College Algebra Textbook

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This textbook is designed as the primary resource for instruction of a traditional College Algebra course at Framingham State University. Each section follows closely with its respective lesson(s) in the accompanying course pack, but offers more detailed explanations and additional worked out examples.

Although largely free of mathematical errors and "typos", students who identify any errors/typos in either the textbook or course pack are encouraged to report them to the instructor, and the reporting of any mathematical errors will be rewarded with small incentives in the form of additional course homework, quiz, or exam points.

The following chapters make up the first half of the course and cover the following content.

- Linear Equations and Inequalities
- Systems of Linear Equations
- Introduction to Functions
- Quadratic Equations and Inequalities

The following chapters make up the second half of the course and cover the following content.

- Advanced Function Concepts
- Polynomials
- Rational Functions

This text contains original content, as well as content adapted from each of the following open-source texts.

Beginning and Intermediate Algebra by T. Wallace. CC-BY
College Algebra by C. Stitz and J. Zeager. CC-BY-NC-SA

Contents

Measurable Outcomes

Below is a comprehensive list of the anticipated measurable outcomes and some essential prerequisite skills needed for successful completion of the College Algebra course. This list is based off of the course description and exit list topics of MATH 123 College Algebra at Framingham State University. Each outcome number aligns to its respective lesson in the accompanying course pack.

1 Solve general linear equations with variables on both sides of the equation.

2 Solve an equation that contains one or more absolute value(s).

3 Graph a linear equation by creating a table of values for x.
Identify the slope of a linear equation both graphically and algebraically.

4 Write the equation of a line in slope-intercept and point-slope form.

5 Write the equation of a line given a line parallel or perpendicular.

6 Solve, graph, and give interval notation for the solution to a linear inequality.
Create a sign diagram to identify those intervals where a linear expression is positive or negative.

7 Solve, graph, and give interval notation to the solution of a compound inequality.

8 Solve, graph, and give interval notation to the solution of an inequality containing absolute values.

9 Solve linear systems by graphing.

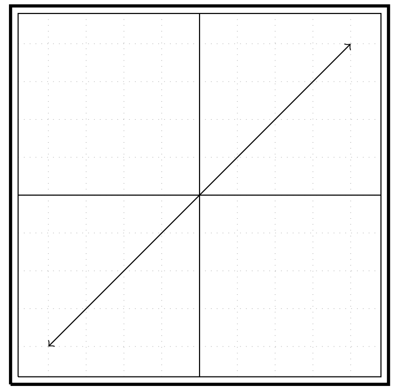
10 Solve linear systems by substitution.

11 Solve linear systems by addition and elimination.

12 Define a relation and a function; determine if a relation is a function.

13 Evaluate functions using appropriate notation.

14 Find the domain and range of a function from its graph.



Chapter 1

Linear Equations and Inequalities

Solving Linear Equations

One-Step Equations

Objective: Solve one-step linear equations by balancing using inverse operations.

Solving linear equations is an important and fundamental skill in algebra. In algebra, we are often presented with a problem where the answer is known, but part of the problem is missing. The missing part of the problem is what we seek to find. An example of such a problem is shown below.

Example 1.

$$4x + 16 = -4$$

Notice the above problem has a missing part, or unknown, that is marked by x. If we are given that the solution to this equation is $x = -5$, it could be plugged into the equation, replacing the x with -5. This is shown in Example 2.

Example 2.

$$\begin{aligned} 4(-5) + 16 &= -4 && \text{Multiply } 4(-5) \\ -20 + 16 &= -4 && \text{Add } -20 + 16 \\ -4 &= -4 && \text{True!} \end{aligned}$$

Now the equation comes out to a true statement! Notice also that if another number, for example, $x = 3$, was plugged in, we would not get a true statement as seen in Example 3.

Example 3.

$$\begin{aligned} 4(3) + 16 &= -4 && \text{Multiply } 4(3) \\ 12 + 16 &= -4 && \text{Add } 12 + 16 \\ 28 &\neq -4 && \text{False!} \end{aligned}$$

Due to the fact that this is not a true statement, this demonstrates that $x = 3$ is not the solution. However, depending on the complexity of the problem, this "guess and check" method is not very efficient. Thus, we take a more algebraic approach to solving equations. Here we will focus on what are called "one-step equations" or equations that only require one step to solve. While these equations often seem very fundamental, it is important to master the pattern for solving these problems so we can solve more complex problems.

Addition Problems

To solve equations, the general rule is to do the opposite, as demonstrated in the following example.

Example 4.

$$\begin{array}{rl} x + 7 = -5 & \text{The 7 is added to the } x \\ \underline{\mathbf{-7 \quad -7}} & \text{Subtract 7 from both sides to get rid of it} \\ x = -12 & \text{Our solution} \end{array}$$

It is important for the reader to recognize the benefit of checking an answer by plugging it back into the given equation, as we did with examples 2 and 3 above. This is a step that often gets overlooked by many individuals who may be eager to attempt the next problem. As is the case with most textbooks, we will often omit this step from this point forward, with the understanding that it will usually be an exercise that is left to the reader to verify the validity of each answer.

The same process is used in each of the following examples.

$4 + x = 8$	$7 = x + 9$	$5 = 8 + x$
$\underline{\mathbf{-4 \qquad -4}}$	$\underline{\mathbf{-9 \quad -9}}$	$\underline{\mathbf{-8 \quad -8}}$
$x = 4$	$-2 = x$	$-3 = x$

Table 1.1: Addition Examples

Subtraction Problems

In a subtraction problem, we get rid of negative numbers by adding them to both sides of the equation, as demonstrated in the following example.

Example 5.

$$\begin{array}{rl} x - 5 = 4 & \text{The 5 is negative, or subtracted from } x \\ \underline{\mathbf{+5 \quad +5}} & \text{Add 5 to both sides} \\ x = 9 & \text{Our solution} \end{array}$$

The same process is used in each of the following examples. Notice that each time we are getting rid of a negative number by adding.

In every example, we introduce the opposite operation of what is shown, in order to solve the given equation. This notion of opposites is more commonly referred to as an *inverse* operation. The inverse operation of addition is subtraction, and vice versa. Similarly, the inverse operation of multiplication is division, and vice versa, which we will see momentarily.

$\begin{array}{r} -6 + x = -2 \\ \underline{\mathbf{+6} \qquad \mathbf{+6}} \\ x = 4 \end{array}$	$\begin{array}{r} -10 = x - 7 \\ \underline{\mathbf{+7} \qquad \mathbf{+7}} \\ -3 = x \end{array}$	$\begin{array}{r} 5 = -8 + x \\ \underline{\mathbf{+8} \quad \mathbf{+8}} \\ 13 = x \end{array}$

Table 1.2: Subtraction Examples

Multiplication Problems

With a multiplication problem, we get rid of the number by dividing on both sides, as demonstrated in the following examples.

Example 6.

$$\begin{array}{rl} 4x = 20 & \text{Variable is multiplied by } 4 \\ \overline{\mathbf{4}} \quad \overline{\mathbf{4}} & \text{Divide both sides by } 4 \\ x = 5 & \text{Our solution} \end{array}$$

With multiplication problems it is very important that care is taken with signs. If x is multiplied by a negative then we will divide by a negative. This is shown in example 7.

Example 7.

$$\begin{array}{rl} -5x = 30 & \text{Variable is multiplied by } -5 \\ \overline{\mathbf{-5}} \quad \overline{\mathbf{-5}} & \text{Divide both sides by } -5 \\ x = -6 & \text{Our solution} \end{array}$$

The same process is used in each of the following examples. Notice how negative and positive numbers are handled as each problem is solved.

$\begin{array}{r} 8x = -24 \\ \overline{\mathbf{8}} \qquad \overline{\mathbf{8}} \\ x = -3 \end{array}$	$\begin{array}{r} -4x = -20 \\ \overline{\mathbf{-4}} \quad \overline{\mathbf{-4}} \\ x = 5 \end{array}$	$\begin{array}{r} 42 = 7x \\ \overline{\mathbf{7}} \quad \overline{\mathbf{7}} \\ 6 = x \end{array}$

Table 1.3: Multiplication Examples

Division Problems

In division problems, we get rid of the denominator by multiplying on both sides, since multiplication is the opposite, or *inverse*, operation of division. This is demonstrated in the examples shown below.

Example 8.

$$\begin{array}{rl} \dfrac{x}{5} = -3 & \text{Variable is divided by 5} \\ (\mathbf{5})\dfrac{x}{5} = -3(\mathbf{5}) & \text{Multiply both sides by 5} \\ x = -15 & \text{Our solution} \end{array}$$

$\frac{x}{-7} = -2$	$\frac{x}{8} = 5$	$\frac{x}{-4} = 9$
$(\mathbf{-7})\frac{x}{-7} = -2(\mathbf{-7})$	$(\mathbf{8})\frac{x}{8} = 5(\mathbf{8})$	$(\mathbf{-4})\frac{x}{-4} = 9(\mathbf{-4})$
$x = 14$	$x = 40$	$x = -36$

Table 1.4: Division Examples

The process described above is fundamental to solving equations. Once this process is mastered, the problems we will see have several more steps. These problems may seem more complex, but the process and patterns used will remain the same.

Two-Step Equations

Objective: Solve two-step equations by balancing and using inverse operations.

After mastering the technique for solving one-step equations, we are ready to consider two-step equations. As we solve two-step equations, the important thing to remember is that everything works backwards! When working with one-step equations, we learned that in order to clear a "plus five" in the equation, we would subtract five from both sides. We learned that to clear "divided by seven" we multiply by seven on both sides. The same pattern applies to the order of operations. When solving for our variable x, we use order of operations backwards as well. This means we will add or subtract first, then multiply or divide second (then exponents, and finally any parentheses or grouping symbols, but that's another lesson).

Example 9.

$$4x - 20 = -8$$

We have two numbers on the same side as the x. We need to move the 4 and the 20 to the other side. We know to move the 4 we need to divide, and to move the 20 we will add 20 to both sides. If order of operations is done backwards, we will add or subtract first. Therefore we will add 20 to both sides first. Once we are done with that, we will divide both sides by 4. The steps are shown below.

$$\begin{array}{rl} 4x - 20 = -8 & \text{Start by focusing on the subtract 20} \\ \underline{\mathbf{+20} \quad \mathbf{+20}} & \text{Add 20 to both sides} \\ 4x = 12 & \text{Now we focus on the 4 multiplied by } x \\ \overline{\mathbf{4}} \quad \overline{\mathbf{4}} & \text{Divide both sides by 4} \\ x = 3 & \text{Our solution} \end{array}$$

Notice in our next example when we replace the x with 3 we get a true statement.

$$\begin{aligned} 4(3) - 20 &= -8 && \text{Multiply } 4(3) \\ 12 - 20 &= -8 && \text{Subtract } 12 - 20 \\ -8 &= -8 && \text{True!} \end{aligned}$$

The same process is used to solve any two-step equation. Add or subtract first, then multiply or divide.

Example 10.

$$\begin{array}{rl} 5x + 7 = 7 & \text{Start by focusing on the plus 7} \\ \underline{\mathbf{-7 \quad -7}} & \text{Subtract 7 from both sides} \\ 5x = 0 & \text{Now focus on the multiplication by 5} \\ \overline{\mathbf{5}} \quad \overline{\mathbf{5}} & \text{Divide both sides by 5} \\ x = 0 & \text{Our solution} \end{array}$$

Notice the seven subtracted out completely! Many students get stuck on this point, do not forget that we have a number for "nothing left", and that number is zero. With this in mind the process is almost identical to our first example.

A common error students make with two-step equations is with negative signs. Remember the sign always stays with the number. Consider the following example.

Example 11.

$$\begin{array}{rl} 4 - 2x = 10 & \text{Start by focusing on the positive 4} \\ \underline{\mathbf{-4 \qquad -4}} & \text{Subtract 4 from both sides} \\ -2x = 6 & \text{Negative (subtraction) stays on the } 2x \\ \overline{\mathbf{-2}} \quad \overline{\mathbf{-2}} & \text{Divide by } -2 \\ x = -3 & \text{Our solution} \end{array}$$

The same is true even if there is no apparent coefficient in front of the variable. The coefficient is 1 or -1 in this case. Consider the next example.

Example 12.

$$\begin{array}{rl} 8 - x = 2 & \text{Start by focusing on the positive 8} \\ \underline{\mathbf{-8 \qquad -8}} & \text{Subtract 8 from both sides} \\ -x = -6 & \text{Negative(subtraction) stays on the } x \\ -1x = -6 & \text{Remember, no number in front of variable means 1} \\ \overline{\mathbf{-1}} \quad \overline{\mathbf{-1}} & \text{Divide both sides by } -1 \\ x = 6 & \text{Our solution} \end{array}$$

$$\begin{array}{rcl} -3x+7 & = & -8 \\ \underline{\mathbf{-7}} & & \underline{\mathbf{-7}} \\ -3x & = & -15 \\ \overline{\mathbf{-3}} & & \overline{\mathbf{-3}} \\ x & = & 5 \end{array} \qquad \begin{array}{rcl} -2+9x & = & 7 \\ \underline{\mathbf{+2}} & & \underline{\mathbf{+2}} \\ 9x & = & 9 \\ \overline{\mathbf{9}} & & \overline{\mathbf{9}} \\ x & = & 1 \end{array} \qquad \begin{array}{rcl} 8 & = & 2x+10 \\ \underline{\mathbf{-10}} & & \underline{\mathbf{-10}} \\ -2 & = & 2x \\ \overline{\mathbf{2}} & & \overline{\mathbf{2}} \\ -1 & = & x \end{array}$$

$$\begin{array}{rcl} 7-5x & = & 17 \\ \underline{\mathbf{-7}} & & \underline{\mathbf{-7}} \\ -5x & = & 10 \\ \overline{\mathbf{-5}} & & \overline{\mathbf{-5}} \\ x & = & -2 \end{array} \qquad \begin{array}{rcl} -5-3x & = & -5 \\ \underline{\mathbf{+5}} & & \underline{\mathbf{+5}} \\ -3x & = & 0 \\ \overline{\mathbf{-3}} & & \overline{\mathbf{-3}} \\ x & = & 0 \end{array} \qquad \begin{array}{rcl} -3 & = & \frac{x}{5}-4 \\ \underline{\mathbf{+4}} & & \underline{\mathbf{+4}} \\ & & \\ (\mathbf{5})(1) & = & \frac{x}{5}(\mathbf{5}) \\ 5 & = & x \end{array}$$

Table 1.5: Two-Step Equation Examples

Solving two-step equations is a very important skill to master, as we study algebra. The first step is to add or subtract, the second is to multiply or divide. This pattern is seen in each of our examples thus far.

As problems in algebra become more complex the process covered here will remain the same. In fact, as we solve problems like those in the next example, each one of them will have several steps to solve, but the last two steps will resemble solving a two-step equation. This is why it is very important to master two-step equations now!

Example 13.

$$3x^2+4-x=6 \qquad \frac{1}{x-8}+\frac{1}{x}=\frac{1}{3}$$

$$\sqrt{5x-5}+1=x \qquad \log_5(2x-4)=1$$

General Equations

Objective: Solve general linear equations with variables on both sides.

Often as we are solving linear equations we will need to do some work to set them up into a form we are familiar with solving. This section will focus on manipulating an equation we are asked to solve in such a way that we can use our pattern for solving two-step equations to ultimately arrive at the solution.

One such issue that needs to be addressed is parentheses. Often the parentheses can get in the way of solving an otherwise easy problem. As you might expect we can get rid of the unwanted parentheses by using the distributive property. This is shown in the following example. Notice the first step is distributing, then it is solved like any other two-step equation.

Example 14.

$$\begin{array}{rl}
4(2x-6)=16 & \text{Distribute 4 through parentheses} \\
8x-24=16 & \text{Focus on the subtraction first} \\
\underline{\mathbf{+24}\ \ \mathbf{+24}} & \text{Add 24 to both sides} \\
8x=40 & \text{Now focus on the multiply by 8} \\
\overline{\mathbf{8}}\qquad\overline{\mathbf{8}} & \text{Divide both sides by 8} \\
x=5 & \text{Our solution}
\end{array}$$

Often after we distribute there will be some like terms on one side of the equation. Example 15 shows distributing to clear the parentheses and then combining like terms next. Notice we only combine like terms on the same side of the equation. Once we have done this, our next example solves just like any other two-step equation.

Example 15.

$$\begin{array}{rl}
3(2x-4)+9=15 & \text{Distribute the 3 through the parentheses} \\
6x-12+9=15 & \text{Combine like terms, } -12+9 \\
6x-3=15 & \text{Focus on the subtraction first} \\
\underline{\mathbf{+3}\ \ \mathbf{+3}} & \text{Add 3 to both sides} \\
6x=18 & \text{Now focus on multiply by 6} \\
\overline{\mathbf{6}}\qquad\overline{\mathbf{6}} & \text{Divide both sides by 6} \\
x=3 & \text{Our solution}
\end{array}$$

A second type of problem that becomes a two-step equation after a bit of work is one where we see the variable on both sides. This is shown in the following example.

Example 16.

$$4x-6=2x+10$$

Notice here the x is on both the left and right sides of the equation. This can make it difficult to decide which side to work with. We fix this by moving one of the terms with x to the other side, much like we moved a constant term. It doesn't matter which term gets moved, $4x$ or $2x$, however, it would be the author's suggestion to move the smaller term (to avoid negative coefficients). For this reason we begin this problem by clearing the positive $2x$ by subtracting $2x$ from both sides.

$$\begin{array}{rl}
4x-6=2x+10 & \text{Notice the variable on both sides} \\
\underline{\boldsymbol{-2x}\qquad\boldsymbol{-2x}} & \text{Subtract } 2x \text{ from both sides} \\
2x-6=10 & \text{Focus on the subtraction first} \\
\underline{\mathbf{+6}\ \ \mathbf{+6}} & \text{Add 6 to both sides} \\
2x=16 & \text{Focus on the multiplication by 2} \\
\overline{\mathbf{2}}\qquad\overline{\mathbf{2}} & \text{Divide both sides by 2} \\
x=8 & \text{Our solution}
\end{array}$$

The previous example shows the check on this solution. Here the solution is plugged into the x on both the left and right sides before simplifying.

Example 17.

$$\begin{array}{rl}
4(8) - 6 = 2(8) + 10 & \text{Multiply } 4(8) \text{ and } 2(8) \text{ first} \\
32 - 6 = 16 + 10 & \text{Add and Subtract} \\
26 = 26 & \text{True!}
\end{array}$$

The next example illustrates the same process with negative coefficients. Notice first the smaller term with the variable is moved to the other side, this time by adding because the coefficient is negative.

Example 18.

$$\begin{array}{rl}
-3x + 9 = 6x - 27 & \text{Notice the variable on both sides, } -3x \text{ is smaller} \\
\underline{\boldsymbol{+3x \qquad + 3x}} & \text{Add } 3x \text{ to both sides} \\
9 = 9x - 27 & \text{Focus on the subtraction by 27} \\
\underline{\mathbf{+27 \qquad + 27}} & \text{Add 27 to both sides} \\
36 = 9x & \text{Focus on the multiplication by 9} \\
\overline{\mathbf{9}} \quad \overline{\mathbf{9}} & \text{Divide both sides by 9} \\
4 = x & \text{Our solution}
\end{array}$$

Linear equations can become particularly interesting when the two processes are combined. In the following problems we have parentheses and the variable on both sides. Notice in each of the following examples we distribute, then combine like terms, then move the variable to one side of the equation.

Example 19.

$$\begin{array}{rl}
2(x - 5) + 3x = x + 18 & \text{Distribute the 2 through parentheses} \\
2x - 10 + 3x = x + 18 & \text{Combine like terms } 2x + 3x \\
5x - 10 = x + 18 & \text{Notice the variable is on both sides} \\
\underline{\boldsymbol{-x \qquad - x}} & \text{Subtract } x \text{ from both sides} \\
4x - 10 = 18 & \text{Focus on the subtraction of 10} \\
\underline{\mathbf{+10 \;\; + 10}} & \text{Add 10 to both sides} \\
4x = 28 & \text{Focus on multiplication by 4} \\
\overline{\mathbf{4}} \quad \overline{\mathbf{4}} & \text{Divide both sides by 4} \\
x = 7 & \text{Our solution}
\end{array}$$

Sometimes we may have to distribute more than once to clear several parentheses. Remember to combine like terms after you distribute!

Example 20.

$$\begin{array}{rl}
3(4x-5)-4(2x+1)=5 & \text{Distribute 3 and } -4 \text{ through parentheses} \\
12x-15-8x-4=5 & \text{Combine like terms } 12x-8x \text{ and } -15-4 \\
4x-19=5 & \text{Focus on subtraction of 19} \\
\underline{\mathbf{+19\quad +19}} & \text{Add 19 to both sides} \\
4x=24 & \text{Focus on multiplication by 4} \\
\overline{\mathbf{4}}\qquad\overline{\mathbf{4}} & \text{Divide both sides by 4} \\
x=6 & \text{Our solution}
\end{array}$$

This leads to a 5-step process to solve any linear equation. While all five steps aren't always needed, this can serve as a guide to solving equations.

1. Distribute through any parentheses.
2. Combine like terms on each side of the equation.
3. Get the variables on one side by adding or subtracting
4. Solve the remaining 2-step equation (add or subtract then multiply or divide)
5. Check your answer by plugging it back in for x to find a true statement. If your resulting statement is false, repeat the procedure, beginning with the first step.

The order of these steps is very important.

We can see each of the above five steps worked through our next example.

Example 21.

$$\begin{array}{rl}
4(2x-6)+9=3(x-7)+8x & \text{Distribute 4 and 3 through parentheses} \\
8x-24+9=3x-21+8x & \text{Combine like terms } -24+9 \text{ and } 3x+8x \\
8x-15=11x-21 & \text{Notice the variable is on both sides} \\
\underline{\boldsymbol{-8x\qquad\quad -8x}} & \text{Subtract } 8x \text{ from both sides} \\
-15=3x-21 & \text{Focus on subtraction of 21} \\
\underline{\mathbf{+21\qquad +21}} & \text{Add 21 to both sides} \\
6=3x & \text{Focus on multiplication by 3} \\
\overline{\mathbf{3}}\quad\overline{\mathbf{3}} & \text{Divide both sides by 3} \\
2=x & \text{Our solution}
\end{array}$$

Check:

$$\begin{array}{rl}
4[2(2)-6]+9=3[(2)-7]+8(2) & \text{Plug 2 in for each } x\text{. Multiply inside parentheses} \\
4[4-6]+9=3[-5]+8(2) & \text{Finish parentheses on left, multiply on right} \\
4[-2]+9=-15+8(2) & \text{Finish multiplication on both sides} \\
-8+9=-15+16 & \text{Add} \\
1=1 & \text{True!}
\end{array}$$

When we check our solution of $x = 2$ we found a true statement, $1 = 1$. Therefore, we know our solution $x = 2$ is the correct solution for the problem.

There are two special cases that can come up as we are solving these linear equations. The first is illustrated in the next two examples. Notice we start by distributing and moving the variables all to the same side.

Example 22.

$$\begin{array}{rl}
3(2x-5) = 6x-15 & \text{Distribute 3 through parentheses} \\
6x-15 = 6x-15 & \text{Notice the variable on both sides} \\
\underline{\boldsymbol{-6x \qquad -6x}} & \text{Subtract } 6x \text{ from both sides} \\
-15 = -15 & \text{Variable is gone! True!}
\end{array}$$

Here the variable subtracted out completely! We are left with a true statement, $-15 = -15$. If the variables subtract out completely and we are left with a true statement, this indicates that the equation is always true, no matter what x is. Thus, for our solution we say "all real numbers" or $\mathbb{R}$.

It is worth mentioning that in both the previous and following examples, we are still *solving* a given equation for all possible values of x. When the variable is eliminated entirely, this can sometimes be confused with *checking* a solution.

Example 23.

$$\begin{array}{rl}
2(3x-5)-4x = 2x+7 & \text{Distribute 2 through parentheses} \\
6x-10-4x = 2x+7 & \text{Combine like terms } 6x-4x \\
2x-10 = 2x+7 & \text{Notice the variable is on both sides} \\
\underline{\boldsymbol{-2x \qquad -2x}} & \text{Subtract } 2x \text{ from both sides} \\
-10 \neq 7 & \text{Variable is gone! False!}
\end{array}$$

Again, the variable subtracted out completely! However, this time we are left with a false statement, this indicates that the equation is never true, no matter what x is. Thus, for our solution we say "no solutions" or $\varnothing$.

Equations Containing Fractions (L1)

Objective: Solve linear equations with rational coefficients by multiplying by the least common multiple of the denominators to clear the fractions.

Often when solving linear equations we will need to work with an equation with fraction coefficients. We can solve these problems as we have in the past. This is demonstrated in our next example.

Example 24.

$$
\begin{array}{rl}
\dfrac{3}{4}x - \dfrac{7}{2} = \dfrac{5}{6} & \text{Focus on subtraction} \\
\underline{+\mathbf{\dfrac{7}{2}} \quad +\mathbf{\dfrac{7}{2}}} & \text{Add } \dfrac{7}{2} \text{ to both sides}
\end{array}
$$

Notice we will need to get a common denominator to add $\frac{5}{6} + \frac{7}{2}$. We have a common denominator of 6. So we build up the denominator, $\frac{7}{2}\left(\frac{3}{3}\right) = \frac{21}{6}$, and we can now add the fractions:

$$
\begin{array}{rl}
\dfrac{3}{4}x - \dfrac{21}{6} = \dfrac{5}{6} & \text{Same problem, with common denominator 6} \\
\underline{+\mathbf{\dfrac{21}{6}} \quad +\mathbf{\dfrac{21}{6}}} & \text{Add } \dfrac{21}{6} \text{ to both sides} \\
\dfrac{3}{4}x = \dfrac{26}{6} & \text{Reduce } \dfrac{26}{6} \text{ to } \dfrac{13}{3} \\
\dfrac{3}{4}x = \dfrac{13}{3} & \text{Focus on multiplication by } \dfrac{3}{4}
\end{array}
$$

We can get rid of $\frac{3}{4}$ by dividing both sides by $\frac{3}{4}$.

Dividing by a fraction is the same as multiplying by the reciprocal, so we will multiply both sides by $\frac{4}{3}$.

$$
\begin{array}{rl}
\left(\mathbf{\dfrac{4}{3}}\right)\dfrac{3}{4}x = \dfrac{13}{3}\left(\mathbf{\dfrac{4}{3}}\right) & \text{Multiply by reciprocal} \\
x = \dfrac{52}{9} & \text{Our solution}
\end{array}
$$

While this process does help us arrive at the correct solution, the fractions can make the process quite difficult. This is why we have an alternate method for dealing with fractions - clearing fractions. Clearing fractions is nice as it gets rid of the fractions for the majority of the problem. We can easily clear the fractions by finding the least common multiple (LCM) of the denominators and multiplying each term by the LCM. This is shown in the next example, the same problem as our first example, but this time we will solve by clearing fractions.

Example 25.

$$\begin{array}{rl}
\frac{3}{4}x - \frac{7}{2} = \frac{5}{6} & \text{LCM} = 12, \text{multiply each term by } 12 \\
\frac{(\mathbf{12})3}{4}x - \frac{(\mathbf{12})7}{2} = \frac{(\mathbf{12})5}{6} & \text{Reduce each 12 with denominators} \\
(\mathbf{3})3x - (\mathbf{6})7 = (\mathbf{2})5 & \text{Multiply out each term} \\
9x - 42 = 10 & \text{Focus on subtraction by 42} \\
\underline{\mathbf{+42 \quad + 42}} & \text{Add 42 to both sides} \\
9x = 52 & \text{Focus on multiplication by 9} \\
\overline{\mathbf{9}} \quad \overline{\mathbf{9}} & \text{Divide both sides by 9} \\
x = \frac{52}{9} & \text{Our solution}
\end{array}$$

The next example illustrates this as well. Notice the 2 isn't a fraction in the original equation, but to solve it we put the 2 over 1 to make it a fraction.

Example 26.

$$\begin{array}{rl}
\frac{2}{3}x - 2 = \frac{3}{2}x + \frac{1}{6} & \text{LCM} = 6, \text{multiply each term by } 6 \\
\frac{(\mathbf{6})2}{3}x - \frac{(\mathbf{6})2}{1} = \frac{(\mathbf{6})3}{2}x + \frac{(\mathbf{6})1}{6} & \text{Reduce 6 with each denominator} \\
(\mathbf{2})2x - (\mathbf{6})2 = (\mathbf{3})3x + (\mathbf{1})1 & \text{Multiply out each term}
\end{array}$$

$$\begin{array}{rl}
4x - 12 = 9x + 1 & \text{Notice variable on both sides} \\
\underline{\mathbf{-4x \qquad - 4x}} & \text{Subtract } 4x \text{ from both sides} \\
-12 = 5x + 1 & \text{Focus on addition of 1} \\
\underline{\mathbf{-1 \qquad - 1}} & \text{Subtract 1 from both sides} \\
-13 = 5x & \text{Focus on multiplication of 5} \\
\overline{\mathbf{5}} \quad \overline{\mathbf{5}} & \text{Divide both sides by 5} \\
-\frac{13}{5} = x & \text{Our solution}
\end{array}$$

We can use this same process if there are parenthesis in the problem. We will first distribute the coefficient in front of the parenthesis, then clear the fractions. This is seen in the following example.

Example 27.

$$
\begin{array}{rl}
\frac{3}{2}\left(\frac{5}{9}x+\frac{4}{27}\right)=3 & \text{Distribute } \frac{3}{2} \text{ through parenthesis, reducing if possible} \\
\frac{5}{6}x+\frac{2}{9}=3 & \text{LCM} = 18\text{, multiply each term by } 18 \\
\frac{(\mathbf{18})5}{6}x+\frac{(\mathbf{18})2}{9}=(\mathbf{18})3 & \text{Reduce 18 with each denominator} \\
(\mathbf{3})5x+(\mathbf{2})2=(\mathbf{18})3 & \text{Multiply out each term} \\
15x+4=54 & \text{Focus on addition of 4} \\
\underline{\mathbf{-4}\quad\mathbf{-4}} & \text{Subtract 4 from both sides} \\
15x=50 & \text{Focus on multiplication by 15} \\
\overline{\mathbf{15}}\qquad\overline{\mathbf{15}} & \text{Divide both sides by 15, reduce on right side} \\
x=\frac{10}{3} & \text{Our solution}
\end{array}
$$

While the problem can take many different forms, the pattern to clear the fraction is the same, after distributing through any parentheses we multiply each term by the LCM and reduce. This will give us a problem with no fractions that is much easier to solve. The following example again illustrates this process.

Example 28.

$$
\begin{array}{rl}
\frac{3}{4}x-\frac{1}{2}=\frac{1}{3}(\frac{3}{4}x+6)-\frac{7}{2} & \text{Distribute } \frac{1}{3}\text{, reduce if possible} \\
\frac{3}{4}x-\frac{1}{2}=\frac{1}{4}x+2-\frac{7}{2} & \text{LCM} = 4\text{, multiply each term by } 4 \\
\frac{(\mathbf{4})3}{4}x-\frac{(\mathbf{4})1}{2}=\frac{(\mathbf{4})1}{4}x+\frac{(\mathbf{4})2}{1}-\frac{(\mathbf{4})7}{2} & \text{Reduce 4 with each denominator} \\
(\mathbf{1})3x-(\mathbf{2})1=(\mathbf{1})1x+(\mathbf{4})2-(\mathbf{2})7 & \text{Multiply out each term} \\
3x-2=x+8-14 & \text{Combine like terms } 8-14
\end{array}
$$

$$
\begin{array}{rl}
3x-2=x-6 & \text{Notice variable on both sides} \\
\underline{\boldsymbol{-x}\qquad\boldsymbol{-x}} & \text{Subtract } x \text{ from both sides} \\
2x-2=-6 & \text{Focus on subtraction by 2} \\
\underline{\mathbf{+2}\quad\mathbf{+2}} & \text{Add 2 to both sides} \\
2x=-4 & \text{Focus on multiplication by 2} \\
\overline{\mathbf{2}}\qquad\overline{\mathbf{2}} & \text{Divide both sides by 2} \\
x=-2 & \text{Our solution}
\end{array}
$$

Linear Equations Containing an Absolute Value (L2)

Objective: Solve linear equations containing an absolute value.

When solving equations with absolute value we can end up with more than one possible answer. This is because what is in the absolute value can be either negative or positive and we must account for both possibilities when solving equations. This is illustrated in the following example.

Example 29.

$$\begin{array}{rl} |x| = 7 & \text{Absolute value can be positive or negative} \\ x = 7 \text{ or } x = -7 & \text{Our solution} \end{array}$$

Notice that we have considered two possibilities, both the positive and negative. Either way, the absolute value of our number will be positive 7.

When we have absolute values in our problem it is important to first isolate the absolute value, then remove the absolute value by considering both the positive and negative solutions. Notice in the next two examples, all the numbers outside of the absolute value are moved to the other side first before we remove the absolute value bars and consider both positive and negative solutions.

Example 30.

$$\begin{array}{rl} 5 + |x| = 8 & \text{Notice absolute value is not alone} \\ \underline{\mathbf{-5} \qquad \mathbf{-5}} & \text{Subtract 5 from both sides} \\ |x| = 3 & \text{Absolute value can be positive or negative} \\ x = 3 \text{ or } x = -3 & \text{Our solution} \end{array}$$

Example 31.

$$\begin{array}{rl} -4|x| = -20 & \text{Notice absolute value is not alone} \\ \overline{\mathbf{-4}} \qquad \overline{\mathbf{-4}} & \text{Divide both sides by } -4 \\ |x| = 5 & \text{Absolute value can be positive or negative} \\ x = 5 \text{ or } x = -5 & \text{Our solution} \end{array}$$

Notice we never combine what is inside the absolute value with what is outside the absolute value. This is very important as it will often change the final result to an incorrect solution. The next example requires two steps to isolate the absolute value. The idea is the same as a two-step equation, add or subtract, then multiply or divide.

Example 32.

$$\begin{array}{rl}
5|x| - 4 = 26 & \text{Notice the absolute value is not alone} \\
\underline{+\mathbf{4} \ \ + \mathbf{4}} & \text{Add 4 to both sides} \\
5|x| = 30 & \text{Absolute value still not alone} \\
\overline{\mathbf{5}} \qquad \overline{\mathbf{5}} & \text{Divide both sides by 5} \\
|x| = 6 & \text{Absolute value can be positive or negative} \\
x = 6 \ \text{ or } \ x = -6 & \text{Our solution}
\end{array}$$

Again we see the same process, get the absolute value alone first, then consider the positive and negative solutions. Often the absolute value will have more than just a variable in it. In this case we will have to solve the resulting equations when we consider the positive and negative possibilities. This is shown in the next example.

Example 33.

$$\begin{array}{rl}
|2x - 1| = 7 & \text{Absolute value can be positive or negative} \\
2x - 1 = 7 \ \text{ or } \ 2x - 1 = -7 & \text{Two equations to solve}
\end{array}$$

Now notice we have two equations to solve, each equation will give us a different solution. Both equations solve like any other two-step equation.

$$\begin{array}{lcl}
2x - 1 = 7 & & 2x - 1 = -7 \\
\quad \underline{+\mathbf{1} \ \ + \mathbf{1}} & & \quad \underline{+\mathbf{1} \ \ \ + \mathbf{1}} \\
2x = 8 & \text{or} & 2x = -6 \\
\overline{\mathbf{2}} \quad \ \overline{\mathbf{2}} & & \overline{\mathbf{2}} \qquad \overline{\mathbf{2}} \\
x = 4 & & x = -3
\end{array}$$

Thus, from our previous example we have two solutions, $x = 4$ or $x = -3$.

Again, it is important to remember that the absolute value must be alone first before we consider the positive and negative possibilities. This is illustrated below.

Example 34.

$$2 - 4|2x + 3| = -18$$

To get the absolute value alone we first need to get rid of the 2 by subtracting, then divide by -4. Notice we cannot combine the 2 and -4 because they are not like terms, the -4 has the absolute value connected to it. Also notice we do not distribute the -4 into the absolute value. This is because the numbers outside cannot be combined with the numbers inside the absolute value. Thus we get the absolute value alone in the following way:

$$\begin{array}{rl}
2 - 4|2x + 3| = -18 & \text{Notice absolute value is not alone} \\
\underline{-\mathbf{2} \qquad\qquad\qquad - \mathbf{2}} & \text{Subtract 2 from both sides} \\
-4|2x + 3| = -20 & \text{Absolute value still not alone} \\
\overline{-\mathbf{4}} \qquad \overline{-\mathbf{4}} & \text{Divide both sides by } -4 \\
|2x + 3| = 5 & \text{Absolute value can be positive or negative} \\
2x + 3 = 5 \ \text{ or } \ 2x + 3 = -5 & \text{Two equations to solve}
\end{array}$$

Now we just solve these two remaining equations to find our solutions.

$$\begin{array}{lcl} 2x+3=5 & & 2x+3=-5 \\ \underline{\quad\mathbf{-3}\quad\mathbf{-3}} & & \underline{\quad\mathbf{-3}\quad\mathbf{-3}} \\ 2x=2 & \text{or} & 2x=-8 \\ \overline{\mathbf{2}}\quad\overline{\mathbf{2}} & & \overline{\mathbf{2}}\quad\overline{\mathbf{2}} \\ x=1 & & x=-4 \end{array}$$

We now have our two solutions, $x = 1$ and $x = -4$.

As we are solving absolute value equations it is important to be aware of special cases. Remember the result of an absolute value must always be positive. Notice what happens in the next example.

Example 35.

$$\begin{array}{rl} 7+|2x-5|=4 & \text{Notice absolute value is not alone} \\ \underline{\mathbf{-7}\qquad\qquad\quad\mathbf{-7}} & \text{Subtract 7 from both sides} \\ |2x-5|=-3 & \text{Result of absolute value is negative!} \end{array}$$

Notice the absolute value equals a negative number! This is impossible with an absolute value. When this occurs we say there is "no solution" or $\varnothing$.

One other type of absolute value problem is when two absolute values are equal to each other. We still will consider both the positive and negative result, the difference here will be that we will have to distribute a negative into the second absolute value for the negative possibility.

Example 36.

$$\begin{array}{rl} |2x-7|=|4x+6| & \text{Absolute value can be} \\ & \quad\text{positive or negative} \\ 2x-7=4x+6 & \text{Make second part of} \\ \text{or}\quad 2x-7=-(4x+6) & \quad\text{second equation negative} \end{array}$$

Notice the first equation is the positive possibility and has no significant difference other than the missing absolute value bars. The second equation considers the negative possibility. For this reason we have a negative in front of the expression which will be distributed through the equation on the first step of solving. So we solve both these equations as follows:

$$\begin{array}{lcl} 2x-7=4x+6 & & 2x-7=-(4x+6) \\ \underline{\boldsymbol{-2x}\qquad\boldsymbol{-2x}} & & 2x-7=-4x-6 \\ -7=2x+6 & & \underline{\boldsymbol{+4x}\qquad\boldsymbol{+4x}} \\ \underline{\mathbf{-6}\qquad\mathbf{-6}} & & 6x-7=-6 \\ -13=2x & \text{or} & \underline{\quad\mathbf{+7}\quad\mathbf{+7}} \\ \overline{\mathbf{2}}\quad\overline{\mathbf{2}} & & 6x=1 \\ -\frac{13}{2}=x & & \overline{\mathbf{6}}\quad\overline{\mathbf{6}} \\ & & x=\frac{1}{6} \end{array}$$

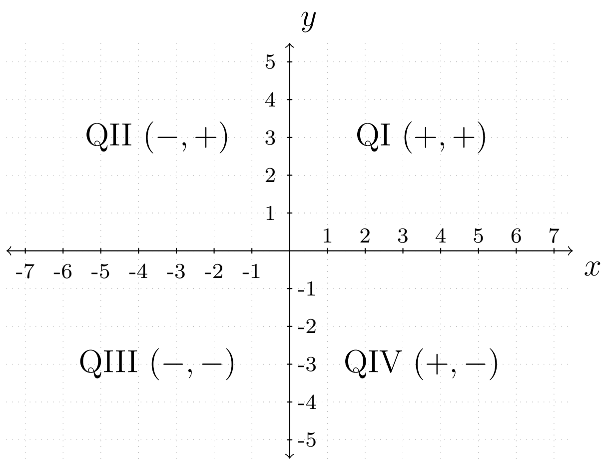
This gives us our two solutions, $x = -\frac{13}{2}$ or $x = \frac{1}{6}$.

Graphing Linear Equations

The Cartesian Plane

Objective: Locate and graph points using xy-coordinates

Often, to get an idea of the behavior of an equation we will make a picture that represents the solutions to the equation. Before we spend much time on making a visual representation of an equation, we first have to understand the basics of graphing. A *graph* is a set of points in the $xy-$plane, also known as the Cartesian plane. In most cases, a graph can simply be thought of as a "picture" of the points. We will see shortly that the graph of a linear equation is a visualization of the solutions to the equation. The following is an example of the Cartesian or $xy-$coordinate plane.



The plane is divided into four *quadrants*, or sections, by a horizontal number line (x-axis) and a vertical number line (y-axis).

Where the two lines, or axes, meet in the center is called the origin. This center origin is where $x = 0$ and $y = 0$.

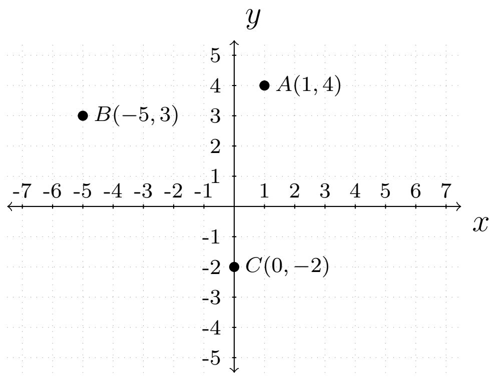
The quadrants are numbered using the roman numerals I, II, III, and IV, beginning with the top-right quadrant (where both x and y are positive) and moving counter-clockwise.

As we move to the right the numbers count up from zero, representing $x = 1, 2, 3 \ldots$. To the left the numbers count down from zero, representing $x = -1, -2, -3, \ldots$. Similarly, as we move up the numbers count up from zero, $y = 1, 2, 3, \ldots$, and as we move down count down from zero, $y = -1, -2, -3 \ldots$.

We can put dots on the graph which we will call points. Each point has an "address" that defines its location. The first number will be the value on the $x-$axis or horizontal number line. This is the distance the point moves left/right from the origin. The second number will represent the value on the $y-$axis or vertical number line. This is the distance the point moves up/down from the origin. The points are given as an ordered pair (x, y).

The following example finds the address or coordinate pair for each of several points on the coordinate plane.

Example 37. Give the coordinates of each point.



Tracing from the origin, point A is right 1, up 4. This becomes $A(1, 4)$.

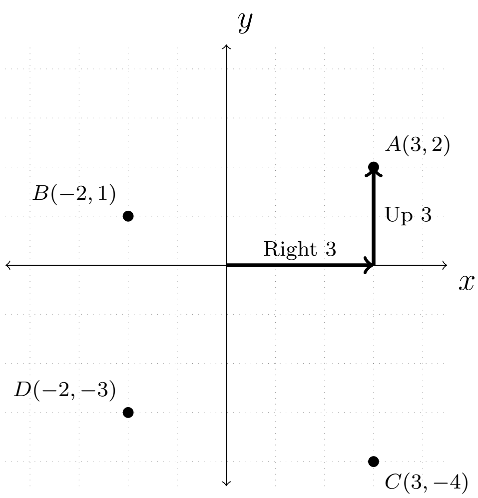
Point B is left 5, up 3. Left is backwards or negative so we have $B(-5, 3)$.

Point C is straight down 2 units. There is no left or right. This means we go right zero so the point is $C(0, -2)$.

Our solution is $A(1, 4), B(-5, 3), C(0, -2)$.

Just as we can give the coordinates for a set of points, we can take a set of points and plot them on the plane.

Example 38. Graph the set of points:
$\{A(3, 2), B(-2, 1), C(3, -4), D(-2, -3), E(-3, 0), F(0, 2), G(0, 0)\}$

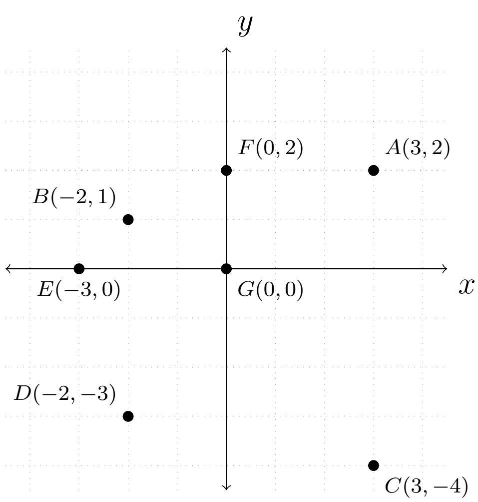


The first point, A is at $(3, 2)$ this means $x = 3$ (right 3) and $y = 2$ (up 2). Following these instructions, starting from the origin, we get our point. This is also illustrated on the graph.

The second point, $B(-2, 1)$, is left 2 (negative moves backwards), up 1.

The third point, $C(3, -4)$ is right 3, down 4 (negative moves backwards).

The fourth point, $D(-2, -3)$ is left 2, down 3 (both negative, both move backwards).



The last three points have zeros in them. We still treat these points just like the other points. If there is a zero there is just no movement.

First is $E(-3, 0)$. This is left 3, and up zero, right on the $x-$axis.

Then is $F(0, 2)$. This is right zero, and up two, right on the $y-$axis.

Finally is $G(0, 0)$. This point has no movement, and thus is right on the origin.

Graphing Lines (L3)

Objective: Graph lines using xy-coordinates.

The main purpose of graphs is not to plot random points, but rather to give a picture of the solutions to an equation. We may have an equation such as $y = 2x - 3$. We may be interested in what type of solution are possible in this equation. We can visualize the solution by making a graph of possible x and y combinations that make this equation a true statement. We will have to start by finding possible x and y combinations. We will do this using a table of values.

Example 39.

Graph $y = 2x - 3$ We make a table of values

x	y
-1	
0	
1	

We will test three values for x. Any three can be used

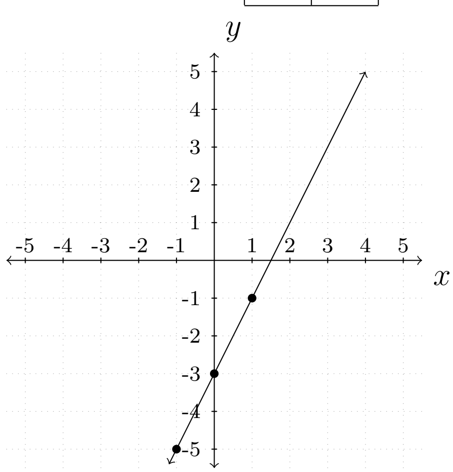
x	y
-1	-5
0	-3
1	-1

Evaluate each by replacing x with the given value

$x = -1$ $\quad y = 2(-1) - 3 = -2 - 3 = -5$

$x = 0$ $\quad y = 2(0) - 3 = 0 - 3 = -3$

$x = 1$ $\quad y = 2(1) - 3 = 2 - 3 = -1$



$(-1, -5), (0, -3)$, and $(1, -1)$

These become the points from our equation which we will plot on our graph.

Once the point are on the graph, connect the dots to make a line.

The graph is our solution.

What this line tells us is that any point on the line will work in the equation $y = 2x - 3$. For example, notice the graph also goes through the point $(2, 1)$. If we use $x = 2$, we should get $y = 1$. Sure enough, $y = 2(2) - 3 = 4 - 3 = 1$, just as the graph suggests. Thus we have the line is a picture of all the solutions for $y = 2x - 3$. We can use this table of values method to draw a graph of any linear equation.

Example 40.

Graph $2x - 3y = 6$	We will use a table of values

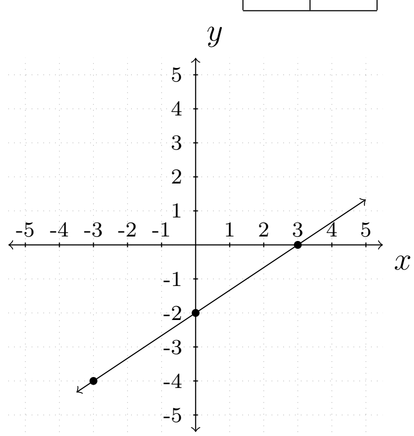
x	y
-3	
0	
3	

We will test three values for x. Any three can be used .

$2(-3) - 3y = 6$	Substitute each value in for x and solve for y
$-6 - 3y = 6$	Start with $x = -3$, multiply first
$\underline{\mathbf{+6} \qquad \mathbf{+6}}$	Add 6 to both sides
$-3y = 12$	Divide both sides by -3
$\overline{\mathbf{-3}} \quad \overline{\mathbf{-3}}$	
$y = -4$	solution for y when $x = -3$, add this to table
$2(0) - 3y = 6$	Next $x = 0$
$-3y = 6$	Multiplying clears the constant term
$\overline{\mathbf{-3}} \quad \overline{\mathbf{-3}}$	Divide each side by -3
$y = -2$	solution for y when $x = 0$, add this to table
$2(3) - 3y = 6$	Next $x = 3$
$6 - 3y = 6$	Multiply
$\underline{\mathbf{-6} \qquad \mathbf{-6}}$	Subtract 9 from both sides
$-3y = 0$	Divide each side by -3
$\overline{\mathbf{-3}} \quad \overline{\mathbf{-3}}$	
$y = 0$	solution for y when $x = -3$, add this to table

x	y
-3	-4
0	-2
3	0

Our completed table



The coordinate points from our table are then $(-3, -4)$, $(0, -2)$, and $(3, 0)$

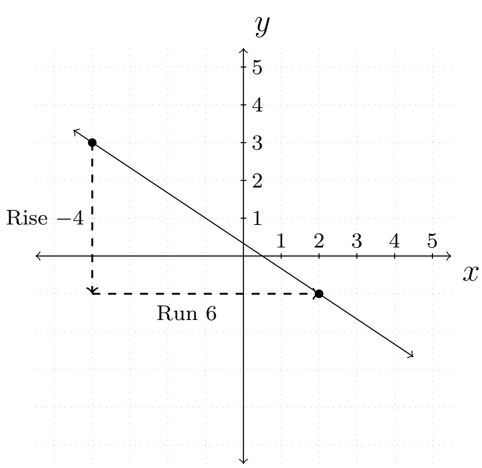
After we plot these points, we connect them to form our graph.

The Slope of a Line

Objective: Find the slope of a line given a graph or two points.

As we graph lines, we will want to be able to identify different properties of the lines we graph. One of the most important properties of a line is its slope. *Slope* is a measure of steepness. A line with a large slope, such as 25, is very steep or increases quickly. A line with a small slope, such as $\frac{1}{10}$ is very flat or increases gradually. We will also use slope to describe the direction of the line. A line that goes up from left to right will have a positive slope and a line that goes down from left to right will have a negative slope.

As we measure steepness we are interested in how fast the line rises compared to how far the line runs. For this reason we will describe slope as the fraction $\frac{\text{rise}}{\text{run}}$. Rise would be a vertical change, or a change in the y-values. Run would be a horizontal change, or a change in the x-values. So another way to describe slope would be the fraction $\frac{\text{change in } y}{\text{change in } x}$. It turns out that if we have a graph we can draw vertical and horizontal lines from one point to another to make what is called a slope triangle. The sides of the slope triangle give us our slope. Using this idea, we find the corresponding slopes for each of the lines that follow.

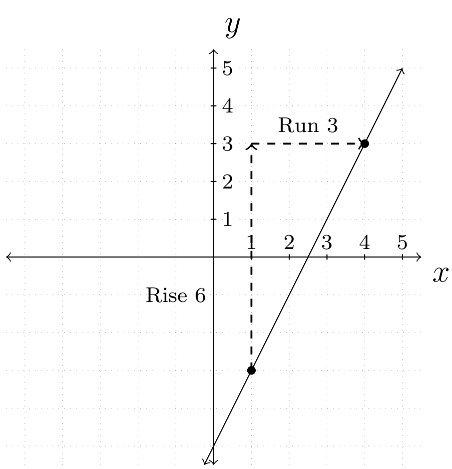


To find the slope of this line we will consider the rise, or vertical change and the run or horizontal change.

Drawing these lines in creates a triangle that we can use to count from one point to the next:

the graph goes down 4, right 6. This is a rise of -4 and a run 6.

As a fraction, we have, $\frac{-4}{6}$, or $-\frac{2}{3}$ when reduced, which is our slope.



To find the slope of this line, the rise is up 6, the run is right 3.

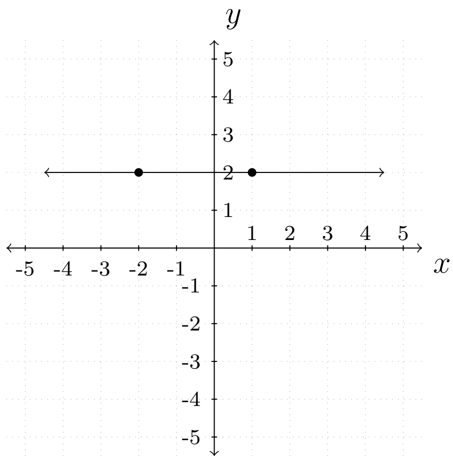
Our slope is then written as a fraction:

$\frac{\text{rise}}{\text{run}} = \frac{6}{3}$.

This fraction reduces to 2.

A slope of 2 is our solution.

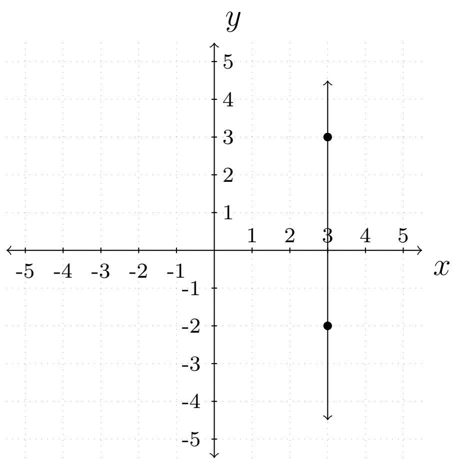
There are two special lines that have unique slopes that we need to be aware of. They are illustrated in the following examples.



In this graph there is no rise, but the run is 3 units.

This slope becomes $\frac{0}{3} = 0$.

This, and all *horizontal* lines have a slope of zero.



This line has a rise of 5, but no run.

The slope becomes $\frac{5}{0}$ = undefined, or $\varnothing$.

This, and all *vertical* lines have an undefined slope.

As you can see there is a big difference between having a zero slope and having no slope or undefined slope. Remember, slope is a measure of steepness. The first slope is not steep at all, in fact it is flat. Therefore it has a zero slope. The second slope can't get any steeper. It is so steep that there is no number large enough to express how steep it is. This is an undefined slope.

We can find the slope of a line through two points without seeing the points on a graph. We can do this using a slope formula. If the rise is the change in y values, we can calculate this by subtracting the y values of a point. Similarly, if run is a change in the x values, we can calculate this by subtracting the x values of a point. In this way we get the following equation for slope.

$$\text{The slope of a line through } (x_1, y_1) \text{ and } (x_2, y_2) \text{ is } \frac{y_2 - y_1}{x_2 - x_1}.$$

When mathematicians began working with slope, it was called the modular slope. For this reason we often represent the slope with the variable m. Now we have the following for slope.

$$\text{Slope} = m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

As we subtract the y values and the x values when calculating slope it is important we subtract them in the same order. This process is shown in the following examples.

Example 41.

$$\begin{array}{rl}
\text{Find the slope between} (-4,3) \text{ and } (2,-9) & \text{Identify } x_1, y_1, x_2, y_2 \\
(x_1, y_1) \text{ and } (x_2, y_2) & \text{Use slope formula, } m = \dfrac{y_2 - y_1}{x_2 - x_1} \\
m = \dfrac{-9-3}{2-(-4)} & \text{Simplify} \\
m = \dfrac{-12}{6} & \text{Reduce} \\
m = -2 & \text{Our solution}
\end{array}$$

Example 42.

$$\begin{array}{rl}
\text{Find the slope between} (4,6) \text{ and } (2,-1) & \text{Identify } x_1, y_1, x_2, y_2 \\
(x_1, y_1) \text{ and } (x_2, y_2) & \text{Use slope formula, } m = \dfrac{y_2 - y_1}{x_2 - x_1} \\
m = \dfrac{-1-6}{2-4} & \text{Simplify} \\
m = \dfrac{-7}{-2} & \text{Reduce, dividing by } -1 \\
m = \dfrac{7}{2} & \text{Our solution}
\end{array}$$

We may come up against a problem that has a zero slope (horizontal line) or no slope (vertical line) just as with using the graphs.

