions.

II - Demo/Discussion Problems:

Solve each of the following equations containing absolute values.

1. $5|x| - 4 = 26$

2. $2 - 4|2x + 3| = -18$

3. $7 + |2x - 5| = 4$

4. $|2x - 7| = |4x + 6|$

III - Practice Problems:

Solve each equation.

1) $|x| = 8$

2) $|n| = 7$

3) $|b| = 1$

4) $|x| = 2$

5) $|5 + 8a| = 53$

6) $|9n + 8| = 46$

7) $|3k + 8| = 2$

8) $|3 - x| = 6$

9) $|9 + 7x| = 30$

10) $|5n + 7| = 23$

11) $|8 + 6m| = 50$

12) $|9p + 6| = 3$

13) $|6 - 2x| = 24$

14) $|3n - 2| = 7$

15) $-7| - 3 - 3r| = -21$

16) $|2 + 2b| + 1 = 3$

17) $7| - 7x - 3| = 21$

18) $\dfrac{| - 4 - 3n|}{4} = 2$

19) $\dfrac{| - 4b - 10|}{8} = 3$

20) $8|5p + 8| - 5 = 11$

21) $8|x + 7| - 3 = 5$

22) $3 - |6n + 7| = -40$

23) $5|3 + 7m| + 1 = 51$

24) $4|r + 7| + 3 = 59$

25) $3 + 5|8 - 2x| = 63$

26) $5 + 8| - 10n - 2| = 101$

27) $|6b - 2| + 10 = 44$

28) $7|10v - 2| - 9 = 5$

29) $-7 + 8| - 7x - 3| = 73$

30) $8|3 - 3n| - 5 = 91$

31) $|5x + 3| = |2x - 1|$

32) $|2 + 3x| = |4 - 2x|$

33) $|3x - 4| = |2x + 3|$

34) $\left|\dfrac{2x - 5}{3}\right| = \left|\dfrac{3x + 4}{2}\right|$

35) $\left|\dfrac{4x - 2}{5}\right| = \left|\dfrac{6x + 3}{2}\right|$

36) $\left|\dfrac{3x + 2}{2}\right| = \left|\dfrac{2x - 3}{3}\right|$

Lesson 3: Graphing Lines

Objective: Graph a linear equation by creating a table of values for x. Identify the slope of a linear equation both graphically and algebraically.

Students will be able to:

- Create and populate a table of points for a given equation or graph.
- Calculate the slope of a line when given two points.

Prerequisite Knowledge:

- Plotting points on a coordinate plane.
- Identifying $x-$ and $y-$coordinates.
- Conceptually understanding of slope.

Lesson:

Given a linear equation, such as $y = 2x - 3$, one may be interested in what solution(s) are possible for a given x or y. We can visualize the set of solutions by making a graph of all possible x and y combinations, or *coordinate pairs*, that satisfy this equation. Our corresponding graph will be a line, and any point on this line will make the equation $y = 2x - 3$ true. We will do this using a table of values.

Additionally, the slope of a line will be extremely useful for drawing conclusions about a linear equation and/or its graph.

Given two points (x_1, y_1) and (x_2, y_2), the slope of the line through these points is defined as follows.

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Whenever we calculate a slope, as we subtract the corresponding y and x coordinates from one another, it is important that we subtract them in the correct order.

I - Motivating Example(s):

Example:

Graph $y = 2x - 3$. Make a table of values. Any test values may be used.

x	y
-1	-5
0	-3
1	-1

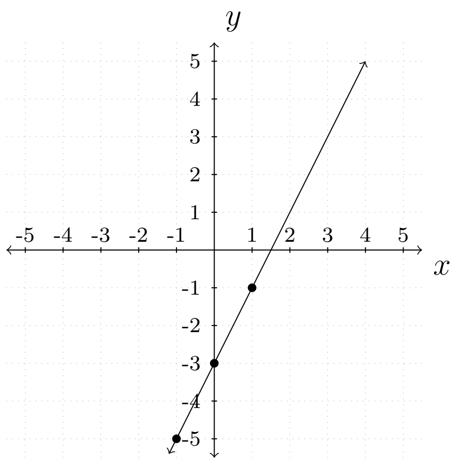
Evaluate each test value by replacing x with the given value.

$x = -1$ $\quad y = 2(-1) - 3 = -2 - 3 = -5$

$x = 0$ $\quad y = 2(0) - 3 = 0 - 3 = -3$

$x = 1$ $\quad y = 2(1) - 3 = 2 - 3 = -1$

$(-1, -5), (0, -3), (1, -1)$ These become our points to graph for our equation.



$(-1, -5), (0, -3)$, and $(1, -1)$
These become the points from our equation which we will plot on our graph.
Once the point are on the graph, connect the dots to make a line.
The graph is our solution.

Notice the graph also goes through the point $(2, 1)$. This means that the pair $(x, y) = (2, 1)$ will also satisfy the equation $y = 2x - 3$, which one can easily check.

Notice also that the slope of the line above is $m = 2$ or $\frac{2}{1}$. We can check this by using any two points from our table. We will use $(-1, -5)$ and $(0, -3)$.

$$m = \frac{-5 - (-3)}{-1 - 0} = \frac{-5 + 3}{-1} = \frac{-2}{-1} = 2 \checkmark$$

Example: Find the slope of the line through the given points.

$(-4, 3)$ and $(2, -9)$. Identify $x_1, y_1, x_2,$ and y_2.

(x_1, y_1) and (x_2, y_2) Use the slope formula, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

$m = \dfrac{-9 - 3}{2 - (-4)}$ Simplify.

$m = \dfrac{-12}{6}$ Reduce.

$m = -2$ Our solution.

II - Demo/Discussion Problems:

1. Make a table of points and use it to graph the linear equation $2x - 3y = 6$.
2. Find the slope of the line through the points $(-4, -1)$ and $(-4, -5)$.
3. Find the slope of the line through the points $(3, 1)$ and $(-2, 1)$.

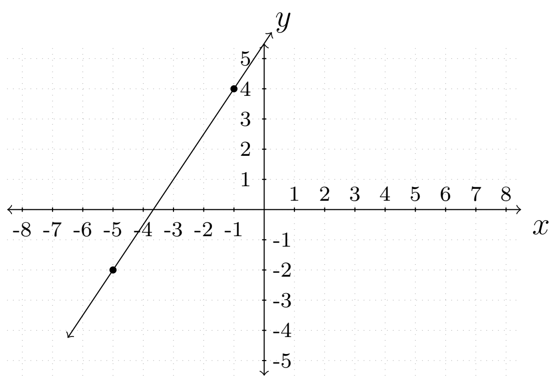
III - Practice Problems:

For each linear equation below, make a table of points and use it to graph the equation.

1. $y = -\frac{1}{4}x - 3$
2. $y = x - 1$
3. $y = -\frac{5}{4}x - 4$
4. $y = -\frac{3}{5}x + 1$
5. $y = -4x + 2$
6. $y = \frac{5}{3}x + 4$
7. $y = \frac{3}{2}x - 5$
8. $y = -x - 2$
9. $y = -\frac{4}{5}x - 3$
10. $y = \frac{1}{2}x$
11. $x + 5y = -15$
12. $8x - y = 5$
13. $4x + y = 5$
14. $3x + 4y = 16$
15. $2x - y = 2$
16. $7x + 3y = -12$
17. $x + y = -1$
18. $3x + 4y = 8$
19. $x - y = -3$
20. $9x - y = -4$

Find the slope of each of the following lines.

21.



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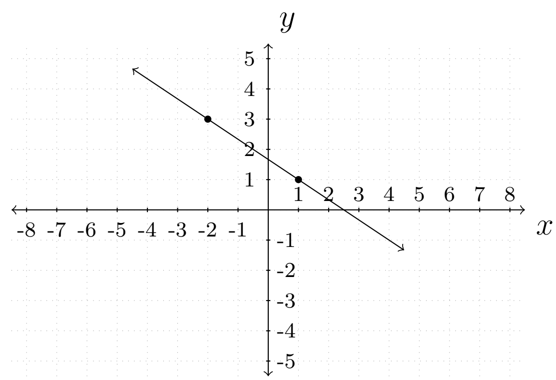

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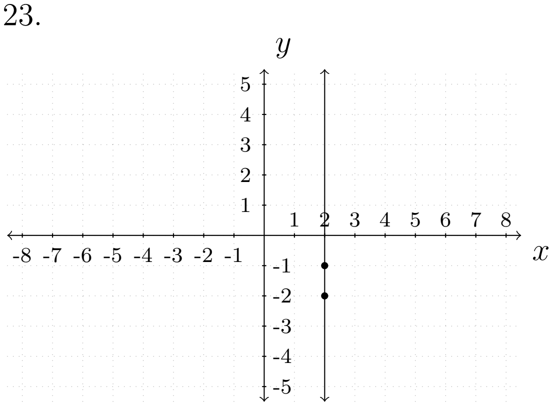

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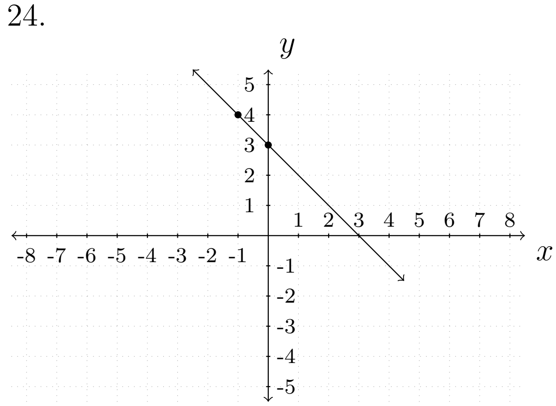

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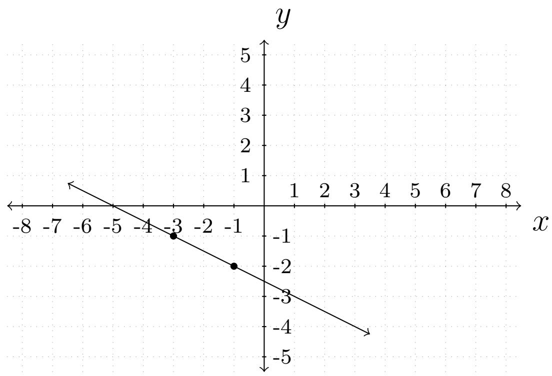
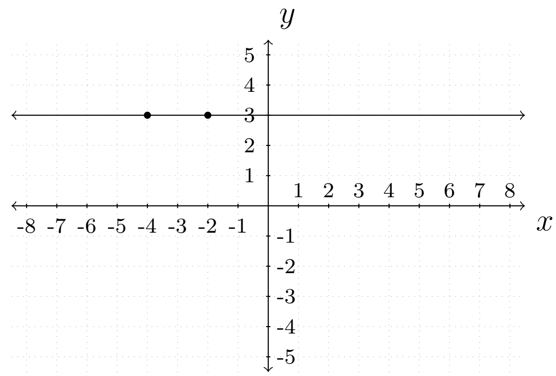


y
x
5
4
3
2
1
-1
-2
-3
-4
-5
-8 -7 -6 -5 -4 -3 -2 -1
1 2 3 4 5 6 7 8

26.



y
x
5
4
3
2
1
-1
-2
-3
-4
-5
-8 -7 -6 -5 -4 -3 -2 -1
1 2 3 4 5 6 7 8

Lesson 4: Two Forms of a Linear Equation

Objective: Write the equation of a line in slope-intercept and point-slope form.

Students will be able to:

- Find the slope of a line having certain characteristics.
- Identify a $y-$intercept.
- Convert between point-slope and slope-intercept forms.

Prerequisite Knowledge:

- Definitions of slope of a line and $y-$intercept.
- Graphing points on a coordinate plane .
- Point-testing.
- Multiplying and dividing fractions.

Lesson:
The two forms for a linear equation are:

$$\begin{aligned} \text{slope-intercept form:} \quad & y = mx + b \\ \text{point-slope form:} \quad & y - y_1 = m(x - x_1) \end{aligned}$$

I - Motivating Example(s):

Find the equation of the line through the points $(-3, 4)$ and $(-1, -2)$. Express your answer in slope-intercept form.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{Use the given points to find the slope.}$$

$$m = \frac{-2 - 4}{-1 - (-3)} = \frac{-6}{2} = -3 \qquad \text{Substitute the } x- \text{ and } y- \text{ coordinates and simplify.}$$

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Use the point-slope form. Substitute } m \text{ and either point.} \\ y - 4 &= -3(x - (-3)) && \text{Simplify to obtain slope-intercept form. Distribute } m. \\ y - 4 &= -3x - 9 && \text{Solve for } y. \\ y &= -3x - 5 && \text{Our solution, in slope-intercept form.} \end{aligned}$$

II - Demo/Discussion Problems:

For each problem, find the equation of a line having the given characteristics. In each case, find both the point-slope and slope-intercept forms.

1. A line through the point $(-6, 2)$ and having a slope of $-\frac{2}{3}$.
2. A line through the points $(-2, 5)$ and $(4, -3)$.

III - Practice Problems:

Find the point-slope form of the line through the given point with the given slope.

1) through $(2, 3)$, slope is undefined
2) through $(1, 2)$, slope is undefined
3) through $(2, 2)$, slope is $\frac{1}{2}$
4) through $(2, 1)$, slope is $-\frac{1}{2}$
5) through $(-1, -5)$, slope is 9
6) through $(2, -2)$, slope is -2
7) through $(-4, 1)$, slope is $\frac{3}{4}$
8) through $(4, -3)$, slope is -2
9) through $(0, -2)$, slope is -3
10) through $(-1, 1)$, slope is 4
11) through $(0, -5)$, slope is $-\frac{1}{4}$
12) through $(0, 2)$, slope is $-\frac{5}{4}$
13) through $(-5, -3)$, slope is $\frac{1}{5}$
14) through $(-1, -4)$, slope is $-\frac{2}{3}$
15) through $(-1, 4)$, slope is $-\frac{5}{4}$
16) through $(1, -4)$, slope is $-\frac{3}{2}$

Find the slope-intercept form of the line through the given point with the given slope.

17) through $(-1, -5)$, slope is 2
18) through $(2, -2)$, slope is -2
19) through $(5, -1)$, slope is $-\frac{3}{5}$
20) through $(-2, -2)$, slope is $-\frac{2}{3}$
21) through $(-4, 1)$, slope is $\frac{1}{2}$
22) through $(4, -3)$, slope is $-\frac{7}{4}$
23) through $(4, -2)$, slope is $-\frac{3}{2}$
24) through $(-2, 0)$, slope is $-\frac{5}{2}$
25) through $(-5, -3)$, slope is $-\frac{2}{5}$
26) through $(3, 3)$, slope is $\frac{7}{3}$
27) through $(2, -2)$, slope is 1
28) through $(-4, -3)$, slope is 0
29) through$(-3, 4)$, slope is undefined
30) through $(-2, -5)$, slope is 2
31) through $(-4, 2)$, slope is $-\frac{1}{2}$
32) through $(5, 3)$, slope is $\frac{6}{5}$

Find the point-slope form of the line through the given points.

33) through $(-4, 3)$ and $(-3, 1)$
34) through $(1, 3)$ and $(-3, 3)$
35) through $(5, 1)$ and $(-3, 0)$
36) through $(-4, 5)$ and $(4, 4)$
37) through $(-4, -2)$ and $(0, 4)$
38) through $(-4, 1)$ and $(4, 4)$
39) through $(3, 5)$ and $(-5, 3)$
40) through $(-1, -4)$ and $(-5, 0)$
41) through $(3, -3)$ and $(-4, 5)$
42) through $(-1, -5)$ and $(-5, -4)$

Find the slope-intercept form of the line through the given points.

43) through $(-5, 1)$ and $(-1, -2)$
44) through $(-5, -1)$ and $(5, -2)$
45) through $(-5, 5)$ and $(2, -3)$
46) through $(1, -1)$ and $(-5, -4)$
47) through $(4, 1)$ and $(1, 4)$
48) through $(0, 1)$ and $(-3, 0)$

49) through $(0, 2)$ and $(5, -3)$

50) through $(0, 2)$ and $(2, 4)$

51) through $(0, 3)$ and $(-1, -1)$

52) through $(-2, 0)$ and $(5, 3)$

Lesson 5: Parallel and Perpendicular Lines

Objective: Write the equation of a line given a line parallel or perpendicular.

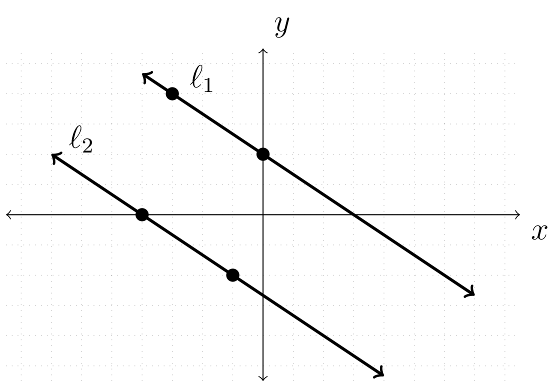
Students will be able to:

- Find a perpendicular slope, given a slope or linear equation.
- Find a parallel slope, given a slope or linear equation.
- Find the equation of a line parallel or perpendicular to a given linear equation.

Prerequisite Knowledge:

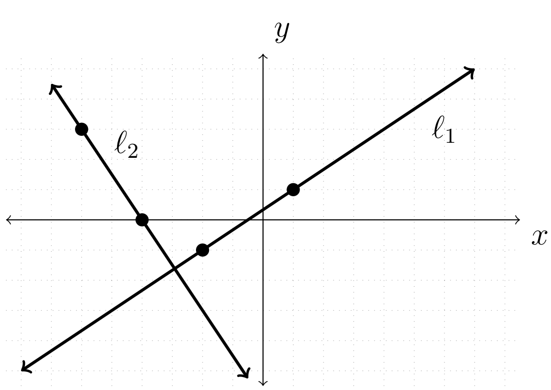
- Identify the slope of a line.
- Work with the slope-intercept form of a line.
- Work with the point-slope form of a line.

Lesson:
I - Motivating Example(s):



This graph shows two parallel lines.

The slope (rise over run) of each line is "down 2, right 3," or $m_1 = m_2 = -\frac{2}{3}$.



This graph shows two perpendicular lines.

The slope (rise over run) of the more gradual line is "up 2, right 3," or $m_1 = \frac{2}{3}$.

The slope of the steeper line is "down 3, right 2," or $m_2 = -\frac{3}{2}$.

As the first graph illustrates, **parallel lines** have the **same slope**, $m_1 = m_2$.

On the other hand, the second graph shows us that **perpendicular lines** have **negative reciprocal slopes**, $m_2 = -\frac{1}{m_1}$ (and so, $m_1 \cdot m_2 = -1$).

We can use these properties to make conclusions about whether two lines are parallel, perpendicular, or neither.

II - Demo/Discussion Problems:

1. Find the equation of the line through (6,-9) perpendicular to the line $y = -\frac{3}{5}x + 4$.
2. Find the equation of a line through (4,-5) and parallel to the line $2x - 3y = 6$.
3. Find the equation of the line through (3,4) and perpendicular to the line $x = -2$.

III - Practice Problems:

Find the slope of a line parallel to each given line.

1) $y = 2x + 4$	3) $y = 4x - 5$	5) $x - y = 4$	7) $7x + y = -2$
2) $y = -\frac{2}{3}x + 5$	4) $y = -\frac{10}{3}x - 5$	6) $6x - 5y = 20$	8) $3x + 4y = -8$

Find the slope of a line perpendicular to each given line.

9) $x = 3$	11) $y = -\frac{1}{3}x$	13) $x - 3y = -6$	15) $x + 2y = 8$
10) $y = -\frac{1}{2}x - 1$	12) $y = \frac{4}{5}x$	14) $3x - y = -3$	16) $8x - 3y = -9$

Write the point-slope form of the equation of the line described.

17) through $(2, 5)$, parallel to $x = 0$
18) through $(5, 2)$, parallel to $y = \frac{7}{5}x + 4$
19) through $(3, 4)$, parallel to $y = \frac{9}{2}x - 5$
20) through $(1, -1)$, parallel to $y = -\frac{3}{4}x + 3$
21) through $(2, 3)$, parallel to $y = \frac{7}{5}x + 4$
22) through $(-1, 3)$, parallel to $y = -3x - 1$
23) through $(4, 2)$, parallel to $x = 0$
24) through $(1, 4)$, parallel to y $= \frac{7}{5}x + 2$
25) through $(1, -5)$, perpendicular to $-x + y = 1$
26) through $(1, -2)$, perpendicular to $-x + 2y = 2$
27) through $(5, 2)$, perpendicular to $5x + y = -3$
28) through $(1, 3)$, perpendicular to $-x + y = 1$
29) through $(4, 2)$, perpendicular to $-4x + y = 0$
30) through $(-3, -5)$, perpendicular to $3x + 7y = 0$
31) through $(2, -2)$, perpendicular to $3y - x = 0$
32) through $(-2, 5)$, perpendicular to $y - 2x = 0$

Write the slope-intercept form of the equation of the line described.

33) through $(4, -3)$, parallel to $y = -2x$
34) through $(-5, 2)$, parallel to $y = \frac{3}{5}x$
35) through $(-3, 1)$, parallel to $y = -\frac{4}{3}x - 1$
36) through $(-4, 0)$, parallel to $y = -\frac{5}{4}x + 4$
37) through $(-4, -1)$, parallel to $y = -\frac{1}{2}x + 1$
38) through $(2, 3)$, parallel to $y = \frac{5}{2}x - 1$
39) through $(-2, -1)$, parallel to $y = -\frac{1}{2}x - 2$
40) through $(-5, -4)$, parallel to $y = \frac{3}{5}x - 2$
41) through $(4, 3)$, perpendicular to $x + y = -1$
42) through $(-3, -5)$, perpendicular to $x + 2y = -4$
43) through $(5, 2)$, perpendicular to $x = 0$
44) through $(5, -1)$, perpendicular to $-5x + 2y = 10$
45) through $(-2, 5)$, perpendicular to $-x + y = -2$
46) through $(2, -3)$, perpendicular to $-2x + 5y = -10$
47) through $(4, -3)$, perpendicular to $-x + 2y = -6$
48) through $(-4, 1)$, perpendicular to $4x + 3y = -9$

Lesson 6: Linear Inequalities

Objective: Solve, graph, and give interval notation for the solution to a linear inequality. Create a sign diagram to identify those intervals where a linear expression is positive or negative.

Students will be able to:

- Solve a linear inequality by isolating the variable.
- Recognize the need to change the direction of an inequality when multiplying or dividing by a negative value.
- Graph a linear inequality on a one-dimensional axis.
- Express solutions using interval notation.

Prerequisite Knowledge:

- Apply the distributive property.
- Verify the accuracy of a solution to an inequality by checking.

Lesson:
I - Motivating Example(s):

Solve the linear inequality $4x - 3 \geq 5$.

$$\begin{array}{rll} 4x - 3 \geq & 5 & \\ \underline{\mathbf{+3}} \quad & \underline{\mathbf{+3}} & \text{Add 3 to both sides} \\ 4x \geq & 8 & \\ \overline{\mathbf{4}} \quad & \overline{\mathbf{4}} & \text{Divide both sides by 4} \\ x \geq & 2 & \text{Our solution} \end{array}$$

Our solution can be expressed as follows.

1. Verbally: "The set of all values of x that are greater than or equal to (at least) 2".
2. Inequality: $\{x | x \geq 2\}$
3. Interval: $[2, \infty)$
4. Real-number Line (Graphically):

Note: A closed (shaded) circle at $x = 2$ is also acceptable in place of a bracket.

Check:

Test Location	Test Value	Unsimplified	Simplified	Result
Shaded region	$x = 3$	$4(3) - 3 \geq 5$	$9 \geq 5$	True
Boundary value	$x = 2$	$4(2) - 3 \geq 5$	$5 \geq 5$	True
Unshaded region	$x = 0$	$4(0) - 3 \geq 5$	$-3 \geq 5$	False

II - Demo/Discussion Problems:

Solve the linear inequality $-1 - 2(x - 3) \leq 5x - 9$.

III - Practice Problems:

Draw a graph for each inequality below and provide interval notation.

1) $n > -5$
2) $n > 4$
3) $-2 \geq k$
4) $1 \geq k$
5) $5 \geq x$
6) $-5 < x$

Solve each inequality, graph each solution, and provide interval notation.

7) $\dfrac{x}{11} \geq 10$
8) $-2 \leq \dfrac{n}{13}$
9) $2 + r < 3$
10) $\dfrac{m}{5} \leq -\dfrac{6}{5}$
11) $8 + \dfrac{n}{3} \geq 6$
12) $11 > 8 + \dfrac{x}{2}$
13) $2 > \dfrac{a-2}{5}$
14) $\dfrac{v-9}{-4} \leq 2$
15) $\dfrac{6+x}{12} \leq -1$

16) $-47 \geq 8 - 5x$

17) $-2(3 + k) < -44$

18) $-7n - 10 \geq 60$

19) $18 < -2(-8 + p)$

20) $5 \geq \dfrac{x}{5} + 1$

21) $24 \geq -6(m - 6)$

22) $-8(n - 5) \geq 0$

23) $-r - 5(r - 6) < -18$

24) $-60 \geq -4(-6x - 3)$

25) $24 + 4b < 4(1 + 6b)$

26) $-8(2 - 2n) \geq -16 + n$

27) $-5v - 5 < -5(4v + 1)$

28) $-36 + 6x > -8(x + 2) + 4x$

29) $4 + 2(a + 5) < -2(-a - 4)$

30) $3(n + 3) + 7(8 - 8n) < 5n + 5 + 2$

31) $-(k - 2) > -k - 20$

32) $-(4 - 5p) + 3 \geq -2(8 - 5p)$

Lesson 7: Compound Inequalities

Objective: Solve, graph, and give interval notation to the solution of a compound inequality.

Students will be able to:

- Understand the distinctions between "OR" and "AND" inequalities.
- Recognize a double inequality as an "AND" inequality.
- Represent the solution to a compound inequality using the three notations (algebraically, graphically, and using interval notation).

Prerequisite Knowledge:

- Solving linear inequalities.
- Graphing on a number line.
- Interval notation, including unions and intersections.

Lesson:

Several inequalities can be combined together to form what are called compound inequalities. There are three types of compound inequalities which we will investigate in this lesson.

- The first type of a compound inequality is an "OR" inequality. A solution for this type of inequality will produce a true statement for either one inequality OR the other inequality OR both. Solutions to OR inequalities will often (but not always) consist of a union of two intervals, denoted by a $\cup$.
- The second type of compound inequality is an "AND" inequality. These inequalities require *both* statements to be true for a given solution. Solutions to AND inequalities equal the intersection (or overlap) of two intervals. Intersections are denoted by a $\cap$, but can be always be simplified.
- The third type of compound inequality is a special type of AND inequality. When a variable (or expression containing the variable) is between two numbers, we can write this as a single mathematical sentence with three parts, such as $5 < x \leq 8$, to show x is greater than 5 AND x is less than or equal to 8. Since this sentence contains two inequalities (both always pointing in the same direction), we refer to it as a *double inequality*.

I - Motivating Example(s):

Example: Solve the inequality below, graph the solution, and provide the interval notation of your solution.

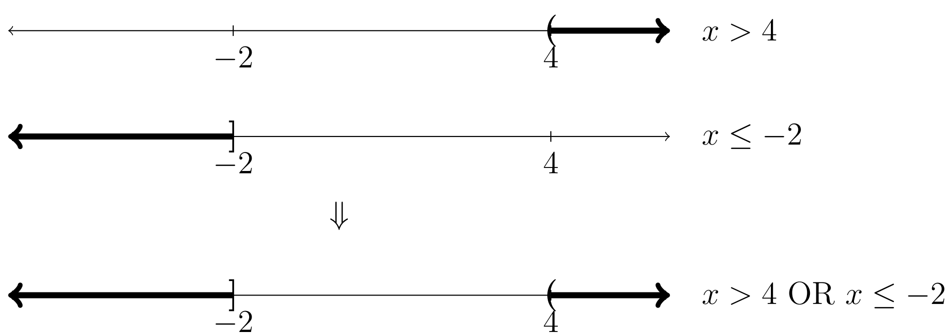
$$2x - 5 > 3 \ \text{ OR } \ 4 - x \geq 6$$

Here, we isolate the variable for each inequality. In the first case, adding 5 and dividing by 2 produces $x > 4$.

For the second inequality, we subtract 4 and multiply (or divide) by a -1, which will change the direction of our inequality. Here, we get $x \leq -2$.

$$x > 4 \ \text{ OR } \ x \leq -2$$

To express our answer graphically, we will sketch three separate intervals: one for the first inequality, one for the second inequality, and one for the union of the two, which will be our answer.



Since our graphical answer includes two pieces, we use a union in our interval notation.

$$(-\infty, -2] \cup (4, \infty)$$

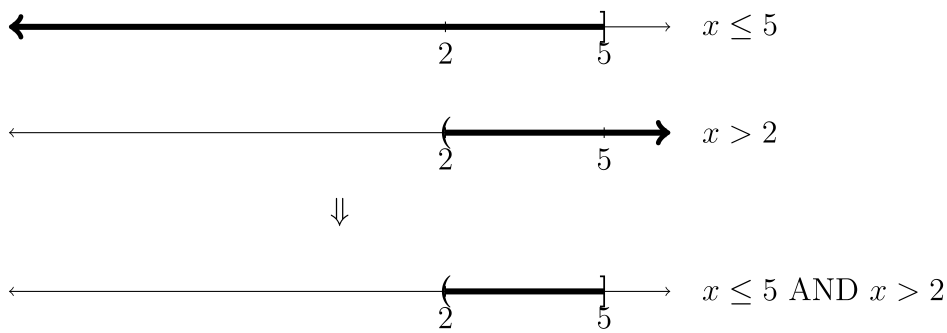
Example: Solve the inequality below, graph the solution, and provide the interval notation of your solution.

$$2x + 8 \geq 5x - 7 \ \text{ AND } \ 5x - 3 > 3x + 1$$

Solving each inequality separately gives us the following.

$$\begin{aligned} 2x + 8 &\geq 5x - 7 \\ 2x &\geq 5x - 15 \\ -3x &\geq -15 \\ x &\leq 5 \end{aligned} \qquad\qquad \begin{aligned} 5x - 3 &> 3x + 1 \\ 5x &> 3x + 4 \\ 2x &> 4 \\ x &> 2 \end{aligned}$$

Next, we graph the two inequalities separately, and take their intersection (overlap) for our final answer.



Our answer, in interval notation is $(2, 5]$.

II - Demo/Discussion Problems:

Graph each compound inequality on a real number line and provide the corresponding interval notation.

1. $x \le -3$ OR $x < -1$
2. $x \ge -3$ OR $x < -1$
3. $x < -1$ AND $x < -2$
4. $x > 2$ AND $x < -1$

Solve each of the given compound inequalities. Graph each solution on a real number line and provide the corresponding interval notation.

5. $9 + n < 2$ OR $5n > 40$
6. $\dfrac{v}{8} > -1$ AND $v - 2 < 1$
7. $-6 \le -4x + 2 < 2$
8. $-4 < 8 - 3m \le 11$

III - Practice Problems:

Solve each of the given compound inequalities. Graph each solution on a real number line and provide the corresponding interval notation.

1. $\dfrac{n}{3} \le -3$ OR $-5n \le -10$
2. $6m \ge -24$ OR $m - 7 < -12$
3. $x + 7 \ge 12$ OR $9x < -45$
4. $10r > 0$ OR $r - 5 < -12$
5. $x - 6 < -13$ OR $6x \le -60$
6. $-9x < 63$ AND $\dfrac{x}{4} < 1$
7. $-8 + b < -3$ AND $4b < 20$
8. $-6n \le 12$ AND $\dfrac{n}{3} \le 2$
9. $a + 10 \ge 3$ AND $8a \le 48$
10. $-6 + v \ge 0$ AND $2v > 4$
11. $3 + 7r > 59$ OR $-6r - 3 > 33$
12. $-6 - 8x \ge -6$ OR $2 + 10x > 82$

13. $3 \leq 9 + x \leq 7$

16. $-11 \leq n - 9 \leq -5$

19. $1 \leq \dfrac{p}{8} \leq 0$

14. $0 \geq \dfrac{x}{9} \geq -1$

17. $-3 < x - 1 < 1$

20. $-22 \leq 2n - 10 \leq -16$

15. $11 < 8 + k \leq 12$

18. $-2 < 1 - 3x \leq 10$

21. $\dfrac{1}{2} < 5 - \dfrac{x}{3} \leq 4$

22. $-5b + 10 \leq 30$ AND $7b + 2 \leq -40$

23. $n + 10 \geq 15$ OR $4n - 5 < -1$

24. $3x - 9 < 2x + 10$ AND $5 + 7x \leq 10x - 10$

25. $4n + 8 < 3n - 6$ OR $10n - 8 \geq 9 + 9n$

26. $-8 - 6v \leq 8 - 8v$ AND $7v + 9 \leq 6 + 10v$

27. $5 - 2a \geq 2a + 1$ OR $10a - 10 \geq 9a + 9$

28. $1 + 5k \leq 7k - 3$ OR $k - 10 > 2k + 10$

29. $8 - 10r \leq 8 + 4r$ OR $-6 + 8r < 2 + 8r$

30. $2x + 9 \geq 10x + 1$ AND $3x - 2 < 7x + 2$

31. $-9m + 2 < -10 - 6m$ OR $-m + 5 \geq 10 + 4m$

Lesson 8: Inequalities Containing Absolute Values

Objective: Solve, graph, and give interval notation to the solution of an inequality containing absolute values.

Students will be able to:

- Recognize and correctly interpret the two cases for inequalities containing absolute values.
- Represent the solution to an inequality containing absolute values using the three notations (algebraically, graphically, and using interval notation).

Prerequisite Knowledge:

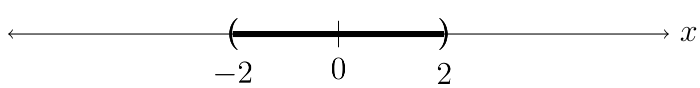
- Solving linear inequalities.
- Solving compound inequalities.
- Graphing on a number line.
- Interval notation, including unions and intersections.

Lesson:

When an inequality contains an absolute value we will have to remove the absolute value in order to graph the solution or provide interval notation. The way we remove the absolute value depends on the direction of the inequality symbol.

Consider $|x| < 2$.

Absolute value is defined as the distance from zero. Another way to read this inequality would be the distance that the variable x is from zero is less than 2. So on a number line we will shade all points that are less than 2 units away from zero.

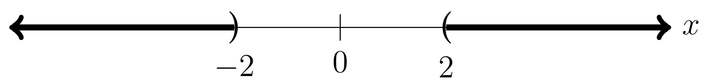


Interval Notation: $(-2, 2)$

This graph looks just like the graphs of a double inequality. When an absolute value is *less than* a number we will remove the absolute value by changing the problem to a double inequality, with the negative value on the left and the positive value on the right. So $|x| < 2$ becomes $-2 < x < 2$, as the previous graph illustrates.

Alternatively, let's consider $|x| > 2$.

Another way to read this inequality would be the distance that x is from zero is greater than 2. So on the number line we shade all points that are *more than* 2 units away from zero.



Interval Notation: $(-\infty, -2) \cup (2, \infty)$

This graph looks just like the graph of an OR inequality. When an absolute value is *greater than* a number we will remove the absolute value by changing the problem to an OR inequality, the first inequality looking just like the problem with no absolute value, the second flipping the inequality symbol and changing the value to a negative. So $|x| > 2$ becomes $x > 2$ OR $x < -2$, as the graph above illustrates.

For all absolute value inequalities we can also express our answers in interval notation which is done the same way as for compound inequalities.

As with an equation, our first step to solving an inequality containing an absolute value will be to isolate the absolute value. Next we will remove the absolute value by either changing to a double inequality if the absolute value is less than a number, or changing to an OR inequality if the absolute value is greater than a number. Then we solve the newly created compound inequality.

II - Demo/Discussion Problems:

Solve each of the given inequalities. Graph each solution on a real number line and provide the corresponding interval notation.

1. $|4x - 5| \geq 6$
2. $-4 - 3|x| \leq -16$
3. $9 - 2|4x + 1| > 3$
4. $12 + 4|6x - 1| < 4$
5. $5 - 6|x + 7| \leq 17$

III - Practice Problems:

Solve each of the given inequalities. Graph each solution on a real number line and provide the corresponding interval notation.

1. $|x| < 3$
2. $|x| \leq 8$
3. $|2x| < 6$
4. $|x+3| < 4$
5. $|x-2| < 6$
6. $|x-8| < 12$
7. $|x-7| < 3$
8. $|x+3| \leq 4$
9. $|3x-2| < 9$
10. $|2x+5| < 9$
11. $1+2|x-1| \leq 9$
12. $10-3|x-2| \geq 4$
13. $6-|2x-5| \geq 3$
14. $|x| > 5$
15. $|3x| > 5$
16. $|x-4| > 5$
17. $|x-3| \geq 3$
18. $|2x-4| > 6$
19. $|3x-5| \geq 3$
20. $3-|2-x| < 1$
21. $4+3|x-1| \geq 10$
22. $3-2|3x-1| \geq -7$
23. $3-2|x-5| \leq -15$
24. $4-6|6+3x| \leq -5$
25. $-2-3|4-2x| \geq -8$
26. $-3-2|4x-5| \geq 1$
27. $4-5|2x+7| < -1$
28. $-2+3|5-x| \leq 4$
29. $3-2|4x-5| \geq 1$
30. $-2-3|3x+5| \geq -5$
31. $-5-2|3x-6| < -8$
32. $6-3|1-4x| < -3$
33. $4-4|-2x+6| > -4$
34. $-3-4|2x+5| \geq -7$
35. $|-10+x| \geq 8$

Lesson 9: Graphing Systems of Linear Equations

Objective: Solve linear systems by graphing.

Students will be able to:

- Solve linear systems by graphing both equations on one coordinate plane.
- Write system solutions as ordered pairs in the form (x, y).
- Verify the accuracy of a solution by plugging it into each equation in the system.

Prerequisite Knowledge:

- Find the slope-intercept form of a linear equation.
- Graph linear equations in slope-intercept form.
- Plot points on the coordinate plane.