Example 43.

$$\begin{array}{rl}
\text{Find the slope between } (-4,-1) \text{ and } (-4,-5) & \text{Identify } x_1, y_1, x_2, y_2 \\
(x_1, y_1) \text{ and } (x_2, y_2) & \text{Use slope formula, } m = \dfrac{y_2 - y_1}{x_2 - x_1} \\
m = \dfrac{-5-(-1)}{-4-(-4)} & \text{Simplify} \\
m = \dfrac{-4}{0} & \text{Can't divide by zero} \\
\text{Slope } m \text{ is undefined} & \text{Our solution}
\end{array}$$

Example 44.

Find the slope between $(3,1)$ and $(-2,1)$	Identify x_1, y_1, x_2, y_2
(x_1, y_1) and (x_2, y_2)	Use slope formula, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$
$m = \dfrac{1-1}{-2-3}$	Simplify
$m = \dfrac{0}{-5}$	Reduce
$m = 0$	Our solution

Again, there is a big difference between no slope and a zero slope. Zero is an integer and it has a value, the slope of a flat horizontal line. No slope has no value, it is undefined, the slope of a vertical line.

Using the slope formula we can also find missing points if we know what the slope is. This is shown in the following two examples.

Example 45. Find the value of y between the points (2, y) and (5, - 1) with slope -3.

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$	We will plug values into the slope formula
$-3 = \dfrac{-1-y}{5-2}$	Simplify
$-3 = \dfrac{-1-y}{3}$	Multiply both sides by 3
$-3(\mathbf{3}) = \dfrac{-1-y}{3}(\mathbf{3})$	Simplify
$-9 = -1 - y$	Add 1 to both sides
$\underline{\mathbf{+1} \quad \mathbf{+1}}$	
$-8 = -y$	Divide both sides by -1
$\overline{\mathbf{-1}} \quad \overline{\mathbf{-1}}$	
$8 = y$	Our solution

Example 46. Find the value of x between the points (- 3, 2) and (x, 6) with slope $\dfrac{2}{5}$.

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$	We will plug values into slope formula
$\dfrac{2}{5} = \dfrac{6-2}{x-(-3)}$	Simplify
$\dfrac{2}{5} = \dfrac{4}{x+3}$	Multiply both sides by $(x+3)$
$\dfrac{2}{5}(x+3) = 4$	Multiply by 5 to clear fraction

$$\begin{aligned}
(\mathbf{5})\frac{2}{5}(x+3) &= 4(\mathbf{5}) && \text{Simplify}\\
2(x+3) &= 20 && \text{Distribute}\\
2x+6 &= 20 &&\\
\underline{\mathbf{-6} \quad} & \underline{\;\mathbf{-\,6}} && \text{Subtract 6 from both sides}\\
2x &= 14 && \text{Divide each side by 2}\\
\overline{\mathbf{2}} \quad & \;\;\overline{\mathbf{2}} &&\\
x &= 7 && \text{Our solution}
\end{aligned}$$

The Two Forms of a Linear Equation (L4)

Slope-Intercept Form

Objective: Find the equation of a line with a known slope and y-intercept.

When graphing a line we found one method we could use is to make a table of values. However, if we can identify some properties of the line, we may be able to make a graph much quicker and easier. One such method is finding the slope and the y-intercept of the equation. The slope can be represented by m and the y-intercept, where it crosses the axis and $x = 0$, can be represented by $(0, b)$ where b is the value where the graph crosses the vertical y-axis. Any other point on the line can be represented by (x, y). Using this information we will look at the slope formula and solve the formula for y.

Example 47.

$$\begin{aligned}
m,\ (0,b),\ (x,y) & && \text{Use the slope formula}\\
\frac{y-b}{x-0} &= m && \text{Simplify}\\
\frac{y-b}{x} &= m && \text{Multiply both sides by } x\\
y-b &= mx && \text{Add } b \text{ to both sides}\\
\underline{+\boldsymbol{b} \quad} & \underline{\;+\,\boldsymbol{b}} &&\\
y &= mx+b && \text{Our solution}
\end{aligned}$$

This equation, $y = mx + b$ can be thought of as the equation of any line that has a slope of m and a y-intercept of b. This formula is known as the slope-intercept formula or equation.

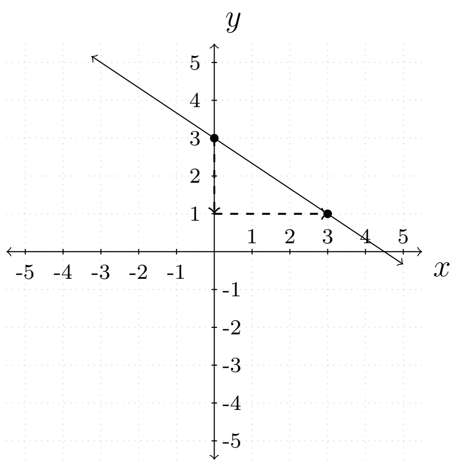
$$\boxed{\text{Slope-intercept equation: } y = mx + b}$$

If we know the slope and the y-intercept we can easily find the equation that represents the line.

Example 48.

$$\begin{array}{rl}
\text{Slope} = \dfrac{3}{4},\quad y-\text{intercept} = -3 & \text{Use the slope} - \text{intercept equation}\\
y = mx + b & m\,\text{is the slope},\quad b\,\text{is the}\,y - \text{intercept}\\
y = \dfrac{3}{4}x - 3 & \text{Our solution}
\end{array}$$

We can also find the equation by looking at a graph and finding the slope and y-intercept.



Identify the point where the graph crosses the y-axis (0,3).

This means the y-intercept is 3.

Identify one other point and draw a slope triangle to find the slope.

The slope is $m = -\frac{2}{3}$.

Slope-intercept form: $y = mx + b$

Our Equation: $y = -\frac{2}{3}x + 3$

We can also move the opposite direction, using the equation identify the slope and y-intercept and graph the equation from this information. However, it will be important for the equation to first be in slope intercept form. If it is not, we will have to solve it for y so we can identify the slope and the y-intercept.

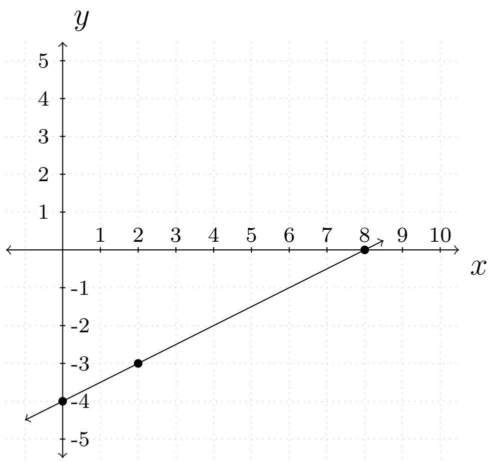
Example 49. Write the equation $2x - 4y = 6$ in slope-intercept form.

$$\begin{array}{rl}
2x - 4y = 6 & \text{Solve for}\,y\\
\underline{\mathbf{-2x \qquad - 2x}} & \text{Subtract}\,2x\,\text{from both sides}\\
-4y = -2x + 6 & \text{Put}\,x\,\text{term first}\\
\overline{\mathbf{-4}}\quad \overline{\mathbf{-4}}\ \overline{\mathbf{-4}} & \text{Divide each term by}\,-4\\
y = \dfrac{1}{2}x - \dfrac{3}{2} & \text{Our solution}
\end{array}$$

Once we have an equation in slope-intercept form we can graph it by first plotting the y-intercept, then using the slope, finding a second point and connecting the dots.

Example 50. Graph $y = \dfrac{1}{2}x - 4$.

$$\begin{array}{rl}
y = mx + b & \text{Slope} - \text{intercept equation}\\
m = \dfrac{1}{2},\ b = -4 & \text{Identify the slope},\ m, \text{and the}\,y - \text{intercept},\ b
\end{array}$$



Start with a point at the y-intercept of -4, $(0, -4)$.

Then use the slope $\frac{\text{rise}}{\text{run}}$ to find the next point, $(2, -3)$.

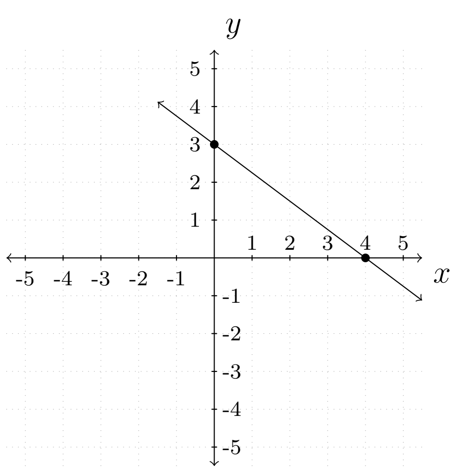
Once we have both points, connect the dots to get our graph.

Here, we have also identified the $x-$intercept $(8, 0)$, by setting $y = 0$ and solving for x:

$\frac{1}{2}x - 4 = 0$ when $x = 8$.

Example 51. Graph $3x + 4y = 12$.

$3x + 4y = 12$	Not in slope-intercept form
$\underline{\boldsymbol{-3x} \qquad \boldsymbol{-\ 3x}}$	Subtract $3x$ from both sides
$4y = -3x + 12$	Put the x term first
$\overline{\mathbf{4}} \qquad \overline{\mathbf{4}} \qquad \overline{\mathbf{4}}$	Divide each term by 4
$y = -\dfrac{3}{4}x + 3$	Now in slope $-$ intercept form
$m = -\dfrac{3}{4},\ b = 3$	Identify m and b



Start with a point at the $y-$intercept, $(0, 3)$.

Then use the slope $\frac{\text{rise}}{\text{run}}$. Since the slope is negative, the graph will decrease from left to right. So we will drop 3 units and run 4 units to the right to find the next point.

Notice that our next point is also the $x-$intercept, $(4, 0)$.

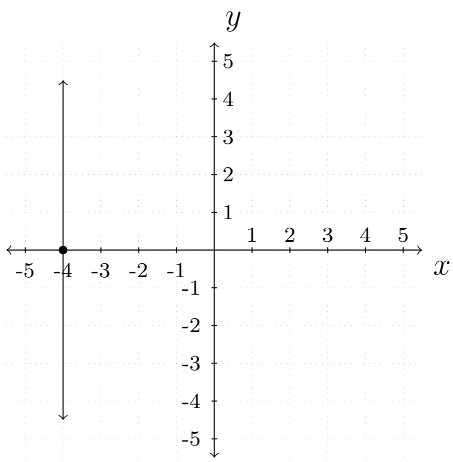
Once we have both points, connect the dots to get our graph.

We want to be very careful not to confuse using slope to find the next point with use a coordinate such as $(4, -2)$ to find an individual point. Coordinates such as $(4, -2)$ start from the origin and move horizontally first, and vertically second. Slope starts from a point on the line that could be anywhere on the graph. The numerator is the vertical change and the denominator is the horizontal change.

Lines with zero slope or no slope can make a problem seem very different. Such lines are either horizontal ($m = 0$) or vertical (m is undefined).

A horizontal line will have a slope of zero which when multiplied by x gives zero. So the equation simply becomes $y = 0x + b$ or just $y = b$. Remember that in this case, b also refers to where the line crosses the $y-$axis.

If we have no slope ($m = \varnothing$), our line is vertical, and the corresponding equation cannot be written in slope-intercept form. In this case, there is no y in our equation, and we simply write $x = a$, where a refers to the $x-$coordinate for the point where the line crosses the $x-$axis.



In this graph, because we have a vertical line (m is undefined), we do not use the slope-intercept form of a linear equation.

Rather, we set x equal to the $x-$coordinate of the $x-$intercept.

Our corresponding equation is $x = -4$.

Point-Slope Form

Objective: Find the equation of a line with a known slope and passing through a given point.

The slope-intercept form has the advantage of being simple to remember and use, however, it has one major disadvantage: we must know the y-intercept in order to use it! Generally we do not know the y-intercept, we only know one or more points (that are not the y-intercept). In these cases we can't use the slope intercept equation, so we will use a different, more flexible formula. If we let the slope of an equation be m, and a specific point on the line be (x_1, y_1), and any other point on the line be (x, y). We can use the slope formula to make a second equation.

Example 52.

$$
\begin{aligned}
m,\ (x_1, y_1),\ (x, y) &\qquad \text{Recall slope formula} \\
\frac{y_2 - y_1}{x_2 - x_1} = m &\qquad \text{Plug in values} \\
\frac{y - y_1}{x - x_1} = m &\qquad \text{Multiply both sides by} (x - x_1) \\
y - y_1 = m(x - x_1) &\qquad \text{Our equation}
\end{aligned}
$$

If we know the slope, m of an equation and any point on the line (x_1, y_1) we can easily plug these values into the equation above which will be called the point-slope formula or equation.

Point-slope equation: $y - y_1 = m(x - x_1)$

Example 53. Write the equation of the line through the point $(3, -4)$ with a slope of $\frac{3}{5}$.

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Plug values into point} - \text{slope formula} \\ y - (-4) &= \frac{3}{5}(x - 3) && \text{Simplify signs} \\ y + 4 &= \frac{3}{5}(x - 3) && \text{Our solution} \end{aligned}$$

Often, we will prefer final answers be written in slope-intercept form. If the directions ask for the answer in slope-intercept form we will simply distribute the slope, then solve for y.

Example 54.

Write the equation of the line through the point $(-6, 2)$ with a slope of $-\frac{2}{3}$ in slope-intercept form.

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Plug values into point} - \text{slope formula} \\ y - 2 &= -\frac{2}{3}\left(x - (-6)\right) && \text{Simplify signs} \\ y - 2 &= -\frac{2}{3}(x + 6) && \text{Distribute slope} \\ y - 2 &= -\frac{2}{3}x - 4 && \text{Solve for } y \text{ by adding 2 to both sides} \\ \underline{\mathbf{+2}} &\quad \underline{\mathbf{+\ 2}} && \\ y &= -\frac{2}{3}x - 2 && \text{Our solution} \end{aligned}$$

An important thing to observe about the point slope formula is that the operation between the x's and y's is subtraction. This means when you simplify the signs you will have the opposite of the numbers in the point. We need to be very careful with signs as we use the point-slope formula.

In order to find the equation of a line we will always need to know the slope. If we don't know the slope to begin with we will have to do some work to find it first before we can get an equation.

Example 55. Find the equation of the line through the points $(-2, 5)$ and $(4, -3)$.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{First we must find the slope} \\ m = \frac{-3 - 5}{4 - (-2)} &= \frac{-8}{6} = -\frac{4}{3} && \text{Plug values in slope formula and evaluate} \\ y - y_1 &= m(x - x_1) && \text{Use point} - \text{slope formula,} \\ & && \quad \text{plugging in slope and either point} \end{aligned}$$

$$
\begin{aligned}
y - 5 &= -\frac{4}{3}(x - (-2)) && \text{Simplify signs} \\
y - 5 &= -\frac{4}{3}(x + 2) && \text{Our solution}
\end{aligned}
$$

Example 56.

Find the equation of the line through the points $(-3, 4)$ and $(-1, -2)$ in slope-intercept form.

$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{First we must find the slope} \\
m = \frac{-2 - 4}{-1 - (-3)} &= \frac{-6}{2} = -3 && \text{Plug values in slope formula and evaluate} \\
y - y_1 &= m(x - x_1) && \text{Use point} - \text{slope formula,} \\
& && \quad \text{plugging in slope and either point} \\
y - 4 &= -3(x - (-3)) && \text{Simplify signs} \\
y - 4 &= -3(x + 3) && \text{Distribute slope} \\
y - 4 &= -3x - 9 && \text{Solve for } y \\
\underline{\mathbf{+4}} &\underline{\quad\quad\ \mathbf{+\,4}} && \text{Add 4 to both sides} \\
y &= -3x - 5 && \text{Our solution}
\end{aligned}
$$

Example 57.

Find the equation of the line through the points $(6, -2)$ and $(-4, 1)$ in slope-intercept form.

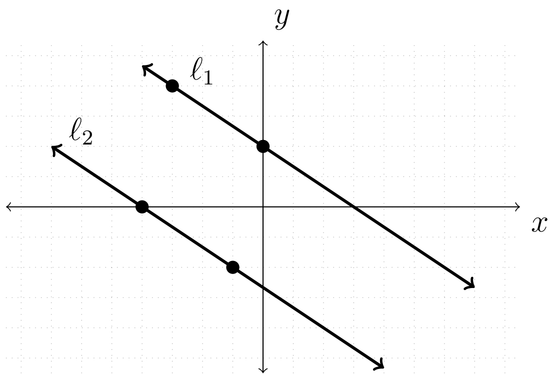
$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{First we must find the slope} \\
m = \frac{1 - (-2)}{-4 - 6} &= \frac{3}{-10} = -\frac{3}{10} && \text{Plug values into slope formula and evaluate}
\end{aligned}
$$

$$
\begin{aligned}
y - y_1 &= m(x - x_1) && \text{Use point} - \text{slope formula,} \\
& && \quad \text{plugging in slope and either point} \\
y - (-2) &= -\frac{3}{10}(x - 6) && \text{Simplify signs} \\
y + 2 &= -\frac{3}{10}(x - 6) && \text{Distribute slope} \\
y + 2 &= -\frac{3}{10}x + \frac{9}{5} && \text{Solve for } y\text{, by subtracting 2 from both sides} \\
\underline{\mathbf{-2}} &\underline{\quad\quad\quad\ \mathbf{-\,\frac{10}{5}}} && \text{Use } \frac{10}{5} \text{ on right so we have a common denominator} \\
y &= -\frac{3}{10}x - \frac{1}{5} && \text{Our solution}
\end{aligned}
$$

Parallel and Perpendicular Lines (L5)

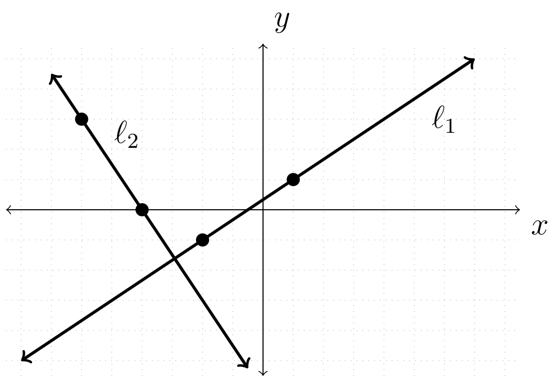
Objective: Identify the equation of a line that is either parallel or perpendicular to a given line.

There is an interesting connection between the slopes of lines that are parallel, as well as the slopes of lines that are perpendicular (meet at a right angle). This is shown in the following example.



This graph shows two parallel lines.

The slope (rise over run) of each line is "down 2, right 3," or $m_1 = m_2 = -\frac{2}{3}$.



This graph shows two perpendicular lines.

The slope (rise over run) of the more gradual line is "up 2, right 3," or $m_1 = \frac{2}{3}$.

The slope of the steeper line is "down 3, right 2," or $m_2 = -\frac{3}{2}$.

As the first graph above illustrates, parallel lines have the same slope.

On the other hand, perpendicular lines are said to have slopes that are *negative reciprocals* of one another. More precisely, if two lines with slopes m_1 and m_2 are known to be perpendicular, then $m_2 = -\dfrac{1}{m_1}$ (and so, $m_1 m_2 = -1$).

We can use these properties to make conclusions about parallel and perpendicular lines.

Example 58. Find the slope of a line parallel to $5y - 2x = 7$.

$$\begin{array}{ll}
5y - 2x = 7 & \text{To find the slope we will put equation in slope} - \text{intercept form} \\
\underline{\quad \boldsymbol{+2x} \quad \boldsymbol{+\,2x}} & \text{Add } 2x \text{ to both sides} \\
5y = 2x + 7 & \text{Put } x \text{ term first} \\
\overline{\mathbf{5}} \quad \overline{\mathbf{5}} \quad \overline{\mathbf{5}} & \text{Divide each term by } 5
\end{array}$$

$$
\begin{array}{rl}
y = \frac{2}{5}x + \frac{7}{5} & \text{The slope is the coefficient of } x \\
m = \frac{2}{5} & \text{Slope of given line} \\
 & \text{Parallel lines have the same slope} \\
m = \frac{2}{5} & \text{Our solution}
\end{array}
$$

Example 59. Find the slope of a line perpendicular to $3x - 4y = 2$.

$$
\begin{array}{rl}
3x - 4y = 2 & \text{To find slope we will put equation in slope} - \text{intercept form} \\
\underline{\mathbf{-3x} \qquad \mathbf{-3x}} & \text{Subtract } 3x \text{ from both sides} \\
-4y = -3x + 2 & \text{Put } x \text{ term first} \\
\overline{\mathbf{-4}} \quad \overline{\mathbf{-4}} \quad \overline{\mathbf{-4}} & \text{Divide each term by } -4 \\
y = \frac{3}{4}x - \frac{1}{2} & \text{The slope is the coefficient of } x \\
m = \frac{3}{4} & \text{Slope of given line} \\
 & \text{Perpendicular lines have negative reciprocal slopes} \\
m = -\frac{4}{3} & \text{Our solution}
\end{array}
$$

Once we have a slope, it is possible to find the complete equation of the desired line, if we know one point on it.

Example 60. Find the equation of a line through $(4, -5)$ and parallel to $2x - 3y = 6$.

$$
\begin{array}{rl}
2x - 3y = 6 & \text{We first need slope of parallel line} \\
\underline{\mathbf{-2x} \qquad \mathbf{-2x}} & \text{Subtract } 2x \text{ from each side} \\
-3y = -2x + 6 & \text{Put } x \text{ term first} \\
\overline{\mathbf{-3}} \quad \overline{\mathbf{-3}} \quad \overline{\mathbf{-3}} & \text{Divide each term by } -3 \\
y = \frac{2}{3}x - 2 & \text{Identify the slope, the coefficient of } x \\
m = \frac{2}{3} & \text{Parallel lines have the same slope} \\
m = \frac{2}{3} & \text{We will use this slope and our point} (4, -5) \\
y - y_1 = m(x - x_1) & \text{Plug this information into point} - \text{slope formula} \\
y - (-5) = \frac{2}{3}(x - 4) & \text{Simplify signs} \\
y + 5 = \frac{2}{3}(x - 4) & \text{Our solution}
\end{array}
$$

Example 61. Find the equation of the line through $(6,-9)$ perpendicular to $y=-\frac{3}{5}x+4$ in slope-intercept form.

$$\begin{aligned}
y &= -\frac{3}{5}x+4 && \text{Identify the slope, coefficient of } x\\
m &= -\frac{3}{5} && \text{Perpendicular lines have negative reciprocal slopes}\\
m &= \frac{5}{3} && \text{We will use this slope and our point}(6,-9)\\
y-y_1 &= m(x-x_1) && \text{Plug this information into point} - \text{slope formula}\\
y-(-9) &= \frac{5}{3}(x-6) && \text{Simplify signs}\\
y+9 &= \frac{5}{3}(x-6) && \text{Distribute slope}
\end{aligned}$$

$$\begin{array}{rl}
y+9=\frac{5}{3}x-10 & \text{Solve for } y\\
\underline{\mathbf{-9}\qquad\quad \mathbf{-\ 9}} & \text{Subtract 9 from both sides}\\
y=\frac{5}{3}x-19 & \text{Our solution}
\end{array}$$

Zero slopes and undefined slopes may seem like opposites (one is a horizontal line, one is a vertical line). Because a horizontal line is perpendicular to a vertical line we can say that an undefined slope and a zero slope are actually perpendicular slopes!

Example 62. Find the equation of the line through (3, 4) perpendicular to $x=-2$.

$$\begin{array}{rl}
x=-2 & \text{This equation has an undefined slope, a vertical line}\\
\text{Undefined slope} & \text{Perpendicular line then would have a zero slope}\\
m=0 & \text{Use this and our point}(3,4)\\
y-y_1=m(x-x_1) & \text{Plug this information into point} - \text{slope formula}\\
y-4=0(x-3) & \text{Distribute slope}\\
y-4=0 & \text{Solve for } y\\
\underline{\mathbf{+4}\ \ \mathbf{+4}} & \text{Add 4 to each side}\\
y=4 & \text{Our solution}
\end{array}$$

Being aware that to be perpendicular to a vertical line means we have a horizontal line through a y value of 4, thus we could have jumped from this point right to the solution, $y=4$.

Linear Inequalities and Sign Diagrams (L6)

Objective: Solve, graph, and give interval notation for the solution to a linear inequality. Create a sign diagram to identify those intervals where a linear expression is positive or negative.

Linear Inequalities

When given a linear equation such as $x+2=5$, one can solve to obtain *one* solution ($x=3$). Although the method for solving an inequality is, in general, very similar to that for solving an equation, we will see that the solution to a inequality will usually include an entire range of values.

In order to solve any inequality, we must first understand the accompanying notation and respective terminology.

Symbol	Meaning
$<$	less than
$>$	greater than
$\leq$	less than or equal to
$\geq$	greater than or equal to
$\neq$	not equal to

For a more in-depth treatment of set notation (graphical, interval, or inequality notation) including unions and intersections, a review of the following open resource is strongly recommended: *Logic and Set Notation*

Notice that the "equals" symbol = is not listed in the table above, as we will be working with *inequalities*, rather than equations. It is also worth mentioning that there are several alternate ways of describing the same symbol. For example, the phrases "at most" or "no more than" can easily be interchanged with "less than or equal to", and similarly for "at least", "no less than", and "greater than or equal to". Because of this, one needs to use a bit of caution, when faced with any problem that is presented verbally.

$$2<5, \qquad 1>-1, \qquad 5\leq 10, \qquad 3\leq 3,$$
$$7\geq -2, \qquad 4\geq 4, \qquad -1\neq 1$$

The examples above, though true, do not contain a variable. We now will work with inequalities containing one (or more) variable(s). Following the previous sections of this chapter, we will first concern ourselves with linear inequalities, followed by compound inequalities and inequalities that contain an absolute value. The solution to an inequality is the set of all real numbers that make the inequality true.

Example 63. Solve the linear inequality $x + 2 < 5$.

$$\begin{array}{rrl} x + 2 & < 5 & \\ \underline{\mathbf{-2}} & \underline{\mathbf{-2}} & \text{Subtract 2 from both sides} \\ x & < 3 & \text{Our solution} \end{array}$$

Notice that we solve the previous inequality using the same method that one would use to solve the equation $x + 2 = 5$. Some differences will be seen later.

When describing the solution to a given inequality, it will often be useful to graph the solution on a number line and shade the section(s) of the number line that coincide with the solution set. The number line below illustrates our previous example.

Note that an open (unshaded) circle is often used in place of the parenthesis above. In each case, this notation denotes an *exclusion* of the value $x = 3$, since it is *not* a valid solution to the given inequality. Alternatively, a closed (shaded) circle or bracket would be used to denote *inclusion* of the boundary value, in the event that it *is* a valid solution.

It is also a good idea to test a few values in order to check our work.

<u>Check</u>:

Test Location	Test Value	Unsimplified	Simplified	Result
Shaded region	$x = 1$	$1 + 2 < 5$	$3 < 5$	True
Boundary value	$x = 3$	$3 + 2 < 5$	$5 < 5$	False
Unshaded region	$x = 5$	$5 + 2 < 5$	$7 < 5$	False

A common misconception that many students have with an inequality such as $x < 3$ and is worth mentioning has to do with the values between $x = 2$ and $x = 3$. Although we have seen that x cannot equal 3 in the given inequality, this does not mean that the solution set has a largest value at $x = 2$ (the largest *integer* solution to the inequality). In fact, there are infinitely many *real-number* solutions to the inequality between the integers 2 and 3. For example, 2.5, 2.7, 2.9, 2.99, 2.999, and 2.9999999999999999 are all valid solutions to $x < 3$. Because of this, one could say that the inequality is *bounded above by* $x = 3$, but there is no *largest* solution that satisfies it.

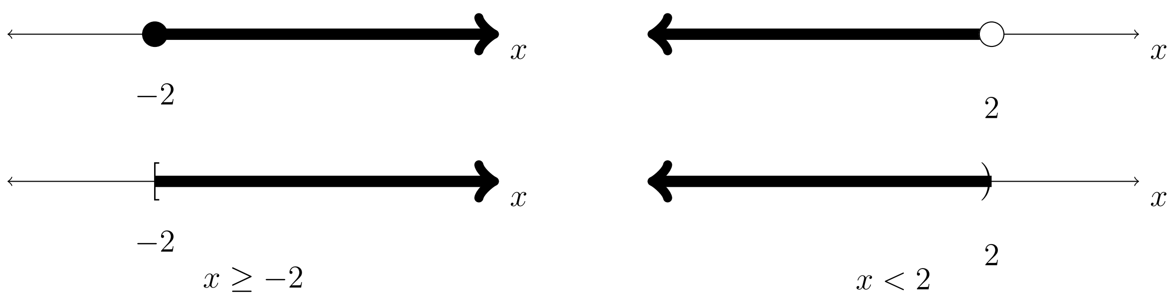
There are four primary ways of presenting the solution to an inequality:

1. In words (verbally): "All real numbers x greater than or equal to 4."
2. Using inequality (and set-builder) notation: $\{x | x \geq 4\}$.

3. Using interval notation: $[4, \infty)$.
4. Using real-number line notation (graphically):

In many of our examples, it will be acceptable to exclude the set-builder notation $\{x| \quad \}$ altogether, and instead simply present the inequality $x \geq 4$. Still, it is important that students recognize the meaning behind the notation ("The set of all real numbers x such that...").

Recall that for interval notation we use brackets [or] to denote *inclusion* of a boundary value, and parentheses (and) to denote *exclusion*. This notation can therefore be interchanged with a closed circle (inclusion) or an open circle (exclusion), when graphing a given solution set on the real-number line. As a convention, from this point forward we will adopt brackets and parentheses instead of closed and open circles for graphical representations of solution sets, since it presents a nice connection between interval and real-number line notation. Both notations, however, are generally accepted. An example is shown below.



Next, we will solve and present the solution to a linear equality using all four presentation methods.

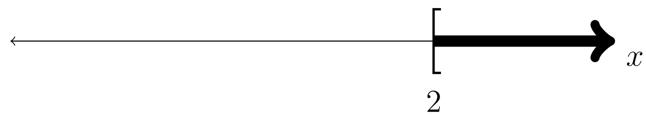
Example 64. Solve the linear inequality $4x - 3 \geq 5$.

$$\begin{array}{rcll}
4x - 3 & \geq & 5 & \\
\underline{\mathbf{+3}} & & \underline{\mathbf{+3}} & \text{Add 3 to both sides} \\
4x & \geq & 8 & \\
\overline{\mathbf{4}} & & \overline{\mathbf{4}} & \text{Divide both sides by 4} \\
x & \geq & 2 & \text{Our solution}
\end{array}$$

Our solution can be expressed as follows.

1. Verbally: "The set of all values of x that are greater than or equal to (at least) 2".

2. Inequality: $\{x | x \geq 2\}$
3. Interval: $[2, \infty)$
4. Real-number Line (Graphically):



Note: A closed (shaded) circle at $x = 2$ is also acceptable in place of a bracket.

Check:

Test Location	Test Value	Unsimplified	Simplified	Result
Shaded region	$x = 3$	$4(3) - 3 \geq 5$	$9 \geq 5$	True
Boundary value	$x = 2$	$4(2) - 3 \geq 5$	$5 \geq 5$	True
Unshaded region	$x = 0$	$4(0) - 3 \geq 5$	$-3 \geq 5$	False

Next, we would like to closely examine the impact that each of the four main operations ($+$, $-$, $\times$, $\div$) has on a given inequality. This will shed more light on one of the fundamental differences between solving an equation and solving an inequality. To demonstrate this, we will repeatedly use an obvious true statement, $4 < 10$.

Original Inequality: $4 < 10$

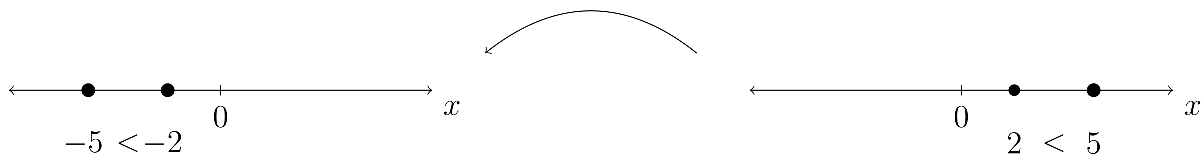
Action	Resulting Inequality	Outcome
Add 5	$9 < 15$	True
Subtract 5	$-1 < 5$	True
Add -3	$1 < 7$	True
Subtract -3	$7 < 13$	True

Note that since addition and subtraction are closely related, we see that the original inequality is also preserved when negative values are either added or subtracted. In other words, adding (or subtracting) -3 will also preserve the validity of the inequality. It is also worth noting that the action of adding -3 is analogous with that of subtracting 3, so there are no surprises. Later on, we will use the term *inverse* to describe the relationship between these two operations.

Original Inequality: $4 < 10$

Action	Resulting Inequality	Outcome
Multiply by 3	$12 < 30$	True
Divide by 2	$2 < 5$	True
Multiply by -3	$-12 < -30$	**False**
Divide by -2	$-2 < -5$	**False**

Here, we see that multiplication, and consequently division, by a negative value forces us to change the direction of the inequality ($-2 < -5$ changes to $-2 > -5$) in order to preserve its validity. This is best illustrated by the following diagram.



Multiply by -1

Note that as with addition and subtraction, the *inverse* relationship between the operations of multiplication and division is again at work, since for example, division by -2 is analogous to multiplication by $-1/2$.

We conclude our treatment of linear inequalities with a more complicated example. All our solution steps will be identical to those for solving a linear equation, with the only exception being those steps related to multiplication or division by a negative number.

Example 65. Solve the linear inequality $-1 - 2(x-3) \leq 5x - 9$.

$$
\begin{array}{rcll}
-1 - 2(x-3) & \leq & 5x - 9 & \\
-1 - 2x + 6 & \leq & 5x - 9 & \text{Distribute } -2 \\
5 - 2x & \leq & 5x - 9 & \text{Combine like terms} \\
\underline{\mathbf{-5}} \quad & & \underline{\mathbf{-5}} & \text{Subtract 5 from both sides} \\
-2x & \leq & 5x - 14 & \\
\underline{\boldsymbol{-5x}} & & \underline{\boldsymbol{-5x}} & \text{Subtract } 5x \text{ from both sides} \\
-7x & \leq & -14 & \\
\overline{\mathbf{-7}} & & \overline{\mathbf{-7}} & \text{Divide both sides by } -7 \\
x & \geq & 2 & \text{Our solution}
\end{array}
$$

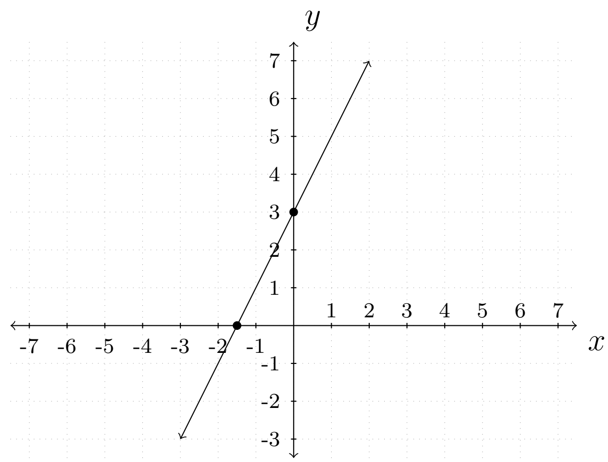
We leave it as an exercise to the reader to check that our solution is correct.

Introduction to Sign Diagrams

In a later chapter we will define a *function*, providing several examples of y as a function of x, and discuss in detail the processes associated with graphing certain families of functions. As both linear and quadratic functions present the most basic examples of polynomials, we will take this opportunity to introduce a tool, called a *sign diagram* (or chart), that will be incredibly useful for graphing these and more complicated functions. For the sake of the mathematics, it should be noted that the usefulness of the sign diagram for graphing is a direct consequence of the *continuity* of a function and the *Intermediate Value Theorem* (IVT). These concepts will be studied more closely in subsequent courses (e.g. Calculus)

Example 66. Graph the linear equation $y = 2x + 3$.

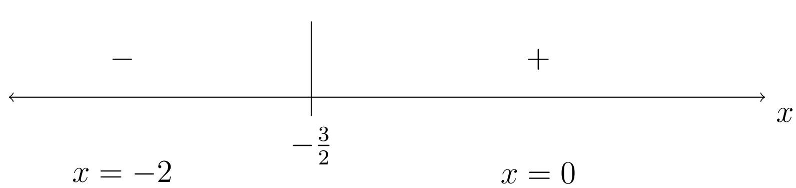
Our graph will have a y-intercept at the point $(0, 3)$. By setting $y = 0$, we obtain an x-intercept at the point $(-3/2, 0)$. We then obtain the following graph by plotting these two intercepts and connecting them.



When graphing any equation, it will be of particular interest to identify any x-intercepts on the graph. Though this will sometimes prove a daunting and even impossible task, as we have seen, it is relatively straightforward when faced with a linear equation. Recall that all lines which are not horizontal will have exactly one x-intercept. Horizontal lines will either have no x-intercepts or, in the case of the horizontal line $y = 0$, will have infinitely many x-intercepts. Once we know the x-intercept of the graph of our linear equation, we can easily determine the sign ($+$ or $-$) of the y-coordinate for every point to the left or right of our x-intercept. Since all lines are by their nature straight, this amounts to testing our equation, by plugging in a single *test value* for each interval on either side of our x-intercept.

In the case of our example, though we are free to choose any real-numbered test values we would like, we will make the more common selections of $x = -2$ and $x = 0$. Note that $x = -1$ would have been a perfectly fine value instead of $x = 0$, but it is often easier to plug $x = 0$ into a function than any other value. After plugging each test value into the equation, we determine the sign of the y-coordinate associated with $x = -2$ is negative $(-)$, since $2(-2) + 3 < 0$, and the sign of the y-coordinate associated with $x = 0$ is positive $(+)$, since $2(0) + 3 > 0$. Note that here we are *not* concerned with the actual values of the y-coordinates, just their respective signs. This point will be reiterated as we encounter more complicated mathematical expressions. The results of our calculations are presented on the real-number line shown below.

Example 67. Sign Diagram for $y = 2x + 3$.



Note that if constructed correctly, our sign diagram should be consistent with the graph of $y = 2x + 3$. Specifically, a plus $(+)$ corresponds to those points on the graph that sit *above* the x-axis, and a minus $(-)$ corresponds to those points that sit *below* the x-axis.

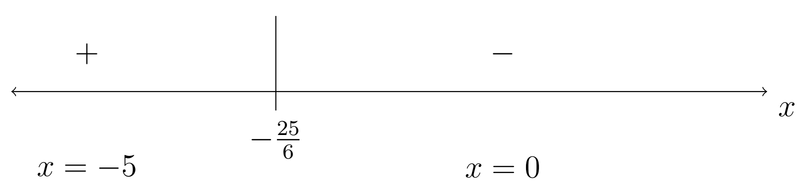
We now will summarize the steps for constructing a sign diagram for a linear equation (or function) with a nonzero slope.

1. If not provided, put the equation in slope-intercept form.
2. Determine the x-intercept of the graph of the equation. Mark this value (call it x_0) on a real-number line by placing a symbol $|$ directly above it that divides the line into two intervals, $(-\infty, x_0)$ and (x_0, ∞).
3. Identify a test value for each interval. Write your test values below their respective test intervals.
4. Determine the sign (either positive or negative) of the y-coordinate for each test value. Mark this on the real-number line by placing either a $+$ or $-$ above the interval.

Example 68. Construct a sign diagram for the linear equation $y = -12x - 50$.

By setting $y = 0$, we get $x = -\frac{50}{12} = -\frac{25}{6} = -4.1\overline{6}$. For test values, we will use $x = -5$ and $x = 0$.

Test Value	Resulting y-coordinate	Sign
$x = -5$	$-12(-5) - 50 = 60 - 50 > 0$	$+$
$x = 0$	$-12(0) - 50 = 0 - 50 < 0$	$-$



Note that in the instance of a horizontal line $m = 0$, our sign diagram will only require us to test a single value for the entire interval $(-\infty, \infty)$. It therefore suffices to just identify the sign of the y-intercept for the graph of our equation. Lastly, if the y-intercept is zero, then our sign diagram will have no test intervals to check, since all points on our graph will be of the form $(x, 0)$.

It is worth mentioning that here we have only sought to "set the table" for the construction of sign diagrams, using linear equations as a very basic introduction. Once we are exposed to more complicated equations and functions, we will see how the construction of a sign diagram will become more involved. In short, more complicated examples will include more x-intercepts, which will result in more test intervals to check. The process, however, will essentially remain the same as we have outlined, and the resulting sign diagram will be critical in understanding the graph of a function and solving any related inequalities.

Compound and Absolute Value Inequalities

Compound Inequalities (L7)

Objective: Solve, graph and give interval notation to the solution of compound inequalities and inequalities containing absolute values.

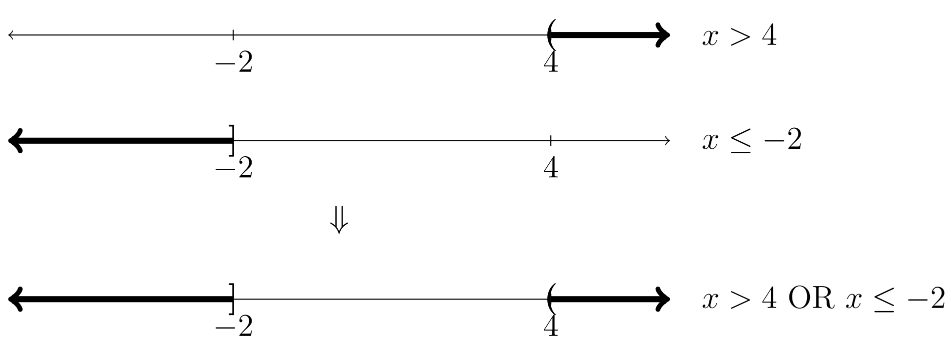
Several inequalities can be combined together to form what are called compound inequalities. There are three types of compound inequalities which we will investigate in this section.

The first type of a compound inequality is an OR inequality. For this type of inequality we want a true statement from either one inequality OR the other inequality OR both. When we are graphing these type of inequalities we will graph each individual inequality above the number line, then combine them together on the number line for our graph.

When we provide interval notation for our solution, if there are two different intervals to the graph we will put a $\cup$ between the two intervals. The $\cup$ symbol represents a *union* of the two intervals in our final answer.

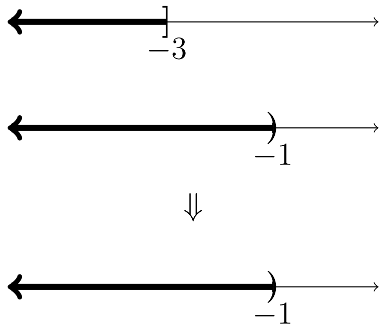
Example 69. Solve each inequality, graph the solution, and provide the interval notation of your solution.

$2x-5>3$	or	$4-x\geq 6$	Solve each inequality
$\underline{+5\ \ +5}$		$\underline{-4\qquad -4}$	Add or subtract first
$2x>8$	or	$-x\geq 2$	Divide
$\overline{\mathbf{2}}\quad \overline{\mathbf{2}}$		$\overline{-\mathbf{1}}\quad \overline{-\mathbf{1}}$	Dividing by negative flips sign
$x>4$	or	$x\leq -2$	Graph the inequalities separately, then combine



Our answer is $(-\infty,-2]\cup(4,\infty)$.

There are several different results that could result from an OR statement. The graphs could be pointing different directions, as in the graph above. The graphs could also be pointing in the same direction, as in the graph below on the left. Lastly, the graphs could be pointing in opposite directions, but overlapping, as in the graph below on the right. Notice how interval notation works for each of these cases.

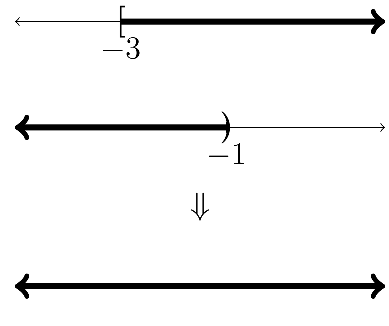


$x \leq -3$ or $x < -1$

$(-\infty, -3] \cup (-\infty, -1) = (-\infty, -1)$

As the graphs overlap, we take the largest graph for our solution.

Interval notation: $(-\infty, -1)$



$x \geq -3$ or $x < -1$

$[-3, \infty) \cup (-\infty, -1) = (-\infty, \infty)$

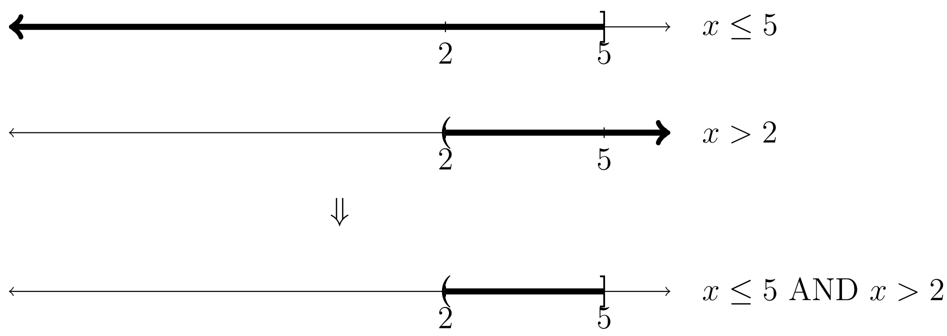
When the graphs are combined they cover the entire number line.

Interval notation: $(-\infty, \infty)$ or $\mathbb{R}$

The second type of compound inequality is an AND inequality. These inequalities require *both* statements to be true. If one is false, they both are false. When we graph these inequalities we can follow a similar process. First, graph both inequalities above the number line. This time, however, we will only consider where they overlap on the number line for our final graph. When our solution is given in interval notation it will be expressed in a manner very similar to single inequalities. The symbol that can be used for simplifying an AND inequality is known as an *intersection*, denoted by a $\cap$. When simplifying, the $\cap$ should not be needed or expressed in our final answer.

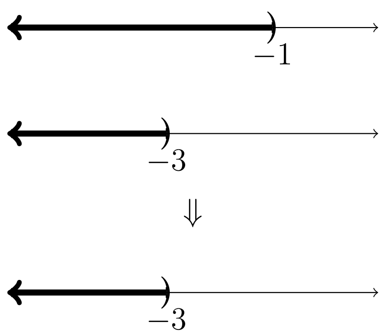
Example 70. Solve each inequality, graph the solution, and provide the interval notation of your solution.

$$\begin{array}{cccl}
2x + 8 \geq 5x - 7 & \text{and} & 5x - 3 > 3x + 1 & \text{Move variables to one side} \\
\underline{\boldsymbol{-2x \qquad - 2x}} & & \underline{\boldsymbol{-3x \qquad - 3x}} & \\
8 \geq 3x - 7 & \text{and} & 2x - 3 > 1 & \text{Add 7 or 3 to both sides} \\
\underline{\boldsymbol{+7 \qquad + 7}} & & \underline{\boldsymbol{+3 \quad + 3}} & \\
15 \geq 3x & \text{and} & 2x > 4 & \text{Divide} \\
\overline{\mathbf{3}} \quad \overline{\mathbf{3}} & & \overline{\mathbf{2}} \quad \overline{\mathbf{2}} & \\
5 \geq x & \text{and} & x > 2 & \text{Graph the inequalities} \\
 & & & \quad \text{separately, then combine}
\end{array}$$



Our answer is $(-\infty, 5] \cap (2, \infty) = (2, 5]$.

Again, as we graph AND inequalities, only the overlapping parts of the individual graphs makes it to the final number line. There are three different types of possibilities we could encounter when analyzing the overlap of an AND inequality. The first is shown in the above example; both intervals have some overlap, but point in opposite directions. The second occurs when the arrows both point in the same direction, as shown below on the left. The third occurs when the arrows point in opposite directions, but do not overlap, as shown below on the right. Notice how interval notation is expressed in each case.

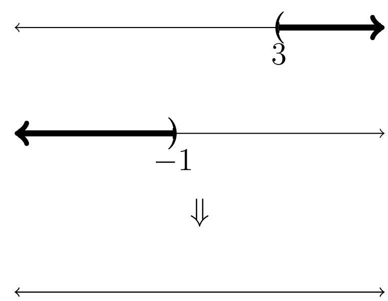


$x < -1$ and $x < -3$

$(-\infty, -1) \cap (-\infty, -3) = (-\infty, -3)$

In this graph, the overlap is only the smaller graph ($x < -2$), so this is what makes it to the final number line.

Interval notation: $(-\infty, -2)$



$x > 3$ and $x < -1$

$(3, \infty) \cap (-\infty, -1) = \varnothing$

In this graph there is no overlap of the parts. Because there is no overlap, no values make it to the final number line.

Interval notation: No solution or $\varnothing$

The third type of compound inequality is a special type of AND inequality, and occurs when our variable (or expression containing the variable) is between two numbers. We can write this as a single mathematical sentence with three parts, such as $5 < x \leq 8$, to show x is between 5 and 8 (or equal to 8). This type of inequality is often referred to as a *double inequality*, since it contains two inequalities. When solving these types of inequalities, as there are three parts to work with, in order to stay balanced we will do the same thing to all three parts (rather than just two sides), and eventually isolate the variable in the middle.

The resulting graph will contain all values between the numbers on either side of the double inequality, with appropriate brackets or parentheses on the ends.

Example 71. Solve each inequality, graph the solution, and provide the interval notation of your solution.

$$\begin{array}{rl}
-6 \leq -4x + 2 < 2 & \text{Subtract 2 from all three parts} \\
\underline{\mathbf{-2} \qquad \mathbf{-2} \quad \mathbf{-2}} & \\
-8 \leq -4x < 0 & \text{Divide all three parts by } -4 \\
\overline{\mathbf{-4}} \quad \overline{\mathbf{-4}} \quad \overline{\mathbf{-4}} & \text{Dividing by a negative flips the symbols} \\
2 \geq x > 0 & \text{Flip entire statement so values get larger left to right} \\
0 < x \leq 2 & \text{Graph } x \text{ between 0 and 2}
\end{array}$$

$0 < x \leq 2$

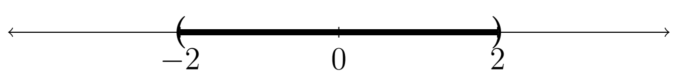
Our answer is $(0, 2]$.

Inequalities Containing Absolute Values (L8)

When an inequality contains an absolute value we will look to isolate the absolute value and eventually remove it, in order to graph the solution and express it using interval notation. The way that we treat the absolute value during this process depends on the direction of the inequality symbol.

Consider $|x| < 2$.

We define the absolute value as the distance from zero. Another way to read this inequality would be *the distance that the variable x is from zero is less than 2.* So on a number line we will shade all values of x that are less than 2 units away from zero. Alternatively stated, we will shade all values of x that are within 2 units of zero.



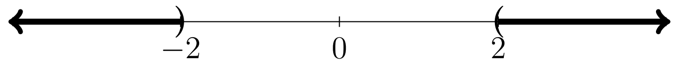
$-2 < x < 2$

Our solution set is all x in the interval $(-2, 2)$.

This graph looks just like the graphs of the double (compound) inequalities from the previous subsection! When an isolated absolute value is *less than* (or $\leq$) a number we will remove the absolute value by changing the problem to a double inequality, with the negative value on the left and the positive value on the right. So $|x| < 2$ becomes $-2 < x < 2$, as the graph above illustrates.

Consider $|x| > 2$.

Similarly, another way to read this inequality would be *the distance that x is from zero is greater than 2.* So on the number line we will shade all values of x that are more than 2 units away from zero.



$x < -2$ OR $x > 2$

Our solution set is all x in the union $(-\infty, -2) \cup (2, \infty)$.

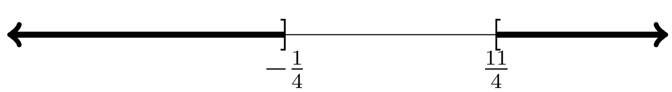
This graph looks just like the graphs of the OR compound inequalities from the previous subsection! When an isolated absolute value is *greater than* (or $\geq$) a number we will remove the absolute value by changing the problem to an OR inequality: the first inequality will look just like the original inequality, but with no absolute value; the second inequality will reverse the direction of the original inequality symbol, and changing the value to a negative. So $|x| > 2$ becomes $x > 2$ or $x < -2$, as the graph above illustrates.

For all absolute value inequalities we can also express our answers in interval notation, which is done the same way as for standard compound inequalities.

We can solve absolute value inequalities much like we solved absolute value equations. Our first step will be to isolate the absolute value. Next we will remove the absolute value by either making a double inequality if the absolute value is less than a number, or making an OR inequality if the absolute value is greater than a number. Then we will solve these inequalities. Remember, if we multiply or divide by a negative number during the process, the inequality symbol(s) must switch directions!

Example 72. Solve, graph, and provide interval notation for the following inequality.

$$\begin{array}{rcrl}
 & |4x-5| \geq 6 & & \text{Absolute value is greater, use OR} \\
4x-5 \geq 6 & \text{OR} & 4x-5 \leq -6 & \text{Solve} \\
\underline{\mathbf{+5 \quad +5}} & & \underline{\mathbf{+5 \quad +5}} & \text{Add } 5 \text{ to both sides} \\
4x \geq 11 & \text{OR} & 4x \leq -1 & \\
\overline{\mathbf{4}} \quad \overline{\mathbf{4}} & & \overline{\mathbf{4}} \quad \overline{\mathbf{4}} & \text{Divide both sides by } 4 \\
x \geq \dfrac{11}{4} & \text{OR} & x \leq -\dfrac{1}{4} & \text{Graph}
\end{array}$$

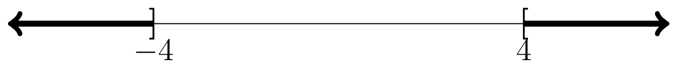


$x \leq -\frac{1}{4}$ OR $x \geq \frac{11}{4}$

Our solution set is all x in the union $(-\infty, -\frac{1}{4}] \cup [\frac{11}{4}, \infty)$.

Example 73. Solve, graph, and provide interval notation for the following inequality.

$$\begin{array}{rl}
-4-3|x| \leq -16 & \\
\underline{\mathbf{+4 \qquad\qquad +4}} & \text{Add } 4 \text{ to both sides} \\
-3|x| \leq -12 & \text{Divide both sides by } -3 \\
\overline{\mathbf{-3}} \quad \overline{\mathbf{-3}} & \text{Dividing by a negative switches the inequality} \\
|x| \geq 4 & \text{Absolute value is greater, use OR} \\
x \geq 4 \quad \text{OR} \quad x \leq -4 & \text{Graph}
\end{array}$$



$x \leq -4$ OR $x \geq 4$

Our solution set is all x in the union $(-\infty, -4] \cup [4, \infty)$.

In the previous example, we cannot combine -4 and -3 because there are no like terms, the -3 is being multiplied by an absolute value. So we must first clear the -4 by adding 4, then divide both sides by by -3. The next example is similar.

Example 74. Solve, graph, and provide interval notation for the solution.

$$\begin{array}{rl}
9-2|4x+1|>3 & \\
\underline{\mathbf{-9 \qquad\qquad -9}} & \text{Subtract 9 from both sides} \\
-2|4x+1|>-6 & \text{Divide both sides by } -2 \\
\overline{\mathbf{-2}} \qquad \overline{\mathbf{-2}} & \text{Dividing by negative switches the inequality} \\
|4x+1|<3 & \text{Absolute value is less, use double inequality} \\
-3<4x+1<3 & \text{Solve} \\
\underline{\mathbf{-1}} \qquad \underline{\mathbf{-1}} \quad \underline{\mathbf{-1}} & \text{Subtract 1 from all three parts} \\
-4<4x<2 & \text{Divide all three parts by } 4 \\
\overline{\mathbf{4}} \quad \overline{\mathbf{4}} \quad \overline{\mathbf{4}} & \\
-1<x<\dfrac{1}{2} & \text{Graph}
\end{array}$$

$-1 < x < \frac{1}{2}$

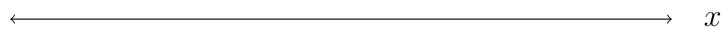
Our solution set is all x in the interval $(-1, \frac{1}{2})$.

In the previous example, we cannot distribute the -2 into the absolute value. In general, it is never recommended to distribute values inside or factor values outside of an absolute value. Our best way to solve is to first isolate the absolute value by clearing the values around it, then convert to the appropriate compound inequality (either a double inequality or an OR inequality) and solve.

It is important to remember that as we are solving these equations, an absolute value is always positive. If we end up with an absolute value that is less than a negative number, then we will have no solution because the absolute value will always be positive, and therefore greater than a negative. Similarly, if an absolute value is greater than a negative, this will always happen. Here our answer will be all real numbers. The next two examples demonstrate these special cases.