Lesson:
I - Motivating Example(s):

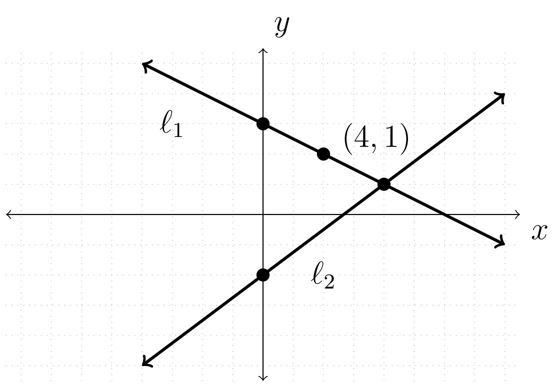
Solve the following system of equations.

$$\begin{cases} y = -\frac{1}{2}x + 3 \\ y = \frac{3}{4}x - 2 \end{cases}$$

First identify slopes and $y-$intercepts.

$$\begin{array}{ll} \text{Line 1}: & m = -\frac{1}{2}, \quad b = 3 \\ \text{Line 2}: & m = \frac{3}{4}, \quad b = -2 \end{array}$$

Next graph both lines on the same plane.



To graph each equation, we start at the y-intercept and use the slope $\left(\frac{\text{rise}}{\text{run}}\right)$ to get the next point, then connect the dots.

Remember a line with a negative slope decreases from left to right.

Use slopes to find the intersection point, $(4, 1)$. This is our solution.

Often our equations won't be in slope-intercept form and we will first have to solve both for y so we can identify the slope and y-intercept.

II - Demo/Discussion Problems:

Solve each of the following systems of linear equations by graphing.

1. $\begin{cases} y = 2x - 4 \\ y = -2x + 4 \end{cases}$

2. $\begin{cases} 2y - 3x = -8 \\ 2y - 3x = 2 \end{cases}$

3. $\begin{cases} 6x - 3y = -9 \\ 2x + 2y = -6 \end{cases}$

III - Practice Problems:

Solve each of the following systems of linear equations by graphing.

1. $\begin{cases} y = -x + 1 \\ y = -5x - 3 \end{cases}$

2. $\begin{cases} y = -\frac{3}{4}x + 1 \\ y = -\frac{3}{4}x + 2 \end{cases}$

3. $\begin{cases} y = \frac{5}{3}x + 4 \\ y = -\frac{2}{3}x - 3 \end{cases}$

4. $\begin{cases} x - y = 4 \\ 2x + y = -1 \end{cases}$

5. $\begin{cases} 2x + y = 2 \\ x - y = 4 \end{cases}$

6. $\begin{cases} 9y + 6x = 36 \\ 3y - 6x = -12 \end{cases}$

7. $\begin{cases} 3 + y = -x \\ -4 - 6x = -y \end{cases}$

8. $\begin{cases} y = -\frac{5}{4}x - 2 \\ y = -\frac{1}{4}x + 2 \end{cases}$

9. $\begin{cases} y = 2x + 2 \\ y = -x - 4 \end{cases}$

10. $\begin{cases} y = \frac{1}{2}x + 4 \\ y = \frac{1}{2}x + 1 \end{cases}$

11. $\begin{cases} 6x + y = -3 \\ x + y = 2 \end{cases}$

12. $\begin{cases} x + 2y = 6 \\ 5x - 4y = 16 \end{cases}$

13. $\begin{cases} -2y + x = 4 \\ 2 = -x + \frac{1}{2}y \end{cases}$

14. $\begin{cases} 16 = -x - 4y \\ -2x = -4 - 4y \end{cases}$

15. $\begin{cases} y = -3 \\ y = -x - 4 \end{cases}$

16. $\begin{cases} y = \frac{1}{3}x + 2 \\ y = -\frac{5}{3}x - 4 \end{cases}$

17. $\begin{cases} x + 3y = -9 \\ 5x + 3y = 3 \end{cases}$

18. $\begin{cases} 2x + 3y = -6 \\ 2x + y = 2 \end{cases}$

19. $\begin{cases} 2x + y = -2 \\ x + 3y = 9 \end{cases}$

20. $\begin{cases} 2x - y = -1 \\ 3 = -2x - y \end{cases}$

21. $\begin{cases} -y + 7x = 4 \\ -y + 7x = 3 \end{cases}$

22. $\begin{cases} y = -x - 2 \\ y = \frac{2}{3}x + 3 \end{cases}$

23. $\begin{cases} y = 2x - 4 \\ y = \frac{1}{2}x + 2 \end{cases}$

24. $\begin{cases} x + 4y = -12 \\ 2x + y = 4 \end{cases}$

25. $\begin{cases} 3x + 2y = 2 \\ 3x + 2y = -6 \end{cases}$

26. $\begin{cases} x - y = 3 \\ 5x + 2y = 8 \end{cases}$

27. $\begin{cases} -2y = -4 - x \\ -2y = -5x + 4 \end{cases}$

28. $\begin{cases} -4 + y = x \\ x + 2 = -y \end{cases}$

Lesson 10: Solving Systems of Linear Equations by Substitution

Objective: Solve linear systems by substitution.

Students will be able to:

- Identify a lone variable.
- Solve linear systems by substitution.
- Write system solutions as ordered pairs in the form (x, y).
- Verify the accuracy of a solution by plugging it into each equation in the system.

Prerequisite Knowledge:

- Solving a linear equation.
- Applying the distributive property.
- Combining like terms.

Lesson:
I - Motivating Example(s):

We present the steps for solving a system of linear equations by substitution alongside an example.

Steps for Substitution	System: $\begin{cases} 4x - 2y = 2 \\ 2x + y = -5 \end{cases}$
1. Identify a lone variable.	The lone variable is y, in the second equation: $2x + \boxed{y} = -5$
2. Solve for the lone variable.	Subtract $2x$ from both sides. $y = -5 - 2x$
3. Substitute into the untouched equation.	$4x - 2(-5 - 2x) = 2$
4. Solve for the remaining variable.	$\begin{aligned} 4x + 10 + 4x &= 2 \\ 8x + 10 &= 2 \\ \underline{-10} &\quad \underline{-10} \\ \frac{8x}{8} &= \frac{-8}{8} \\ x &= -1 \end{aligned}$
5. Plug into lone variable equation and evaluate.	$\begin{aligned} y &= -5 - 2(-1) \\ y &= -5 + 2 \\ y &= -3 \end{aligned}$
Our solution, as a coordinate pair.	$(x, y) = (-1, -3)$

II - Demo/Discussion Problems:

Solve each of the following systems of linear equations by substitution.

1. $\begin{cases} 2x - 3y = 7 \\ y = 3x - 7 \end{cases}$

2. $\begin{cases} 3x + 2y = 1 \\ x - 5y = 6 \end{cases}$

3. $\begin{cases} x - y = 2 \\ 8x - 3y = 16 \end{cases}$

III - Practice Problems:

Solve each of the following systems of linear equations by substitution.

1. $\begin{cases} y = -3x \\ y = 6x - 9 \end{cases}$

2. $\begin{cases} y = 6x + 4 \\ y = -3x - 5 \end{cases}$

3. $\begin{cases} y = 2x - 3 \\ y = -2x + 9 \end{cases}$

4. $\begin{cases} y = -6 \\ 3x - 6y = 30 \end{cases}$

5. $\begin{cases} -2x + 2y = 18 \\ y = 7x + 15 \end{cases}$

6. $\begin{cases} 7x - 2y = -7 \\ y = 7 \end{cases}$

7. $\begin{cases} -2x - y = -5 \\ x - 8y = -23 \end{cases}$

8. $\begin{cases} 3x + y = 9 \\ 2x + 8y = -16 \end{cases}$

9. $\begin{cases} x + 5y = 15 \\ -3x + 2y = 6 \end{cases}$

10. $\begin{cases} y = x + 5 \\ y = -2x - 4 \end{cases}$

11. $\begin{cases} y = 3x + 13 \\ y = -2x - 22 \end{cases}$

12. $\begin{cases} y = 7x - 24 \\ y = -3x + 16 \end{cases}$

13. $\begin{cases} 6x - 4y = -8 \\ y = -6x + 2 \end{cases}$

14. $\begin{cases} y = x + 4 \\ 3x - 4y = -19 \end{cases}$

15. $\begin{cases} x - 2y = -13 \\ 4x + 2y = 18 \end{cases}$

16. $\begin{cases} 6x + 4y = 16 \\ -2x + y = -3 \end{cases}$

17. $\begin{cases} -5x - 5y = -20 \\ -2x + y = 7 \end{cases}$

18. $\begin{cases} 2x + 3y = -10 \\ 7x + y = 3 \end{cases}$

19. $\begin{cases} y = -2x - 9 \\ y = 2x - 1 \end{cases}$

20. $\begin{cases} y = 3x + 2 \\ y = -3x + 8 \end{cases}$

21. $\begin{cases} y = 6x - 6 \\ -3x - 3y = -24 \end{cases}$

22. $\begin{cases} y = -5 \\ 3x + 4y = -17 \end{cases}$

23. $\begin{cases} y = -8x + 19 \\ -x + 6y = 16 \end{cases}$

24. $\begin{cases} x - 5y = 7 \\ 2x + 7y = -20 \end{cases}$

25. $\begin{cases} -6x + y = 20 \\ -3x - 3y = -18 \end{cases}$

26. $\begin{cases} 2x + y = 2 \\ 3x + 7y = 14 \end{cases}$

27. $\begin{cases} -2x + 4y = -16 \\ y = -2 \end{cases}$

28. $\begin{cases} y = -6x + 3 \\ y = 6x + 3 \end{cases}$

29. $\begin{cases} y = -2x - 9 \\ y = -5x - 21 \end{cases}$

30. $\begin{cases} -x + 3y = 12 \\ y = 6x + 21 \end{cases}$

31. $\begin{cases} 7x + 2y = -7 \\ y = 5x + 5 \end{cases}$

32. $\begin{cases} y = -2x + 8 \\ -7x - 6y = -8 \end{cases}$

33. $\begin{cases} 3x - 4y = 15 \\ 7x + y = 4 \end{cases}$

34. $\begin{cases} 7x + 5y = -13 \\ x - 4y = -16 \end{cases}$

35. $\begin{cases} 2x + y = -7 \\ 5x + 3y = -21 \end{cases}$

36. $\begin{cases} -2x + 2y = -22 \\ -5x - 7y = -19 \end{cases}$

Lesson 11: Solving Systems of Equations by Addition/Elimination

Objective: Solve linear systems by addition and elimination.

Students will be able to:

- Solve linear systems by addition and elimination of one variable.
- Write system solutions as ordered pairs in the form (x, y).
- Verify the accuracy of a solution by plugging it into each equation in the system.

Prerequisite Knowledge:

- Solving a linear equation.
- Applying the distributive property.
- Combining like terms.

Lesson:

I - Motivating Example(s):

We present the steps for solving a system of linear equations by addition/elimination alongside an example.

Steps for Addition/Elmination	System: $\begin{cases} 2x - 5y = -13 \\ -3y + 4 = -5x \end{cases}$
1. Line up the variables and constants.	Rearrange the second equation $2x - 5y = -13$ $5x - 3y = -4$
2. Multiply to get opposites (use LCM).	First Equation: multiply by -5 $-5 \cdot (2x - 5y) = (-13) \cdot (-5)$ $-10x + 25y = 65$ Second Equation: multiply by 2 $2 \cdot (5x - 3y) = (-4) \cdot 2$ $10x - 6y = -8$
3. Add equations to eliminate a variable.	$\begin{array}{r} -10x + 25y = 65 \\ \underline{10x - 6y = -8} \\ 19y = 57 \end{array}$

4. Solve for the remaining variable.	$\frac{19y}{19} = \frac{57}{19}$ $y = 3$
5. Plug back into either of the given equations and solve.	$2x - 5(3) = -13$ $2x - 15 = -13$ $+15 \quad +15$ $\frac{2x}{2} = \frac{2}{2}$ $x = 1$
Our solution, as a coordinate pair.	$(x, y) = (1, 3)$

II - Demo/Discussion Problems:

Solve each of the following systems of linear equations by addition/elimination.

1. $\begin{cases} 3x + 6y = -9 \\ 2x + 9y = -26 \end{cases}$
2. $\begin{cases} 2x - 5y = 3 \\ -6x + 15y = -9 \end{cases}$
3. $\begin{cases} 4x - 6y = 8 \\ 6x - 9y = 15 \end{cases}$

III - Practice Problems:

Solve each of the following systems of linear equations by addition/elimination.

1. $\begin{cases} 4x + 2y = 0 \\ -4x - 9y = -28 \end{cases}$
2. $\begin{cases} -6x + 9y = 3 \\ 6x - 9y = -9 \end{cases}$
3. $\begin{cases} -x - 5y = 28 \\ -x + 4y = -17 \end{cases}$
4. $\begin{cases} 10x + 6y = 24 \\ -6x + y = 4 \end{cases}$
5. $\begin{cases} -7x + 4y = -4 \\ 10x - 8y = -8 \end{cases}$
6. $\begin{cases} -7x - 3y = 12 \\ -6x - 5y = 20 \end{cases}$
7. $\begin{cases} 9x + 6y = -21 \\ -10x - 9y = 28 \end{cases}$
8. $\begin{cases} -8x - 8y = -8 \\ 10x + 9y = 1 \end{cases}$
9. $\begin{cases} 0 = 9x + 5y \\ y = \frac{2}{7}x \end{cases}$
10. $\begin{cases} -7x + y = -10 \\ -9x - y = -22 \end{cases}$
11. $\begin{cases} 5x - 5y = -15 \\ 5x - 5y = -15 \end{cases}$
12. $\begin{cases} -10x - 5y = 0 \\ 10x + 10y = 30 \end{cases}$
13. $\begin{cases} x + 3y = -1 \\ 10x + 6y = -10 \end{cases}$
14. $\begin{cases} -6x + 4y = 4 \\ -3x - y = 26 \end{cases}$
15. $\begin{cases} -5x + 4y = 4 \\ -7x - 10y = -10 \end{cases}$
16. $\begin{cases} -4x - 5y = 12 \\ -10x + 6y = 30 \end{cases}$
17. $\begin{cases} -7x + 10y = 13 \\ 4x + 9y = 22 \end{cases}$
18. $\begin{cases} -6 - 42y = -12x \\ x - \frac{7}{2}y = \frac{1}{2} \end{cases}$
19. $\begin{cases} -9x + 5y = -22 \\ 9x - 5y = 13 \end{cases}$
20. $\begin{cases} 4x - 6y = -10 \\ 4x - 6y = -14 \end{cases}$
21. $\begin{cases} 2x - y = 5 \\ 5x + 2y = -28 \end{cases}$
22. $\begin{cases} 2x + 4y = 24 \\ 4x - 12y = 8 \end{cases}$
23. $\begin{cases} 5x + 10y = 20 \\ -6x - 5y = -3 \end{cases}$
24. $\begin{cases} 9x - 2y = -18 \\ 5x - 7y = -10 \end{cases}$

25. $\begin{cases} -7x + 5y = -8 \\ -3x - 3y = 12 \end{cases}$

26. $\begin{cases} 9y = 7 - x \\ -18y + 4x = -26 \end{cases}$

27. $\begin{cases} -x - 2y = -7 \\ x + 2y = 7 \end{cases}$

28. $\begin{cases} -3x + 3y = -12 \\ -3x + 9y = -24 \end{cases}$

29. $\begin{cases} -5x + 6y = -17 \\ x - 2y = 5 \end{cases}$

30. $\begin{cases} -6x + 4y = 12 \\ 12x + 6y = 18 \end{cases}$

31. $\begin{cases} -9x - 5y = -19 \\ 3x - 7y = -11 \end{cases}$

32. $\begin{cases} 3x + 7y = -8 \\ 4x + 6y = -4 \end{cases}$

33. $\begin{cases} 8x + 7y = -24 \\ 6x + 3y = -18 \end{cases}$

34. $\begin{cases} 21 = -9x + 12y \\ \frac{4}{3}y + \frac{7}{3}x = -1 \end{cases}$

Lesson 12: Functions and Relations

Objective: Define a relation and a function; determine if a relation is a function.

Students will be able to:

- Apply the Vertical Line Test to a graph to determine whether a relation represents a function.
- Determine whether a relation given as either a set, an equation, or a table of points represents a function.

Prerequisite Knowledge:

- Graph a set or table of points on the coordinate plane.
- Isolate a variable in an equation.

Lesson:

A **relation** R is a set of points in the xy-plane. A relation in which each x-coordinate is paired with exactly one y-coordinate is said to describe y as a **function** of x. Relations which represent functions of x will often be denoted by f, or $f(x)$, rather than R. The set of all x-coordinates of the points in a function f is called the **domain** of f, and the set of all y-coordinates of the points in f is called the **range** of f.

One major test that is used to determine whether or not a graph of a relation represents y as a function of x is known as the Vertical Line Test. We will now state the Vertical Line Test as a mathematical theorem and then demonstrate its use.

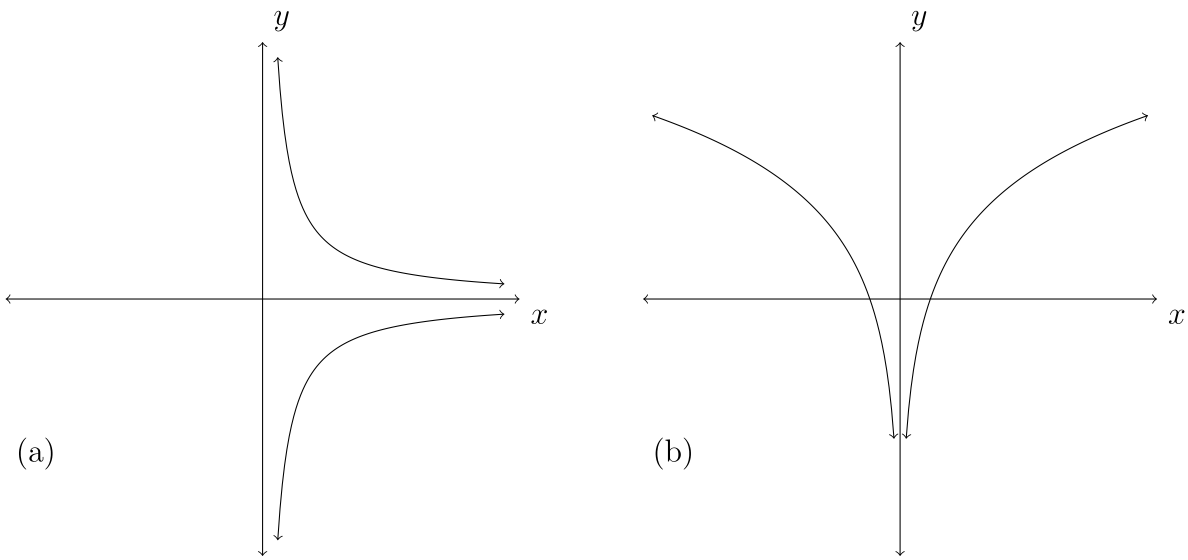
Vertical Line Test: A set of points in the xy-plane represents y as a function of x if and only if no two points lie on the same vertical line.

Alternatively stated, if a graph is known to represent y as a function of x, then there can be no vertical line that intersects the graph in more than one point. Conversely, if a known graph has the property that no vertical line intersects it in more than one point, then the given graph represents y as a function of x.

When we are presented with an equation, instead of a graph, we can still determine whether or not the equation for our relation represents y as a function of x by solving the equation for y and carefully considering the result. When solving for y, the existence of the $\pm$ in our solution will cause our corresponding graph to fail the VLT.

I - Motivating Example(s):

Example: Use the Vertical Line Test to determine whether each of the following graphs represent y as a function of x.



Graph (a) above fails the VLT, since any vertical line drawn in the right half-plane (where $x > 0$) intersects the relation at two points. Graph (b) passes the VLT, since no vertical line intersects the graph at more than one point.

Example: Determine whether the following equation represents y as a function of x.

$$x^2 + y^2 = 9$$

Solve the equation for y.

$$\begin{array}{rll}
x^2 + y^2 = 9 & & \text{Solve for } y \\
\underline{-\cancel{x^2}} \qquad\quad \underline{-x^2} & & \text{Subtract } x^2 \\
y^2 = 9 - x^2 & & \\
\sqrt{y^2} = \pm\sqrt{9 - x^2} & & \text{Introduce a square root} \\
 & & \quad \text{include a } \pm \text{ on right side} \\
y = \pm\sqrt{9 - x^2} & & y \text{ is not a function of } x
\end{array}$$

Due to the $\pm$, we can conclude that the equation does *not* represent y as a function of x.

II - Demo/Discussion Problems:

Determine whether each of the following relations represent y as a function of x. Use Desmos to sketch a graph of each relation.

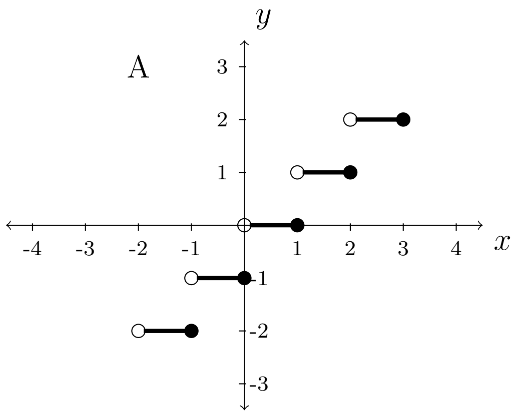
1. $\{(1,1),(2,-3),(2,0),(0,3),(-2,1/2)\}$
2. $\{(x,y) \mid x > 3 \text{ and } y \leq 2\}$
3. $x^2 = 1 - y^2$
4. $x = y^2$
5. $y = x^2$
6. $y = 3 - 2x$

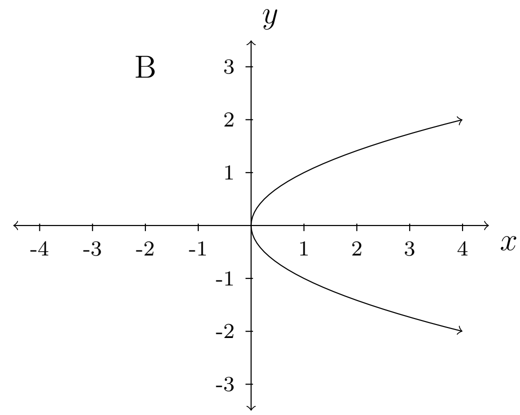
III - Practice Problems:

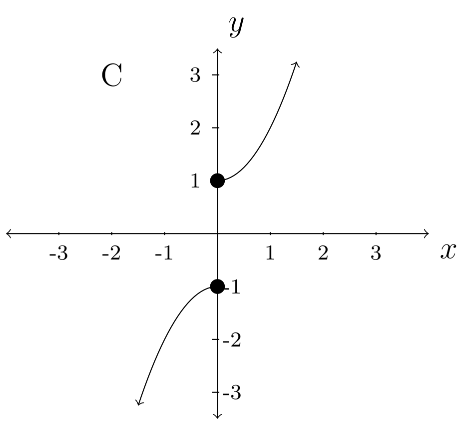
Determine if the following relations represent y as a function of x by making a table of values and graphing. Explain your reasoning. Use Desmos to confirm your results.

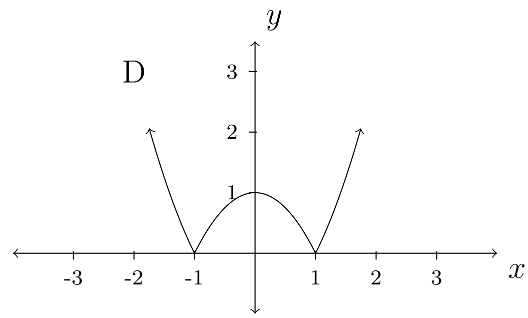
1. $x = y^3$
2. $y = x$
3. $xy = 1$
4. $y = (x-3)^2$
5. $x = (y-3)^2$
6. $y < 2x - 5$

Circle the letter of each graph/table below that represents y as a function of x.









E

x	y
3	-3
2	-2
1	-1
0	0
1	1
2	2
3	3

F

x	y
-3	3
-2	2
-1	1
0	0
1	1
2	2
3	3

G

x	y
-3	0
-2	0
-1	0
0	0
1	0
2	0
3	0

H

x	y
3	8
2	4
1	2
0	1
-1	$1/2$
-2	$1/4$
-3	$1/8$

Lesson 13: Evaluating Functions

Objective: Evaluate functions using appropriate notation.

Students will be able to:

- Evaluate a function by substituting a given value as an input and simplifying.

Prerequisite Knowledge:

- Order of operations.

Lesson:

A function-related skill we will want to quickly master is evaluating functions at certain values of the independent variable (usually x). This is accomplished by substituting the specified value into the function for x and simplifying the resulting expression to find $f(x)$.

I - Motivating Example(s):

Example: Find $f(-2)$, where $f(x) = 3x^2 - 4x$.

$f(x) = 3x^2 - 4x$	Evaluate; Substitute -2 for each x
$f(-2) = 3(-2)^2 - 4(-2)$	Simplify using order of operations; exponent first
$f(-2) = 3(4) - 4(-2)$	Multiply and add
$f(-2) = 20$	Our solution

II - Demo/Discussion Problems:

1. Find $h(4)$, where $h(x) = 3^{2x-6}$.
2. Find $k(-7)$, where $k(a) = 2|a + 4|$.
3. Find $p(t + 1)$, where $p(t) = t^2 - t$.

III - Practice Problems:

Find $f(0)$ for each of the given functions.

1. $f(x) = 2x - 1$
2. $f(x) = 3 - \dfrac{2x}{5}$
3. $f(x) = 2x^2 - 6$
4. $f(x) = x^2 - x - 12$
5. $f(x) = \sqrt{x + 4}$
6. $f(x) = \sqrt{1 - 2x}$
7. $f(x) = \dfrac{3}{4 - x}$
8. $f(x) = \dfrac{3x^2 - 12x}{4 - x^2}$

Use the functions listed below to evaluate each of the following.

- $f(x) = -4x^2$
- $g(x) = x - 9$
- $k(x) = 2x^3 + |x - 1|$
- $m(a) = 3a^2$

9. $f(-2)$
10. $g(7)$
11. $k(-2)$
12. $m(-2)$
13. $f(a+2)$
14. $g(2y)$
15. $k(2)$
16. $m(a-2)$

For each problem, use the given function f to find and simplify each of the **nine** related values/expressions listed below.

- $f(1)$
- $f(-3)$
- $f\left(\frac{3}{2}\right)$
- $f(4x)$
- $4f(x)$
- $f(-x)$
- $f(x-4)$
- $f(x) - 4$
- $f\left(x^2\right)$

17. $f(x) = 2x + 1$
18. $f(x) = 3 - 4x$
19. $f(x) = 2 - x^2$
20. $f(x) = x^2 - 3x + 2$
21. $f(x) = \sqrt{x - 1}$
22. $f(x) = \dfrac{x}{x - 1}$
23. $f(x) = 6$
24. $f(x) = 0$

For each problem, use the given function f to find and simplify each of the **nine** related values/expressions listed below.

- $f(2)$
- $f(-2)$
- $f(2a)$
- $2f(a)$
- $f(a+2)$
- $f(a) + f(2)$
- $f\left(\frac{2}{a}\right)$
- $\frac{f(a)}{2}$
- $f(a+h)$

25. $f(x) = 2x - 5$
26. $f(x) = 5 - 2x$
27. $f(x) = 2x^2 - 1$
28. $f(x) = 3x^2 + 3x - 2$
29. $f(x) = \sqrt{2x + 1}$
30. $f(x) = 1$
31. $f(x) = \dfrac{x}{2}$
32. $f(x) = \dfrac{2}{x}$

Lesson 14: Finding Domain and Range from a Graph

CC attribute: *College Algebra* by C. Stitz and J. Zeager.

Objective: Find the domain and range of a function from its graph.

Students will be able to:

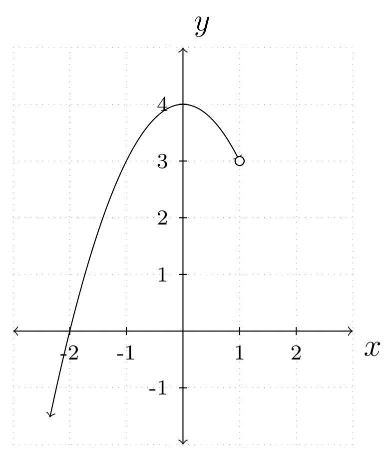
- Find domain graphically.
- Find range graphically.

Prerequisite Knowledge:

- Interval notation
- The definition of a function.
- Graphing a function on the coordinate plane.

Lesson:
I - Motivating Example(s):

Example: Find the domain and range of the function f whose graph is given below.

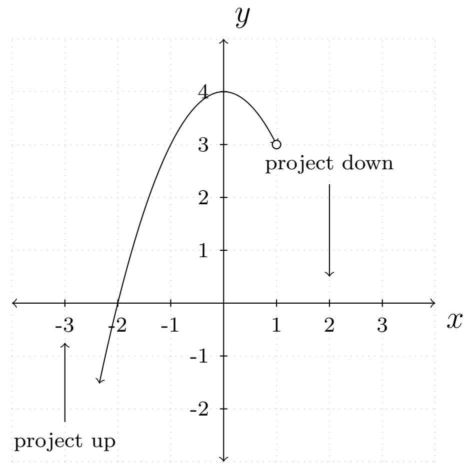


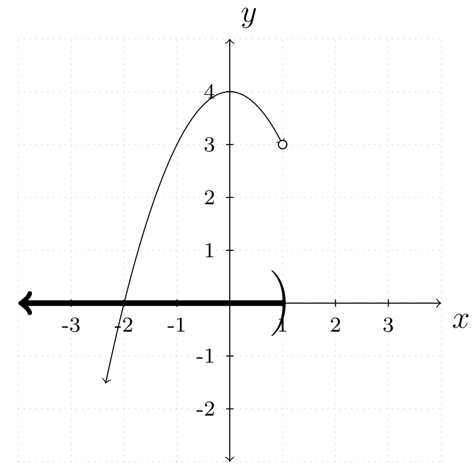
The graph of f

To determine the domain and range of f, we need to determine which x and y-values respectively occur as coordinates of points on the given graph.

To find the domain, it will be helpful to imagine collapsing the curve onto the x-axis and determining the portion of the x-axis that gets covered. This is often described as **projecting** the curve onto the x-axis.

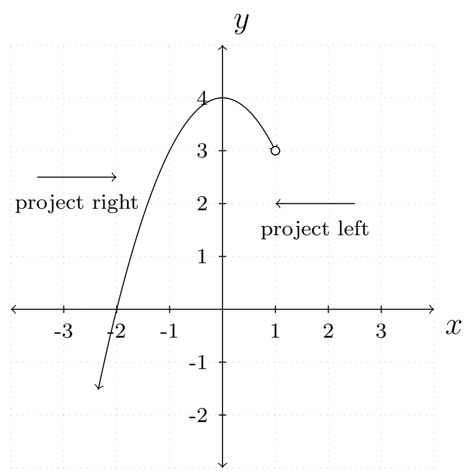
Before we project, we need to pay attention to two subtle notations on the graph: the arrowhead on the lower left corner of the graph indicates that the graph continues to curve downwards to the left forever; and the open circle at $(1, 3)$ indicates that the point $(1, 3)$ is *not* on the graph, but all the points on the curve leading up to $(1, 3)$ are on the graph.

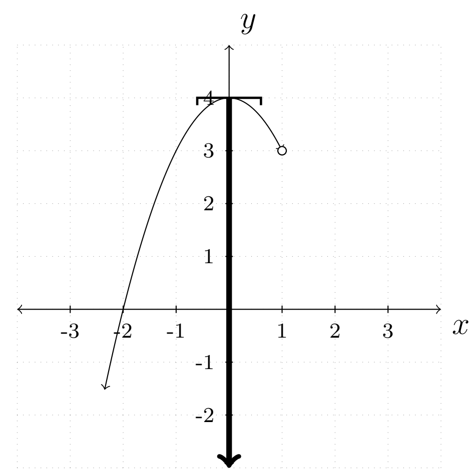




We see from the figure that if we project the graph of f to the x-axis, we get all real numbers less than 1. Using interval notation, we write the domain of f as $(-\infty, 1)$.

To determine the range of f, we use a similar method, projecting the curve onto the y-axis as follows.

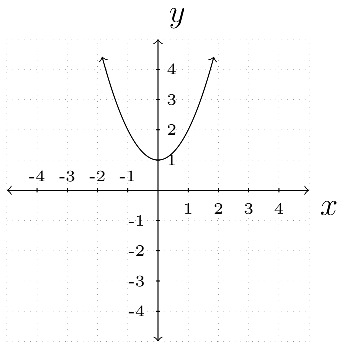




Note that even though there is an open circle at $(1, 3)$, we still include the y value of 3 in our range, since the point $(-1, 3)$ is on the graph of f. We also include $y = 4$ in our answer, since the point $(0, 4)$ is also on our graph. Consequently, the range of f is all real numbers less than or equal to 4, or $(-\infty, 4]$.

II - Demo/Discussion Problems:

For each of the following graphs, identify the corresponding domain and range. Express your answers using interval notation.

1.


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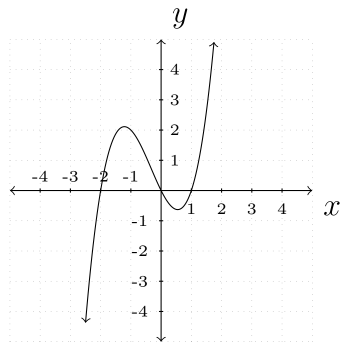

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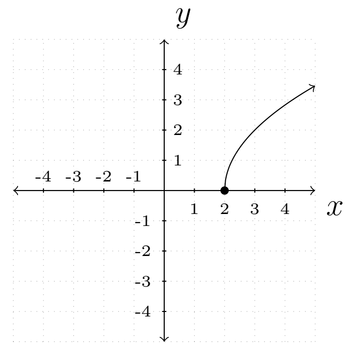

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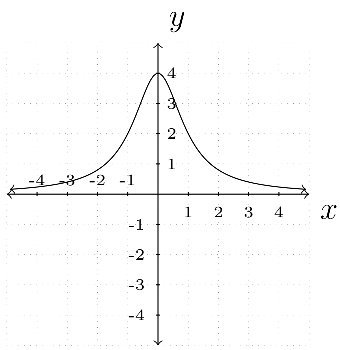

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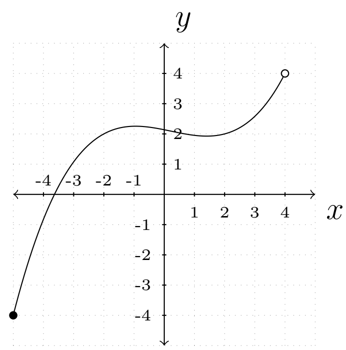

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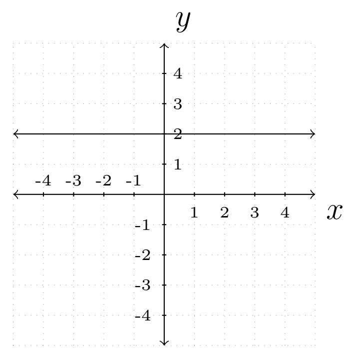

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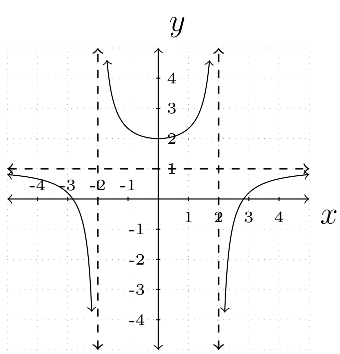

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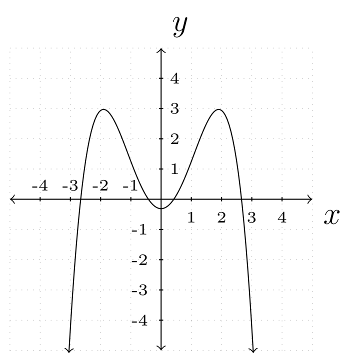

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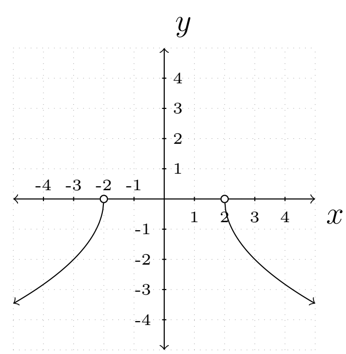
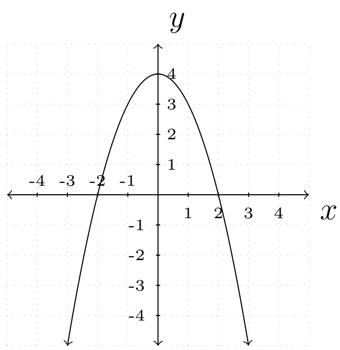
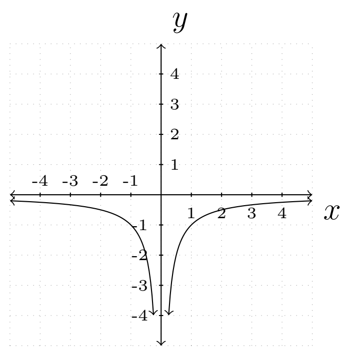
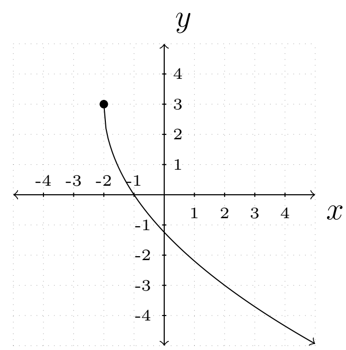


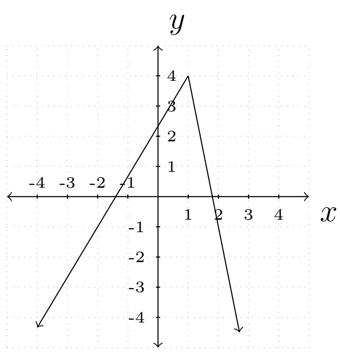
III - Practice Problems:

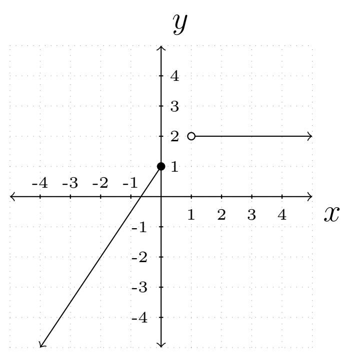
For each of the following graphs, identify the corresponding domain and range. Express your answers using interval notation.

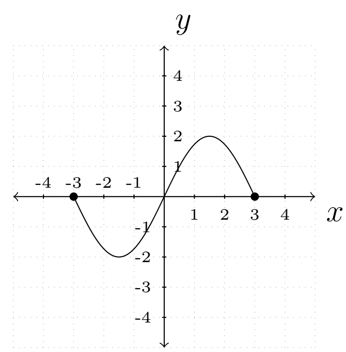
1. 

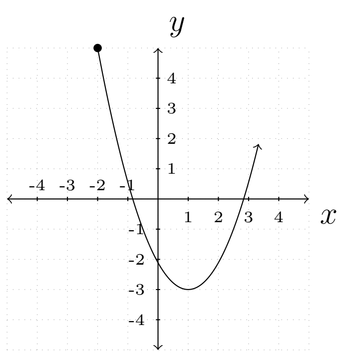
2. 

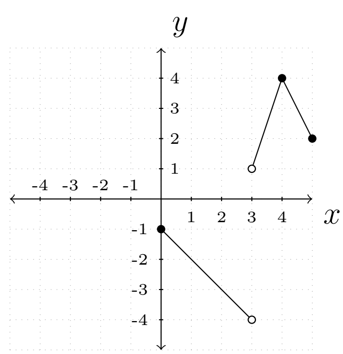
3. 

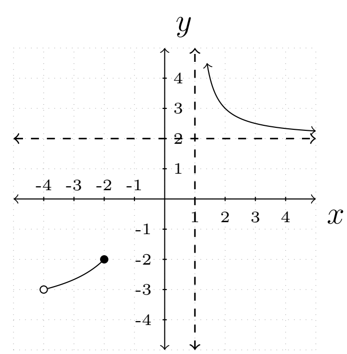
4. 

5. 

6. 

7. 

8. 

9. 

Lesson 15: Fundamental Functions

Objective: Graph and identify the domain, range, and intercepts of any of the ten fundamental functions.

Students will be able to:

- Identify (as well as produce) the graph of a variety of fundamental functions.
- Identify the domain and range of a variety of fundamental functions using a graph.

Prerequisite Knowledge:

- Definitions of domain and range of a function.
- Graph a function by plotting points.

Lesson:

In this lesson, we focus on ten fundamental function types which will be referenced throughout the rest of the course, as well as one example of each. Each type of function represents a "building block" for understanding the concepts of a traditional algebra course.

Students should be able to both identify and sketch a graph of each function, as well as identify its intercepts, domain (both graphically and algebraically), and range (graphically). Each representative form in the table below includes some element of generalization to reinforce understanding.

Function Type	Representative Form	Example
Linear	$mx + b$	$f(x) = 3x - 4$
Quadratic	$ax^2 + bx + c$	$g(x) = x^2$
Square Root	$\sqrt{x-h}$	$k(x) = \sqrt{x}$
Absolute Value	$\lvert x-h \rvert$	$\ell(x) = \lvert x \rvert$
Cubic	$(x-h)^3$	$m(x) = x^3$
Cube Root	$\sqrt[3]{x-h}$	$n(x) = \sqrt[3]{x}$
Reciprocal (Rational)	$\frac{1}{x-h}$	$p(x) = \frac{1}{x}$

Function Type	Representative Form	Example
Semicircular	$\sqrt{r^2 - x^2},\ r > 0$	$q(x) = \sqrt{9 - x^2}$
Exponential*	$a^x,\ a > 0, a \neq 1$	$r(x) = 2^x$
Logarithmic*	$\log_a(x),\ a > 0, a \neq 1$	$s(x) = \log_2(x)$

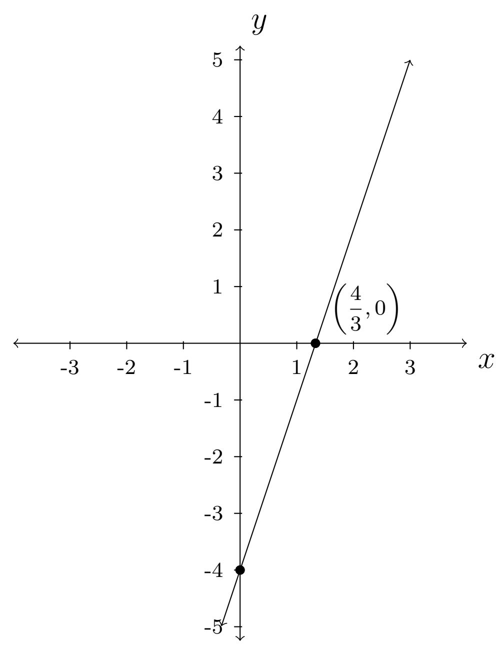
*We have included Exponential and Logarithmic functions for a more complete list. These functions are more formally treated in a Precalculus setting.

I - Motivating Example(s):

Example:

Function Type: Linear ($m \neq 0$)
Representative: $f(x) = 3x - 4$

x	$f(x)$
-3	-13
-2	-10
-1	-7
0	-4
1	-1
$\frac{4}{3}$	0
2	2
3	5



Graph of $f(x) = 3x - 4$

$y-$intercept: $(0, -4)$
$x-$intercept(s): $\left(\frac{4}{3}, 0\right)$

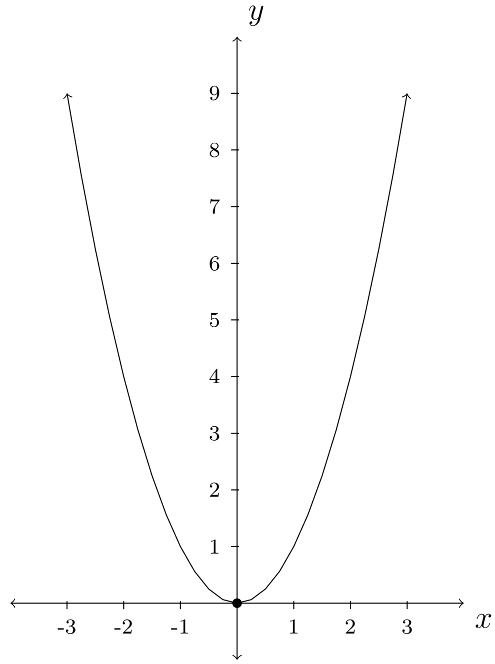
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Notes: If $m = 0$, then the corresponding graph of $f(x) = b$ is a horizontal line. The domain of f is still $(-\infty, \infty)$, but the range consists of a single value, $\{b\}$.

Example:

Function Type: Quadratic
Representative: $g(x) = x^2$

x	$g(x)$
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9



Graph of $g(x) = x^2$

$y-$intercept: $(0, 0)$
$x-$intercept(s): $(0, 0)$

Domain: $(-\infty, \infty)$
Range: $[0, \infty)$, or $y \geq 0$

Notes: The domain of any quadratic function is $(-\infty, \infty)$. If $g(x) = a(x-h)^2 + k$, is a quadratic function in vertex form, then if $a > 0$, the corresponding parabola will be concave *up*, and the range of g will be $[k, \infty)$. If $a < 0$, then the corresponding parabola will be concave *down*, and the range of g will be $(-\infty, k]$.

II - Demo/Discussion Problems:

Complete each of the following for the functions listed below.

- Use Desmos to sketch a complete graph of the function.
- Construct a table of points from your graph. Check your table by evaluating the function at the given $x-$coordinates.
- Identify the domain and range of your graph.
- Identify any $x-$ and $y-$intercepts of your graph.

1. $m(x) = x^3$
2. $n(x) = \sqrt[3]{x}$
3. $q(x) = \sqrt{9 - x^2}$
4. $r(x) = 2^x$

III - Practice Problems:

Complete each of the following for the functions listed below.

- Use Desmos to sketch a complete graph of the function.
- Construct a table of points from your graph. Check your table by evaluating the function at the given $x-$coordinates.
- Identify the domain and range of your graph.
- Identify any $x-$ and $y-$intercepts of your graph.

1. $k(x) = \sqrt{x}$
2. $\ell(x) = |x|$
3. $p(x) = \dfrac{1}{x}$
4. $s(x) = \log_2(x)$

Lesson 16: Introduction to Quadratics

Objective: Recognize a quadratic equation and graphically.

Students will be able to:

- Determine if an equation is linear or quadratic.
- Determine if the corresponding graph of a quadratic is concave up or down.
- Identify the $y-$intercept for the graph of a quadratic.
- Recognize the vertex form of a quadratic.

Prerequisite Knowledge:

- Application of the distributive property.
- Order of operations and combining like terms.

Lesson:

A quadratic equation is an equation of the form

$$y = ax^2 + bx + c,$$

where the *coefficients* of a,b, and c are real numbers and $a \neq 0$. This form is most commonly referred to as the *standard form* of a quadratic. We call a the *leading coefficient*, ax^2 the *leading term* (also known as the *quadratic term*), bx the *linear term* and c the *constant term*. The quadratic term ax^2, must have a nonzero coefficient in order for the equation to be a quadratic (otherwise f would be linear, in slope-intercept form). The most fundamental quadratic equation is $y = x^2$ and its graph, like all quadratics, is known as a *parabola*.

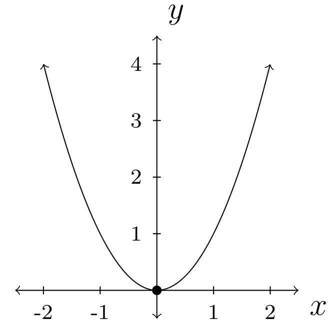
The most useful form for graphing a quadratic equation is the *vertex form*. A quadratic equation is said to be in vertex form if it is represented as

$$y = a(x - h)^2 + k,$$

where h and k are real numbers. The vertex form, unlike the standard form, allows us to immediately identify the vertex of the resulting parabola, which will be the point (h, k).

I - Motivating Example(s):

Example: $y = x^2$

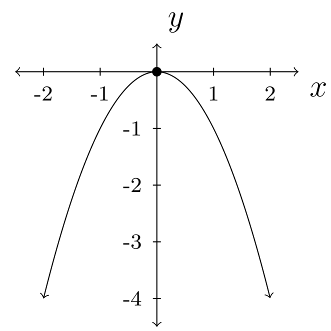


From the standard form, since $a > 0$, the graph opens upwards and is said to be *concave up*.

As a result, there is a minimum point, known as the *vertex*, located at the origin, $(0, 0)$.

Notice the symmetry over the y-axis.

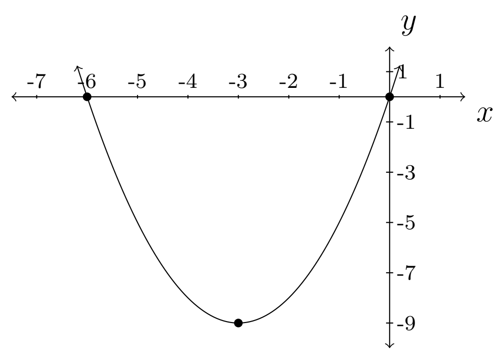
Example: $y = -x^2$



Since $a = -1$, the graph opens downward or we say that it is *concave down*.

Every parabola with a negative leading coefficient ($a < 0$) will be concave down with a maximum value at its vertex.

Example: $y = (x + 3)^2 - 9$



The vertex is at $(-3, -9)$ and the graph can be realized as the graph of $y = x^2$ shifted left 3 units and down 9 units from the origin.

Since our graph is concave up there will be two x-intercepts as the curve opens upward from below the x-axis.

II - Demo/Discussion Problems:

Describe the following equations as either linear or quadratic. If quadratic, identify the $y-$intercept and determine whether the corresponding graph is concave up or down.

1. $y = x^2 - 4x + 2x^2 - 3x + 5$
2. $y = -2(x - 3)(x + 5) - 6$
3. $y = -2x^2 - 4x + 2x^2 - 7x + 7$

Identify the vertex, $y-$intercept, and concavity of each of the following quadratic equations. Find the standard form for each quadratic.

4. $y = -3(x-1)^2 + 2$
5. $y = -\frac{1}{2}(x+2)^2$
6. $y = x^2 + 3$

III - Practice Problems:

Simplifying each of the following equations and classify each as linear, quadratic, or neither. If the equation is a quadratic, identify its $y-$intercept and concavity.

1. $y = x^2 + 9$
2. $y = 5 - 2x + x^2$
3. $y = x + 6 - 3x$
4. $y = 5x + x^2 - 3x - 3x^2$
5. $y = -5x + 3 + 2x - 3x^2 + 6$
6. $y = 3x^2 + -x + x - 3x^2 + 6$
7. $y = (x-1)(x+2) + 3$
8. $y = (x-5)(2x+3) - 2(x-3)$
9. $y = (x-4)(x+4) - (x+1)^2$
10. $y = (2x-4)(x-1) - 2(x+3)^2 + 3x^2$

Identify the vertex and concavity of each of the following quadratics.

11. $y = (x-3)^2 + 4$
12. $y = (x-2)^2 + 5$
13. $y = 6(x+3)^2 + 4$
14. $y = -2(x-3)^2 + 4$
15. $y = -2(x-1)^2 - 7$
16. $y = -(x+1)^2$
17. $y = 7x^2 + 4$
18. $y = -\frac{1}{2}(x-8)^2 + 5$
19. $y = x^2 + 4$
20. $y = 5x^2 + 23$

Lesson 17: Identifying a Greatest Common Factor

Objective: Find the greatest common factor (GCF) and factor it out of an expression.

Students will be able to:

- Identify common factors between two or more terms.
- Identify a greatest common factor.
- Factor out a greatest common factor.

Prerequisite Knowledge:

- Multiplication properties of exponents.
- Application of the distributive property.
- Multiplication and division of algebraic expressions.

Lesson:

In working with polynomial expressions, there are many benefits to identifying both expanded and factored forms. Specifically, we will use factored polynomials to help us solve equations, learn behaviors of graphs, and understand more complicated rational expressions. Because so many concepts in algebra depend on being able to factor polynomials, it is critical that we establish strong factorization skills.

In this first lesson on factoring, we will focus on identifying the greatest common factor or GCF of a polynomial. When multiplying polynomials, we employ the distributive property, as demonstrated below.

$$4x^2(2x^2 - 3x + 8) = 8x^4 - 12x^3 + 32x$$

Here, we will work with the same expression, but with a backwards approach, starting with the expanded form and obtaining one that is partially (or completely) factored.

We will start with $8x^2 - 12x^3 + 32x$ and try and work backwards to reach $4x^2(2x - 3x + 8)$.

To do this we have to be able to first identify what the GCF of a polynomial is. To find a GCF of two or more integers, we must find the largest integer d that divides nicely into each of the given integers. Alternatively stated, d should be the largest factor of each of the integers in our set. When there are variables in our problem we can first find the GCF of the numbers, then we can identify any variables that appear in every term and factor them out, taking the smallest exponent in each case.

I - Motivating Example(s):

Example: Find the GCF of 15, 24, and 27.

$$\frac{15}{3} = 5, \quad \frac{24}{3} = 8, \quad \frac{27}{3} = 9 \qquad \text{Each of the numbers can be divided by 3}$$
$$\text{GCF} = 3 \qquad \text{Our solution}$$

Example: Find the GCF of $24x^4y^2z$, $18x^2y^4$, and $12x^3yz^5$.

$$\frac{24}{6} = 4, \quad \frac{18}{6} = 3, \quad \frac{12}{6} = 2 \qquad \text{Each number can be divided by 6}$$
$$x^2y \qquad x\text{and } y \text{ appears in all three terms, taking the lowest exponent for each variable}$$
$$\text{GCF} = 6x^2y \qquad \text{Our solution}$$

II - Demo/Discussion Problems:

Identify and factor out the GCF from each of the given polynomial expressions.

1. $4x^2 - 20x + 16$
2. $25x^4 - 15x^3 + 20x^2$
3. $3x^3y^2z + 5x^4y^3z^5 - 4xy^4$
4. $21x^3 + 14x^2 + 7x$
5. $12x^5y^2 - 6x^4y^4 + 8x^3y^5$
6. $18a^4b^3 - 27a^3b^3 + 9a^2b^3$

III - Practice Problems:

Identify and factor out the GCF from each of the given polynomial expressions.

1. $4 + 8b^2$
2. $x - 5$
3. $45x^2 - 25$
4. $-n - 2n^2$
5. $56 - 35p$
6. $50x - 80y$
7. $7ab - 35a^2b$
8. $27x^2y^5 - 72x^3y^2$
9. $-3a^2b + 6a^3b^2$
10. $8x^3y^2 + 4x^3$
11. $-5x^2 - 5x^3 - 15x^4$
12. $-32n^9 + 32n^6 + 40n^5$
13. $20x^4 - 30x + 30$
14. $21p^6 + 30p^2 + 27$
15. $28m^4 + 40m^3 + 8$
16. $-10x^4 + 20x^2 + 12x$
17. $30b^9 + 5ab - 15a^2$
18. $27y^7 + 12y^2x + 9y^2$
19. $-48a^2b^2 - 56a^3b - 56a^5b$
20. $30m^6 + 15mn^2 - 25$
21. $20x^8y^2z^2 + 15x^5y^2z + 35x^3y^3z$
22. $3p + 12q - 15q^2r^2$
23. $50x^2y + 10y^2 + 70xz^2$
24. $30y^4z^3x^5 + 50y^4z^5 - 10y^4z^3x$
25. $30qpr - 5qp + 5q$
26. $28b + 14b^2 + 35b^3 + 7b^5$
27. $-18n^5 + 3n^3 - 21n + 3$
28. $30a^8 + 6a^5 + 27a^3 + 21a^2$
29. $-40x^{11} - 20x^{12} + 50x^{13} - 50x^{14}$
30. $-24x^6 - 4x^4 + 12x^3 + 4x^2$
31. $-32mn^8 + 4m^6n + 12mn^4 + 16mn$
32. $-10y^7 + 6y^{10} - 4y^{10}x - 8y^8x$

Lesson 18: Factor by Grouping

Objective: Factor a tetranomial (four-term) expression by grouping.

Students will be able to:

- Split a tetranomial expression into a pair of binomial expressions that share a common factor.
- Factor out a GCF from each binomial expression, and group them using the distributive property.

Prerequisite Knowledge:

- Multiplication properties of exponents.
- Application of the distributive property.
- Multiplication and division of algebraic expressions.
- Identifying a greatest common factor.

Lesson:

In this lesson, we will introduce another useful factorization strategy, known as *grouping*. Grouping is typically employed when faced with an expression containing four terms. Here, it is important to reinforce the fact that factoring is essentially expansion (multiplication) done in reverse.

The first thing we will always do when factoring is try to factor out a GCF. This GCF is often a monomial (a single term) like in the expression $5xy + 10xz$. Here, the GCF is the monomial $5x$, so we would have $5x(y + 2z)$ as our answer. However, a GCF does not have to be a monomial. It could, in fact, be a binomial and contain two terms, as we will see with grouping.

When attempting to factor by grouping, we will always divide an expression into two parts, or groups: group one will usually contain the first two terms of our expression and group two will contain the last two terms. Then we can identify and factor the GCF out of each group. In doing this, our hope is that what is left over in each group will be the same binomial expression. If the resulting expressions match, we can then factor out this matching expression from both of our designated groups, writing what remains in a new set of parentheses.

The key for grouping to be successful is for the two binomials to match exactly, once the GCF has been factored out of both groups. If there is any difference between the two binomials, we either have to choose two different groups or we cannot factor by grouping.

I - Motivating Example(s):

Example: Find and factor out the GCF of the given expression.

$3ax - 7bx$	Both terms have an x in common, factor it out
$x(3a - 7b)$	Our solution

Now we will work with the same expression, replacing x with $(2a+5b)$. In the same way that we factored out a GCF of x we can factor out a GCF which is a binomial, such as $(2a+5b)$.

Example: Find and factor out the GCF of the given expression.

$3a(2a+5b) - 7b(2a+5b)$	Both terms have $(2a+5b)$ in common, factor it out
$(2a+5b)(3a-7b)$	Our solution

Example: Write the expanded form for the given expression.

$(2a+3)(5b+2)$	Distribute $(2a+3)$ into the second set of parentheses
$5b(2a+3) + 2(2a+3)$	Distribute each monomial
$10ab + 15b + 4a + 6$	Our solution

Our solution above has four terms in it. We arrived at this solution by focusing on the two parts, $5b(2a+3)$ and $2(2a+3)$. Reversing the process above is the central idea behind grouping.

Example: Factor the given expression by grouping.

$10ab + 15b + 4a + 6$	Split the expression into two groups
$(10ab + 15b) + (4a + 6)$	GCF on the left is $5b$, on the right is 2
$5b(2a+3) + 2(2a+3)$	$(2a+3)$ appears twice! Factor out this GCF.
$(2a+3)(5b+2)$	Our solution

II - Demo/Discussion Problems:

Factor each of the given expressions by grouping.

1. $5xy - 8x - 10y + 16$
2. $12ab - 14a - 6b + 7$
3. $6x^2 + 9xy - 14x - 21y$
4. $6x^3 - 15x^2 + 2x - 5$
5. $4a^2 - 21b^3 + 6ab - 14ab^2$
6. $7 + y - 3xy - 21x$
7. $8xy - 12y + 15 - 10$

III - Practice Problems:

Factor each of the given expressions by grouping.

1. $40r^3 - 8r^2 - 25r + 5$
2. $35x^3 - 10x^2 - 56x + 16$
3. $3n^3 - 2n^2 - 9n + 6$
4. $14v^3 + 10v^2 - 7v - 5$
5. $15b^3 + 21b^2 - 35b - 49$
6. $6x^3 - 48x^2 + 5x - 40$
7. $3x^3 + 15x^2 + 2x + 10$
8. $28p^3 + 21p^2 + 20p + 15$
9. $35x^3 - 28x^2 - 20x + 16$
10. $7n^3 + 21n^2 - 5n - 15$
11. $7xy - 49x + 5y - 35$
12. $42r^3 - 49r^2 + 18r - 21$
13. $32xy + 40x^2 + 12y + 15x$
14. $15ab - 6a + 5b^3 - 2b^2$
15. $16xy - 56x + 2y - 7$
16. $3mn - 8m + 15n - 40$
17. $2xy - 8x^2 + 7y^3 - 28y^2x$
18. $5mn + 2m - 25n - 10$
19. $40xy + 35x - 8y^2 - 7y$
20. $8xy + 56x - y - 7$
21. $32uv - 20u + 24v - 15$
22. $4uv + 14u^2 + 12v + 42u$
23. $10xy + 30 + 25x + 12y$
24. $24xy + 25y^2 - 20x - 30y^3$
25. $3uv + 14u - 6u^2 - 7v$
26. $56ab + 14 - 49a - 16b$
27. $16xy - 3x - 6x^2 + 8y$

Lesson 19: Factoring Trinomials with a Leading Coefficient of One

Objective: Factor a trinomial with a leading coefficient of one.

Students will be able to:

- Identify two integer values that add to b and multiply to $a \cdot c = c$ in a trinomial expression with ordered coefficients a, b, and c.
- Multiply binomials to verify the accuracy of a factorization.
- Recognize the relationship between factoring and expanding an expression.

Prerequisite Knowledge:

- Identifying a greatest common factor.
- Factor by grouping.
- Application of the distributive property.
- Multiplication and division of algebraic expressions.

Lesson:
I - Motivating Example(s):

Example: Write the expanded form for the given expression.

$(x+6)(x-4)$	Distribute $(x+6)$ through the second set of parentheses.
$x(x+6)-4(x+6)$	Distribute each monomial through parentheses.
$x^2+6x-4x-24$	Combine like terms.
$x^2+2x-24$	Our solution.

Notice that if we reverse the last three steps of the previous example, the process resembles grouping. This is because it is grouping! In the second-to-last line, the GCF of the first two terms is x and the GCF of the last two terms is -4. In this manner, we will factor trinomials by writing them as a polynomial containing four terms, splitting up the middle term, and then factor by grouping. This is demonstrated in the following example, which is the previous one done in reverse.

Example: Factor the given expression.

$x^2+2x-24$	Split middle (linear) term into $+6x-4x$, since $6+(-4)=2$ and $6\cdot(-4)=-24$.
$x^2+6x-4x-24$	Grouping: GCF on left is x, on right is -4.
$x(x+6)-4(x+6)$	$(x+6)$ appears twice, factor out this GCF.
$(x+6)(x-4)$	Our solution.

II - Demo/Discussion Problems:

Factor each of the given trinomial expressions.

1. $x^2 + 9x + 18$
2. $x^2 - 4x + 3$
3. $x^2 - 13x + 30$
4. $x^2 + 13x - 30$
5. $5x^2 - 40x - 100$
6. $x^2 - 9xy + 14y^2$

III - Practice Problems:

Factor each of the given trinomial expressions.

1. $p^2 + 17p + 72$
2. $x^2 + x - 72$
3. $n^2 - 9n + 8$
4. $x^2 + x - 30$
5. $x^2 - 9x - 10$
6. $x^2 + 13x + 40$
7. $b^2 + 12b + 32$
8. $b^2 - 17b + 70$
9. $x^2 + 3x - 70$
10. $x^2 + 3x - 18$
11. $n^2 - 8n + 15$
12. $a^2 - 6a - 27$
13. $p^2 + 15p + 54$
14. $p^2 + 7p - 30$
15. $n^2 - 15n + 56$
16. $m^2 - 15mn + 50n^2$
17. $u^2 - 8uv + 15v^2$
18. $m^2 - 3mn - 40n^2$
19. $m^2 + 2mn - 8n^2$
20. $x^2 + 10xy + 16y^2$
21. $x^2 - 11xy + 18y^2$
22. $u^2 - 9uv + 14v^2$
23. $x^2 + xy - 12y^2$
24. $x^2 + 14xy + 45y^2$
25. $x^2 + 4xy - 12y^2$
26. $4x^2 + 52x + 168$
27. $5a^2 + 60a + 100$
28. $5n^2 - 45n + 40$
29. $6a^2 + 24a - 192$
30. $5v^2 + 20v - 25$
31. $6x^2 + 18xy + 12y^2$
32. $5m^2 + 35mn - 90n^2$
33. $6x^2 + 96xy + 378y^2$
34. $6m^2 - 36mn - 162n^2$

Lesson 20: Factoring Trinomials with a Leading Coefficient of $a \neq 1$

Objective: Factor a trinomial with a leading coefficient of $a \neq 1$.

Students will be able to:

- Identify two integer values that add to b and multiply to $a \cdot c$ in a trinomial expression with ordered coefficients a, b, and c.
- Multiply binomials to verify the accuracy of a factorization.
- Recognize the relationship between factoring and expanding an expression.

Prerequisite Knowledge:

- Identifying a greatest common factor.
- Factor by grouping.
- Application of the distributive property.
- Multiplication and division of algebraic expressions.

Lesson:

When factoring trinomials we use the ac-method to split the middle (or linear) term and then factor by grouping. The ac-method gets its name from the general trinomial expression, $ax^2 + bx + c$, where a, b, and c are the leading coefficient, linear coefficient, and constant term, respectively.

The ac-method is named as such because we will use the product $a \cdot c$ to help find out what two numbers we will need for grouping later on. In the previous lesson, we always found two numbers whose product was equal to c, since the leading coefficient a was 1 in our expression (so $a \cdot c = 1 \cdot c = c$). Now we will be working with trinomials where $a \neq 1$, so we will need to identify two numbers that multiply to ac and add to b. Aside from this adjustment, the process will be the same as before.

When $a = 1$, we were able to use a shortcut, using the numbers that split the middle coefficient for our factors. As we will see in our examples, this shortcut will not work when $a \neq 1$. Therefore, we must go through all the steps of grouping in order to factor the expression.

I - Motivating Example(s):

Example: Factor the given expression.

$$\begin{aligned} 3x^2+11x+6 &\quad \text{Multiply to } a\cdot c \text{ or } 3\cdot 6=18, \text{ add to } b=11.\\ 3x^2+9x+2x+6 &\quad \text{The numbers are 9 and 2, split the linear term.}\\ 3x(x+3)+2(x+3) &\quad \text{Factor by grouping.}\\ (x+3)(3x+2) &\quad \text{Our solution.} \end{aligned}$$

Example: Factor the given expression.

$$\begin{aligned} 8x^2-2x-15 &\quad \text{Multiply to } a\cdot c \text{ or } 8\cdot(-15)=-120, \text{ add to } b=-2.\\ 8x^2-12x+10x-15 &\quad \text{The numbers are } -12 \text{ and } 10, \text{ split the linear term.}\\ 4x(2x-3)+5(2x-3) &\quad \text{Factor by grouping.}\\ (2x-3)(4x+5) &\quad \text{Our solution.} \end{aligned}$$

II - Demo/Discussion Problems:

Factor each of the given trinomial expressions.

1. $10x^2-27x+5$
2. $4x^2-xy-5y^2$
3. $18x^3+33x^2-30x$

III - Practice Problems:

Factor each of the given trinomial expressions.

1. $7x^2-48x+36$
2. $7n^2-44n+12$
3. $7b^2+15b+2$
4. $7v^2-24v-16$
5. $5a^2-13a-28$
6. $5n^2-7n-24$
7. $2x^2-5x+2$
8. $3r^2-4r-4$
9. $2x^2+19x+35$
10. $7x^2+29x-30$
11. $2b^2-b-3$
12. $5x^2-26x+24$
13. $5x^2+13x+6$
14. $3r^2+16r+21$
15. $3x^2-17x+20$
16. $3u^2+13uv-10v^2$
17. $3x^2+17xy+10y^2$
18. $7x^2-2xy-5y^2$
19. $5x^2+28xy-49y^2$
20. $5u^2+31uv-28v^2$
21. $6x^2-39x-21$
22. $10a^2-54a-36$
23. $21x^2-87x-90$
24. $21n^2+45n-54$
25. $14x^2-60x+16$
26. $4r^2+r-3$
27. $6x^2+29x+20$
28. $6p^2+11p-7$
29. $4x^2-17x+4$
30. $4r^2+3r-7$
31. $4x^2+9xy+2y^2$
32. $4m^2+6mn+6n^2$
33. $4m^2-9mn-9n^2$
34. $4x^2-6xy+30y^2$
35. $4x^2+13xy+3y^2$
36. $18u^2-3uv-36v^2$
37. $12x^2+62xy+70y^2$
38. $16x^2+60xy+36y^2$
39. $24x^2-52xy+8y^2$
40. $12x^2+50xy+28y^2$

Lesson 21: Solving by Factoring

Objective: Solve polynomial equations by factoring and using the Zero Factor Property.

Students will be able to:

- Apply the Zero Factor Property to a factored equation that has been set equal to zero.
- Verify solutions by substituting into an original equation.

Prerequisite Knowledge:

- Combining like terms.
- Factoring Techniques: GCF, grouping, $ac-$method for trinomials.
- Evaluate polynomial expressions.

Lesson:

When solving linear equations such as $2x - 5 = 21$ we can isolate the variable directly by adding 5 and dividing by 2 to get 13. When working with quadratic equations (or higher degree polynomials), however, we must factor first, before isolating the variable in this way. One property that we will use to solve for the variable is known as the Zero Factor Property.

Zero Factor Property: If $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

The Zero Factor Property tells us that if the product of two factors is zero, then one of the factors must be zero. We can use this property to help us solve factored polynomial equations. It is important to stress that for the Zero Factor Property to work we must first set our equation equal to zero and factor the resulting expression.

I - Motivating Example(s):

Example: Solve the given equation for all possible values of x.

$12x^2 - 10x - 4 = 2x^2 + 3x - 1$	Set right side equal to zero; combine like terms.
$10x^2 - 13x - 3 = 0$	Factor using the $ac-$method.
$10x^2 - 15x + 2x - 3 = 0$	$(-15) \cdot 2 = (10) \cdot (-3) = -30$ ✓ and $-15 + 2 = -13$ ✓
$(2x - 3)(5x + 1) = 0$	One factor must equal zero for the equation to hold.

$2x - 3 = 0$ or $5x + 1 = 0$	Set each factor equal to zero and solve for x.
$2x = 3$ or $5x = -1$	Simplify; add, then divide.
$x = \frac{3}{2}$ or $-\frac{1}{5}$	Our solution.

II - Demo/Discussion Problems:

Solve each of the following equations for the given variable.

1. $4x^2 + x - 3 = 0$
2. $x^2 = 8x - 15$
3. $(x - 7)(x + 3) = -9$
4. $3x^2 + 4x - 5 = 7x^2 + 4x - 14$

III - Practice Problems:

Solve each of the following equations for the given variable.

1. $2b^2 = 32 = 0$
2. $45x^2 - 20 = 0$
3. $n = -2n^2$
4. $56 - 35p = 0$
5. $20r^3 - 4r^2 - 25r + 5 = 0$
6. $9x^3 - 72x^2 = 4x - 32$
7. $3n^3 = 2n^2 + 9n - 6$
8. $30p^3 + 25p^2 - 3p - 3 = 2p^3 + 4p^2 + p$
9. $p^2 + 17p = -72$
10. $x^2 + x - 72 = 0$
11. $n^2 - 9n + 8 = 0$
12. $x^2 + x - 30 = 0$
13. $x^2 = 9x + 10$
14. $x^2 + 13x + 40 = 0$
15. $b^2 + 32 = -12b$
16. $b^2 - 17b + 70 = 0$
17. $x^2 + 3x = 70$
18. $x^2 + 3x - 18 = 0$
19. $n^2 - 8n + 15 = 0$
20. $a^2 - 6a - 27 = 0$
21. $p^2 + 15p = -54$
22. $2p^2 + 10p - 32 = p^2 + 3p + 2$
23. $n^2 - 15n + 56 = 0$
24. $4x^2 + 52x + 168 = 0$
25. $5a^2 + 60a + 100 = 0$
26. $5n^2 - 45n + 40 = 0$
27. $6a^2 - 192 = -24a$
28. $5v^2 + 20v - 25 = 0$
29. $7x^2 - 48x + 36 = 0$
30. $7n^2 - 44n + 12 = 0$
31. $7b^2 + 15b + 2 = 0$
32. $11v^2 - 24v + 4 = 4v^2 + 20$
33. $5a^2 - 13a - 28 = 0$
34. $5n^2 - 7n - 24 = 0$
35. $2 = -2x^2 + 5x$
36. $3r^2 - 4r - 4 = 0$
37. $2x^2 + 19x + 35 = 0$
38. $7x^2 + 29x - 30 = 0$
39. $2b^2 - 8 = b - 5$
40. $5x^2 - 20x + 10 = 6x - 14$
41. $5x^2 + 13x + 6 = 0$
42. $3r^2 + 16r + 21 = 0$
43. $3x^2 - 17x + 20 = 0$
44. $6x^2 - 21 = 39x$
45. $10a^2 - 54a - 36 = 0$
46. $21x^2 - 87x - 90 = 0$
47. $21n^2 + 45n = 54$
48. $14x^2 - 60x + 16 = 0$
49. $4r^2 = 3 - r$
50. $6x^2 + 29x + 20 = 0$
51. $6p^2 + 11p - 7 = 0$
52. $4x^2 - 17x + 4 = 0$
53. $4r^2 - 4r + 4 = -7r + 11$

Lesson 22: Square Roots

Objective: Simplify and evaluate expressions involving square roots.

Students will be able to:

- Simplify square roots.

Prerequisite Knowledge:

- Prime factorization of a number.
- Perfect square numbers.
- Find square roots of square numbers.

Lesson:
I - Motivating Example(s):

Example:

$\sqrt{0} = 0$	$\sqrt{121} = 11$
$\sqrt{1} = 1$	$\sqrt{625} = 25$
$\sqrt{4} = 2$	$\sqrt{-81} =$ undefined

The final example of $\sqrt{-81}$ is currently considered to be undefined, since the square root of a negative number does not equal a real number. This is because if we square a positive or a negative number, the answer will be positive, not to mention that $0^2 = 0$. Thus we can only take square roots of nonnegative numbers (positive numbers or zero). In a future lesson, we will define a method we can use to work with and evaluate negative square roots. For now we will simply say they are undefined.

Not all numbers have a "nice" (or *rational*) square root. For example, if we found $\sqrt{8}$ on our calculator, the answer would be 2.828427124746190097..., and even this number is a rounded approximation of the square root. To be as accurate as possible, we will never use the calculator to find decimal approximations of square roots. Instead we will express roots in simplest radical form. We will do this using a property known as the product rule of radicals (in this case, square roots).

$$\textbf{Product Rule of Square Roots}: \sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$$

More generally,

$$\textbf{Product Rule of Radicals}: \sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$

We can use the product rule of square roots to simplify an expression such as $\sqrt{180} = \sqrt{36 \cdot 5}$ by splitting it into two roots, $\sqrt{36} \cdot \sqrt{5}$, and simplifying the first root, $6\sqrt{5}$. The trick in this process is being able to recognize that an expression like $\sqrt{180}$ may be rewritten as $\sqrt{36 \cdot 5}$, since $180 = 36 \cdot 5$. In the case of $\sqrt{8}$, we may write $\sqrt{8} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}$.

There are several ways of applying the product rule of square roots. The most common and, with a bit of practice, fastest method is to find perfect squares that divide nicely into the radicand (the number under the radical). This is demonstrated in the next example.

Example: Completely simplify the given radical.

$\sqrt{75}$	75 is divisible by 25, a perfect square
$\sqrt{25 \cdot 3}$	Split into factors
$\sqrt{25} \cdot \sqrt{3}$	Product rule, take the square root of 25
$5\sqrt{3}$	Our solution

II - Demo/Discussion Problems:

Completely simplify the given radicals.

1. $5\sqrt{63}$
2. $\sqrt{72}$
3. $-5\sqrt{72x^3y^4}$
4. $-5\sqrt{18x^4y^6z^{10}}$

III - Practice Problems:

Completely simplify each of the following square roots completely.

1. $\sqrt{245}$
2. $\sqrt{125}$
3. $\sqrt{36}$
4. $\sqrt{196}$
5. $\sqrt{12}$
6. $\sqrt{72}$
7. $3\sqrt{12}$
8. $5\sqrt{32}$
9. $6\sqrt{128}$
10. $7\sqrt{128}$
11. $-8\sqrt{392}$
12. $-7\sqrt{63}$
13. $\sqrt{192n}$
14. $\sqrt{343b}$
15. $\sqrt{196v^2}$
16. $\sqrt{100n^3}$
17. $\sqrt{252x^2}$
18. $\sqrt{200a^3}$
19. $-\sqrt{100k^4}$
20. $-4\sqrt{175p^4}$
21. $-7\sqrt{64x^4}$
22. $-2\sqrt{128n}$
23. $-5\sqrt{36m}$
24. $8\sqrt{112p^2}$
25. $\sqrt{45x^2y^2}$
26. $\sqrt{72a^3b^4}$
27. $\sqrt{16x^3y^3}$
28. $\sqrt{512a^4b^2}$
29. $\sqrt{320x^4y^4}$
30. $\sqrt{512m^4n^3}$
31. $6\sqrt{80xy^2}$
32. $8\sqrt{98mn}$
33. $5\sqrt{245x^2y^3}$
34. $2\sqrt{72x^2y^2}$
35. $-2\sqrt{180u^3v}$
36. $-5\sqrt{96x^4y^3}$
37. $-8\sqrt{180x^4y^2z^4}$
38. $6\sqrt{50a^4bc^2}$
39. $2\sqrt{80hj^4k}$
40. $-\sqrt{32xy^2z^3}$
41. $-4\sqrt{54mnp^2}$
42. $-8\sqrt{32m^2p^4q}$

Lesson 23: Complex Numbers

Objective: Simplify expressions involving complex numbers.