Example 75. Solve, graph, and provide interval notation for the solution.

$$\begin{array}{rl}
12+4|6x-1|<4 & \text{Subtract } 12 \text{ from both sides} \\
\underline{\mathbf{-12 \qquad\qquad -12}} & \\
4|6x-1|<-8 & \text{Divide both sides by } 4 \\
\overline{\mathbf{4}} \qquad \overline{\mathbf{4}} & \\
|6x-1|<-2 & \text{Absolute value cannot be less than a negative}
\end{array}$$



Our solution set is *no solutions* or $\varnothing$.

Example 76. Solve, graph, and provide interval notation for the solution.

$$\begin{array}{rl}
5-6|x+7|\leq 17 & \\
\underline{\mathbf{-5 \qquad\qquad -5}} & \text{Subtract } 5 \text{ from both sides} \\
-6|x+7|\leq 12 & \text{Divide both sides by } -6 \\
\overline{\mathbf{-6}} \qquad \overline{\mathbf{-6}} & \text{Dividing by a negative flips the symbol} \\
|x+7|\geq -2 & \text{Absolute value is always greater than a negative}
\end{array}$$

Our solution set is *all real numbers* or $(-\infty,\infty)$.

Practice Problems

Solving Linear Equations

One-Step Equations

Solve each equation.

1) $v+9=16$
2) $14=b+3$
3)$x-11=-16$
4) $-14=x-18$
5) $30=a+20$
6) $-1+k=5$
7) $x-7=-26$
8) $-13+p=-19$
9) $13=n-5$
10) $22=16+m$
11) $340=-17x$
12) $4r=-28$
13) $-9=\frac{n}{12}$
14) $\frac{5}{9}=\frac{b}{9}$
15) $20v=-160$
16) $-20x=-80$
17) $340=20n$
18) $\frac{1}{2}=\frac{a}{8}$
19) $16x=320$
20) $\frac{k}{13}=-16$
21) $-16+n=-13$
22) $21=x+5$
23) $p-8=-21$
24) $m-4=-13$
25) $180=12x$
26) $3n=24$
27) $20b=-200$
28) $-17=\frac{x}{12}$
29) $\frac{r}{14}=\frac{5}{14}$
30) $n+8=10$
31) $-7=a+4$
32) $v-16=-30$
33) $10=x-4$
34) $-15=x-16$
35) $13a=-143$
36) $-8k=120$
37) $\frac{p}{20}=-12$
38) $-15=\frac{x}{9}$
39) $9+m=-7$
40) $-19=\frac{n}{20}$

Two-Step Equations

Solve each equation.

1) $5 + \frac{n}{4} = 4$
2) $-2 = -2m + 12$
3) $102 = -7r + 4$
4) $27 = 21 - 3x$
5) $-8n + 3 = -77$
6) $-4 - b = 8$
7) $0 = -6v$
8) $-2 + \frac{x}{2} = 4$
9) $-8 = \frac{x}{5} - 6$
10) $-5 = \frac{a}{4} - 1$
11) $0 = -7 + \frac{k}{2}$
12) $-6 = 15 + 3p$
13) $-12 + 3x = 0$
14) $-5m + 2 = 27$
15) $24 = 2n - 8$
16) $-37 = 8 + 3x$
17) $2 = -12 + 2r$
18) $-8 + \frac{n}{12} = -7$
19) $\frac{b}{3} + 7 = 10$
20) $\frac{x}{1} - 8 = -8$
21) $152 = 8n + 64$
22) $-11 = -8 + \frac{v}{2}$
23) $-16 = 8a + 64$
24) $-2x - 3 = -29$
25) $56 + 8k = 64$
26) $-4 - 3n = -16$
27) $-2x + 4 = 22$
28) $67 = 5m - 8$
29) $-20 = 4p + 4$
30) $9 = 8 + \frac{x}{6}$
31) $-5 = 3 + \frac{n}{2}$
32) $\frac{m}{4} - 1 = -2$
33) $\frac{r}{8} - 6 = -5$
34) $-80 = 4x - 28$
35) $-40 = 4n - 32$
36) $33 = 3b + 3$
37) $87 = 3 - 7v$
38) $3x - 3 = -3$
39) $-x + 1 = -11$
40) $4 + \frac{a}{3} = 1$

General Linear Equations

Solve each equation.

1) $2 - (-3a - 8) = 1$
2) $2(-3n + 8) = -20$
3) $-5(-4 + 2v) = -50$
4) $2 - 8(-4 + 3x) = 34$
5) $66 = 6(6 + 5x)$
6) $32 = 2 - 5(-4n + 6)$
7) $0 = -8(p - 5)$
8) $-55 = 8 + 7(k - 5)$
9) $-2 + 2(8x - 7) = -16$
10) $-(3 - 5n) = 12$
11) $-21x + 12 = -6 - 3x$
12) $-3n - 27 = -27 - 3n$
13) $-1 - 7m = -8m + 7$
14) $56p - 48 = 6p + 2$
15) $1 - 12r = 29 - 8r$
16) $4 + 3x = -12x + 4$
17) $20 - 7b = -12b + 30$
18) $-16n + 12 = 39 - 7n$
19) $-32 - 24v = 34 - 2v$
20) $17 - 2x = 35 - 8x$

21) $-2 - 5(2 - 4m) = 33 + 5m$
22) $-25 - 7x = 6(2x - 1)$
23) $-4n + 11 = 2(1 - 8n) + 3n$
24) $-7(1 + b) = -5 - 5b$
25) $-6v - 29 = -4v - 5(v + 1)$
26) $-8(8r - 2) = 3r + 16$
27) $2(4x - 4) = -20 - 4x$
28) $-8n - 19 = -2(8n - 3) + 3n$
29) $-a - 5(8a - 1) = 39 - 7a$
30) $-4 + 4k = 4(8k - 8)$
31) $-57 = -(-p + 1) + 2(6 + 8p)$
32) $16 = -5(1 - 6x) + 3(6x + 7)$
33) $-2(m - 2) + 7(m - 8) = -67$
34) $7 = 4(n - 7) + 5(7n + 7)$
35) $50 = 8(7 + 7r) - (4r + 6)$
36) $-8(6 + 6x) + 4(-3 + 6x) = -12$
37) $-8(n - 7) + 3(3n - 3) = 41$
38) $-76 = 5(1 + 3b) + 3(3b - 3)$
39) $-61 = -5(5r - 4) + 4(3r - 4)$
40) $-6(x - 8) - 4(x - 2) = -4$
41) $-2(8n - 4) = 8(1 - n)$
42) $-4(1 + a) = 2a - 8(5 + 3a)$
43) $-3(-7v + 3) + 8v = 5v - 4(1 - 6v)$
44) $-6(x - 3) + 5 = -2 - 5(x - 5)$
45) $-7(x - 2) = -4 - 6(x - 1)$
46) $-(n + 8) + n = -8n + 2(4n - 4)$
47) $-6(8k + 4) = 8(6k + 3) + 12$
48) $-5(x + 7) = 4(-8x - 2)$
49) $-2(1 - 7p) = 8(p - 7)$
50) $8(-8n + 4) = 4(-7n + 8)$

Equations Containing Fractions

Solve each equation.

1) $\frac{3}{5}(1+p)=\frac{21}{20}$
2) $-\frac{1}{2}=\frac{3}{2}k+\frac{3}{2}$
3) $0=-\frac{5}{4}(x-\frac{6}{5})$
4) $\frac{3}{2}n-\frac{8}{3}=-\frac{29}{12}$
5) $\frac{3}{4}-\frac{5}{4}m=\frac{113}{24}$
6) $\frac{11}{4}+\frac{3}{4}r=\frac{163}{32}$
7) $\frac{635}{72}=-\frac{5}{2}(-\frac{11}{4}+x)$
8) $-\frac{16}{9}=-\frac{4}{3}(\frac{5}{3}+n)$
9) $2b+\frac{9}{5}=-\frac{11}{5}$
10) $\frac{3}{2}-\frac{7}{4}v=-\frac{9}{8}$
11) $\frac{3}{2}(\frac{7}{3}n+1)=\frac{3}{2}$
12) $\frac{41}{9}=\frac{5}{2}(x+\frac{2}{3})-\frac{1}{3}x$
13) $-a-\frac{5}{4}(-\frac{8}{3}a+1)=-\frac{19}{4}$
14) $\frac{1}{3}(-\frac{7}{4}k+1)-\frac{10}{3}k=-\frac{13}{8}$
15) $\frac{55}{6}=-\frac{5}{2}(\frac{3}{2}p-\frac{5}{3})$
16) $-\frac{1}{2}(\frac{2}{3}x-\frac{3}{4})-\frac{7}{2}x=-\frac{83}{24}$
17) $\frac{16}{9}=-\frac{4}{3}(-\frac{4}{3}n-\frac{4}{3})$
18) $\frac{2}{3}(m+\frac{9}{4})-\frac{10}{3}=-\frac{53}{18}$
19) $-\frac{5}{8}=\frac{5}{4}(r-\frac{3}{2})$
20) $\frac{1}{12}=\frac{4}{3}x+\frac{5}{3}(x-\frac{7}{4})$
21) $-\frac{11}{3}+\frac{3}{2}b=\frac{5}{2}(b-\frac{5}{3})$
22) $\frac{7}{6}-\frac{4}{3}n=-\frac{3}{2}n+2(n+\frac{3}{2})$
23) $-(-\frac{5}{2}x-\frac{3}{2})=-\frac{3}{2}+x$
24) $-\frac{149}{16}-\frac{11}{3}r=-\frac{7}{4}r-\frac{5}{4}(-\frac{4}{3}r+1)$
25) $\frac{45}{16}+\frac{3}{2}n=\frac{7}{4}n-\frac{19}{16}$
26) $-\frac{7}{2}(\frac{5}{3}a+\frac{1}{3})=\frac{11}{4}a+\frac{25}{8}$
27) $\frac{3}{2}(v+\frac{3}{2})=-\frac{7}{4}v-\frac{19}{6}$
28) $-\frac{8}{3}-\frac{1}{2}x=-\frac{4}{3}x-\frac{2}{3}(-\frac{13}{4}x+1)$
29) $\frac{47}{9}+\frac{3}{2}x=\frac{5}{3}(\frac{5}{2}x+1)$
30) $\frac{1}{3}n+\frac{29}{6}=2(\frac{4}{3}n+\frac{2}{3})$

Equations Containing an Absolute Value

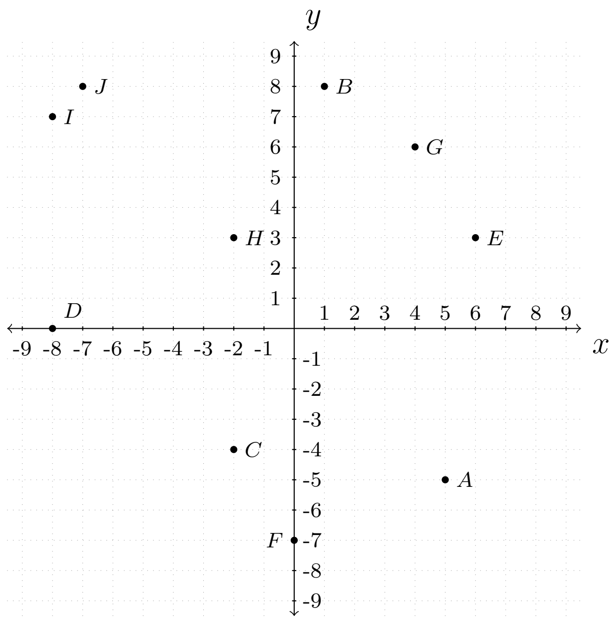
Solve each equation.

1) $|x|=8$
2) $|n|=7$
3) $|b|=1$
4) $|x|=2$
5) $|5+8a|=53$
6) $|9n+8|=46$
7) $|3k+8|=2$
8) $|3-x|=6$
9) $|9+7x|=30$
10) $|5n+7|=23$
11) $|8+6m|=50$
12) $|9p+6|=3$
13) $|6-2x|=24$
14) $|3n-2|=7$
15) $-7|-3-3r|=-21$
16) $|2+2b|+1=3$
17) $7|-7x-3|=21$
18) $\frac{|-4-3n|}{4}=2$
19) $\frac{|-4b-10|}{8}=3$
20) $8|5p+8|-5=11$
21) $8|x+7|-3=5$
22) $3-|6n+7|=-40$
23) $5|3+7m|+1=51$
24) $4|r+7|+3=59$
25) $3+5|8-2x|=63$
26) $5+8|-10n-2|=101$
27) $|6b-2|+10=44$
28) $7|10v-2|-9=5$
29) $-7+8|-7x-3|=73$
30) $8|3-3n|-5=91$

Graphing Linear Equations

The Cartesian Plane

1) **Find the coordinates of each point.**



2) **Graph each point on the $xy-$plane.**

$A(0,4)$	$F(-4,2)$
$B(0,3)$	$G(-3,0)$
$C(3,-2)$	$H(-3,4)$
$D(-2,-2)$	$I(1,0)$
$E(4,-2)$	$J(-5,5)$

Graphing Lines from Points

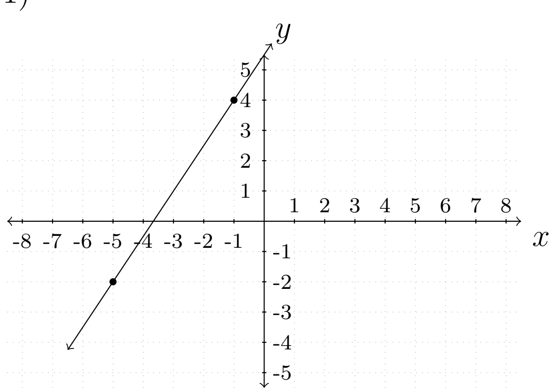
Sketch the graph of each line.

1) $y = -\frac{1}{4}x - 3$
2) $y = x - 1$
3) $y = -\frac{5}{4}x - 4$
4) $y = -\frac{3}{5}x + 1$
5) $y = -4x + 2$
6) $y = \frac{5}{3}x + 4$
7) $y = \frac{3}{2}x - 5$
8) $y = -x - 2$
9) $y = -\frac{4}{5}x - 3$
10) $y = \frac{1}{2}x$
11) $x + 5y = -15$
12) $8x - y = 5$
13) $4x + y = 5$
14) $3x + 4y = 16$
15) $2x - y = 2$
16) $7x + 3y = -12$
17) $x + y = -1$
18) $3x + 4y = 8$
19) $x - y = -3$
20) $9x - y = -4$

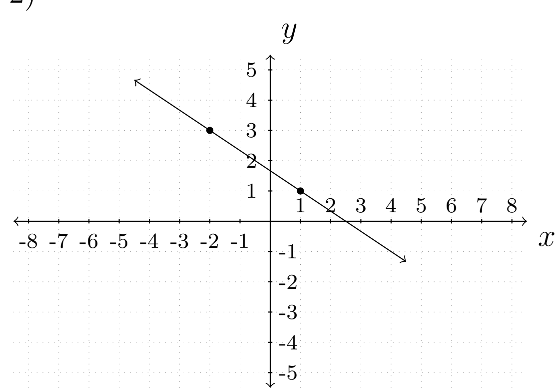
The Slope of a Line

Find the slope of each line.

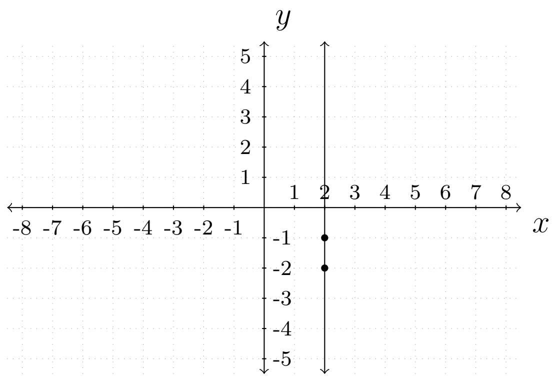
1)



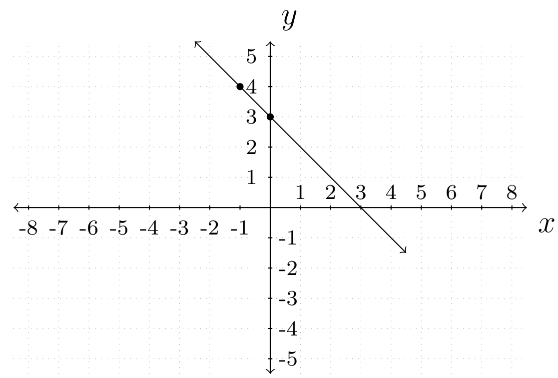
2)



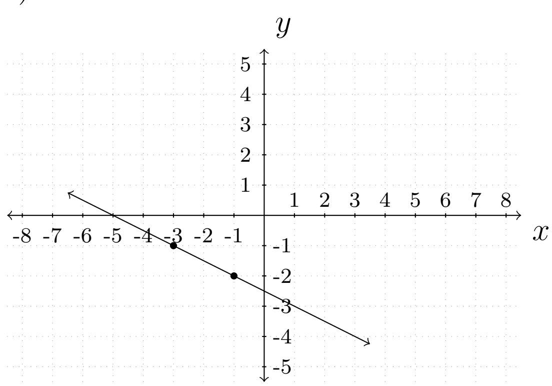
3)



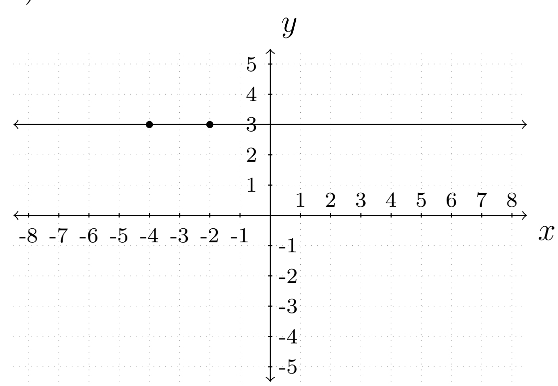
4)



5)



6)



Find the slope of the line through each pair of points.

7) $(-2, 10), (-2, -15)$
8) $(1, 2), (-6, -14)$
9) $(-15, 10), (16, -7)$
10) $(13, -2), (7, 7)$
11) $(10, 18), (-11, -10)$
12) $(-3, 6), (-20, 13)$
13) $(-16, -14), (11, -14)$
14) $(13, 15), (2, 10)$
15) $(-4, 14), (-16, 8)$
16) $(9, -6), (-7, -7)$
17) $(12, -19), (6, 14)$
18) $(-16, 2), (15, -10)$
19) $(-5, -10), (-5, 20)$
20) $(8, 11), (-3, -13)$
21) $(-17, 19), (10, -7)$
22) $(11, -2), (1, 17)$
23) $(7, -14), (-8, -9)$
24) $(-18, -5), (14, -3)$
25) $(-5, 7), (-18, 14)$
26) $(19, 15), (5, 11)$

Find the value of x or y so that the line through the points has the given slope.

27) $(2, 6)$ and $(x, 2)$; slope : $\frac{4}{7}$
28) $(8, y)$ and $(-2, 4)$; slope : $-\frac{1}{5}$
29) $(-3, -2)$ and $(x, 6)$; slope : $-\frac{8}{5}$
30) $(-2, y)$ and $(2, 4)$; slope : $\frac{1}{4}$
31) $(-8, y)$ and $(-1, 1)$; slope : $\frac{6}{7}$
32) $(x, -1)$ and $(-4, 6)$; slope : $-\frac{7}{10}$
33) $(x, -7)$ and $(-9, -9)$; slope : $\frac{2}{5}$
34) $(2, -5)$ and $(3, y)$; slope : 6
35) $(x, 5)$ and $(8, 0)$; slope : $-\frac{5}{6}$
36) $(6, 2)$ and $(x, 6)$; slope : $-\frac{4}{5}$

The Two Forms of a Linear Equation

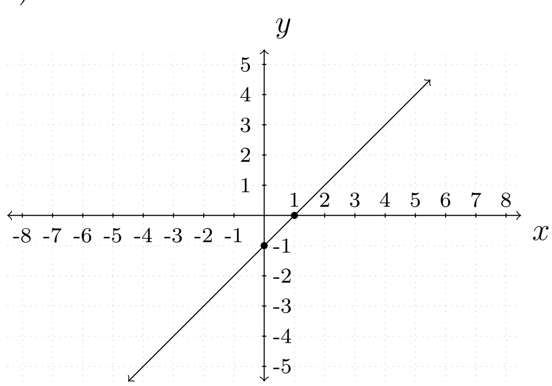
Slope-Intercept Form

Write the slope-intercept form of the equation of each line given the slope and the $y-$intercept.

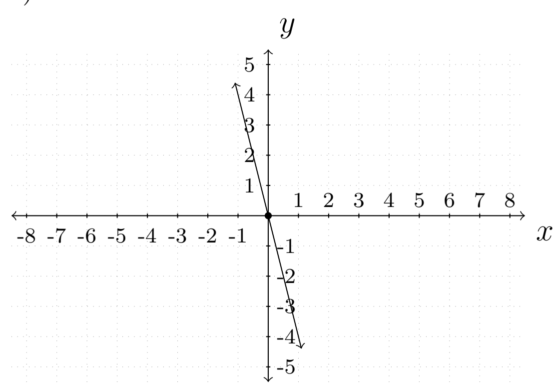
1) Slope = 2, y-intercept = 5
2) Slope = -6, y-intercept = 4
3) Slope = 1, y-intercept = -4
4) Slope = -1, y-intercept = -2
5) Slope = $-\frac{3}{4}$, y-intercept = -1
6) Slope = $-\frac{1}{4}$, y-intercept = 3
7) Slope = $\frac{1}{3}$, y-intercept = 1
8) Slope = $\frac{2}{5}$, y-intercept = 5

Write the slope-intercept form of the equation of each line.

1)



2)



Write each linear equation in slope-intercept form.

11) $x + 10y = -37$
12) $x - 10y = 3$
13) $2x + y = -1$
14) $6x - 11y = -70$
15) $7x - 3y = 24$
16) $4x + 7y = 28$
17) $x = -8$
18) $x - 7y = -42$
19) $y - 4 = -(x + 5)$
20) $y - 5 = \frac{5}{2}(x - 2)$
21) $y - 4 = 4(x - 1)$
22) $y - 3 = -\frac{2}{3}(x + 3)$
23) $y + 5 = -4(x - 2)$
24) $0 = x - 4$
25) $y + 1 = -\frac{1}{2}(x - 4)$
26) $y + 2 = \frac{6}{5}(x + 5)$

Sketch the graph of each line.

27) $y = \frac{1}{3}x + 4$
28) $y = -\frac{1}{5}x - 4$
29) $y = \frac{6}{5}x - 5$
30) $y = -\frac{3}{2}x - 1$
31) $y = \frac{3}{2}x$
32) $y = -\frac{3}{4}x + 1$
33) $x - y + 3 = 0$
34) $4x + 5 = 5y$
35) $-y - 4 + 3x = 0$
36) $-8 = 6x - 2y$
37) $-3y = -5x + 9$
38) $-3y = 3 - \frac{3}{2}x$

Point-Slope Form

Write the point-slope form of the equation of the line through the given point with the given slope.

1) through$(2, 3)$, slope = undefined
2) through$(1, 2)$, slope =undefined
3) through$(2, 2)$, slope $= \frac{1}{2}$
4) through$(2, 1)$, slope $= -\frac{1}{2}$
5) through$(-1, -5)$, slope $=9$
6) through$(2, -2)$, slope $= -2$
7) through$(-4, 1)$, slope $= \frac{3}{4}$
8) through$(4, -3)$, slope $= -2$
9) through$(0, -2)$, slope $= -3$
10) through$(-1, 1)$, slope $= 4$
11) through$(0, -5)$, slope $= -\frac{1}{4}$
12) through$(0, 2)$, slope $= -\frac{5}{4}$
13) through$(-5, -3)$, slope $= \frac{1}{5}$
14) through$(-1, -4)$, slope $= -\frac{2}{3}$
15) through$(-1, 4)$, slope $= -\frac{5}{4}$
16) through$(1, -4)$, slope $= -\frac{3}{2}$

Write the slope-intercept form of the equation of the line through the given point with the given slope.

17) through: $(-1, -5)$, slope $= 2$
18) through: $(2, -2)$, slope $= -2$
19) through: $(5, -1)$, slope $= -\frac{3}{5}$
20) through: $(-2, -2)$, slope $= -\frac{2}{3}$
21) through: $(-4, 1)$, slope $= \frac{1}{2}$
22) through: $(4, -3)$, slope $= -\frac{7}{4}$
23) through: $(4, -2)$, slope $= -\frac{3}{2}$
24) through: $(-2, 0)$, slope $= -\frac{5}{2}$
25) through: $(-5, -3)$, slope $= -\frac{2}{5}$
26) through: $(3, 3)$, slope $= \frac{7}{3}$
27) through: $(2, -2)$, slope $= 1$
28) through: $(-4, -3)$, slope $=0$
29) through:$(-3, 4)$, slope=undefined
30) through: $(-2, -5)$, slope $=2$
31) through: $(-4, 2)$, slope $= -\frac{1}{2}$
32) through: $(5, 3)$, slope $= \frac{6}{5}$

Write the point-slope form of the equation of the line through the given points.

33) through: $(-4, 3)$ and$(-3, 1)$
34) through: $(1, 3)$ and$(-3, 3)$
35) through: $(5, 1)$ and$(-3, 0)$
36) through: $(-4, 5)$ and$(4, 4)$
37) through: $(-4, -2)$ and$(0, 4)$
38) through: $(-4, 1)$ and$(4, 4)$
39) through: $(3, 5)$ and$(-5, 3)$
40) through: $(-1, -4)$ and$(-5, 0)$
41) through: $(3, -3)$ and$(-4, 5)$
42) through: $(-1, -5)$ and$(-5, -4)$

Write the slope-intercept form of the equation of the line through the given points.

43) through: $(-5, 1)$ and$(-1, -2)$
44) through: $(-5, -1)$ and$(5, -2)$
45) through: $(-5, 5)$ and$(2, -3)$
46) through: $(1, -1)$ and$(-5, -4)$
47) through: $(4, 1)$ and$(1, 4)$
48) through: $(0, 1)$ and$(-3, 0)$
49) through: $(0, 2)$ and$(5, -3)$
50) through: $(0, 2)$ and$(2, 4)$
51) through: $(0, 3)$ and$(-1, -1)$
52) through: $(-2, 0)$ and$(5, 3)$

Parallel and Perpendicular Lines

Find the slope of a line parallel to each given line.

1) $y = 2x + 4$
2) $y = -\frac{2}{3}x + 5$
3) $y = 4x - 5$
4) $y = -\frac{10}{3}x - 5$
5) $x - y = 4$
6) $6x - 5y = 20$
7) $7x + y = -2$
8) $3x + 4y = -8$

Find the slope of a line perpendicular to each given line.

9) $x = 3$
10) $y = -\frac{1}{2}x - 1$
11) $y = -\frac{1}{3}x$
12) $y = \frac{4}{5}x$
13) $x - 3y = -6$
14) $3x - y = -3$
15) $x + 2y = 8$
16) $8x - 3y = -9$

Write the point-slope form of the equation of the line described.

17) through : $(2, 5)$, parallel to $x = 0$
18) through: (5, 2), parallel to $y = \frac{7}{5}x + 4$
19) through : $(3, 4)$, parallel to $y = \frac{9}{2}x - 5$
20) through: $(1, -1)$, parallel to $y = -\frac{3}{4}x + 3$
21) through : $(2, 3)$, parallel to $y = \frac{7}{5}x + 4$
22) through : $(-1, 3)$, parallel to $y = -3x - 1$
23) through : $(4, 2)$, parallel to $x = 0$
24) through : $(1, 4)$, parallel to $y = \frac{7}{5}x + 2$
25) through: (1, −5), perpendicular to $-x + y = 1$
26) through : $(1, -2)$, perpendicular to $-x + 2y = 2$
27) through : $(5, 2)$, perpendicular to $5x + y = -3$
28) through: (1, 3), perpendicular to $-x + y = 1$
29) through : $(4, 2)$, perpendicular to $-4x + y = 0$
30) through: $(-3, -5)$, perpendicular to $3x + 7y = 0$
31) through : $(2, -2)$ perpendicular to $3y - x = 0$
32) through: $(-2, 5)$. perpendicular to $y - 2x = 0$

Write the slope-intercept form of the equation of the line described.

33) through : $(4, -3)$, parallel to $y = -2x$
34) through : $(-5, 2)$, parallel to $y = \frac{3}{5}x$
35) through : $(-3, 1)$, parallel to $y = -\frac{4}{3}x - 1$
36) through : $(-4, 0)$, parallel to $y = -\frac{5}{4}x + 4$
37) through : $(-4, -1)$, parallel to $y = -\frac{1}{2}x + 1$
38) through : $(2, 3)$, parallel to $y = \frac{5}{2}x - 1$
39) through : $(-2, -1)$, parallel to $y = -\frac{1}{2}x - 2$
40) through : $(-5, -4)$, parallel to $y = \frac{3}{5}x - 2$
41) through : $(4, 3)$, perpendicular to $x + y = -1$
42) through : $(-3, -5)$, perpendicular to $x + 2y = -4$
43) through : $(5, 2)$, perpendicular to $x = 0$
44) through : $(5, -1)$, perpendicular to $-5x + 2y = 10$
45) through : $(-2, 5)$, perpendicular to $-x + y = -2$
46) through : $(2, -3)$, perpendicular to $-2x + 5y = -10$

47) through : $(4, -3)$, perpendicular to $-x + 2y = -6$
48) through : $(-4, 1)$, perpendicular to $4x + 3y = -9$

Linear Inequalities and Sign Diagrams

Draw a graph for each inequality and provide interval notation.

1) $n > -5$
2) $n > 4$
3) $-2 \geq k$
4) $1 \geq k$
5) $5 \geq x$
6) $-5 < x$

Solve each inequality, graph each solution, and provide interval notation.

7) $\frac{x}{11} \geq 10$
8) $-2 \leq \frac{n}{13}$
9) $2 + r < 3$
10) $\frac{m}{5} \leq -\frac{6}{5}$
11) $8 + \frac{n}{3} \geq 6$
12) $11 > 8 + \frac{x}{2}$
13) $2 > \frac{a-2}{5}$
14) $\frac{v-9}{-4} \leq 2$
15) $-47 \geq 8 - 5x$
16) $\frac{6+x}{12} \leq -1$
17) $-2(3 + k) < -44$
18) $-7n - 10 \geq 60$
19) $18 < -2(-8 + p)$
20) $5 \geq \frac{x}{5} + 1$
21) $24 \geq -6(m - 6)$

22) $-8(n - 5) \geq 0$
23) $-r - 5(r - 6) < -18$
24) $-60 \geq -4(-6x - 3)$
25) $24 + 4b < 4(1 + 6b)$
26) $-8(2 - 2n) \geq -16 + n$
27) $-5v - 5 < -5(4v + 1)$
28) $-36 + 6x > -8(x + 2) + 4x$
29) $4 + 2(a + 5) < -2(-a - 4)$
30) $3(n + 3) + 7(8 - 8n) < 5n + 5 + 2$
31) $-(k - 2) > -k - 20$
32) $-(4 - 5p) + 3 \geq -2(8 - 5p)$

Construct a sign diagram for each of following graphs/linear equations referenced below.

33)-38): Graphs (1) through (6) on page 62.

39)-50): Linear equations (27) through (38) on page 64.

Compound and Absolute Value Inequalities

Compound Inequalities

Solve each compound inequality, graph its solution, and provide interval notation.

1) $\frac{n}{3} \leq -3$ or $-5n \leq -10$
2) $6m \geq -24$ or $m - 7 < -12$
3) $x + 7 \geq 12$ or $9x < -45$
4) $10r > 0$ or $r - 5 < -12$
5) $x - 6 < -13$ or $6x \leq -60$
6) $9 + n < 2$ or $5n > 40$
7) $\frac{v}{8} > -1$ and $v - 2 < 1$
8) $-9x < 63$ and $\frac{x}{4} < 1$
9) $-8 + b < -3$ and $4b < 20$
10) $-6n \leq 12$ and $\frac{n}{3} \leq 2$

11) $a + 10 \geq 3$ and $8a \leq 48$
12) $-6 + v \geq 0$ and $2v > 4$
13) $3 \leq 9 + x \leq 7$
14) $0 \geq \frac{x}{9} \geq -1$
15) $11 < 8 + k \leq 12$
16) $-11 \leq n - 9 \leq -5$
17) $-3 < x - 1 < 1$
18) $1 \leq \frac{p}{8} \leq 0$
19) $-4 < 8 - 3m \leq 11$
20) $3 + 7r > 59$ or $-6r - 3 > 33$
21) $-22 \leq 2n - 10 \leq -16$
22) $-6 - 8x \geq -6$ or $2 + 10x > 82$

23) $-5b + 10 \leq 30$ and $7b + 2 \leq -40$
24) $n + 10 \geq 15$ or $4n - 5 < -1$
25) $3x - 9 < 2x + 10$ and $5 + 7x \leq 10x - 10$
26) $4n + 8 < 3n - 6$ or $10n - 8 \geq 9 + 9n$
27) $-8 - 6v \leq 8 - 8v$ and $7v + 9 \leq 6 + 10v$
28) $5 - 2a \geq 2a + 1$ or $10a - 10 \geq 9a + 9$
29) $1 + 5k \leq 7k - 3$ or $k - 10 > 2k + 10$
30) $8 - 10r \leq 8 + 4r$ or $-6 + 8r < 2 + 8r$
31) $2x + 9 \geq 10x + 1$ and $3x - 2 < 7x + 2$
32) $-9m + 2 < -10 - 6m$ or $-m + 5 \geq 10 + 4m$

Inequalities Containing an Absolute Value

Solve each inequality, graph its solution, and provide interval notation.

1) $|x| < 3$
2) $|x| \leq 8$
3) $|2x| < 6$
4) $|x + 3| < 4$
5) $|x - 2| < 6$
6) $|x - 8| < 12$
7) $|x - 7| < 3$
8) $|x + 3| \leq 4$
9) $|3x - 2| < 9$
10) $|2x + 5| < 9$
11) $1 + 2|x - 1| \leq 9$
12) $10 - 3|x - 2| \geq 4$
13) $6 - |2x - 5| \geq 3$
14) $|x| > 5$
15) $|3x| > 5$
16) $|x - 4| > 5$
17) $|x - 3| \geq 3$
18) $|2x - 4| > 6$
19) $|3x - 5| \geq 3$
20) $3 - |2 - x| < 1$
21) $4 + 3|x - 1| \geq 10$
22) $3 - 2|3x - 1| \geq -7$
23) $3 - 2|x - 5| \leq -15$
24) $4 - 6|-6 - 3x| \leq -5$
25) $-2 - 3|4 - 2x| \geq -8$
26) $-3 - 2|4x - 5| \geq 1$
27) $4 - 5|-2x - 7| < -1$
28) $-2 + 3|5 - x| \leq 4$
29) $3 - 2|4x - 5| \geq 1$
30) $-2 - 3|-3x - 5| \geq -5$
31) $-5 - 2|3x - 6| < -8$
32) $6 - 3|1 - 4x| < -3$
33) $4 - 4|-2x + 6| > -4$
34) $-3 - 4|-2x - 5| \geq -7$
35) $|-10 + x| \geq 8$

Selected Answers

Solving Linear Equations

One-Step Equations

1) $v = 7$	9) $n = 18$	17) $n = 17$	25) $x = 15$	33) $x = 14$
5) $a = 10$	13) $n = -108$	21) $n = 3$	29) $r = 5$	37) $p = -240$

Two-Step Equations

1) $n = -4$	9) $x = -10$	17) $r = 7$	25) $k = 1$	33) $r = 8$
5) $n = 10$	13) $x = 4$	21) $n = 11$	29) $p = -6$	37) $v = -12$

General Linear Equations

1) $a = -3$	13) $m = 8$	25) $v = 8$	37) $n = -6$	49) $p = -9$
5) $x = 1$	17) $b = 2$	29) $a = -1$	41) $n = 0$	
9) $x = 0$	21) $m = 3$	33) $m = -3$	45) $x = 12$	

Equations Containing Fractions

1) $p = \frac{3}{4}$	9) $b = -2$	17) $n = 0$	25) $n = 16$
5) $m = -\frac{19}{6}$	13) $a = -\frac{3}{2}$	21) $b = \frac{1}{2}$	29) $x = \frac{4}{3}$

Equations Containing an Absolute Value

1) $x = \pm 8$	9) $x = -\frac{39}{7}, 3$	17) $x = 0, \frac{6}{7}$	25) $x = -2, 10$
5) $a = -\frac{29}{4}, 6$	13) $x = -9, 15$	21) $x = -8, -6$	29) $x = -\frac{13}{7}, 1$

Graphing Linear Equations

The Cartesian Plane

1) $\{A(5,-5),\ B(1,8),\ C(-2,-4),\ D(-8,0),\ E(6,3),$
$F(0,-7),\ G(4,6),\ H(-2,3),\ I(-8,7),\ J(-7,8)\}$

The Slope of a Line

1) $m = \frac{3}{2}$	9) $m = -\frac{17}{31}$	17) $m = -\frac{33}{6}$	25) $m = -\frac{7}{13}$	31) $y = -5$
5) $m = -\frac{1}{2}$	13) $m = 0$	21) $m = -\frac{26}{27}$	27) $x = -5$	35) $x = 2$

The Two Forms of a Linear Equation

Slope-Intercept Form

1) $y = 2x + 5$
3) $y = x - 4$
5) $y = -\frac{3}{4}x - 1$
7) $y = \frac{1}{3}x + 1$
9) $y = x - 1$
11) $y = -\frac{1}{10}x - \frac{37}{10}$
13) $y = -2x - 1$
15) $y = \frac{7}{3}x - 8$
17) $x = -8$ (m undefined)
19) $y = -x - 1$
21) $y = 4x$
23) $y = -4x + 3$
25) $y = -\frac{1}{2}x + 1$

Point-Slope Form

1) $x = 2$
5) $y + 5 = 9(x + 1)$
9) $y + 2 = -3(x - 0)$
13) $y + 3 = \frac{1}{5}(x + 5)$
17) $y = 2x - 3$
21) $y = \frac{1}{2}x + 3$
25) $y = -\frac{2}{5}x - 5$
29) $x = -3$
33) $y - 3 = -2(x + 4)$
37) $y + 2 = \frac{3}{2}(x + 4)$
41) $y + 3 = -\frac{8}{7}(x - 3)$
45) $y = -\frac{8}{7}x - \frac{5}{7}$
49) $y = -x + 2$

Parallel and Perpendicular Lines

1) $m = 2$
3) $m = 4$
5) $m = 1$
7) $m = -7$
9) $m = 0$
11) $m = 3$
13) $m = -3$
15) $m = 2$
17) $x = 2$

21) $y - 3 = \frac{7}{5}(x - 2)$
25) $y + 5 = -(x - 1)$
29) $y - 2 = -\frac{1}{4}(x - 4)$
33) $y = -2x + 5$
37) $y = -\frac{1}{2}x - 3$
41) $y = x - 1$
45) $y = -x + 3$

Linear Inequalities and Sign Diagrams

9) $(-\infty, 1)$
13) $(-\infty, 12)$
17) $(19, \infty)$
21) $[2, \infty)$
25) $(1, \infty)$
29) No Solution, $\varnothing$

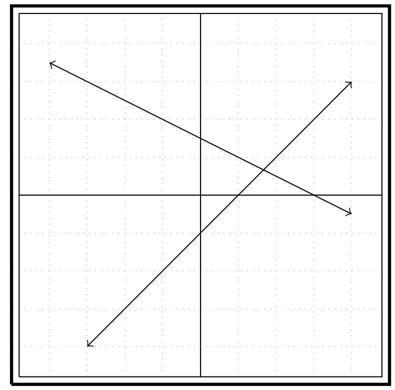
Compound and Absolute Value Inequalities

Compound Inequalities

1) $(-\infty, -9] \cup [2, \infty)$
5) $(-\infty, -7)$
9) $(-\infty, 5)$
13) $[-6, -2]$
17) $(-2, 2)$
21) $[-6, -3]$
25) $[5, 19)$
29) $(-\infty, -20) \cup [2, \infty)$

Inequalities Containing an Absolute Value

1) $(-3, 3)$
5) $(-4, 8)$
9) $(-\frac{7}{3}, \frac{11}{3})$
13) $[1, 4]$
15) $(-\infty, -\frac{5}{3}) \cup (\frac{5}{3}, \infty)$
17) $(-\infty, 0] \cup [6, \infty)$
19) $(-\infty, \frac{2}{3}] \cup [\frac{8}{3}, \infty)$
21) $(-\infty, -1] \cup [3, \infty)$
25) $[1, 3]$
29) $[1, \frac{3}{2}]$
33) $(2, 4)$



Chapter 2

Systems of Linear Equations

Introduction and Graphing (L9)

Objective: Solve a system of linear equations by graphing and identifying the point of intersection.

We have solved problems like $3x - 4 = 11$ by adding 4 to both sides and then dividing by 3 (solution is $x = 5$). We also have methods to solve equations with more than one variable in them. It turns out that to solve for more than one variable we will need the same number of equations as variables. For example, to solve for two variables such as x and y we will need two equations. When we have two (or more) equations we are working with, we call the set of equations a *system*. When solving a system of equations we are looking for a solution that satisfies each equation simultaneously. If our system consists of two equations in terms of x and y, this solution is usually described as an ordered pair (x, y). The following example demonstrates a solution for a system of two linear equations.

Example 77. Show (x, y)=(2,1) is the solution to the system

$$3x - y = 5 \qquad x + y = 3$$

$$\begin{array}{rl} (x,y) = (2,1) & \text{Identify } x \text{ and } y \text{ from the ordered pair} \\ x = 2,\ y = 1 & \text{Plug these values into each equation} \\ & \\ 3(2) - (1) = 5 & \text{First equation} \\ 6 - 1 = 5 & \text{Evaluate} \\ 5 = 5 & \text{True} \\ & \\ (2) + (1) = 3 & \text{Second equation, evaluate} \\ 3 = 3 & \text{True} \end{array}$$

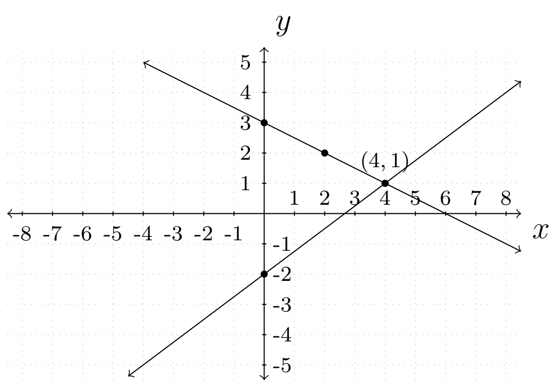
As we found a true statement for both equations we know $(2, 1)$ is the solution to the system. It is in fact the only combination of numbers that works in both equations. In this section, we will attempt to identify a (simultaneous) solution to two equations, if such a solution exists. It stands to reason that if we use points to describe the solution, we can use graphs to find the solution.

If the graph of a line is a picture of all the solutions to its equation, we can graph two lines on the same coordinate plane to see the solutions of both equations. In particular, we are interested in finding all points that are a solution for both equations. This will be the point(s) where the two lines intersect! If we can find the intersection of the lines we have found the solution that works in both equations.

Example 78. Solve the following system of equations.

$$\begin{array}{rl} y = -\frac{1}{2}x + 3 & \\ y = \frac{3}{4}x - 2 & \end{array} \qquad \text{First identify slopes and } y - \text{intercepts}$$

$$\begin{array}{ll} \text{First Line}: & m = -\frac{1}{2}, \quad b = 3 \\ \text{Second Line}: & m = \frac{3}{4}, \quad b = -2 \end{array} \qquad \text{Next graph both lines on the same plane}$$



To graph each equation, we start at the y-intercept and use the slope to get the next point, then connect the dots.

Remember a line with a negative slope points downhill from left to right!

Find the intersection point, $(4, 1)$. This is our solution.

Often our equations won't be in slope-intercept form and we will have to solve both equations for y first so we can identify the slope and $y-$intercept.

Example 79. Solve the following system of equations.

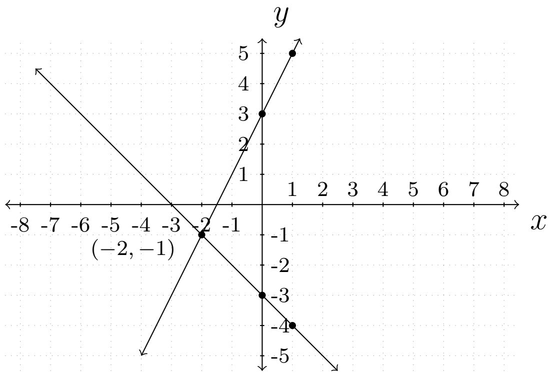
$$\begin{array}{r} 6x - 3y = -9 \\ 2x + 2y = -6 \end{array} \qquad \text{Solve each equation for } y$$

$$\begin{array}{rcr} 6x - 3y = -9 & & 2x + 2y = -6 \\ \underline{\mathbf{-6x \qquad\quad -6x}} & & \underline{\mathbf{-2x \qquad\quad -2x}} \end{array} \qquad \text{Subtract } x \text{ terms}$$

$$\begin{array}{ccc} \dfrac{-3y}{\mathbf{-3}} = \dfrac{-6x}{\mathbf{-3}} \dfrac{-9}{\mathbf{-3}} & & \dfrac{2y}{\mathbf{2}} = \dfrac{-2x}{\mathbf{2}} \dfrac{-6}{\mathbf{2}} \end{array} \qquad \begin{array}{l} \text{Rearrange equations} \\ \text{Divide by coefficient of } y \end{array}$$

$$y = 2x + 3 \qquad y = -x - 3 \quad \text{Identify slope and } y - \text{intercepts}$$

$$\begin{array}{ll} \text{First Line :} & m = 2, \quad b = 3 \\ \text{Second Line :} & m = -1, \quad b = -3 \end{array} \qquad \text{Next graph both lines on the same plane}$$



To graph each equation, we start at the y-intercept and use the slope to get the next point, then connect the dots.

Remember a line with a negative slope decreases from left to right!

Using our slopes, we can find the intersection point, $(-2, -1)$. This is our solution.

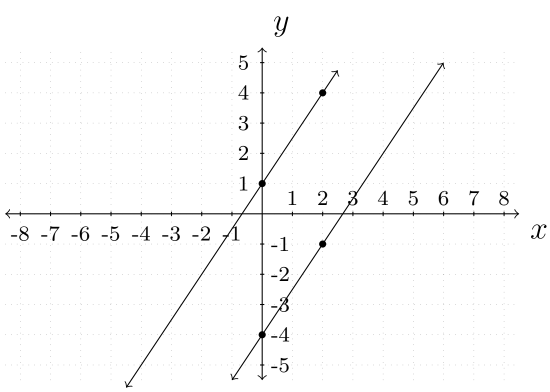
As we are graphing our lines, it is possible to have one of two unexpected results. These are shown and discussed in the next two examples.

Example 80. Solve the following system of equations.

$$y = \tfrac{3}{2}x - 4 \qquad\qquad y = \tfrac{3}{2}x + 1$$

Identify the slope and $y-$intercept of each equation.

$$\begin{array}{ll} \text{First Line :} & m = \frac{3}{2}, \quad b = -4 \\ \text{Second Line :} & m = \frac{3}{2}, \quad b = 1 \end{array} \qquad \text{Next graph both lines on the same plane}$$



To graph each equation, we start at the y-intercept and use the slope to get the next point, then connect the dots.

The two lines do not intersect; they are parallel!

Since the lines do not intersect, we know that there is no point that will satisfy both equations.

There is no solution, or $\varnothing$.

Notice that we could also have recognized that both lines had the same slope. Remembering that parallel lines have the same slope one could conclude that there is no solution without having to graph the lines.

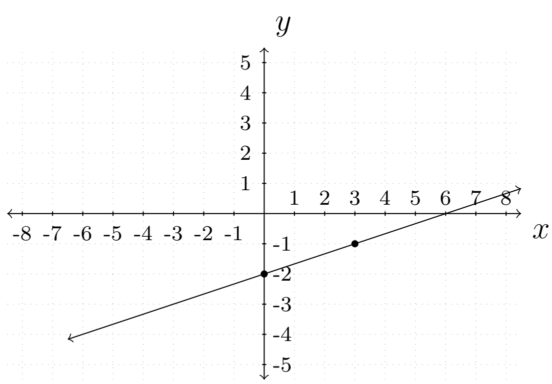
Example 81. Solve the following system of equations.

$$\begin{array}{l} 2x-6y=12 \\ 3x-9y=18 \end{array} \qquad \text{Solve each equation for } y$$

$$\begin{array}{ccc|ccc|l}
2x & -6y= & 12 & 3x & -9y= & 18 & \\
\mathbf{-2x} & & \mathbf{-2x} & \mathbf{-3x} & & \mathbf{-3x} & \text{Subtract } x \text{ terms} \\
\end{array}$$

$$\begin{array}{ll}
-6y=-2x+12 \qquad -9y=-3x+18 & \text{Put } x \text{ terms first} \\
\overline{\mathbf{-6}} \quad \overline{\mathbf{-6}} \quad \overline{\mathbf{-6}} \qquad \overline{\mathbf{-9}} \quad \overline{\mathbf{-9}} \quad \overline{\mathbf{-9}} & \text{Divide by coefficient of } y \\
y=\frac{1}{3}x-2 \qquad y=\frac{1}{3}x-2 & \text{Identify the slopes and } y-\text{intercepts}
\end{array}$$

$$\begin{array}{ll} \text{First Line}: & m=\frac{1}{3}, \quad b=-2 \\ \text{Second Line}: & m=\frac{1}{3}, \quad b=-2 \end{array} \qquad \text{Next graph both lines on the same plane}$$



To graph each equation, we start at the y-intercept and use the slope to get the next point, then connect the dots.

Both equations are the same line!

As one line is directly on top of the other line, we can say that the lines intersect at every point on the line!

Here we say there are infinitely many solutions.

Notice that once we had both equations in slope-intercept form we could have recognized that the equations were the same. At this point one could state that there are infinitely many solutions without having to go through the work of graphing the equations.

The Substitution Method (L10)

Objective: Solve systems of equations using substitution.

Solving a system of equations by graphing has several limitations. First, it requires the graph to be perfectly drawn, if the lines are not straight we may arrive at the wrong answer. Second, graphing is not a great method to use if the answer is really large, over 100 for example, or a decimal, since a graph will not help us find an answer such as 3.2134. For these reasons we will rarely use graphs to solve a given system of equations. Instead, an algebraic approach will be used.

The first algebraic approach is called substitution. We will build the concepts of substitution through several examples, then end with a five-step process to solve problems using this method.

Example 82. Solve the following system of equations.

$$x = 5 \qquad\qquad y = 2x - 3$$

We already know x must equal 5, so we can substitute $x = 5$ into the other equation.

$$\begin{array}{rl} y = 2(\mathbf{5}) - 3 & \text{Evaluate: Multiply first} \\ y = 10 - 3 & \text{Next subtract} \\ y = 7 & \text{We now also have } y \\ (x, y) = (5, 7) & \text{Our solution} \end{array}$$

When we know what one variable equals we can plug that value (or expression) in for the variable in the other equation. It is very important that when we substitute, the substituted value goes in parentheses. The reason for this is shown in the next example.

Example 83. Solve the following system of equations.

$$2x - 3y = 7 \qquad\qquad y = 3x - 7$$

We begin by substituting $y = 3x - 7$ into the other equation.

$$\begin{array}{rl} 2x - 3(\mathbf{3x - 7}) = 7 & \text{Solve for } x\text{, distributing } -3 \text{ first} \\ 2x - 9x + 21 = 7 & \text{Combine like terms } 2x - 9x \\ -7x + 21 = 7 & \\ \underline{\mathbf{-21 \quad -21}} & \text{Subtract 21} \\ -7x = -14 & \\ \overline{\mathbf{-7}} \quad \overline{\mathbf{-7}} & \text{Divide by } -7 \\ x = 2 & \text{We now have our } x. \\ & \text{Substitute back in equation to find } y. \\ y = 3(\mathbf{2}) - 7 & \text{Evaluate: Multiply first} \\ y = 6 - 7 & \text{Next subtract} \\ y = -1 & \text{We now also have } y \\ (x, y) = (2, -1) & \text{Our solution} \end{array}$$

By using the entire expression $3x - 7$ to replace y in the other equation we were able to reduce the system to a single linear equation which we can easily solve for our first variable. However, the "lone" variable (a variable with a coefficient of 1) is not always alone on one side of the equation. If this happens we can isolate the lone variable by solving for it.

Example 84. Solve the following system of equations.

$$3x + 2y = 1 \qquad \boldsymbol{x} - 5y = 6$$

The lone variable is x. Isolate the lone variable by adding $5y$ to both sides.

$$\begin{array}{rl}
x - 5y = 6 & \\
\underline{\mathbf{+5}\boldsymbol{y} \quad \mathbf{+5}\boldsymbol{y}} & \\
x = 6 + 5y & \text{Substitute this into the untouched equation} \\
 & \\
3(\mathbf{6} + \mathbf{5}\boldsymbol{y}) + 2y = 1 & \text{Solve this equation, distributing 3 first} \\
18 + 15y + 2y = 1 & \text{Combine like terms } 15y + 2y \\
18 + 17y = 1 & \\
\underline{\mathbf{-18} \qquad \mathbf{-18}} & \text{Subtract 18 from both sides} \\
17y = -17 & \\
\overline{\mathbf{17}} \quad \overline{\mathbf{17}} & \text{Divide both sides by 17} \\
y = -1 & \text{We have our } y. \\
 & \text{Substitute back in equation to find } x \\
x = 6 + 5(\mathbf{-1}) & \text{Evaluate: Multiply first, then subtract} \\
x = 1 & \text{We now also have } x \\
(x, y) = (1, -1) & \text{Our solution}
\end{array}$$

The process in the previous example is known as solving by substitution. This process is described and illustrated in the following table which lists the five steps to solving by substitution.

Problem	$4x - 2y = 2$ $2x + y = -5$
1. Find the lone variable.	Lone variable is y, in the second equation: $2x + \boldsymbol{y} = -5$
2. Solve for the lone variable.	Subtract $2x$ from both sides. $\boldsymbol{y = -5 - 2x}$
3. Substitute into the untouched equation.	$4x - 2(\mathbf{-5 - 2}\boldsymbol{x}) = 2$
4. Solve.	$4x + 10 + 4x = 2$ $8x + 10 = 2$ $\underline{\mathbf{-10} \quad \mathbf{-10}}$ $8x = -8$ $\overline{\mathbf{8}} \quad \overline{\mathbf{8}}$ $\boldsymbol{x = -1}$
5. Plug into lone variable equation and evaluate.	$y = -5 - 2(\mathbf{-1})$ $y = -5 + 2$ $\boldsymbol{y = -3}$
Our solution	$(x, y) = (-1, -3)$

Sometimes we have several lone variables in a problem. In this case we will have the choice on which lone variable we wish to solve for, either will give the same final result.

Example 85. Solve the following system of equations.

$$x + y = 5 \qquad x - y = -1$$

Find the lone variable: either x or y in the first equation, or x in the second equation. We will choose x in the first equation.

$$\begin{array}{rl}
\boldsymbol{x} + y = 5 & \text{Solve for the lone variable } x \\
\underline{\boldsymbol{-y \quad -y}} & \text{Subtract } y \text{ from both sides} \\
x = 5 - y & \text{Plug into the untouched equation, the second equation} \\
\boldsymbol{(5 - y)} - y = -1 & \text{Combine like terms. Parentheses may be removed} \\
5 - 2y = -1 & \\
\underline{\boldsymbol{-5 \qquad -5}} & \text{Subtract 5 from both sides} \\
-2y = -6 & \\
\overline{\boldsymbol{-2}} \quad \overline{\boldsymbol{-2}} & \text{Divide both sides by } -2 \\
y = 3 & \text{We have our } y! \\
x = 5 - (3) & \text{Plug into lone variable equation and evaluate} \\
x = 2 & \text{Now we have our } x \\
(x, y) = (2, 3) & \text{Our solution}
\end{array}$$

Just as with graphing it is possible to have no solution $\varnothing$ (parallel lines) or infinite solutions (same line) with the substitution method. While we won't have a parallel line or the same line to look at and conclude if it is one or the other, the process takes an interesting turn as shown in the following example.

Example 86. Solve the following system of equations.

$$y + 4 = 3x \qquad 2y - 6x = -8$$

Find the lone variable: y in the first equation.

$$\begin{array}{rl}
\boldsymbol{y} + 4 = 3x & \text{Solve for the lone variable } y \\
\underline{\boldsymbol{-4 \quad -4}} & \text{Subtract 4 from both sides} \\
y = 3x - 4 & \text{Plug into second equation} \\
2\boldsymbol{(3x - 4)} - 6x = -8 & \text{Solve, distribute through parentheses} \\
6x - 8 - 6x = -8 & \text{Combine like terms } 6x - 6x \\
-8 = -8 & \text{Variables are gone!}
\end{array}$$

Since we are left with a true statement ($-8 = -8$), we conclude that there are infinitely many solutions.