Students will be able to:

- Define the form of a complex number.
- Simplify square roots with negative radicands.
- Add, subtract, multiply, rationalize, and simplify expressions using complex numbers.

Prerequisite Knowledge:

- Properties of exponents.
- Combining like terms.
- Polynomial arithmetic.

Lesson:

To work with the square root of a negative number, mathematicians have defined what we now know as imaginary and complex numbers.

Imaginary Number $i: \; i^2 = -1$ (thus $i = \sqrt{-1}$)

Examples of imaginary numbers include $3i$, $-6i$, $\frac{3}{5}i$, and $3\sqrt{5}i$. A *complex number* is one that contains both a real and imaginary part, such as $2 + 5i$.

Complex Number: $a + bi$, where a and b are real numbers, $i = \sqrt{-1}$

With this definition, the square root of a negative number will no longer be considered undefined. We now will be able to perform basic operations with the square root of a negative number.

First, we consider powers of the imaginary number i. As the exponents of i^n increase, our simplified value for i^n will cycle through the simplified values for i, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$. As there are 4 different possible answers in this cycle, if we divide the exponent n by 4 and consider the remainder, we can easily simplify any power of i by knowing the following four values:

Cyclic Property of Powers of i

$$\begin{aligned} i^0 &= 1 \\ i^1 &= i \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 = i^0 &= 1 \end{aligned}$$

I - Motivating Example(s):

Example: Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl} (2+5i)+(4-7i) & \text{Combine like terms, } 2+4 \text{ and } 5i-7i \\ 6-2i & \text{Our solution} \end{array}$$

Example: Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl} (2-4i)(3+5i) & \text{Expand} \\ 6+10i-12i-20i^2 & \text{Simplify, } i^2=-1 \\ 6+10i-12i-20(-1) & \text{Multiply} \\ 6+10i-12i+20 & \text{Combine like terms } 6+20 \text{ and } 10i-12i \\ 26-2i & \text{Our solution} \end{array}$$

Example: Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl} \dfrac{2-6i}{4+8i} & \text{Binomial in denominator,} \\ & \quad \text{multiply by conjugate}, 4-8i \\ \dfrac{2-6i}{4+8i}\left(\dfrac{4-8i}{4-8i}\right) & \text{Expand the numerator,} \\ & \quad \text{denominator is a difference of two squares} \\ \dfrac{8-16i-24i+48i^2}{16-64i^2} & \text{Simplify } i^2=-1 \\ \dfrac{8-16i-24i+48(-1)}{16-64(-1)} & \text{Multiply} \\ \dfrac{8-16i-24i-48}{16+64} & \text{Combine like terms} \\ \dfrac{-40-40i}{80} & \text{Reduce, factor out 40 and divide} \\ \dfrac{-1-i}{2} & \text{Rewrite as } a+bi \\ -\dfrac{1}{2}-\dfrac{1}{2}i & \text{Our solution} \end{array}$$

II - Demo/Discussion Problems:

Rewrite each of the following complex numbers in the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$.

1. $\sqrt{-16}$
2. $\sqrt{-24}$
3. $\sqrt{-6}\sqrt{3}$
4. $\dfrac{-15 - \sqrt{-200}}{20}$
5. i^{35}
6. i^{124}
7. $(4 - 8i) - (3 - 5i)$
8. $5i - (3 + 8i) + (-4 + 7i)$
9. $(3i)(7i)$
10. $5i(3i - 7)$
11. $(3i)(6i)(2 - 3i)$
12. $(4 - 5i)^2$
13. $\dfrac{7 + 3i}{-5i}$

III - Practice Problems:

Rewrite each of the following complex numbers in the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$.

1. $\sqrt{-81}$
2. $\sqrt{-45}$
3. $\sqrt{-10}\sqrt{-2}$
4. $\sqrt{-12}\sqrt{-2}$
5. $\dfrac{3 + \sqrt{-27}}{6}$
6. $\dfrac{-4 - \sqrt{-8}}{-4}$
7. $\dfrac{8 - \sqrt{-16}}{4}$
8. $\dfrac{6 + \sqrt{-32}}{4}$
9. i^{73}
10. i^{251}
11. i^{48}
12. i^{68}
13. i^{62}
14. i^{181}
15. i^{154}
16. i^{51}
17. $3 - (-8 + 4i)$
18. $3i - (7i)$
19. $7i - (3 - 2i)$
20. $5 + (-6 - 6i)$
21. $-6i - (3 + 7i)$
22. $-8i - 7i - (5 - 3i)$
23. $(3 - 3i) + (-7 - 8i)$
24. $(-4 - i) + (1 - 5i)$
25. $-6 + i - (2 + 3i)$
26. $(5 - 4i) + (8 - 4i)$
27. $(6i)(-8i)$
28. $(3i)(-8i)$
29. $(-5i)(8i)$
30. $(8i)(-4i)$
31. $(-7i)^2$
32. $(-i)(7i)(4 - 3i)$
33. $(6 + 5i)^2$
34. $(8i)(-2i)(-2 - 8i)$
35. $(-7 - 4i)(-8 + 6i)$
36. $(3i)(-3i)(4 - 4i)$
37. $(-4 + 5i)(2 - 7i)$
38. $-8(4 - 8i) - 2(-2 - 6i)$
39. $(-8 - 6i)(-4 + 2i)$
40. $(-6i)(3 - 2i) - (7i)(4i)$
41. $(1 + 5i)(2 + i)$
42. $(-2 + i)(3 - 5i)$
43. $\dfrac{-9 + 5i}{i}$
44. $\dfrac{-3 + 2i}{-3i}$
45. $\dfrac{-10 - 9i}{6i}$
46. $\dfrac{-4 + 2i}{3i}$
47. $\dfrac{-3 - 6i}{4i}$
48. $\dfrac{-5 + 9i}{9i}$
49. $\dfrac{10 - i}{-i}$
50. $\dfrac{10}{5i}$
51. $\dfrac{4i}{-10 + i}$
52. $\dfrac{9i}{1 - 5i}$
53. $\dfrac{8}{7 - 6i}$
54. $\dfrac{4}{4 + 6i}$
55. $\dfrac{7}{10 - 7i}$
56. $\dfrac{9}{-8 - 6i}$
57. $\dfrac{5i}{-6 - i}$
58. $\dfrac{8i}{6 - 7i}$

Lesson 24: Vertex Form and Graphing

Objective: Graph quadratic equations in both standard and vertex forms.

Students will be able to:

- Find the vertex, axis of symmetry, and $x-$intercept(s) of a quadratic equation in vertex or standard form.

Prerequisite Knowledge:

- State the concavity of the graph of a quadratic equation.
- Find the $y-$intercept of the graph of an equation.
- Evaluate an equation and create a table of values.
- Plot points on the Cartesian $xy-$coordinate plane.

Lesson:

Recall the two forms of a quadratic equation, shown below. In both forms, we assume $a \neq 0$.

Standard Form: $y = ax^2 + bx + c$, where $a, b,$ and c are real numbers

Vertex Form: $y = a(x - h)^2 + k$, where $a, h,$ and k are real numbers

Unlike the standard form, a quadratic equation written in vertex form allows for immediate recognition of the vertex (h, k), which will always coincide with either a maximum (if $a < 0$) or a minimum (if $a > 0$) on the accompanying graph, called a parabola. Additionally, using the vertex form, we can easily identify the *axis of symmetry* for the parabola, which is a vertical line $x = h$ that passes through the x-coordinate of the vertex and "splits" the graph into two identical halves.

If $y = ax^2 + bx + c$ $(a \neq 0)$, we can identify the x-coordinate for the vertex (and consequently the equation for the axis of symmetry) using the following formula.

$$h = -\frac{b}{2a}$$

Consequently, the y-coordinate for our vertex can be found by plugging $x = h$ back into the given equation for our quadratic, and simplifying to find the y-coordinate, which we will relabel as k.

Once we have h and k, we can use them, along with a, to write the vertex form for our quadratic,

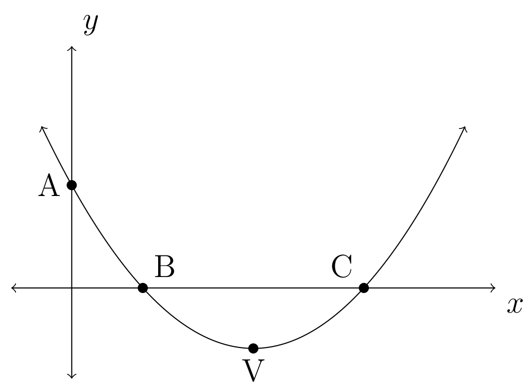
$$y = a(x - h)^2 + k.$$

On the other hand, if we are given a quadratic equation in vertex form, we can expand it to obtain the corresponding vertex form, as shown below.

$$\begin{aligned} y &= a(x - h)^2 + k \\ &= a(x - h)(x - h) + k \\ &= a(x^2 - 2hx + h^2) + k \\ &= ax^2 - 2ahx + ah^2 + k \end{aligned}$$

If we set the linear coefficient $-2ah$ above equal to b and solve for h, we can see the connection with the formula for the vertex.

It can be difficult to find a sufficient collection of points to determine the overall shape of our parabola. For this reason, we will now formally identify several key points on a parabola, which will enable us to always determine a complete graph. These points are the $y-$intercept, $x-$intercepts, and the vertex.



Point A: y-intercept; where the graph crosses the vertical y-axis (when $x = 0$).

Points B and C: x-intercepts; where the graph crosses the horizontal x-axis (when $y = 0$)

Point V: vertex (h, k); The point of the minimum (or maximum) value, where the graph changes direction.

We will use the following method to find each of the key points on our parabola.

Steps for graphing a quadratic in standard form, $y = ax^2 + bx + c$.

1. Identify and plot the vertex: $h = -\dfrac{b}{2a}$. Plug h into the equation to find k. Resulting point is (h, k).
2. Identify and plot the y-intercept: Set $x = 0$ and solve. The y-intercept will correspond to the constant term c. Resulting point is $(0, c)$.
3. Identify and plot the x-intercept(s): Set $y = 0$ and solve for x. Depending on the expression, we will end up with zero, one or two $x-$intercepts.
4. After plotting these points we can connect them with a smooth curve.

Important: Up until now, we have only discussed how to solve a quadratic equation for x by factoring. If an expression is not easily factorable, we may not be able to identify the x-intercepts. In our next few lessons, we will learn of two additional methods for finding x-intercepts, which will prove especially useful, when an equation is not easily factorable.

I - Motivating Example(s):

Example: Consider $y = -2(x+1)^2 + 3$.

In this example we can see that the vertex is at $(-1, 3)$. It is important that we not overlook the negative value for h. The axis of symmetry, passes through the x-coordinate for the vertex, $x = -1$. Expanding our vertex form gives us the corresponding standard form, shown below.

$$\begin{aligned} y &= -2(x+1)(x+1) + 3 \\ &= -2(x^2 + 2x + 1) + 3 \\ &= -2x^2 - 4x + 1 \end{aligned}$$

The value of c from our standard form gives us or $y-$intercept of $(0, 1)$, which we also confirm using the vertex form.

$$y = -2(0+1)^2 + 3 = -2(1) + 3 = 1$$

Example: Identify the vertex and axis of symmetry for the parabola represented by the given quadratic equation.

$$\begin{array}{rl} y = x^2 + 8x - 12 & \text{Given an equation in standard form} \\ a = 1, \quad b = 8, \quad c = -12 & \text{Identify } a, b, \text{ and } c \\ h = -\dfrac{b}{2a} = -\dfrac{8}{2(1)} = -4 & \text{Identify } h \\ x = -4 & \text{Use } h \text{ for axis of symmetry, a vertical line} \\ k = (-4)^2 + 8(-4) - 12 & \text{Plug in } h \text{ to find } k \\ k = 16 - 32 - 12 = -28 & \text{Simplify} \\ (-4, -28) & \text{Write the vertex as an ordered pair } (h, k) \end{array}$$

II - Demo/Discussion Problems:

Identify the vertex and axis of symmetry for each quadratic equation below. Write the vertex form for each equation. Sketch a complete graph of the corresponding parabola, labeling any $x-$ and $y-$intercepts.

1. $y = x^2 - 4x + 3$
2. $y = x^2 + 4x + 3$
3. $y = -3x^2 + 12x - 9$
4. $y = x^2 - 6x + 9$
5. $y = -x^2 + 16$
6. $y = -x^2 - 25$
7. $y = x^2 + 1$
8. $y = -3x^2 + 6x - 1$

III - Practice Problems:

Expand each equation below to find the corresponding standard form. Sketch a graph of the corresponding parabola, identifying the vertex, $y-$intercept, and axis of symmetry. If possible, identify any $x-$intercepts by factoring.

1. $y = (x-3)^2 + 4$
2. $y = (x-2)^2 + 5$
3. $y = 6(x+3)^2 + 4$
4. $y = -2(x-3)^2 + 4$
5. $y = -2(x-1)^2 - 7$
6. $y = -(x+1)^2$
7. $y = -\dfrac{1}{5}(x+1)^2$

Identify whether the quadratic is in vertex form, standard form, or both. If it is in vertex form, then identify the vertex (h, k).

8. $y = (x-12)^2 + 5$
9. $y = -3(x-3)^2 + 5$
10. $y = x^2 + 8$
11. $y = 2(x-4)^2$
12. $y = -4(x-1)^2 + 2$
13. $y = -5(x-7)^2$
14. $y = x^2 + 3x + 4$
15. $y = x^2 - 1$
16. $y = x^2 - 3$
17. $y = (x-1)^2 - 3$
18. $y = (x-1)^2$
19. $y = x^2$

Each quadratic equation below has been given in standard form. Rewrite each equation in vertex form.

20. $y = x^2 + 2x - 1$
21. $y = -3x^2 - 12x - 5$
22. $y = 3x^2 + 12x - 1$
23. $y = x^2 + 2x$
24. $y = x^2 + 6$
25. $y = -5x^2 - 40x$
26. $y = x^2 + 8x$
27. $y = x^2$
28. $y = x^2 + 4x - 2$
29. $y = x^2 + 16x - 2$
30. $y = 4x^2 + 10x$

Find the vertex and any intercepts ($x-$ and $y-$) of the following quadratics. Use this information to graph the resulting parabola. Identify the axis of symmetry on your graph, and write the corresponding vertex form.

31. $y = x^2 - 2x - 8$
32. $y = x^2 - 2x - 3$
33. $y = 2x^2 - 12x + 10$
34. $y = 2x^2 - 12x + 16$
35. $y = -2x^2 + 12x - 18$
36. $y = -2x^2 + 12x - 10$
37. $y = -3x^2 + 24x - 45$
38. $y = -3x^2 + 12x - 9$
39. $y = -x^2 + 4x + 5$
40. $y = -x^2 + 4x - 3$
41. $y = -x^2 + 6x - 5$
42. $y = -2x^2 + 16x - 30$
43. $y = -2x^2 + 16x - 24$
44. $y = 2x^2 + 4x - 6$
45. $y = 3x^2 + 12x + 9$
46. $y = 5x^2 + 30x + 45$
47. $y = 5x^2 - 40x + 75$
48. $y = 5x^2 + 20x + 15$
49. $y = -5x^2 - 60x - 175$
50. $y = -5x^2 + 20x - 15$

Lesson 25: Solve by Square Roots

Objective: Solve quadratic equations of the form $ax^2 + c = 0$ by introducing a square root.

Students will be able to:

- Solve equations by taking a square root.

Prerequisite Knowledge:

- Simplifying radicals.
- Solving by isolating a quadratic term.
- Complex numbers.

Lesson:

Up until now, when attempting to solve an equation such as $x^2 - 4 = 0$, we have had no choice but to factor the expression on the left and set each factor equal to zero.

$$\begin{aligned} x^2 - 4 &= 0 \\ (x-2)(x+2) &= 0 \\ x &= \pm 2 \end{aligned}$$

In this lesson, we will look at solving the expression $ax^2 + c = 0$, and instead will introduce a square root, when solving for x. In our next lesson, we will refer to this method as *extracting square roots*.

I - Motivating Example(s):

Example: Solve the equation $x^2 - 4 = 0$ for all possible values of x.

$$\begin{aligned} x^2 - 4 &= 0 && \text{Isolate } x^2\text{; Add 4 to both sides.} \\ x^2 &= 4 && \text{Introduce a square root, include a } \pm. \\ \sqrt{x^2} &= \pm\sqrt{4} && \text{Simplify.} \\ x &= \pm 2 && \text{Our solution} \end{aligned}$$

The values $x = 2$ and $x = -2$ are known as the *zeros* or *roots* of the equation $y = x^2 - 4$.

II - Demo/Discussion Problems:

Solve each of the following equations for all possible x. Classify each solution as either real or imaginary.

1. $x^2 - 9 = 0$
2. $3x^2 - 20 = x^2 + 20$
3. $x^2 + 16 = 0$
4. $x^3 + 16x = 0$
5. $x^2 + 12 = 0$

III - Practice Problems:

Solve each of the following equations for all possible x. Classify each solution as either real or imaginary.

1. $x^2 - 100 = 0$
2. $x^2 + 1 = 0$
3. $3x^2 - 48 = 0$
4. $x^2 + 81 = 0$
5. $x^2 + 20 = 0$
6. $3x^2 - 27 = 0$
7. $4x^3 - 32x = 0$
8. $-5x^2 = 80$
9. $x^3 = 6x$
10. $10x^2 - 70 = 0$
11. $-5x^2 = 20$
12. $\frac{1}{4}x^3 = 100x$

Lesson 26: Solve by Extracting Square Roots

Objective: Solve quadratic equations using the method of extracting square roots.

Students will be able to:

- Solve quadratic equations in vertex form as an alternative to factoring or when factoring fails.
- Approximate an irrational root to a quadratic equation for the purposes of graphing.

Prerequisite Knowledge:

- Simplifying radicals.
- Solving by isolating a quadratic term.
- Complex numbers.

Lesson:

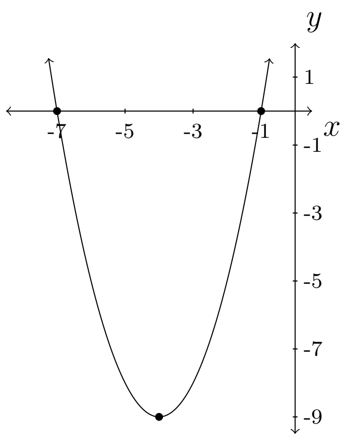
We now introduce a new technique for solving quadratic equations, known as *extracting square roots*. This method will only be employed once we have identified the vertex form for a given quadratic, $y = a(x-h)^2 + k$. The general steps for extracting square roots are shown in our first example, and the requirement of the vertex form will be essential.

I - Motivating Example(s):

Example: Determine the zeros of the quadratic equation $y = ax^2 + bx + c$, where $a \neq 0$.
First obtain the vertex form: $h = -\dfrac{b}{2a}$, set $x = h$ to find k.

$$
\begin{array}{rll}
a(x-h)^2 + \cancel{k} = 0 & & \text{Vertex form} \\
\underline{-\cancel{k}} \quad \underline{-k} & & \text{Subtract } k \text{ from both sides} \\
\dfrac{\cancel{a}(x-h)^2}{\cancel{a}} = \dfrac{-k}{a} & & \text{Divide both sides by } a \\
(x-h)^2 = -\dfrac{k}{a} & & \\
\sqrt{(x-h)^2} = \pm\sqrt{-\dfrac{k}{a}} & & \text{Take square root of both sides} \\
 & & \quad \text{to extract radicand, } x-h \\
x - \cancel{h} = \pm\sqrt{-\dfrac{k}{a}} & & \\
\underline{+\cancel{h}} \qquad \underline{+h} & & \text{Add } h \text{ to both sides} \\
x = h \pm \sqrt{-\dfrac{k}{a}} & & \text{Our solution}
\end{array}
$$

Example: Use the method of extracting square roots to find the zeros of the equation $y = (x+4)^2 - 9$.



$$\begin{array}{rl}
0 = (x+4)^2 - 9 & \text{Set equal to zero} \\
\underline{\mathbf{+9}} \qquad\qquad \underline{\mathbf{+9}} & \text{Isolate the square} \\
9 = (x+4)^2 & \text{Introduce a} \\
\pm\sqrt{9} = \sqrt{(x+4)^2} & \quad \text{square root; include } \pm \\
\pm 3 = x + 4 & \text{Solve for } x \\
\underline{\mathbf{-4}} \quad \underline{\mathbf{-4}} & \\
x = \pm 3 - 4 & \text{Two solutions} \\
x = 3 - 4 \ \Rightarrow x = -1 & \text{First solution} \\
x = -3 - 4 \Rightarrow x = -7 & \text{Second solution}
\end{array}$$

Our zeros are $x = -7$ and -1. The corresponding $x-$intercepts are at $(-7, 0)$ and $(-1, 0)$.

Example: Use the method of extracting square roots to find the zeros of the equation $y = -2(x+3)^2 + 48$.

We show our solution below, this time omitting each step in the overall simplification.

$$\begin{aligned}
-2(x+3)^2 + 48 &= 0 \\
-2(x+3)^2 &= -48 \\
(x+3)^2 &= 24 \\
\sqrt{(x+3)^2} &= \pm\sqrt{24} \\
x + 3 &= \pm\sqrt{4}\sqrt{6} \\
x &= -3 \pm 2\sqrt{6}
\end{aligned}$$

Note that we can approximate our two roots, by realizing that

$$2 = \sqrt{4} < \sqrt{6} < \sqrt{9} = 3.$$

Since $\sqrt{6} \approx 2.4$, we can say that our two roots are $x \approx -3 \pm 4.8$. This reduces to $x \approx 1.8$ and $x \approx -7.8$. We may conclude that our $x-$intercepts are approximately located at $(1.8, 0)$ and $(-7.8, 0)$.

II - Demo/Discussion Problems:

Solve each of the following equations for all possible x. Classify each solution as either real or imaginary. If your answer includes a square root, find a decimal approximation for your answer(s).

1. $y = -3(x-1)^2 + 12$
2. $y = (x+5)^2 - 8$
3. $y = \frac{1}{8}(x-6)^2 - 5$
4. $y = -1(x+3)^2 - 4$
5. $y = -\frac{1}{4}(x+3)^2 + 27$

III - Practice Problems:

Solve each of the following equations for all possible x. Classify each solution as either real or imaginary. If your answer includes a square root, find a decimal approximation for your answer(s).

1. $y = 2(x-4)^2 - 200$
2. $y = -2(x-7)^2 + 50$
3. $y = (x-4)^2 - 98$
4. $y = (x-12)^2 - 5$
5. $y = x^2 + 18$
6. $y = (x-16)^2$
7. $y = -3(x-3)^2 + 30$
8. $y = -4(x-1)^2 + 20$
9. $y = -4(x+6)^2 + 8$
10. $y = \frac{1}{20}(x-1)^2 - 15$
11. $y = (x+2)^2 + 12$
12. $y = 9(x-11)^2 - 81$

Lesson 27: The Discriminant

Objective: Use the discriminant to determine the number of real solutions to a quadratic equation.

Students will be able to:

- Find, simplify, and interpret the discriminant of a quadratic equation in standard form.

Prerequisite Knowledge:

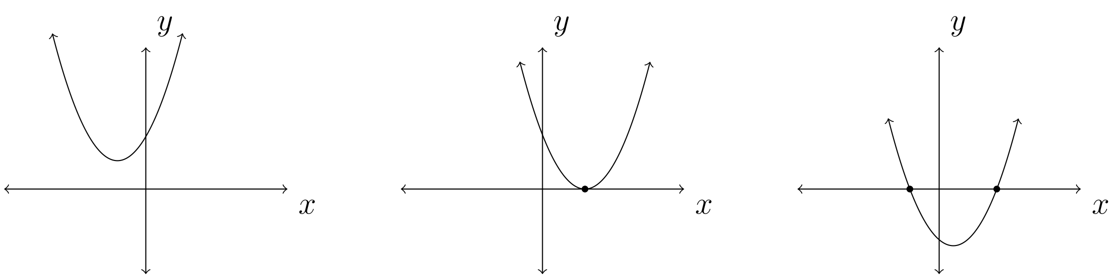
- Identifying coefficients of a quadratic in standard form.
- Order of operations.

Lesson:

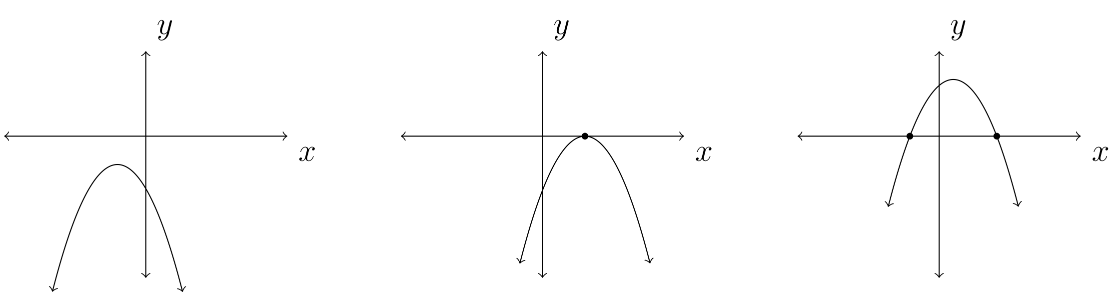
The *discriminant* of a quadratic expression $ax^2 + bx + c$ is defined as the real number $D = b^2 - 4ac$. In the next lesson, we will see that the discriminant is one piece of the larger quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

which is used for identifying the roots of the equation $y = ax^2+bx+c$. Since the discriminant appears underneath of a square root in the quadratic formula, whether it is positive, negative, or zero will determine the number of real roots of a quadratic, and consequently the number of $x-$intercepts on its corresponding parabola.



Negative Discriminant	Zero Discriminant	Positive Discriminant
$b^2 - 4ac < 0$	$b^2 - 4ac = 0$	$b^2 - 4ac > 0$
No Real Solutions	One Real Solution	Two Real Solutions



I - Motivating Example(s):

Example: Determine the number of real roots for the quadratic equation below.

$$y = 3x^2 - 5x + 2$$

$$\begin{aligned} D &= b^2 - 4ac \\ &= (-5)^2 - 4(3)(2) \\ &= 25 - 24 \\ &= 1 \end{aligned}$$

Since $D > 0$, the given equation has two real roots.

II - Demo/Discussion Problems:

Determine the number of real roots for each of the quadratic equations below.

1. $y = -4x^2 - 15$
2. $y = 3x^2 - 12x - 15$
3. $y = 2x^2 + 4x + 2$
4. $y = 10x^2 + 31x + 24$
5. $y = x^2 - 4x + 13$

III - Practice Problems:

Determine the number of real roots for each of the quadratic equations below.

1. $y = x^2 + 6$
2. $y = x^2 + 2x - 1$
3. $y = -3x^2 - 12x - 5$
4. $y = 3x^2 + 12x - 1$
5. $y = -5x^2 - 40x$
6. $y = x^2 - 8x + 15$
7. $y = x^2 + 4x - 2$
8. $y = x^2 + 16x - 2$
9. $y = 4x^2 + 10x$
10. $y = 5x^2 - 4x + 1$
11. $y = -x^2 + 3x - 9$
12. $y = x^2 + 6x + 9$

Lesson 28: The Quadratic Formula

Objective: Solve quadratic equations using the quadratic formula.

Students will be able to:

- Use the quadratic formula to solve a quadratic equation.
- Fully simplify solutions to quadratic equations obtained using the quadratic formula.
- Approximate a decimal solution to a quadratic equation for graphing purposes.

Prerequisite Knowledge:

- Identifying coefficients of a quadratic in standard form.
- Order of operations.
- Simplifying radicals.
- Evaluating expressions.

Lesson:

The *Quadratic Formula* states that the solutions to the equation $ax^2 + bx + c = 0$ are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

I - Motivating Example(s):

Example: Solve the given equation for all values of x.

$$\begin{aligned}
x^2 - 4x - 1 &= 0 && \\
a = 1,\ b = -4,\ c &= -1 && \text{Identify } a, b, \text{ and } c \\
x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-1)}}{2(1)} && \text{Use quadratic formula} \\
x &= \frac{4 \pm \sqrt{16 + 4}}{2} && \text{Simplify} \\
x &= \frac{4 \pm \sqrt{20}}{2} && \text{Discriminant is 20 (positive)} \\
x &= \frac{4}{2} \pm \frac{2\sqrt{5}}{2} && \text{Split up fraction} \\
x &= 2 \pm \sqrt{5} && \text{Our solutions} \\
x &\approx 2 \pm 2.2 && \sqrt{5} \approx 2.2 \\
x \approx 4.2 \ \text{ or } \ x &\approx -0.2 && \text{Decimal approximations}
\end{aligned}$$

II - Demo/Discussion Problems:

Use the quadratic formula to find the roots of each of the following equations. If your answer contains a square root, find a decimal approximation.

1. $y = x^2 + 7x - 8$
2. $y = x^2 - 13x - 30$
3. $y = 25x^2 - 30x - 11$
4. $y = 4x^2 - 12x + 9$
5. $y = x^2 - 6x + 25$

III - Practice Problems:

Use the quadratic formula to find the roots of each of the following equations. If your answer contains a square root, find a decimal approximation.

1. $y = x^2 + 6$
2. $y = x^2 + 2x - 1$
3. $y = -3x^2 - 12x - 5$
4. $y = 3x^2 + 12x - 1$
5. $y = -5x^2 - 40x$
6. $y = x^2 - 8x + 15$
7. $y = x^2 + 4x - 2$
8. $y = x^2 + 16x - 2$
9. $y = 4x^2 + 10x$
10. $y = 5x^2 - 4x + 1$
11. $y = -x^2 + 3x - 9$
12. $y = x^2 + 6x + 9$

Solve each of the following equations using any means.

13. $4a^2 + 6 = 0$
14. $3k^2 + 2 = 0$
15. $2x^2 - 8x - 2 = 0$
16. $6n^2 - 1 = 0$
17. $2m^2 - 3 = 0$
18. $5p^2 + 2p + 6 = 0$
19. $3r^2 - 2r - 1 = 0$
20. $2x^2 - 2x - 15 = 0$
21. $4n^2 - 36 = 0$
22. $3b^2 + 6 = 0$
23. $v^2 - 4v - 5 = -8$
24. $2x^2 + 4x + 12 = 8$
25. $2a^2 + 3a + 14 = 6$
26. $6n^2 - 3n + 3 = -4$
27. $3k^2 + 3k - 4 = 7$
28. $4x^2 - 14 = -2$
29. $7x^2 + 3x - 16 = -2$
30. $4n^2 + 5n = 7$
31. $2p^2 + 6p - 16 = 4$
32. $m^2 + 4m - 48 = -3$
33. $3n^2 + 3n = -3$
34. $3b^2 - 3 = 8b$
35. $2x^2 = -7x + 49$
36. $3r^2 + 4 = -6r$
37. $5x^2 = 7x + 7$
38. $6a^2 = -5a + 13$
39. $8n^2 = -3n - 8$
40. $6v^2 = 4 + 6v$
41. $2x^2 + 5x = -3$
42. $x^2 = 8$
43. $4a^2 - 64 = 0$
44. $2k^2 + 6k - 16 = 2k$
45. $4p^2 + 5p - 36 = 3p^2$
46. $12x^2 + x + 7 = 5x^2 + 5x$
47. $-5n^2 - 3n - 52 = 2 - 7n^2$
48. $7m^2 - 6m + 6 = -m$
49. $7r^2 - 12 = -3r$
50. $3x^2 - 3 = x^2$
51. $2n^2 - 9 = 4$
52. $6b^2 = b^2 + 7 - b$

Lesson 29: Quadratic Inequalities

Objective: Solve quadratic inequalities using a sign diagram.

Students will be able to:

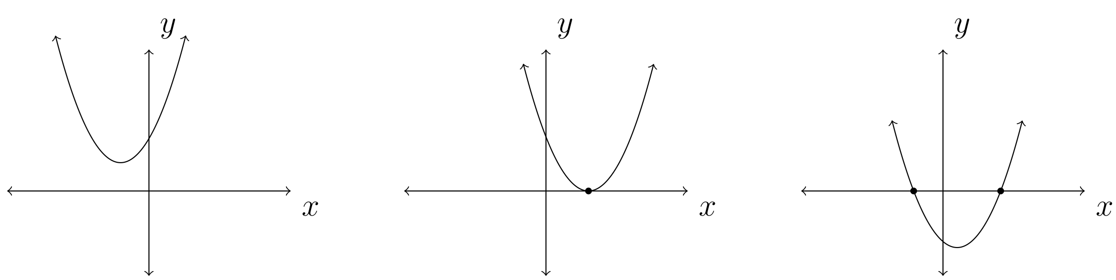
- Construct a sign diagram for a quadratic expression.
- Solve a quadratic inequality.
- Represent a solution to a quadratic inequality using interval notation.

Prerequisite Knowledge:

- Solving and factoring quadratic equations.
- Finding $x-$intercepts.
- Evaluating an expression for x.

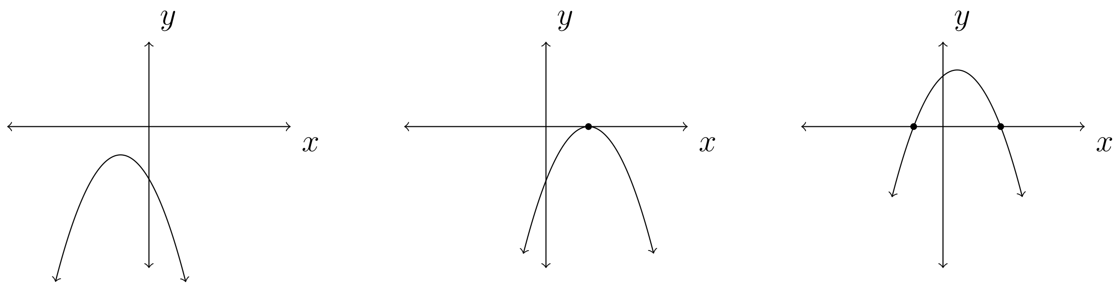
Lesson:

Recall that the *vertex form* for a quadratic equation is $y = a(x-h)^2 + k$, where $a \neq 0$ and (h, k) is the *vertex* of the corresponding parabola. If $a > 0$, then the parabola opens upward, and if $a < 0$, then the parabola opens downward. With any quadratic equation, we know that there are three possibilities for the number of roots, namely $0, 1$, or 2. Assuming $a > 0$, we illustrate these possibilities in the graphs below.



Notice also that each of these three graphs lie above the x-axis over different intervals. In the case of the parabola on the left, the entire graph lies above the x-axis, whereas the middle parabola lies above the x-axis everywhere *except* at its x-intercept (where $y = 0$). Even more interesting is the parabola on the right, which contains two *separate* intervals where its graph lies above the x-axis.

Considering the case where $a < 0$, we see three similar graphs as those appearing above, with the only major difference being the opening of each parabola downward instead of upward (when $a > 0$). When we consider again those intervals where each graph lies above the x-axis, each parabola exhibits a different behavior than those where $a > 0$.



Now, each of the first two graphs have no points that lie above the x-axis, whereas the last graph, on the right, lies above the x-axis over the interval that is between its x-intercepts.

Each of these six graphs exhibit all of the various possibilities for the *sign* of a quadratic expression $ax^2 + bx + c$, where $a \neq 0$. We can determine the general shape of the graph of a quadratic equation (or function) through identification of its roots and construction of a sign diagram. As a consequence, we will also use a sign diagram to solve a quadratic inequality.

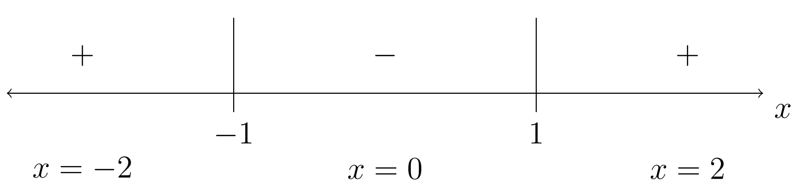
I - Motivating Example(s):

Example: Solve the quadratic inequality $x^2 - 1 > 0$.

Unlike with solving linear inequalities, one should not attempt to solve for the variable x, but rather set the given inequality equal to zero and attempt to *factor* the resulting expression on the other side. In doing this, we obtain $(x+1)(x-1) > 0$. Recalling that $x = \pm 1$ are roots of the given expression, we can therefore rule them out of our solution. Next, we will *test* the expression on the left by plugging in three values for x: one that is less than -1, one that is between -1 and 1, and one that is greater than 1.

Case	Test Value	Unsimplified	Simplified	Result
i	$x = -2$	$(-2+1)(-2-1)$	$(-)\cdot(-)$	$(+)$
ii	$x = 0$	$(0+1)(0-1)$	$(+)\cdot(-)$	$(-)$
iii	$x = 2$	$(2+1)(2-1)$	$(+)\cdot(+)$	$(+)$

Our end result can be summarized in the following *sign diagram*.



From our sign diagram, we can conclude that $x^2 - 1 > 0$ when $x < -1$ or $x > 1$. Using interval notation, our answer is $(-\infty, -1) \cup (1, \infty)$.

II - Demo/Discussion Problems:

Solve each of the following inequalities. Express your answers using interval notation.

1. $-x^2 + 4x + 5 \geq 0$
2. $x^2 - 17x \geq -60$
3. $x^2 + 4x + 1 > 0$
4. $-(x-1)^2 + 9 \geq 0$

Solve each of the following inequalities. Express your answers using interval notation.

5. $x^2 + 4x + 4 > 0$
6. $x^2 + 4x + 4 \geq 0$
7. $x^2 + 4x + 4 < 0$
8. $x^2 + 4x + 4 \leq 0$

III - Practice Problems:

Solve each of the following inequalities. Express your answers using interval notation.

1. $x^2 - 4 > 0$
2. $x^2 - 5x - 6 \leq 0$
3. $-(3x-2)(x+4) \geq 0$
4. $3x^2 + 2x - 8 > 0$
5. $2x^2 - 16 < 0$
6. $-x^2 - 13x + 30 \geq 0$
7. $x^2 - 5x + 6 > 0$
8. $2x^2 - 3x - 14 \leq 0$
9. $x^2 + 2x - 9 > 0$
10. $-2x^2 + 12x - 18 < 0$

Lesson 30: Finding Domain Algebraically

Objective: Find the domain of a function by algebraic methods.

Students will be able to:

- Determine the appropriate course of action for identifying the domain of a variety of algebraic functions (polynomial, rational, radical, etc.).
- Identify the domain of an arbitrary algebraic function.

Prerequisite Knowledge:

- Solving basic inequalities.
- Interval notation.

Lesson:

When trying to identify the domain of a function that has been described algebraically or whose graph is not known, we will often need to consider what is *not* permissible for the function, then exclude any values of x that will make the function undefined from the interval $(-\infty, \infty)$. What is left will be our domain. With virtually every algebraic function, this amounts to avoiding the following situations.

- Negatives under an even radical $\left(\sqrt{\ },\ \sqrt[4]{\ },\ \sqrt[6]{\ },\ \ldots\right)$
- Zero in a denominator

I - Motivating Example(s):

Example: Find the domain of $f(x) = \frac{1}{3}x^2 - x$.

$$\begin{aligned} f(x) = \frac{1}{3}x^2 - x & \qquad \text{No radicals or variables in a denominator} \\ & \qquad \text{No values of } x \text{ need to be excluded} \\ \text{All real numbers or } (-\infty, \infty) & \qquad \text{Our solution} \end{aligned}$$

Our next example will be of a *rational function*, which is defined as a ratio of two polynomial functions. We will explore rational functions and their graphs in a later lesson. Since rational functions usually include expressions in a denominator, their domains will often require us to exclude one or more values of x.

Example: Find the domain of the function $f(x) = \dfrac{3x-1}{x^2+x-6}$.

$$\begin{aligned} f(x) = \frac{3x-1}{x^2+x-6} &\qquad \text{Cannot have zero in a denominator} \\ x^2+x-6 \neq 0 &\qquad \text{Solve by factoring} \\ (x+3)(x-2) \neq 0 &\qquad \text{Set each factor not equal to zero} \\ x+3 \neq 0 \text{ and } x-2 \neq 0 &\qquad \text{Solve each inequality} \\ x \neq -3, 2 &\qquad \text{Our solution as an inequality} \\ (-\infty,-3) \cup (-3,2) \cup (2,\infty) &\qquad \text{Our solution using interval notation} \end{aligned}$$

Although one can easily see that $x = \frac{1}{3}$ will make the numerator equal zero, since $x = \frac{1}{3}$ does not coincide with the two values obtained above (either -3 or 2), we should not exclude it from our domain. Whenever we are finding the domain of a rational function, we need not be concerned at all with the numerator, and instead must restrict our domain to exclude any value for x that would make the *denominator* equal to zero.

Example: Find the domain of $f(x) = \sqrt{-2x+3}$.

$$\begin{aligned} f(x) = \sqrt{-2x+3} &\qquad \text{Even radical; cannot have negative underneath} \\ -2x+3 \geq 0 &\qquad \text{Set greater than or equal to zero and solve} \\ -2x \geq -3 &\qquad \text{Remember to switch direction of inequality} \\ x \leq \frac{3}{2} \text{ or } \left(-\infty, \frac{3}{2}\right] &\qquad \text{Our solution as an inequality or an interval} \end{aligned}$$

II - Demo/Discussion Problems:

Find the domain of each of the following functions. Express your answers using interval notation.

1. $f(x) = |3x-2|$
2. $g(x) = (3x-2)^2$
3. $h(x) = \dfrac{1}{3x-2}$
4. $k(x) = \sqrt{3x-2}$
5. $k(x) = \sqrt[3]{3x-2}$
6. $\ell(x) = \sqrt[4]{2-3x}$
7. $m(x) = \dfrac{x-2}{\sqrt{3x-2}}$
8. $n(x) = \dfrac{\sqrt{3x-2}}{x-2}$

III - Practice Problems:

Find the domain of each of the following functions. Express your answers using interval notation.

1. $g(x) - 4x^2$
2. $f(x) = x^4 - 13x^3 + 56x^2 - 19$
3. $g(x) = x^2 - 4$
4. $k(x) = \dfrac{x}{x-8}$
5. $h(x) = \dfrac{x-5}{x+4}$
6. $h(x) = \dfrac{x-2}{x+1}$
7. $k(x) = \dfrac{x-2}{x-2}$
8. $k(x) = \dfrac{3x}{x^2+x-2}$
9. $g(x) = \dfrac{2x}{x^2-9}$
10. $f(x) = \dfrac{2x}{x^2+9}$
11. $h(x) = \dfrac{x+4}{x^2-36}$
12. $f(x) = \sqrt{3-x}$
13. $g(x) = \sqrt{2x+5}$
14. $f(x) = 5\sqrt{x-1}$
15. $h(x) = 9x\sqrt{x+3}$
16. $k(x) = \dfrac{\sqrt{7-x}}{x^2+1}$
17. $f(x) = \sqrt{6x-2}$
18. $g(x) = \dfrac{6}{\sqrt{6x-2}}$
19. $k(x) = \dfrac{4}{\sqrt{x-3}}$
20. $g(x) = \dfrac{x}{\sqrt{x-8}}$
21. $h(x) = \sqrt[3]{6x-2}$
22. $k(x) = \dfrac{6}{4-\sqrt{6x-2}}$
23. $f(x) = \dfrac{\sqrt{6x-2}}{x^2-36}$
24. $g(x) = \dfrac{\sqrt[3]{6x-2}}{x^2+36}$
25. $h(x) = \sqrt{x-7} + \sqrt{9-x}$
26. $h(t) = \dfrac{\sqrt{t}-8}{5-t}$
27. $f(r) = \dfrac{\sqrt{r}}{r-8}$
28. $k(v) = \dfrac{1}{4-\dfrac{1}{v^2}}$
29. $f(y) = \sqrt[3]{\dfrac{y}{y-8}}$
30. $k(w) = \dfrac{w-8}{5-\sqrt{w}}$

Lesson 31: Solving Functions

CC attribute: *College Algebra* by C. Stitz and J. Zeager.

Objective: Solve functions using appropriate notation.

Students will be able to:

- Solve an equation to determine a set of inputs that produce a given output.

Prerequisite Knowledge:

- Order of operations.
- Isolating a variable.

Lesson:

In a previous lesson we discussed evaluating functions for a given input, $x = a$. In this lesson, we focus on *solving* functions for a particular output, $y = a$, or $f(x) = a$. There is a fundamental difference between evaluating, $f(a)$, and solving functions, $f(x) = a$. To see this, we will pay close attention to when $a = 0$ and focus on the graphical implications.

When we are evaluating $f(0)$, we are finding the $y-$coordinate associated with when $x = 0$. This corresponds to the $y-$intercept in our graph of f. Alternatively, when we are solving $f(x) = 0$, we are finding *any and all* $x-$coordinates associated with when $y = 0$. This corresponds to the (set of) $x-$intercept(s) in the graph of f.

Whereas evaluating functions requires us to use the standard order of operations, PEMDAS, solving a function usually requires us to use the *reverse* order of operations, SADMEP, in addition to other methods, such as factoring. We do this in order to *isolate* the variable x.

I - Motivating Example(s):

Example: Given $f(x) = x^2 + 3x + 5$, find all x such that $f(x) = 5$.

$$\begin{aligned} f(x) &= x^2 + 3x + 5 && \text{Substitute 5 in for } f(x) \\ 5 &= x^2 + 3x + 5 && \text{Solve for } x \text{ by factoring} \\ 0 &= x^2 + 3x && \text{Set equal to 0} \\ 0 &= x(x+3) && \text{Factor} \\ x = 0 \text{ or } x &= -3 && \text{Our solutions} \end{aligned}$$

The above answer can be verified by checking. When we input $x = 0$ into the function, we simplify to find that $f(0) = 5$. Similarly, we see that when $x = -3$, $f(-3) = 5$.

II - Demo/Discussion Problems:

1. Given $h(x) = 4x - 1$, find all x such that $h(x) = -3$.
2. Given $g(x) = \dfrac{1}{4x-1}$, find all x such that $g(x) = 0$.
3. Given $k(x) = 2|2x - 3| + 3$, find all x such that $k(x) = 11$.
4. Given $\ell(x) = \sqrt{3 - 5x}$, find all x such that $\ell(x) = 0$
5. Given $m(x) = -\sqrt{25 - x^2}$, find all x such that $m(x) = 0$
6. Given $f(x) = x^2 - 10x + 28$, find all x such that $f(x) = 5$.
   Hint: Use either the Vertex Form or the Quadratic Formula.

III - Practice Problems:

Find $f(0)$ and solve $f(x) = 0$ for each of the given functions.

1. $f(x) = 2x - 1$
2. $f(x) = 3 - \dfrac{2x}{5}$
3. $f(x) = 2x^2 - 6$
4. $f(x) = x^2 - x - 12$
5. $f(x) = \sqrt{x + 4}$
6. $f(x) = \dfrac{1}{2}\sqrt{1 - 2x}$
7. $f(x) = \sqrt{20 - x^2}$
8. $f(x) = \dfrac{3}{4 - x}$
9. $f(x) = \dfrac{3x^2 - 12x}{4 - x^2}$

Find when $f(x) = 2$ for each of the functions above.

Lesson 32: Function Arithmetic

Objective: Add, subtract, multiply, and divide functions.

Students will be able to:

- Evaluate functions that are added, subtracted, multiplied, or divided by substituting a value into each function, then applying the operation and simplifying.
- Apply the four basic operations to functions of the same variable.

Prerequisite Knowledge:

- Order of operations.
- Evaluating functions.
- Parentheses and grouping.

Lesson:

The notation for the four basic function operations is as follows.

$$\begin{array}{rl}
\text{Addition} & (f+g)(x) = f(x) + g(x) \\
\text{Subtraction} & (f-g)(x) = f(x) - g(x) \\
\text{Multiplication} & (f \cdot g)(x) = f(x) \cdot g(x) \\
\text{Division} & \left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}
\end{array}$$

I - Motivating Example(s):

Example: Find $f + g$, where $f(x) = x^2 - x - 2$ and $g(x) = x + 1$.

$$\begin{array}{rl}
(f+g)(x) & \text{Consider the problem} \\
f(x) + g(x) & \text{Rewrite as a sum of two functions} \\
(x^2 - x - 2) + (x + 1) & \text{Substitute functions, inserting parentheses} \\
x^2 - x - 2 + x + 1 & \text{Simplify; remove the parentheses} \\
x^2 - x + x - 2 + 1 & \text{Combine like terms} \\
(f+g)(x) = x^2 - 1 & \text{Our solution} \\
= (x-1)(x+1) & \text{Our solution in factored form}
\end{array}$$

Generally, either form (expanded or factored) would be considered acceptable.

Example: Find $g - f$, where $f(x) = x^2 - x - 2$ and $g(x) = x + 1$.

$(g-f)(x)$	Consider the problem
$g(x) - f(x)$	Rewrite as a difference of two functions
$(x+1) - (x^2 - x - 2)$	Substitute functions, inserting parentheses
$x + 1 - x^2 + x + 2$	Simplify; distribute the negative sign
$-x^2 + x + x + 1 + 2$	Combine like terms
$(g-f)(x) = -x^2 + 2x + 3$	Our solution
$= -(x-3)(x+1)$	Our solution in factored form

Example: Find $h \cdot k$, where $h(x) = 3x^2 - 4x$ and $k(x) = x - 2$.

$(h \cdot k)(x)$	Consider the problem
$h(x) \cdot k(x)$	Rewrite as a product of two functions
$(3x^2 - 4x)(x - 2)$	Substitute functions, inserting parentheses
$3x^3 - 6x^2 - 4x^2 + 8x$	Expand by distributing
$3x^3 - 10x^2 + 8x$	Combine like terms
$(h \cdot k)(x) = 3x^3 - 10x^2 + 8x$	Our solution
$= x(3x-4)(x-2)$	Our solution in factored form

Example: Find $\frac{g}{f}$, where $f(x) = x^2 - x - 2$ and $g(x) = x + 1$.

$\left(\frac{g}{f}\right)(x)$	Consider the problem
$\frac{g(x)}{f(x)}$	Rewrite as a quotient of two functions
$\frac{x+1}{x^2 - x - 2}$	Substitute functions, parentheses unnecessary
$\frac{x+1}{(x+1)(x-2)}$	Factor (if possible)
$x \neq -1 \quad \text{and} \quad x \neq 2$	Restrict denominator: $g(x) \neq 0$
$\frac{\cancel{x+1}}{\cancel{(x+1)}(x-2)}$	Simplify: reduce $\frac{x+1}{x+1}$
$\left(\frac{g}{f}\right)(x) = \frac{1}{x-2}, \quad x \neq -1$	Our solution with added restriction

II - Demo/Discussion Problems:

Use $h(x) = 2x - 4$ and $k(x) = -3x + 1$ to find each of the following values for $x = 3$ in two ways:

a. Evaluate both h and k at $x = 3$, then combine and simplify the two values accordingly.
b. Find a simplified expression for the desired combined function, then evaluate it at $x = 3$.

1. $(h + k)(3)$
2. $(h \cdot k)(3)$
3. $(h - k)(3)$
4. $(k - h)(3)$
5. $\left(\dfrac{h}{k}\right)(3)$
6. $\left(\dfrac{k}{h}\right)(3)$

III - Practice Problems:

In each problem, use the pair of functions f and g to find the following values, if they exist.

- $(f + g)(2)$
- $(f - g)(-1)$
- $(g - f)(1)$
- $(fg)\left(\dfrac{1}{2}\right)$
- $\left(\dfrac{f}{g}\right)(0)$
- $\left(\dfrac{g}{f}\right)(-2)$

1. $f(x) = 3x + 1 \quad g(x) = 4 - x$
2. $f(x) = x^2 \quad g(x) = -2x + 1$
3. $f(x) = x^2 - x \quad g(x) = 12 - x^2$
4. $f(x) = 2x^3 \quad g(x) = -x^2 - 2x - 3$
5. $f(x) = \sqrt{x + 3} \quad g(x) = 2x - 1$
6. $f(x) = \sqrt{4 - x} \quad g(x) = \sqrt{x + 2}$
7. $f(x) = 2x \quad g(x) = \dfrac{1}{2x + 1}$
8. $f(x) = x^2 \quad g(x) = \dfrac{3}{2x - 3}$
9. $f(x) = x^2 \quad g(x) = \dfrac{1}{x^2}$
10. $f(x) = x^2 + 1 \quad g(x) = \dfrac{1}{x^2 + 1}$

In each problem, use the pair of functions f and g to find the domain of the indicated function then find and simplify an expression for it.

- $(f + g)(x)$
- $(f - g)(x)$
- $(fg)(x)$
- $\left(\dfrac{f}{g}\right)(x)$

11. $f(x) = 2x + 1 \quad g(x) = x - 2$
12. $f(x) = 1 - 4x \quad g(x) = 2x - 1$
13. $f(x) = x^2 \quad g(x) = 3x - 1$
14. $f(x) = x^2 - x \quad g(x) = 7x$
15. $f(x) = x^2 - 4 \quad g(x) = 3x + 6$
16. $f(x) = -x^2 + x + 6 \quad g(x) = x^2 - 9$
17. $f(x) = \dfrac{x}{2} \quad g(x) = \dfrac{2}{x}$
18. $f(x) = x - 1 \quad g(x) = \dfrac{1}{x - 1}$
19. $f(x) = x \quad g(x) = \sqrt{x + 1}$
20. $f(x) = g(x) = \sqrt{x - 5}$

Lesson 33: Composite Functions

Objective: Construct, evaluate, and interpret composite functions.

Students will be able to:

- Evaluate composite functions for at a specified value, $x = c$.
- Find and simplify composite functions in terms of a variable.

Prerequisite Knowledge:

- Order of operations.
- Evaluating Functions.
- Function Arithmetic.

Lesson:

In addition to the four basic arithmetic operations $(+, -, \cdot, \div)$, we will now discuss a fifth operation, known as a *composition* and denoted by $\circ$ (not to be confused with a product, $\cdot$). The result of a composition is called a *composite function* and is defined as follows.

$$(f \circ g)(x) = f(g(x))$$

The notation $(f \circ g)(x)$ above should always be interpreted as "f of g of x". In this situation, we consider g to be the *inner* function, since it is being substituted into f for x. Consequently, we refer to f as the *outer* function.

Similarly, if we reversed the order of the two functions f and g, then the resulting composite function $(g \circ f)(x) = g(f(x))$ will have inner function f and outer function g, and should be interpreted as "g of f of x". As we will see, one should never assume that the two composite functions $f \circ g$ and $g \circ f$ will be equal.

We will begin by evaluating a composite function at a single value. This is accomplished by first evaluating the inner function at the specified value, and then substituting ("plugging in") the corresponding *output* into the outer function.

We can also identify a composite function in terms of the variable. In our second example, we will substitute the inner function into the outer function for every instance of the variable and then simplify. This approach is sometimes referred to as the "inside-out" approach.

It is important to note that $(f \circ g)(x)$ usually will *not* equal $(g \circ f)(x)$, as our first Demo/Discussion problem will show.

I - Motivating Example(s):

Example: Find $(f \circ g)(3)$, where $f(x) = x^2 - 2x + 1$ and $g(x) = x - 5$.

$$\begin{aligned}
(f \circ g)(3) &= f(g(3)) && \text{Rewrite } f \circ g \text{ as inner and outer functions} \\
\\
g(3) &= (3) - 5 = -2 && \text{Evaluate inner function at } x = 3 \\
&&& \text{Use output of } -2 \text{ as input for } f \\
f(-2) &= (-2)^2 - 2(-2) + 1 && \text{Evaluate outer function at } x = -2 \\
&= 4 + 4 + 1 && \text{Simplify} \\
(f \circ g)(3) &= 9 && \text{Our solution}
\end{aligned}$$

Example: Find $(f \circ g)(x)$, where $f(x) = x^2 - x$ and $g(x) = x + 3$.

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) && \text{Rewrite } f \circ g \text{ as inner and outer functions} \\
&&& \text{Our inner function is } g(x) = x + 3 \\
& f(x+3) && \text{Replace each } x \text{ in } f \text{ with } (x+3) \\
&&& \text{Make sure to include parentheses!} \\
& (x+3)^2 - (x+3) && \text{Simplify; expand binomial} \\
& (x^2 + 6x + 9) - (x+3) && \text{Distribute negative} \\
& x^2 + 6x + 9 - x - 3 && \text{Combine like terms} \\
\\
(f \circ g)(x) &= x^2 + 5x + 6 && \text{Our solution} \\
&= (x+3)(x+2) && \text{in factored form}
\end{aligned}$$

II - Demo/Discussion Problems:

Find each of the following.

1. $(g \circ f)(x)$, where $f(x) = x^2 - x$ and $g(x) = x + 3$
2. $(g \circ g)(x)$, where $g(x) = x^2 - 2x$

In each problem below, use the given pair of functions to find the following six composite values if they exist.

- $(g \circ f)(0)$
- $(g \circ f)(-3)$
- $(f \circ g)(-1)$
- $(f \circ g)\left(\frac{1}{2}\right)$
- $(f \circ f)(2)$
- $(f \circ f)(-2)$

3. $f(x) = 4 - x, \quad g(x) = 1 - x^2$
4. $f(x) = |x - 1|, \quad g(x) = x^2 - 5$
5. $f(x) = 4x + 5, \quad g(x) = \sqrt{x}$
6. $f(x) = \dfrac{3}{1 - x}, \quad g(x) = \dfrac{4x}{x^2 + 1}$

Use the given pair of functions to find and simplify expressions for the following three composite functions. Then state the domain of each using interval notation.

- $(g \circ f)(x)$
- $(f \circ g)(x)$
- $(f \circ f)(x)$

7. $f(x) = 3x - 5, \quad g(x) = \sqrt{x}$
8. $f(x) = 3 - x^2, \quad g(x) = \sqrt{x+1}$
9. $f(x) = 3x - 1, \quad g(x) = \dfrac{1}{x+3}$
10. $f(x) = \dfrac{2x}{x^2 - 4}, \quad g(x) = \sqrt{1-x}$

Write each of the given functions as a composition of two or more non-identity functions. (There are several correct answers, so check your answer using function composition.)

11. $p(x) = (2x+3)^3$
12. $H(x) = |7 - 3x|$
13. $Q(x) = \dfrac{2x^3 + 1}{x^3 - 1}$
14. $w(x) = \dfrac{x^2}{x^4 + 1}$

III - Practice Problems:

In each problem below, use the given pair of functions to find the following six composite values if they exist.

- $(g \circ f)(0)$
- $(f \circ g)(-1)$
- $(f \circ f)(2)$
- $(g \circ f)(-3)$
- $(f \circ g)\left(\frac{1}{2}\right)$
- $(f \circ f)(-2)$

1. $f(x) = x^2, \quad g(x) = 2x + 1$
2. $f(x) = 4 - 3x, \quad g(x) = |x|$
3. $f(x) = \sqrt{3-x}, \quad g(x) = x^2 + 1$
4. $f(x) = \dfrac{x}{x+5}, \quad g(x) = \dfrac{2}{7 - x^2}$

Use the given pair of functions to find and simplify expressions for the following three composite functions. Then state the domain of each using interval notation.

- $(g \circ f)(x)$
- $(f \circ g)(x)$
- $(f \circ f)(x)$

5. $f(x) = 2x + 3, \quad g(x) = x^2 - 9$
6. $f(x) = x^2 - x + 1, \quad g(x) = 3x - 5$
7. $f(x) = x^2 - 4, \quad g(x) = |x|$
8. $f(x) = |x + 1|, \quad g(x) = \sqrt{x}$
9. $f(x) = |x|, \quad g(x) = \sqrt{4-x}$
10. $f(x) = x^2 - x - 1, \quad g(x) = \sqrt{x-5}$

11. $f(x) = \dfrac{3x}{x-1}, \quad g(x) = \dfrac{x}{x-3}$

12. $f(x) = \dfrac{x}{2x+1}, \quad g(x) = \dfrac{2x+1}{x}$

Write each of the given functions as a composition of two or more non-identity functions. (There are several correct answers, so check your answer using function composition.)

13. $P(x) = (x^2 - x + 1)^5$

14. $h(x) = \sqrt{2x-1}$

15. $r(x) = \dfrac{2}{5x+1}$

16. $R(x) = \dfrac{7}{x^2-1}$

17. $q(x) = \dfrac{|x|+1}{|x|-1}$

18. $v(x) = \dfrac{2x+1}{3-4x}$

19. Let $g(x) = -x$, $h(x) = x+2$, $j(x) = 3x$ and $k(x) = x-4$. In what order must these functions be composed with $f(x) = \sqrt{x}$ to create $F(x) = 3\sqrt{-x+2} - 4$?

20. What linear functions could be used to transform $f(x) = x^3$ into $F(x) = -\frac{1}{2}(2x-7)^3 + 1$? What is the proper order of composition?

Let f be the function defined by

$$f = \{(-3, 4), (-2, 2), (-1, 0), (0, 1), (1, 3), (2, 4), (3, -1)\}$$

and let g be the function defined

$$g = \{(-3, -2), (-2, 0), (-1, -4), (0, 0), (1, -3), (2, 1), (3, 2)\}$$

. Find each composite value if it exists.

21. $(f \circ g)(3)$

22. $f(g(-1))$

23. $(f \circ f)(0)$

24. $(f \circ g)(-3)$

25. $(g \circ f)(3)$

26. $g(f(-3))$

27. $(g \circ g)(-2)$

28. $(g \circ f)(-2)$

29. $g(f(g(0)))$

Lesson 34: Inverse Functions - Definition and the HLT

Objective: Understand the definition of an inverse function and graphical implications. Determine whether a function is invertible.

Students will be able to:

- Define an inverse function.
- Obtain the graph of an inverse function from the graph of a function.
- Use the Horizontal Line Test (HLT) to determine if a function is invertible.

Prerequisite Knowledge:

- Plotting points on the Cartesian plane.
- Graphing horizontal lines.
- Graph functions by creating a table.

Lesson:

One often considers the operations of addition and subtraction to be "opposites" of one another, and similarly for multiplication and division. The reason for this, naturally, is because each of these operations "undoes" the other. In mathematics, since the term "opposite" can take on different meanings, we instead consider addition and subtraction (or multiplication and division) to be *inverse operations* of one another. This notion of an inverse can be applied to entire functions, which we will now discuss.

We start by analyzing a very basic function which is reversible, a linear function. Consider the function $f(x) = 3x + 4$. Thinking of f as a process, we start with an input x and apply two steps, in order:

1. multiply by 3
2. add 4.

To reverse this process, we seek a function g which will undo each of these steps, by taking the output from f, $3x + 4$, and returning the original input x. If we think of the real-world reversible two-step process of first putting on socks then putting on shoes, to reverse the process, we first take off the shoes, and then we take off the socks. In much the same way, the function g should undo the last step of f first. That is, the function g should:

1. subtract 4, then
2. divide by 3.

Following this procedure, we get $g(x) = \dfrac{x-4}{3}$.

Now we can test our function to see if it conceptually agrees with our "feet, socks, and shoes" analogy. Just as in the first part of the process we began with our bare feet and ended up in shoes, the reverse process brings us back, in the end, to our bare feet. We can see if this holds for f and g by using what we already know about functions.

For example, if $x = 5$, then

$$f(5) = 3(5) + 4 = 15 + 4 = 19.$$

Substituting the output 19 from f as our new input for g, we get our original input for f.

$$g(19) = \frac{19-4}{3} = \frac{15}{3} = 5$$

To check that g does this for all x in the domain of f (not just a single value), we will need to find and simplify the composite function $(g \circ f)(x) = g(f(x))$.

$$g(f(x)) = g(3x+4) = \frac{(3x+\cancel{4}) - \cancel{4}}{3} = \frac{\cancel{3}x}{\cancel{3}} = x$$

Not only does g "undo" f, but f also undoes g, which we can verify by once again looking at a composite function. This time we will find and simplify $(f \circ g)(x) = f(g(x))$.

$$f(g(x)) = f\left(\frac{x-4}{3}\right) = \cancel{3}\left(\frac{x-4}{\cancel{3}}\right) + 4 = (x - \cancel{4}) + \cancel{4} = x$$

Two functions f and g which are related in this manner are defined to be *inverse functions*, or simply *inverses*, of each other. More precisely, using the language of function composition, two functions f and g are said to be inverses if both:

- $g(f(x)) = x$ for all x in the domain of f, and
- $f(g(x)) = x$ for all x in the domain of g.

We say that a function f is *invertible* if an inverse function of f exists. If two functions g and f are inverses of each other, then we denote this by $g(x) = f^{-1}(x)$, and similarly $f(x) = g^{-1}(x)$. This notation can be a bit "gnarly" at first, since an inverse function f^{-1} of f must not be confused with the reciprocal function, $1/f$. The primary difference between these two functions is that a reciprocal function satisfies the property that

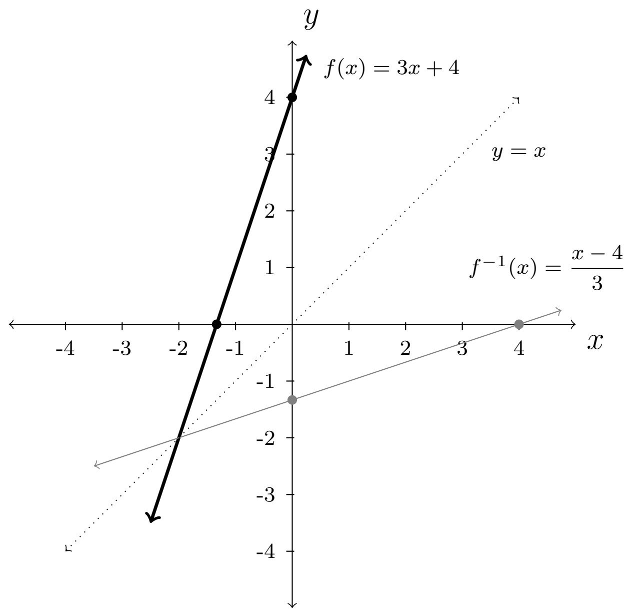
$$f(x) \cdot (1/f)(x) = 1,$$

whereas for inverses,

$$(f \circ f^{-1})(x) = x \quad \text{and} \quad (f^{-1} \circ f)(x) = x.$$

Properties of Inverse Functions:
Let f and f^{-1} be inverse functions of one another.

- The range of f is the domain of f^{-1} and the domain of f is the range of f^{-1}.
- $f(a) = b$ if and only if $f^{-1}(b) = a$.
- The point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}.



Graphically, we can identify one-to-one functions using the following test.

The Horizontal Line Test (HLT):
A function f is one-to-one if and only if no horizontal line intersects the graph of f more than once.

We say that the graph of a function **passes** the Horizontal Line Test if no horizontal line intersects the graph more than once; otherwise, we say the graph of the function **fails** the Horizontal Line Test.

Lastly, we have argued that if f is invertible, then f must be one-to-one, since otherwise the reflection of the graph of $y = f(x)$ about the line $y = x$ will fail the Vertical Line Test. It turns out that being one-to-one is also enough to guarantee invertibility of a function f. To see this, we can think of f as the set of ordered pairs which constitute its graph. If switching the x- and y-coordinates of the points results in a function (i.e., passes the VLT), then f is invertible and we have found the graph of its inverse, f^{-1}. This is precisely what the Horizontal Line Test does for us: it checks to see whether or not a set of points describes x as a function of y.

We can now summarize our results.

Equivalent Conditions for Invertibility:
Suppose f is a function. The following statements are equivalent.

- f is invertible (f^{-1} exists).
- f is one-to-one.
- The graph of f passes the Horizontal Line Test.

II - Demo/Discussion Problems:

Graph each function. Use the HLT to determine if the given function is invertible. If so, graph both the function and its inverse on the same set of axes. Identify at least three reference points for your function and its inverse.

1. $f(x) = 4x - 3$
2. $g(x) = \frac{1}{2}|x|$
3. $h(x) = (x+1)^2 - 4$
4. $k(x) = x^3$
5. $\ell(x) = \sqrt{x} - 5$
6. $m(x) = \dfrac{1}{x-2}$
7. $n(x) = x^2 - 10x,\ x \geq 5$
8. $p(x) = 3(x+4)^2 - 5,\ x \leq -4$

III - Practice Problems:

Graph each function in Desmos. Use the HLT to determine if the given function is invertible. If so, graph both the function and its inverse on the same set of axes. Identify at least three reference points for your function and its inverse.

1. $f(x) = 2 - 6x$
2. $f(x) = \dfrac{x-2}{3} + 4$
3. $f(x) = 1 - \dfrac{4+3x}{5}$
4. $f(x) = \sqrt{3x-1} + 5$
5. $f(x) = 2 - \sqrt{x-5}$
6. $f(x) = 3\sqrt{x-1} - 4$
7. $f(x) = 1 - 2\sqrt{2x+5}$
8. $f(x) = \sqrt[3]{3x-1}$
9. $f(x) = 3 - \sqrt[3]{x-2}$
10. $f(x) = x^2 - 6x + 5,\ x \leq 3$
11. $f(x) = 4x^2 + 4x + 1,\ x < -1$
12. $f(x) = \dfrac{3}{4-x}$
13. $f(x) = \dfrac{x}{1-3x}$
14. $f(x) = \dfrac{2x-1}{3x+4}$
15. $f(x) = \dfrac{4x+2}{3x-6}$
16. $f(x) = \dfrac{-3x-2}{x+3}$
17. $f(x) = \dfrac{x-2}{2x-1}$

Lesson 35: Inverse Functions - Finding an Inverse Function, f^{-1}

Objective: Find the inverse of a given function.

Students will be able to:

- Find the inverse of a function algebraically.

Prerequisite Knowledge:

- Definition and properties of inverse function.

Lesson:

Steps for finding the Inverse of a Function

1. Rewrite $f(x)$ as y.
2. Switch x and y.
3. Solve for y.
4. Rewrite y as $f^{-1}(x)$.

I - Motivating Example(s):

Example: Find the inverse f^{-1} of the function $f(x) = \dfrac{1-2x}{5}$.

We replace $f(x)$ with y and proceed to switch x and y

$$\begin{aligned}
y &= \frac{1-2x}{5} && \\
x &= \frac{1-2y}{5} && \text{Switch } x \text{ and } y \\
5x &= 1-2y && \text{Solve for } y \\
5x-1 &= -2y && \\
\frac{5x-1}{-2} &= y && \\
y &= -\frac{5}{2}x + \frac{1}{2} &&
\end{aligned}$$

We have $f^{-1}(x) = -\dfrac{5}{2}x + \dfrac{1}{2}$.

To verify this answer, we leave it as an exercise to the reader to check that $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f, and $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}. Note that since f and f^{-1} are both linear functions, the domain and range for each function is $(-\infty, \infty)$.

Example: Find the inverse g^{-1} of the function $g(x) = \dfrac{2x}{1-x}$.

Notice that the domain of g is $(-\infty, 1) \cup (1, \infty)$. One can verify graphically, that the range of g is $(-\infty, -2) \cup (-2, \infty)$.

To find $g^{-1}(x)$, we start by replacing $g(x)$ with y.

$$
\begin{array}{rcll}
y & = & \dfrac{2x}{1-x} & \\
x & = & \dfrac{2y}{1-y} & \text{Switch } x \text{ and } y \\
x(1-y) & = & 2y & \text{Solve for } y\text{; clear denominator} \\
x - xy & = & 2y & \text{Distribute } x \\
x & = & xy + 2y & \text{Move } y \text{ terms to one side} \\
x & = & y(x+2) & \text{Factor out } y \\
y & = & \dfrac{x}{x+2} & \text{Divide by } x+2
\end{array}
$$

We have $g^{-1}(x) = \dfrac{x}{x+2}$.

Notice that the domain of g^{-1} matches the range of g from earlier, $(-\infty, -2) \cup (-2, \infty)$. Again, we can use the graph of g^{-1} to verify that the range of g^{-1} also matches the domain of g, $(-\infty, 1) \cup (1, \infty)$.

We leave it as an exercise to show that $\left(g^{-1} \circ g\right)(x) = x$ and $\left(g \circ g^{-1}\right)(x) = x$.

II - Demo/Discussion Problems:

Find the inverse function for each of the following functions. Graph both the original function and your answer using Desmos to confirm your results and compare the domains and ranges for your pair of functions.

1. $f(x) = 3\sqrt{x} + 4$
2. $g(x) = 2\sqrt{x-1} - 4$
3. $h(x) = -x^2 - 1$, $x \geq 0$
4. $k(x) = -x^2 - 1$, $x \leq 0$
5. $\ell(x) = x^2 + 10x + 15$, where $x \geq -5$
6. $m(x) = x^2 + 10x + 15$, where $x \leq -5$

III - Practice Problems:

Find the inverse function for each of the following functions. Check your answer algebraically by finding $f \circ f^{-1}$ and $f^{-1} \circ f$. Graph both the original function and your answer using Desmos to confirm your results and compare the domains and ranges for your pair of functions.

1. $f(x) = 2 - 6x$
2. $f(x) = \dfrac{x-2}{3} + 4$
3. $f(x) = 1 - \dfrac{4+3x}{5}$
4. $f(x) = \sqrt{3x-1} + 5$
5. $f(x) = 2 - \sqrt{x-5}$
6. $f(x) = 3\sqrt{x-1} - 4$
7. $f(x) = 1 - 2\sqrt{2x+5}$
8. $f(x) = \sqrt[3]{3x-1}$
9. $f(x) = 3 - \sqrt[3]{x-2}$
10. $f(x) = 8(x-2)^3$
11. $f(x) = (x+3)^2 - 6,\ x \geq -3$
12. $f(x) = 2(x-1)^2 + 4,\ x < 1$
13. $f(x) = x^2 - 6x + 5,\ x \leq 3$
14. $f(x) = 4x^2 + 4x + 1,\ x < -1$
15. $f(x) = \dfrac{3}{4-x}$
16. $f(x) = \dfrac{x}{1-3x}$
17. $f(x) = \dfrac{2x-1}{3x+4}$
18. $f(x) = \dfrac{4x+2}{3x-6}$
19. $f(x) = \dfrac{-3x-2}{x+3}$
20. $f(x) = \dfrac{x-2}{2x-1}$

Lessons 36-39 cover *transformations* of functions. These consist of translations (or “shifts”), reflections, and scalings, which include both compressions and expansions (or “shrinks” and “stretches”). The following in-class activity handout will be used to treat these lessons.

Transformations Summary Worksheet

The following is a list of functions and their respective **transformation** of the graph of a function $f(x)$ (assume $k > 0$). For each example, use $f(x) = \sqrt{x}$ to help sketch the graph of the given example. Then identify the domain, range, and any intercepts.

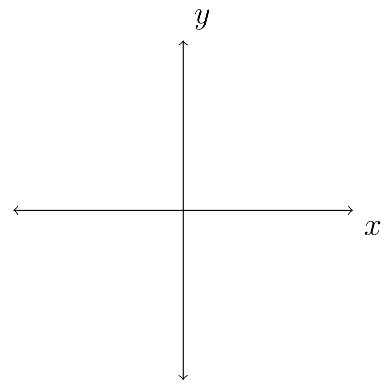
$f(x) = \sqrt{x}$

Domain: $x \geq 0$ or $[0, \infty)$

Range: $y \geq 0$ or $[0, \infty)$

$x-$intercept at $(0, 0)$

$y-$intercept at $(0, 0)$



Function	Transformation	Example	Graph
$-f(x)$	Reflection about the $x-$axis $(y = 0)$	$-\sqrt{x}$	
$f(-x)$	Reflection about the $y-$axis $(x = 0)$	$\sqrt{-x}$	

Function	Transformation	Example	Graph
$f(x) + k$	Vertical shift up k units	$\sqrt{x} + 3$	
$f(x) - k$	Vertical shift down k units	$\sqrt{x} - 3$	

Function	Transformation	Example	Graph
$f(x-k)$	Horizontal shift right k units	$\sqrt{x-3}$	
$f(x+k)$	Horizontal shift left k units	$\sqrt{x+3}$	

Function	Transformation	Example	Graph
$kf(x)$, $k > 1$	Vertical stretch by a factor of k	$3\sqrt{x}$	
$kf(x)$, $0 < k < 1$	Vertical shrink by a factor of $\frac{1}{k}$	$\dfrac{\sqrt{x}}{3}$	

Function	Transformation	Example	Graph
$f(kx)$, $k>1$	Horizontal shrink by a factor of k	$\sqrt{3x}$	
$f\left(kx\right)$, $0<k<1$	Horizontal stretch by a factor of $\frac{1}{k}$	$\sqrt{\dfrac{x}{3}}$	

Lesson 40: Piecewise-Defined Functions

CC attribute: *College Algebra* by C. Stitz and J. Zeager.