Because we had a true statement, and no variables, we know that anything that works in the first equation, will also work in the second equation. However, we do not always end up with a true statement.

Example 87. Solve the following system of equations.

$$6x - 3y = -9 \qquad -2x + y = 5$$

Find the lone variable: y in the second equation.

$$\begin{array}{rl}
-2x + \boldsymbol{y} = 5 & \text{Solve for the lone variable} \\
\underline{\boldsymbol{+2x} \qquad \boldsymbol{+2x}} & \text{Add } 2x \text{ to both sides} \\
y = 5 + 2x & \text{Plug into untouched equation} \\
6x - 3(\boldsymbol{5+2x}) = -9 & \text{Solve, distribute through parentheses} \\
6x - 15 - 6x = -9 & \text{Combine like terms } 6x - 6x \\
-15 \neq -9 & \text{Variables are gone!}
\end{array}$$

Since we are left with a false statement ($-15 \neq -9$) and no variables, we know that nothing will work in both equations and we may conclude that there are no solutions, or $\varnothing$.

One more question needs to be considered: what if there is no lone variable? If there is no lone variable substitution can still work, we will just have to select one variable to solve for, and introduce fractions.

Example 88. Solve the following system of equations.

$$5x - 6y = -14 \qquad -2x + 4y = 12$$

There is no lone variable, so we will solve the first equation for x.

$$\begin{array}{rl}
5\boldsymbol{x} - 6y = -14 & \text{Solve for our variable } x \\
\underline{\boldsymbol{+6y} \quad \boldsymbol{+6y}} & \text{Add } 6y \text{ to both sides} \\
5x = -14 + 6y & \\
\overline{\boldsymbol{5}} \qquad \overline{\boldsymbol{5}} \quad \overline{\boldsymbol{5}} & \text{Divide each term by } 5 \\
x = \dfrac{-14}{5} + \dfrac{6y}{5} & \text{Plug into untouched equation} \\
-2\left(\dfrac{-14}{5} + \dfrac{6y}{5}\right) + 4y = 12 & \text{Solve, distribute through parentheses} \\
\dfrac{28}{5} - \dfrac{12y}{5} + 4y = 12 & \text{Clear fractions by multiplying by } 5 \\
\dfrac{28(\boldsymbol{5})}{5} - \dfrac{12y(\boldsymbol{5})}{5} + 4y(\boldsymbol{5}) = 12(\boldsymbol{5}) & \text{Reduce fractions and multiply} \\
28 - 12y + 20y = 60 & \text{Combine like terms } -12y + 20y \\
28 + 8y = 60 & \\
\underline{\boldsymbol{-28} \qquad \boldsymbol{-28}} & \text{Subtract } 28 \text{ from both sides} \\
8y = 32 & \\
\overline{\boldsymbol{8}} \quad \overline{\boldsymbol{8}} & \text{Divide both sides by } 8 \\
y = 4 & \text{We have our } y
\end{array}$$

$$
\begin{array}{ll}
x = -\dfrac{14}{5} + \dfrac{6(4)}{5} & \text{Plug into lone variable equation, multiply} \\
x = -\dfrac{14}{5} + \dfrac{24}{5} & \text{Add fractions} \\
x = \dfrac{10}{5} & \text{Reduce fraction} \\
x = 2 & \text{Now we have our } x \\
(x, y) = (2, 4) & \text{Our solution}
\end{array}
$$

Using the fractions does make the problem a bit trickier. This is why we have yet another method for solving systems of equations that will be discussed in the next section.

The Addition Elimination Method (L11)

Objective: Solve systems of equations using the addition/elimination method.

When solving systems we have found that graphing is very limited when solving equations. We then considered a second method known as substitution. This is probably the most used idea in solving systems in various areas of algebra. However, substitution can get ugly if we don't have a lone variable. This leads us to our second method for solving systems of equations. This method is known as either Elimination or Addition. We will set up the process in the following examples, then define the five step process we can use to solve by elimination.

Example 89. Solve the following system of equations.

$$x - 4y = 8 \qquad 5x + 4y = -24$$

$$
\begin{array}{rl}
3x - 4y = 8 & \text{Notice opposite signs in front of } y \\
\underline{+ \quad 5x + 4y = -24} & \text{Add columns to eliminate } y \\
8x \qquad = -16 & \text{Solve for } x \\
\overline{\mathbf{8}} \qquad\quad \overline{\mathbf{8}} & \text{Divide by } 8 \\
x = -2 & \text{We have our } x! \\
5(-2) + 4y = -24 & \text{Plug into either original equation} \\
-10 + 4y = -24 & \text{Simplify} \\
\underline{\mathbf{+10} \qquad\quad \mathbf{+10}} & \text{Add } 10 \text{ to both sides} \\
4y = -14 & \\
\overline{\mathbf{4}} \qquad \overline{\mathbf{4}} & \text{Divide by } 4 \\
y = -\dfrac{7}{2} & \text{Now we have our } y! \\
(x, y) = \left(-2, -\dfrac{7}{2}\right) & \text{Our solution}
\end{array}
$$

In the previous example one variable had opposites in front of it, $-4y$ and $4y$. Adding these together eliminated the y completely. This allowed us to solve for the x. This is the idea behind the addition method. However, generally we won't have opposites in front of one of the variables. In this case we will manipulate the equations to get the opposites we want by multiplying one or both equations (on both sides!). This is shown in the next example.

Example 90. Solve the following system of equations.

$$-6x + 5y = 22 \qquad 2x + 3y = 2$$

Notice that we can obtain "opposite" coefficients (one positive and one negative) in front of x by multiplying both sides of the second equation by 3.

$3(2x + 3y) = (2)3$	Distribute to get new second equation
$6x + 9y = \ \ 6$	New second equation
$\underline{-6x + 5y = 22}$	Add equations to eliminate x
$14y = 28$	
$\overline{\mathbf{14}} \quad \overline{\mathbf{14}}$	Divide both sides by 14
$y = 2$	We have our y!
$2x + 3(2) = 2$	Plug into one of the original equations
$2x + 6 = 2$	Simplify
$\underline{\mathbf{-6} \ \ \mathbf{-6}}$	Subtract 6 from both sides
$2x = -4$	
$\overline{\mathbf{2}} \qquad \overline{\mathbf{2}}$	Divide both sides by 2
$x = -2$	We also have our x!
$(x, y) = (-2, 2)$	Our solution

When we looked at the x terms, $-6x$ and $2x$ we decided to multiply the $2x$ by 3 to get the opposites we were looking for. What we are looking for with our opposites is the least common multiple (LCM) of the coefficients. We also could have solved the above problem by looking at the terms with y, $5y$ and $3y$. The LCM of 3 and 5 is 15. So we would want to multiply both equations, the $5y$ by 3, and the $3y$ by-5 to get opposites, $15y$ and $-15y$. This illustrates an important point: for some problems we will have to multiply both equations by a constant (on both sides) to get the opposites we are looking for.

Example 91. Solve the following system of equations.

$$3x + 6y = -9 \qquad 2x + 9y = -26$$

Here, we can obtain opposite coefficients in front of y by finding the least common multiple (LCM) of 6 and 9, which is 18. We will therefore multiply both sides of both equations by the appropriate values to get $18y$ and $-18y$.

$$\begin{aligned} 3(3x+6y) &= (-9)3 && \text{Multiply the first equation by 3} \\ 9x+18y &= -27 \end{aligned}$$

$$\begin{aligned} -2(2x+9y) &= (-26)(-2) && \text{Multiply the second equation by } -2 \\ -4x-18y &= 52 \end{aligned}$$

$$\begin{aligned} 9x+18y &= -27 && \text{Add two new equations together} \\ \underline{-4x-18y} &\underline{= 52} && \quad \text{to eliminate } y \\ 5x &= 25 \\ \overline{\mathbf{5}} \quad & \quad \overline{\mathbf{5}} && \text{Divide both sides by } 5 \\ x &= 5 && \text{We have our solution for } x \end{aligned}$$

$$\begin{aligned} 3(5)+6y &= -9 && \text{Plug into either original equation} \\ 15+6y &= -9 && \text{Simplify} \\ \underline{\mathbf{-15}} \quad & \underline{\quad \mathbf{-15}} && \text{Subtract } 15 \text{ from both sides} \\ 6y &= -24 \\ \overline{\mathbf{6}} \quad & \quad \overline{\mathbf{6}} && \text{Divide both sides by } 6 \\ y &= -4 && \text{Now we have our solution for } y \\ (x,y) &= (5,-4) && \text{Our solution} \end{aligned}$$

As we get started, it is important for each problem that all variables and constants are aligned before we begin multiplying and adding equations. This is illustrated in the next example which includes the five steps we will go through to solve a problem using elimination.

Problem	$2x - 5y = -13$ $-3y + 4 = -5x$
1. Line up the variables and constants.	Rearrange the second equation $2x - 5y = -13$ $5x - 3y = -4$
2. Multiply to get opposites (use LCM).	First Equation : multiply by -5 $-5(2x - 5y) = (-13)(-5)$ $-10x + 25y = 65$ Second Equation : multiply by 2 $2(5x - 3y) = (-4)2$ $10x - 6y = -8$
3. Add equations to eliminate a variable.	$\begin{array}{r} -10x + 25y = 65 \\ \underline{10x - 6y = -8} \\ 19y = 57 \end{array}$
4. Solve.	$\frac{19y}{\mathbf{19}} = \frac{57}{\mathbf{19}}$ $y = 3$
5. Plug back into either of the given equations and solve.	$2x - 5(3) = -13$ $2x - 15 = -13$ $\underline{\mathbf{+15} \quad \mathbf{+15}}$ $\frac{2x}{\mathbf{2}} = \frac{2}{\mathbf{2}}$ $x = 1$
Solution	$(x, y) = (1, 3)$

Just as with graphing and substitution, it is possible to have no solution or infinitely many solutions with elimination. If the variables all disappear from our problem, a true statement will always indicate infinitely many solutions and a false statement will always indicate no solutions.

Example 92. Solve the following system of equations.

$$2x - 5y = 3 \qquad -6x + 15y = -9$$

In order to obtain opposite coefficients in front of x, multiply the first equation by 3.

$$\begin{array}{rl} 3(2x - 5y) = (3)3 & \\ 6x - 15y = 9 & \text{Distribute} \\ & \\ 6x - 15y = 9 & \\ \underline{-6x + 15y = -9} & \text{Add equations together} \\ 0 = 0 & \text{True statement} \end{array}$$

Since we are left with a true statement, we conclude that there are infinitely many solutions.

Example 93. Solve the following system of equations.

$$4x - 6y = 8 \qquad 6x - 9y = 15$$

Here, we will seek to obtain opposite coefficients for x. This means we must find the LCM of 4 and 6, which is 12. We will multiply both sides of both equations by the appropriate values in order to get $12x$ and $-12x$.

$$\begin{aligned} 3(4x - 6y) &= (8)3 && \text{Multiply first equation by } 3 \\ 12x - 18y &= 24 \end{aligned}$$

$$\begin{aligned} -2(6x - 9y) &= (15)(-2) && \text{Multiply second equation by } -2 \\ -12x + 18y &= -30 \end{aligned}$$

$$\begin{aligned} 12x - 18y &= 24 \\ \underline{-12x + 18y} &\underline{= -30} && \text{Add both new equations together} \\ 0 &\neq -6 && \text{False statement} \end{aligned}$$

Since we are left with a false statement, we conclude that there are no solutions, or $\varnothing$.

We have now covered three different methods that can be used to solve a system of two equations with two variables: graphing, substitution, and addition/elimination. While all three can be used to solve any system, graphing works well for small integer solutions. Substitution works best when we have a lone variable, and addition/elimination works best when the other two methods fail. As each method has its own strengths, it is important that students become familiar with all three methods.

Three Variables

Objective: Solve systems of equations with three variables.

Recall that the graph of an equation containing two variables is a (two-dimensional) line. If we increase the number of variables in an equation to three, then the resulting graph will be a three-dimensional plane. This particular section deals with solving a system of equations containing three variables. Whereas the solution for a system of *two* equations is the set of points where their respective *lines* intersect, the solution for a system of *three* equations will be the set of points where all three respective *planes* intersect. Although we do not intend to undertake the arduous task of graphing even a single equation containing three variables in this setting, the visual is sometimes helpful in justifying a particular outcome, and is often critical to understanding in more advanced mathematics courses such as multivariate calculus and linear algebra.

The method for solving a system of equations with three (or more) variables is very similar to that for solving a system with two variables. When we had two variables we reduced the system down to one equation with one variable (by either substitution or addition/elimination).

With three variables we will reduce the system down to one equation with two variables (usually by addition/elimination), which we can then solve by either substitution or addition/elimination.

To reduce from three variables down to two it is very important to keep the work organized by lining up the variables vertically and using enough space to carefully keep track of everything. We will use addition/elimination with two equations to eliminate one variable. This new equation we will call (A). Then we will use a different pair of equations and use addition/elimination to eliminate the *same* variable. This second new equation we will call (B). Once we have done this we will have a system of two equations, (A) and (B), with the same two variables that we can solve using either method, substitution or elimination, depending on the context of the problem. This is demonstrated in the following examples.

Example 94. Solve the following system of equations.

$$\begin{aligned} 3x + 2y \ - z &= -1 \\ -2x - 2y + 3z &= 5 \\ 5x + 2y \ - z &= 3 \end{aligned}$$

Our strategy will be to first eliminate y using two different pairs of equations from those provided.

$$\begin{array}{rl} 3x + 2y \ - z = -1 & \text{Using the first two equations,} \\ \underline{-2x - 2y + 3z = \ \ 5} & \text{Add} \\ x + 2z = \ \ 4 & \text{Call this equation (A)} \end{array}$$

$$\begin{array}{rl} -2x - 2y + 3z = 5 & \text{Using the second two equations} \\ \underline{5x + 2y \ - \ z = 3} & \text{Add} \\ 3x + 2z = 8 & \text{Call this equation (B)} \end{array}$$

$$\begin{array}{rl} x + 2z = 4 & \text{Equation (A)} \\ 3x + 2z = 8 & \text{Equation (B)} \end{array}$$

$$\begin{array}{ll} -1(x + 2z) = (4)(-1) & \text{Multiply equation (A) by } -1 \\ \quad -x - 2z = -4 & \text{Simplify} \end{array}$$

$$\begin{array}{rl} -x - 2z = -4 & \\ \underline{3x + 2z = \ \ 8} & \text{Add the two equations} \\ 2x = 4 & \\ \overline{\mathbf{2}} \quad \overline{\mathbf{2}} & \text{Divide by } 2 \\ x = 2 & \text{We now have } x! \\ & \text{Plug } x \text{ into either (A) or (B)} \end{array}$$

$$
\begin{array}{rl}
(2)+2z=4 & \text{We will use (A)} \\
\underline{\mathbf{-2} \qquad \mathbf{-2}} & \text{Subtract 2} \\
2z=2 & \\
\overline{\mathbf{2}} \quad \overline{\mathbf{2}} & \text{Divide by 2} \\
z=1 & \text{We now have } z! \\
 & \text{Plug } x \text{ and } z \text{ into any of the original equations} \\
3(2)+2y-(1)=-1 & \text{We will use the first equation} \\
 & \text{Simplify; reduce and combine constant terms} \\
2y+5=-1 & \text{Solve for } y \\
\underline{\mathbf{-5} \quad \mathbf{-5}} & \text{Subtract 5} \\
2y=-6 & \\
\overline{\mathbf{2}} \qquad \overline{\mathbf{2}} & \text{Divide by 2} \\
y=-3 & \text{We now have } y! \\
(x,y,z)=(2,-3,1) & \text{Our solution}
\end{array}
$$

As we are solving for x, y, and z we will have an ordered triplet (x, y, z) instead of just the ordered pair (x, y). In the previous problem, y was easily eliminated using the addition method. Sometimes, however, we may have to do a bit of work to eliminate a variable. Just as with the addition of two equations, we may have to multiply the equations by a constant on both sides in order to get the opposites we want and eliminate the variable. As we do this, remember that it is important to eliminate the *same variable each time*, using two *different* pairs of equations.

Example 95. Solve the following system of equations.

$$
\begin{aligned}
4x-3y+2z&=-29 \\
6x+2y-z&=-16 \\
-8x-y+3z&=23
\end{aligned}
$$

Notice that no variable will easily eliminate. Although we are free to choose any variable to eliminate, we will choose x here. Remember, we will be eliminating x *twice*, using two different equations each time.

$$
\begin{array}{rl}
4x-3y+2z=-29 & \text{Begin with the first two equations} \\
6x+2y-z=-16 & \text{The LCM of 4 and 6 is 12}
\end{array}
$$

We will multiply both sides of the first equation by 3 to obtain $12x$. Similarly, we will multiply both sides of the second equation by -2 to obtain $-12x$.

$$\begin{aligned} 3(4x - 3y + 2z) &= (-29)3 && \text{Multiply the first equation by } 3 \\ 12x - 9y + 6z &= -87 \end{aligned}$$

$$\begin{aligned} -2(6x + 2y - z) &= (-16)(-2) && \text{Multiply the second equation by } -2 \\ -12x - 4y + 2z &= 32 \end{aligned}$$

$$\begin{array}{rl} 12x - 9y + 6z = -87 & \\ \underline{-12x - 4y + 2z = \ \ 32} & \text{Add these two equations together} \\ -13y + 8z = -55 & \text{Call this equation (A)} \end{array}$$

Next, we will use a different pair of equations.

$$\begin{aligned} 6x + 2y - z &= -16 && \text{Now use the second pair of equations} \\ -8x - y + 3z &= \ \ 23 && \text{The LCM of 6 and } -8 \text{ is } 24 \end{aligned}$$

Now, we will multiply both sides of the first equation by 4 to obtain $24x$, and both sides of the second equation by 3 to obtain $-24x$.

$$\begin{aligned} 4(6x + 2y - z) &= (-16)4 && \text{Multiply the first equation by } 4 \\ 24x + 8y - 4z &= -64 \end{aligned}$$

$$\begin{aligned} 3(-8x - y + 3z) &= (23)3 && \text{Multiply the second equation by } 3 \\ -24x - 3y + 9z &= 69 \end{aligned}$$

$$\begin{array}{rl} 24x + 8y - 4z = -64 & \\ \underline{-24x - 3y + 9z = \ \ 69} & \text{Add these two equations together} \\ 5y + 5z = \ \ 5 & \text{Call this equation (B)} \end{array}$$

Now, using equations (A) and (B), we will solve the given system.

$$\begin{aligned} -13y + 8z &= -55 && \text{Equation (A)} \\ 5y + 5z &= \ \ 5 && \text{Equation (B)} \end{aligned}$$

$$\begin{array}{rl} 5y + 5z = 5 & \text{Solve equation (B) for } z \\ \underline{\boldsymbol{-5y} \qquad \boldsymbol{-\ 5y}} & \text{Subtract } 5y \\ 5z = 5 - 5y & \\ \overline{\mathbf{5}} \quad \overline{\mathbf{5}} \quad \overline{\mathbf{5}} & \text{Divide both sides by } 5 \\ z = 1 - y & \text{Equation for } z \end{array}$$

Next, substitute z into equation (A).

$$\begin{aligned} -13y + 8\boldsymbol{(1 - y)} &= -55 && \text{Simplify} \\ -13y + 8 - 8y &= -55 && \text{Distribute} \end{aligned}$$

$$\begin{aligned} -21y + 8 &= -55 && \text{Combine like terms} \\ \underline{\mathbf{-8} \quad\;\; \mathbf{-8}} & && \text{Subtract } 8 \\ -21y &= -63 \\ \overline{\mathbf{-21}} \quad \overline{\mathbf{-21}} & && \text{Divide by } -21 \\ y &= 3 && \text{We have our } y! \end{aligned}$$

Now plug y into the equation for z.

$$\begin{aligned} z &= 1 - (3) && \text{Evaluate} \\ z &= -2 && \text{We have } z! \end{aligned}$$

Now, we can find x from one of our original equations. We will use the first equation.

$$\begin{aligned} 4x - 3(3) + 2(-2) &= -29 && \text{Simplify} \\ 4x - 13 &= -29 && \text{Combine like terms} \\ \underline{\mathbf{+13} \quad \mathbf{+13}} & && \text{Add } 13 \\ 4x &= -16 \\ \overline{\mathbf{4}} \qquad \overline{\mathbf{4}} & && \text{Divide by } 4 \\ x &= -4 && \text{We have our } x! \\ (x, y, z) &= (-4, 3, -2) && \text{Our solution} \end{aligned}$$

Just as with two variables and two equations, we can have special cases come up with three variables and three equations. Specifically, it is possible to encounter a system of equations that has infinitely many solutions, or none at all. The way we handle such systems is identical to that for a system containing only two equations/variables.

Example 96. Solve the following system of equations.

$$\begin{aligned} 5x - 4y + 3z &= -4 \\ -10x + 8y - 6z &= 8 \\ 15x - 12y + 9z &= -12 \end{aligned}$$

Again, we will choose to eliminate x.

$$\begin{aligned} 5x - 4y + 3z &= -4 && \text{Begin with the first two equations} \\ -10x + 8y - 6z &= 8 && \text{The LCM of 5 and } -10 \text{ is } 10 \end{aligned}$$

We will multiply both sides of the first equation by 2 to obtain $10x$. Since the second equation contains $-10x$, we do not need to multiply it by a constant.

$$\begin{aligned} 2(5x - 4y + 3z) &= -4(2) && \text{Multiply the first equation by } 2 \\ 10x - 8y + 6z &= -8 \end{aligned}$$

$$\begin{aligned} 10x - 8y + 6z &= -8 \\ \underline{-10x + 8y - 6 = 8} & && \text{Add the two equations} \\ 0 &= 0 && \text{A true statement} \end{aligned}$$

Since we are left with a true statement, we conclude that there are infinitely many solutions to the first two equations.

Remember, that our usual procedure requires us to eliminate a variable (x in this case) *twice*, using two different equations each time. Even though we have concluded that there are infinitely many simultaneous solutions to the first two equations, we still must consider two different equations. In this particular example, we will obtain the same outcome by choosing *any* two equations, and it is left as an exercise for the reader to show this.

Hint: What do you notice about the set of coefficients for each equation, in relation to each of the other two equations? Do you think our results are related to this?

Once we have eliminated the same variable *twice* and drawn the same conclusions as above, we can conclude that there are infinitely many simultaneous solutions (x, y, z) to *all three* equations, i.e., the entire system.

There are, in fact, cases where two equations will share infinitely many solutions, but the entire system of equations might *fail* to have any simultaneous solutions. This is why it is critical that we not rush to an incorrect conclusion. These more subtle cases will usually be treated in detail in a multivariate calculus or a linear algebra course.

Our last example demonstrates the only time when it is permissible to eliminate a variable from two equations in our system once.

Example 97. Solve the following system of equations.

$$\begin{aligned} 3x - 4y + z &= 2 \\ -9x + 12y - 3z &= -5 \\ 4x - 2y - z &= 3 \end{aligned}$$

Here, it will be slightly easier to eliminate z.

$$\begin{aligned} 3x - 4y + z &= 2 && \text{Begin with the first two equations} \\ -9x + 12y - 3z &= -5 && \text{The LCM of 1 and } -3 \text{ is } 3 \end{aligned}$$

We will multiply both sides of the first equation by 3 to obtain $3z$. Since the second equation contains $-3z$, we do not need to multiply it by a constant.

$$\begin{aligned} 3(3x - 4y + z) &= (2)3 && \text{Multiply the first equation by } 3 \\ 9x - 12y + 3z &= 6 \end{aligned}$$

$$\begin{array}{rl} 9x - 12y + 3z = 6 & \\ \underline{-9x + 12y - 3z = -5} & \text{Add the two equations} \\ 0 \neq 1 & \text{A false statement} \end{array}$$

Since we are left with a false statement, we conclude that there are no solutions to the given system.

Again, our usual procedure requires us to eliminate a variable (z in this case) *twice*, using two different equations each time. In this particular case, however, we need only eliminate the variable once. Since we obtained a false statement, which implies that there can be no solution to the first *two* equations in the system, it will be impossible to obtain a simultaneous solution to *all three* equations.

Equations with three (or more) variables are no more difficult to attempt to solve than those containing two variables, if we are careful to keep our information organized and eliminate the same variable twice, each time using two different pairs of equations. As with many problems, it is possible to solve each system several different ways. We can use different pairs of equations or eliminate variables in different orders. But as long as our information is organized and our algebra is correct, we should always arrive at the same conclusion.

Matrices

Objective: Represent a system of linear equations as an augmented matrix. Solve a system of linear equations using matrix row reduction.

In this section, we will solve systems of linear equations using matrices and row operations. The first step will be to represent a system as an augmented matrix, as in the following example.

Example 98.

System	Augmented Matrix
$\begin{aligned} x - 2y + z &= 7 \\ 3x - 5y + z &= 14 \\ 2x - 2y - z &= 3 \end{aligned}$	$\left[\begin{array}{rrr\|l} 1 & -2 & 1 & 7 \\ 3 & -5 & 1 & 14 \\ 2 & -2 & -1 & 3 \end{array}\right]$

In our example the entries in the first three columns of the matrix are given by the coefficients of each of the variables in their corresponding equations; the first column contains the coefficients of x, the second column contains the coefficients of y, and the third the coefficients of z. The last column of the matrix will always contain the constant term from each equation, and is separated from the coefficient columns by a vertical line. Each row of the matrix should also match its respective equation in the ordered system.

The following row operations may be used to reduce an augmented matrix.

1. Interchange two rows.
2. Multiply all entries of a row by a nonzero constant.
3. Add one row to another row.

Furthermore, multiple row operations may be used in combination, as our first example will demonstrate.

Initially, our goal will be to transform (or reduce) the given augmented matrix using the row operations specified above into a matrix in *triangular form*. A matrix obtained from our original matrix that is in triangular form will have a solution that equals the solution for our original matrix, but which will be easier to identify.

We will now use the specified row reduction operations to transform our given matrix to a matrix in triangular form.

Example 99.

Original Matrix $\left[\begin{array}{rrr|r} 1 & -2 & 1 & 7 \\ 3 & -5 & 1 & 14 \\ 2 & -2 & -1 & 3 \end{array}\right]$

Multiply Row 1 by -3 and
add to Row 2 (replacing Row 2)
Symbolic: R2+(-3)R1 $\Rightarrow$ R2 $\left[\begin{array}{rrr|r} 1 & -2 & 1 & 7 \\ 0 & 1 & -2 & -7 \\ 2 & -2 & -1 & 3 \end{array}\right]$

Multiply Row 1 by -2 and
add to Row 3
Symbolic: R3+(-2)R1 $\Rightarrow$ R3 $\left[\begin{array}{rrr|r} 1 & -2 & 1 & 7 \\ 0 & 1 & -2 & -7 \\ 0 & 2 & -3 & -11 \end{array}\right]$

Multiply Row 2 by -2 and
add to Row 3
Symbolic: R3+(-2)R2 $\Rightarrow$ R3 $\left[\begin{array}{rrr|r} 1 & -2 & 1 & 7 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 1 & 3 \end{array}\right]$

The new matrix is now in triangular form, with resulting system of equations listed below.

$$\begin{aligned} x - 2y + z &= 7 \\ y - 2z &= -7 \\ z &= 3 \end{aligned}$$

At this point, we can easily solve the new system by first substituting $z = 3$ into the second equation to find y, and then substituting both known values for z and y into the first equation to find x. This results in the following solution, which the reader can easily verify.

$$(x, y, z) = (2, -1, 3)$$

It is also worth mentioning that, just as with every problem we have encountered, it is straightforward to check whether a particular answer is correct. In our previous example, this will amount to plugging $(x, y, z) = (2, -1, 3)$ into each equation and simplifying. Although this can be a tedious process, it is important to do every so often, in order to ensure accuracy. In the previous example, we see below that the answer checks out.

$$\begin{array}{rcccccc} x-2y+z=7 & = & 2-2(-1)+3 & = & 2+2+3 & = & 7 \\ 3x-5y+z=14 & = & 3(2)-5(-1)+3 & = & 6+5+3 & = & 14 \\ 2x-2y-z=3 & = & 2(2)-2(-1)-3 & = & 4+2-3 & = & 3 \end{array}$$

The last matrix obtained in the previous example is said to be in *row echelon form.* A matrix is in row echelon form if the following conditions are satisfied.

1. Any row consisting entirely of zeros (if any exist) is listed at the bottom of the matrix.
2. The first coefficient entry of any nonzero row (i.e., a row that does not consist entirely of zeros) is 1. We will call such an entry a "leading one".
3. The leading ones indent. In other words, the column number for the leading ones increases from left to right as the row numbers increase from top to bottom.

In fact, if we continue to apply the permissible row operations to the row echelon form of a matrix, we can obtain a matrix in which all the columns that contain a leading one will have zeros elsewhere. This particular type of matrix is known as the *reduced row echelon form* of a matrix.

Continuing with our previous example, we will obtain the reduced row echelon form for our original augmented matrix.

Example 100.

Row Echelon Form
$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 7 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Multiply Row 2 by 2 and
add to Row 1
Symbolic: R1+(2)R2 $\Rightarrow$ R1
$$\left[\begin{array}{rrr|r} 1 & 0 & -3 & -7 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Multiply Row 3 by 2 and
add to Row 2
Symbolic: R2+(2)R3 $\Rightarrow$ R2
$$\left[\begin{array}{rrr|r} 1 & 0 & -3 & -7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Multiply Row 3 by 3 and
add to Row 1
Symbolic: R1+(3)R3 $\Rightarrow$ R1
$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Our resulting equations are shown below.

$$\begin{aligned} x &= 2 \\ y &= -1 \\ z &= 3 \end{aligned}$$

Consequently, no additional work is needed to obtain our solution!

This example helps demonstrate the benefit to solving a given system of equations by row reducing its corresponding augmented matrix. And, although the row echelon form was certainly helpful in completing our task, by continuing our row reduction to obtain the *reduced* row echelon form of the matrix we completely eliminated the requirement to directly solve any equations.

This is because the applied row operations have done the work of solving the equations for us. In fact, throughout our reduction process, it would not be difficult for us to "translate" each step into an application of the addition/elimination procedure learned earlier in the chapter. So, although row reducing an augmented matrix may appear somewhat as 'mathematical magic', it is nothing more than a prescribed arithmetic manipulation of coefficients and constants to achieve a solution to a system of equations.

We continue with our next example.

Example 101.

System	Augmented Matrix
$x+y+z=3$ $2x+y+4z=8$ $x+2y-z=1$	$\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 3 \\ 2 & 1 & 4 & 8 \\ 1 & 2 & -1 & 1 \end{array}\right]$
Multiply Row 1 by -2 and add to Row 2 Symbolic: R2+(-2)R1 $\Rightarrow$ R2	$\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 2 \\ 1 & 2 & -1 & 1 \end{array}\right]$
Multiply Row 1 by -1 and add to Row 3 Symbolic: R3+(-1)R1 $\Rightarrow$ R3	$\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 2 \\ 0 & 1 & -2 & -2 \end{array}\right]$
Add Row 2 to Row 3 Symbolic: R3+R2 $\Rightarrow$ R3	$\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$
Add Row 2 to Row 1 Symbolic: R1+R2 $\Rightarrow$ R1	$\left[\begin{array}{ccc\|c} 1 & 0 & 3 & 5 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$
Multiply Row 2 by -1 Symbolic: (-1)R2 $\Rightarrow$ R2	$\left[\begin{array}{ccc\|c} 1 & 0 & 3 & 5 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Our last matrix is in reduced row echelon form, since the row containing all zeros occurs at the bottom and the two columns that contain leading ones also contain zeros elsewhere. The resulting system of equations is shown below.

$$\begin{aligned} x + 3z &= \ \ 5 \\ y - 2z &= -2 \\ 0 &= \ \ 0 \end{aligned}$$

The last equation in our system ($0 = 0$) above can be interpreted to mean that the variable z in this example is an *independent variable.* In other words, we are free to choose any real number for z (since $0 = 0$ is a true statement). On the other hand, the variables x and y in this case are *dependent variables*, since they depend on the choice of z. Specifically, solving for both x and y, we get $x = 5 - 3z$ and $y = -2 + 2z$. Since we are free to choose any value for z, we may conclude that there are infinitely many solutions to the given system of equations. Moreover, a solution to the given system must be of the following form.

$$(x, y, z) = (5 - 3z, -2 + 2z, z)$$

Furthermore, we may once again check that our solution makes sense by plugging it back into the original system.

$$\begin{aligned} x + y + z &= (5 - 3z) + (-2 + 2z) + z \\ &= 5 - 3z - 2 + 2z + z \\ &= (5 - 2) + (-3z + 2z + z) \\ &= 3 \end{aligned}$$

$$\begin{aligned} 2x + y + 4z &= 2(5 - 3z) + (-2 + 2z) + 4z \\ &= 10 - 6z - 2 + 2z + 4z \\ &= (10 - 2) + (-6z + 2z + 4z) \\ &= 8 \end{aligned}$$

$$\begin{aligned} x + 2y - z &= (5 - 3z) + 2(-2 + 2z) - z \\ &= 5 - 3z - 4 + 4z - z \\ &= (5 - 4) + (-3z + 4z - z) \\ &= 1 \end{aligned}$$

For out last example, we will work with a system of equations that will have no solution.

Example 102.

System	Augmented Matrix
$\begin{aligned} x + y + 3z &= 2 \\ 3x + 4y + 10z &= 5 \\ x + 2y + 4z &= 3 \end{aligned}$	$\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 2 \\ 3 & 4 & 10 & 5 \\ 1 & 2 & 4 & 3 \end{array}\right]$
Multiply Row 1 by -3 and add to Row 2 Symbolic: R2+(-3)R1 $\Rightarrow$ R2	$\left[\begin{array}{ccc\|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & 4 & 3 \end{array}\right]$

Multiply Row 1 by -1 and
add to Row 3
Symbolic: R3+(-1)R1 $\Rightarrow$ R3

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

Multiply Row 2 by -1 and
add to Row 3
Symbolic: R3+(-1)R2 $\Rightarrow$ R3

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

The resulting matrix is in row echelon form, but not reduced row echelon form. Notice that the last row of our matrix has corresponding equation $0 = 2$, which is false. Since our row reduction has resulted in a false statement, we conclude that the given system of equations has no solution. Therefore, we have no need to continue row reducing in order to obtain the reduced row echelon form.

We have now seen three examples of how matrices can be used to solve a system of equations containing three variables: one example with a single solution, one with infinitely many solutions, and one with no solution. Naturally, we can apply this approach to simpler systems, containing just two variables/equations, as well as to more complicated systems.

Practice Problems

Graphing

Solve each system by graphing.

1. $\begin{cases} y = -x + 1 \\ y = -5x - 3 \end{cases}$

2. $\begin{cases} y = -\frac{3}{4}x + 1 \\ y = -\frac{3}{4}x + 2 \end{cases}$

3. $\begin{cases} y = \frac{5}{3}x + 4 \\ y = -\frac{2}{3}x - 3 \end{cases}$

4. $\begin{cases} x - y = 4 \\ 2x + y = -1 \end{cases}$

5. $\begin{cases} 2x + y = 2 \\ x - y = 4 \end{cases}$

6. $\begin{cases} 9y + 6x = 36 \\ 3y - 6x = -12 \end{cases}$

7. $\begin{cases} 3 + y = -x \\ -4 - 6x = -y \end{cases}$

8. $\begin{cases} y = -\frac{5}{4}x - 2 \\ y = -\frac{1}{4}x + 2 \end{cases}$

9. $\begin{cases} y = 2x + 2 \\ y = -x - 4 \end{cases}$

10. $\begin{cases} y = \frac{1}{2}x + 4 \\ y = \frac{1}{2}x + 1 \end{cases}$

11. $\begin{cases} 6x + y = -3 \\ x + y = 2 \end{cases}$

12. $\begin{cases} x + 2y = 6 \\ 5x - 4y = 16 \end{cases}$

13. $\begin{cases} -2y + x = 4 \\ 2 = -x + \frac{1}{2}y \end{cases}$

14. $\begin{cases} 16 = -x - 4y \\ -2x = -4 - 4y \end{cases}$

15. $\begin{cases} y = -3 \\ y = -x - 4 \end{cases}$

16. $\begin{cases} y = \frac{1}{3}x + 2 \\ y = -\frac{5}{3}x - 4 \end{cases}$

17. $\begin{cases} x + 3y = -9 \\ 5x + 3y = 3 \end{cases}$

18. $\begin{cases} 2x + 3y = -6 \\ 2x + y = 2 \end{cases}$

19. $\begin{cases} 2x + y = -2 \\ x + 3y = 9 \end{cases}$

20. $\begin{cases} 2x - y = -1 \\ 3 = -2x - y \end{cases}$

21. $\begin{cases} -y + 7x = 4 \\ -y + 7x = 3 \end{cases}$

22. $\begin{cases} y = -x - 2 \\ y = \frac{2}{3}x + 3 \end{cases}$

23. $\begin{cases} y = 2x - 4 \\ y = \frac{1}{2}x + 2 \end{cases}$

24. $\begin{cases} x + 4y = -12 \\ 2x + y = 4 \end{cases}$

25. $\begin{cases} 3x + 2y = 2 \\ 3x + 2y = -6 \end{cases}$

26. $\begin{cases} x - y = 3 \\ 5x + 2y = 8 \end{cases}$

27. $\begin{cases} -2y = -4 - x \\ -2y = -5x + 4 \end{cases}$

28. $\begin{cases} -4 + y = x \\ x + 2 = -y \end{cases}$

Substitution

Solve each system by substitution.

1. $\begin{cases} y = -3x \\ y = 6x - 9 \end{cases}$

2. $\begin{cases} y = 6x + 4 \\ y = -3x - 5 \end{cases}$

3. $\begin{cases} y = 2x - 3 \\ y = -2x + 9 \end{cases}$

4. $\begin{cases} y = -6 \\ 3x - 6y = 30 \end{cases}$

5. $\begin{cases} -2x + 2y = 18 \\ y = 7x + 15 \end{cases}$

6. $\begin{cases} 7x - 2y = -7 \\ y = 7 \end{cases}$

7. $\begin{cases} -2x - y = -5 \\ x - 8y = -23 \end{cases}$

8. $\begin{cases} 3x + y = 9 \\ 2x + 8y = -16 \end{cases}$

9. $\begin{cases} x + 5y = 15 \\ -3x + 2y = 6 \end{cases}$

10. $\begin{cases} y = x + 5 \\ y = -2x - 4 \end{cases}$

11. $\begin{cases} y = 3x + 13 \\ y = -2x - 22 \end{cases}$

12. $\begin{cases} y = 7x - 24 \\ y = -3x + 16 \end{cases}$

13. $\begin{cases} 6x - 4y = -8 \\ y = -6x + 2 \end{cases}$

14. $\begin{cases} y = x + 4 \\ 3x - 4y = -19 \end{cases}$

15. $\begin{cases} x - 2y = -13 \\ 4x + 2y = 18 \end{cases}$

16. $\begin{cases} 6x + 4y = 16 \\ -2x + y = -3 \end{cases}$

17. $\begin{cases} -5x - 5y = -20 \\ -2x + y = 7 \end{cases}$

18. $\begin{cases} 2x + 3y = -10 \\ 7x + y = 3 \end{cases}$

19. $\begin{cases} y = -2x - 9 \\ y = 2x - 1 \end{cases}$

20. $\begin{cases} y = 3x + 2 \\ y = -3x + 8 \end{cases}$

21. $\begin{cases} y = 6x - 6 \\ -3x - 3y = -24 \end{cases}$

22. $\begin{cases} y = -5 \\ 3x + 4y = -17 \end{cases}$

23. $\begin{cases} y = -8x + 19 \\ -x + 6y = 16 \end{cases}$

24. $\begin{cases} x - 5y = 7 \\ 2x + 7y = -20 \end{cases}$

25. $\begin{cases} -6x + y = 20 \\ -3x - 3y = -18 \end{cases}$

26. $\begin{cases} 2x + y = 2 \\ 3x + 7y = 14 \end{cases}$

27. $\begin{cases} -2x + 4y = -16 \\ y = -2 \end{cases}$

28. $\begin{cases} y = -6x + 3 \\ y = 6x + 3 \end{cases}$

29. $\begin{cases} y = -2x - 9 \\ y = -5x - 21 \end{cases}$

30. $\begin{cases} -x + 3y = 12 \\ y = 6x + 21 \end{cases}$

31. $\begin{cases} 7x + 2y = -7 \\ y = 5x + 5 \end{cases}$

32. $\begin{cases} y = -2x + 8 \\ -7x - 6y = -8 \end{cases}$

33. $\begin{cases} 3x - 4y = 15 \\ 7x + y = 4 \end{cases}$

34. $\begin{cases} 7x + 5y = -13 \\ x - 4y = -16 \end{cases}$

35. $\begin{cases} 2x + y = -7 \\ 5x + 3y = -21 \end{cases}$

36. $\begin{cases} -2x + 2y = -22 \\ -5x - 7y = -19 \end{cases}$

Addition Elimination

Solve each system by elimination.

1. $\begin{cases} 4x + 2y = 0 \\ -4x - 9y = -28 \end{cases}$

2. $\begin{cases} -6x + 9y = 3 \\ 6x - 9y = -9 \end{cases}$

3. $\begin{cases} -x - 5y = 28 \\ -x + 4y = -17 \end{cases}$

4. $\begin{cases} 10x + 6y = 24 \\ -6x + y = 4 \end{cases}$

5. $\begin{cases} -7x + 4y = -4 \\ 10x - 8y = -8 \end{cases}$

6. $\begin{cases} -7x - 3y = 12 \\ -6x - 5y = 20 \end{cases}$

7. $\begin{cases} 9x + 6y = -21 \\ -10x - 9y = 28 \end{cases}$

8. $\begin{cases} -8x - 8y = -8 \\ 10x + 9y = 1 \end{cases}$

9. $\begin{cases} 0 = 9x + 5y \\ y = \frac{2}{7}x \end{cases}$

10. $\begin{cases} -7x + y = -10 \\ -9x - y = -22 \end{cases}$

11. $\begin{cases} 5x - 5y = -15 \\ 5x - 5y = -15 \end{cases}$

12. $\begin{cases} -10x - 5y = 0 \\ 10x + 10y = 30 \end{cases}$

13. $\begin{cases} x + 3y = -1 \\ 10x + 6y = -10 \end{cases}$

14. $\begin{cases} -6x + 4y = 4 \\ -3x - y = 26 \end{cases}$

15. $\begin{cases} -5x + 4y = 4 \\ -7x - 10y = -10 \end{cases}$

16. $\begin{cases} -4x - 5y = 12 \\ -10x + 6y = 30 \end{cases}$

17. $\begin{cases} -7x + 10y = 13 \\ 4x + 9y = 22 \end{cases}$

18. $\begin{cases} -6 - 42y = -12x \\ x - \frac{7}{2}y = \frac{1}{2} \end{cases}$

19. $\begin{cases} -9x + 5y = -22 \\ 9x - 5y = 13 \end{cases}$

20. $\begin{cases} 4x - 6y = -10 \\ 4x - 6y = -14 \end{cases}$

21. $\begin{cases} 2x - y = 5 \\ 5x + 2y = -28 \end{cases}$

22. $\begin{cases} 2x + 4y = 24 \\ 4x - 12y = 8 \end{cases}$

23. $\begin{cases} 5x + 10y = 20 \\ -6x - 5y = -3 \end{cases}$

24. $\begin{cases} 9x - 2y = -18 \\ 5x - 7y = -10 \end{cases}$

25. $\begin{cases} -7x + 5y = -8 \\ -3x - 3y = 12 \end{cases}$

26. $\begin{cases} 9y = 7 - x \\ -18y + 4x = -26 \end{cases}$

27. $\begin{cases} -x - 2y = -7 \\ x + 2y = 7 \end{cases}$

28. $\begin{cases} -3x + 3y = -12 \\ -3x + 9y = -24 \end{cases}$

29. $\begin{cases} -5x + 6y = -17 \\ x - 2y = 5 \end{cases}$

30. $\begin{cases} -6x + 4y = 12 \\ 12x + 6y = 18 \end{cases}$

31. $\begin{cases} -9x - 5y = -19 \\ 3x - 7y = -11 \end{cases}$

32. $\begin{cases} 3x + 7y = -8 \\ 4x + 6y = -4 \end{cases}$

33. $\begin{cases} 8x + 7y = -24 \\ 6x + 3y = -18 \end{cases}$

34. $\begin{cases} 21 = -9x + 12y \\ \frac{4}{3}y + \frac{7}{3}x = -1 \end{cases}$

Three Variables

Solve each of the following systems of equation.

1. $\begin{cases} 2x + y = z \\ 4x + z = 4y \\ y = x + 1 \end{cases}$

2. $\begin{cases} 3x + 2y = z + 2 \\ y = 1 - 2x \\ 3z = -2y \end{cases}$

3. $\begin{cases} x + y - z = 0 \\ x - y - z = 0 \\ x + y + 2z = 0 \end{cases}$

4. $\begin{cases} x + y - z = 0 \\ x + 2y - 4z = 0 \\ 2x + y + z = 0 \end{cases}$

5. $\begin{cases} m + 6n + 3p = 8 \\ 3m + 4n = -3 \\ 5m + 7n = 1 \end{cases}$

6. $\begin{cases} 2x + 3y = z - 1 \\ 3x = 8z - 1 \\ 5y + 7z = -1 \end{cases}$

7. $\begin{cases} x + 2y - z = 4 \\ 4x - 3y + z = 8 \\ 5x - y = 12 \end{cases}$

8. $\begin{cases} 4x + 12y + 16z = 0 \\ 3x + 4y + 5z = 0 \\ x + 8y + 11z = 0 \end{cases}$

9. $\begin{cases} 2x + y - 3z = 0 \\ x - 4y + z = 0 \\ 4x + 16y + 4z = 0 \end{cases}$

10. $\begin{cases} a - 2b + c = 5 \\ 2a + b - c = -1 \\ 3a + 3b - 2c = -4 \end{cases}$

11. $\begin{cases} x + y + z = 2 \\ 6x - 4y + 5z = 31 \\ 5x + 2y + 2z = 13 \end{cases}$

12. $\begin{cases} x + y + z = 6 \\ 2x - y - z = -3 \\ x - 2y + 3z = 6 \end{cases}$

13. $\begin{cases} p + q + r = 1 \\ p + 2q + 3r = 4 \\ 4p + 5q + 6r = 7 \end{cases}$

14. $\begin{cases} x - y + 2z = 0 \\ x - 2y + 3z = -1 \\ 2x - 2y + z = -3 \end{cases}$

15. $\begin{cases} x + 6y + 3z = 4 \\ 2x + y + 2z = 3 \\ 3x - 2y + z = 0 \end{cases}$

16. $\begin{cases} -2x + y - 3z = 1 \\ x - 4y + z = 6 \\ 4x + 16y + 4z = 24 \end{cases}$

17. $\begin{cases} x - 2y + 3z = 4 \\ 2x - y + z = -1 \\ 4x + y + z = 1 \end{cases}$

18. $\begin{cases} 4x - 7y + 3z = 1 \\ 3x + y - 2z = 4 \\ 4x - 7y + 3z = 6 \end{cases}$

19. $\begin{cases} 3x + y - z = 11 \\ x + 3y = z + 13 \\ x + y - 3z = 11 \end{cases}$

20. $\begin{cases} x - y + 2z = -3 \\ x + 2y + 3z = 4 \\ 2x + y + z = -3 \end{cases}$

21. $\begin{cases} 4x + 12y + 16z = 4 \\ 3x + 4y + 5z = 3 \\ x + 8y + 11z = 1 \end{cases}$

22. $\begin{cases} 3x + 2y + 2z = 3 \\ x + 2y - z = 5 \\ 2x - 4y + z = 0 \end{cases}$

23. $\begin{cases} x + 2y - 3z = 9 \\ 2x - y + 2z = -8 \\ 3x - y - 4z = 3 \end{cases}$

24. $\begin{cases} 4x - 3y + 2z = 40 \\ 5x + 9y - 7z = 47 \\ 9x + 8y - 3z = 97 \end{cases}$

25. $\begin{cases} 3x + y - z = 10 \\ 8x - y - 6z = -3 \\ 5x - 2y - 5z = 1 \end{cases}$

26. $\begin{cases} 3x + 3y - 2z = 13 \\ 6x + 2y - 5z = 13 \\ 5x - 2y - 5z = -1 \end{cases}$

27. $\begin{cases} 2x - 3y + 5z = 1 \\ 3x + 2y - z = 4 \\ 4x + 7y - 7z = 7 \end{cases}$

28. $\begin{cases} 3x - 4y + 2z = 1 \\ 2x + 3y - 3z = -1 \\ x + 10y - 8z = 7 \end{cases}$

29. $\begin{cases} 2w - 2x - 2y + 2z = 10 \\ w + x + y + z = -5 \\ 3w + 2x + 2y + 4z = -11 \\ w + 3x - 2y + 2z = -6 \end{cases}$

30. $\begin{cases} w - 2x + 3y - z = 8 \\ w - x - y + z = 4 \\ w + x + y + z = 22 \\ w - x + y + z = 14 \end{cases}$

31. $\begin{cases} w + x + y + z = 2 \\ w + 2x + 2y + 4z = 1 \\ w - x + y + z = 6 \\ w - 3x - y + z = 2 \end{cases}$

32. $\begin{cases} w + x - y + z = 0 \\ -w + 2x + 2y + z = 5 \\ w - 3x - y + z = 4 \\ 2w - x - y + 3z = 7 \end{cases}$

Matrix Notation

Construct an augmented matrix for each of the systems of equations referenced below. Then row reduce your matrix to its row echelon form and determine if the given system has (1) no solution, (2) infinitely many solutions, or (3) exactly one solution. If one solution exists, determine the reduced row echelon form for your matrix and use it to find the solution to the given system.

1) - 5): Systems (1) through (5) on page 97.

6) - 10): Systems (11) through (15) on page 97.

11) - 20): Systems (1) through (10) on page 98.

21) - 24): Systems (29) through (32) on page 98.

Selected Answers

Graphing

1) $(-1, 2)$
3) $(-3, -1)$
5) $(2, -2)$
7) $(-1, -2)$
9) $(-2, -2)$
11) $(-1, 3)$
13) $(-4, -4)$
15) $(-1, -3)$
17) $(3, -4)$
19) $(-3, 4)$
21)No solution, $\varnothing$
23) $(4, 4)$
25)No solution, $\varnothing$
27) $(2, 3)$

Substitution

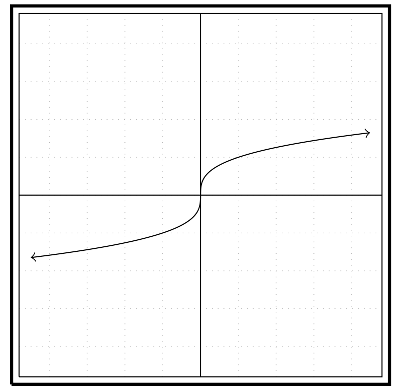
1) $(1, -3)$
3) $(3, 3)$
5) $(-1, 8)$
7) $(1, 3)$
9) $(0, 3)$
11) $(-7, -8)$
13) $(0, 2)$
15) $(1, 7)$
17) $(-1, 5)$
19) $(-2, -5)$
21) $(2, 6)$
23) $(2, 3)$
25) $(-2, 8)$
27) $(4, -2)$
29) $(-4, -1)$
31) $(-1, 0)$
33) $(1, -3)$
35) $(0, -7)$

Addition Elimination

1) $(-2, 4)$
3) $(-3, -5)$
5) $(4, 6)$
7) $(-1, -2)$
9) $(0, 0)$
11) Infinite
13) $(-1, 0)$
15) $(0, 1)$
17) $(1, 2)$
19) No solution, $\varnothing$
21) $(-2, -9)$
23) $(-2, 3)$
25) $(-1, -3)$
27) Infinite
29) $(1, -2)$
31) $(1, -2)$
33) $(-3, 0)$

Three Variables

3) $(0, 0, 0)$
9) $(0, 0, 0)$
10) $(1, -1, 2)$
15) $(-2, -1, 4)$
17) $(-1, 2, 3)$
24) $(10, 2, 3)$
28) No solution, $\varnothing$
29) $(1, -3, -2, -1)$



Chapter 3

Introduction to Functions

Notation and Basic Examples

Definitions and the Vertical Line Test (L12)

Objective: Identify functions and use correct notation to evaluate and solve functions for specific values.

A *relation* R is a set of points in the xy-plane. A relation in which each x-coordinate is paired with exactly one y-coordinate is said to describe y as a *function* of x. Relations which represent functions of x will often be denoted by f, or $f(x)$, rather than R. The set of all x-coordinates of the points in a function f is called the *domain* of f, and the set of all y-coordinates of the points in f is called the *range* of f.

Example 103. The following examples represent relations. Examples (5) and (6) also represent y as a *function* of x, $y = f(x)$, since each $x-$coordinate is paired with exactly one $y-$coordinate.

1. $\{(1,1),(2,-3),(2,0),(0,3),(-2,1/2)\}$
2. $\{(x,y) \mid x > 3 \text{ and } y \leq 2\}$
3. $x^2 + y^2 = 9$
4. $x = y^2$
5. $y = x^2$
6. $y = 3 - 2x$

Alternatively, one can define a function as a rule that assigns to each element of one set (the domain) exactly one element of a second set (the range). This definition is essentially the same as that given above, but avoids the term "relation" entirely. In each definition, however, the critical phrase that cannot be overlooked is "exactly one". This means that the first four relations given above cannot represent y as a function of x, since, for example, the third relation contains the points $(0, 3)$ and $(0, -3)$. On the other hand, each of the last two relations above can be considered to represent y as a function of x. Furthermore, their graphs should also look familiar, since they represent a quadratic equation ($y = x^2$) and a linear equation ($y = -2x + 3$).

In each of the last two examples above, we refer to the variable x as the *independent variable*, since we are free to choose any real number for x. We consequently refer to y as the *dependent variable*, since its value depends on the choice of value for x. One can also more simply refer to x as the *input* of the function and y as the *output*. This terminology naturally lends itself to what is the standard function notation of $f(x)$, read as "f of x". In the following example, we will use the given function to complete a table of values for x and $f(x)$. Each pair $(x, f(x))$ corresponds to a point (x, y) on the graph of f.

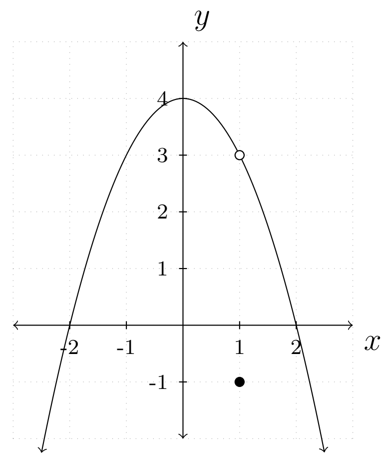
Example 104. $f(x) = x^2 - 4x - 5$

x	$f(x)$
-2	$(-2)^2 - 4(-2) - 5 = 7$
-1	$(-1)^2 - 4(-1) - 5 = 0$
0	$(0)^2 - 4(0) - 5 = -5$
1	$(1)^2 - 4(1) - 5 = -8$
2	$(2)^2 - 4(2) - 5 = -9$

To complete each row of the table, we simply substitute the specified value for x into the given equation and simplify. So, if we wanted to complete another row in the table, we could substitute $x = 3$ into the equation to obtain $f(3) = (3)^2 - 4(3) - 5 = -8$.

In the previous example, the y-coordinates for the relation $y = x^2 - 4x - 5$ are represented by $f(x)$, or more simply $y = f(x)$. It is important to note that the parentheses in function notation do not represent multiplication. This is a common misconception among students. Instead, one should consider the parentheses as an identifier, enclosing the value of x that the rule f is applied to. This will be especially important as we discuss composite functions later in the chapter.

In the following examples we will answer a variety of questions related to functions and their graphs. First, we will consider the case where we are presented with the graph of a particular function and asked to identify specific values of x or $f(x)$ from it.



The graph of f

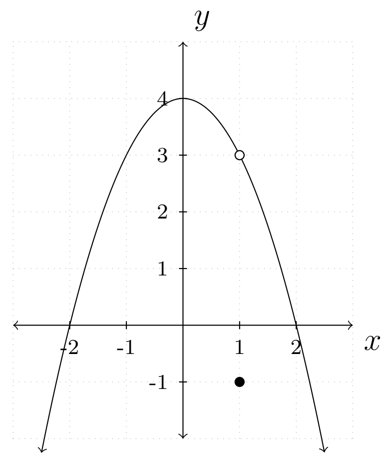
In our first scenario, we will be provided with an input x and asked to find the output $f(x)$. To find an output when given a specific input, locate the input value on the x-axis and follow the vertical line (above and below) the input value until it intersects, or "hits", the graph. The corresponding $y-$coordinate for the point of intersection will be the desired output, $y = f(x)$.

Example 105. Use the graph of f provided to find the desired outputs.

$$\begin{aligned} f(2) &= \ ? && \text{What is } y \text{ when } x = 2? \\ f(2) &= 0 && \text{Our answer} \\ \\ f(0) &= \ ? && \text{What is } y \text{ when } x = 0? \\ f(0) &= 4 && \text{Our answer} \\ \\ f(1) &= \ ? && \text{What is } y \text{ when } x = 1? \\ f(1) &= -1 && \text{Our answer} \end{aligned}$$

It is important to point out that many students will misinterpret the last example and incorrectly conclude that $f(1) = 3$, since the open circle at $(1, 3)$ appears to coincide with the rest of the graph of f. An *open* circle, however, is used to identify a *break* in the graph of f, also known as a point of *discontinuity*. In fact, the given function is not defined at $(1, 3)$, but rather at the *solid* (or *closed*) point $(1, -1)$. Hence, we get a corresponding value of $y = -1$ for $f(x)$.

Next, we will be provided with an output y and asked to find all corresponding inputs x such that $f(x) = y$. To find all possible inputs, we will make a simple adjustment to the method used in the previous example. Now, we will locate the output value on the y-axis and follow the horizontal line (left and right) of the output value until it intersects, or "hits", the graph. All corresponding $x-$coordinates for the points of intersection will represent the set of all values of x such that $f(x)$ equals our given output y and should be included as part of our final answer.



The graph of f

Example 106. Use the graph above to find all possible inputs that correspond to the specified output.

$$\begin{aligned} \text{Find } x \text{ where } f(x) = 0. &\quad \text{Which inputs for } x \text{ have an output of } y = 0? \\ x = -2, 2 &\quad \text{Our answers} \\ \text{Find } x \text{ where } f(x) = 3. &\quad \text{Which inputs for } x \text{ have an output of } y = 3? \\ x = -1 &\quad \text{Our answer; We should not include } x = 1. \end{aligned}$$

Similarly, if we were also asked to find all possible x such that $f(x) = -1$, then we would end up with three values, since there are three points that intersect the horizontal line $y = -1$, namely $x \approx -2.2$, $x = 1$, and $x \approx 2.2$.

There are four major representations of functions: verbal (in words), numerical (using a table), symbolic (with an algebraic expression), and visual (with a graph). In many cases, we will be asked to identify one representation of y as a function of x when given a different representation. The next two examples demonstrate this.

Example 107. Provide the symbolic form for each of the following verbal descriptions of a function.

1. Add 2 to a value and then take the square root of the resulting value.

$$\text{Our answer} \qquad f(x) = \sqrt{2+x}$$

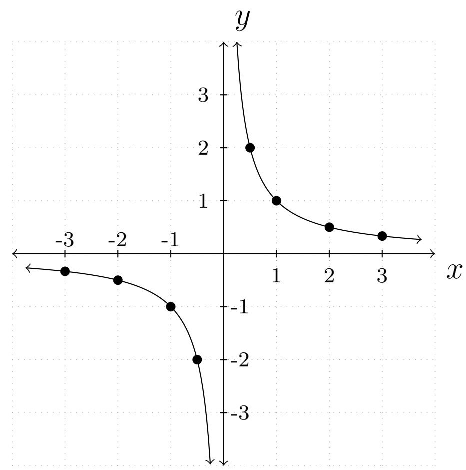
2. Take the square root of a value and then add 2 to the resulting value.

$$\text{Our answer} \qquad g(x) = \sqrt{x} + 2$$

Note: It is often beneficial to rewrite $g(x)$ in the previous example as $g(x) = 2 + \sqrt{x}$, so as not to accidentally extend the radical to include the $+2$.

Example 108. Provide a graphical representation for the function given by the table of values.

x	$f(x)$
-3	-1/3
-2	-1/2
-1	-1
-1/2	-2
1/2	2
1	1
2	1/2
3	1/3

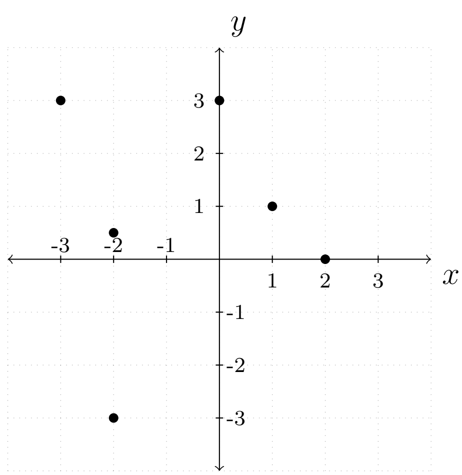


Since there are often advantages to working with either symbolic or graphical representations of functions, we will focus our attention on working with these two representations. One major test that is used to determine whether or not a graph of a relation represents y as a function of x is known as the Vertical Line Test. We will now state the Vertical Line Test as a mathematical theorem and then demonstrate its use.

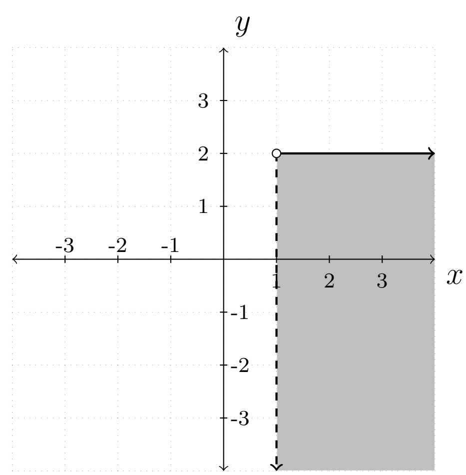
Vertical Line Test: A set of points in the xy-plane represents y as a function of x if and only if no two points lie on the same vertical line.

Alternatively stated, if a graph is known to represent y as a function of x, then there can be no vertical line that intersects the graph in more than one point. Conversely, if a known graph has the property that no vertical line intersects it in more than one point, then the given graph represents y as a function of x.

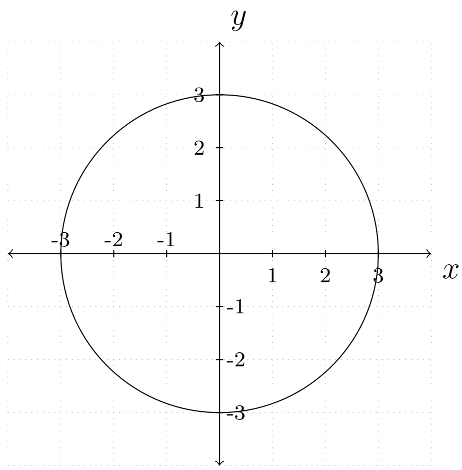
Example 109. Use the Vertical Line Test to determine whether or not each of the following graphs represent y as a function of x.



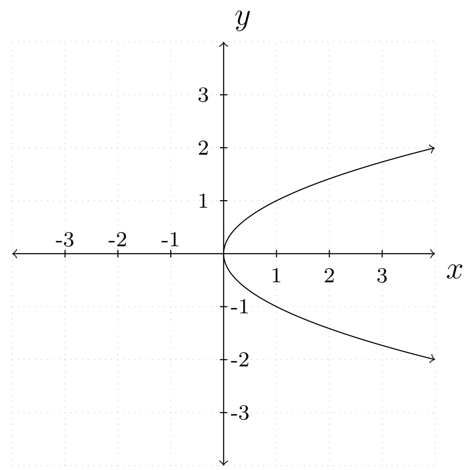
$\{(1,1),(-3,3),(-2,-3),(2,0),(0,3),(-2,\frac{1}{2})\}$



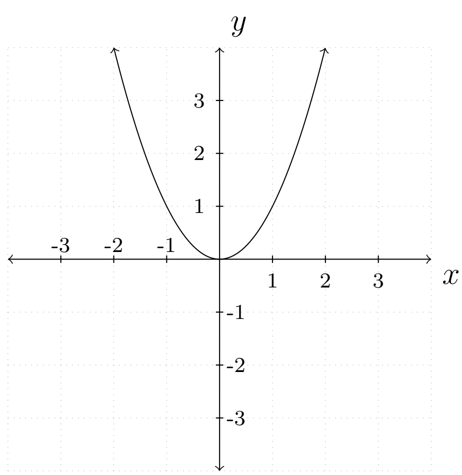
$\{(x,y) \mid x>1, y\leq 2\}$



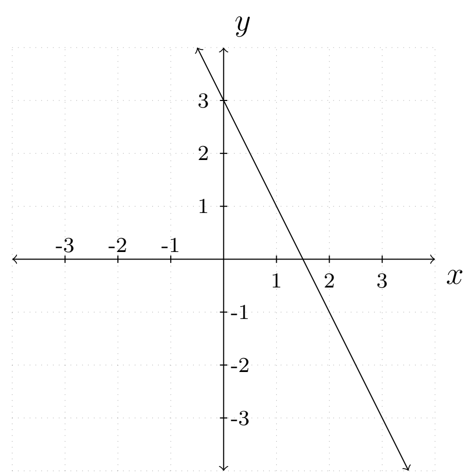
$x^2 + y^2 = 9$



$x = y^2$



$y = x^2$



$y = 3 - 2x$

In each example, we utilize the Vertical Line Test by "slicing" through each graph with several vertical lines, located at various values along the x-axis. Consequently, we can see that each of the first four examples at the beginning of the section *do not* represent y as a function of x, since in each case there exists at least one vertical line that intersects the graph in two (or possibly more) points. The last two examples *do* represent y as a function of x, since no such vertical line exists. As a result, we say that the first four examples *fail* the Vertical Line Test, and the last two examples *pass* the Vertical Line Test.

When we are presented with an equation, instead of a graph, we can still determine whether or not the equation represents y as a function of x by solving the equation for y and carefully considering the result. For example, if we consider the equation $x = y^2$, which corresponds to the fourth graph in our previous example (a sideways parabola), we know from the graph that $x = y^2$ cannot represent y as a function of x, since it fails the VLT.

If, instead of looking at the graph of $x = y^2$, we were to solve the equation for y, we would get $y = \pm\sqrt{x}$. The existence of the $\pm$ in this equation is the cause of our failure to have y

as a function of x, since, for example, $y = \pm 2$ when $x = 4$. This means that the points $(4, 2)$ and $(4, -2)$ will both be on our graph, which we cannot have for y to be a function of x. We conclude this section with two examples that demonstrate this algebraic analysis of an equation, in order to determine whether y represents a function of x.

Example 110. Determine whether the following equation represents y as a function of x.

$$x^2 + y^2 = 9$$

Solve the equation for y.

$$\begin{array}{rll}
x^2 + y^2 &= 9 & \text{Solve for } y \\
\underline{-\cancel{x^2}} \qquad & \underline{-x^2} & \text{Subtract } x^2 \\
y^2 &= 9 - x^2 & \\
\sqrt{y^2} &= \pm\sqrt{9 - x^2} & \text{Introduce a square root} \\
& & \quad \text{include a } \pm \text{ on right side} \\
y &= \pm\sqrt{9 - x^2} & y \text{ is not a function of } x
\end{array}$$

Due to the $\pm$, we can conclude that the equation does *not* represent y as a function of x.

Example 111. Determine whether the following equation represents y as a function of x.

$$2x + \frac{1}{y-3} = 0$$

Solve the equation for y.

$$\begin{array}{rll}
\cancel{2x} + \dfrac{1}{y-3} &= 0 & \text{Solve for } y \\
\underline{-\cancel{2x}} \qquad & \underline{-2x} & \text{Subtract } 2x \\
\dfrac{1}{y-3} &= -2x & \\
\cancel{(y-3)} \cdot \dfrac{1}{\cancel{y-3}} &= (-2x) \cdot (y-3) & \text{Multiply by } y - 3 \\
1 &= (-2x)(y-3) & \\
\dfrac{1}{-2x} &= \dfrac{(\cancel{-2x})(y-3)}{\cancel{-2x}} & \text{Divide by } -2x \\
-\dfrac{1}{2x} &= y - 3 & \\
\underline{+3} \qquad & \underline{+\cancel{3}} & \text{Add } 3 \\
y &= 3 - \dfrac{1}{2x} & y \text{ as a function of } x
\end{array}$$

We can conclude that y represents a function of x.

Evaluating (L13) and Solving (L31) Functions

Objective: Evaluate functions using appropriate notation.

Another function-related skill we will want to quickly master is evaluating functions at certain values of the independent variable (usually x). This is accomplished by substituting the specified value into the function for x and simplifying the resulting expression to find $f(x)$. This idea of "plugging in" values of x to find $f(x)$ is demonstrated in the following examples.

Example 112. Find $f(-2)$, where $f(x) = 3x^2 - 4x$.

$$\begin{aligned} f(x) &= 3x^2 - 4x && \text{Evaluate; Substitute } -2 \text{ for each } x \\ f(-2) &= 3(-2)^2 - 4(-2) && \text{Simplify using order of operations; exponent first} \\ f(-2) &= 3(4) - 4(-2) && \text{Multiply} \\ f(-2) &= 12 + 8 && \text{Add} \\ f(-2) &= 20 && \text{Our solution} \end{aligned}$$

Example 113. Find $h(4)$, where $h(x) = 3^{2x-6}$.

$$\begin{aligned} h(x) &= 3^{2x-6} && \text{Evaluate; Substitute } 4 \text{ for } x \\ h(4) &= 3^{2(4)-6} && \text{Simplify exponent, multiplying first} \\ h(4) &= 3^{8-6} && \text{Subtract in exponent} \\ h(4) &= 3^2 && \text{Evaluate exponent} \\ h(4) &= 9 && \text{Our solution} \end{aligned}$$

Example 114. Find $k(-7)$, where $k(a) = 2|a + 4|$.

$$\begin{aligned} k(a) &= 2|a+4| && \text{Evaluate; Substitute } -7 \text{ for } a \\ k(-7) &= 2|-7+4| && \text{Simplify, add inside absolute value} \\ k(-7) &= 2|-3| && \text{Evaluate absolute value} \\ k(-7) &= 2(3) && \text{Multiply} \\ k(-7) &= 6 && \text{Our solution} \end{aligned}$$

As the previous examples show, a function can take many different forms, but the method to evaluate the function is always the same: replace each instance of the variable with the specified value and simplify.

We can also substitute entire expressions into functions using this same process. This idea is known as a *composition* of two functions or expressions, and will be formally outlined in a later section. We present the following two examples as a preview of this concept.

Example 115. Find $g(3x)$, where $g(x) = x^4 + 1$.

$$\begin{aligned} g(x) &= x^4 + 1 && \text{Replace } x \text{ in the function with } (3x) \\ g(3x) &= (3x)^4 + 1 && \text{Simplify exponent} \\ g(3x) &= 81x^4 + 1 && \text{Our solution} \end{aligned}$$

Example 116. Find $p(t+1)$, where $p(t) = t^2 - t$.

$$
\begin{array}{rl}
p(t) = t^2 - t & \text{Replace each } t \text{ in } p(t) \text{ with } (t+1) \\
p(t+1) = (t+1)^2 - (t+1) & \text{Simplify; square binomial} \\
p(t+1) = t^2 + 2t + 1 - (t+1) & \text{Distribute negative sign} \\
p(t+1) = t^2 + 2t + 1 - t - 1 & \text{Combine like terms} \\
p(t+1) = t^2 + t & \text{Our solution} \\
p(t+1) = t(t+1) & \text{Our solution in factored form}
\end{array}
$$

As is the case with each of the previous examples, it is important to keep in mind that each expression (or function) will often use the same variable. Hence, it is critical that we recognize that each variable must be replaced by whatever expression appears in parentheses.

So far, all of the previous examples have shown how to find an output when given a specific input. Next, we will demonstrate how one can also algebraically find which input(s) x yield a required output $f(x)$. This is often referred to as *solving* a function (for a specific output).

Example 117. Given $f(x) = x^2 + 3x + 5$, find all x such that $f(x) = 5$.

$$
\begin{array}{rl}
f(x) = x^2 + 3x + 5 & \text{Substitute 5 in for } f(x) \\
5 = x^2 + 3x + 5 & \text{Solve for } x \text{ by factoring} \\
0 = x^2 + 3x & \text{Set equal to 0} \\
0 = x(x+3) & \text{Factor} \\
x = 0 \text{ or } x = -3 & \text{Our solutions}
\end{array}
$$

The above answer can be verified by checking. When we input $x = 0$ into the function, we simplify to find that $f(0) = 5$. Similarly, we see that when $x = -3$, $f(-3) = 5$.

Example 118. Given $h(x) = 4x - 1$, find all x such that $h(x) = -3$.

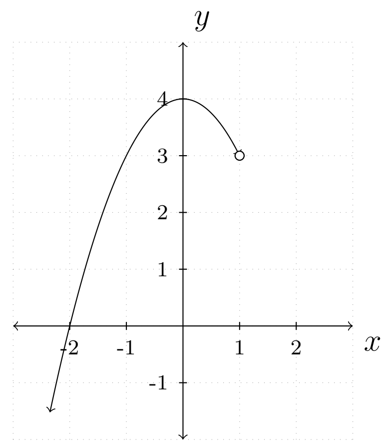
$$
\begin{array}{rl}
h(x) = 4x - 1 & \text{Substitute } -3 \text{ for } h(x) \\
-3 = 4x - 1 & \text{Solve for } x \\
-2 = 4x & \text{Divide} \\
x = -\dfrac{1}{2} & \text{Our solution}
\end{array}
$$

It is important that we become comfortable with function notation and how to use it, as we begin to transition to more advanced algebraic concepts.

Identifying Domain and Range Graphically (L14)

Objective: Identify the domain and range of a function that is described graphically. In this section, we will first discuss how one can identify the domain and range of a function using its graph. Later, we will explore finding the domain of a function using algebraic methods. As finding the range of a function using algebraic methods can often prove quite challenging, we will postpone this topic for another time.

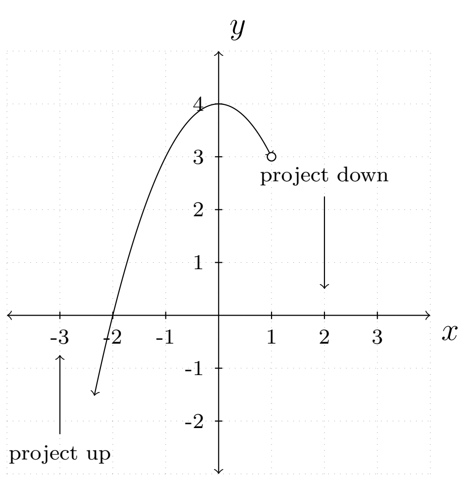
Example 119. Find the domain and range of the function f whose graph is given below.

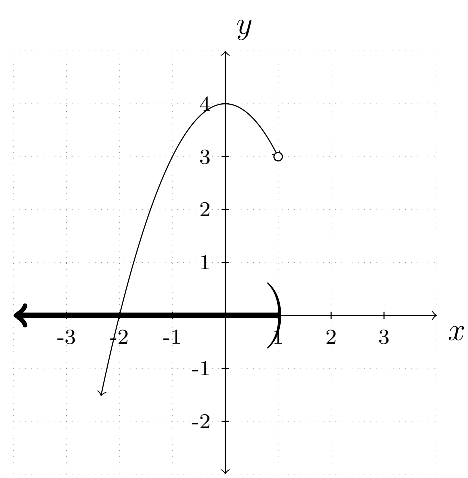


The graph of f

To determine the domain and range of f, we need to determine which x and y-values respectively occur as coordinates of points on the given graph.

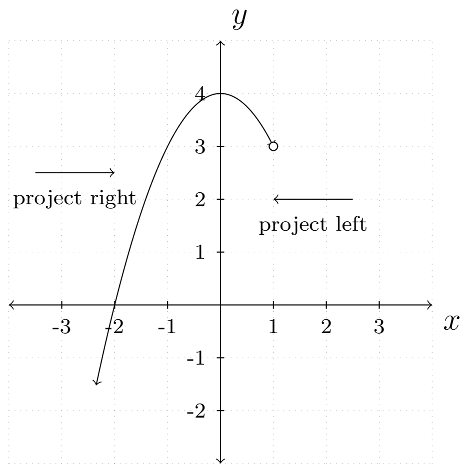
To find the domain, it will be helpful to imagine collapsing the curve onto the x-axis and determining the portion of the x-axis that gets covered. This is often described as *projecting* the curve onto the x-axis. Before we project, we need to pay attention to two subtle notations on the graph: the arrowhead on the lower left corner of the graph indicates that the graph continues to curve downwards to the left forever; and the open circle at $(1, 3)$ indicates that the point $(1, 3)$ is *not* on the graph, but all the points on the curve leading up to $(1, 3)$ are on the graph.

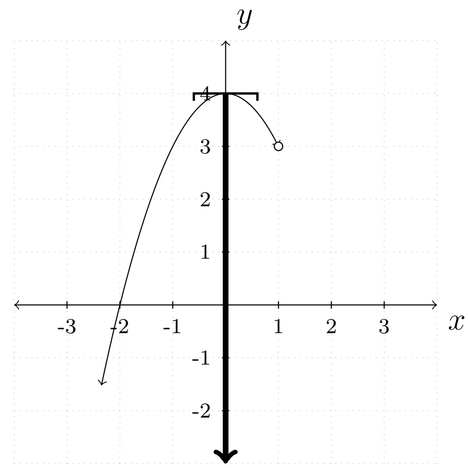




We see from the figure that if we project the graph of f to the x-axis, we get all real numbers less than 1. Using interval notation, we write the domain of f as $(-\infty, 1)$.

To determine the range of f, we use a similar method, projecting the curve onto the y-axis as follows.





Note that even though there is an open circle at $(1,3)$, we still include the y value of 3 in our range, since the point $(-1,3)$ is on the graph of f. We also include $y=4$ in our answer, since the point $(0,4)$ is also on our graph. Consequently, the range of f is all real numbers less than or equal to 4, or $(-\infty,4]$.

Fundamental Functions (L15)

Objective: Graph and identify the domain, range, and intercepts of any of the ten fundamental functions.

In this section, we have listed ten fundamental function types which will be referenced throughout the rest of the text, as well as one example of each. Each type of function represents a "building block" for understanding the concepts of a traditional algebra course.

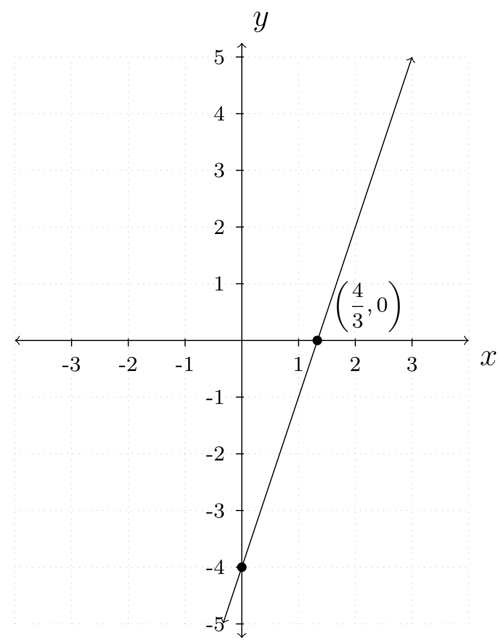
Students should be able to both identify and sketch a graph of each function, as well as identify its intercepts, domain (both graphically and algebraically), and range (graphically). Each representative form in the table below includes some element of generalization to reinforce understanding.

Function Type	Representative Form	Example
Linear	$mx + b$	$f(x) = 3x - 4$
Quadratic	$ax^2 + bx + c$	$g(x) = x^2$
Square Root	$\sqrt{x - h}$	$k(x) = \sqrt{x}$
Absolute Value	$\|x - h\|$	$\ell(x) = \|x\|$
Cubic	$(x - h)^3$	$m(x) = x^3$
Cube Root	$\sqrt[3]{x - h}$	$n(x) = \sqrt[3]{x}$
Reciprocal (Rational)	$\dfrac{1}{x - h}$	$p(x) = \dfrac{1}{x}$
Semicircular	$\sqrt{r^2 - x^2},\ r > 0$	$q(x) = \sqrt{9 - x^2}$
Exponential*	$a^x,\ a > 0, a \neq 1$	$r(x) = 2^x$
Logarithmic*	$\log_a(x),\ a > 0, a \neq 1$	$s(x) = \log_2(x)$

*We have included Exponential and Logarithmic functions for a more complete list. These functions are more formally treated in a Precalculus setting.

Function Type: **Linear** $(m \neq 0)$
Example: $f(x) = 3x - 4$

x	$f(x)$
-3	-13
-2	-10
-1	-7
0	-4
1	-1
$\frac{4}{3}$	0
2	2
3	5



Graph of $f(x) = 3x - 4$

$y-$intercept: $(0, -4)$

$x-$intercept(s): $\left(\frac{4}{3}, 0\right)$

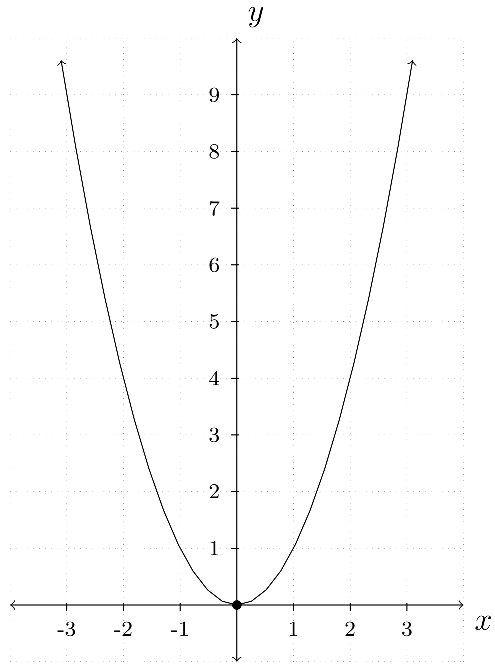
Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

Notes: If $m = 0$, then the corresponding graph of $f(x) = b$ is a horizontal line. The domain of f is still $(-\infty, \infty)$, but the range consists of a single value, $\{b\}$.

Function Type: **Quadratic**
Example: $g(x) = x^2$

x	$g(x)$
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9



Graph of $g(x) = x^2$

$y-$intercept: $(0, 0)$

$x-$intercept(s): $(0, 0)$

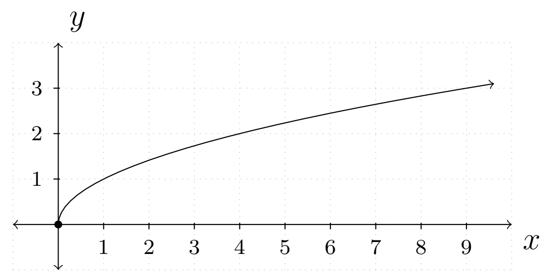
Domain: $(-\infty, \infty)$

Range: $[0, \infty)$, or $y \geq 0$

Notes: The domain of any quadratic function is $(-\infty, \infty)$. If $g(x) = a(x-h)^2 + k$, is a quadratic function in vertex form, then if $a > 0$, the corresponding parabola will be concave *up*, and the range of g will be $[k, \infty)$. If $a < 0$, then the corresponding parabola will be concave *down*, and the range of g will be $(-\infty, k]$. Quadratics will be covered extensively in the next chapter.

Function Type: **Square Root**
Example: $k(x) = \sqrt{x}$

x	$k(x)$
-1	undefined
0	0
1	1
2	$\sqrt{2} \approx 1.41$
3	$\sqrt{3} \approx 1.73$
4	2
9	3



Graph of $k(x) = \sqrt{x}$

$y-$intercept: $(0, 0)$

$x-$intercept(s): $(0, 0)$

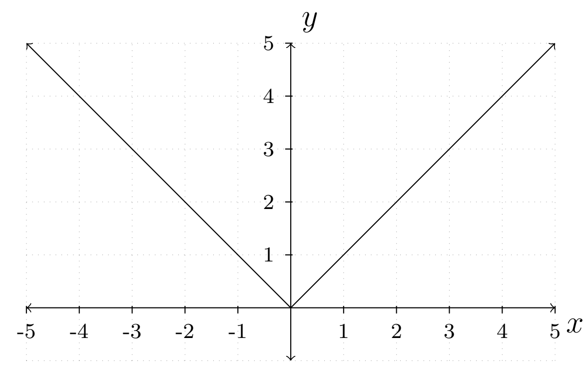
Domain: $[0, \infty)$, or $x \geq 0$

Range: $[0, \infty)$, or $y \geq 0$

Notes: The domain of a square root function of the form $k(x) = \sqrt{x-h}$ will be $x > h$. The range will be the same as in the example, $[0, \infty)$. The $x-$intercept will be $(h, 0)$.

Function Type: **Absolute Value**
Example: $\ell(x) = |x|$

x	$\ell(x)$
-3	3
-2	2
-1	1
0	0
1	1
2	2
3	3



Graph of $\ell(x) = |x|$

$y-$intercept: $(0, 0)$

$x-$intercept(s): $(0, 0)$

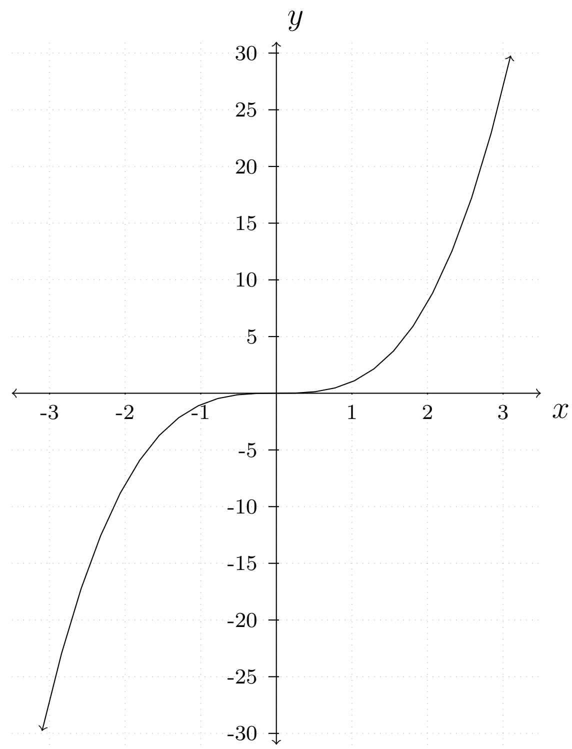
Domain: $(-\infty, \infty)$

Range: $[0, \infty)$, or $y \geq 0$

Notes: The domain and range of an absolute value function of the form $\ell(x) = |x - h|$ will remain the same as above. The $x-$intercept will be $(h, 0)$.

Function Type: **Cubic**
Example: $m(x) = x^3$

x	$m(x)$
-3	-27
-2	-8
-1	-1
0	0
1	1
2	8
3	27



Graph of $m(x) = x^3$

$y-$intercept: $(0, 0)$

$x-$intercept(s): $(0, 0)$

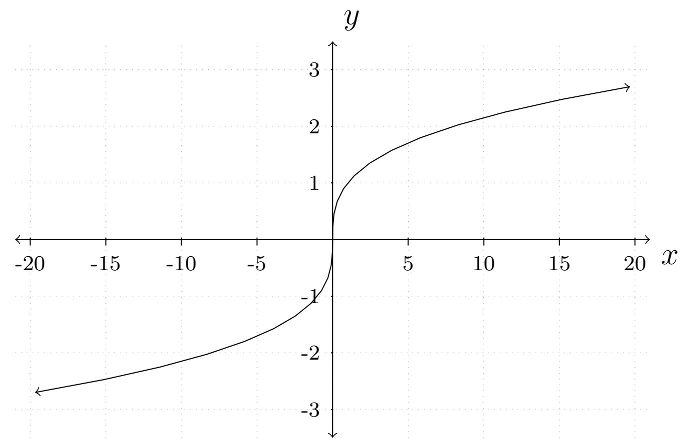
Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

Notes: The domain and range of a cubic function of the form $m(x) = (x - h)^3$ will remain the same as above. The $x-$intercept will be $(h, 0)$. The $y-$intercept will be $(0, -h^3)$.

Function Type: **Cube Root**
Example: $n(x) = \sqrt[3]{x}$

x	$n(x)$
-27	-3
-8	-2
-1	-1
0	0
1	1
8	2
27	3



Graph of $n(x) = \sqrt[3]{x}$

$y-$intercept: $(0, 0)$

$x-$intercept(s): $(0, 0)$

Domain: $(-\infty, \infty)$

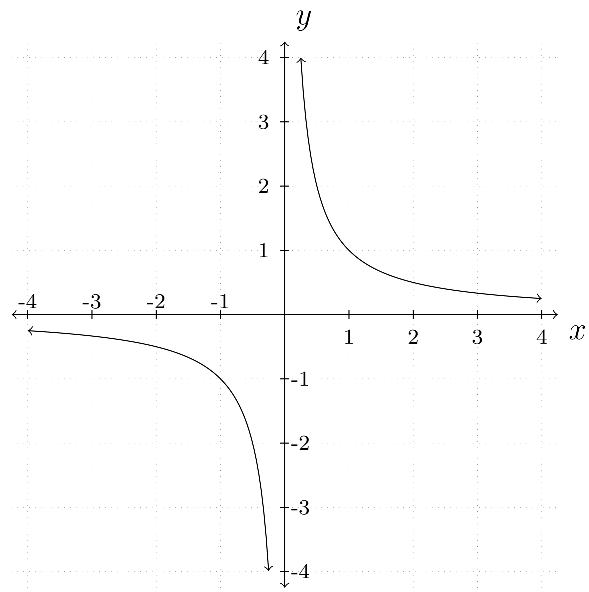
Range: $(-\infty, \infty)$

Notes: The domain and range of a cube root function of the form $n(x) = \sqrt[3]{x-h}$ will remain the same as above. The $x-$intercept will be $(h, 0)$. The $y-$intercept will be $(0, -\sqrt[3]{h})$.

Function Type: **Reciprocal (Rational)**

Example: $p(x) = \dfrac{1}{x}$

x	$p(x)$
-3	$-\dfrac{1}{3}$
-2	$-\dfrac{1}{2}$
-1	-1
0	undefined
1	1
2	$\dfrac{1}{2}$
3	$\dfrac{1}{3}$



Graph of $p(x) = \dfrac{1}{x}$

$y-$intercept: None

$x-$intercept(s): None

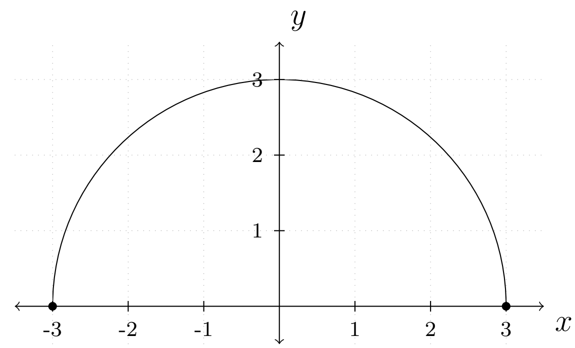
Domain: $(-\infty, 0) \cup (0, \infty)$, or $x \neq 0$

Range: $(-\infty, 0) \cup (0, \infty)$, or $y \neq 0$

Notes: The reciprocal function $\dfrac{1}{x}$ gets its name since each $y-$coordinate is the reciprocal of its corresponding $x-$coordinate, and vice versa. Although the more general representative function $\dfrac{1}{x-h}$ does not uphold this reciprocal property, we can still categorize both the reciprocal form and the more general form as specific types of *rational* functions. The domain of a function of the form $p(x) = \dfrac{1}{x-h}$ will be $(-\infty, h) \cup (h, \infty)$, or $x \neq h$. The range, however, will remain the same as the reciprocal function, $\dfrac{1}{x}$. The graph of $p(x) = \dfrac{1}{x-h}$ will have no $x-$intercept. The $y-$intercept will be at $\left(0, -\dfrac{1}{h}\right)$.

Function Type: **Semicircular**
Example: $q(x) = \sqrt{9 - x^2}$

x	$q(x)$
-3	0
-2	$\sqrt{5}$
-1	$\sqrt{8}$
0	0
1	$\sqrt{8}$
2	$\sqrt{5}$
9	0



Graph of $q(x) = \sqrt{9 - x^2}$

$y-$intercept: $(0, 3)$

$x-$intercept(s): $(-3, 0)$ and $(3, 0)$

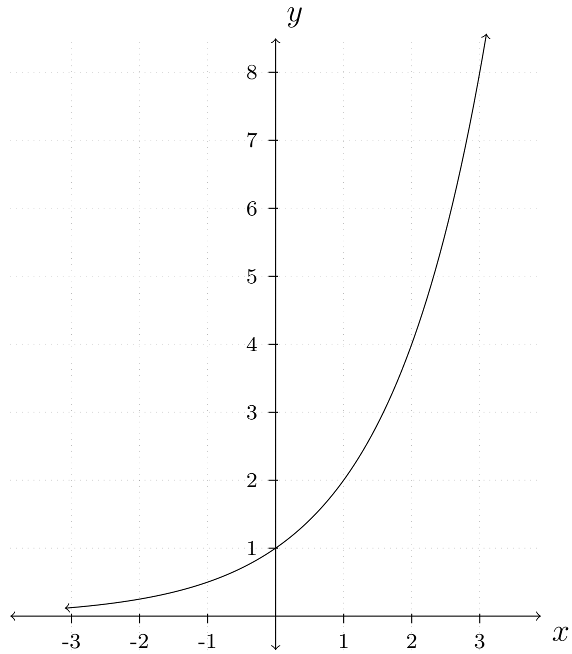
Domain: $[-3, 3]$, or $-3 \leq x \leq 3$

Range: $[0, 3]$, or $0 \leq y \leq 3$

Notes: The domain of a semicircular function of the form $q(x) = \sqrt{r^2 - x^2}$ will be $[-r, r]$, or $-r \leq x \leq r$. The range will be $[0, r]$, or $0 \leq y \leq r$. The graph of $q(x) = \sqrt{r^2 - x^2}$ will have $x-$intercepts at $(\pm r, 0)$ and a $y-$intercept at $(0, r)$.

Function Type: **Exponential**
Example: $r(x) = 2^x$

x	$r(x)$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8



Graph of $r(x) = 2^x$

$y-$intercept: $(0, 1)$

$x-$intercept(s): None

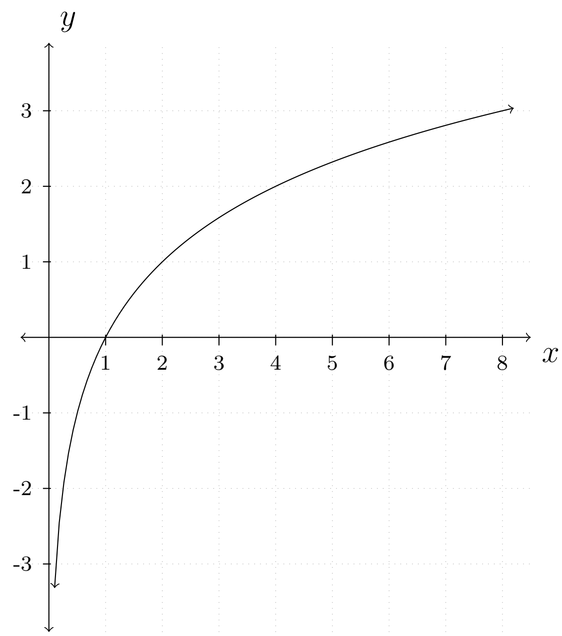
Domain: $(-\infty, \infty)$

Range: $(0, \infty)$, or $y > 0$

Notes: The domain, range, $x-$ and $y-$ intercepts of an exponential function of the form $r(x) = a^x$, where a is positive ($a \neq 1$) will all be the same as above.

Function Type: **Logarithmic**
Example: $s(x) = \log_2 x$

x	$s(x)$
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3



Graph of $s(x) = \log_2 x$

$y-$intercept: None

$x-$intercept(s): $(1, 0)$

Domain: $(0, \infty)$, or $x > 0$

Range: $(-\infty, \infty)$

Notes: The domain, range, $x-$ and $y-$ intercepts of a logarithmic function of the form $s(x) = \log_a x$, where a is positive $(a \neq 1)$ will all be the same as above.

Practice Problems

Notation and Basic Examples

Determine whether or not each relation represents y as a function of x.

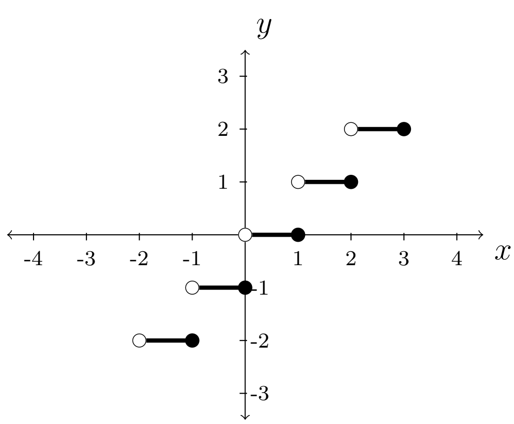
1. $\{(-3,9),\ (-2,4),\ (-1,1),\ (0,0),\ (1,1),\ (2,4),\ (3,9)\}$
2. $\{(-3,0),(1,6),(2,-3),(4,2),(-5,6),(4,-9),(6,2)\}$
3. $\{(-3,0),(-7,6),(5,5),(6,4),(4,9),(3,0)\}$
4. $\{(1,2),(4,4),(9,6),(16,8),(25,10),(36,12),\ldots\}$
5. $\{(x,y)\,|\,x \text{ is an odd integer, and } y \text{ is an even integer}\}$
6. $\{(x,1)\,|\,x \text{ is an irrational number}\}$
7. $\{(1,0),\ (2,1),\ (4,2),\ (8,3),\ (16,4),\ (32,5),\ \ldots\}$
8. $\{\ldots,\ (-3,9),\ (-2,4),\ (-1,1),\ (0,0),\ (1,1),\ (2,4),\ (3,9),\ \ldots\}$
9. $\{(-2,y)\,|\,-3<y<4\}$
10. $\{(x,3)\,|\,-2\le x<4\}$

Determine if the following relations represent y as a function of x by making a table of values and graphing. Explain your reasoning. Use Desmos to confirm your results.

11. $x=y^3$
12. $y=x$
13. $xy=1$
14. $y=(x-3)^2$
15. $x=(y-3)^2$
16. $y<2x-5$

Determine whether each of the following relations represents y as a function of x.

17.



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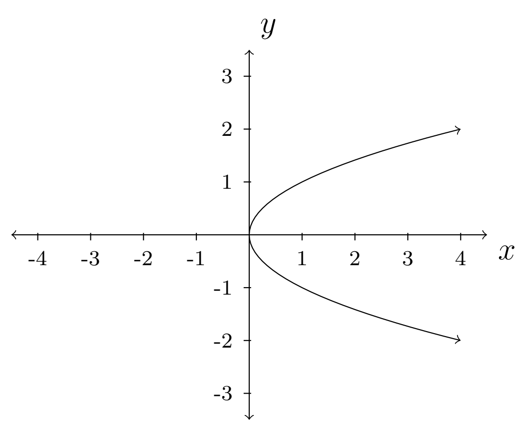

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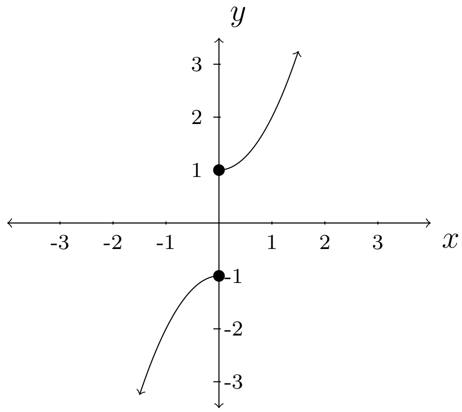

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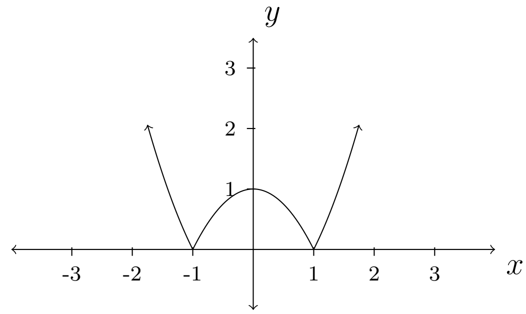


21.

x	y
3	-3
2	-2
1	-1
0	0
1	1
2	2
3	3

22.

x	y
-3	3
-2	2
-1	1
0	0
1	1
2	2
3	3

23.

x	y
-3	0
-2	0
-1	0
0	0
1	0
2	0
3	0

24.

x	y
3	8
2	4
1	2
0	1
-1	1/2
-2	1/4
-3	1/8

Determine whether each of the following equations represents y as a function of x.

25. $y = x^3 - x$
26. $y = \sqrt{x-2}$
27. $3x + 2y = 6$
28. $x^2 - y^2 = 1$
29. $y = \dfrac{x}{x^2 - 9}$
30. $x = -6$
31. $x = y^2 + 4$
32. $y = x^2 + 4$
33. $x^2 + y^2 = 4$

For each of the following statements, find an expression for $f(x)$.

34. f is a function that takes a real number x and performs the following three steps in the order given: (1) multiply by 2; (2) add 3; (3) divide by 4.
35. f is a function that takes a real number x and performs the following three steps in the order given: (1) add 3; (2) multiply by 2; (3) divide by 4.
36. f is a function that takes a real number x and performs the following three steps in the order given: (1) divide by 4; (2) add 3; (3) multiply by 2.
37. f is a function that takes a real number x and performs the following three steps in the order given: (1) multiply by 2; (2) add 3; (3) take the square root.
38. f is a function that takes a real number x and performs the following three steps in the order given: (1) add 3; (2) multiply by 2; (3) take the square root.

39. f is a function that takes a real number x and performs the following three steps in the order given: (1) add 3; (2) take the square root; (3) multiply by 2.

40. f is a function that takes a real number x and performs the following three steps in the order given: (1) take the square root; (2) subtract 13; (3) make the quantity the denominator of a fraction with numerator 4.

41. f is a function that takes a real number x and performs the following three steps in the order given: (1) subtract 13; (2) take the square root; (3) make the quantity the denominator of a fraction with numerator 4.

42. f is a function that takes a real number x and performs the following three steps in the order given: (1) take the square root; (2) make the quantity the denominator of a fraction with numerator 4; (3) subtract 13.

43. f is a function that takes a real number x and performs the following three steps in the order given: (1) make the quantity the denominator of a fraction with numerator 4; (2) take the square root; (3) subtract 13.

For each exercise, use the given function f to find and simplify each of the **nine** related values/expressions listed below.

- $f(1)$
- $f(-3)$
- $f\left(\frac{3}{2}\right)$
- $f(4x)$
- $4f(x)$
- $f(-x)$
- $f(x-4)$
- $f(x)-4$
- $f\left(x^2\right)$

44. $f(x) = 2x + 1$
45. $f(x) = 3 - 4x$
46. $f(x) = 2 - x^2$
47. $f(x) = x^2 - 3x + 2$
48. $f(x) = \sqrt{x-1}$
49. $f(x) = \dfrac{x}{x-1}$
50. $f(x) = 6$
51. $f(x) = 0$

For each exercise, use the given function f to find and simplify each of the **nine** related values/expressions listed below.

- $f(2)$
- $f(-2)$
- $f(2a)$
- $2f(a)$
- $f(a+2)$
- $f(a)+f(2)$
- $f\left(\frac{2}{a}\right)$
- $\frac{f(a)}{2}$
- $f(a+h)$

52. $f(x) = 2x - 5$
53. $f(x) = 5 - 2x$
54. $f(x) = 2x^2 - 1$
55. $f(x) = 3x^2 + 3x - 2$
56. $f(x) = \sqrt{2x+1}$
57. $f(x) = 1$

58. $f(x) = \dfrac{x}{2}$

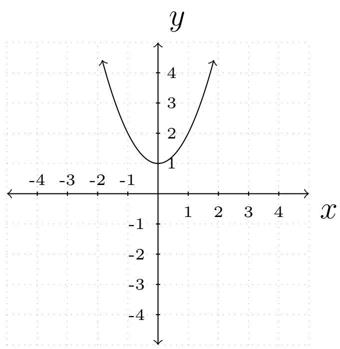
59. $f(x) = \dfrac{2}{x}$

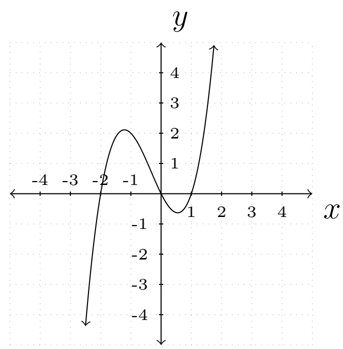
In each of the following exercises, use the given function f to find $f(0)$ and solve $f(x) = 0$

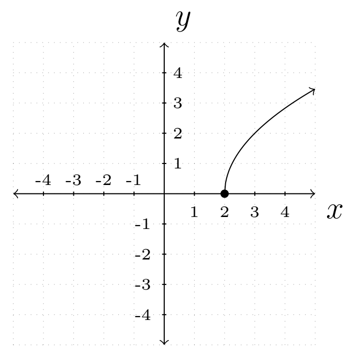
60. $f(x) = 2x - 1$
61. $f(x) = 3 - \frac{2}{5}x$
62. $f(x) = 2x^2 - 6$
63. $f(x) = x^2 - x - 12$
64. $f(x) = \sqrt{x+4}$
65. $f(x) = \sqrt{1-2x}$
66. $f(x) = \dfrac{3}{4-x}$
67. $f(x) = \dfrac{3x^2 - 12x}{4 - x^2}$

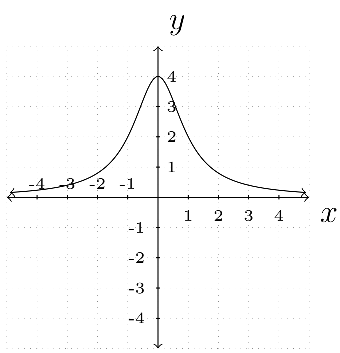
Identifying Domain and Range Graphically

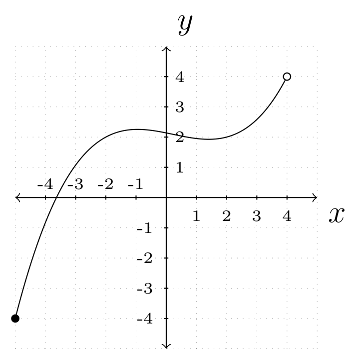
For each of the following graphs, identify the corresponding domain and range. Express your answers using interval notation.

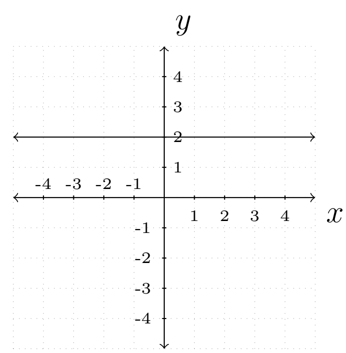
1. 

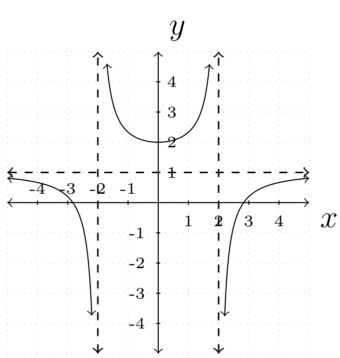
2. 

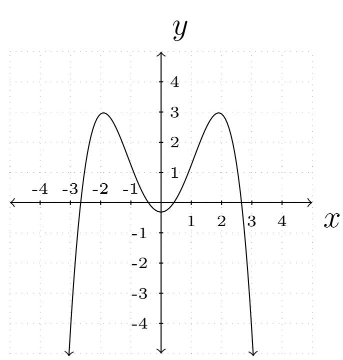
3. 

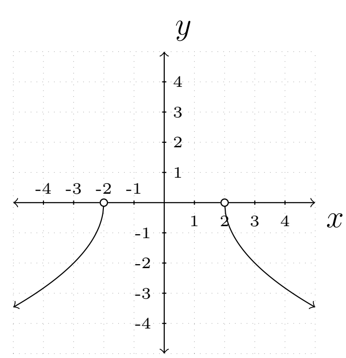
4. 

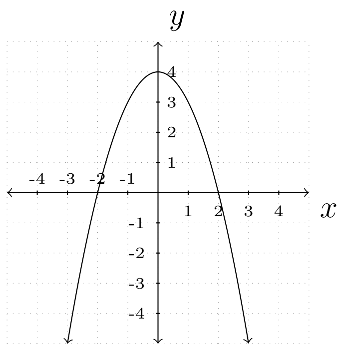
5. 

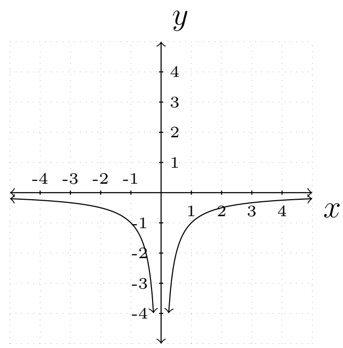
6. 

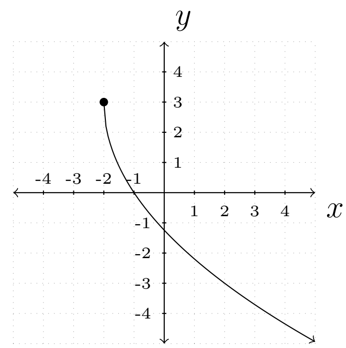
7. 

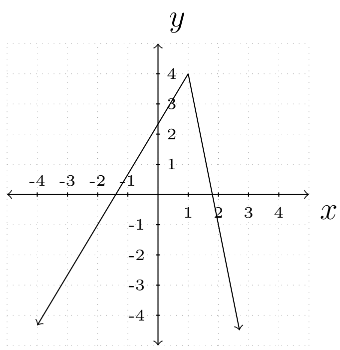
8. 

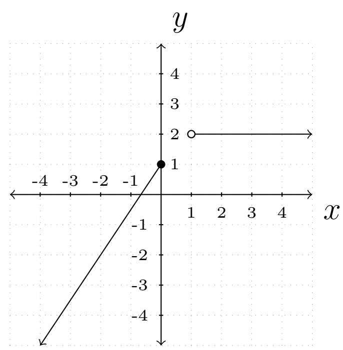
9. 

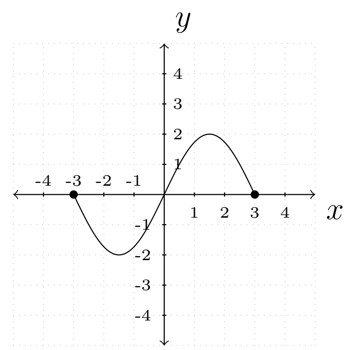
10. 

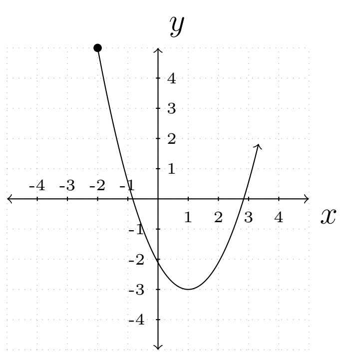
11. 

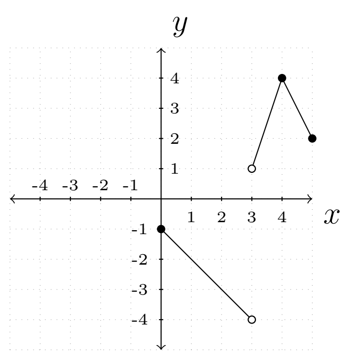
12. 

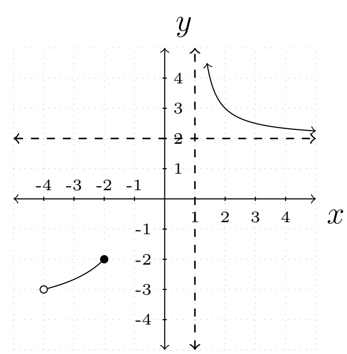
13. 

14. 

15. 

16. 

17. 

18. 