Objective: Define, evaluate, and solve piecewise functions.

Students will be able to:

- Correctly evaluate a piecewise function.
- Graph piecewise functions.
- Solve piecewise functions, while taking domain restrictions into account.

Prerequisite Knowledge:

- Evaluating functions.
- Understanding domain both graphically and algebraically.
- Graphing by making a table.

Lesson:

A *piecewise-defined* (or simply, a *piecewise*) function is a function that is defined in pieces. More precisely, a piecewise-defined function is a function that is presented using one or more expressions, each defined over non-intersecting intervals. An example of a piecewise-defined function is shown below.

$$f(x) = \begin{cases} 2x - 1 & \text{if } x > 0 \\ x^2 - 1 & \text{if } x \leq 0 \end{cases}$$

To evaluate a piecewise-defined function at a particular value of the variable, we must first compare our value to the various intervals (or domains) applied to each piece, and then substitute our value into the piece that coincides with the correct domain. For example, since $x = 1$ is greater than zero, we would use the expression $2x - 1$ to evaluate $f(1)$,

$$f(1) = 2(1) - 1 = 2 - 1 = 1.$$

Similarly, since $x = -1$ is less than zero, we would use the expression $x^2 - 1$ to evaluate $f(-1)$,

$$f(-1) = (-1)^2 - 1 = 1 - 1 = 0.$$

Next, we address the issue of solving a piecewise function. For some constant k, to find all x such that $f(x) = k$, we will use the strategy outlined below, which will be the same for any piecewise-defined function.

- Set each separate piece equal to k and solve for x.
- Compare your answers for x to the domain applied to each piece. Only keep those solutions that coincide with the specified domain.

I - Motivating Example(s):

Example: Consider the piecewise function

$$f(x) \;=\; \begin{cases} 2x-1 & \text{if } \; x>0 \\ x^2-1 & \text{if } \; x\leq 0 \end{cases}$$

Below is a table of points obtained from f.

x	$f(x)$
2	$2(2)-1=3$
1	$2(1)-1=1$
0	$(0)^2-1=-1$
-1	$(-1)^2-1=0$
-2	$(-2)^2-1=3$

We have included an extra line between the values of $x=0$ and $x=1$ in the table above, in order to emphasize the changeover from one piece of our function $(2x-1)$ to another (x^2-1).

The value of $x=0$ is very important, since it is an endpoint for the two domains of our function, $(0,\infty)$ and $(-\infty,0]$.

A common misconception among students is to evaluate $f(0)$ at both $2x-1$ and x^2-1 because it seems to "straddle" both individual domains. And although the values for both pieces are equal at $x=0$,

$$2(0)-1=-1=0^2-1$$

this will often not be the case. Regardless, we must be careful to *always* associate $x=0$ with x^2-1, since it is contained in our second piece's domain $(0\leq 0)$ and not in our first.

Example: Find the set of all zeros of

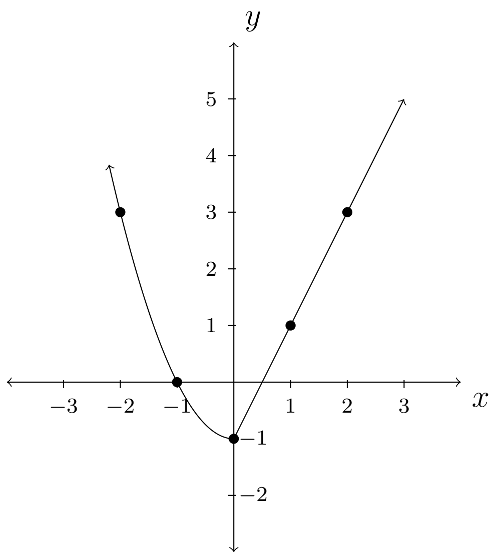
$$f(x) \;=\; \begin{cases} 2x-1 & \text{if } \; x>0 \\ x^2-1 & \text{if } \; x\leq 0 \end{cases}$$

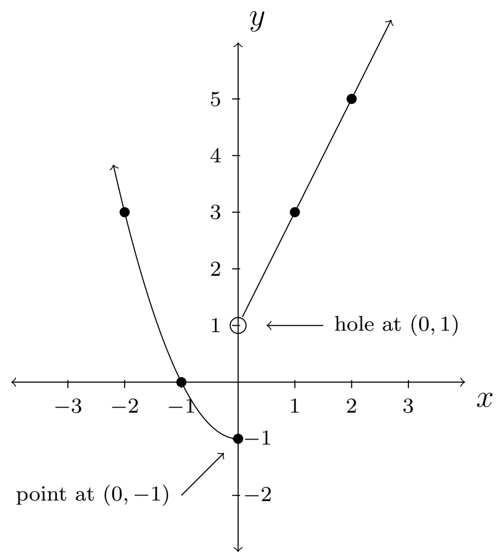
.

$f(x)=0$	Apply to each piece separately
$2x-1=0,\; x>0$	First piece; solve for x
$x=\frac{1}{2},\; x>0$	One solution; coincides with domain
$x^2-1=0\;,\quad x\leq 0$	Second piece; solve for x
$(x-1)(x+1)=0,\; x\leq 0$	Solve by factoring
$x=\pm 1,\; x\leq 0$	Two potential solutions
$x=-1,\; x\leq 0$	Exclude $x=1$; does not coincide with domain
$f(x)=0$ when $x=-1,\;\frac{1}{2}$	Our answer

Example: Below are the graphs of piecewise functions f and g.

$$f(x) = \begin{cases} 2x - 1 & \text{if } x > 0 \\ x^2 - 1 & \text{if } x \leq 0 \end{cases} \qquad g(x) = \begin{cases} 2x + 1 & \text{if } x > 0 \\ x^2 - 1 & \text{if } x \leq 0 \end{cases}$$





In this example, we see that both pieces for $g(x)$ do not "match up", since the values we obtain for both pieces at $x = 0$ do not agree:

$$g(0) = 0^2 - 1 = -1, \text{ but } 2(0) + 1 = 1.$$

Remember that when evaluating any function at a value of x in its domain, we should always only have a *single value* for $g(x)$, since this is how we defined a function. Furthermore, if we were to associate two values ($g(0) = \pm 1$) to $x = 0$, our graph would consequently contain points at $(0, -1)$ and $(0, 1)$, and therefore fail the Vertical Line Test.

When we consider the graphs of both f and g, since both pieces of f seem to "match up" at $x = 0$, we will see that the graph of f will be one *continuous* graph, with no breaks or separations appearing. On the other hand, since both pieces of g do not "match up" at $x = 0$, we will see that the graph of g will contain a break at $x = 0$, known as a *discontinuity* in the graph. It is important that our graph also identifies the precise location of the ends of each piece.

II - Demo/Discussion Problems:

1. Find the set of all zeros of $g(x) = \begin{cases} 2x + 1 & \text{if } x > 0 \\ x^2 - 1 & \text{if } x \leq 0 \end{cases}$.

2. Make a table of values and graph the function $h(x) = \begin{cases} 3 & \text{if } x > 0 \\ 1 & \text{if } x = 0 \\ x & \text{if } x < 0 \end{cases}$.

3. Use the piecewise function below to complete the following.

$$f(x) \;=\; \begin{cases} \frac{x}{2}-2 & \text{if } \; x \geq 3 \\ -2x^2+x+1 & \text{if } \; -2 < x < 2 \\ x-8 & \text{if } \; x \leq -2 \end{cases}$$

(a) Make a table of values for f.
(b) Solve when $f(x) = 0$.
(c) Sketch a complete graph of f, making sure to identify both the $x-$ and $y-$coordinates for the ends of each piece of your graph.

III - Practice Problems:

1. Let $f(x) = \begin{cases} x+5 & \text{if } \; x \leq -3 \\ \sqrt{9-x^2} & \text{if } \; -3 < x \leq 3 \\ -x+5 & \text{if } \; x > 3 \end{cases}$.

   Compute the following function values.

   (a) $f(-4)$
   (b) $f(-3)$
   (c) $f(-3.01)$
   (d) $f(2)$
   (e) $f(3)$
   (f) $f(3.1)$

2. Let $f(x) = \begin{cases} x^2 & \text{if } \; x \leq -1 \\ \sqrt{1-x^2} & \text{if } \; -1 < x \leq 1 \\ x & \text{if } \; x > 1 \end{cases}$.

   Compute the following function values.

   (a) $f(4)$
   (b) $f(-3)$
   (c) $f(1)$
   (d) $f(0)$
   (e) $f(-1)$
   (f) $f(-0.99)$

For each of the following items, find all possible x such that $f(x) = 0$. Then sketch the graph of the given piecewise-defined function. Check your answer using Desmos. Use your graph to identify the domain and range of each function.

3. $f(x) = \begin{cases} 4-x & \text{if } \; x \leq 3 \\ 2 & \text{if } \; x > 3 \end{cases}$

4. $f(x) = \begin{cases} x^2 & \text{if } \; x \leq 0 \\ 2x & \text{if } \; x > 0 \end{cases}$

5. $f(x) = \begin{cases} -2x-4 & \text{if } \; x < 0 \\ 3x & \text{if } \; x \geq 0 \end{cases}$

6. $f(x) = \begin{cases} x-2 & \text{if } \; x \geq -2 \\ -x-6 & \text{if } \; x < -2 \end{cases}$

7. $f(x) = \begin{cases} -3 & \text{if } x < 0 \\ 2x - 3 & \text{if } 0 \leq x \leq 3 \\ 3 & \text{if } x > 3 \end{cases}$

8. $f(x) = \begin{cases} x^2 - 4 & \text{if } x \leq -2 \\ 4 - x^2 & \text{if } -2 < x < 2 \\ x^2 - 4 & \text{if } x \geq 2 \end{cases}$

9. $f(x) = \begin{cases} x^2 & \text{if } x \leq -2 \\ 3 - x & \text{if } -2 < x < 2 \\ 4 & \text{if } x \geq 2 \end{cases}$

10. $f(x) = \begin{cases} \dfrac{1}{x} & \text{if } -6 < x < -1 \\ x & \text{if } -1 < x < 1 \\ \sqrt{x} & \text{if } 1 < x < 9 \end{cases}$

Lesson 41: Functions Containing Absolute Values

Objective: Graph a variety of functions that contain an absolute value.

Students will be able to:

- Create a table of values for a function containing an absolute value.
- Identify the intercepts of a function containing an absolute value.
- Graph a function that contains an absolute value.
- Solve equations containing an absolute value graphically.

Prerequisite Knowledge:

- Creating a table of values for a function.
- Definition of an absolute value.

Lesson:

The most basic of functions containing an absolute value is $\ell(x) = |x|$. Later, when we cover transformations of functions, we will see more general forms of such functions. In particular, if we consider the function

$$f(x) = a|x - h| + k,$$

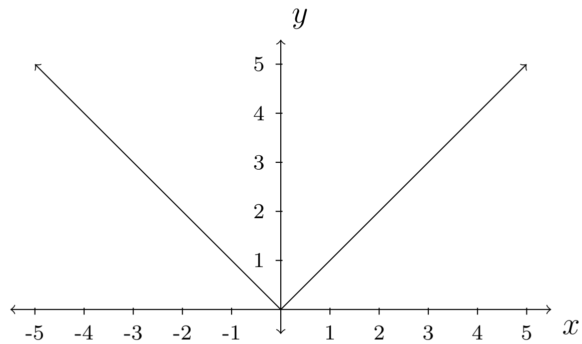
we can make some simple observations about the graph of f. For example, if $a > 0$, the graph of f will point upwards. In this case, the graph of f will have a *minimum* at $y = k$, corresponding to the point (h, k). Alternatively, if $a < 0$, the graph of f will point downwards, and the graph will achieve its *maximum* value at the point (h, k). The magnitude of the coefficient a (i.e. its absolute value) will also determine whether the graph of f is wide or narrow.

I - Motivating Example(s):

Example:

Function Type: Absolute Value
Representative: $\ell(x) = |x|$

x	$\ell(x)$
-3	3
-2	2
-1	1
0	0
1	1
2	2
3	3



Graph of $\ell(x) = |x|$

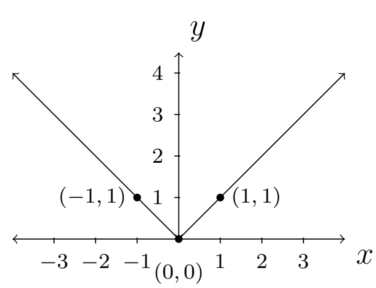
$y-$intercept: $(0, 0)$
$x-$intercept(s): $(0, 0)$

Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Notes: The domain of an absolute value function of the form $f(x) = a|x - h| + k$ will remain the same as above. If $a > 0$, the range of f will be $[k, \infty)$. If $a < 0$, the range of f will be $(-\infty, k]$.

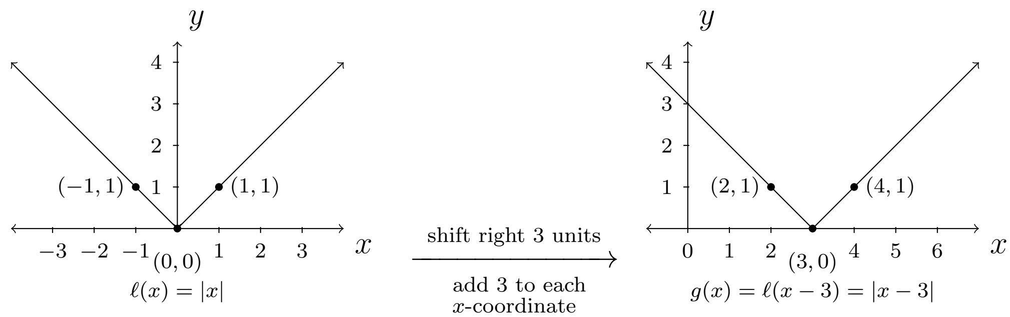
Example: Use the graph of $\ell(x) = |x|$ to graph the function $g(x) = |x - 3|$.

We begin by graphing $\ell(x) = |x|$ and labeling three reference points: $(-1, 1)$, $(0, 0)$ and $(1, 1)$.



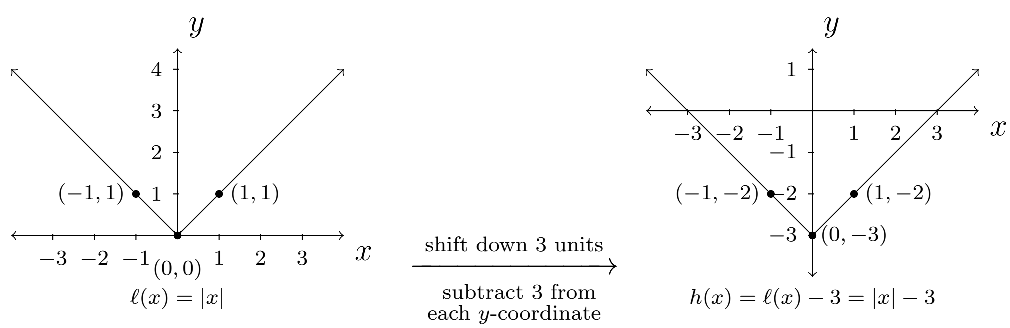
$\ell(x) = |x|$

Since $g(x) = |x - 3| = \ell(x - 3)$, we will add 3 to each of the $x-$coordinates of the points on the graph of $y = \ell(x)$ to obtain the graph of $y = g(x)$. This shifts the graph of $y = \ell(x)$ to the *right* by 3 units and moves the points $(-1,1)$ to $(2,1)$, $(0,0)$ to $(3,0)$ and $(1,1)$ to $(4,1)$. Connecting these points in the classic '$\vee$' fashion produces the graph of $y = g(x)$.



Example: Use the graph of $\ell(x) = |x|$ to graph the function $h(x) = |x| - 3$.

Since $h(x) = |x| - 3 = f(x) - 3$, we will subtract 3 from each of the $y-$coordinates of the points on the graph of $y = \ell(x)$ to obtain the graph of $y = h(x)$. This shifts the graph of $y = \ell(x)$ *down* by 3 units and moves the points $(-1,1)$ to $(-1,-2)$, $(0,0)$ to $(0,-3)$ and $(1,1)$ to $(1,-2)$. Connecting these points with the '$\vee$' shape produces our graph of $y = h(x)$.



II - Demo/Discussion Problems:

Graph each of the following functions. In each case, make a table of values. Then graph $\ell(x) = |x|$ and f on Desmos and compare the two graphs. Find the intercepts, domain, and range of f.

1. $f(x) = |x+3|$
2. $f(x) = 2|x+3|$
3. $f(x) = -2|x-3|$
4. $f(x) = \frac{1}{2}|3x+1|$
5. $f(x) = \frac{1}{2}|3x+1| - 5$
6. $f(x) = 4 - 2|3x+1|$

III - Practice Problems:

Graph each of the following functions. Make a table of values if necessary. Find the intercepts, domain, and range of the function.

1. $f(x) = -|x| - 7$
2. $g(x) = 4|x| + 2$
3. $h(x) = 5|x-3| + 6$
4. $k(x) = -2|x+1| + 10$
5. $m(x) = |x-3|$
6. $n(x) = -4|x-1| + 1$

Lesson 42: Absolute-Value as a Piecewise Function

CC attribute: *College Algebra* by C. Stitz and J. Zeager.

Objective: Interpret a function containing an absolute value as a piecewise-defined function.

Students will be able to:

- Express an absolute value function as a piecewise function.

Prerequisite Knowledge:

- Evaluate and solve absolute value expressions.
- Graphing piecewise functions.
- Finding $x-$ and $y-$intercepts.

Lesson:

By definition, we know that

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}.$$

If $m \neq 0$ and b is a real number, we may generalize the definition above as follows.

$$\begin{aligned} |mx+b| &= \begin{cases} -(mx+b) & \text{if } mx+b < 0 \\ mx+b & \text{if } mx+b \geq 0 \end{cases} \\ &= \begin{cases} -mx-b & \text{if } mx+b < 0 \\ mx+b & \text{if } mx+b \geq 0 \end{cases}. \end{aligned}$$

Notice that since we have never specified whether m is positive or negative above, it would not be wise to attempt to simplify either inequality in our new definition. Once we are given a value for m, as in our next example, we will be able to simplify our piecewise representation completely.

I - Motivating Example(s):

Example: Express $g(x) = |x-3|$ as a piecewise-defined function.

$$g(x) = |x-3| = \begin{cases} -(x-3) & \text{if } x-3 < 0 \\ x-3 & \text{if } x-3 \geq 0 \end{cases}$$

Simplifying, we get

$$g(x) = \begin{cases} -x+3 & \text{if } \ x < 3 \\ x-3 & \text{if } \ x \geq 3 \end{cases}.$$

Our piecewise answer above should begin to make sense, when one considers the graph of g as a horizontal shift of $y = |x|$ to the right by 3 units.

Example: Express $h(x) = |x| - 3$ as a piecewise-defined function.

Since the variable within the absolute value remains unchanged, the domains for each piece in our resulting function will not change. Instead, we need only subtract 3 from each piece of our answer. Thus, we get the following representation.

$$h(x) = \begin{cases} -x-3 & \text{if } \ x < 0 \\ x-3 & \text{if } \ x \geq 0 \end{cases}.$$

Similarly, this answer again seems reasonable, as the graph of $h(x) = |x| - 3$ represents a vertical shift of $y = |x|$ down by 3 units.

II - Demo/Discussion Problems:

Express each function as a piecewise-defined function and identify any $x-$ and $y-$intercepts on its graph. Determine the domain and range of of the function from its graph. Use Desmos to confirm your answers.

1. $r(x) = |x - 10|$
2. $s(x) = \frac{1}{2}|x - 10| + 5$
3. $g(x) = |3x + 1|$
4. $h(x) = -2|3x + 1|$
5. $k(x) = 4 - 2|3x + 1|$

III - Practice Problems:

Express each function below as a piecewise-defined function. Graph the function and use Desmos to confirm your answers.

1. $f(x) = -|x| - 7$
2. $g(x) = 4|x| + 2$
3. $h(x) = 5|x - 3| + 6$
4. $k(x) = -2|x + 1| + 10$
5. $m(x) = |x - 3|$
6. $n(x) = -4|x - 1| + 1$

Lesson 43: Polynomials Introduction

Objective: Identify key features of and classify a polynomial by degree and number of nonzero terms.

Students will be able to:

- Identify a polynomial from its definition, and arrange a polynomial in descending-power order.
- Identify the degree, set of coefficients, leading coefficient, leading and constant term of a polynomial.
- Classify a polynomial by both its degree and number of nonzero terms.

Prerequisite Knowledge:

- Order of operations.

Lesson:

A *polynomial* in terms of a variable x is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0,$$

where each *coefficient*, a_i, is a real number ($a_n \neq 0$) and the exponent, or *degree* of the polynomial, n, is a nonnegative integer.
Examples of polynomials include: $f(x) = x^2 + 5$, $f(x) = x$ and $f(x) = -3x^7 + 4x^3 - 5x$. For our general polynomial above, the:

degree	is	n
set of coefficients	is	$\{a_n, a_{n-1}, \ldots, a_1, a_0\}$
leading coefficient	is	a_n
leading term	is	$a_n x^n$
constant term	is	$a_0 x^0 = a_0$.

We will categorize polynomials based upon their degree, as well as the number of terms, after all necessary simplification.

Polynomials Classification by Degree

Degree	Type	Example
0	Constant	-1
1	Linear	$2x + \sqrt{5}$
2	Quadratic	$5x^2 - 32x + 2$
3	Cubic	$-0.5x^3$
4	Quartic	$-3x^4 + 2x^2 + 3x + 1$
5	Quintic	$-2x^5$
6 or more	n^{th} Degree	$-2x^7 + 52x^6 + 12$

One point of note in the table above is the appearance of both rational and irrational coefficients $\left(-0.5 \text{ and } \sqrt{5}\right)$. The appearance of such coefficients is permissible in polynomials, since our coefficients a_i are only required to be real numbers. A coefficient containing the imaginary number $i = \sqrt{-1}$, on the other hand, is not permitted.

Polynomial Classification by Number of Nonzero Terms

Number of Terms	Name	Example
1	Monomial	$4x^5$
2	Binomial	$2x^3 + 1$
3	Trinomial	$-23x^{18} + 4x^2 + 3x$
4	Tetranomial	$-23x^{18} + 4x^2 + 3x + 1$
5 or more	Polynomial	$-2x^4 + x^3 + 15x^2 - 41x + 12$

I - Motivating Example(s):

Example: Identify the degree, set of coefficients, leading coefficient, leading term and constant term for the polynomial

$$f(x) = -19x^5 + 4x^4 - 6x + 21.$$

Classify the polynomial by both degree and number of nonzero terms.

- The degree of this polynomial is $n = 5$, since five is the greatest exponent.
- The leading term, which is the term that contains the greatest exponent (degree), is $a_n x^n = -19x^5$.
- The leading coefficient is the real number being multiplied by x^n in the leading term, namely $a_n = -19$.
- The constant term is $a_0 = 21$, which also represents the $y-$intercept for the graph of the given polynomial.
- The complete set of coefficients for the given polynomial is

$$\{a_5 = -19,\ a_4 = 4,\ a_3 = 0,\ a_2 = 0,\ a_1 = -6,\ a_0 = 21\}.$$

- The polynomial is a quintic tetranomial, as it is degree 5 and consists of 4 nonzeroterms.

II - Demo/Discussion Problems:

Identify the degree, set of coefficients, leading coefficient, leading term and constant term for each of the polynomials listed. Classify each polynomial by both degree and number of nonzero terms. If it is not already provided, write the polynomial in descending-power order.

1. $f(x) = 1$
2. $g(x) = x^3 - x^2$
3. $h(x) = x^2 - x^3$
4. $k(x) = \sqrt{2}x^4 + \pi x^2 - e$
5. $\ell(x) = 21x^4 + 12x^2 - 3x^2 - 9x^2 - 22x^4$
6. $m(x) = 3(x+1)(x-1) + 2x + 4x^3 + 3$

III - Practice Problems:

Identify the degree, set of coefficients, leading coefficient, leading term and constant term for each of the polynomials listed. Classify each polynomial by both degree and number of nonzero terms. If it is not already provided, write the polynomial in descending-power order.

1. $f(x) = -2x^3 - 1$
2. $f(x) = -2x^4 + 4x + 1$
3. $f(x) = 40 - x^3$
4. $f(x) = (x-1)^2$
5. $f(x) = 32x^5 + x^2 + x$
6. $f(x) = 4x^2 - 3x^4$
7. $f(x) = -2x^4 - 4x^2 - 6x - 8$
8. $f(x) = 5x + 3x^2 + x^3 + \sqrt{3}$
9. $f(x) = \frac{1}{2}x^4 - 5x^2 - \frac{1}{2}$
10. $f(x) = 12 - 6x + 3x^2 - 2x^3 - x^6$
11. $f(x) = -3x^4 + 15x^3 + x^2 - 27x^3 - x^2 - 13$

Lesson 44: Sign Diagrams for Polynomials

Objective: Construct a sign diagram for a given polynomial expression.

Students will be able to:

- Evaluate a factored polynomial expression at specified test values in order to determine its sign.

Prerequisite Knowledge:

- Factoring.
- Identifying roots of a factored polynomial expression.
- Evaluating functions.
- Order of operations.

Lesson:

If a polynomial function or expression is completely factored, it will be beneficial to us to construct a sign diagram for the polynomial, in order to answer questions about its graph and confirm any other findings. Therefore, we devote this lesson to the construction of a sign diagram for a factored polynomial. Note that expanded polynomials first require us to find a complete factorization prior to constructing a sign diagram.

Recall that the roots of a quadratic expression represent the dividers in its corresponding sign diagram. This carries over directly to a polynomial expression.

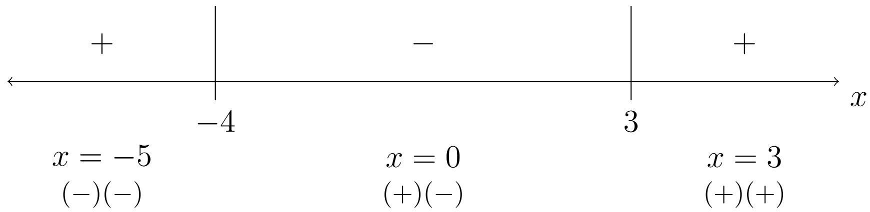
I - Motivating Example(s):

Example: Construct a sign diagram for the polynomial function $f(x) = 2x^2 + 3x - 20$.

Although our first example is not factored, we can apply the *ac*-method to quickly factor our function.

$$\begin{aligned} f(x) &= 2x^2 + 3x - 20 \\ &= 2x^2 + 8x - 5x - 20 \\ &= 2x(x+4) - 5(x+4) \\ &= (x+4)(2x-5) \end{aligned}$$

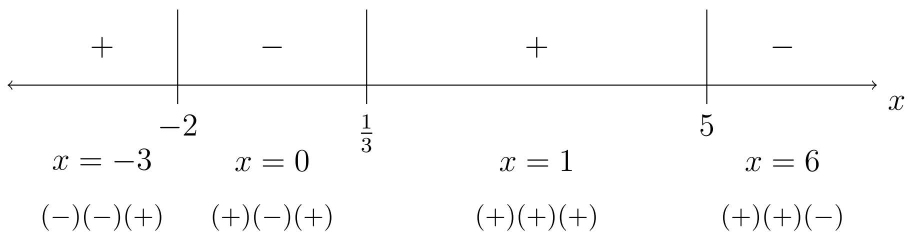
This gives us two roots, $x = -4$ and $x = \frac{5}{2}$, which serve as the dividers in our accompanying diagram. For our three test values, we will use $x = -5, 0,$ and 3.



Example: Construct a sign diagram for the factored polynomial function

$$g(x) = (x + 2)(3x - 1)(5 - x).$$

Our roots are $x = -2, \frac{1}{3}$, and 5. Consequently, the following diagram shows three dividers.



II - Demo/Discussion Problems:

Construct a sign diagram for the factored polynomial functions below. Use Desmos to graph each function and check the accuracy of your diagram. Identify the interval(s) where the function is positive and where it is negative.

1. $h(x) = (x + 2)^2(3x - 1)(5 - x)$
2. $f(x) = x(x + 1)(x - 2)^2(x^2 + 4)$

III - Practice Problems:

Construct a sign diagram for the factored polynomial functions below. Use Desmos to graph each function and check the accuracy of your diagram. Identify the interval(s) where the function is positive and where it is negative.

1. $f(x) = x^3(x - 2)(x + 2)$
2. $g(x) = (x^2 + 1)(1 - x)$
3. $h(x) = x(x - 3)^2(x + 3)$
4. $k(x) = (3x - 4)^3$
5. $\ell(x) = (x^2 + 2)(x^2 + 3)$
6. $m(x) = -2(x + 7)^2(1 - 2x)^2$
7. $f(x) = (x^2 - 1)(x + 4)$
8. $g(x) = (x^2 - 1)(x^2 - 16)$
9. $h(x) = -2x^3(3x - 1)(2 - x)$
10. $k(x) = (x^2 - 4x + 1)(x + 2)^2$

Lesson 45: Factoring Summary

Objective: Factor a general polynomial expression using one or more of factorization methods.

Students will be able to:

- Recognize and factor sums and differences of cubes.
- Apply the appropriate factorization method from those previously learned to an arbitrary polynomial expression.

Prerequisite Knowledge:

- GCF, grouping, ac-method, and quadratic type factorization methods.
- Properties of exponents.
- The distributive property.

Lesson:

When factoring polynomials there are a few special products that, if we can recognize them, can be easily broken down. The first is one we have seen before, when factoring a quadratic in which there is no linear term.

Difference of Two Squares: $a^2 - b^2 = (a+b)(a-b)$

It is important to note that, unlike differences, a *sum* of squares will never factor over the real numbers. Such expressions only factor over the complex numbers. Hence, we say that they are *irreducible* over the reals.

Sum of Two Squares: $a^2 + b^2 = (a+bi)(a-bi)$

In many cases, we can also recognize an entire expression as a perfect square (or a squared binomial).

Perfect Square: $a^2 + 2ab + b^2 = (a+b)^2$

While it might seem difficult to recognize a perfect square at first glance, by employing the $ac-$method, we can see that in the case where $m = n$, the resulting factorization will be a perfect square. In this case, we can factor by identifying the square roots of the first and last terms and using the sign from the middle term.

Another factoring shortcut can be applied to sums and differences of cubes, which have very similar factorizations.

$$\text{Sum of Cubes: } a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$$
$$\text{Difference of Cubes: } a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$$

Comparing the formulas for the sum and difference of cubes, one may notice that the only difference resides in the signs between the terms. One way to remember these two formulas is to think of "**SOAP**":

S	The first sign in our factorization is the **Same** sign as the given expression.
O	The second sign in our factorization is the **Opposite** sign as the given expression.
AP	The last sign in our factorization is **Always Positive**.

We are now ready to summarize the many factoring methods we have seen thus far. An important part of the process for factoring any polynomial expression is the identification of the number of terms in the simplified equation.

Factoring Summary

- **GCF** - Always look for a GCF first!
- **2 terms** - Sum or difference of squares or cubes.
  - $a^2 - b^2 = (a+b)(a-b)$
  - $a^2 + b^2$, Irreducible over the reals
  - $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
  - $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
- **3 terms** - Factor; watch for a perfect square.
  - $ax^2 + bx + c$, Apply ac-method
  - $a^2 + 2ab + b^2 = (a+b)^2$
- **4 terms** - Grouping
- **Special case** - Quadratic type: used in cases with polynomials having even degree and containing 2 or 3 terms.

I - Motivating Example(s):

Example: The expression $m^3 - 27$ is a difference of cubes, with cube roots of m and 3. Using **SOAP** we obtain a factorization of

$$(m-3)(m^2 + 3m + 9).$$

Example: The expression $125p^3 + 8r^3$ is a sum of cubes, with cube roots of $5p$ and $2r$. Using **SOAP** we obtain a factorization of

$$(5p + 2r)(25p^2 - 10pr + 4r^2).$$

II - Demo/Discussion Problems:

Completely factor each of the following polynomial expressions.

1. $100x^2 - 400$
2. $5 + 625y^3$
3. $4x^2 + 56xy + 196y^2$
4. $5x^2y + 15xy - 35x^2 - 105x$
5. $108x^3y^2 - 39x^2y^2 + 3xy^2$

III - Practice Problems:

Completely factor each of the following polynomial expressions.

1. $2x^2 - 11x + 15$
2. $5n^3 + 7n^2 - 6n$
3. $54u^3 - 16$
4. $54 - 128x^3$
5. $n^2 - n$
6. $2x^4 - 21x^2 - 11$
7. $24az - 18ah + 60yz - 45yh$
8. $5u^2 - 9uv + 4v^2$
9. $16x^2 + 48xy + 36y^2$
10. $-2x^3 + 128y^3$
11. $20uv - 60u^3 - 5xv + 15xu^2$
12. $2x^3 + 5x^2y + 3y^2x$

Lesson 46: Factoring Polynomials of Quadratic Type

Objective: Recognize and factor a polynomial expression of quadratic type.

Students will be able to:

- Identify polynomial expressions of quadratic type.
- Apply the appropriate substitution, $y = x^n$, to obtain a factorable quadratic polynomial in terms of y.
- Find the set of roots of a polynomial expression of quadratic type.

Prerequisite Knowledge:

- Properties of exponents.
- Factoring a difference of squares.
- The ac-method for factoring quadratics.

Lesson:

Recall that a quadratic expression in terms of a variable x is an expression of the form

$$ax^2 + bx + c.$$

If y is any algebraic expression, we say that the expression

$$ay^2 + by + c$$

is an expression of *quadratic type.*

In just about every case we will see, we will consider y as a power of x, $y = x^n$, so that our expression of quadratic type will appear as follows.

Quadratic Type:

$$ax^{2n} + bx^n + c \ = \ a\left[x^n\right]^2 + b\left[x^n\right] + c$$

If $y = x^3$, then the expression

$$ay^2 + by + c = ax^6 + bx^3 + c$$

would be an expression of quadratic type.

Similarly, if $y = x^4$, then the expression

$$ay^2 + by + c = ax^8 + bx^4 + c$$

would be an expression of quadratic type.

In each of these last two examples, notice the exponential pattern, where the middle term has an exponent that is half that of the leading term's. This will always be apparent, as long as the middle coefficient b is nonzero.

By viewing certain expressions as quadratic type, we can often apply more familiar methods, such as the $ac-$method, to obtain a complete factorization.

I - Motivating Example(s):

Example: If we let $y = x^2$, then the difference of fourth powers $x^4 - 16$ can be rewritten as a difference of squares, $y^2 - 4^2$, leading us to the complete factorization over the real numbers shown below.

$$\begin{aligned} x^4 - 16 &= \left(x^2\right)^2 - 4^2 \\ &= y^2 - 4^2, \quad y = x^2 \\ &= (y+4)(y-4) \\ &= \left(x^2+4\right)\left(x^2-4\right) \\ &= \left(x^2+4\right)(x+2)(x-2) \end{aligned}$$

Example: The trinomial expression $x^4 + 2x^2 - 24$ exhibits quadratic type characteristics, since the degree of four is double the exponent appearing in the middle term. Consequently, we will let $y = x^2$ and rewrite the expression in terms of y.

$$y^2 + 2y - 24$$

Applying the $ac-$method, we see the following.

$$\begin{aligned} y^2 + 2y - 24 &= y^2 + 6y - 4y - 24 \\ &= y(y+6) - 4(y+6) \\ &= (y+6)(y-4) \end{aligned}$$

Substituting back for x, we have $(x^2+6)(x^2-4)$. The first factor is a sum of squares, which is irreducible over the reals. The second factor of $x^2 - 4$ is a difference of perfect squares, which we know is factorable as $(x+2)(x-2)$. Our final factorization is

$$x^4 + 2x^2 - 24 = \left(x^2+6\right)(x+2)(x-2).$$

II - Demo/Discussion Problems:

Completely factor each of the following polynomials over the real numbers and identify the set of all real roots.

1. $x^4 - 12x^2 + 27$
2. $x^8 + 2x^4 - 24$
3. $x^6 + 2x^3 - 24$
4. $x^4 - 49$
5. $x^6 - 4x^3 - 5$

III - Practice Problems:

Completely factor each of the following polynomials over the real numbers and identify the set of all real roots.

1. $x^4 + 13x^2 + 40$
2. $x^4 - 5x^2 + 4$
3. $x^4 - 17x^2 + 16$
4. $x^4 - 3x^2 - 40$
5. $3x^4 - 32x^2 + 45$
6. $x^4 + x^2 - 12$
7. $x^4 - 3x^2 - 10$
8. $x^6 - 82x^3 + 81$
9. $8x^4 + 2x^2 - 3$
10. $2x^4 - 19x^2 + 9$

Lesson 47: Polynomial Division

CC attribute: *College Algebra* by C. Stitz and J. Zeager.

Objective: Apply polynomial division.

Students will be able to:

- Divide polynomials of varying degrees.
- Correctly label a divisor, dividend, quotient, and remainder in a polynomial division equation.

Prerequisite Knowledge:

- Polynomial definition and terminology.
- Combining like terms.
- Distributive property.
- Properties of exponents.

Lesson:

Let's recall the terminology and format associated with division.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

Alternatively, multiplying both sides of the above equation by the divisor, we have the following.

$$\frac{\text{dividend}}{\cancel{\text{divisor}}} \cdot \cancel{\text{divisor}} = \text{quotient} \cdot \text{divisor} + \frac{\text{remainder}}{\cancel{\text{divisor}}} \cdot \cancel{\text{divisor}}$$
$$\text{dividend} = \text{quotient} \cdot \text{divisor} + \text{remainder}$$

The general process for division of polynomials follows closely with that for dividing integers.

General Steps for Polynomial (Long) Division

Let $D(x)$ and $d(x)$ represent two nonzero polynomial functions. The steps for simplifying the rational expression $\dfrac{D(x)}{d(x)}$ are as follows.

1. Divide the leading term of the dividend D by the leading term of the divisor d. Label the resulting term $a_n x^n$, and write it above the dividend. This will be the leading term of the quotient, $q(x)$.
2. Multiply $a_n x^n$ by the divisor, distribute, and simplify. Label this as $d_1(x)$ and write it directly below the dividend, D, making sure to align terms according to exponents.
3. Subtract the resulting terms from the dividend. Label the new expression D_1.