Selected Answers

Notation and Basic Examples

1) Function
3) Function
5) Not a function
7) Function
9) Not a function
11) Function
13) Function
15) Function
17) Not a function
19) Function
21) Not a function
23) Function
25) Function
27) Function
29) Function
31) Not a function
33) Not a function

35) $f(x)=\dfrac{x+3}{2}$

37) $f(x)=\sqrt{2x+3}$

39) $f(x)=2\sqrt{x+3}$

41) $f(x)=\dfrac{4}{\sqrt{x-13}}$

43) $f(x)=\sqrt{\frac{4}{x}-13}$

45) $f(x)=3-4x$

- $f(1)=-1$
- $f(-3)=15$
- $f\left(\frac{3}{2}\right)=-3$
- $f(4x)=3-16x$
- $4f(x)=12-16x$
- $f(-x)=3+4x$
- $f(x-4)=19-4x$
- $f(x)-4=-1-4x$
- $f\left(x^2\right)=3-4x^2$

47) $f(x)=x^2-3x+2$

- $f(1)=0$
- $f(-3)=20$
- $f\left(\frac{3}{2}\right)=-\frac{1}{4}$
- $f(4x)=16x^2-12x+2$
- $4f(x)=4x^2-12x+8$
- $f(-x)=x^2+3x+2$
- $f(x-4)=x^2-11x+30$
- $f(x)-4=x^2-3x-2$
- $f\left(x^2\right)=x^4-3x^2+2$

49) $f(x)=\dfrac{x}{x-1}$

- $f(1)=$undefined
- $f(-3)=\frac{3}{4}$
- $f\left(\frac{3}{2}\right)=3$
- $f(4x)=\dfrac{4x}{4x-1}$
- $4f(x)=\dfrac{4x}{x-1}$
- $f(-x)=\dfrac{x}{x+1}$
- $f(x-4)=\dfrac{x-4}{x-5}$
- $f(x)-4=\dfrac{-3x+4}{x-1}$
- $f\left(x^2\right)=\dfrac{x^2}{x^2-1}$

51) $f(x)=0$

- $f(1)=0$
- $f(-3)=0$
- $f\left(\frac{3}{2}\right)=0$
- $f(4x)=0$
- $4f(x)=0$
- $f(-x)=0$
- $f(x-4)=0$
- $f(x)-4=-4$
- $f\left(x^2\right)=0$

53) $f(x)=5-2x$

- $f(2)=1$
- $f(-2)=9$
- $f(2a)=5-4a$
- $2f(a)=10-4a$
- $f(a+2)=1-2a$
- $f(a)+f(2)=6-2a$

- $f\left(\dfrac{2}{a}\right) = \dfrac{5a-4}{a}$
- $\dfrac{f(a)}{2} = \dfrac{5}{2} - a$
- $f(a+h) = 5 - 2a - 2h$

55) $f(x) = 3x^2 + 3x - 2$

- $f(2) = 16$
- $f(-2) = 4$
- $f(2a) = 12a^2 + 6a - 2$
- $2f(a) = 6a^2 + 6a - 4$
- $f(a+2) = 3a^2 + 15a + 16$
- $f(a) + f(2) = 3a^2 + 3a + 14$
- $f\left(\dfrac{2}{a}\right) = \dfrac{-2a^2 + 6a + 12}{a^2}$
- $\dfrac{f(a)}{2} = \dfrac{3}{2}a^2 + \dfrac{3}{2}a - 1$
- $f(a+h) = 3a^2 + 6ah + 3h^2 + 3a + 3h - 2$

57) $f(x) = 1$

- $f(2) = 1$
- $f(-2) = 1$
- $f(2a) = 1$
- $2f(a) = 2$
- $f(a+2) = 1$
- $f(a) + f(2) = 2$
- $f\left(\dfrac{2}{a}\right) = 1$
- $\dfrac{f(a)}{2} = \dfrac{1}{2}$
- $f(a+h) = 1$

59) $f(x) = \dfrac{2}{x}$

- $f(2) = 1$
- $f(-2) = -1$
- $f(2a) = \dfrac{1}{a}$
- $2f(a) = \dfrac{4}{a}$
- $f(a+2) = \dfrac{2}{a+2}$
- $f(a) + f(2) = \dfrac{2+a}{a}$
- $f\left(\dfrac{2}{a}\right) = a$
- $\dfrac{f(a)}{2} = \dfrac{1}{a}$
- $f(a+h) = \dfrac{2}{a+h}$

61) $f(0) = 3$; $f(x) = 0$ for $x = 15/2$

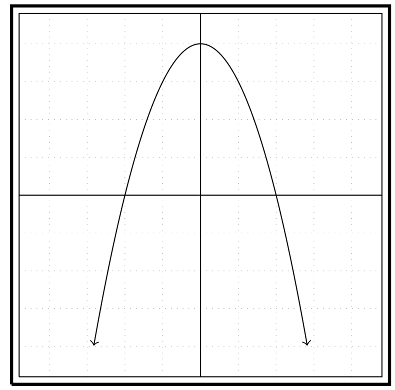
63) $f(0) = -12$; $f(x) = 0$ for $x = -3, 4$

65) $f(0) = 1$; $f(x) = 0$ for $x = 1/2$

67) $f(0) = 0$; $f(x) = 0$ for $x = 0, 4$

Domain and Range

Exercise	Domain	Range
1)	$(-\infty, \infty)$	$[1, \infty)$
3)	$[2, \infty)$	$[0, \infty)$
5)	$[-5, 4)$	$[-4, 4)$
7)	$(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$	$(-\infty, 1) \cup [2, \infty)$
9)	$(-\infty, -2) \cup (2, \infty)$	$(-\infty, 0)$
11)	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0)$
13)	$(-\infty, \infty)$	$(-\infty, 4]$
15)	$[-3, 3]$	$[-2, 2]$
17)	$[0, 3) \cup (3, 5]$	$(-4, -1] \cup (1, 4]$



Chapter 4

Quadratic Equations and Inequalities

Introduction (L16)

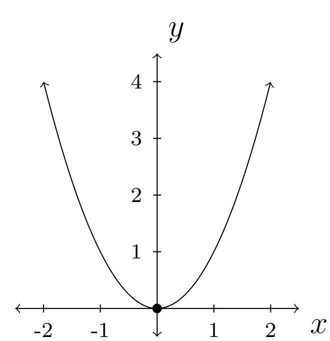
Objective: Recognize and classify a quadratic equation algebraically and graphically.

A quadratic equation is an equation of the form

$$y = ax^2 + bx + c,$$

where the *coefficients* of a,b, and c are real numbers and $a \neq 0$. This form is most commonly referred to as the *standard form* of a quadratic. We call a the *leading coefficient*, ax^2 the *leading term* (also known as the *quadratic term*), bx the *linear term* and c the *constant term* of the equation. The quadratic term ax^2, must have a nonzero coefficient in order for the equation to be a quadratic (otherwise y would be linear, in slope-intercept form). The most fundamental quadratic equation is $y = x^2$ and its graph, like all quadratics, is known as a *parabola*.

Example 120. $y = x^2$

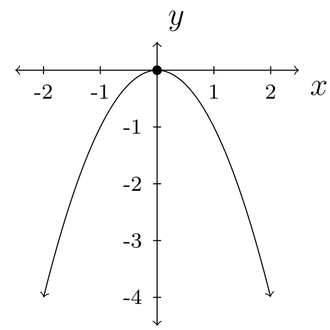


From the standard form, since $a > 0$, the graph opens upwards and is said to be *concave up*.

As a result, there is a minimum point, known as the *vertex*, located at the origin, $(0, 0)$.

Notice the symmetry over the y-axis.

Example 121. $y = -x^2$



Since $a = -1$, the graph opens downward or we say that it is *concave down*.

Every parabola with a negative leading coefficient ($a < 0$) will be concave down with a maximum value at its vertex.

The graph above has the same vertex as that in the previous example, but is a reflection of the previous graph about the y-axis. This flip of the graph is known as a transformation and will be discussed in the next chapter.

Aside from the shape and concavity, there is little else that the standard form immediately provides for graphing a quadratic. Additional aspects related to graphing quadratics will be covered a bit later in the chapter. Following this introduction, we will primarily focus on factoring quadratics from standard form. With all of the algebraic material that will follow, however, it will help to have a graphical sense of a quadratic equation.

An Introduction to the Vertex Form

Objective: Recognize and utilize the vertex form to graph a quadratic.

The most useful form for graphing a quadratic equation is the *vertex form*. A quadratic equation is said to be in vertex form if it is represented as

$$y = a(x - h)^2 + k,$$

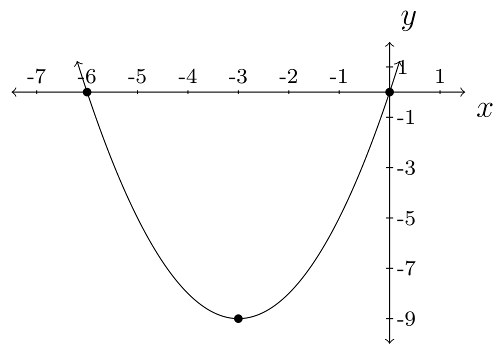
where h and k are real numbers.

It is important to note that the value a appearing above is also the leading coefficient from the standard form for a quadratic. Later, we will see the relationships between the coefficients a, b, and c in the standard form with h and k above.

When $a = 1$, the graph of a quadratic equation given in vertex form can be represented as a *shift*, or translation, of the original or "parent equation" $y = x^2$ presented earlier. The vertex form, unlike the standard form, allows us to immediately identify the vertex of the resulting parabola, which will be the point (h, k).

Next, we will see a few examples of quadratics in vertex form, the last of which is a bit surprising.

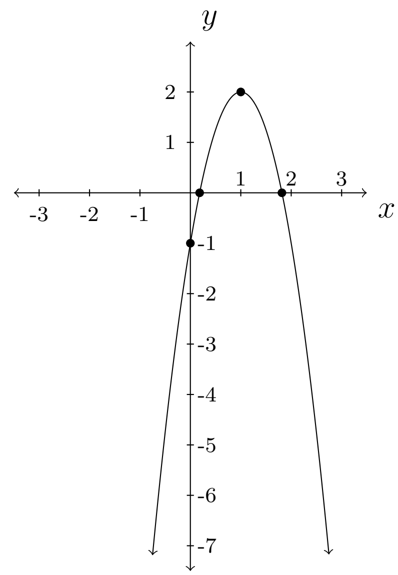
Example 122. $y = (x + 3)^2 - 9$



The vertex is at $(-3, -9)$ and the graph can be realized as the graph of $y = x^2$ shifted left 3 units and down 9 units from the origin.

Since our graph is concave up there will be two x-intercepts as the function opens upward from below the x-axis.

Example 123. $y = -3(x - 1)^2 + 2$



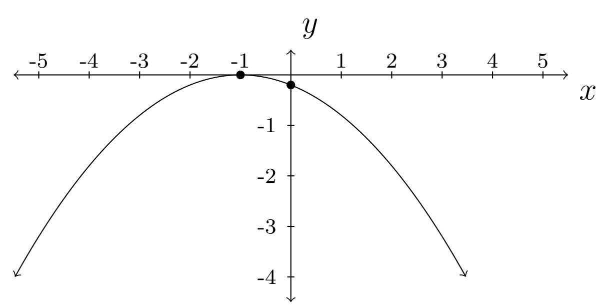
The vertex is at $(1, 2)$ and represents a translation of the vertex for the graph of $y = x^2$ right 1 unit and up 2 units.

This graph is also concave down, since the leading coefficient $a = -3$ is less than zero.

Moreover, since $|a| > 1$, the shape of the graph is narrower than those which we have seen thus far.

Just like the previous example, this graph will have two x-intercepts as its vertex is above the x-axis and it opens downward.

Example 124. $y = -\frac{1}{5}(x + 1)^2$



The vertex is at $(-1, 0)$ and represents a translation of the vertex for the graph of $y = x^2$ left 1 unit.

There is no vertical shift, since there is no addition of a constant outside of the given expression.

Our graph is concave down and is much wider than any example we have seen thus far. This is on account of the fact that a is both negative and $|a| < 1$.

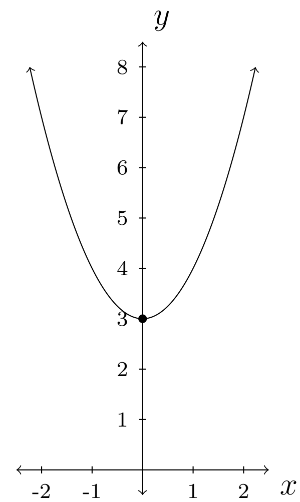
The following example shows an equation represented in both vertex and standard forms.

Example 125. $y = x^2 + 3$

The vertex is at $(0, 3)$ and our graph is a shift of the graph of $y = x^2$ up 3 units.

Since our graph is concave up with a vertex above the x-axis, there will be no real x-intercepts.

Notice that there is no horizontal shift because no number has been added or subtracted to x prior to it being squared.



This final example above may be recognized as a quadratic equation in standard form, where $b = 0$. Since there is no linear term, this quadratic is also in vertex form.

More generally, the graph of any equation of the form

$$y = ax^2 + c$$

has a y-intercept and vertex at $(0, c)$, since the resulting parabola represents a only a vertical shift of the graph of $y = x^2$ by c units and no horizontal shift.

Factoring Methods

Greatest Common Factors (L17)

Objective: Find the greatest common factor (GCF) and factor it out of an expression.

In order to discuss the factorization methods of this section, it will be necessary to introduce some of the terminology a bit early. In particular, in this section we will be working with *polynomial expressions.* While most of our work will be with polynomials containing a single variable, it will be helpful to see a few examples of polynomials that contain two (or more) variables.

Both linear and quadratic expressions of a variable x are basic examples of polynomials. A more general description of a polynomial in terms of the variable x is

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

where n is a nonnegative integer and $a_0, a_1, \ldots, a_{n-1}, a_n$ represent real coefficients ($a_n \neq 0$).

A basic interpretation of this description is a sum of n terms, each containing a real coefficient (possibly equal to 0), where the associated power of the variable is a positive integer (or possibly 0, in the case of the constant term $a_0 = a_0 x^0$).

The expression $8x^4 - 12x^3 + 32x$ would be an example of a polynomial, in which the power n (known as the *degree* of the polynomial) equals 4, and the coefficients are as follows.

$$a_4 = 8, \quad a_3 = -12, \quad a_2 = 0, \quad a_1 = 32, \quad a_0 = 0$$

If we inserted another variable(s) into each of the terms of our expression, we could create a polynomial expression in terms of two (or more) variables. An example of this would be

$$8x^4y - 12x^3y^2 + 32x.$$

While there is much more that we could say about this important concept of algebra, we will postpone a more in-depth treatment of polynomials until a later chapter, and move on to the topic of factorization.

Factoring a polynomial could be considered as the "opposite" action of multiplying (or expanding) polynomials together. In working with polynomial expressions, there are many benefits to identifying both its expanded and factored forms. Specifically, we will use factored polynomials to help us solve equations, learn behaviors of graphs, and understand more complicated rational expressions. Because so many concepts in algebra depend on being able to factor polynomials, it is critical that we establish strong factorization skills.

In this first part of the section, we will focus on factoring using the greatest common factor or GCF of a polynomial. When multiplying polynomials, we employ the distributive property, as demonstrated below.

$$4x^2(2x^2 - 3x + 8) = 8x^4 - 12x^3 + 32x$$

Here, we will work with the same expression, but with a backwards approach, starting with the expanded form and obtaining one that is partially (or completely) factored.

We will start with $8x^2 - 12x^3 + 32x$ and try and work backwards to reach $4x^2(2x - 3x + 8)$.

To do this we have to be able to first identify what the GCF of a polynomial is. We will first introduce this concept by finding the GCF of a set of integers. To find a GCF of two or more integers, we must find the largest integer d that divides nicely into each of the given integers. Alternatively stated, d should be the largest factor of each of the integers in our set. This can often be determined with quick "mental math", as shown in the following example.

Example 126. Find the GCF of 15, 24, and 27.

$$\begin{aligned} \frac{15}{3} = 5, \quad \frac{24}{3} = 8, \quad \frac{27}{3} &= 9 && \text{Each of the numbers can be divided by 3} \\ \text{GCF} &= 3 && \text{Our solution} \end{aligned}$$

When there are variables in our problem we can first find the GCF of the numbers, then we can identify any variables that appear in every term and factor them out, taking the smallest exponent in each case. This is shown in the next example.

Example 127. Find the GCF of $24x^4y^2z$, $18x^2y^4$, and $12x^3yz^5$.

$$\frac{24}{6} = 4, \quad \frac{18}{6} = 3, \quad \frac{12}{6} = 2 \qquad \text{Each number can be divided by 6}$$

$$x^2y \qquad x \text{ and } y \text{ appear in all three terms, taking the lowest exponent for each variable}$$

$$\text{GCF} = 6x^2y \qquad \text{Our solution}$$

To factor out a GCF from a polynomial we first need to identify the GCF of all the terms, this is the part that goes in front of the parentheses, then we divide each term by the GCF in order to determine what should appear inside of the parentheses. This is demonstrated in the following examples.

Example 128. Find and factor out the GCF of the given polynomial expression.

$$4x^2 - 20x + 16 \qquad \text{GCF is 4, divide each term by 4}$$

$$\frac{4x^2}{4} = x^2, \quad \frac{-20x}{4} = -5x, \quad \frac{16}{4} = 4 \qquad \text{This is what is left inside the parentheses}$$

$$4(x^2 - 5x + 4) \qquad \text{Our solution}$$

With factoring we can always check our solutions by expanding or multiplying out the answer. As in the example above, this usually will involve some form of the distributive property. Our end result upon checking should match the original expression.

Example 129. Find and factor out the GCF of the given polynomial expression.

$$25x^4 - 15x^3 + 20x^2 \qquad \text{GCF is } 5x^2\text{, divide each term by } 5x^2$$

$$\frac{25x^4}{5x^2} = 5x^2, \quad \frac{-15x^3}{5x^2} = -3x, \quad \frac{20x^2}{5x^2} = 4$$

This is what is left inside the parentheses.

$$5x^2(5x^2 - 3x + 4) \qquad \text{Our solution}$$

Example 130. Find and factor out the GCF of the given polynomial expression.

$$3x^3y^2z + 5x^4y^3z^5 - 4xy^4 \qquad \text{GCF is } xy^2\text{, divide each term by } xy^2$$

$$\frac{3x^3y^2z}{xy^2} = 3x^2z, \quad \frac{5x^4y^3z^5}{xy^2} = 5x^3yz^5, \quad \frac{-4xy^4}{xy^2} = -4y^2$$

This is what is left inside the parentheses.

$$xy^2(3x^2z + 5x^3yz^5 - 4y^2) \qquad \text{Our solution}$$

Example 131. Find and factor out the GCF of the given polynomial expression.

$$21x^3 + 14x^2 + 7x \qquad \text{GCF is } 7x, \text{divide each term by } 7x$$

$$\frac{21x^3}{7x} = 3x^2, \quad \frac{14x^2}{7x} = 2x, \quad \frac{7x}{7x} = 1$$

This is what is left inside the parentheses.

$$7x(3x^2 + 2x + 1) \qquad \text{Our solution}$$

It is important to note that in the previous example, the GCF of $7x$ was also one of the original terms. Dividing this term by the GCF left us with 1. A common mistake is to try to factor out the $7x$ and leave a value of zero. Factoring, however, will never make terms disappear completely. Any (nonzero) number or term that is divided by itself will always equal 1. Therefore, we must make certain to not forget to include a 1 in our solution.

Often the line showing the division is not written in the work of factoring the GCF, and we will simply identify the GCF and put it in front of the parentheses. This step is one that will eventually be understood, and can therefore be omitted once the skill has been mastered. The following two examples demonstrate this.

Example 132. Find and factor out the GCF of the given polynomial expression.

$$12x^5y^2 - 6x^4y^4 + 8x^3y^5$$

Notice, the GCF is $2x^3y^2$. Write $2x^3y^2$ in front of the parentheses and divide each term by it, writing the resulting terms inside the parentheses.

$$2x^3y^2(6x^2 - 3xy^2 + 4y^3) \qquad \text{Our solution}$$

Example 133. Find and factor out the GCF of the given polynomial expression.

$$18a^4b^3 - 27a^3b^3 + 9a^2b^3$$

Notice, the GCF is $9a^2b^3$. Write $9a^2b^3$ in front of the parentheses and divide each term by it, writing the resulting terms inside the parentheses.

$$9a^2b^3(2a^2 - 3a + 1) \qquad \text{Our solution}$$

Again, in the previous problem, when dividing $9a^2b^3$ by itself, the resulting term is 1, not zero. Be very careful that each term is accounted for in your final solution, and never forget that we can easily check our answers by expanding.

Factor by Grouping (L18)

Objective: Factor a tetranomial (four-term) expression by grouping.

The first thing we will always do when factoring is try to factor out a GCF. A GCF is often a *monomial* (a single term) like in the expression $5xy+10xz$. Here, the GCF is the monomial $5x$, so we would have $5x(y+2z)$ as our answer. However, a GCF does not have to be a monomial. It could, in fact, be a *binomial* and contain two terms. To see this, consider the following two examples.

Example 134. Find and factor out the GCF of the given expression.

$$\begin{array}{rl} 3ax-7bx & \text{Both terms have } x \text{ in common, factor it out} \\ x(3a-7b) & \text{Our solution} \end{array}$$

Now we will work with the same expression, replacing x with $(2a+5b)$.

Example 135. Find and factor out the GCF of the given expression.

$$\begin{array}{rl} 3a(2a+5b)-7b(2a+5b) & \text{Both terms have } (2a+5b) \text{ in common,} \\ & \quad\text{factor it out} \\ (2a+5b)(3a-7b) & \text{Our solution} \end{array}$$

In the same way that we factored out a GCF of x we can factor out a GCF which is a binomial, such as $(2a+5b)$ in the example above. This process can be extended to factoring expressions in which there is either no apparent GCF or there is more factoring that can be done after the GCF has been factored. At this point, we will introduce another useful factorization strategy, known as *grouping.* Grouping is typically employed when faced with an expression containing four terms.

Throughout this section, it is important to reinforce the fact that factoring is essentially expansion (multiplication) done in reverse. Therefore, we will first look at problem which requires us to multiply two expressions, and then try to reverse the process.

Example 136. Write the expanded form for the given expression.

$$\begin{array}{rl} (2a+3)(5b+2) & \text{Distribute } (2a+3) \text{ into second parentheses} \\ 5b(2a+3)+2(2a+3) & \text{Distribute each monomial} \\ 10ab+15b+4a+6 & \text{Our solution} \end{array}$$

Our solution above has four terms in it. We arrived at this solution by focusing on the two parts, $5b(2a+3)$ and $2(2a+3)$.

When attempting to factor by grouping, we will always divide an expression into two parts, or groups: group one will contain the first two terms of our expression and group two will contain the last two terms. Then we can identify and factor the GCF out of each group. In doing this, our hope is that what is left over in each group will be the same expression. If the resulting expressions match, we can then factor out this matching GCF from both of our designated groups, writing what is left in a new set of parentheses.

Although the description of this method can sound rather complicated, the next few examples will help to clear up any lingering questions. We will start by working through the last example in reverse, factoring instead of multiplying.

Example 137. Factor the given expression.

$$10ab + 15b + 4a + 6 \qquad \text{Split expression into two groups}$$

$$\boxed{10ab + 15b \,|\, +4a + 6} \qquad \text{GCF on left is } 5b\text{, on the right is } 2$$

$$\boxed{5b(2a + 3) \,|\, +2(2a + 3)} \qquad (2a + 3) \text{ appears twice! Factor out this GCF}$$

$$(2a + 3)(5b + 2) \qquad \text{Our solution}$$

The key for grouping to be successful is for the two binomials to match exactly, once the GCF has been factored out of both groups. If there is any difference between the two binomials, we either have to do some adjusting or we cannot factor by grouping. Consider the following example.

Example 138. Factor the given expression.

$$6x^2 + 9xy - 14x - 21y \qquad \text{Split expression into two groups}$$

$$\boxed{6x^2 + 9xy \,|\, -14x - 21y} \qquad \text{GCF on left is } 3x\text{, on right is } 7$$

$$\boxed{3x(2x + 3y) \,|\, +7(-2x - 3y)} \qquad \text{The signs in the parentheses do not match!}$$

When the signs on both terms do not match, we can easily make them match by factoring a negative out of the GCF on the right side. Instead of 7 we will use -7. This will change the signs inside the second set of parentheses.

$$\boxed{3x(2x + 3y) \,|\, -7(2x + 3y)} \qquad (2x + 3y) \text{ appears twice! Factor out this GCF}$$

$$(2x + 3y)(3x - 7) \qquad \text{Our solution}$$

It will often be easy to recognize if we will need to factor out a negative sign when grouping. Specifically, if the first term of the first binomial is positive, the first term of the second binomial will also need to be positive. Similarly, if the first term of the first binomial is negative, the first term of the second binomial will also need to be negative.

Example 139. Factor the given expression.

$5xy - 8x - 10y + 16$	Split the expression into two groups
$\boxed{5xy - 8x}\boxed{-10y + 16}$	GCF on left is x, on right we need to factor out a negative, we will use -2
$\boxed{x(5y - 8)}\boxed{-2(5y - 8)}$	$(5y - 8)$ appears twice! Factor out this GCF
$(5y - 8)(x - 2)$	Our solution

Occasionally, when factoring out a GCF from either group, it will appear as though there is nothing that can be factored out. In this case a GCF of either 1 or -1 should be used. Often this will be all that is required, in order to match up the two binomials.

Example 140. Factor the given expression.

$12ab - 14a - 6b + 7$	Split the expression into two groups
$\boxed{12ab - 14a}\boxed{-6b + 7}$	GCF on left is $2a$, on right use GCF of -1
$\boxed{2a(6b - 7)}\boxed{-1(6b - 7)}$	$(6b - 7)$ appears twice! Factor out this GCF
$(6b - 7)(2a - 1)$	Our solution

Example 141. Factor the given expression.

$6x^3 - 15x^2 + 2x - 5$	Split expression into two groups
$\boxed{6x^3 - 15x^2}\boxed{+2x - 5}$	GCF on left is $3x^2$, on right use GCF of 1
$\boxed{3x^2(2x - 5)}\boxed{+1(2x - 5)}$	$(2x - 5)$ appears twice! Factor out this GCF
$(2x - 5)(3x^2 + 1)$	Our solution

When grouping, the selection or assignment of terms for each group can also be an area of concern. In particular, if after factoring out the GCF from the preassigned groups, the binomials do not match *and* cannot be adjusted as in the previous examples, a change in the group assignments may be necessary. In the next example we will demonstrate this by eventually moving the second term to the end of the given expression, to see if grouping may still be used.

Example 142. Factor the given expression.

$$4a^2 - 21b^3 + 6ab - 14ab^2 \quad \text{Split the expression into two groups}$$

$$\boxed{4a^2 - 21b^3}\boxed{+6ab - 14ab^2} \quad \text{GCF on left is } 1, \text{ on right is } 2ab$$

$$\boxed{1(4a^2 - 21b^3)}\boxed{+2ab(3 - 7b)} \quad \text{Binomials do not match! Move second term to end}$$

$$4a^2 + 6ab - 14ab^2 - 21b^3 \quad \text{Start over, split expression into two groups}$$

$$\boxed{4a^2 + 6ab}\boxed{-14ab^2 - 21b^3} \quad \text{GCF on left is } 2a, \text{ on right is } -7b^2$$

$$\boxed{2a(2a + 3b)}\boxed{-7b^2(2a + 3b)} \quad (2a + 3b) \text{ appears twice! Factor out this GCF}$$

$$(2a + 3b)(2a - 7b^2) \quad \text{Our solution}$$

When rearranging terms the expression might still appear to be out of order. Sometimes after factoring out the GCF the resulting binomials appear "backwards". There are two scenarios where this can happen: one with addition and one with subtraction. In the first scenario, if the binomials are say $(a + b)$ and $(b + a)$, then we do not have to do any extra work. This is because addition is a *commutative* operation. This means that the sum of two terms is the same, regardless of their order. For example, $5 + 3 = 3 + 5 = 8$.

Example 143. Factor the given expression.

$$7 + y - 3xy - 21x \quad \text{Split the expression into two groups}$$

$$\boxed{7 + y}\boxed{-3xy - 21x} \quad \text{GCF on left is } 1, \text{ on the right is } -3x$$

$$\boxed{1(7 + y)}\boxed{-3x(y + 7)} \quad y + 7 \text{ and } 7 + y \text{ are equal, use either one}$$

$$(y + 7)(1 - 3x) \quad \text{Our solution}$$

In the second scenario, if the binomials contain subtraction, then we need to be a bit more careful. For example, if the binomials are $(a - b)$ and $(b - a)$, we will factor a negative sign out of either group (usually the second). Notice what happens when we factor out a -1 in the following example.

Example 144. Factor the given expression.

$(b-a)$	Factor out a -1
$-1(-b+a)$	Resulting binomial contains addition, we may switch the order
$-1(a-b)$	The order of the subtraction has been switched!

Generally we will not show all of the steps in the previous example when simplifying. Instead, we will simply factor out a negative sign and switch the order of the subtraction to make the resulting binomials. As with previous concepts, this omission should only be made by the student when the skill has been mastered. We conclude our discussion of grouping with one final example.

Example 145. Factor the given expression.

$8xy-12y+15-10x$	Split the expression into two groups
$\boxed{8xy-12y}\boxed{15-10x}$	GCF on left is $4y$, on right is 5
$\boxed{4y(2x-3)}\boxed{+5(3-2x)}$	Need to switch order, Factor negative sign out of second binomial
$\boxed{4y(2x-3)}\boxed{-5(2x-3)}$	$(2x-3)$ appears twice! Factor out this GCF
$(2x-3)(4y-5)$	Our solution

Trinomials with Leading Coefficient $a=1$ (L19)

Objective: Factor a trinomial with a leading coefficient of one.

Factoring polynomial expressions that contain three terms, or *trinomials*, is the most essential factorization skill to algebra. Consequently, it is also the most important factorization skill to master. Again, since factoring is basically multiplication performed in reverse, we will start with a multiplication example and look at how we can reverse the process.

Example 146. Write the expanded form for the given expression.

$(x+6)(x-4)$	Distribute $(x+6)$ through second parentheses
$x(x+6)-4(x+6)$	Distribute each monomial through parentheses
$x^2+6x-4x-24$	Combine like terms
$x^2+2x-24$	Our solution

Notice that if we reverse the last three steps of the previous example, the process looks like grouping. This is because it is grouping! In the second-to-last line, the GCF of the first two terms is x and the GCF of the last two terms is -4. In this manner, we will factor trinomials by writing them as a polynomial containing four terms, and then factor by grouping. This is demonstrated in the following example, which is the previous one done in reverse.

Example 147. Factor the given expression.

$$\begin{array}{rl} x^2+2x-24 & \text{split middle (linear) term into} +6x-4x \\ x^2+6x-4x-24 & \text{Grouping : GCF on left is } x, \text{ on right is } -4 \\ x(x+6)-4(x+6) & (x+6) \text{ appears twice, factor out this GCF} \\ (x+6)(x-4) & \text{Our solution} \end{array}$$

The trick to make these problems work resides in how we split the middle (or linear) term. Why did we choose $+6x-4x$ and not $+5x-3x$? The reason is because $6x-4x$ is the only combination that will allow grouping to work! So how do we know what is the one combination that we need? To find the correct way to split the middle term we will use what is called the *ac*-method. Later, we will discuss why it is called the *ac*-method.

The idea behind the *ac*-method is that we must find a pair of numbers that *multiply* to get the last (or constant) term in the expression and *add* to get the coefficient of the middle (or linear) term. In the previous example, we would want two numbers whose product is -24 and sum is $+2$. The only numbers that can do this are 6 and -4, since $6\cdot -4=-24$ and $6+(-4)=2$. This method is demonstrated in the next few examples.

Example 148. Factor the given expression.

$$\begin{array}{rl} x^2+9x+18 & \text{Need to multiply to 18, add to 9} \\ x^2+6x+3x+18 & \text{Use } 6 \text{ and } 3, \text{ split the middle term} \\ x(x+6)+3(x+6) & \text{Factor by grouping} \\ (x+6)(x+3) & \text{Our solution} \end{array}$$

Example 149. Factor the given expression.

$$\begin{array}{rl} x^2-4x+3 & \text{Need to multiply to 3, add to } -4 \\ x^2-3x-x+3 & \text{Use } -3 \text{ and } -1, \text{ split the middle term} \\ x(x-3)-1(x-3) & \text{Factor by grouping} \\ (x-3)(x-1) & \text{Our solution} \end{array}$$

Example 150. Factor the given expression.

$$\begin{array}{rl} x^2-8x-20 & \text{Need to multiply to } -20, \text{ add to } -8 \\ x^2-10x+2x-20 & \text{Use } -10 \text{ and } 2, \text{ split the middle term} \\ x(x-10)+2(x-10) & \text{Factor by grouping} \\ (x-10)(x+2) & \text{Our solution} \end{array}$$

Often when factoring we are faced with an expression containing two variables. These expressions are treated just like those containing only one variable. As in the next example, we will still use the coefficients to decide how to split the linear term.

Example 151. Factor the given expression.

$$\begin{array}{rl} a^2-9ab+14b^2 & \text{Need to multiply to } 14, \text{add to } -9 \\ a^2-7ab-2ab+14b^2 & \text{Use } -7 \text{ and } -2, \text{ split the middle term} \\ a(a-7b)-2b(a-7b) & \text{Factor by grouping} \\ (a-7b)(a-2b) & \text{Our solution} \end{array}$$

As the past few examples has shown, it is very important to be aware of negatives in finding the right pair of numbers used to split the linear term. Consider the following example, done *incorrectly*, ignoring negative signs.

Example 152. Factor the given expression.

$$\begin{array}{rl} x^2+5x-6 & \text{Need to multiply to } 6, \text{add to } 5 \\ x^2+2x+3x-6 & \text{Use } 2 \text{ and } 3, \text{ split the middle term} \\ x(x+2)+3(x-2) & \text{Factor by grouping} \\ ??? & \text{Binomials do not match!} \end{array}$$

Because we did not consider the negative sign with the constant term of -6 to find our pair of numbers, the binomials did not match and grouping was unsuccessful. Now we show factorization done correctly.

Example 153. Factor the given expression.

$$\begin{array}{rl} x^2+5x-6 & \text{Need to multiply to } -6, \text{add to } 5 \\ x^2+6x-x-6 & \text{Use } 6 \text{ and } -1, \text{ split the middle term} \\ x(x+6)-1(x+6) & \text{Factor by grouping} \\ (x+6)(x-1) & \text{Our solution} \end{array}$$

At this point, one might notice a shortcut for factoring such expressions. Once we identify the two numbers that are used to split the linear term, these will be the two numbers in each of our factors! In the previous example, the numbers used to split the linear term were 6 and -1, our factors turned out to be $(x+6)(x-1)$.

This shortcut will not always work out, as we will see momentarily. We can use it, however, when we have a leading coefficient of $a=1$ for our quadratic term ax^2, which has been the case for all of the trinomials we have factored thus far. This shortcut is employed in the next few examples.

Example 154. Factor the given expression.

$$\begin{array}{rl} x^2-7x-18 & \text{Need to multiply to } -18, \text{add to } -7 \\ & \text{Use } -9 \text{ and } 2, \text{write the factors} \\ (x-9)(x+2) & \text{Our solution} \end{array}$$

Example 155. Factor the given expression.

$m^2 - mn - 30n^2$	Need to multiply to -30, add to -1
	Use 5 and -6, write the factors
	Do not forget second variable!
$(m+5n)(m-6n)$	Our solution

It is also certainly possible to have a trinomial that does not factor using the *ac*-method. If there is no combination of numbers that multiplies and adds to the correct numbers, then we say that we cannot factor the polynomial "nicely", or easily. Later on in the chapter, we will learn of some other methods and terminology for factoring quadratic expressions of this type. The next example is of a quadratic expression that is not easily factorable.

Example 156. Factor the given expression.

$x^2 + 2x + 6$	Need to multiply to 6, add to 2
$1 \cdot 6$ and $2 \cdot 3$	Only possibilities to multiply to 6, none add to 2
Not easily factorable	Our solution

Later, we will discover that the quadratic expression above cannot be factored over the real numbers. In other words, there exist no real numbers r and s such that

$$x^2 + 2x + 6 = (x - r)(x - s)$$

Such expressions are said to be *irreducible over the reals*, and any factorization will require us to use *complex* numbers. Complex numbers will be discussed later on in the chapter.

When factoring any expression, it is important to not forget about first identifying a GCF of all the given terms. If all the terms in an expression have a common factor we will want to first factor out the GCF before using any other method.

Example 157. Factor the given expression.

$3x^2 - 24x + 45$	GCF of all terms is 3, factor this out first
$3(x^2 - 8x + 15)$	Need to multiply to 15, add to -8
	Use -5 and -3, write the factors
$3(x-5)(x-3)$	Our solution

Again it is important to comment on the shortcut of jumping right to the factors, this only works if the leading coefficient $a = 1$. In the example above, we applied the shortcut only *after* we factored out a GCF of 3. Next, we will look at how this process changes when $a \neq 1$.

Trinomials with Leading Coefficient $a \neq 1$ (L20)

Objective: Factor a trinomial with a leading coefficient of $a \neq 1$.

When factoring trinomials we used the ac-method to split the middle (or linear) term and then factor by grouping. The ac-method gets its name from the general trinomial expression, $ax^2 + bx + c$, where a, b, and c are the leading coefficient, linear coefficient, and constant term, respectively.

The ac-method is named as such because we will use the product $a \cdot c$ to help find out what two numbers we will need for grouping later on. Previously, we always found two numbers whose product was equal to c, since the leading coefficient a was 1 in our expression (so $ac = 1c = c$). Now we will be working with trinomials where $a \neq 1$, so we will need to identify two numbers that multiply to ac and add to b. Aside from this adjustment, the process will be the same as before.

Example 158. Factor the given expression.

$$\begin{array}{rl}
3x^2 + 11x + 6 & \text{Multiply to } ac \text{ or } (3)(6) = 18, \text{ add to } 11 \\
3x^2 + 9x + 2x + 6 & \text{The numbers are } 9 \text{ and } 2, \text{ split the linear term} \\
3x(x+3) + 2(x+3) & \text{Factor by grouping} \\
(x+3)(3x+2) & \text{Our solution}
\end{array}$$

When $a = 1$, we were able to use a shortcut, using the numbers that split the linear term for our factors. The previous example illustrates an important point: the shortcut does not work when $a \neq 1$. Therefore, we must go through all the steps of grouping in order to factor the expression.

Example 159. Factor the given expression.

$$\begin{array}{rl}
8x^2 - 2x - 15 & \text{Multiply to } ac \text{ or } (8)(-15) = -120, \text{ add to } -2 \\
8x^2 - 12x + 10x - 15 & \text{The numbers are } -12 \text{ and } 10, \text{ split the linear term} \\
4x(2x-3) + 5(2x-3) & \text{Factor by grouping} \\
(2x-3)(4x+5) & \text{Our solution}
\end{array}$$

Example 160. Factor the given expression.

$$\begin{array}{rl}
10x^2 - 27x + 5 & \text{Multiply to } ac \text{ or } (10)(5) = 50, \text{ add to } -27 \\
10x^2 - 25x - 2x + 5 & \text{The numbers are } -25 \text{ and } -2, \text{ split the linear term} \\
5x(2x-5) - 1(2x-5) & \text{Factor by grouping} \\
(2x-5)(5x-1) & \text{Our solution}
\end{array}$$

The same process will work for trinomials containing two variables.

Example 161. Factor the given expression.

$$\begin{array}{rl} 4x^2-xy-5y^2 & \text{Multiply to } ac \text{ or } (4)(-5)=-20, \text{ add to } -1 \\ 4x^2+4xy-5xy-5y^2 & \text{The numbers are 4 and } -5, \text{ split the middle term} \\ 4x(x+y)-5y(x+y) & \text{Factor by grouping} \\ (x+y)(4x-5y) & \text{Our solution} \end{array}$$

As always, when factoring we will first look for a GCF before using any other method, including the ac-method. Factoring out the GCF first also has the added bonus of making the coefficients smaller, so other methods become easier.

Example 162. Factor the given expression.

$$\begin{array}{rl} 18x^3+33x^2-30x & \text{GCF is } 3x, \text{ factor this out first} \\ 3x(6x^2+11x-10) & \text{Multiply to } ac \text{ or} (6)(-10)=-60, \text{ add to } 11 \\ 3x(6x^2+15x-4x-10) & \text{The numbers are 15 and } -4, \text{ split the linear term} \\ 3x[3x(2x+5)-2(2x+5)] & \text{Factor by grouping} \\ 3x(2x+5)(3x-2) & \text{Our solution} \end{array}$$

As was the case with trinomials when $a=1$, not all trinomials can be factored easily. If there are no combinations that multiply and add correctly, then we can say the trinomial is not easily factorable. In such cases, the expression will require a new method of factorization, and may even be shown to be irreducible over the real numbers (the factorization will require complex numbers). We will encounter such expressions and learn how to properly handle them before the end of this chapter. We conclude this section with one such example.

Example 163. Factor the given expression.

$$\begin{array}{rl} 3x^2+2x-7 & \text{Multiply to } ac \text{ or } (3)(-7)=-21, \text{ add to } 2 \\ -3(7) \text{ and } -7(3) & \text{Only two ways to multiply to } -21, \text{ neither adds to } 2 \\ \text{Not easily factorable} & \text{Our solution} \end{array}$$

It turns out that the previous example *is* factorable over the real numbers, but we will postpone this discovery until later.

Solving by Factoring (L21)

Objective: Solve polynomial equations by factoring and using the Zero Factor Property.

When solving linear equations such as $2x-5=21$ we can solve for the variable directly by adding 5 and dividing by 2 to get 13. When working with quadratic equations (or higher degree polynomials), however, we cannot simply isolate the variable as we did with linear equations. One property that we can use to solve for the variable is known as the zero factor property.

Zero Factor Property : If $\boldsymbol{ab = 0}$ then either $\boldsymbol{a = 0}$ or $\boldsymbol{b = 0}$.

The zero factor property tells us that if the product of two factors is zero, then one of the factors must be zero. We can use this property to help us solve factored polynomials as in the following example.

Example 164. Solve the given equation for all possible values of x.

$$\begin{array}{rl}
(2x-3)(5x+1)=0 & \text{One factor must be zero} \\
2x-3=0 \text{ or } 5x+1=0 & \text{Set each factor equal to zero} \\
\underline{\mathbf{+3 \quad +3}} \qquad \underline{\mathbf{-1 \quad -1}} & \text{Solve each equation} \\
2x=3 \quad \text{or} \quad 5x=-1 & \\
\overline{\mathbf{2}} \quad \overline{\mathbf{2}} \qquad \overline{\mathbf{5}} \quad \overline{\mathbf{5}} & \\
x=\dfrac{3}{2} \text{ or } -\dfrac{1}{5} & \text{Our solution}
\end{array}$$

For the zero factor property to work we must have factors to set equal to zero. This means if an expression is not already factored, we must first factor it.

Example 165. Solve the given equation for all possible values of x.

$$\begin{array}{rl}
4x^2+x-3=0 & \text{Factor using the } ac\text{-method,} \\
 & \quad \text{multiply to } -12, \text{ add to } 1 \\
4x^2-3x+4x-3=0 & \text{The numbers are } -3 \text{ and } 4, \text{ split the linear term} \\
x(4x-3)+1(4x-3)=0 & \text{Factor by grouping}
\end{array}$$

$$\begin{array}{rl}
(4x-3)(x+1)=0 & \text{One factor must be zero} \\
4x-3=0 \text{ or } x+1=0 & \text{Set each factor equal to zero} \\
\underline{\mathbf{+3 \quad +3}} \qquad \underline{\mathbf{-1 \quad -1}} & \text{Solve each equation} \\
4x=3 \quad \text{or} \quad x=-1 & \\
\overline{\mathbf{4}} \quad \overline{\mathbf{4}} \qquad & \\
x=\dfrac{3}{4} \text{ or } -1 & \text{Our solution}
\end{array}$$

Another important aspect of the zero factor property is that before we factor, our equation must equal zero. If it does not, we must move terms around so it does equal zero. Although it is not necessary, it will generally be easier to keep our leading term ax^2 positive.

Example 166. Solve the given equation for all possible values of x.

$$
\begin{array}{rl}
x^2 = 8x - 15 & \text{Set equal to zero by moving terms to the left} \\
x^2 - 8x + 15 = 0 & \text{Factor using the } ac\text{-method,} \\
 & \quad \text{multiply to } 15, \text{ add to } -8 \\
(x-5)(x-3) = 0 & \text{The numbers are } -5 \text{ and } -3 \\
x - 5 = 0 \text{ or } x - 3 = 0 & \text{Set each factor equal to zero} \\
\underline{\mathbf{+5 \quad +5}} \qquad \underline{\mathbf{+3 \quad +3}} & \text{Solve each equation} \\
x = 5 \text{ or } 3 & \text{Our solution}
\end{array}
$$

Example 167. Solve the given equation for all possible values of x.

$$
\begin{array}{rl}
(x-7)(x+3) = -9 & \text{Not equal to zero, multiply first} \\
x^2 - 7x + 3x - 21 = -9 & \text{Combine like terms} \\
x^2 - 4x - 21 = -9 & \text{Move } -9 \text{ to other side so equation equals zero} \\
\underline{\mathbf{+9 \quad +9}} & \\
x^2 - 4x - 12 = 0 & \text{Factor using the } ac\text{-method,} \\
 & \quad \text{multiply to } -12, \text{ add to } -4 \\
(x-6)(x+2) = 0 & \text{The numbers are } 6 \text{ and } -2 \\
x - 6 = 0 \text{ or } x + 2 = 0 & \text{Set each factor equal to zero} \\
\underline{\mathbf{+6 \quad +6}} \qquad \underline{\mathbf{-2 \quad -2}} & \text{Solve each equation} \\
x = 6 \text{ or } -2 & \text{Our solution}
\end{array}
$$

Example 168. Solve the given equation for all possible values of x.

$$
\begin{array}{rl}
3x^2 + 4x - 5 = 7x^2 + 4x - 14 & \text{Set equal to zero by} \\
 & \quad \text{moving terms to the right} \\
0 = 4x^2 - 9 & \text{Factor using difference of squares} \\
0 = (2x+3)(2x-3) & \text{One factor must be zero} \\
2x + 3 = 0 \text{ or } 2x - 3 = 0 & \text{Set each factor equal to zero} \\
\underline{\mathbf{-3 \quad -3}} \qquad \underline{\mathbf{+3 \quad +3}} & \text{Solve each equation} \\
2x = -3 \quad \text{or} \quad 2x = 3 & \\
\overline{\mathbf{2}} \quad \overline{\mathbf{2}} \qquad \overline{\mathbf{2}} \quad \overline{\mathbf{2}} & \\
x = -\frac{3}{2} \text{ or } \frac{3}{2} & \text{Our solution}
\end{array}
$$

Most quadratic equations will have two unique real solutions. It is possible, however, to have only one real solution as the next example illustrates.

Example 169. Solve the given equation for all possible values of x.

$$
\begin{array}{rl}
4x^2 = 12x - 9 & \text{Set equal to zero by moving terms to left} \\
4x^2 - 12x + 9 = 0 & \text{Factor using the } ac\text{-method,} \\
 & \quad \text{multiply to } 36, \text{ add to } -12 \\
4x^2 - 6x - 6x + 9 = 0 & \text{Use } -6 \text{ and } -6, \text{ split the linear term} \\
2x(2x-3) - 3(2x-3) = 0 & \text{Factor by grouping} \\
(2x-3)^2 = 0 & \text{A perfect square!} \\
2x - 3 = 0 & \text{Set this factor equal to zero} \\
\underline{\mathbf{+3} \quad \mathbf{+3}} & \text{Solve the equation} \\
2x = 3 & \\
\overline{\mathbf{2}} \quad \overline{\mathbf{2}} & \\
x = \dfrac{3}{2} & \text{Our solution}
\end{array}
$$

As always, it will be important to factor out the GCF first if we have one. This GCF is also a factor, and therefore must also be set equal to zero using the zero factor property. The next example illustrates this.

Example 170. Solve the given equation for all possible values of x.

$$
\begin{array}{rl}
4x^2 = 8x & \text{Set equal to zero by moving the terms to left} \\
 & \text{Be careful, } 4x^2 \text{ and } 8x \text{ are not like terms!} \\
4x^2 - 8x = 0 & \text{Factor out the GCF of } 4x \\
4x(x-2) = 0 & \text{One factor must be zero} \\
4x = 0 \text{ or } x - 2 = 0 & \text{Set each factor equal to zero} \\
\overline{\mathbf{4}} \quad \overline{\mathbf{4}} \qquad \underline{\mathbf{+2} \quad \mathbf{+2}} & \text{Solve each equation} \\
x = 0 \text{ or } 2 & \text{Our solution}
\end{array}
$$

If our polynomial is not a quadratic, as in the next example, we may end up with more than two solutions.

Example 171. Solve the given equation for all possible values of x.

$$
\begin{array}{rl}
2x^3 - 14x^2 + 24x = 0 & \text{Factor out the GCF of } 2x \\
2x(x^2 - 7x + 12) = 0 & \text{Factor with } ac\text{-method,} \\
 & \quad \text{multiply to } 12, \text{ add to } -7 \\
2x(x-3)(x-4) = 0 & \text{The numbers are } -3 \text{ and } -4 \\
2x = 0 \text{ or } x - 3 = 0 \text{ or } x - 4 = 0 & \text{Set each factor equal to zero} \\
\overline{\mathbf{2}} \quad \overline{\mathbf{2}} \qquad \underline{\mathbf{+3} \quad \mathbf{+3}} \qquad \underline{\mathbf{+4} \quad \mathbf{+4}} & \text{Solve each equation} \\
x = 0 \text{ or } 3 \text{ or } 4 & \text{Our solution}
\end{array}
$$

Example 172. Solve the given equation for all possible values of x.

$$\begin{aligned}
6x^2+21x-27&=0 && \text{Factor out the GCF of 3}\\
3(2x^2+7x-9)&=0 && \text{Factor with } ac\text{-method,}\\
& && \quad\text{multiply to } -18\text{, add to } 7\\
3(2x^2+9x-2x-9)&=0 && \text{The numbers are 9 and } -2\\
3[x(2x+9)-1(2x+9)]&=0 && \text{Factor by grouping}\\
3(2x+9)(x-1)&=0 && \text{One factor must be zero}
\end{aligned}$$

$$\begin{array}{lll}
3=0 \ \text{ or } \ 2x+9=0 \ \text{ or } \ x-1=0 & & \text{Set each factor equal to zero}\\
\mathbf{3\neq 0} \qquad \underline{\mathbf{-9 \quad -9}} \qquad \underline{\mathbf{+1 \quad +1}} & & \text{Solve each equation}\\
\qquad\qquad 2x=-9 \quad \text{or} \quad x=1 & & \\
\qquad\qquad \overline{\mathbf{2}} \qquad \overline{\mathbf{2}} & & \\
\qquad\qquad\qquad x=-\dfrac{9}{2} \ \text{ or } \ 1 & & \text{Our solution}
\end{array}$$

In the previous example, the GCF did not have a variable in it. When we set this factor equal to zero we got a false statement. No solutions come from this factor. We can only disregard setting the GCF factor equal to zero if it is a constant.

Just as not all polynomials can be easily factored, all equations cannot be easily solved by factoring. If an equation does not factor easily, we will have to solve it using another method. These other methods are saved for another section.

It is a common question to ask if it is permissible to get rid of the square on the variable x^2 by taking the square root of both sides of the equation. Although it is sometimes possible, there are a few properties of square roots that we have not covered yet, and thus it is more common to inadvertently break a rule of roots that we may not yet be aware of. Because of this, we will postpone a discussion of roots until we see how they can be employed properly to solve quadratic equations. For now, we will advise to *not* take the square root of both sides of an equation!

Square Roots and the Imaginary Number i

Square Roots (L22)

Objective: Simplify and evaluate expressions involving square roots.

Recall that we define a radical (or n^{th} root) as follows.

$$\sqrt[n]{a}=a^{1/n},$$

where a is a nonnegative real number and n a positive integer.

We refer to n as the *index* of the radical and a as the *radicand.* Square roots (when $n = 2$) are the most common type of radical used in mathematics. A square root "un-squares" a number. In other words, if $a^2 = b$, then $\sqrt[2]{b} = a$. This relationship between a square and a square root is similar to the relationship between multiplication and division, as well as the relationship between addition and subtraction. In each case, the two operations are said to be *inverse* operations of each other. The idea behind inverses and the notion of an inverse function is one that will be discussed in detail in a later chapter.

Note that although we have written the index of 2 for the square root of b in the previous paragraph, in general, the index of a square root is usually omitted ($\sqrt[2]{b} = \sqrt{b}$). Using numbers, since $5^2 = 25$ we say the square root of 25 is 5, and write $\sqrt{25} = 5$.

While a great deal more could be said about radicals and how they fit in with the properties of exponents, for now we will focus our attention on properly working with expressions that contain a square root.

The following example gives several square roots.

Example 173.

$\sqrt{0} = 0$	$\sqrt{121} = 11$
$\sqrt{1} = 1$	$\sqrt{625} = 25$
$\sqrt{4} = 2$	$\sqrt{-81} = \text{Undefined}$

The final example of $\sqrt{-81}$ is currently considered to be undefined, since the square root of a negative number does not equal a real number. This is because if we square a positive or a negative number, the answer will be positive, not to mention that $0^2 = 0$. Thus we can only take square roots of nonnegative numbers (positive numbers or zero). In the second part of this section, we will define a method we can use to work with and evaluate negative square roots. For now we will simply say they are undefined.

Not all numbers have a "nice" (or *rational*) square root. For example, if we found $\sqrt{8}$ on our calculator, the answer would be 2.828427124746190097... , and even this number is a rounded approximation of the square root. To be as accurate as possible, we will never use the calculator to find decimal approximations of square roots. Instead we will express roots in simplest radical form. We will do this using a property known as the product rule of radicals (in this case, square roots).

$$\textbf{Product Rule of Square Roots : } \boldsymbol{\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}}$$

More generally,

$$\textbf{Product Rule of Radicals : } \boldsymbol{\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}}$$

We can use the product rule of square roots to simplify an expression such as $\sqrt{180} = \sqrt{36 \cdot 5}$ by splitting it into two roots, $\sqrt{36} \cdot \sqrt{5}$, and simplifying the first root, $6\sqrt{5}$. The trick in this process is being able to recognize that an expression like $\sqrt{180}$ may be rewritten as $\sqrt{36 \cdot 5}$, since $180 = 36 \cdot 5$. In the case of $\sqrt{8}$, we may write $\sqrt{8} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}$.

There are several ways of applying the product rule of square roots. The most common and, with a bit of practice, fastest method is to find perfect squares that divide nicely into the radicand (the number under the radical). This is demonstrated in the next example.

Example 174. Completely simplify the given radical.

$\sqrt{75}$	75 is divisible by 25, a perfect square
$\sqrt{25\cdot 3}$	Split into factors
$\sqrt{25}\cdot\sqrt{3}$	Product rule, take the square root of 25
$5\sqrt{3}$	Our solution

If there is a coefficient in front of the radical to begin with, the problem merely becomes a big multiplication problem, as seen in the next example.

Example 175. Completely simplify the given radical.

$5\sqrt{63}$	63 is divisible by 9, a perfect square
$5\sqrt{9\cdot 7}$	Split into factors
$5\sqrt{9}\cdot\sqrt{7}$	Product rule, take the square root of 9
$5\cdot 3\sqrt{7}$	Multiply coefficients
$15\sqrt{7}$	Our solution

As we simplify radicals using this method it is important to be sure our final answer can not be simplified further, as seen in the next example.

Example 176. Completely simplify the given radical.

$\sqrt{72}$	72 is divisible by 9, a perfect square
$\sqrt{9\cdot 8}$	Split into factors
$\sqrt{9}\cdot\sqrt{8}$	Product rule, take the square root of 9
$3\sqrt{8}$	But 8 is also divisible by a perfect square, 4
$3\sqrt{4\cdot 2}$	Split into factors
$3\sqrt{4}\cdot\sqrt{2}$	Product rule, take the square root of 4
$3\cdot 2\sqrt{2}$	Multiply
$6\sqrt{2}$	Our solution

The previous example could also have been done in fewer steps if we had noticed that $72=36\cdot 2$, but often it can take longer to discover the larger perfect square than to simplify in several steps.

Variables often are part of the radicand as well. When taking the square roots of one (or more) variable(s), we can divide the associated exponent of the variable by two, and write the new exponent outside of the root. For example, $\sqrt{x^{10}}=x^5$. This follows from a familiar property of exponents, shown below.

$$(x^m)^n = x^{mn}$$

Applying this to a square root, we have

$$\sqrt{x^m} = (x^m)^{1/2} = x^{m/2}.$$

So, $\sqrt{x^{10}} = x^{10/2} = x^5$. This makes sense, since

$$\begin{aligned}
\left(x^5\right)^2 &= x^5 \cdot x^5 \\
&= \underbrace{x \cdot x \cdot \ldots \cdot x}_{10 \text{ times}} \\
&= x^{10} \\
&= x^{5 \cdot 2}.
\end{aligned}$$

In summary, when squaring, we multiply the exponent by two. So, when taking a square root, we divide the exponent by two. The following example demonstrates this property.

Example 177. Completely simplify the given radical.