4. Repeat steps (1)-(3) for the divisor d and the new expression D_i until the degree of D_i is *less than* the degree of the divisor. Relabel the final new dividend as the remainder, $r(x)$. The entire polynomial expression appearing above the original dividend is the quotient, $q(x)$.

$$
\begin{array}{rl}
 & \quad q(x) \\ \cline{2-2}
d(x)\,\big) & \quad D(x) \\
- & \quad \underline{d_1(x)} \\
 & \qquad D_1(x) \\
 & \quad - \;\underline{d_2(x)} \\
 & \qquad\qquad \ddots \\
 & \qquad\quad - \;\underline{d_i(x)} \\
 & \qquad\qquad D_i(x) = r(x)
\end{array}
$$

Step (3) often tends to pose the greatest challenge for students. It is important to keep in mind that we are are always subtracting the top term from the bottom term, which is why we must change the signs of the term(s) on the bottom. In most cases, we will need to utilize the distributive property.

I - Motivating Example(s):

Example: Divide $9x^5 + 6x^4 - 18x^3 - 24x^2$ by $3x^2$. Simplify and express your answer in the form

$$\frac{D(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

$$
\begin{array}{rrrrrl}
 & & 3x^3 & +\,2x^2 & -\,6x - 8 & \\ \cline{2-5}
3x^2\big) & 9x^5 & +\,6x^4 & -\,18x^3 & -\,24x^2 & \\
 & -\,9x^5 & & & & \\ \cline{2-2}
 & & 6x^4 & & & \\
 & & -\,6x^4 & & & \\ \cline{3-3}
 & & & -\,18x^3 & & \\
 & & & 18x^3 & & \\ \cline{4-4}
 & & & & -\,24x^2 & \\
 & & & & 24x^2 & \\ \cline{5-5}
 & & & & 0 &
\end{array}
$$

We set up our division process by first writing the dividend and the divisor in the appropriate locations. Next, we identify the leading term for our quotient, $3x^3$. Multiplying and subtracting produces our new expression, $D_1(x) = 6x^4$. While it is perfectly fine to carry down $-18x^3 - 24x^2$, it is not necessary until these terms play a role in the subtraction step.

Repeating our division steps gives us the second term in our quotient, $2x^2$. Multiplying, subtracting, and carrying down the next term gives us our new expression of $D_2(x) = -18x^3$. Repeating our steps again produces the third term in our quotient, $-6x$. Multiplying and subtracting produces another new expression, $D_3(x) = -24x^2$. Since the degree of D_3 equals that of our divisor, $d(x) = 3x^2$, we will need to apply our steps for division one final time. After our fourth and final round of steps, our new expression produces a remainder of $r(x) = 0$.

We express the results of our division in the required form as follows.

$$\frac{9x^5 + 6x^4 - 18x^3 - 24x^2}{3x^2} = 3x^3 + 2x^2 - 6x - 8 + \frac{0}{3x^2}$$

Example: Divide $3x^3 - 5x^2 - 32x + 7$ by $x - 4$. Simplify and express your answer in the form

$$\frac{D(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

$$\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & 3x^2 & +7x & -4 & \\ \cline{2-5}
x-4 & 3x^3 & -5x^2 & -32x & +7 \\
 & -3x^3 & +12x^2 & & \\ \cline{2-3}
 & & 7x^2 & -32x & \\
 & & -7x^2 & +28x & \\ \cline{3-4}
 & & & -4x & +7 \\
 & & & 4x & -16 \\ \cline{4-5}
 & & & & -9
\end{array}$$

Since our quotient, $3x^2 + 7x - 4$, is a trinomial, we must apply the steps for division three times.

In this second example, our remainder is the constant term $r(x) = -9$, which has one degree less than our linear divisor, $d(x) = x - 4$.

Our answer is $\dfrac{3x^3 - 5x^2 - 32x + 7}{x - 4} = 3x^2 + 7x - 4 + \dfrac{-9}{x - 4}$.

II - Demo/Discussion Problems:

Use polynomial long division to divide and simplify each of the given expressions. Express each answer in the form below.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

1. $\dfrac{8x^3 + 4x^2 - 2x + 6}{4x^2}$
2. $\dfrac{n^2 + 7n + 15}{n + 4}$
3. $\dfrac{x^3 - 46x + 22}{x + 7}$
4. $\dfrac{6x^3 - 8x^2 + 10x + 103}{4 + 2x}$
5. $\dfrac{2x^3 - 4x + 42}{x + 3}$

III - Practice Problems:

Use polynomial long division to divide and simplify each of the given expressions. Express each answer in the form below.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

1. $\dfrac{20x^4 + x^3 + 2x^2}{4x^3}$
2. $\dfrac{5x^4 + 45x^3 + 4x^2}{9x}$
3. $\dfrac{20x^4 + x^3 + 40x^2}{10x}$
4. $\dfrac{3x^3 + 4x^2 + 2x}{8x}$
5. $\dfrac{12x^4 + 24x^3 + 3x^2}{6x}$
6. $\dfrac{5x^4 + 16x^3 + 16x^2}{4x}$
7. $\dfrac{10x^4 + 50x^3 + 2x^2}{10x^2}$
8. $\dfrac{3x^4 + 18x^3 + 27x^2}{9x^2}$
9. $\dfrac{x^2 - 2x - 71}{x + 8}$
10. $\dfrac{x^2 - 3x - 53}{x - 9}$
11. $\dfrac{x^2 + 13x + 32}{x + 5}$
12. $\dfrac{x^2 - 10x + 16}{x - 7}$
13. $\dfrac{x^2 - 2x - 89}{x - 10}$
14. $\dfrac{x^2 + 4x - 26}{x + 7}$
15. $\dfrac{x^2 - 4x - 38}{x - 8}$
16. $\dfrac{x^2 - 4}{x - 2}$
17. $\dfrac{x^3 + 15x^2 + 49x - 55}{x + 7}$
18. $\dfrac{x^3 - 26x - 41}{x + 4}$
19. $\dfrac{3x^3 + 9x^2 - 64x - 68}{x + 6}$
20. $\dfrac{9x^3 + 45x^2 + 27x - 5}{9x + 9}$
21. $\dfrac{x^3 - x^2 - 16x + 8}{x - 4}$
22. $\dfrac{x^2 - 10x + 22}{x - 4}$
23. $\dfrac{x^3 - 16x^2 + 71x - 56}{x - 8}$
24. $\dfrac{x^3 - 4x^2 - 6x + 4}{x - 1}$
25. $\dfrac{8x^3 - 66x^2 + 12x + 37}{x - 8}$
26. $\dfrac{3x^2 + 9x - 9}{3x - 3}$
27. $\dfrac{2x^2 - 5x - 8}{2x + 3}$
28. $\dfrac{3x^2 - 32}{3x - 9}$
29. $\dfrac{4x^2 - 23x - 38}{4x + 5}$
30. $\dfrac{2x^3 + 21x^2 + 25x}{2x + 3}$
31. $\dfrac{4x^3 - 21x^2 + 6x + 19}{4x + 3}$
32. $\dfrac{8x^3 - 57x^2 + 42}{8x + 7}$
33. $\dfrac{2x^3 + 12x^2 + 4x - 37}{2x + 6}$
34. $\dfrac{45x^2 + 56x + 19}{9x + 4}$
35. $\dfrac{10x^2 - 32x + 9}{10x - 2}$
36. $\dfrac{4x^2 - x - 1}{4x + 3}$
37. $\dfrac{27x^2 + 87x + 35}{3x + 8}$
38. $\dfrac{4x^2 - 33x + 28}{4x - 5}$
39. $\dfrac{48x^2 - 70x + 16}{6x - 2}$
40. $\dfrac{12x^3 + 12x^2 - 15x - 4}{2x + 3}$
41. $\dfrac{24x^3 - 38x^2 + 29x - 60}{4x - 7}$

Lesson 48: Synthetic Division

CC attribute: *College Algebra* by C. Stitz and J. Zeager.

Objective: Apply synthetic division.

Students will be able to:

- Quickly divide polynomials by a linear factor.
- Correctly label a divisor, dividend, quotient, and remainder in a polynomial division equation.

Prerequisite Knowledge:

- Polynomial definition and terminology.

Lesson:

Due in large part to its speed, for some students synthetic division is often the preferred method over standard long division, when dividing polynomials by expressions of the form $x - c$. It is worth mentioning that when a polynomial (of degree at least one) is divided by $x - c$, the result will be a quotient polynomial of exactly one degree less than the original dividend. This is a direct result of the divisor being a linear expression.

The method of synthetic division focuses primarily on the coefficients of both the divisor and dividend. We must also pay careful attention, however, to the values of our exponents, which will serve as placeholders throughout the process. To start, we will write our coefficients in what we is sometimes referred to as a *synthetic division tableau* prior to dividing. This is illustrated in the motivating examples.

It is important to stress that synthetic division will *only* work for linear divisors with leading coefficient one. Hence, we will need to use long division for divisors having degree larger than one. For a more complete understanding of the relationship between long and synthetic division, students are encouraged to trace each step in synthetic division back to its corresponding long division step.

I - Motivating Example(s):

Example: Divide $x^3 + 4x^2 - 5x - 14$ by $x - 2$ using synthetic division.

To divide $x^3 + 4x^2 - 5x - 14$ by $x - 2$, we first write 2 in the place of the divisor since 2 is zero of the factor $x - 2$ and we write the coefficients of $x^3 + 4x^2 - 5x - 14$ in for the dividend. As our next step, we "bring down" the first coefficient of the dividend. We will then multiply and add repeatedly.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & & & \\ \hline & & & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & & & \\ \hline & 1 & & & \end{array}$$

Next, take the 2 from the divisor and multiply by the 1 that was brought down to get 2. Write this underneath the 4, then add to get 6.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & & \\ \hline & 1 & & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & & \\ \hline & 1 & 6 & & \end{array}$$

Now multiply the 2 from the divisor by the 6 to get 12, and add it to the -5 to get 7.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & 12 & \\ \hline & 1 & 6 & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & 12 & \\ \hline & 1 & 6 & 7 & \end{array}$$

Finally, multiply the 2 in the divisor by the 7 to get 14, and add it to the -14 to get 0.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \mathbf{0} \end{array}$$

The first three numbers in the last row of our tableau will be the coefficients of the desired quotient polynomial. Remember, we started with a third degree polynomial and divided by a first degree polynomial, so the quotient will be a second degree polynomial. Hence the quotient is $x^2 + 6x + 7$. The number appearing in bold represents the remainder, which is zero in this case.

$$\frac{x^3 + 4x^2 - 5x - 14}{x - 2} = x^2 + 6x + 7$$

Example: Divide $5x^3 - 2x^2 + 1$ by $x - 3$ using synthetic division.

Setting up and working through the tableau gives us the following result.

$$\begin{array}{r|rrrr} 3 & 5 & -2 & 0 & 1 \\ & & 15 & 39 & 117 \\ \hline & 5 & 13 & 39 & \mathbf{118} \end{array}$$

Since the dividend is a degree-3 polynomial, the quotient is a quadratic polynomial with coefficients 5, 13 and 39. Our quotient is then $q(x) = 5x^2 + 13x + 39$ and the remainder is $r(x) = 118$. Putting this all together, we have the following equation.

$$\frac{5x^3 - 2x^2 + 1}{x - 3} = 5x^2 + 13x + 39 + \frac{118}{x - 3}$$

II - Demo/Discussion Problems:

Use synthetic division to divide and simplify each of the given expressions. Express each answer in the form below.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

1. $\dfrac{x^3+8}{x+2}$
2. $\dfrac{-12x^2-8x+4}{2x-3}$
3. $\dfrac{x^2+7x+15}{x+4}$
4. $\dfrac{x^3-46x+22}{x+7}$
5. $\dfrac{2x^3-4x+42}{x+3}$

III - Practice Problems:

Use synthetic division to divide and simplify each of the given expressions. Express each answer in the form below.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

1. $\dfrac{x^4-4x^3+2x^2-x+1}{x+2}$
2. $\dfrac{x^4-2x^3+7x^2-6x+3}{x-2}$
3. $\dfrac{2x^4-2x^3-10x^2+1}{x+2}$
4. $\dfrac{5x^4-2x^3+4x^2-5x}{x-1}$
5. $\dfrac{-x^4-x^3+x^2+x+1}{x+5}$
6. $\dfrac{x^4-3x^3+2x^2-x+1}{x-4}$
7. $\dfrac{12x^4-x^3+x^2-3x+1}{x+2}$
8. $\dfrac{3x^4+3x^3+13x^2-4x+14}{x+1}$
9. $\dfrac{1x^4-3x^3+5x^2-14x+2}{x-2}$
10. $\dfrac{2x^4-2x+1}{x+3}$
11. $\dfrac{x^4-3x-4}{x-3}$
12. $\dfrac{x^4-4x^3+13x^2-5x+7}{x-4}$

Use synthetic division to divide and simplify each of the given expression from Practice Problems 9-41 from the previous lesson.

Lesson 49: Polynomial End Behavior

Objective: Determine the end behavior of the graph of a polynomial function.

Students will be able to:

- Identify the degree, leading coefficient, and constant term of a factored polynomial.
- Use the degree and leading coefficient of a polynomial to determine the end behavior of its graph.
- Describe the end behavior of a polynomial in a mathematical sentence.

Prerequisite Knowledge:

- Definition of a polynomial and associated terminology.
- Order of operations.

Lesson:

The *end behavior* of any function refers to what happens near the extreme ends of its graph. We also often refer to these as the "tails" of the graph. The ends of the graph of a function correspond to points having large positive or negative $x-$coordinates. Because of this, we can associate the expressions

$$x \to \infty \qquad \text{and} \qquad x \to -\infty$$

to the end behavior of a function. For example, the sentence

$$\text{As } x \to \infty, \;\; f(x) \to \infty.$$

describes a function for which the right-hand side of its graph, i.e. when $x \to \infty$, points upward. Alternatively, the sentence

$$\text{As } x \to \infty, \;\; f(x) \to -\infty.$$

describes a function for which the right-hand side of its graph points downward.

In each of the above mathematical statements, we are identifying both a horizontal direction and a vertical direction:

1. the independent variable x getting large (either positively or negatively),
2. and the effect this has on the values of $f(x)$.

For each algebraic function, the corresponding graph will describe two such statements: one for the left-hand side of the graph ($x \to -\infty$) and one for the right-hand side of the graph ($x \to \infty$). In the case of polynomials, there are only four cases for these two statements, summarized as follows.

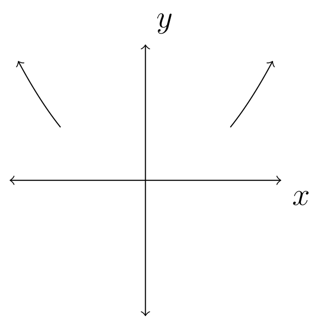
Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$$

be a polynomial function with degree n and nonzero leading coefficient a_n.

The end behavior of f is described by one of the following four cases.

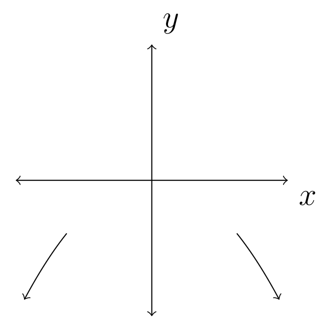
I. n even, $a_n > 0$



As $x \to \infty,\ f(x) \to \infty$

As $x \to -\infty,\ f(x) \to \infty$

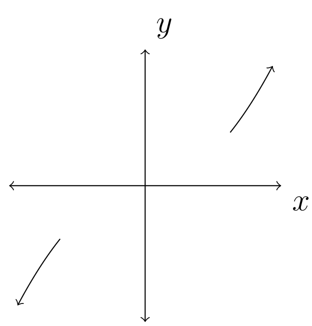
II. n even, $a_n < 0$



As $x \to \infty,\ f(x) \to -\infty$

As $x \to -\infty,\ f(x) \to -\infty$

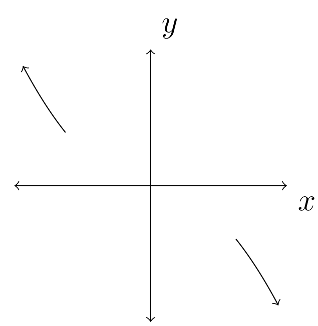
III. n odd, $a_n > 0$



As $x \to \infty,\ f(x) \to \infty$

As $x \to -\infty,\ f(x) \to -\infty$

IV. n odd, $a_n < 0$



As $x \to \infty,\ f(x) \to -\infty$

As $x \to -\infty,\ f(x) \to \infty$

I - Motivating Example(s):

Example: Find the leading and constant terms for the following function, and use them to identify the end behavior and $y-$intercept of its graph.

$$f(x) = 3(-2x+1)^2(x-2)^2(x-5)$$

First, we boldface the contributors for the leading term.

$$f(x) = \mathbf{3}(\mathbf{-2x}+1)^{\mathbf{2}}(\mathbf{x}-2)^{\mathbf{2}}(\mathbf{x}-5)$$

This gives us the following.

$$\begin{aligned} a_n x^n &= 3(-2x)^2(x)^2(x) \\ &= 3(4x^2)x^3 \\ &= 12x^5 \end{aligned}$$

Next, we boldface the contributors for the constant term.

$$f(x) = \mathbf{3}(-2x+\mathbf{1})^{\mathbf{2}}(x-\mathbf{2})^{\mathbf{2}}(x-\mathbf{5})$$

This gives us the following.

$$\begin{aligned} a_0 &= 3(1)^2(-2)^2(-5) \\ &= 3(1)(4)(-5) \\ &= -60 \end{aligned}$$

Hence, we have that

$$f(x) = 12x^5 + \ldots + (-60),$$

with middle terms unknown.

Since our degree, $n = 5$, is odd, and our leading coefficient, $a_n = 12$, is positive, we are in case III for end behavior.

As $x \to -\infty,\ f(x) \to -\infty.$ As $x \to \infty,\ f(x) \to +\infty.$

Our constant term also tells us that the graph of f has a $y-$intercept at $(0, -60)$.

II - Demo/Discussion Problems:

Determine the end behavior of each of the following functions. Write your answers as mathematical sentences. Graph each function on Desmos to check your answers.

1. $f(x) = 1 - 3x^4$
2. $g(x) = -x^3 + 3x - 2$
3. $h(x) = -2x^3 + 10000x^2 + 1000$
4. $k(x) = x(2x-1)(x-5)^2$
5. $\ell(x) = -2(1-3x)^2(x+1)(x-1)(x^2+1)$

III - Practice Problems:

Determine the end behavior of each of the following functions. Write your answers as mathematical sentences. Graph each function on Desmos to check your answers.

1. $f(x) = -2x^3 + 4x + 1$
2. $g(x) = 32x^5 + x^2 + 15$
3. $h(x) = -3x^4 + 4x^2$
4. $k(x) = 15x^4 - 32x^2 - x - 14$
5. $\ell(x) = x^5 + 40$
6. $m(x) = 5x^5 + 3x^2 + x + 14$
7. $n(x) = 123x^4 - 7x^3 - 5x^2 - 3x + 1$
8. $p(x) = x^3 - 1$
9. $q(x) = -23x^6 + x^3 + x^2 + x + 1$

Identify the degree, leading coefficient, and constant term of each polynomial function below. Use the degree and leading coefficient to identify the end behavior of the graph of each function. Write your answers as mathematical sentences. Graph each function on Desmos to check your answers.

10. $f(x) = x^3(x-2)(x+2)$
11. $g(x) = (x^2+1)(1-x)$
12. $h(x) = x(x-3)^2(x+3)$
13. $k(x) = (3x-4)^3$
14. $\ell(x) = (x^2+2)(x^2+3)$
15. $m(x) = -2(x+7)^2(1-2x)^2$
16. $f(x) = (x^2-1)(x+4)$
17. $g(x) = (x^2-1)(x^2-16)$
18. $h(x) = -2x^3(3x-1)(2-x)$
19. $k(x) = (x^2-4x+1)(x+2)^2$

Lesson 50: Polynomial Local Behavior; Roots and Multiplicities

Objective: Identify all real roots and their corresponding multiplicities for a polynomial function.

Students will be able to:

- Identify the multiplicity of a root of a polynomial f, and use it to describe the nature of the graph of f at the corresponding $x-$intercept.

Prerequisite Knowledge:

- Factoring.
- Properties of exponents.

Lesson:

In this lesson, we seek to classify $x-$intercepts for the graph of a polynomial as either "turnaround" or "crossover" points. This can be parsed down to one basic concept, known as the *multiplicity* of a root, defined as follows.

Definition: Suppose f is a polynomial function with real root $x = c$. For some positive integer k, if $(x - c)^k$ is a factor of f but $(x - c)^{k+1}$ is not, then we say $x = c$ is a root of f having associated multiplicity k.

Another way of describing the multiplicity k of a root $x = c$ is that k represents the maximum number of factors of $(x - c)$ that divide the polynomial f (with a remainder of 0). That is,

$$f(x) = (x - c)^k \cdot q(x),$$

where $(x - c)$ is *not* a factor of the quotient $q(x)$.

The multiplicity of a root of a polynomial tells us the following information about the corresponding $x-$intercept.

Let f be a polynomial function with a real root at $x = c$ having multiplicity k.

- If k is *even*, the corresponding $x-$intercept $(c, 0)$ is a *turnaround point*. In other words, the graph of f touches and rebounds from the x-axis at $(c, 0)$, leaving the $y-$values to maintain the same sign on either side of the root $x = c$.
- If k is *odd*, the corresponding $x-$intercept $(c, 0)$ is a *crossover point*. In other words, the graph of f crosses through the x-axis at $(c, 0)$, leaving the $y-$values to change signs on either side of the root $x = c$.

I - Motivating Example(s):

Example: Determine the set of roots and corresponding multiplicities for the following functions. In each case, classify the corresponding $x-$intercept as either a turnaround or crossover point.

1. $f(x) = x^6 - 2x^5 - 15x^4$
2. $g(x) = (x-6)^5(x+2)^2(x^2+1)$

1. Factoring f gives us the following.

$$\begin{aligned} f(x) &= x^6 - 2x^5 - 15x^4 \\ &= x^4(x^2 - 2x - 15) \\ &= x^4(x-5)(x+3) \end{aligned}$$

   We then can easily see that f has a root at $x = 0$ with multiplicity four, and roots at $x = 5$ and $x = -3$, each with multiplicity one. Hence, the graph of f has a turnaround point at the origin and crossover points at the $x-$intercepts $(-3, 0)$ and $(5, 0)$.

2. Since g is already factored, we see that $x = 6$ is a root having multiplicity five, and $x = -2$ is a root having multiplicity two. The factor of $x^2 + 1$ is meant to throw us off, since its roots are the imaginary numbers $\pm i$. Hence, the graph of g has a crossover point at $x-$intercept $(6, 0)$ and a turnaround point at $x-$intercept $(-2, 0)$.

II - Demo/Discussion Problems:

Determine the set of roots and corresponding multiplicities for the following functions. In each case, classify the corresponding $x-$intercept as either a turnaround or crossover point. Use Desmos to check your answers.

1. $f(x) = \frac{1}{2}(x-2)^2(x+5)(x-3)$
2. $g(x) = (x+2)^2(3x-1)(5-x)$
3. $h(x) = -(x-3)^2(x+1)(x+5)^2$

III - Practice Problems:

Determine the set of roots and corresponding multiplicities for the following functions. In each case, classify the corresponding $x-$intercept as either a turnaround or crossover point. Use Desmos to check your answers.

1. $f(x) = x^3(x-2)(x+2)$
2. $g(x) = (x^2+1)(1-x)$
3. $h(x) = x(x-3)^2(x+3)$
4. $k(x) = (3x-4)^3$
5. $\ell(x) = (x^2+2)(x^2+3)$
6. $m(x) = -2(x+7)^2(1-2x)^2$
7. $f(x) = (x^2-1)(x+4)$
8. $g(x) = (x^2-1)(x^2-16)$
9. $h(x) = -2x^3(3x-1)(2-x)$
10. $k(x) = (x^2-4x+1)(x+2)^2$

Lesson 51: The Rational Root Theorem

CC attribute: *College Algebra* by C. Stitz and J. Zeager.

Objective: Apply the Rational Root Theorem to determine a set of possible rational roots for and a factorization of a given polynomial.

Students will be able to:

- Identify a set of possible rational roots for a given polynomial.
- Test the values from a set of possible rational roots to determine if they are actual roots of a given polynomial.

Prerequisite Knowledge:

- Polynomial definition and terminology.
- Evaluate a function for a given x.
- Synthetic (or polynomial) division.

Lesson:

The Rational Root Theorem is used to identify a list of all possible rational roots for a given polynomial.

> **Rational Root Theorem:** Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial of degree n with $n \geq 1$, and a_0, a_1, $\ldots a_n$ are integers. If r is a rational root of f, then r is of the form $\pm\frac{p}{q}$, where p is a factor of the constant term a_0, and q is a factor of the leading coefficient a_n.

The Rational Root Theorem gives us a list of numbers to test as roots of a given polynomial using synthetic division, which is a nicer approach than simply guessing at possible roots. If none of the numbers in the list turn out to be roots, then either the polynomial has no real roots at all, or all of the real roots will be irrational numbers.

I - Motivating Example(s):

Example: Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. Use the Rational Root Theorem to list all of the possible rational roots of f.

To generate a complete list of rational roots, we need to take each of the factors of the constant term, $a_0 = -3$, and divide them by each of the factors of the leading coefficient $a_4 = 2$.

The factors of -3 are ± 1 and ± 3. Since the Rational Root Theorem tacks on a $\pm$ anyway, for the moment, we consider only the positive factors 1 and 3. The factors of 2 are 1 and 2, so the Rational Root Theorem gives the list $\left\{\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{3}{1}, \pm\frac{3}{2}\right\}$ or $\left\{\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 3\right\}$.

Additionally, we can evaluate f at each of the eight potential rational roots in our list, to see if any of them are indeed roots. Starting with ± 1, we see that

$$f(1) = 2 + 4 - 1 - 6 - 3 = -4 \neq 0 \qquad \text{and} \qquad f(-1) = 2 - 4 - 1 + 6 - 3 = 0.$$

Hence, we can conclude that $x = -1$ is a root of f and $x = 1$ is not. Using synthetic division, we can then divide f by the linear factor $x + 1$ as follows.

$$\begin{array}{r|rrrrr} -1 & 2 & 4 & -1 & -6 & -3 \\ & & -2 & -2 & 3 & 3 \\ \hline & 2 & 2 & -3 & -3 & \mathbf{0} \end{array}$$

We can then begin to factor f,

$$2x^4 + 4x^3 - x^2 - 6x - 3 = (x + 1)(2x^3 + 2x^2 - 3x - 3)$$

The resulting quotient polynomial is then factorable by grouping,

$$2x^3 + 2x^2 - 3x - 3 = (2x^2 - 3)(x + 1).$$

Factoring out a 2 from the expression $2x^2 - 3$, allows us to factor it as the difference of two squares,

$$\begin{aligned} 2x^2 - 3 &= 2\left(x^2 - \frac{3}{2}\right) \\ &= 2\left(x - \sqrt{\frac{3}{2}}\right)\left(x + \sqrt{\frac{3}{2}}\right) \\ &= 2\left(x - \frac{\sqrt{6}}{2}\right)\left(x + \frac{\sqrt{6}}{2}\right) \end{aligned}$$

So, a complete factorization for f would be

$$2x^4 + 4x^3 - x^2 - 6x - 3 = 2\left(x - \frac{\sqrt{6}}{2}\right)\left(x + \frac{\sqrt{6}}{2}\right)(x + 1)^2,$$

and the set of real roots for f is $\left\{-1, \pm\frac{\sqrt{6}}{2}\right\}$.

II - Demo/Discussion Problems:

Use the Rational Root Theorem to identify a set of possible rational roots for each of the polynomial functions below. Evaluate the function at at least two of your possible roots, in order to determine if they are actual roots of the polynomial. If successful, divide your polynomial by the respective factor. Use Desmos to help determine the actual set of real roots.

1. $f(x) = x^4 - 9x^2 - 4x + 12$
2. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
3. $f(x) = 6x^3 + 19x^2 - 6x - 40$

III - Practice Problems:

Use the Rational Root Theorem to identify a set of possible rational roots for each of the polynomial functions below. Evaluate the function at $x = 1$. If $x = 1$ is a real root, divide the polynomial by $x - 1$ and factor the resulting quotient. If $x = 1$ is not a real root, evaluate the function at at least one of your remaining possible roots, in order to determine if they are actual roots of the polynomial. If successful, divide your polynomial by the respective factor and factor the remaining quotient. Use Desmos to help determine the actual set of real roots.

1. $f(x) = x^3 - 2x^2 - 5x + 6$
2. $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$
3. $f(x) = x^5 - x^4 - 37x^3 + 37x^2 + 36x - 36$
4. $f(x) = 3x^3 + 3x^2 - 11x - 10$

Use the Rational Root Theorem to identify a set of possible rational roots for each of the polynomial functions below. Evaluate the function at at least two of your possible roots, in order to determine if they are actual roots of the polynomial. If successful, divide your polynomial by the respective factor. Use Desmos to help determine the actual set of real roots.

5. $f(x) = x^4 - 2x^3 + 5x^2 - 8x + 4$
6. $f(x) = x^3 + 4x^2 - 11x + 6$
7. $f(x) = -2x^3 + 19x^2 - 49x + 20$
8. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$

Lesson 52: Polynomials Graphing Summary

Objective: Graph a polynomial function in its entirety.

Students will be able to:

- Identify all important aspects of the graph of a polynomial function: $y-$intercept, $x-$intercepts (including their nature), and end behavior.
- Sketch a complete graph of a polynomial function.

Prerequisite Knowledge:

- Evaluating a function.
- Polynomial terminology and end behavior.
- Factoring.
- The Rational Root Theorem.
- Polynomial and /or Synthetic Division.
- Sign Diagrams.

Lesson:

At this point, we have addressed all key features of polynomials individually. This lesson pulls each of these aspects together, for a detailed analysis of a polynomial, culminating in a complete sketch of its graph. Along the way, we will need to address each of the following aspects for our polynomial $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$. It is important to note that there is no universally accepted order to this checklist.

- Find the $y-$intercept of the graph of f, $(0, f(0)) = (0, a_0)$.
- Use the degree n and leading coefficient a_n to determine the end behavior of the graph of f.
- Identify a complete factorization of f, and use it to find any $x-$intercepts of the graph of f. Using multiplicities, classify each $x-$intercept as a crossover or turnaround ("bounce") point.
- Using the $x-$intercepts, construct a sign diagram for f.

In each polynomial we encounter, we will carefully examine the function, making sure not to omit any of the checklist items above and to compare each item to those that precede it along the way for accuracy. Although the process will take some time, if we are thorough, our end result should be a complete, accurate sketch of the given polynomial.

I - Motivating Example(s):

Example: Sketch a complete graph of $f(x) = 14x^4 - 17x^3 - 6x^2 + 7x + 2$.

We will start with the $y-$intercept, which is $(0, 2)$.

Next, we see that f has even degree and positive leading coefficient. So, the tails of the graph of f both point upwards. In other words, as $x \to \pm\infty, f(x) \to \infty$.

Since f is degree-4, contains more than four terms, and is not of quadratic type, we will apply the Rational Root Theorem. In this case, our set of possible rational roots is

$$\left\{\pm 1, \pm 2, \pm\frac{1}{2}, \pm\frac{1}{7}, \pm\frac{1}{14}, \pm\frac{2}{7}\right\}$$

Fortunately, we see that $f(1) = 14 - 17 - 6 + 7 + 2 = 0$. So, $x - 1$ is a factor of f. Dividing, we get:

$$\begin{array}{r l}
 & 14x^3 - 3x^2 - 9x - 2 \\
x-1) & 14x^4 - 17x^3 - 6x^2 + 7x + 2 \\
 & -14x^4 + 14x^3 \\
 & \qquad\quad -3x^3 - 6x^2 \\
 & \qquad\quad\;\; 3x^3 - 3x^2 \\
 & \qquad\qquad\quad -9x^2 + 7x \\
 & \qquad\qquad\quad\;\; 9x^2 - 9x \\
 & \qquad\qquad\qquad\quad -2x + 2 \\
 & \qquad\qquad\qquad\quad\;\; 2x - 2 \\
 & \qquad\qquad\qquad\qquad\quad 0
\end{array}
\qquad
\begin{array}{c|ccccc}
1 & 14 & -17 & -6 & 7 & 2 \\
 & & 14 & -3 & -9 & -2 \\
\hline
 & 14 & -3 & -9 & -2 & \mathbf{0}
\end{array}$$

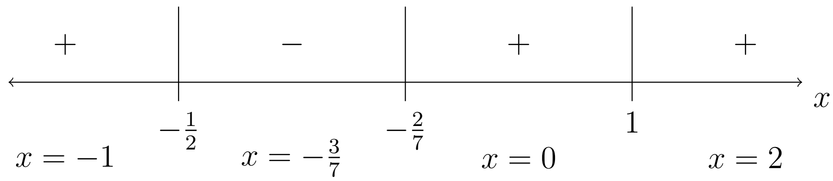
So, $f(x) = (x - 1)(14x^3 - 3x^2 - 9x - 2)$. Applying the Rational Root Theorem a second time, we can see that $x = 1$ is also a root of the cubic factor of f, since $14 - 3 - 9 - 2 = 0$. Again, we can divide to factor f further.

$$\begin{array}{r l}
 & 14x^2 + 11x + 2 \\
x-1) & 14x^3 - 3x^2 - 9x - 2 \\
 & -14x^3 + 14x^2 \\
 & \qquad\quad 11x^2 - 9x \\
 & \qquad -11x^2 + 11x \\
 & \qquad\qquad\quad 2x - 2 \\
 & \qquad\qquad\;\; -2x + 2 \\
 & \qquad\qquad\qquad\quad 0
\end{array}
\qquad
\begin{array}{c|cccc}
1 & 14 & -3 & -9 & -2 \\
 & & 14 & 11 & 2 \\
\hline
 & 14 & 11 & 2 & \mathbf{0}
\end{array}$$

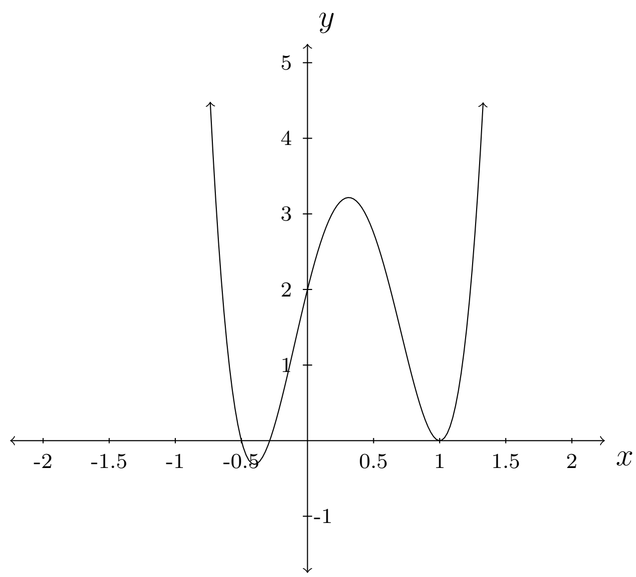
So, $f(x) = (x - 1)^2(14x^2 + 11x + 2)$. Factoring the remaining quadratic, we have $f(x) = (x - 1)^2(7x + 2)(2x + 1)$, with set of roots $\{1, -\frac{1}{2}, -\frac{2}{7}\}$.

Using multiplicities, we conclude that the $x-$intercept $(1,0)$ is a turnaround point, and the intercepts $\left(-\frac{1}{2},0\right)$ and $\left(-\frac{2}{7},0\right)$ are crossover points.

Though not necessary for graphing, a sign diagram confirms our end and local behavior findings.



Putting all of this information together results in the following graph.



II - Demo/Discussion Problems:

Factor each polynomial below, and sketch a complete graph of the function, making sure to have a clearly defined scale and label any intercepts. Use Desmos to compare your results.

1. $f(x) = -x^3 + 7x^2 - x + 7$
2. $f(x) = 3x^4 - 5x^3 - 12x^2$
3. $f(x) = 2x^3 - 5x^2 - x$
4. $f(x) = -x^4 - 2x^2 + 15$

Use the Rational Root Theorem and polynomial division to get a complete factorization of each polynomial function below. Then sketch a graph of the function, making sure to have a clearly defined scale and label any intercepts. Use Desmos to compare your results.

5. $f(x) = x^4 + 4x^3 - x - 4$
6. $f(x) = 2x^3 - 5x^2 - 52x + 60$
7. $f(x) = -x^3 - x^2 + 39x + 45$
8. $f(x) = -2x^4 + 7x^3 + 17x^2 - 28x - 36$
9. $f(x) = x^7 - 5x^6 - 24x^5 + 120x^4 - 25x^3 + 125x^2$

III - Practice Problems:

Factor each polynomial below, and sketch a complete graph of the function, making sure to have a clearly defined scale and label any intercepts. Use Desmos to compare your results.

1. $f(x) = -17x^3 + 5x^2 + 34x - 10$
2. $f(x) = x^4 - 9x^2 + 14$
3. $f(x) = 3x^4 - 14x^2 - 5$
4. $f(x) = 2x^4 - 7x^2 + 6$
5. $f(x) = x^5 - 2x^4 - x + 2$
6. $f(x) = 2x^5 + 3x^4 - 32x - 48$
7. $f(x) = x^6 - 6x^3 - 16$
8. $f(x) = 2x^6 - 7x^3 + 5$

Get a complete factorization of each polynomial below by first dividing the function by $x-1$. Then sketch a graph of the function, making sure to have a clearly defined scale and label any intercepts. Use Desmos to compare your results.

9. $f(x) = x^3 - 2x^2 - 5x + 6$
10. $f(x) = x^3 + 4x^2 - 11x + 6$
11. $f(x) = x^5 - x^4 - 37x^3 + 37x^2 + 36x - 36$
12. $f(x) = x^4 - 2x^3 + 5x^2 - 8x + 4$

Use the Rational Root Theorem and polynomial division to get a complete factorization of each polynomial function below. Then sketch a graph of the function, making sure to have a clearly defined scale and label any intercepts. Use Desmos to compare your results.

13. $f(x) = x^4 - 9x^2 - 4x + 12$
14. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
15. $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$
16. $f(x) = 3x^3 + 3x^2 - 11x - 10$
17. $f(x) = 6x^3 + 19x^2 - 34x - 40$
18. $f(x) = -2x^3 + 19x^2 - 49x + 20$
19. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$

Lesson 53: Polynomial Inequalities

Objective: Solve a polynomial inequality by constructing a sign diagram.

Students will be able to:

- Identify the solution for a polynomial inequality using interval notation.

Prerequisite Knowledge:

- Sign diagrams.
- Factoring.
- Evaluating a function at a given value.
- Interval notation.

Lesson:
I - Motivating Example(s):

Example: Solve the polynomial inequality

$$x^4 + 6x^2 - 15x \leq x^4 + 2x^3 - 7x^2.$$

Just as with quadratic inequalities, we begin by setting one side equal to zero. This gives us

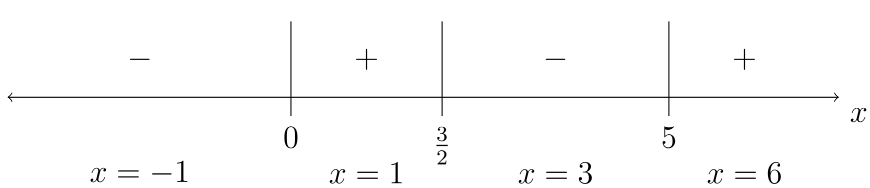
$$2x^3 - 13x^2 + 15x \geq 0.$$

In order to construct a sign diagram, we must find a factorization and identify the roots of the left-hand side of our inequality.

$$2x^3 - 13x^2 + 15x = 2x\left(x - \frac{3}{2}\right)(x - 5)$$

So the dividers in our diagram will be the roots $x = 0, \frac{3}{2}$, and 5. Below is a chart for testing the intervals in our sign diagram, as well as the end result.

Interval	Test Value	Signs	Result
$(-\infty, 0)$	$x = -1$	$(-)(-)(-)$	$-$
$(0, \frac{3}{2})$	$x = 1$	$(+)(-)(-)$	$+$
$(\frac{3}{2}, 5)$	$x = 3$	$(+)(+)(-)$	$-$
$(5, \infty)$	$x = 6$	$(+)(+)(+)$	$+$



So, using our diagram as an aide, we see that the solution to the inequality

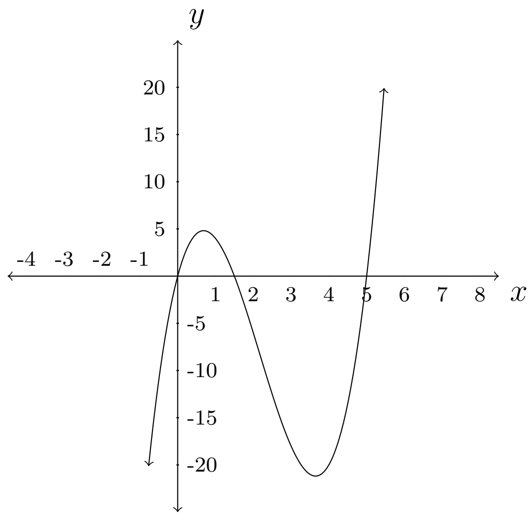
$$2x^3 - 13x^2 + 15x \geq 0,$$

as well as our original inequality

$$x^4 + 6x^2 - 15x \leq x^4 + 2x^3 - 7x^2,$$

will be

$$\left[0, \frac{3}{2}\right] \cup [5, \infty).$$



Since our given inequality was inclusive ($\leq$ or $\geq$), we include the corresponding endpoints in our answer.

We can verify that our answer is correct by comparing it to the graph of the function

$$f(x) = 2x^3 - 13x^2 + 15x,$$

which lies above (or on) the $x-$axis over the intervals in our answer.

II - Demo/Discussion Problems:

Solve each polynomial inequality below, expressing your answers using interval notation. Use Desmos to help confirm that each answer is correct.

1. $x^3 < 4x^2$

2. $x^3 - 7x^2 \leq 12x - 84$

3. $(x-1)^2 \geq 4$

4. $2x^4 > 5x^2 + 3$

III - Practice Problems:

Solve each polynomial inequality below, expressing your answers using interval notation. Use Desmos to help confirm that each answer is correct.

1. $x^4 + x^2 \geq 6$
2. $x^4 - 9x^2 \leq 4x - 12$
3. $4x^3 \geq 3x + 1$
4. $x^4 \leq 16 + 4x - x^3$
5. $3x^2 + 2x < x^4$
6. $\dfrac{x^3 + 2x^2}{2} < x + 2$
7. $\dfrac{x^3 + 20x}{8} \geq x^2 + 2$
8. $19x^2 + 20 > 2x^3 + 49x$

Lesson 54: Rational Function Introduction and Terminology

Objective: Define and Identify key features of rational functions.

Students will be able to:

- Find the y-intercept of a rational function.
- Find the x-intercept(s) of a rational function.
- State the domain of a rational function.

Prerequisite Knowledge:

- Factor polynomials.
- Evaluate functions.
- Write the domain of a function.

Lesson:
I - Motivating Example(s):

Example: $f(x) = \dfrac{(x-2)^2(x+3)}{2(x-6)(1-x)}$

To find the $y-$intercept of the graph of f, we evaluate $f(0)$ below.

$$f(0) = \frac{(0-2)^2(0+3)}{2(0-6)(1-0)} = \frac{(-2)^2(3)}{2(-6)(1)} = \frac{12}{-12} = -1$$

Hence, the $y-$intercept of the graph of f is $(0, -1)$.

To identify the domain of a rational function $f(x) = \frac{p(x)}{q(x)}$, we must eliminate all real numbers x which make the denominator equal to zero. In other words, the domain of f is the set of all x such that $q(x) \neq 0$.

The domain of f is $x \neq 6$ or 1, or, in interval notation, $(-\infty, 1) \cup (1, 6) \cup (6, \infty)$.

To find all possible $x-$intercepts for the graph of $f(x) = \frac{p(x)}{q(x)}$, we set the function equal to zero and solve for all possible x, keeping *only* those values that are also in our domain.

The numerator has two zeros, $x = 2$ and -3, that do not also make the denominator equal to zero. Written as coordinates, the x-intercepts are $(2, 0)$ and $(-3, 0)$.

II - Demo/Discussion Problems:

Find the domain, x-intercept(s), and y-intercept of each function. Write your intercepts as coordinate pairs.

1. $f(x) = \dfrac{x^3 - x^2 - 8x + 12}{-2x^2 + 14x - 12}$

2. $g(x) = \dfrac{(x-2)^2(x+3)}{2(x-6)(1-x)}$

3. $h(x) = \dfrac{3(x+4)(x-2)^2}{(x+3)^2(2x-3)}$

4. $j(x) = \dfrac{-x^2 - 4x + 45}{2x^3 - 5x^2 - 18x + 45}$

III - Practice Problems:

Find the domain, x-intercept(s), and y-intercept of each function. Write your intercepts as coordinate pairs.

1. $f(x) = \dfrac{(x-5)(x-4)}{x+3}$

2. $g(x) = \dfrac{x^2 - 4x}{x^2 - 4}$

3. $h(x) = \dfrac{x^2 + 1}{x^3 + 3x^2 - 10x}$

4. $j(x) = \dfrac{2x^2 - x - 10}{3x^2 + x - 10}$

5. $k(x) = \dfrac{3x^2 + x - 10}{2x^2 - x - 10}$

6. $m(x) = \dfrac{x^3 - 25x}{x^4 - 16x^2}$

7. $n(x) = \dfrac{x(x-3)}{(2x-1)(x+5)}$

Lesson 55: Sign Diagrams for Rational Functions

CC attribute: *College Algebra* by C. Stitz and J. Zeager.

Objective: Solve rational inequalities by constructing a sign diagram.

Students will be able to:

- Create a sign diagram for a rational expression.
- Identify and express the solution to a rational inequality using interval notation.

Prerequisite Knowledge:

- Factoring.
- The Rational Root Theorem.
- Polynomial and /or synthetic division.
- Definition and associated terminology of a rational function.
- Domain of a rational function.
- Interval notation.

Lesson:
I - Motivating Example(s):

Whenever we are asked to find when a rational expression or function f is positive, negative, ≥ 0, or ≤ 0, we can always apply the following steps.

1. Identify a complete factorization of the expression.
2. Construct a sign diagram.
3. Find all intervals that correspond to the desired inequality.
4. In the case of $\geq$ or $\leq$, make sure to include any $x-$intercepts.

Solve the inequality $f(x) > 0$ for the following function.

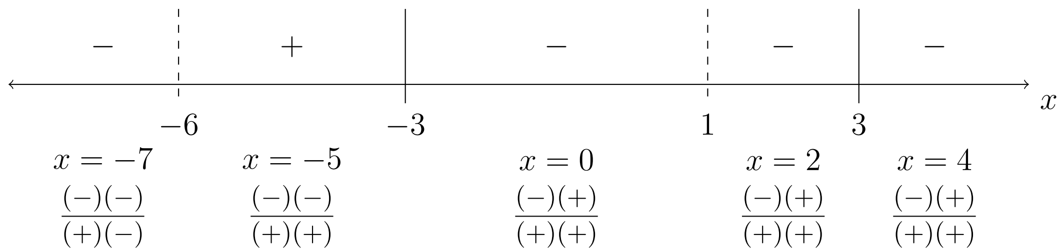
$$f(x) = \frac{-2x^3 + 6x^2 + 18x - 54}{3x^3 + 12x^2 - 33x + 18}$$

Using Desmos to graph our function, we can see that our answer should be $(-6, -3)$. To reach this result algebraically, we will need to find a factored form for f in order to construct a sign diagram. Using our prerequisite knowledge from factoring polynomials, specifically the Rational Root Theorem, one can obtain the following factorization.

$$f(x) = \frac{-2(x+3)(x-3)^2}{3(x+6)(x-1)^2}$$

To find the sign diagram for f, we need to identify our $x-$intercepts, as well as those x not in the domain. From our factorization, we see that this is the set $x = \{-6, -3, 1, 3\}$, with $x \pm 3$ being our intercepts.

Our diagram is shown below.



One important observation in our diagram is in the calculation of each sign. For each test value, we have excluded the *squared* factors in the numerator and denominator, since both $(x-3)^2$ and $(x-1)^2$ will always contribute a positive sign and not affect the end result. For example, when $x=0$, we get

$$\frac{(-)(+)(-)^2}{(+)(+)(-)^2},$$

which reduces to the result that we see above. Similarly, we could have excluded the $(+)$ that appears in the denominator of each test value's sign calculation, since the constant multiplier of 3 will have no impact on sign.

At this point we are essentially done, since the factorization and construction of our diagram has done the bulk of the work for us. Since we are asked to find all x such that $f(x) > 0$, we see that this equals to the union of all intervals that correspond to a $+$ sign. This gives us our anticipated answer of $(-6, -3)$.

Recalling our discussion of the last example, if we wished to answer the follow-up question of when $f(x) \geq 0$, we would just need to include all boundary values in our diagram that correspond to $x-$intercepts (when $f(x) = 0$). From our diagram, this would be any value of x that has a *solid* divider, remembering that dashed dividers correspond to values not in our domain. In this case, the function $f(x) \geq 0$ for all x in the set $(-6, -3] \cup \{3\}$.

II - Demo/Discussion Problems:

Solve each of the following inequalities. Use Desmos to confirm your answers.

1. $\dfrac{4x^2 - 4x + 1}{x^3 - x^2 - 17x - 15} \leq 0$

2. $\dfrac{x-6}{x} \geq \dfrac{-2}{x-1}$

III - Practice Problems:

Solve each of the following inequalities. Use Desmos to confirm your answers.

1. $\dfrac{(x-5)(x+4)}{x-1} \leq 0$
2. $\dfrac{x+1}{x-1} \geq 0$
3. $\dfrac{(x+1)(x-3)}{x+2} \geq 0$
4. $\dfrac{x^2-25}{x^2-1} \leq 0$
5. $\dfrac{x^2+x-12}{x^3-25x} \leq 0$

Lesson 56: Horizontal Asymptotes

Objective: Identify a horizontal asymptote in the graph of a rational function.

Students will be able to:

- Distinguish between and summarize the three cases for the end behavior of the graph of a rational function.
- Identify the equation of a horizontal asymptote in the graph of a rational function.
- Use appropriate notation to describe the end behavior of the graph of a rational function.

Prerequisite Knowledge:

- End behavior of polynomials, including degree and leading coefficient.

Lesson:

In this lesson, we will look at the end (or long run) behavior of the graph of a rational function f, as $x \to \pm\infty$.

Let $f(x) = \dfrac{p(x)}{q(x)}$ be a rational function with leading terms $a_n x^n$ and $b_m x^m$ of $p(x)$ and $q(x)$, respectively.

- If $n = m$, the graph of f will have a horizontal asymptote at $y = \dfrac{a_n}{b_m}$.
- If $n < m$, the graph of f will have a horizontal asymptote at $y = 0$.
- If $n > m$, the graph of f will not have a horizontal asymptote.

Since any polynomial is, by definition, also a rational function, we will begin by including the possibilities that $f(x) \to \infty$ or $f(x) \to -\infty$ for either the left (as $x \to -\infty$) or right (as $x \to \infty$) end behavior of the graph of a rational function f.

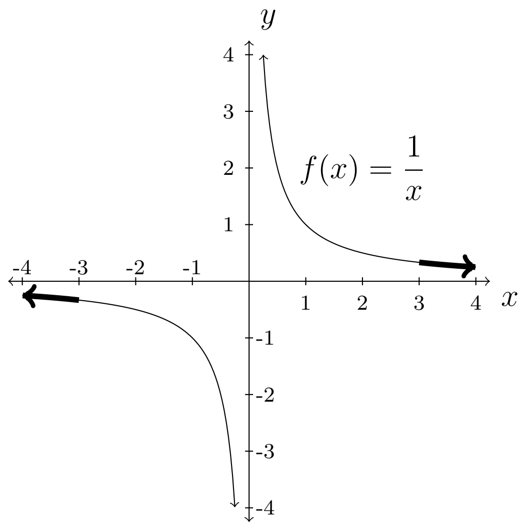
Recall that we used two aspects of a polynomial to identify the end behavior of its graph:

1. the parity of the degree (even or odd), and
2. the sign of the leading coefficient (positive or negative).

As with polynomials, we will use the degree and leading coefficient of both the numerator and denominator of a rational function f, to identify the end behavior of its graph.

I - Motivating Example(s):

Let us consider the graph of the reciprocal function $f(x) = \dfrac{1}{x}$, shown below.



This example presents us with the first instance in which a graph does not tend towards either ∞ or $-\infty$, but instead "levels off" as the values of x grow in either the positive (right) or negative (left) direction.

$$\text{As } x \to \infty,\ f(x) \to 0^+.$$

$$\text{As } x \to -\infty,\ f(x) \to 0^-.$$

Here, we use a $+$ or $-$ in the exponent to further describe how the tails of the graph approach 0, either from *above* $(+)$ or from *below* $(-)$. These identifiers can just as easily be omitted entirely, but provide a bit more insight into the graph of the function f. The tails of the graph are thickened for additional emphasis of this concept.

In fact, for any real number k, we can transform the graph above, by simply adding k to the function, to produce a new rational function whose graph levels off at k. The resulting graph represents a vertical shift of the graph of $\frac{1}{x}$ by k units. The shift is up when $k > 0$ and down when $k < 0$.

II - Demo/Discussion Problems:

Identify the equation of the horizontal asymptote, if one exists, for each of the following functions, and describe the end behavior of the function (As $x \to \pm\infty, \ldots$). Use Desmos to verify your answers.

1. $f(x) = \dfrac{-2x+4}{x-5}$
2. $g(x) = \dfrac{x-1}{x^2-4}$
3. $h(x) = \dfrac{x^2-4}{x-1}$
4. $j(x) = \dfrac{18x^3-4x-1}{6x^3-2x^2-8x}$

III - Practice Problems:

Identify the equation of the horizontal asymptote, if one exists, for each of the following functions, and describe the end behavior of the function (As $x \to \pm\infty, \ldots$). Use Desmos to verify your answers.

1. $a(x) = \dfrac{-x+4}{x-9}$
2. $b(x) = \dfrac{x^4 - x^3 - 2x - 19}{x^5 - 4}$
3. $c(x) = \dfrac{x^2 - 4}{5x^2 - 1}$
4. $d(x) = \dfrac{18x^7 - 4x - 1}{3x^7 - 2x^2 - 8x}$
5. $m(x) = \dfrac{18x^3 - 4x - 1}{3x^6 - 2x^2 - 8x}$
6. $n(x) = \dfrac{(x^2-9)(x-7)}{(x+3)(x-4)(2x-5)}$
7. $p(x) = \dfrac{-x^2+4}{x-9}$
8. $q(x) = \dfrac{x^3}{x^4 - 4}$
9. $r(x) = \dfrac{x^4 - 4}{x^3}$
10. $t(x) = \dfrac{15x - 10}{5x - 19}$

Lesson 57: Slant and Curvilinear Asymptotes

Objective: Identify a slant or curvilinear asymptote in the graph of a rational function.

Students will be able to:

- Determine the existence of a slant or curvilinear asymptote in the graph of a rational function.
- Identify the equation of a slant or curvilinear asymptote for the graph of a rational function.

Prerequisite Knowledge:

- Rational function end behavior criteria.
- Polynomial and /or synthetic division.

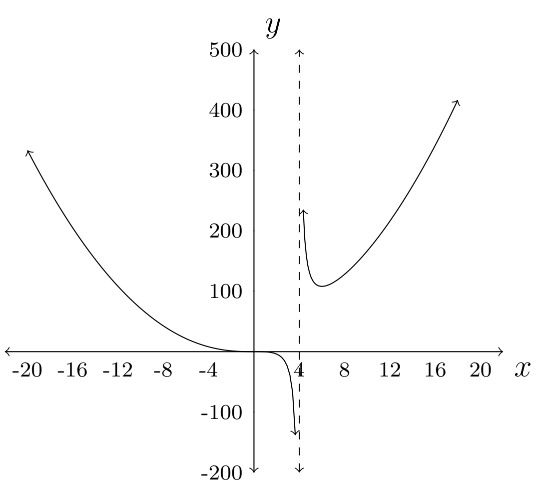
Lesson:

For a rational function

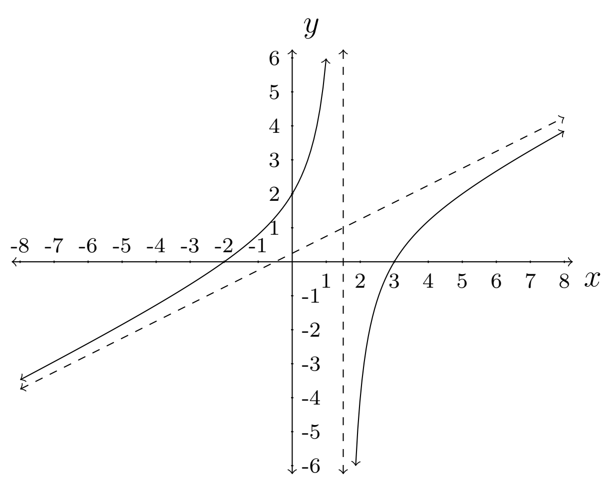
$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0},$$

when $n > m$, we know that the graph of f will have no horizontal asymptotes. Depending upon the difference between n and m, however, there is more to discover about the nature of the graph of f, as $x \to \pm\infty$.

I - Motivating Example(s):



$$f(x) = \frac{x^3}{x-4}$$



$$g(x) = \frac{x^2 - x - 6}{2x - 3}$$

In the case of $g(x) = \dfrac{x^2 - x - 6}{2x - 3}$, we see that as $x \to \pm\infty$, the graph of g actually approaches a linear asymptote. Whereas horizontal asymptotes are horizontal lines, having a slope of zero, this new type of linear asymptote has a non-zero slope and is consequently *slanted*. Hence, we say that the graph of g contains a *slant* or *oblique asymptote*. This occurs when the degree of the numerator is *one* more than the denominator, resulting in a *slanted* or *linear* asymptote.

On the other hand, the graph of $f(x) = \dfrac{x^3}{x - 4}$ does not appear to contain a slant asymptote. In fact, as $x \to \pm\infty$, the graph of f resembles a parabola. In cases such as these, we could say that the graph of f contains a *curvilinear asymptote*. In other words, the graph of f approaches some identifiable non-linear curve, as x approaches $\pm\infty$. This happens because the difference in degree between the numerator and the denominator is 2, hence the asymptote is quadratic.

II - Demo/Discussion Problems:

Use division to identify the equation of the slant or curvilinear asymptote for the graph of each of the following rational functions. Use Desmos to verify your answers.

1. $g(x) = \dfrac{x^2 - x - 6}{2x - 3}$
2. $f(x) = \dfrac{-2x^3 + x^2 - 2x + 3}{x^2 + 1}$
3. $h(x) = \dfrac{x^2 - 5x + 7}{x - 2}$

III - Practice Problems:

Use division to identify the equation of the slant or curvilinear asymptote for the graph of each of the following rational functions. Use Desmos to verify your answers.

1. $a(x) = \dfrac{5x^2 - 1}{x + 3}$
2. $g(x) = \dfrac{x^2 - 1}{x - 4}$
3. $h(x) = \dfrac{x^2 - 4x - 9}{x + 2}$
4. $j(x) = \dfrac{18x^3 - 4x - 1}{3x - 8}$
5. $k(x) = \dfrac{-x^2 + 4}{x - 9}$
6. $p(x) = \dfrac{-x^2 + 4}{x - 9}$
7. $q(x) = \dfrac{x^5}{x^4 - 4x^3 - 2x + 1}$
8. $r(x) = \dfrac{x^4 - 4}{x^3}$
9. $t(x) = \dfrac{15x^2 - 10}{5x - 19}$

Lesson 58: Vertical Asymptotes

CC attribute: *College Algebra* by C. Stitz and J. Zeager.

Objective: Identify one or more vertical asymptotes in the graph of a rational function.

Students will be able to:

- Identify infinite discontinuities and their corresponding vertical asymptotes in the graph of a rational function.

Prerequisite Knowledge:

- Equations of vertical lines.
- Factoring.
- The Rational Root Theorem.
- Polynomial and /or Synthetic Division.

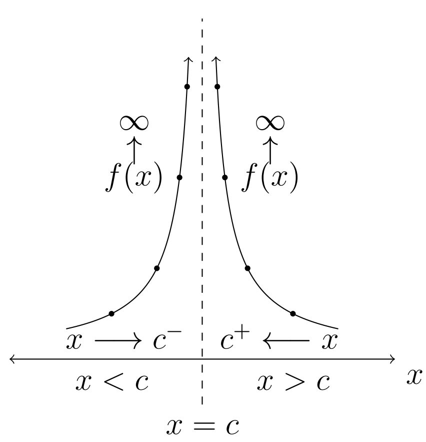
Lesson:

The central idea around a vertical asymptote, say $x = c$, is that as x approaches the value of c, either from the left or the right, the values for the corresponding function $f(x)$ will approach either ∞ or $-\infty$.

$$\text{Approaching from the right:} \quad \text{As } x \to c^+,\ f(x) \to \pm\infty.$$

$$\text{Approaching from the left:} \quad \text{As } x \to c^-,\ f(x) \to \pm\infty.$$

We should be clear here, in that when we say x approaches c *from the right,* what is meant is that we are evaluating the function at values of x that are getting arbitrarily close go c, but are all *greater* than c, i.e., $x > c$. This is precisely why we can write $x \to c^+$ in the statement above. The $+$ in the exponent signifies that $x > c$. The same can be said for when x approaches c from the left. The following graph further illustrates this point.



Our graph above shows that as x approaches c from either direction, the values for $f(x)$ approach $+\infty$. If, instead, we reflected the right-hand side of the graph across the $x-$axis, we would say that as $x \to c^+$, $f(x) \to -\infty$, since the right-hand side would now point downwards.

Up until this point, we have seen several examples of graphs of rational functions that contain vertical asymptotes. We are now ready to formally state the condition for the existence of a vertical asymptote.

> Let $f(x)$ be a rational function and let $g(x)$ represent the simplified expression for f. If $x = c$ is not in the domain of *both* f and g, then the graph of f will have a vertical asymptote at $x = c$.
>
> Alternatively, we could say that a vertical asymptote exists as a zero of a factor in the denominator of $f(x)$, *as long as that zero is it is not also in the numerator.* We will learn later that this would result in a hole.

II - Demo/Discussion Problems:

Identify the equation of any vertical asymptotes for the graph of each of the following rational functions. Use Desmos to verify your answers.

1. $f(x) = \dfrac{-2x+4}{x-5} = \dfrac{-2(x-2)}{x-5}$
2. $g(x) = \dfrac{x^2+25}{x^2-10x+25}$
3. $h(x) = \dfrac{x^2-9x+20}{x^2-3x-10}$

III - Practice Problems:

Identify the equation of any vertical asymptotes for the graph of each of the following rational functions. Use Desmos to verify your answers.

1. $a(x) = \dfrac{5x^2-1}{x+3}$
2. $g(x) = \dfrac{x^2-1}{x-4}$
3. $h(x) = \dfrac{x^2-4x-9}{x+2}$
4. $j(x) = \dfrac{18x^3-4x-1}{x^2-4}$
5. $k(x) = \dfrac{-x^2+4}{x-9}$
6. $p(x) = \dfrac{-x^2+4}{x^2+9}$
7. $q(x) = \dfrac{x^5}{x(x-5)}$
8. $r(x) = \dfrac{x^2-11x+30}{x^2+10x+24}$
9. $t(x) = \dfrac{15x^2-10}{5x-7}$

Lesson 59: Holes

Objective: Identify the precise location of one or more holes in the graph of a rational function.

Students will be able to:

- Identify removable discontinuities and their corresponding holes in the graph of a rational function.

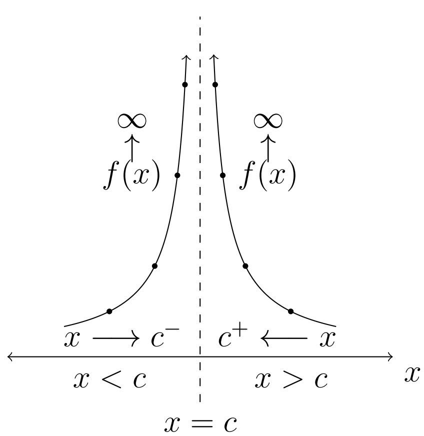
Prerequisite Knowledge:

- Evaluating a function.
- Factoring.
- The Rational Root Theorem.
- Polynomial and /or Synthetic Division.
- Multiplicative identity /inverse.

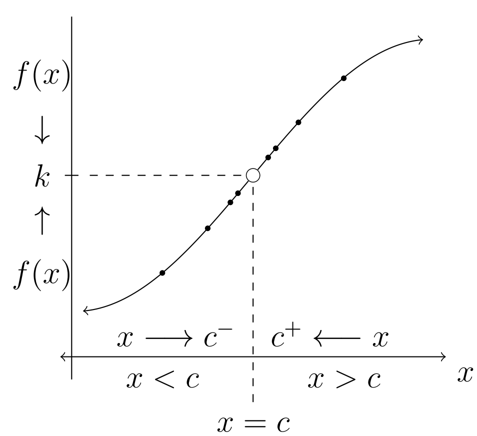
Lesson:

While vertical asymptotes correspond to infinite discontinuities, a hole corresponds to a *removable discontinuity*, since the removal of a single point along a continuous curve creates the hole.

Suppose that the rational function $f(x)$ has a discontinuity at $x = c$, i.e., c is not in the domain of f. If $x = c$ is a vertical asymptote of the graph of f, in the last lesson we saw that as $x \to c$, $f(x) \to \pm\infty$. If $x = c$ represents a hole in the graph of f, however, we will see that as $x \to c$, $f(x) \to k$, for some real number k. This is the fundamental difference between infinite and removable discontinuities.



Infinite Discontinuity



Removable Discontinuity

In the case of the graph of the left, recall that we have the following statements.

$$\text{As } x \to c^+,\ f(x) \to \infty. \qquad\qquad \text{As } x \to c^-,\ f(x) \to \infty.$$

Similarly, in the case of the graph on the right, we employ the same idea, using k^+ and k^- in order to identify whether or not the graph of f approaches k from *above* if $f(x) > k$ and *below* if $f(x) < k$.

$$\text{As } x \to c^+,\ f(x) \to k^+. \qquad\qquad \text{As } x \to c^-,\ f(x) \to k^-.$$

In virtually all cases, however, it will be sufficient enough to simply state that as $x \to c,\ f(x) \to k$, since further analysis will often prove difficult.

We now state the requirement for a hole, which, as with vertical asymptotes, depends on both the rational function f and its simplified expression.

> Let $f(x)$ be a rational function and let $g(x)$ represent the simplified expression for f. If $x = c$ is not in the domain of f, but *is* in the domain of g, then the graph of f will have a hole at $(c, g(c))$.

II - Demo/Discussion Problems:

Identify the coordinates of any holes in the graph of each of the following rational functions. Use Desmos to verify your answers.

1. $f(x) = \dfrac{-2x+4}{4x-8}$
2. $g(x) = \dfrac{x^2+25}{x^2-10x+25}$
3. $h(x) = \dfrac{x^2-9x+20}{x^2-3x-10}$

III - Practice Problems:

Identify the coordinates of any holes in the graph of each of the following rational functions. Use Desmos to verify your answers.

1. $a(x) = \dfrac{2x+6}{x+3}$
2. $g(x) = \dfrac{x^3-16x}{x^2-4x}$
3. $h(x) = \dfrac{x^2-4x-9}{x+2}$
4. $j(x) = \dfrac{18x^3-4x-1}{x^2-4}$
5. $k(x) = \dfrac{x^2-17x+72}{x-9}$
6. $p(x) = \dfrac{x^2+6x+8}{2x+4}$
7. $q(x) = \dfrac{x^5}{x(x-5)}$
8. $r(x) = \dfrac{x^2-11x+30}{x^2-36}$
9. $t(x) = \dfrac{3x^2-12x+12}{x^2+2x-8}$
10. $v(x) = \dfrac{x^2+2x-8}{3x^2-12x+12}$

Lesson 60: Rational Functions Graphing Summary

CC attribute: *College Algebra* by C. Stitz and J. Zeager.

Objective: Graph a rational function in its entirety.

Students will be able to:

- Identify all important aspects of the graph of a rational function: intercepts, holes, and any vertical, horizontal, slant, or curvilinear asymptotes.
- Sketch a complete graph of a rational function using a sign diagram.

Prerequisite Knowledge:

- Evaluating a function.
- Factoring.
- The Rational Root Theorem.
- Polynomial and /or Synthetic Division.
- Multiplicative identity /inverse.
- Sign Diagrams.
- Rational function features.

Lesson:

At this point, we have addressed all key features of rational functions individually. This section pulls each of these aspects together, for a detailed analysis of a rational function, culminating in a complete sketch of its graph. Along the way, we will need to address each of the following aspects for our rational function $f(x) = \dfrac{p(x)}{q(x)}$. It is important to note that there is no universally accepted order to this checklist.

- Find the $y-$intercept of the graph of f, $(0, f(0))$, if it exists.
- Use the degrees and leading coefficients of p and q to determine whether the graph of f has a horizontal asymptote. If the graph of f has a slant asymptote, use polynomial division to find where it is located.
- Identify a complete factorization of f, and use it to find the domain of the function. This is the set of all x, such that $q(x) \neq 0$.
- Find any $x-$intercepts of the graph of f. This is the set of all x in the domain of f, such that $p(x) = 0$. Using multiplicities, classify each $x-$intercept as a crossover or turnaround ("bounce") point.

- Find the simplified expression g for the given function f, and use it to identify any vertical asymptotes or holes in the graph of f. Use multiplicities to help visualize the nature of the graph of f near its vertical asymptotes. If f has a hole at $x = c$, use g to help plot the hole's precise location at $(c, g(c))$.
- Using both the $x-$intercepts and the discontinuities (those x not in the domain), construct a sign diagram for f.

In each rational function we encounter, we will carefully examine the function, making sure not to omit any of the checklist items above and to compare each item to those that precede it along the way for accuracy. Although the process will take some time, if we are thorough, our end result should be a complete, accurate sketch of the given rational function.

I - Motivating Example(s):

Example: Sketch a complete graph of the rational function below, making sure to have a clearly defined scale and label all key features of your graph (intercepts, asymptotes, and holes).

$$f(x) = \frac{4x^2}{x^3 + 3x^2 - 4x}$$

In this example, we see that the graph of f will not have a $y-$intercept, since $f(0) = \frac{0}{0}$, which is undefined.

Since the degree of the numerator is less than the degree of the denominator, we conclude that the graph of f has a horizontal asymptote along the $x-$axis, $y = 0$.

Our graph also has no $x-$intercepts, since our numerator only equals zero when $x = 0$, which we know is not in our domain of f.

Furthermore, using the knowledge from our previous lessons, we can find the following factorization of f.

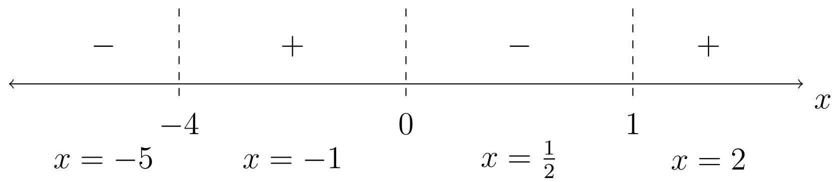
$$f(x) = \frac{4x^2}{x(x+4)(x-1)}$$

Consequently, f has corresponding domain $x \neq -4, 0, 1$, and related simplified expression

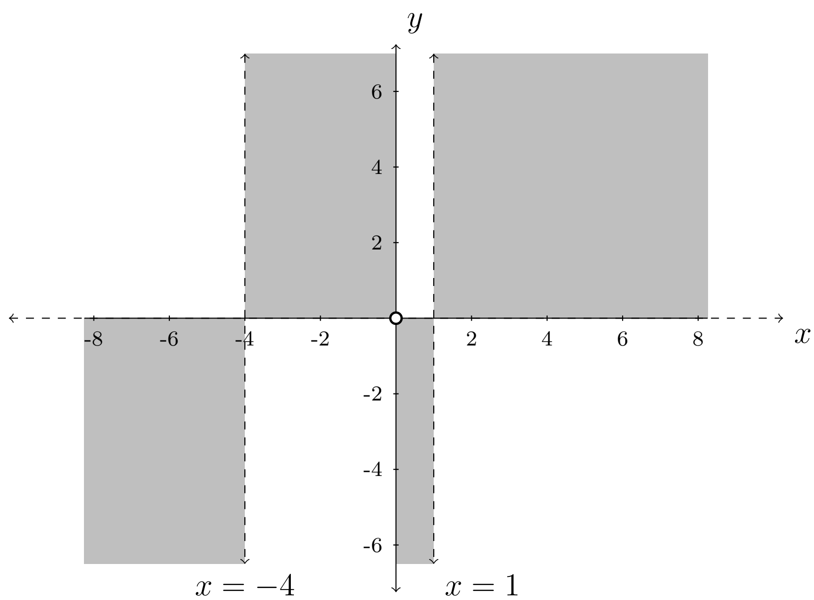
$$g(x) = \frac{4x}{(x+4)(x-1)}.$$

The graph of f has vertical asymptotes at $x = -4$ and $x = 1$ and a hole at $(0, g(0)) = (0, 0)$.

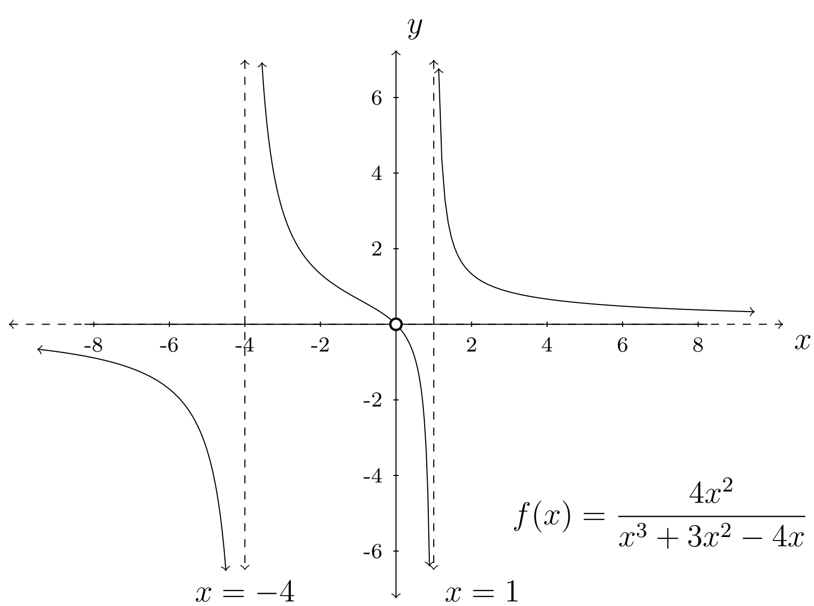
Since the multiplicities of both $x = -4$ and $x = 1$ in the denominator of f are both one (odd), we know that the graph of f will approach each vertical asymptote from opposite sides of the $x-$axis. The following sign diagram confirms this observation.



We are now ready to try our hand at graphing f, and begin our graph by defining a scale for both the $x-$ and $y-$axes, and identifying all intercepts, and asymptotes. This should always be our first step to successfully sketching a decent-looking graph. To emphasize this point, we first show an initial graph that identifies each of these features, and further shades those areas of the $xy-$plane that correspond to our sign diagram above.



We now carefully sketch the graph of f based upon our findings.



II - Demo/Discussion Problems:

Sketch a complete graph of each rational function below, making sure to have a clearly defined scale and label all key features of your graph (intercepts, asymptotes, and holes). Use Desmos to compare your results.

1. $f(x) = \dfrac{x^3 - 16x}{3x^2 - 6x - 24}$
2. $g(x) = \dfrac{4x^2}{x^3 + 3x^2 - 4x}$

III - Practice Problems:

Sketch a complete graph of each rational function below, making sure to have a clearly defined scale and label all key features of your graph (intercepts, asymptotes, and holes). Use Desmos to compare your results.

1. $a(x) = \dfrac{2x + 6}{x + 3}$
2. $g(x) = \dfrac{x^3 - 16x}{x^2 - 4x}$
3. $h(x) = \dfrac{x^2 - 4x - 9}{x + 2}$
4. $j(x) = \dfrac{18x^3 - 4x - 1}{x^2 - 4}$
5. $k(x) = \dfrac{x^2 - 17x + 72}{x - 9}$
6. $p(x) = \dfrac{x^2 + 6x + 8}{2x + 4}$
7. $q(x) = \dfrac{x^5}{x(x - 5)}$
8. $r(x) = \dfrac{x^2 - 11x + 30}{x^2 - 36}$
9. $t(x) = \dfrac{3x^2 - 12x + 12}{x^2 + 2x - 8}$
10. $v(x) = \dfrac{x^2 + 2x - 8}{3x^2 - 12x + 12}$