$-5\sqrt{18x^4y^6z^{10}}$	18 is divisible by 9, a perfect square
$-5\sqrt{9 \cdot 2x^4y^6z^{10}}$	Split into factors
$-5\sqrt{9} \cdot \sqrt{2} \cdot \sqrt{x^4} \cdot \sqrt{y^6} \cdot \sqrt{z^{10}}$	Product rule applied to all parts
$-5 \cdot 3x^2y^3z^5\sqrt{2}$	Simplify roots, divide exponents by 2
$-15x^2y^3z^5\sqrt{2}$	Multiply coefficients, Our solution

We can't always nicely divide the exponent on a variable by two, since sometimes we will have a positive remainder. If there is a positive remainder, this means the remainder is left inside the radical, and the whole number portion (or quotient) represents the exponent that should appear outside of the radical. The next example demonstrates this.

Example 178. Completely simplify the given radical.

$\sqrt{20x^5y^9z^6}$	20 is divisible by 4, a perfect square
$\sqrt{4 \cdot 5x^5y^9z^6}$	Split into factors
$\sqrt{4} \cdot \sqrt{5} \cdot \sqrt{x^5} \cdot \sqrt{y^9} \cdot \sqrt{z^6}$	Simplify, divide exponents by 2 Remainder is left inside
$2x^2y^4z^3\sqrt{5xy}$	Our solution

If we focus on the variable y in the previous example, when we divide the exponent 9 by 2, we get a quotient of 4 and a remainder of 1 ($9 = 2 \cdot 4 + 1$). Consequently, $\sqrt{y^9} = y^4\sqrt{y}$. This same idea also applies to x above, since the exponent 5 is odd and therefore will have a remainder of 1. Since the exponent for z is even, it is divisible by 2, and so the radical in our final answer does not contain z.

Introduction to Complex Numbers (L23)

Objective: Simplify expressions involving complex numbers.

In mathematics, when the current number system does not provide the tools to solve the problems the culture is working with, we tend to develop new ways for solving the problem. Throughout history, this has been the case with the need for a number that represents nothing (0), smaller than zero (negatives), between integers (fractions), and between fractions (irrational numbers). This is also the case for square roots of negative numbers. To work with the square root of a negative number, mathematicians have defined what we now know as imaginary and complex numbers.

$$\textbf{Imaginary Number } \boldsymbol{i}: \ \boldsymbol{i^2 = -1} \ \textbf{(thus } \boldsymbol{i = \sqrt{-1}}\textbf{)}$$

Examples of imaginary numbers include $3i, -6i, \frac{3}{5}i$ and $3i\sqrt{5}$. A *complex number* is one that contains both a real and imaginary part, such as $2 + 5i$.

$$\textbf{Complex Number: } \boldsymbol{a + bi}, \text{ where } \boldsymbol{a} \text{ and } \boldsymbol{b} \text{ are real numbers, } \boldsymbol{i = \sqrt{-1}}$$

With this definition, the square root of a negative number will no longer be considered undefined. We now will be able to perform basic operations with the square root of a negative number. First we will consider powers of imaginary numbers. We will do this by manipulating our definition of $i^2 = -1$. If we multiply both sides of the definition by i, the equation becomes $i^3 = -i$. Then if we multiply both sides of the equation again by i, the equation becomes $i^4 = -i^2 = -(-1) = 1$, or simply $i^4 = 1$. Multiplying again by i gives $i^5 = i$. One more time gives $i^6 = i^2 = -1$.

This pattern continues, and we can see a cycle forming. Specifically, as the exponents on i increase, our simplified value for i^n will cycle through the simplified values for i, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$. As there are 4 different possible answers in this cycle, if we divide the exponent n by 4 and consider the remainder, we can easily simplify any power of i by knowing the following four values:

Cyclic Property of Powers of i

$$\begin{aligned} i^0 &= 1 \\ i^1 &= i \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 = i^0 &= 1 \end{aligned}$$

Example 179. Write the given expression as $a+bi$, where a and b are real numbers.

$$\begin{array}{rl} i^{35} & \text{Divide exponent by 4} \\ 35 = 4\cdot 8+3 & \text{Use remainder as exponent for } i \\ i^3 & \text{Simplify} \\ -i & \text{Our solution} \end{array}$$

Example 180. Write the given expression as $a+bi$, where a and b are real numbers.

$$\begin{array}{rl} i^{124} & \text{Divide exponent by 4} \\ 124 = 4\cdot 31+0 & \text{Use remainder as exponent for } i \\ i^0 & \text{Simplify} \\ 1 & \text{Our solution} \end{array}$$

When performing the basic mathematical operations (addition, subtraction, multiplication, division) we may treat i just like any other variable. This means that when adding and subtracting complex numbers we may simply combine like terms.

Example 181. Write the given expression as $a+bi$, where a and b are real numbers.

$$\begin{array}{rl} (2+5i)+(4-7i) & \text{Combine like terms, } \ 2+4 \ \text{ and } \ 5i-7i \\ 6-2i & \text{Our solution} \end{array}$$

It is important to recognize what operation we are applying. A common mistake in the previous example is to view the parentheses and think that one must distribute. The previous example, however, requires addition. So we simply add (or combine) the like terms.

For problems involving subtraction the idea is the same, but we must first remember to distribute the negative to each term in the parentheses.

Example 182. Write the given expression as $a+bi$, where a and b are real numbers.

$$\begin{array}{rl} (4-8i)-(3-5i) & \text{Distribute the negative} \\ 4-8i-3+5i & \text{Combine like terms, } \ 4-3 \ \text{ and } \ -8i+5i \\ 1-3i & \text{Our solution} \end{array}$$

Addition and subtraction may also appear in a single problem.

Example 183. Write the given expression as $a+bi$, where a and b are real numbers.

$$\begin{array}{rl} (5i)-(3+8i)+(-4+7i) & \text{Distribute the negative} \\ 5i-3-8i-4+7i & \text{Combine like terms, } \ 5i-8i+7i \ \text{ and } \ -3-4 \\ -7+4i & \text{Our solution} \end{array}$$

Multiplying two (or more) complex numbers is similar to the multiplication of two binomials with one key exception. In each problem, we will want to simplify our final answer so that it does not contain any power of i greater than or equal to 2. This will always enable us to write our answer in the standard form of $a+bi$. We now show this in general below, remembering that $i^2=-1$.

$$\begin{array}{rl} (c+di)(g+hi) & \text{Expand} \\ cg+chi+dgi+dhi^2 & \text{Simplify,}\ \ i^2=-1 \\ cg+chi+dgi-dh & \text{Combine like terms} \\ (cg-dh)+(ch+dg)i & \text{Our solution, in standard form} \end{array}$$

Here, $cg-dh$ represents the real part a and $ch+dg$ represents the imaginary part b of our resulting complex number $a+bi$.

Next we will see several examples to reinforce the concept. We will begin with the product of two imaginary numbers.

Example 184. Write the given expression as $a+bi$, where a and b are real numbers.

$$\begin{array}{rl} (3i)(7i) & \text{Multiply,}\ \ 3\cdot 7\ \text{ and }\ i\cdot i \\ 21i^2 & \text{Simplify,}\ \ i^2=-1 \\ 21(-1) & \text{Multiply} \\ -21 & \text{Our solution} \end{array}$$

Example 185. Write the given expression as $a+bi$, where a and b are real numbers.

$$\begin{array}{rl} 5i(3i-7) & \text{Distribute} \\ 15i^2-35i & \text{Simplify,}\ \ i^2=-1 \\ 15(-1)-35i & \text{Multiply} \\ -15-35i & \text{Our solution} \end{array}$$

Example 186. Write the given expression as $a+bi$, where a and b are real numbers.

$$\begin{array}{rl} (2-4i)(3+5i) & \text{Expand} \\ 6+10i-12i-20i^2 & \text{Simplify,}\ \ i^2=-1 \\ 6+10i-12i-20(-1) & \text{Multiply} \\ 6+10i-12i+20 & \text{Combine like terms}\ \ 6+20\ \text{ and }\ 10i-12i \\ 26-2i & \text{Our solution} \end{array}$$

Example 187. Write the given expression as $a+bi$, where a and b are real numbers.

$$\begin{array}{rl} (3i)(6i)(2-3i) & \text{Multiply first two monomials} \\ 18i^2(2-3i) & \text{Simplify,}\ \ i^2=-1 \\ 18(-1)(2-3i) & \text{Multiply} \\ -18(2-3i) & \text{Distribute} \\ -36+54i & \text{Our solution} \end{array}$$

Notice that in the previous example we chose to simplify i^2 before distributing. This could also have been done *after* distributing $18i^2$ through $(2 - 3i)$. The resulting expression of $36i^2 - 54i^3$ will then simplify to match our solution above.

Recall that when squaring a binomial such as $(a - b)^2$, we must be careful to expand *completely*, and not forget the inner and outer terms of the product.

$$\begin{aligned}(a-b)^2 &= (a-b)(a-b)\\ &= a^2 - ab - ab + b^2\\ &= a^2 - 2ab + b^2\end{aligned}$$

The next example demonstrates this using complex numbers.

Example 188. Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl} (4-5i)^2 & \text{Rewrite as a product of two binomials}\\ (4-5i)(4-5i) & \text{Expand}\\ 16-20i-20i+25i^2 & \text{Simplify, } \ i^2=-1\\ 16-20i-20i-25 & \text{Combine like terms}\\ -9-40i & \text{Our solution}\end{array}$$

When simplifying rational expressions (fractions) that contain imaginary or complex numbers in a denominator, we will employ the same strategy as that which is used for eliminating square roots from a denominator. This is a logical progression, since we defined i so that $i^2 = \sqrt{-1}$. We refer to this strategy as *rationalizing the denominator*, since the end result will be an expression in which the denominator is a rational number (it contains no radicals).

As we did with complex multiplication, we will first demonstrate the technique generally, followed by several examples.

$$\begin{array}{rl} \dfrac{c+di}{g+hi} & \text{Multiply top and bottom by } \ g-hi\\[2ex] \dfrac{c+di}{g+hi}\cdot\left(\dfrac{g-hi}{g-hi}\right) & \text{Expand numerator and denominator}\\[2ex] \dfrac{cg-chi+dgi-dhi^2}{g^2-ghi+ghi-h^2i^2} & \text{Simplify, } \ i^2=-1\\[3ex] \dfrac{cg-chi+dgi+dh}{g^2-\cancel{ghi}+\cancel{ghi}+h^2} & \text{Combine like terms in top and bottom}\\[2ex] \dfrac{(cg+dh)+(dg-ch)i}{g^2+h^2} & \text{Rewrite as } \ a+bi\\[2ex] \left(\dfrac{cg+dh}{g^2+h^2}\right)+\left(\dfrac{dg-ch}{g^2+h^2}\right)i & \text{Our solution, in standard form}\end{array}$$

Here, $\dfrac{cg+dh}{g^2+h^2}$ represents the real part a and $\dfrac{dg-ch}{g^2+h^2}$ represents the imaginary part b of our resulting complex number $a+bi$. Remember that c, d, g and h all represent real numbers, so our denominator g^2+h^2 is also a real number.

As shown above, the expression that we will typically choose to rationalize with (in this case $g-hi$) is known as the *complex conjugate* to the original denominator $(g+hi)$. When multiplying two complex numbers that are conjugates to one another, the resulting product in our denominator (g^2+h^2) should have no imaginary part.

For our first example, we will start with a denominator which only contains an imaginary part, $0+bi$. In this case, although the complex conjugate would equal $0-bi$, we only need to multiply the numerator and denominator by i, since multiplying by $-bi$ would result in an eventual cancellation of $-b$ from the entire expression.

Example 189. Write the given expression as $a+bi$, where a and b are real numbers.

$$\frac{7+3i}{-5i} \quad \text{A monomial in denominator, multiply by } i$$

$$\frac{7+3i}{-5i}\left(\frac{i}{i}\right) \quad \text{Distribute } i \text{ in numerator}$$

$$\frac{7i+3i^2}{-5i^2} \quad \text{Simplify } i^2=-1$$

$$\frac{7i+3(-1)}{-5(-1)} \quad \text{Multiply}$$

$$\frac{7i-3}{5} \quad \text{Simplify, split up fraction}$$

$$\frac{7i}{5}-\frac{3}{5} \quad \text{Rewrite as } a+bi$$

$$-\frac{3}{5}+\frac{7}{5}i \quad \text{Our solution}$$

As shown in the previous example, a solution for such problems can be written several different ways, for example $\dfrac{-3+7i}{5}$ or $-\dfrac{3}{5}+\dfrac{7}{5}i$. Although both answers are generally accepted, we will keep our final answers consistent with the definition of a complex number:

$$a+bi = (\text{Real part}) \; + \; (\text{Imaginary part})i.$$

Example 190. Write the given expression as $a+bi$, where a and b are real numbers.

$$\frac{2-6i}{4+8i}$$ Binomial in denominator, multiply by conjugate, $4-8i$

$$\frac{2-6i}{4+8i}\left(\frac{4-8i}{4-8i}\right)$$ Expand the numerator, denominator is a difference of two squares

$$\frac{8-16i-24i+48i^2}{16-64i^2}$$ Simplify $i^2=-1$

$$\frac{8-16i-24i+48(-1)}{16-64(-1)}$$ Multiply

$$\frac{8-16i-24i-48}{16+64}$$ Combine like terms

$$\frac{-40-40i}{80}$$ Reduce, factor out 40 and divide

$$\frac{-1-i}{2}$$ Rewrite as $a+bi$

$$-\frac{1}{2}-\frac{1}{2}i$$ Our solution

By rewriting $\sqrt{-1}$ as i, we can now simplify square roots with negatives underneath. We will use the product rule and simplify the negative as a factor of negative one. This is shown in the following examples.

Example 191. Write the given expression as $a+bi$, where a and b are real numbers.

$$\sqrt{-16}$$ Consider the negative as a factor of -1

$$\sqrt{-1\cdot 16}$$ Take each root, square root of -1 is i

$$4i$$ Our solution

Example 192. Write the given expression as $a+bi$, where a and b are real numbers.

$$\sqrt{-24}$$ Find perfect square factors . Factor out -1

$$\sqrt{-1\cdot 4\cdot 6}$$ Square root of -1 is i, square root of 4 is 2

$$2i\sqrt{6}$$ Move i over

$$(2\sqrt{6})i$$ Our solution

When simplifying complex radicals, it is important that we take the -1 out of the radical (as an i) before we combine radicals.

Notice also that in the previous example our final answer is $(2\sqrt{6})i$ and not $2\sqrt{6i}$. Although the parentheses are not technically needed, they are included because there is a subtle mathematical difference between these two values, since having i *underneath* a square root ($\sqrt{6i}$) is not equivalent to having it *beside* the square root ($\sqrt{6}i$). This common mistake can be easily avoided by taking care not to extend the square root too far when writing our final answer. The parentheses are simply an added precaution. The same care must be made in order to distinguish an expression like $\sqrt{6}x$ from $\sqrt{6x}$.

Example 193. Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl}
\sqrt{-6}\sqrt{-3} & \text{Simplify the negatives, bringing } i \text{ out of radicals} \\
(i\sqrt{6})(i\sqrt{3}) & \text{Multiply, } i^2 = -1 \\
-\sqrt{18} & \text{Simplify the radical} \\
-\sqrt{9 \cdot 2} & \text{Take square root of } 9 \\
-3\sqrt{2} & \text{Our solution}
\end{array}$$

Lastly, when reducing fractions that involve i, as is often the case, we must take extra care to properly simplify and avoid any common mistakes. This is demonstrated in the following example.

Example 194. Write the given expression as $a + bi$, where a and b are real numbers.

$$\begin{array}{rl}
\dfrac{-15 - \sqrt{-200}}{20} & \text{We will simplify the radical first} \\
\\
\sqrt{-200} & \text{Find perfect square factors . Factor out } -1 \\
\sqrt{-1 \cdot 100 \cdot 2} & \text{Take square root of } -1 \text{ and } 100 \\
10i\sqrt{2} & \text{Put this back into original expression} \\
\\
\dfrac{-15 - 10i\sqrt{2}}{20} & \text{Factor out } 5 \text{ and divide} \\
\dfrac{-3 - 2i\sqrt{2}}{4} & \text{Simplify answer, split up fraction} \\
-\dfrac{3}{4} - \dfrac{2i\sqrt{2}}{4} & \text{Reduce, move } i \text{ to side} \\
-\dfrac{3}{4} - \dfrac{\sqrt{2}}{2}i & \text{Our solution}
\end{array}$$

By using $i = \sqrt{-1}$ we will be able to simplify expressions and solve problems that we could not before. In the next few sections, we will see how this will enable us to better understand quadratic equations and their graphs.

Vertex Form and Graphing (L24)

The Vertex Form

Objective: Express a quadratic equation in vertex form.

Recall the two forms of a quadratic equation, shown below. In both forms, assume $a \neq 0$.

Standard Form: $y = ax^2 + bx + c$, where a, b, and c are real numbers

Vertex Form: $y = a(x - h)^2 + k$, where a, h, and k are real numbers

Unlike the standard form, a quadratic equation written in vertex form allows for immediate recognition of the vertex (h, k), which will always coincide with either a maximum (if $a < 0$) or a minimum (if $a > 0$) on the accompanying graph, called a parabola. Additionally, using the vertex form, we can easily identify the *axis of symmetry* for the parabola, which is a vertical line $x = h$ that passes through the x-coordinate of the vertex and "splits" the graph into two identical halves.

When graphing parabolas, it will help to think of the axis of symmetry as a vertical line over which either half of the graph could be "folded", to produce the other half. This will allow us to reflect (by symmetry) any point on the parabola to the other side of the axis of symmetry, and identify another point on the graph. As a result, both points will have the same y-coordinate, and will be (horizontally) equidistant from the axis of symmetry. By reflecting points about the axis of symmetry, we can graph not just one, but two points on the graph, for every single value of x that we plug into the given equation.

Example 195. Consider $y = -2(x + 1)^2 + 3$.

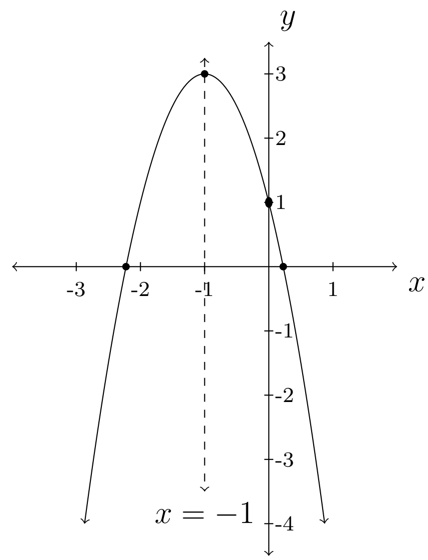
In this example we can see immediately that the vertex is at $(-1, 3)$. It is important that we not overlook the negative value for h. The axis of symmetry, passes through the x-coordinate for the vertex, $x = -1$.

Now to find more points on the parabola we can plug in $x = 0$. We can see that $y = -2(0 + 1)^2 + 3 = 1$, so $(0, 1)$ is a point on our parabola.

Since the point we just located sits one unit to the right of the axis of symmetry, we also know that the point $(-2, 1)$, sitting one unit to the left of the axis of symmetry will also be a point on our graph. We can always check this by plugging $x = -2$ into the equation and solving for y.

Similarly, we can plug in $x = 1$, a coordinate that is two units to the right of the axis of symmetry and get a y-coordinate of -5.

Thus an x-coordinate two units left of the axis, $x = -3$, will also yield a y-coordinate of -5. The accompanying graph shows our parabola, with the axis of symmetry appearing as a dashed vertical line at $x = -1$.



We began the discussion of vertex form in the introductory section of this chapter. It follows naturally to learn how to transform a quadratic equation that is given in standard form into one written in vertex form.

If $y = ax^2 + bx + c$ $(a \neq 0)$, we can identify the x-coordinate for the vertex (and consequently the equation for the axis of symmetry) using the following formula.

$$h = -\frac{b}{2a}$$

After identifying h, we can determine based upon the sign of the leading coefficient a whether the vertex will be a maximum (if a is negative, $a < 0$) or a minimum (if a is positive, $a > 0$). The equation for the vertical line $x = h$ will be our axis of symmetry.

Finally, we know that the y-coordinate for our vertex must occur somewhere on the axis of symmetry. This can easily be found by plugging $x = h$ back into the given equation for our quadratic, and simplifying to find the y-coordinate, which we will relabel as k.

Once we have h and k, we can use them, along with a, to write the vertex form for our quadratic,

$$y = a(x - h)^2 + k.$$

The following examples will clearly demonstrate this process.

Example 196. Identify the vertex and axis of symmetry for the parabola represented by the given quadratic equation.

$$\begin{array}{rl}
y = x^2 + 8x - 12 & \text{Given an equation in standard form} \\
a = 1, \quad b = 8, \quad c = -12 & \text{Identify } a, b, \text{ and } c \\
h = -\dfrac{b}{2a} = -\dfrac{8}{2(1)} = -4 & \text{Identify } h \\
x = -4 & \text{Use } h \text{ for axis of symmetry, a vertical line} \\
k = (-4)^2 + 8(-4) - 12 & \text{Plug in } h \text{ to find } k \\
k = 16 - 32 - 12 = -28 & \text{Simplify} \\
(-4, -28) & \text{Write the vertex as an ordered pair } (h, k)
\end{array}$$

Example 197. Identify the vertex and axis of symmetry for the parabola represented by the given quadratic equation.

$$\begin{array}{rl}
y = -3x^2 + 6x - 1 & \text{Given an equation in standard form} \\
a = -3, \quad b = 6, \quad c = -1 & \text{Identify } a, b, \text{ and } c \\
h = -\dfrac{b}{2a} = -\dfrac{6}{2(-3)} = 1 & \text{Identify } h \\
x = 1 & \text{Use } h \text{ for axis of symmetry, a vertical line} \\
k = -3(1)^2 + 6(1) - 1 & \text{Plug in } h \text{ to find } k \\
k = -3 + 6 - 1 = 2 & \text{Simplify} \\
(1, 2) & \text{Write the vertex as an ordered pair } (h, k)
\end{array}$$

Example 198. Identify the vertex and axis of symmetry for the parabola represented by the given quadratic equation.

$$\begin{array}{rl}
y = -x^2 - 12 & \text{Given an equation in standard form} \\
a = -1, \quad b = 0, \quad c = -12 & \text{Identify } a, b, \text{ and } c \\
h = -\dfrac{b}{2a} = -\dfrac{0}{2(-1)} = 0 & \text{Identify } h \\
x = 0 & \text{Use } h \text{ for axis of symmetry, a vertical line} \\
k = -(0)^2 - 12 & \text{Plug in } h \text{ to find } k \\
k = 0 - 12 = -12 & \text{Simplify} \\
(0, -12) & \text{Write the vertex as an ordered pair } (h, k)
\end{array}$$

There is a more algebraic (and complicated) method of transforming a quadratic equation given in standard form into one that is in vertex form, known as *completing the square*. This method will be explained in detail towards the end of the chapter.

We will also see how the vertex form can be particularly useful when solving a quadratic equation, in order to identify the x-intercepts of the corresponding parabola. Solving a quadratic equation using the vertex form is know as the method of *extracting square roots*, and will be seen once we have had a thorough discussion of square roots, as well as complex numbers.

Graphing Quadratics

Objective: Graph equations in both standard and vertex forms.

Up until now, we have discussed the general shape of the graph of a quadratic equation (known as a *parabola*), but have only seen a few examples. Furthermore, most of our examples have only identified the vertex of the parabola, and perhaps an $x-$ or $y-$intercept of the graph. Although these examples have been able to show us the general shape of each graph (where it is centered, whether it opens up or down, whether it is narrow or wide), our steps for obtaining each graph have not followed a standard procedure. Here, we will define that procedure more precisely, and provide a few examples for reinforcement.

One way that we can always build a picture of the general shape of a graph is to make a table of values, as we will do in our first example.

Example 199. Sketch a graph of the quadratic equation $y = x^2 - 4x + 3$ by making a table of values and plotting points on the graph.

We will test five values to get an idea of the shape of the graph.

x	0	1	2	3	4
y					

$$\begin{aligned}
y = (0)^2 + 4(0) + 3 = 0 - 0 + 3 &= 3 && \text{Plug in 0 for } x \text{ and evaluate.}\\
y = (1)^2 - 4(1) + 3 = 1 - 4 + 3 &= 0 && \text{Plug in 1 for } x \text{ and evaluate.}\\
y = (2)^2 - 4(2) + 3 = 4 - 8 + 3 &= -1 && \text{Plug in 2 for } x \text{ and evaluate.}\\
y = (3)^2 - 4(3) + 3 = 9 - 12 + 3 &= 0 && \text{Plug in 3 for } x \text{ and evaluate.}\\
y = (4)^2 - 4(4) + 3 = 16 - 16 + 3 &= 3 && \text{Plug in 4 for } x \text{ and evaluate.}
\end{aligned}$$

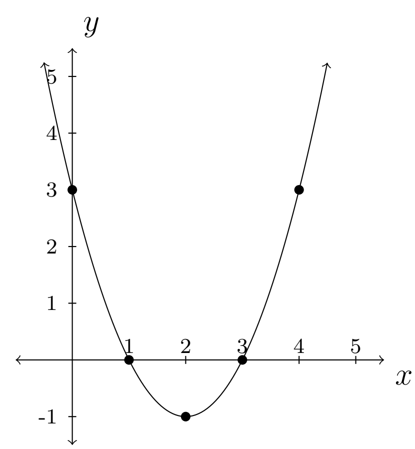
Our completed table is below.

x	0	1	2	3	4
y	3	0	-1	0	3

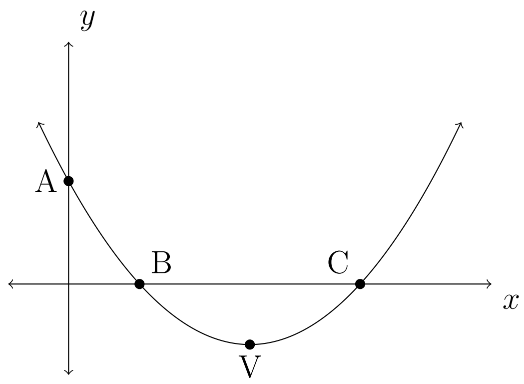
Plot the points on the $xy-$plane.

Plot the points $(0, 3), (1, 0), (2, -1), (3, 0)$, and $(4, 3)$.

Connect the dots with a smooth curve.



The above method to graph a parabola works for any equation, however, it can be very difficult to find a sufficient collection of points in order to identify the overall shape of the complete graph. For this reason, we will now formally identify several key points on the graph of a parabola, which will enable us to always determine a complete graph. These points are the $y-$intercept, $x-$intercepts, and the vertex (h, k).



Point A: y-intercept; where the graph crosses the vertical y-axis (when $x = 0$).

Points B and C: x-intercepts; where the graph crosses the horizontal x-axis (when $y = 0$)

Point V: vertex (h, k); The point of the minimum (or maximum) value, where the graph changes direction.

We will use the following method to find each of the key points on our parabola.

Steps for graphing a quadratic in standard form, $y = ax^2 + bx + c$.

1. Identify and plot the vertex: $h = -\dfrac{b}{2a}$. Plug h into the equation to find k. Resulting point is (h, k).
2. Identify and plot the y-intercept: Set $x = 0$ and solve. The y-intercept will correspond to the constant term c. Resulting point is $(0, c)$.
3. Identify and plot the x-intercept(s): Set $y = 0$ and solve for x. Depending on the expression, we will end up with zero, one or two $x-$intercepts.

Important: Up until now, we have only discussed how to solve a quadratic equation for x by factoring. If an expression is not easily factorable, we may not be able to identify the x-intercepts. Soon, we will learn of two additional methods for finding x-intercepts, which will prove especially useful, when an equation is not easily factorable.

After plotting these points we can connect them with a smooth curve to find a complete sketch of our parabola!

Example 200. Provide a complete sketch of the equation $y = x^2 + 4x + 3$.

$$\begin{aligned}
y &= x^2 + 4x + 3 && \text{Find the key points}\\
\\
h &= -\frac{4}{2(1)} = -\frac{4}{2} = -2 && \text{To find the vertex, use } h = -\frac{b}{2a}\\
k &= (-2)^2 + 4(-2) + 3 && \text{Plug } h \text{ into the equation to find } k\\
k &= 4 - 8 + 3 && \text{Evaluate}\\
k &= -1 && \text{The } y\text{-coordinate of the vertex}\\
&(-2, -1) && \text{Vertex as a point}\\
\\
y &= 3 && (0, c) \text{ is the } y\text{-intercept}
\end{aligned}$$

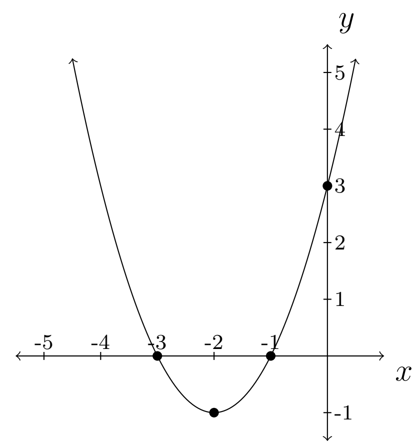
$0 = x^2 + 4x + 3$	To find the x-intercept we solve the equation
$0 = (x+3)(x+1)$	Factor
$x+3=0$ and $x+1=0$	Set each factor equal to zero
$\underline{\mathbf{-3 \quad -3}} \qquad \underline{\mathbf{-1 \quad -1}}$	Solve each equation
$x=-3$ and $x=-1$	Our x-intercepts

Graph the y-intercept at $(0,3)$,

the x-intercepts at $(-3,0)$ and $(-1,0)$,

and the vertex at $(-2,-1)$.

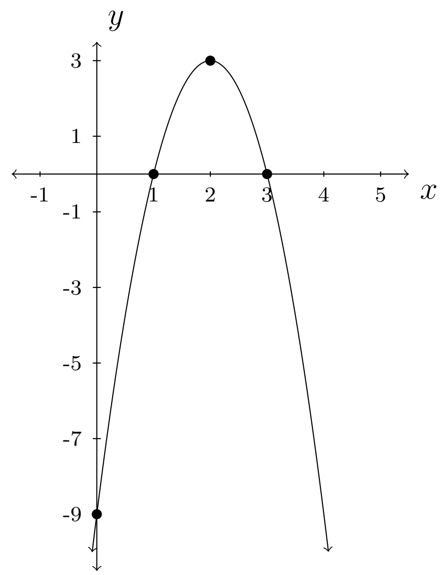
Connect the dots with a smooth curve in a 'U'-shape to get our parabola.



Remember that if $a > 0$, then our parabola will open upwards, as in the previous example. In our next example, $a < 0$, and the resulting parabola will open downwards.

Example 201. Provide a complete sketch of the equation $y = -3x^2 + 12x - 9$.

$y = -3x^2 + 12x - 9$	Find key points
$h = -\dfrac{12}{2(-3)} = -\dfrac{12}{-6} = 2$	To find the vertex, use $h = -\dfrac{b}{2a}$
$k = -3(2)^2 + 12(2) - 9$	Plug h into the equation to find k
$k = -3(4) + 24 - 9$	Evaluate
$k = 3$	The y-coordinate of the vertex
$(2,3)$	Vertex as a point
$y = -9$	$(0,c)$ is the y-intercept
$0 = -3x^2 + 12x - 9$	To find the x-intercept we solve the equation
$0 = -3(x^2 - 4x + 3)$	Factor out GCF first
$0 = -3(x-3)(x-1)$	Factor remaining trinomial
$x-3=0$ and $x-1=0$	Set each factor with a variable equal to zero
$\underline{\mathbf{+3 \quad +3}} \qquad \underline{\mathbf{+1 \quad +1}}$	Solve each equation
$x=3$ and $x=1$	Our x-intercepts



Graph the y-intercept at $(0, -9)$,

the x-intercepts at $(3, 0)$ and $(1, 0)$,

and the vertex at $(2, 3)$.

Connect the dots with a smooth curve in a 'U'-shape to get our parabola.

Remember that the graph of any quadratic is a parabola with the same 'U'-shape (opening up or down). If we plot our points and we cannot connect them in the correct 'U'-shape, then one or more of our points is likely to be incorrect. If this happens, a simple check of our calculations should identify where any mistakes were made! Each of our examples have involved quadratics that were easily factorable. Although we can still graph quadratics such as $y = x^2 - 3$ without actually identifying the $x-$intercepts, being able to identify them by solving $x^2 - 3 = 0$ and other more involved quadratic equations for x is a skill that we will eventually come to master.

The Method of Extracting Square Roots

Solve by Square Roots (L25)

Objective: Solve quadratic equations of the form $ax^2 + c = 0$ by introducing a square root.

Up until now, when attempting to solve an equation such as $x^2 - 4 = 0$, we have had no choice but to factor the expression on the left and set each factor equal to zero.

Example 202. Solve the given equation for all possible values of x.

$$\begin{aligned} x^2 - 4 &= 0 && \text{Factor, difference of two squares} \\ (x-2)(x+2) &= 0 && \text{Use Zero Factor Property} \\ x - 2 = 0 \text{ or } x + 2 &= 0 && \text{Solve} \\ x = 2 \quad \text{or} \quad &-2 && \text{Our solution} \end{aligned}$$

As an alternative method, in this subsection, we will look at solving the expression $ax^2+c = 0$ using an alternative method. Instead of attempting to factor the expression, we will introduce a square root, when solving for x. In each case, when faced with such an expression, our solution can be reached by applying the following three stops, in the specified order.

Solve $ax^2 + c = 0$.

Step	Equation
1. Subtract c.	$ax^2 = -c$
2. Divide by a.	$x^2 = -\frac{c}{a}$
3. Take a square root.	$x = \pm\sqrt{-\frac{c}{a}}$

Example 203. Solve the given equation for all possible values of x.

$$\begin{array}{rl} x^2 - 4 = 0 & \text{Add } 4 \\ x^2 = 4 & \text{Take a square root} \\ x = \pm 2 & \text{Our solution} \end{array}$$

Recall that the values $x = 2$ and $x = -2$ are known as the *zeros* or *roots* of the equation $y = x^2 - 4$. Observe that the graphical interpretation of a zero is an $x-$intercept (when $y = 0$). In this case, the $x-$intercepts of the resulting parabola are at $(2, 0)$ and $(-2, 0)$.

Example 204. Solve the given equation for all possible values of x.

$$\begin{array}{rl} 5x^2 + 60 = 0 & \text{Subtract } 60 \\ 5x^2 = -60 & \text{Divide by } 5 \\ x^2 = -12 & \text{Take a square root} \\ x = \pm\sqrt{-12} & \text{Imaginary roots; Simplify} \\ x = \pm 2\sqrt{3}i & \text{Our solution} \end{array}$$

In this example, we see that our two solutions, $x = 2\sqrt{3}i$ and $x = -2\sqrt{3}i$ are not real. Hence, the corresponding parabola for $y = 5x^2 + 60$ will have no $x-$intercepts.

In what follows, we will refer to the more general form of this method as *extracting square roots.*

Extracting Square Roots (L26)

Objective: Solve quadratic equations using the method of extracting square roots.

We will now introduce a new technique for identifying the zeros of a quadratic equation, known as the method of *extracting square roots.* The method of extracting square roots will only be employed once we have identified the vertex form for a given quadratic, $y = a(x-h)^2 + k$. The general steps for the method are shown below, and the requirement of the vertex form will be essential.

Example 205. Determine the zeros of the quadratic equation $y = ax^2 + bx + c$, where $a \neq 0$.

First obtain the vertex form: $h = -\dfrac{b}{2a}$, set $x = h$ to find k.

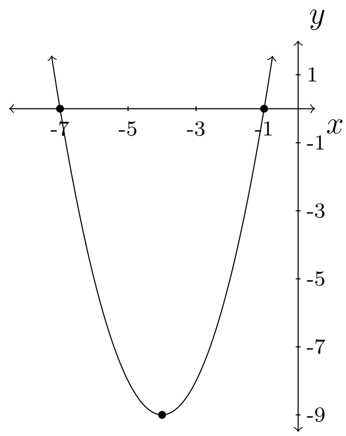
$$\begin{array}{rl}
a(x-h)^2 + k = 0 & \text{Vertex form} \\
\underline{\boldsymbol{-k} \quad \boldsymbol{-k}} & \text{Subtract } k \text{ from both sides} \\
\dfrac{a(x-h)^2}{\boldsymbol{a}} = \dfrac{-k}{\boldsymbol{a}} & \text{Divide both sides by } a \\
(x-h)^2 = -\dfrac{k}{a} & \\
\sqrt{(x-h)^2} = \pm\sqrt{-\dfrac{k}{a}} & \text{Take square root of both sides} \\
& \quad \text{to extract radicand, } x-h \\
x - h = \pm\sqrt{-\dfrac{k}{a}} & \\
\underline{\boldsymbol{+h}} \qquad \underline{\boldsymbol{+h}} & \text{Add } h \text{ to both sides} \\
x = h \pm \sqrt{-\dfrac{k}{a}} & \text{Our solution}
\end{array}$$

In the previous example, there are two important points to consider. First is the introduction of the square root into the equation. This step is the reason behind the name of the method, and its success hinges upon the fact that the vertex form contains a single instance of the variable x. Unlike with the vertex form, if we were to introduce a square root directly to the equation $ax^2 + bx + c = 0$ (using the standard form), we would immediately reach a dead end, and be unable to simplify the resulting equation. This is primarily because we cannot combine the "unlike" terms ax^2 and bx, and we cannot split up sums (and differences) of terms underneath a square root.

Additionally, it is critical that we include a '$\pm$' on the right side of the equation once the square root has been introduced. The justification for this follows from the fact that there are always two values (one positive and one negative) that will equal the value underneath a square root (assuming that value is nonzero, since $\sqrt{0} = 0$). For example, $\sqrt{4} = \pm 2$ and $\sqrt{-9} = \pm 3i$.

We now present a few examples that demonstrate the method, as well as some of the possibilities for the number of zeros, and consequently, the number of $x-$intercepts of the corresponding graph.

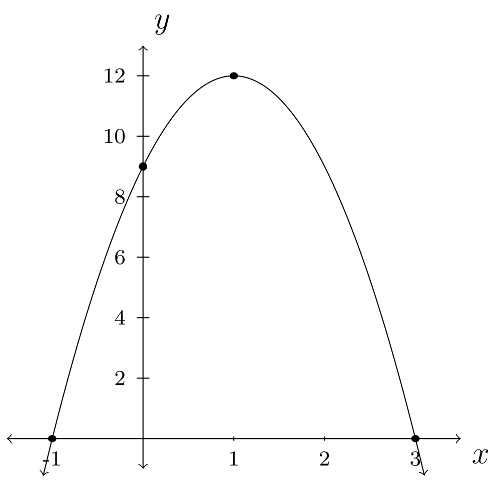
Example 206. Use the method of extracting square roots to find the zeros of the equation $y = (x+4)^2 - 9$.



$$\begin{array}{rl}
0 = (x+4)^2 - 9 & \text{Set equal to zero and solve} \\
\underline{\mathbf{+9} \qquad\qquad \mathbf{+9}} & \text{Isolate the square} \\
9 = (x+4)^2 & \\
\pm\sqrt{9} = \sqrt{(x+4)^2} & \text{Square root both sides} \\
\pm 3 = x + 4 & \text{Solve for } x \\
\underline{\mathbf{-4}} \quad \underline{\mathbf{-4}} & \text{Subtract } 4 \\
x = \pm 3 - 4 & \text{Two solutions} \\
x = 3 - 4 \ \Rightarrow x = -1 & \text{One solution} \\
x = -3 - 4 \Rightarrow x = -7 & \text{The other solution}
\end{array}$$

Our zeros are $x = -7$ and -1. The corresponding $x-$intercepts are at $(-7, 0)$ and $(-1, 0)$.

Example 207. Use the method of extracting square roots to find the zeros of the equation $y = -3(x-1)^2 + 12$.



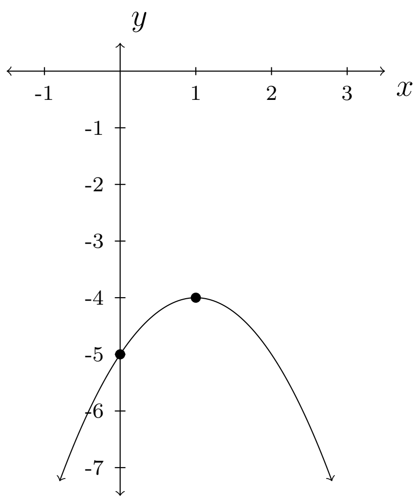
$$\begin{array}{rl}
0 = -3(x-1)^2 + 12 & \text{Set equal to zero and solve} \\
\underline{\mathbf{-12} \qquad\qquad \mathbf{-12}} & \text{Subtract } 12 \\
-12 = -3(x-1)^2 & \text{Isolate the square,} \\
\overline{\mathbf{-3}} \qquad \overline{\mathbf{-3}} & \quad \text{divide both sides by } -3 \\
4 = (x-1)^2 & \\
\pm\sqrt{4} = \sqrt{(x-1)^2} & \text{Square root both sides} \\
\pm 2 = x - 1 & \text{Solve for } x \\
\underline{\mathbf{+1}} \quad \underline{\mathbf{+1}} & \text{Add } 1 \\
x = \pm 2 + 1 & \text{Two solutions} \\
x = 1 - 2 \Rightarrow x = -1 & \text{One solution} \\
x = 1 + 2 \Rightarrow x = 3 & \text{The other solution}
\end{array}$$

Our two zeros are $x = -1$ and $x = 3$.

In some cases, the introduction of a square root results in an imaginary number. This scenario coincides with our corresponding parabola having no $x-$intercepts. In the previous example, if we were to change the sign of k from $+12$ to -12, the corresponding parabola would still open downwards, while having a vertex at $(1, -12)$, located below the $x-$axis. This will result in the appearance of a $\sqrt{-4} = 2i$, rather than a $\sqrt{4}$, in our solution. Consequently, there will be no real zeros for the equation and no $x-$intercepts on its graph.

We conclude this section with a final example, which will also result in no real zeros.

Example 208. Use the method of extracting square roots to find the zeros of the equation $y = -1(x-1)^2 - 4$.



$0 = -1(x-1)^2 - 4$	Set equal to zero and solve
$\underline{\mathbf{+4}} \qquad \underline{\mathbf{+4}}$	Add 4
$4 = -1(x-1)^2$	Isolate the square,
$\overline{\mathbf{-1}} \qquad \overline{\mathbf{-1}}$	divide both sides by -1
$-4 = (x-1)^2$	
$\pm\sqrt{-4} = \sqrt{(x-1)^2}$	Square root both sides
$\pm 2i = x - 1$	Solve for x
$\underline{\mathbf{+1}} \qquad \underline{\mathbf{+1}}$	Add 1
$x = \pm 2i + 1$	Two solutions
$x = 1 - 2i$	One solution
$x = 1 + 2i$	The other solution

Once again, the negative appearing under the square root results in two complex zeros (no real zeros). Graphically, the function never touches or crosses the $x-$axis.

Completing the Square

Objective: Solve quadratic equations by completing the square.

In this section, we will introduce a method for obtaining the vertex form of a quadratic function from the standard form, without having to rely on the vertex formula $h = -\dfrac{b}{2a}$. This method is known as *completing the square*. To complete the square, and convert a quadratic expression $ax^2 + bx + c$ from standard form to the vertex form $a(x-h)^2 + k$ (without our prior knowledge of the relationship between h, a and b), we will first start by considering the expression $ax^2 + bx$.

Observe that if a quadratic is of the form $x^2 + bx + c$, and *is* a perfect square, the constant term, c, can be found by the formula $\left(\frac{1}{2} \cdot b\right)^2$. This is shown in the following examples. In each example, we will find the number needed to complete the perfect square, and then factor it.

Example 209. Identify the constant term c that is needed to factor the given trinomial as a perfect square.

$$\begin{array}{rl} x^2+8x+c & c=\left(\frac{1}{2}\cdot b\right)^2 \text{ and our } b=8 \\ \left(\frac{1}{2}\cdot 8\right)^2=4^2=16 & \text{The necessary constant term is } 16 \\ x^2+8x+16 & \text{Our desired trinomial; factor} \\ (x+4)^2 & \text{Our solution} \end{array}$$

Example 210. Identify the constant term c that is needed to factor the given trinomial as a perfect square.

$$\begin{array}{rl} x^2-7x+c & c=\left(\frac{1}{2}\cdot b\right)^2 \text{ and our } b=7 \\ \left(\frac{1}{2}\cdot 7\right)^2=\left(\frac{7}{2}\right)^2=\frac{49}{4} & \text{The necessary constant term is } \frac{49}{4} \\ x^2-7x+\frac{49}{4} & \text{Our desired trinomial; factor} \\ \left(x-\frac{7}{2}\right)^2 & \text{Our solution} \end{array}$$

Example 211. Identify the constant term c that is needed to factor the given trinomial as a perfect square.

$$\begin{array}{rl} x^2+\frac{5}{3}x+c & c=\left(\frac{1}{2}\cdot b\right)^2 \text{ and our } b=\frac{5}{3} \\ \left(\frac{1}{2}\cdot \frac{5}{3}\right)^2=\left(\frac{5}{6}\right)^2=\frac{25}{36} & \text{The necessary constant term is } \frac{25}{36} \\ x^2+\frac{5}{3}x+\frac{25}{36} & \text{Our desired trinomial; factor} \\ \left(x+\frac{5}{6}\right)^2 & \text{Our solution} \end{array}$$

The process demonstrated in the previous examples may be used to obtain the vertex form of a quadratic. The following set of steps describes the process used to complete the square. Since all three of the previous examples contained a leading coefficient of $a=1$, an example where $a\neq 1$ has been included below to illustrate the special care that must be taken in this case.

Expression	$3x^2+18x-6$
1. Separate constant term from variables	$(3x^2+18x)-6$
2. Factor out a from each term in parentheses	$3(x^2+6x)-6$
3. Determine value to complete the square: $\left(\frac{1}{2}\cdot b\right)^2$	$\left(\frac{1}{2}\cdot 6\right)^2 = 3^2 = 9$
4. Add & subtract value to expression in parentheses	$3(x^2+6x+9-9)-6$
5. Separate subtracted value from other three terms, making sure to multiply by a	$3(x^2+6x+9)-3(9)-6$
6. Combine constant terms outside parentheses	$3(x^2+6x+9)^2-27-6$
7. Factor remaining trinomial	$3(x+3)^2-33$

Example 212. Use the method of completing the square to solve the given equation.

$$\begin{aligned}
4x^2+40x+51&=0 && \text{Equation in standard form}\\
(4x^2+40x)+51&=0 && \text{Separate constant term}\\
4(x^2+10x)+51&=0 && \text{Factor out } a\\
\left(\frac{1}{2}\cdot 10\right)^2&=5^2=25 && \text{Complete the square : find } \left(\frac{1}{2}\cdot b\right)^2\\
4(x^2+10x+25-25)+51&=0 && \text{Add and subtract } 25 \text{ inside parentheses}\\
4(x^2+10x+25)-4(25)+51&=0 && \text{Separate trinomial}\\
4(x^2+10x+25)^2-100+51&=0 && \text{Simplify: combine constant terms,}\\
& && \quad\text{factor trinomial}\\
4(x+5)^2-49&=0 && \text{Solve by extracting square roots}\\
(x+5)^2&=\frac{49}{4} && \text{Isolate the square}\\
\sqrt{(x+5)^2}&=\pm\sqrt{\frac{49}{4}} && \text{Square root both sides}\\
x+5&=\pm\frac{7}{2} && \text{Subtract } 5 \text{ from both sides}\\
\underline{\mathbf{-5}}\quad&\quad\underline{\mathbf{-5}} && \\
x&=-5\pm\frac{7}{2} && \\
x&=-\frac{17}{2} \text{ or } -\frac{3}{2} && \text{Our solution}
\end{aligned}$$

Example 213. Use the method of completing the square to solve the given equation.

$$x^2 - 3x - 2 = 0 \quad \text{Equation in standard form}$$
$$(x^2 - 3x) - 2 = 0 \quad \text{Separate constant term}$$
Leading coefficient is $a = 1$

$$\left(\frac{1}{2} \cdot -3\right)^2 = \left(-\frac{3}{2}\right)^2 = \frac{9}{4} \quad \text{Complete the square : find } \left(\frac{1}{2} \cdot b\right)^2$$
$$\left(x^2 - 3x + \frac{9}{4} - \frac{9}{4}\right) - 2 = 0 \quad \text{Add and subtract } \frac{9}{4} \text{ inside parentheses}$$
$$\left(x^2 - 3x + \frac{9}{4}\right) - \frac{9}{4} - 2 = 0 \quad \text{Separate trinomial}$$
$$\left(x^2 - 3x + \frac{9}{4}\right) - \frac{9}{4} - 2 = 0 \quad \text{Simplify: combine constant terms,}$$
factor trinomial
$$\left(x - \frac{3}{2}\right)^2 - \frac{17}{4} = 0 \quad \text{Solve by extracting square roots}$$
$$\left(x - \frac{3}{2}\right)^2 = \frac{17}{4} \quad \text{Isolate the square}$$
$$\sqrt{\left(x - \frac{3}{2}\right)^2} = \pm\sqrt{\frac{17}{4}} \quad \text{Square root both sides}$$
$$x - \frac{3}{2} = \pm\frac{\sqrt{17}}{2} \quad \text{Reduce square root;}$$
$$\underline{+\mathbf{\frac{3}{2}}} \qquad \underline{+\mathbf{\frac{3}{2}}} \quad \text{Add } \frac{3}{2} \text{ to both sides}$$
$$x = \frac{3}{2} \pm \frac{\sqrt{17}}{2}$$
$$x = \frac{3 + \sqrt{17}}{2} \text{ or } \frac{3 - \sqrt{17}}{2} \quad \text{Our solution}$$

As the previous example shows, completing the square when $a = 1$ can be seen as slightly easier than when $a \neq 1$. Our last example demonstrates how we can more also handle the case when $a \neq 1$ early on in our solution, by simply dividing the equation by a.

Example 214. Use the method of completing the square to solve the given equation.

$$3x^2 - 2x + 7 = 0 \qquad \text{Equation in standard form}$$

$$\overline{\mathbf{3}} \quad \overline{\mathbf{3}} \quad \overline{\mathbf{3}} \quad \overline{\mathbf{3}} \qquad \text{Divide both sides by 3}$$

$$x^2 - \frac{2}{3}x + \frac{7}{3} = 0 \qquad \text{Resulting equation has } a = 1$$

$$\left(\frac{1}{2} \cdot -\frac{2}{3}\right)^2 = \left(-\frac{1}{3}\right)^2 = \frac{1}{9} \qquad \text{Complete the square: find } \left(\frac{1}{2} \cdot b\right)^2$$

$$x^2 - \frac{2}{3}x + \frac{1}{9} - \frac{1}{9} + \frac{7}{3} = 0 \qquad \text{Add and subtract } \frac{1}{9} \text{ to left side}$$

$$\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) - \frac{1}{9} + \frac{7}{3} = 0 \qquad \text{Combine constant terms by}$$

obtaining a common denominator

$$-\frac{1}{9} + \frac{7}{3} = -\frac{1}{9} + \frac{21}{9} = \frac{20}{9}$$

$$\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) + \frac{20}{9} = 0 \qquad \text{Factor trinomial}$$

$$\left(x - \frac{1}{3}\right)^2 + \frac{20}{9} = 0 \qquad \text{Solve by extracting square roots}$$

$$\left(x - \frac{1}{3}\right)^2 = -\frac{20}{9} \qquad \text{Isolate the square}$$

$$\sqrt{\left(x - \frac{1}{3}\right)^2} = \pm\sqrt{\frac{-20}{9}} \qquad \text{Square root both sides}$$

$$x - \frac{1}{3} = \frac{\pm 2i\sqrt{5}}{3} \qquad \text{Simplify the radical}$$

$$\underline{+\frac{\mathbf{1}}{\mathbf{3}}} \qquad \underline{+\frac{\mathbf{1}}{\mathbf{3}}} \qquad \text{Add } \frac{1}{3} \text{ to both sides}$$

$$x = \frac{1}{3} \pm \frac{2\sqrt{5}}{3}i \qquad \text{Our solution}$$

As we mentioned earlier, completing the square is simply an alternative method to the vertex formula for converting a quadratic expression from standard form into vertex form. Still, as many of the previous examples have demonstrated, we will often need to work with fractions and be comfortable finding common denominators when solving quadratic equations using this method. Although this can be intimidating, with enough practice, one should be able to easily solve almost any quadratic equation by completing the square.

In the next section, we will present one final method for determining the zeros of a quadratic.

The Quadratic Formula (L28) and the Discriminant (L27)

Objective: Solve quadratic equations using the quadratic formula. Use the discriminant to determine the number of real solutions to a quadratic equation.

Recall that the general from of a quadratic equation is $y = ax^2 + bx + c$, where $a \neq 0$. We are now ready to solve the general equation $ax^2 + bx + c = 0$ for x by completing the square, which we show in the following example.

Example 215. Solve the equation $ax^2 + bx + c = 0$ for all values of x using the method of completing the square.

$$ax^2 + bx + c = 0 \qquad \text{Divide each term by } a$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad \text{Separate constant term } \frac{c}{a}$$

$$\left(x^2 + \frac{b}{a}x\right) + \frac{c}{a} = 0 \qquad \text{Complete the square}$$

$$\left(\frac{1}{2} \cdot \frac{b}{a}\right)^2 = \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} \qquad \text{Add and subtract } \frac{b^2}{4a^2} \text{ inside parentheses}$$

$$\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + \frac{c}{a} = 0 \qquad \text{Separate trinomial}$$

$$\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a^2} + \frac{c}{a} = 0 \qquad \text{Simplify:}$$

$$-\frac{b^2}{4a^2} + \frac{c}{a}\left(\frac{4a}{4a}\right) = -\frac{b^2}{4a^2} + \frac{4ac}{4a^2} = -\frac{b^2 - 4ac}{4a^2} \qquad \text{(1) Combine constant terms}$$

$$\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) = \left(x + \frac{b}{2a}\right)^2 \qquad \text{(2) Factor trinomial}$$

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} = 0 \qquad \text{Now solve by extracting square roots}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad \text{Isolate the square}$$

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad \text{Square root both sides}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \qquad \text{Subtract } \frac{b}{2a} \text{ from both sides}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{Write as single fraction}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Our solution}$$

This solution is a very important one to us. Since we solved a *general* equation by completing the square, we can now use this formula to solve any quadratic equation. Once we identify what $a, b,$ and c are, we can substitute those values into the equation $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and simplify in order to find our solution to the given quadratic. This formula is known as the *quadratic formula*. We call the expression underneath the square root, $b^2 - 4ac$, the *discriminant* of the quadratic equation $ax^2 + bx + c = 0$, and will see its importance later on in the section.

Quadratic Formula: The solutions to $ax^2 + bx + c = 0$ are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Discriminant: The discriminant of a quadratic equation $ax^2 + bx + c = 0$ is the expression

$$D = b^2 - 4ac.$$

We can use the quadratic formula to solve any quadratic, this is shown in the following examples.

Notice that we focus on calculating the discriminant first, and that it will have a major impact on the type of solutions that we receive.

Example 216. Solve the given equation for all values of x.

$$\begin{array}{rl}
x^2 + 3x + 2 = 0 & \text{Identify } a, b, \text{ and } c \\
a = 1,\ b = 3,\ c = 2 & \text{Use quadratic formula} \\
x = \dfrac{-3 \pm \sqrt{3^2 - 4(1)(2)}}{2(1)} & \text{Substitute } a, b, \text{ and } c \text{ without simplifying} \\
x = \dfrac{-3 \pm \sqrt{3^2 - 4(1)(2)}}{2(1)} & \\
x = \dfrac{-3 \pm \sqrt{9 - 8}}{2} & \text{Simplify} \\
x = \dfrac{-3 \pm \sqrt{1}}{2} & \text{Discriminant is } 1 \\
x = \dfrac{-3 \pm 1}{2} & \text{Evaluate} \pm\text{; write as two equations} \\
x = \dfrac{-3 + 1}{2} \text{ or } \dfrac{-3 - 1}{2} & \text{Simplify} \\
x = \dfrac{-2}{2} \text{ or } \dfrac{-4}{2} & \\
x = -1 \text{ or } -2 & \text{Our solutions}
\end{array}$$

Notice that the previous equation resulted in two real solutions. This is directly related to the discriminant being positive (in this case, 1). If the discriminant had been zero, then we would not have had anything underneath the square root, meaning that the plus or minus ($\pm$) would have had no effect on the rest of the procedure. Consequently, we would have only had one real solution. Furthermore, since the discriminant was a perfect square, we actually could have factored our quadratic from the start.

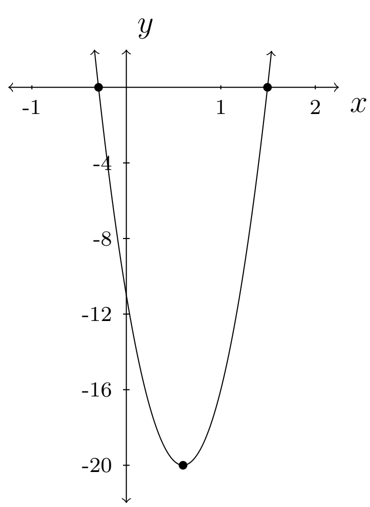
$$x^2 + 3x + 2 = (x + 1)(x + 2)$$

It is important to mention that when solving using the quadratic formula, we must remember to first set the given equation equal to zero and make sure the quadratic is in standard form.

Example 217. Solve the given equation for all values of x.

$$\begin{aligned}
25x^2 &= 30x + 11 && \text{First set equal to zero}\\
25x^2 - 30x - 11 &= 0 && \text{Identify } a, b, \text{ and } c\\
a = 25,\ b = -30,\ c &= -11 && \text{Use quadratic formula}
\end{aligned}$$

$$\begin{aligned}
x &= \frac{30 \pm\sqrt{(-30)^2 - 4(25)(-11)}}{2(25)} && \text{Substitute } a, b, \text{ and } c \text{ without simplifying}\\
x &= \frac{30 \pm\sqrt{(-30)^2 - 4(25)(-11)}}{2(25)} && \\
x &= \frac{30 \pm\sqrt{2000}}{50} && \text{Discriminant is } 2000\\
x &= \frac{30 \pm 20\sqrt{5}}{50} && \text{Divide each term by 10}\\
x &= \frac{3 \pm 2\sqrt{5}}{5} && \text{Our solutions}
\end{aligned}$$

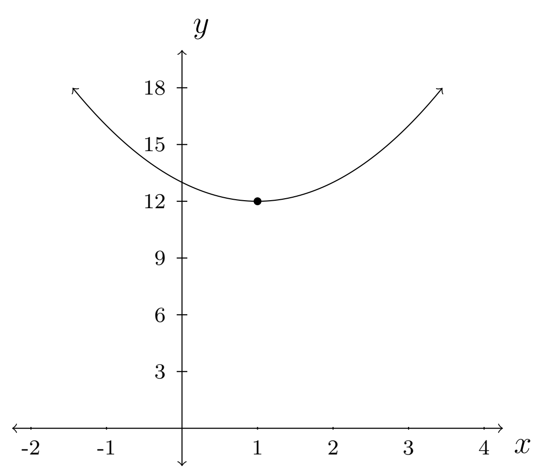


In each of the previous two examples the discriminant was positive, and consequently, there were two real solutions. Graphically, quadratics with a positive discriminant will intersect the x-axis at two distinct points.

The included graph shows the two real solutions to $25x^2 - 30x - 11 = 0$. This example demonstrates the importance of our efforts to relate an algebraic solution to a graphical representation, in order to help internalize the meaning behind the quadratic formula.

Example 218. Solve the given equation for all values of x.

$$
\begin{aligned}
3x^2+4x+8 &= 2x^2+6x-5 && \text{First set equation equal to zero}\\
x^2-2x+13 &= 0 && \text{Identify } a, b, \text{ and } c\\
a=1,\ b=-2,\ c &= 13, && \text{Use quadratic formula}\\
x &= \frac{2 \pm\sqrt{(-2)^2-4(1)(13)}}{2(1)} && \text{Substitute } a, b, \text{ and } c \text{ without simplifying}\\
x &= \frac{2 \pm\sqrt{4-52}}{2} && \text{Simplify}\\
x &= \frac{2 \pm\sqrt{-48}}{2} && \text{Discriminant is } -48\\
x &= \frac{2 \pm 4i\sqrt{3}}{2} && \text{Simplify: reduce radical, divide by 2}\\
x &= 1 \pm 2i\sqrt{3} && \text{Our solutions}
\end{aligned}
$$



The previous example has two complex solutions that are not real. Consequently, we see that graphically our parabola has no $x-$intercepts. This results from the discriminant being negative, -48 in this case.

When using the quadratic formula, it is possible to *not* obtain two unique real (or complex) solutions. If the discriminant under the square root simplifies to zero, we can end up with only *one* real solution.

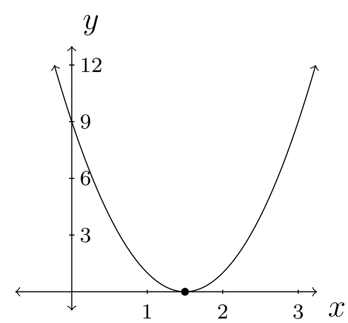
As it turns out, this single solution will coincide with the vertex of our parabola, (h, k). Recalling that $h = -\frac{b}{2a}$, we can verify that this result makes sense, when we consider that a discriminant of zero will eliminate the term $\pm\frac{\sqrt{b^2-4ac}}{2a}$ from our quadratic formula completely. What we are left with is precisely h.

Our next example will result in a single real solution, and will coincide to a parabola that touches the x-axis exactly once, at its vertex.

Example 219. Solve the given equation for all values of x.

$$\begin{aligned}
4x^2 - 12x + 9 &= 0 && \text{Identify } a, b, \text{ and } c\\
a = 4,\ b = -12,\ c &= 9, && \text{Use quadratic formula}\\
x &= \frac{12 \pm \sqrt{(-12)^2 - 4(4)(9)}}{2(4)} && \text{Substitute } a, b, \text{ and } c \text{ without simplifying}\\
x &= \frac{12 \pm \sqrt{144 - 144}}{8} && \text{Simplify}\\
x &= \frac{12 \pm \sqrt{0}}{8} && \text{Discriminant is zero}\\
x &= \frac{12 \pm 0}{8} && \text{We get one real solution}\\
x &= \frac{12}{8} && \text{Reduce fraction}\\
x &= \frac{3}{2} && \text{Our solution}
\end{aligned}$$

A graph of our resulting parabola confirms our previous result of a single zero, and consequently one x-intercept. In this case, the x-intercept should equal the vertex.



If a term is absent from our quadratic, we can still use the quadratic formula and simply use zero in place of the missing coefficient. The order of terms, however, is still important. If, for example, the linear term was absent, we would use $b = 0$. And, if the constant term is missing, we would use $c = 0$.

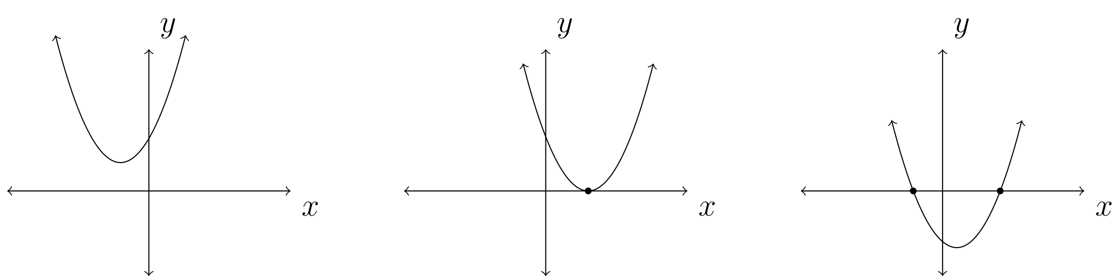
It is necessary that we take extra precautions when using the quadratic formula, since one false step can lead to a substantial amount of time lost. Taking the time to write the quadratic in standard form, set equal to zero, and identify the correct values for a,b, and c is crucial to the success of the quadratic formula.

Example 220. Solve the given equation for all values of x.

$$\begin{aligned}
3x^2+7&=0 && \text{Identify } a, b, \text{ and } c\\
a=3,\ b=0 \text{ (missing term)},\ c&=7 && \text{Use quadratic formula}\\
x=\frac{-0\pm\sqrt{0^2-4(3)(7)}}{2(3)} & && \text{Substitute } a, b, \text{ and } c \text{ without simplifying}\\
x=\frac{\pm\sqrt{-84}}{6} & && \text{Simplify; discriminant is } -84\\
x=\frac{\pm 2i\sqrt{21}}{6} & && \text{Reduce radical and divide by } 2\\
x=\frac{\pm i\sqrt{21}}{3} & && \text{Our solutions}
\end{aligned}$$

We leave it as an exercise to the reader to graph the corresponding parabola and confirm that our solution is correct. Remember, the fact that we have two imaginary solutions means that our parabola should have no $x-$intercepts.

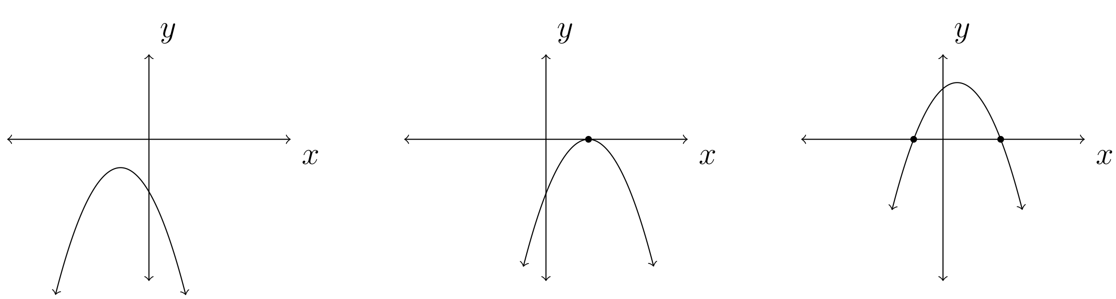
As we have seen in the previous examples, the discriminant determines the nature and quantity of the solutions of the quadratic formula. The following collection of graphs summarizes both the graphical and algebraic consequences for each type of discriminant (negative, zero, or positive).



Negative Discriminant
$b^2-4ac<0$
No Real Solutions

Zero Discriminant
$b^2-4ac=0$
One Real Solution

Positive Discriminant
$b^2-4ac>0$
Two Real Solutions



We have now outlined three different methods to use to solve a quadratic equation: factoring, extracting square roots, and using the quadratic formula. It is important to be familiar with all three methods, since each has its advantages. The following table suggests a procedure to help determine which method might be best to use for solving a given quadratic equation.

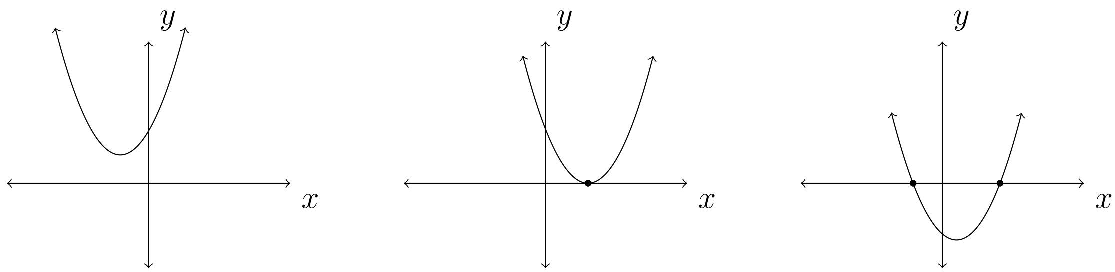
1. If we can easily factor, solve by factoring	$x^2-5x+6=0$ $(x-2)(x-3)=0$ $x=2$ or $x=3$
2. If $a=1$ and b is even, complete the square (or use the vertex formula) and extract square roots	$x^2+2x-4=0$ $\left(\frac{1}{2}\cdot 2\right)^2=1^2=1$ $(x^2+2x+1)-1-4=0$ $(x+1)^2-5=0$ $(x+1)^2=5$ $x+1=\pm\sqrt{5}$ $x=-1\pm\sqrt{5}$
3. As a last resort, apply the quadratic formula	$x^2-3x+4=0$ $x=\dfrac{3\pm\sqrt{(-3)^2-4(1)(4)}}{2(1)}$ $x=\dfrac{3\pm i\sqrt{7}}{2}$

The above table is merely a suggestion for approaching quadratic equations. Recall that completing the square and extracting square roots, as well as the quadratic formula may always be used to solve any quadratic, but often may not be the most efficient or "cleanest" method. Factoring can be very efficient but only works if the given equation can be easily factored.

Quadratic Inequalities and Sign Diagrams (L29)

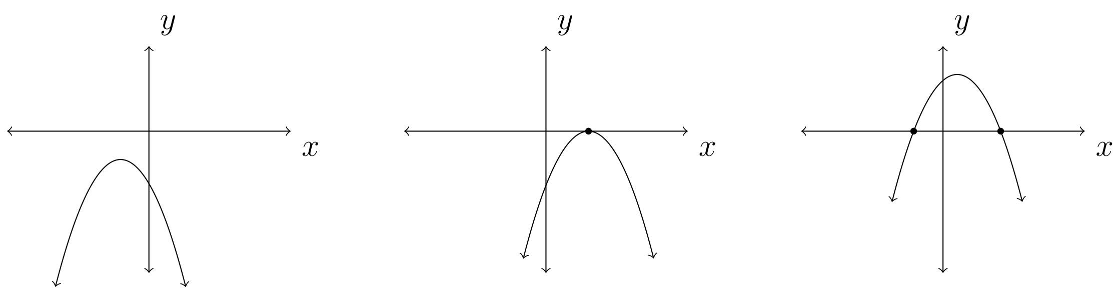
Objective: Solve and give interval notation for the solution to a quadratic inequality. Create a sign diagram to identify those intervals where a quadratic expression is positive or negative.

Recall that the *vertex form* for a quadratic equation is $y=a(x-h)^2+k$, where $a\neq 0$ and (h,k) represents the *vertex* of the corresponding graph, called a *parabola*. If $a>0$, then the parabola opens upward, and if $a<0$, then the parabola opens downward. With any quadratic equation, we have seen that there are three possibilities for the number of *zeros* or *roots* of the equation $(0,1,$ or $2)$. Assuming $a>0$, we illustrate these possibilities in the graphs below.



Notice also that each of these three graphs lie above the x-axis over different intervals. In the case of the parabola on the left, the entire graph lies above the x-axis, whereas the middle parabola lies above the x-axis everywhere *except* at its x-intercept (where $y = 0$). Even more interesting is the parabola on the right, which contains two *separate* intervals where its graph lies above the x-axis.

Considering the case where $a < 0$, we see three similar graphs as those appearing above, with the only major difference being the opening of each parabola downward instead of upward (when $a > 0$). When we consider again those intervals where each graph lies above the x-axis, each parabola below exhibits a different behavior than those where $a > 0$.



Now, each of the first two graphs have no points that lie above the x-axis, whereas the last graph, on the right, lies above the x-axis over the interval that is between its x-intercepts.

Each of these six graphs above exhibit all of the various possibilities for the *sign* of a quadratic expression $ax^2 + bx + c$, where $a \neq 0$. As was the case with linear equations in the previous chapter, we can determine the general shape of the graph of a quadratic equation (or function) through identification of its zeros and construction of a sign diagram. As a consequence, we will also see the care that must be taken when asked to solve a quadratic inequality.

Let us begin with what should be a familiar example, $y = x^2 - 1$, which we can recall has a factorization of $y = (x + 1)(x - 1)$.

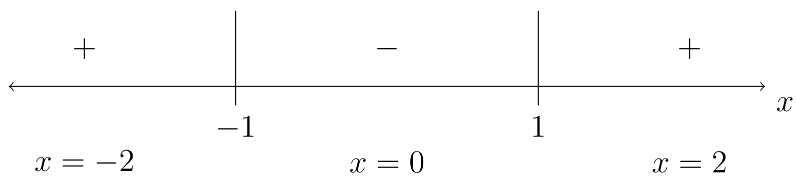
Example 221. Solve the quadratic inequality $x^2 - 1 < 0$.

As has often been the case, our first instinct is to add 1 to both sides of the given inequality, obtaining $x^2 < 1$. Our next guess is most likely to take a square root of both sides of the given inequality. Here, however, is where we encounter a common "pitfall", which begs the question: how does one handle radicals and inequalities?

The answer is that unlike with solving linear inequalities, one should not attempt to solve for the variable x, but rather set the given inequality equal to zero and attempt to *factor* the resulting expression on the other side. In doing this, we obtain $(x+1)(x-1) < 0$. Recalling that $x = \pm 1$ are zeros of the given expression, we can therefore rule them out of our solution. Next, we will *test* the expression on the left by plugging in three values for x: (i) $x < -1$, (ii) $-1 < x < 1$, and (iii) $x > 1$.

Case	Test Value	Unsimplified	Simplified	Result
i	$x = -2$	$(-2+1)(-2-1)$	$(-)\cdot(-)$	$(+)$
ii	$x = 0$	$(0+1)(0-1)$	$(+)\cdot(-)$	$(-)$
iii	$x = 2$	$(2+1)(2-1)$	$(+)\cdot(+)$	$(+)$

Our end result can be summarized in the following *sign diagram*.



From our sign diagram, we can conclude that $x^2 - 1 < 0$ when $-1 < x < 1$, or using interval notation, $(-1, 1)$.

Example 222. Solve the inequality $x^2 \geq 1$.

Here, we need only subtract -1 from both sides of the inequality, to obtain $x^2 - 1 \geq 0$. After factoring the left-hand side, We may then use the sign diagram from our previous example. Our solution set will be the *union* of two intervals, $(-\infty, -1] \cup [1, \infty)$.

Example 223. Solve the inequality $-(x-1)^2 + 9 \geq 0$.

Notice that the left-hand side of our inequality is in vertex form. So we will draw upon our knowledge of the graph of $y = -(x-1)^2 + 9$ later on to confirm our answer.

We start by expanding the left-hand side to obtain

$$-(x^2 - 2x + 1) + 9 \geq 0,$$

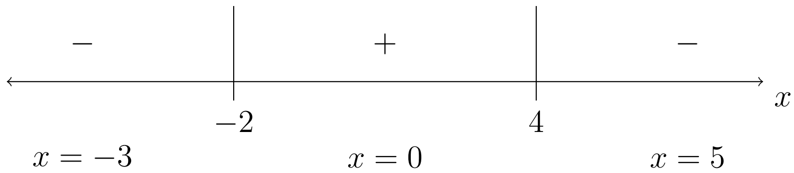
which reduces to

$$-x^2 + 2x + 8 \geq 0.$$

After factoring, we obtain

$$-(x+2)(x-4) \geq 0.$$

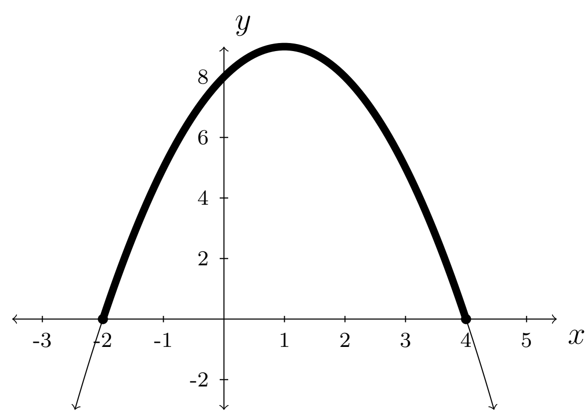
Since both $x = -2$ and $x = 4$ are zeros of the left-hand side, for our sign diagram, we will therefore test $x = -3$, $x = 0$, and $x = 5$. It is important to not overlook the negative sign that appears in front of our inequality when testing our values. Our results are shown in the following diagram.



From our sign diagram, we can determine that

$$-(x-1)^2+9 \geq 0 \text{ when } -2 \leq x \leq 4.$$

Again, the vertex form $y=-(x-1)^2+9$ confirms this, since the corresponding parabola will have a vertex of $(1,9)$, which lies above the x-axis, and will open downward, as the leading coefficient $a=-1$ is negative. This implies that there will be two x-intercepts, which we found to be at the points $(-2,0)$ and $(4,0)$. Hence the graph will be nonnegative over an interval between (and including) the x-intercepts. To reinforce this, we provide the graph below, highlighting the portion that coincides with our desired interval.



In our next example, we will touch upon the notion of the *multiplicity* of a zero for a given equation/function, and how it affects the graph.

Example 224. Solve the inequality $x^2+4x>-4$.

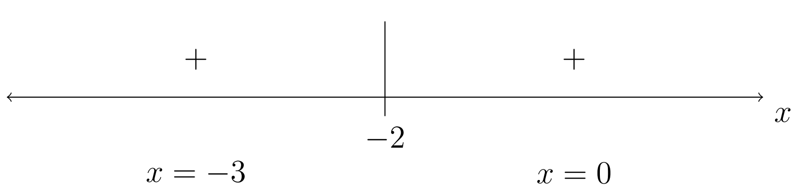
Setting the right-hand side to zero gives us

$$x^2+4x+4>0.$$

Factoring, we then have

$$(x+2)^2>0.$$

Hence, we have only one zero for the left-hand side ($x=-2$), which means that there are only two intervals to test.



Our solution set may be represented as the inequality $x \neq -2$, or as the union of intervals $(-\infty, -2) \cup (-2, \infty)$.

Notice that $x = -2$ was a zero in each of the last two examples. In the first example, a change in sign occurred (negative to positive) as the values of x increased from one side of our zero to the other. In the second example, however, both the values below and above $x = -2$ yield positive signs.

This result has to do with the number of factors of $(x+2)$ appearing in our expression. This number is known as the *multiplicity* of the zero $x = -2$. Briefly stated, the *parity* of a zero's multiplicity (whether the number of factors is even or odd) will determine whether or not the sign of the given expression on either side of the zero remains the same or changes. This notion will be quite useful when graphing complicated functions, and will be revisited in the chapter on polynomial functions.

Example 225. Solve the inequality $x^2 + 4x < -4$.

Since we have only switched the direction of our inequality in the last example, we may conclude that the inequality has no solution set, represented by the empty set, $\varnothing$.

Up until this point, all of our examples have reduced to expressions that can easily be factored. As this is often not the case for quadratic expressions, we will now attempt to solve some more challenging inequalities.

Example 226. Solve the inequality $x^2 - x + 1 > 0$.

After brief inspection, we see that the expression on the left-hand side is not easily factorable. At this point, in order to determine if any real zeros exist for $x^2 - x + 1$, we have a few methods to choose from. We will use the quadratic formula, shown below.

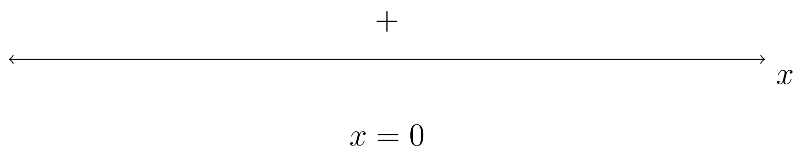
$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$x = \frac{1}{2} \pm \frac{\sqrt{-3}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

Since we are left with a negative under the square root, we conclude that the given expression has no real zeros. Hence, the corresponding parabola will have no x-intercepts. Note: A slightly quicker method would have been to simply calculate the discriminant, shown below.

$$D = (-1)^2 - 4(1)(1) = -3 < 0$$

As our leading coefficient $a = 1$ in the above expression is positive, we know that the corresponding parabola will open upward. Using this information, along with the fact that there are no x-intercepts, we may conclude that the entire parabola must lie above the x-axis. The corresponding sign diagram, with chosen test value of $x = 0$, is included for completeness.



Hence, our solution set for the inequality $x^2 - x + 1 > 0$ is all real numbers, $(-\infty, \infty)$.

Example 227. Solve the inequality $x^2 > 4x - 1$.

Setting the right-hand side to zero, we have $x^2 - 4x + 1 > 0$.

Although we could again resort to the quadratic formula, we will instead identify the vertex form of the expression on the left, shown below.

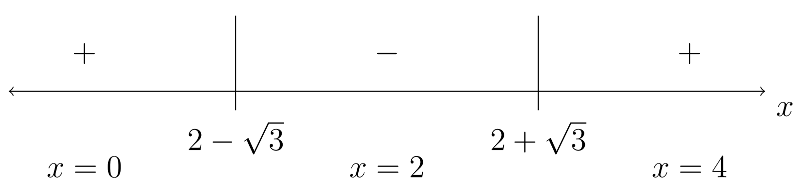
$$h = -\frac{-4}{2(1)} = 2 \qquad\qquad k = 2^2 - 4(2) + 1 = -3$$

$$x^2 - 4x + 1 = (x-2)^2 - 3$$

So, setting $(x-2)^2 - 3$ equal to zero and extracting square roots, we obtain two real zeros at $x = 2 \pm \sqrt{3}$. It then follows that

$$x^2 - 4x + 1 = \left(x - (2 - \sqrt{3})\right)\left(x - (2 + \sqrt{3})\right).$$

Since we have two real zeros, we will construct a sign diagram, using test values on either side of $2 - \sqrt{3} \approx 0.27$ and $2 + \sqrt{3} \approx 3.73$. Our results are shown below.



Note that since we already obtained the vertex of $(2, -3)$, we have chosen $x = 2$ as a test value for our middle interval.

From the above diagram, we conclude that $x^2 > 4x - 1$ precisely on the union of intervals $(-\infty, 2 - \sqrt{3}) \cup (2 + \sqrt{3}, \infty)$.

Practice Problems

Introduction

After simplifying, classify each equation as linear, quadratic, or neither. If the equation is a quadratic, then specify whether it is concave up or down.

1) $y = x^2 + 9$
2) $y = 5 - 2x + x^2$
3) $y = x + 6 - 3x$
4) $y = 5x + x^2 - 3x - 3x^2$
5) $y = -5x + 3 + 2x - 3x^2 + 6$
6) $y = 3x^2 + -x + x - 3x^2 + 6$
7) $y = (x - 1)(x + 2) + 3$
8) $y = (x - 5)(2x + 3) - 2(x - 3)$
9) $y = (x - 4)(x + 4) - (x + 1)^2$
10) $y = (2x - 4)(x - 1) - 2(x + 3)^2 + 3x^2$

An Introduction to the Vertex Form

Identify the vertex and concavity (concave up or down) of each quadratic.

11) $y = (x - 3)^2 + 4$
12) $y = (x - 2)^2 + 5$
13) $y = 6(x + 3)^2 + 4$
14) $y = -2(x - 3)^2 + 4$
15) $y = -2(x - 1)^2 - 7$
16) $y = -(x + 1)^2$
17) $y = 7x^2 + 4$
18) $y = -\frac{1}{23}(x - 8)^2 + 5$
19) $y = x^2 + 4$
20) $y = 5x^2 + 23$

Factoring Methods

Greatest Common Factors

Factor the common factor out of each expression.

1) $4 + 8b^2$
2) $x - 5$
3) $45x^2 - 25$
4) $-n - 2n^2$
5) $56 - 35p$
6) $50x - 80y$
7) $7ab - 35a^2b$
8) $27x^2y^5 - 72x^3y^2$
9) $-3a^2b + 6a^3b^2$
10) $8x^3y^2 + 4x^3$
11) $-5x^2 - 5x^3 - 15x^4$
12) $-32n^9 + 32n^6 + 40n^5$
13) $20x^4 - 30x + 30$
14) $21p^6 + 30p^2 + 27$
15) $28m^4 + 40m^3 + 8$
16) $-10x^4 + 20x^2 + 12x$
17) $30b^9 + 5ab - 15a^2$
18) $27y^7 + 12y^2x + 9y^2$
19) $-48a^2b^2 - 56a^3b - 56a^5b$
20) $30m^6 + 15mn^2 - 25$
21) $20x^8y^2z^2 + 15x^5y^2z + 35x^3y^3z$
22) $3p + 12q - 15q^2r^2$
23) $50x^2y + 10y^2 + 70xz^2$
24) $30y^4z^3x^5 + 50y^4z^5 - 10y^4z^3x$
25) $30qpr - 5qp + 5q$
26) $28b + 14b^2 + 35b^3 + 7b^5$
27) $-18n^5 + 3n^3 - 21n + 3$
28) $30a^8 + 6a^5 + 27a^3 + 21a^2$
29) $-40x^{11} - 20x^{12} + 50x^{13} - 50x^{14}$
30) $-24x^6 - 4x^4 + 12x^3 + 4x^2$
31) $-32mn^8 + 4m^6n + 12mn^4 + 16mn$
32) $-10y^7 + 6y^{10} - 4y^{10}x - 8y^8x$

Factor by Grouping

Factor each expression completely.

33) $40r^3 - 8r^2 - 25r + 5$
34) $35x^3 - 10x^2 - 56x + 16$
35) $3n^3 - 2n^2 - 9n + 6$
36) $14v^3 + 10v^2 - 7v - 5$
37) $15b^3 + 21b^2 - 35b - 49$
38) $6x^3 - 48x^2 + 5x - 40$
39) $3x^3 + 15x^2 + 2x + 10$
40) $28p^3 + 21p^2 + 20p + 15$
41) $35x^3 - 28x^2 - 20x + 16$
42) $7n^3 + 21n^2 - 5n - 15$
43) $7xy - 49x + 5y - 35$
44) $42r^3 - 49r^2 + 18r - 21$
45) $32xy + 40x^2 + 12y + 15x$
46) $15ab - 6a + 5b^3 - 2b^2$
47) $16xy - 56x + 2y - 7$
48) $3mn - 8m + 15n - 40$
49) $2xy - 8x^2 + 7y^3 - 28y^2x$
50) $5mn + 2m - 25n - 10$
51) $40xy + 35x - 8y^2 - 7y$
52) $8xy + 56x - y - 7$
53) $32uv - 20u + 24v - 15$
54) $4uv + 14u^2 + 12v + 42u$
55) $10xy + 30 + 25x + 12y$
56) $24xy + 25y^2 - 20x - 30y^3$
57) $3uv + 14u - 6u^2 - 7v$
58) $56ab + 14 - 49a - 16b$
59) $16xy - 3x - 6x^2 + 8y$

Trinomials with Leading Coefficient $a = 1$

Factor each expression completely.

60) $p^2 + 17p + 72$
61) $x^2 + x - 72$
62) $n^2 - 9n + 8$
63) $x^2 + x - 30$
64) $x^2 - 9x - 10$
65) $x^2 + 13x + 40$
66) $b^2 + 12b + 32$
67) $b^2 - 17b + 70$
68) $x^2 + 3x - 70$
69) $x^2 + 3x - 18$
70) $n^2 - 8n + 15$
71) $a^2 - 6a - 27$
72) $p^2 + 15p + 54$
73) $p^2 + 7p - 30$
74) $n^2 - 15n + 56$
75) $m^2 - 15mn + 50n^2$
76) $u^2 - 8uv + 15v^2$
77) $m^2 - 3mn - 40n^2$
78) $m^2 + 2mn - 8n^2$
79) $x^2 + 10xy + 16y^2$
80) $x^2 - 11xy + 18y^2$
81) $u^2 - 9uv + 14v^2$
82) $x^2 + xy - 12y^2$
83) $x^2 + 14xy + 45y^2$
84) $x^2 + 4xy - 12y^2$
85) $4x^2 + 52x + 168$
86) $5a^2 + 60a + 100$
87) $5n^2 - 45n + 40$
88) $6a^2 + 24a - 192$
89) $5v^2 + 20v - 25$
90) $6x^2 + 18xy + 12y^2$
91) $5m^2 + 30mn - 90n^2$
92) $6x^2 + 96xy + 378y^2$
93) $6m^2 - 36mn - 162n^2$

Trinomials with Leading Coefficient $a \neq 1$

Factor each expression completely.

94) $7x^2 - 48x + 36$
95) $7n^2 - 44n + 12$
96) $7b^2 + 15b + 2$
97) $7v^2 - 24v - 16$
98) $5a^2 - 13a - 28$
99) $5n^2 - 7n - 24$
100) $2x^2 - 5x + 2$
101) $3r^2 - 4r - 4$
102) $2x^2 + 19x + 35$
103) $7x^2 + 29x - 30$
104) $2b^2 - b - 3$
105) $5x^2 - 26x + 24$
106) $5x^2 + 13x + 6$
107) $3r^2 + 16r + 21$
108) $3x^2 - 17x + 20$
109) $3u^2 + 13uv - 10v^2$
110) $3x^2 + 17xy + 10y^2$
111) $7x^2 - 2xy - 5y^2$
112) $5x^2 + 28xy - 49y^2$
113) $5u^2 + 31uv - 28v^2$
114) $6x^2 - 39x - 21$
115) $10a^2 - 54a - 36$
116) $21x^2 - 87x - 90$
117) $21n^2 + 45n - 54$
118) $14x^2 - 60x + 16$
119) $4r^2 + r - 3$
120) $6x^2 + 29x + 20$

121) $6p^2 + 11p - 7$
122) $4x^2 - 17x + 4$
123) $4r^2 + 3r - 7$
124) $4x^2 + 9xy + 2y^2$
125) $4m^2 + 6mn + 6n^2$
126) $4m^2 - 9mn - 9n^2$
127) $4x^2 - 6xy + 30y^2$
128) $4x^2 + 13xy + 3y^2$
129) $18u^2 - 3uv - 36v^2$
130) $12x^2 + 62xy + 70y^2$
131) $16x^2 + 60xy + 36y^2$
132) $24x^2 - 52xy + 8y^2$
133) $12x^2 + 50xy + 28y^2$

Solving by Factoring

Set each of the following expressions equal to zero and solve for the given variable.

1) - 15): Expressions (60) through (74) on page 190.

16) - 30): Expressions (94) through (108) on page 190.

31) - 40): Expressions (114) through (123) on page 190.

Square Roots and the Imaginary Number i

Square Roots

Simplify each of the following square roots completely.

1) $\sqrt{245}$
2) $\sqrt{125}$
3) $\sqrt{36}$
4) $\sqrt{196}$
5) $\sqrt{12}$
6) $\sqrt{72}$
7) $3\sqrt{12}$
8) $5\sqrt{32}$
9) $6\sqrt{128}$
10) $7\sqrt{128}$
11) $-8\sqrt{392}$
12) $-7\sqrt{63}$
13) $\sqrt{192n}$
14) $\sqrt{343b}$
15) $\sqrt{196v^2}$
16) $\sqrt{100n^3}$
17) $\sqrt{252x^2}$
18) $\sqrt{200a^3}$
19) $-\sqrt{100k^4}$
20) $-4\sqrt{175p^4}$
21) $-7\sqrt{64x^4}$
22) $-2\sqrt{128n}$
23) $-5\sqrt{36m}$
24) $8\sqrt{112p^2}$
25) $\sqrt{45x^2y^2}$
26) $\sqrt{72a^3b^4}$
27) $\sqrt{16x^3y^3}$
28) $\sqrt{512a^4b^2}$
29) $\sqrt{320x^4y^4}$
30) $\sqrt{512m^4n^3}$
31) $6\sqrt{80xy^2}$
32) $8\sqrt{98mn}$
33) $5\sqrt{245x^2y^3}$
34) $2\sqrt{72x^2y^2}$
35) $-2\sqrt{180u^3v}$
36) $-5\sqrt{72x^3y^4}$
37) $-8\sqrt{180x^4y^2z^4}$
38) $6\sqrt{50a^4bc^2}$
39) $2\sqrt{80hj^4k}$
40) $-\sqrt{32xy^2z^3}$
41) $-4\sqrt{54mnp^2}$
42) $-8\sqrt{32m^2p^4q}$

Introduction to Complex Numbers

Rewrite each of the following complex numbers in the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$.

1) $3 - (-8 + 4i)$
2) $(3i) - (7i)$
3) $(7i) - (3 - 2i)$
4) $5 + (-6 - 6i)$
5) $(-6i) - (3 + 7i)$
6) $(-8i) - (7i) - (5 - 3i)$
7) $(3 - 3i) + (-7 - 8i)$
8) $(-4 - i) + (1 - 5i)$
9) $(i) - (2 + 3i) - 6$
10) $(5 - 4i) + (8 - 4i)$
11) $(6i)(-8i)$
12) $(3i)(-8i)$
13) $(-5i)(8i)$
14) $(8i)(-4i)$
15) $(-7i)^2$

16) $(-i)(7i)(4-3i)$
17) $(6+5i)^2$
18) $(8i)(-2i)(-2-8i)$
19) $(-7-4i)(-8+6i)$
20) $(3i)(-3i)(4-4i)$
21) $(-4+5i)(2-7i)$
22) $-8(4-8i)-2(-2-6i)$
23) $(-8-6i)(-4+2i)$
24) $(-6i)(3-2i)-(7i)(4i)$
25) $(1+5i)(2+i)$
26) $(-2+i)(3-5i)$
27) $\dfrac{-9+5i}{i}$
28) $\dfrac{-3+2i}{-3i}$
29) $\dfrac{-10-9i}{6i}$
30) $\dfrac{-4+2i}{3i}$
31) $\dfrac{-3-6i}{4i}$
32) $\dfrac{-5+9i}{9i}$
33) $\dfrac{10-i}{-i}$
34) $\dfrac{10}{5i}$
35) $\dfrac{4i}{-10+i}$
36) $\dfrac{9i}{1-5i}$
37) $\dfrac{8}{7-6i}$
38) $\dfrac{4}{4+6i}$
39) $\dfrac{7}{10-7i}$
40) $\dfrac{9}{-8-6i}$
41) $\dfrac{5i}{-6-i}$
42) $\dfrac{8i}{6-7i}$
43) $\sqrt{-81}$
44) $\sqrt{-45}$
45) $\sqrt{-10}\sqrt{-2}$
46) $\sqrt{-12}\sqrt{-2}$
47) $\frac{3+\sqrt{-27}}{6}$
48) $\frac{-4-\sqrt{-8}}{-4}$
49) $\frac{8-\sqrt{-16}}{4}$
50) $\frac{6+\sqrt{-32}}{4}$
51) i^{73}
52) i^{251}
53) i^{48}
54) i^{68}
55) i^{62}
56) i^{181}
57) i^{154}
58) i^{51}

Vertex Form and Graphing

The Vertex Form

Identify whether the quadratic is in vertex form, standard form, or both. If it is in vertex form, then identify the vertex (h, k).

1) $y=(x-12)^2+5$
2) $y=-3(x-3)^2+5$
3) $y=x^2+8$
4) $y=2(x-4)^2$
5) $y=-4(x-1)^2+2$
6) $y=-5(x-7)^2$
7) $y=x^2+3x+4$
8) $y=x^2-1$
9) $y=x^2-3$
10) $y=(x-1)^2-3$
11) $y=(x-1)^2$
12) $y=x^2$

Each quadratic equation below has been given in standard form. Rewrite each equation in vertex form.

13) $y=x^2+2x-1$
14) $y=-3x^2-12x-5$
15) $y=3x^2+12x-1$
16) $y=x^2+2x$
17) $y=x^2+6$
18) $y=-5x^2-40x$
19) $y=x^2+8x$
20) $y=x^2$
21) $y=x^2+4x-2$
22) $y=x^2+16x-2$
23) $y=4x^2+10x$

Graphing Quadratics

Find the vertex and intercepts of the following quadratics. Use this information to graph the resulting parabola.

1) $y = x^2 - 2x - 8$
2) $y = x^2 - 2x - 3$
3) $y = 2x^2 - 12x + 10$
4) $y = 2x^2 - 12x + 16$
5) $y = -2x^2 + 12x - 18$
6) $y = -2x^2 + 12x - 10$
7) $y = -3x^2 + 24x - 45$
8) $y = -3x^2 + 12x - 9$
9) $y = -x^2 + 4x + 5$
10) $y = -x^2 + 4x - 3$
11) $y = -x^2 + 6x - 5$
12) $y = -2x^2 + 16x - 30$
13) $y = -2x^2 + 16x - 24$
14) $y = 2x^2 + 4x - 6$
15) $y = 3x^2 + 12x + 9$
16) $y = 5x^2 + 30x + 45$
17) $y = 5x^2 - 40x + 75$
18) $y = 5x^2 + 20x + 15$
19) $y = -5x^2 - 60x - 175$
20) $y = -5x^2 + 20x - 15$

The Method of Extracting Square Roots

Find the x-intercepts of each quadratic by setting $y = 0$ and using the method of extracting square roots.

1) $y = (x - 12)^2 - 5$
2) $y = -3(x - 3)^2 + 30$
3) $y = x^2 - 16$
4) $y = 2(x - 4)^2 - 200$
5) $y = -4(x - 1)^2 + 20$
6) $y = -2(x - 7)^2 + 50$
7) $y = -4(x + 6)^2 + 8$
8) $y = x^2 - 4$
9) $y = (x - 4)^2 - 9$
10) $y = (x - 1)^2 - 25$
11) $y = (x + 2)^2 + 16$
12) $y = 9(x - 11)^2 - 81$

Completing the Square

Find the value that completes the square and then rewrite the given expression as a perfect square.

1) $x^2 - 30x +$ ______
2) $a^2 - 24a +$ ______
3) $m^2 - 36m +$ ______
4) $x^2 - 34x +$ ______
5) $x^2 - 15x +$ ______
6) $r^2 - \frac{1}{9}r +$ ______
7) $y^2 - y +$ ______
8) $p^2 - 17p +$ ______

Solve each equation by completing the square.

9) $x^2 - 16x + 55 = 0$
10) $n^2 - 8n - 12 = 0$
11) $v^2 - 8v + 45 = 0$
12) $b^2 + 2b + 43 = 0$
13) $6x^2 + 12x + 63 = 0$
14) $3x^2 - 6x + 47 = 0$
15) $5k^2 - 10k + 48 = 0$
16) $8a^2 + 16a - 1 = 0$
17) $x^2 + 10x - 57 = 4$
18) $p^2 - 16p - 52 = 0$
19) $n^2 - 16n + 67 = 4$
20) $m^2 - 8m - 3 = 6$
21) $2x^2 + 4x + 38 = -6$
22) $6r^2 + 12r - 24 = -6$
23) $8b^2 + 16b - 37 = 5$
24) $6n^2 - 12n - 14 = 4$
25) $x^2 = -10x - 29$
26) $v^2 = 14v + 36$

27) $n^2 = -21 + 10n$
28) $a^2 - 56 = -10a$
29) $3k^2 + 9 = 6k$
30) $5x^2 = -26 + 10x$
31) $2x^2 + 63 = 8x$
32) $5n^2 = -10n + 15$
33) $p^2 - 8p = -55$
34) $x^2 + 8x + 15 = 8$
35) $7n^2 - n + 7 = 7n + 6n^2$
36) $n^2 + 4n = 12$
37) $13b^2 + 15b + 44 = -5 + 7b^2 + 3b$
38) $-3r^2 + 12r + 49 = -6r^2$
39) $5x^2 + 5x = -31 - 5x$
40) $8n^2 + 16n = 64$
41) $v^2 + 5v + 28 = 0$
42) $b^2 + 7b - 33 = 0$
43) $7x^2 - 6x + 40 = 0$
44) $4x^2 + 4x + 25 = 0$
45) $k^2 - 7k + 50 = 3$
46) $a^2 - 5a + 25 = 3$
47) $5x^2 + 8x - 40 = 8$
48) $2p^2 - p + 56 = -8$
49) $m^2 = -15 + 9m$
50) $n^2 - n = -41$
51) $8r^2 + 10r = -55$
52) $3x^2 - 11x = -18$
53) $5n^2 - 8n + 60 = -3n + 6 + 4n^2$
54) $4b^2 - 15b + 56 = 3b^2$
55) $-2x^2 + 3x - 5 = -4x^2$
56) $10v^2 - 15v = 27 + 4v^2 - 6v$

The Quadratic Formula and the Discriminant

Use the discriminant in order to determine the number of real roots for each equation. If an equation is shown to have one (or two) real root(s), set $y = 0$ and use the quadratic formula to find them.

1) $y = x^2 + 2x - 1$
2) $y = -3x^2 - 12x - 5$
3) $y = 3x^2 + 12x - 1$
4) $y = x^2 + 2x$
5) $y = x^2 + 6$
6) $y = -5x^2 - 40x$
7) $y = x^2 + 8x$
8) $y = x^2$
9) $y = x^2 + 4x - 2$
10) $y = x^2 + 16x - 2$
11) $y = 4x^2 + 10x$

Solve each equation using the quadratic formula.

12) $4a^2 + 6 = 0$
13) $3k^2 + 2 = 0$
14) $2x^2 - 8x - 2 = 0$
15) $6n^2 - 1 = 0$
16) $2m^2 - 3 = 0$
17) $5p^2 + 2p + 6 = 0$
18) $3r^2 - 2r - 1 = 0$
19) $2x^2 - 2x - 15 = 0$
20) $4n^2 - 36 = 0$
21) $3b^2 + 6 = 0$
22) $v^2 - 4v - 5 = -8$
23) $2x^2 + 4x + 12 = 8$
24) $2a^2 + 3a + 14 = 6$
25) $6n^2 - 3n + 3 = -4$
26) $3k^2 + 3k - 4 = 7$
27) $4x^2 - 14 = -2$
28) $7x^2 + 3x - 16 = -2$
29) $4n^2 + 5n = 7$
30) $2p^2 + 6p - 16 = 4$
31) $m^2 + 4m - 48 = -3$
32) $3n^2 + 3n = -3$
33) $3b^2 - 3 = 8b$
34) $2x^2 = -7x + 49$
35) $3r^2 + 4 = -6r$
36) $5x^2 = 7x + 7$
37) $6a^2 = -5a + 13$
38) $8n^2 = -3n - 8$
39) $6v^2 = 4 + 6v$
40) $2x^2 + 5x = -3$
41) $x^2 = 8$
42) $4a^2 - 64 = 0$
43) $2k^2 + 6k - 16 = 2k$
44) $4p^2 + 5p - 36 = 3p^2$
45) $12x^2 + x + 7 = 5x^2 + 5x$
46) $-5n^2 - 3n - 52 = 2 - 7n^2$
47) $7m^2 - 6m + 6 = -m$
48) $7r^2 - 12 = -3r$
49) $3x^2 - 3 = x^2$
50) $2n^2 - 9 = 4$
51) $6b^2 = b^2 + 7 - b$

Quadratic Inequalities and Sign Diagrams

Construct a sign diagram for each of the following expressions/equations. Then using interval notation, describe the set of values for which the given expression is greater than or equal to zero.

1) - 5): Expressions (60) through (64) on page 190.

6) - 10): Expressions (94) through (98) on page 190.

11) - 15): Expressions (114) through (118) on page 190.

16) - 20): Equations (1) through (5) on page 194.

Selected Answers

Introduction

L=linear, Q=quadratic, N=neither, U=concave up, D=concave down

1) $y = x^2 + 9$, QU
3) $y = -2x + 6$, L
5) $y = -3x^2 - 3x + 9$, QD
7) $y = x^2 + x + 1$, QU
9) $y = -2x - 17$, L

An Introduction to the Vertex Form

U=concave up, D=concave down

11) $(3, 4)$, U
13) $(-3, 4)$, U
15) $(1, -7)$, D
17) $(0, 4)$, U
19) $(0, 4)$, U

Factoring Methods

Greatest Common Factors

1) $4(1 + 2b^2)$
3) $5(9x^2 - 5)$
5) $7(8 - 5p)$
7) $7ab(1 - 5a)$
9) $-3a^2b(1 - 2ab)$
11) $-5x^2(1 + x + 3x^2)$
13) $10(2x^4 - 3x + 3)$
15) $4(7m^4 + 10m^3 + 2)$
17) $5(6b^9 + ab - 3a^2)$
19) $-8a^2b(6b - 7a - 7a^3)$
21) $5x^3y^2z(4x^5z + 3x^2 + 7y)$
23) $10(5x^2y + y^2 + 7xz^2)$
25) $5q(6pr - p + 1)$
27) $-3(6n^5 - n^3 + 7n - 1)$
29) $-10x^{11}(4+2x-5x^2+5x^3)$
31) $-4mn(8n^7-m^5-3n^3-4)$

Factor by Grouping

33) $(8r^2 - 5)(5r - 1) = \left(2\sqrt{2}r - \sqrt{5}\right)\left(2\sqrt{2}r + \sqrt{5}\right)(5r - 1)$
35) $(n^2 - 3)(3n - 2) = \left(n - \sqrt{3}\right)\left(n + \sqrt{3}\right)(3n - 2)$

37) $(3b^2-7)(5b+7) = \left(\sqrt{3}b-\sqrt{7}\right)\left(\sqrt{3}b+\sqrt{7}\right)(5b+7)$
39) $(3x^2+2)(x+5)$
41) $(7x^2-4)(5x-4) = \left(\sqrt{7}x-2\right)\left(\sqrt{7}x+2\right)(5x-4)$

43) $(7x+5)(y-7)$
45) $(8x+3)(4y+5x)$
47) $(8x+1)(2y-7)$
49) $(2x+7y^2)(y-4x)$
51) $(5x-y)(8y+7)$
53) $(4u+3)(8v-5)$
55) $(5x+6)(2y+5)$
57) $(3u-7)(v-2u)$
59) $(8y-3x)(2x+1)$

Trinomials with Leading Coefficient $a = 1$

60) $(p+9)(p+8)$
62) $(n-8)(n-1)$
64) $(x-10)(x+1)$
66) $(b+4)(b+8)$
68) $(x+10)(x-7)$
70) $(n-5)(n-3)$
72) $(p+9)(p+6)$
74) $(n-7)(n-8)$
76) $(u-5v)(u-3v)$
78) $(m+4n)(m-2n)$
80) $(x-9y)(x-2y)$
82) $(x+4y)(x-3y)$
84) $(x+6y)(x-2y)$
86) $5(a+2)(a+10)$
88) $6(a+8)(a-4)$
90) $6(x+2y)(x+y)$
92) $6(x+9y)(x+7y)$

Trinomials with Leading Coefficient $a \neq 1$

94) $(7x-6)(x-6)$
96) $(7b+1)(b+2)$
98) $(5a+7)(a-4)$
100) $(2x-1)(x-2)$
102) $(2x+5)(x+7)$
104) $(2b-3)(b+1)$
106) $(5x+3)(x+2)$
108) $(3x-5)(x-4)$
110) $(3x+2y)(x+5y)$
112) $(5x-7y)(x+7y)$
114) $3(2x+1)(x-7)$
116) $3(7x+6)(x-5)$
118) $3(7x-2)(x-4)$
120) $(6x+5)(x+4)$
122) $(4x-1)(x-4)$
124) $(4x+y)(x+2y)$
126) $(4m+3n)(m-3n)$
128) $(4x+y)(x+3y)$
130) $2(3x+5y)(2x+7y)$
132) $4(6x-y)(x-2y)$

Solving by Factoring

1) $p=-9,-8$
3) $n=1,8$
5) $x=-1,10$
7) $b=-4,-8$
9) $x=-10,7$
11) $n=3,5$
13) $p=-6,-9$
15) $n=7,8$
17) $n=2/7,6$
19) $v=-4/7,4$
21) $n=-8/5,3$
23) $r=-2/3,2$
25) $x=-5,6/7$
27) $x=6/5,4$
29) $r=-3,-7/3$
31) $x=-1/2,7$
33) $x=-6/7,5$
35) $x=2/7,4$
37) $x=-4,-5/6$
39) $x=1/4,4$

Square Roots and the Imaginary Number i

Square Roots

1) $7\sqrt{5}$
3) 6
5) $2\sqrt{3}$
7) $6\sqrt{3}$
9) $48\sqrt{2}$
11) $-112\sqrt{2}$
13) $8\sqrt{3n}$
15) $14v$
17) $6x\sqrt{7}$
19) $-10k^2$
21) $-56x^2$
23) $-30\sqrt{m}$

25) $3xy\sqrt{5}$
27) $4xy\sqrt{xy}$
29) $8x^2y^2\sqrt{5}$
31) $24y\sqrt{5x}$
33) $35xy\sqrt{5y}$
35) $-12u\sqrt{5uv}$
37) $-48x^2yz^2\sqrt{5}$
39) $8j^2\sqrt{5hk}$
41) $-12p\sqrt{6mn}$

Introduction to Complex Numbers

1) $11+4i$
3) $-3+9i$
5) $-3-13i$
7) $-4-11i$
9) $-8-2i$
11) 48
13) 40
15) -49
17) $11+60i$
19) $80-10i$
21) $27+38i$
23) $44+8i$
25) $-3+11i$
27) $5+9i$
29) $-\frac{3}{2}+\frac{5}{3}i$
31) $-\frac{3}{2}+\frac{3}{4}i$
33) $1+10i$
35) $\frac{4}{101}-\frac{40}{101}i$
37) $\frac{56}{85}+\frac{48}{85}i$
39) $\frac{70}{149}+\frac{49}{149}i$
41) $-\frac{5}{37}-\frac{30}{37}i$
43) $9i$
45) $2\sqrt{5}$
47) $\frac{1}{2}+\frac{\sqrt{3}}{2}i$
49) $2-i$
51) i
53) 1
55) -1
57) -1

Vertex Form and Graphing

The Vertex Form

V=vertex form, S=standard form, B=both

1) V, $(12,5)$
3) B, $(0,8)$
5) V, $(1,2)$
7) S
9) B, $(0,-3)$
11) V, $(1,0)$
13) $y=(x+1)^2-2$
15) $y=3(x+2)^2-13$
17) $y=x^2+6$
19) $y=(x+4)^2-16$
21) $y=(x+2)^2-6$
23) $y=4\left(x+\frac{5}{4}\right)^2-\frac{25}{4}$

Graphing Quadratics

No.) y-int, vertex, x-int(s)

1) $(0,-8)$, $(1,-9)$, $(-2,0),(4,0)$
3) $(0,10)$, $(3,-8)$, $(1,0)$, $(5,0)$
5) $(0,-18)$, $(3,0)$, $(3,0)$
7) $(0,-45)$, $(4,3)$, $(3,0),(5,0)$
9) $(0,5)$, $(2,9)$, $(-1,0),(5,0)$
11) $(0,-5)$, $(3,4)$, $(1,0),(5,0)$
13) $(0,-24)$, $(4,8)$, $(2,0),(6,0)$
15) $(0,9)$, $(-2,-3)$, $(-3,0),(-1,0)$
17) $(0,75)$, $(4,-5)$, $(3,0),(5,0)$
19) $(0,-175)$, $(-6,5)$, $(-7,0),(-5,0)$

The Method of Extracting Square Roots

1) $12\pm\sqrt{5}$
3) ± 4
5) $1\pm\sqrt{5}$
7) $-6\pm\sqrt{2}$
9) $1,7$
11) $-2\pm 4i$

Completing the Square

1) $x^2 - 30x + \underline{\ 225\ } = (x-15)^2$
3) $m^2 - 36m + \underline{\ 324\ } = (m-18)^2$
5) $x^2 - 15x + \underline{\ 225/4\ } = (x-15/2)^2$
7) $y^2 - y + \underline{\ 1/4\ } = (y-1/2)^2$

11) $v = 4 \pm \sqrt{29}i$
15) $k = 1 \pm \frac{\sqrt{215}}{5}i$
17) $x = -\frac{5}{2} \pm \sqrt{86}$
21) $x = -1 \pm \sqrt{21}i$
25) $x = -5 \pm 2i$
27) $n = 3,\ 7$
31) $x = 2 \pm \sqrt{29}i$
35) $n = 1,\ 7$
37) $b = -1 \pm \frac{\sqrt{258}}{6}i$
41) $v = -\frac{5}{2} \pm \frac{\sqrt{87}}{2}i$
45) $k = \frac{7}{2} \pm \frac{\sqrt{137}}{2}i$
47) $x = -4,\ \frac{12}{5}$
51) $r = -\frac{5}{8} \pm \frac{\sqrt{415}}{8}i$
55) $x = -\frac{5}{2},\ 1$

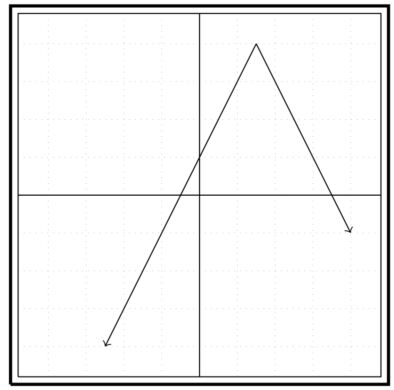
The Quadratic Formula and the Discriminant

1) Two real roots, $x = -1 \pm \sqrt{2}$
3) Two real roots, $x = -2 \pm \frac{\sqrt{39}}{3}$
5) No real roots
7) Two real roots, $x = 0,\ -8$
9) Two real roots, $x = -2 \pm \sqrt{6}$
11) Two real roots, $x = 0,\ -\frac{5}{2}$

13) $k = \pm\frac{\sqrt{6}}{3}i$
15) $n = \pm\frac{\sqrt{6}}{6}$
17) $p = -\frac{1}{5} \pm \frac{\sqrt{29}}{5}i$
21) $b = \pm\sqrt{2}i$
25) $n = \frac{1}{4} \pm \frac{\sqrt{159}}{12}i$
27) $x = \pm\sqrt{3}$
31) $m = 5,\ -9$
35) $r = -1 \pm \frac{\sqrt{3}}{3}i$
37) $a = -\frac{5}{12} \pm \frac{\sqrt{337}}{12}$
41) $x = \pm 2\sqrt{2}$
45) $x = \frac{2}{7} \pm \frac{3\sqrt{5}}{7}i$
47) $m = \frac{5}{14} \pm \frac{\sqrt{143}}{14}i$
49) $x = \pm\frac{\sqrt{6}}{2}$

Quadratic Inequalities and Sign Diagrams

1) $(-\infty, -9] \cup [-8, \infty)$
3) $(-\infty, 1] \cup [8, \infty)$
5) $(-\infty, -1] \cup [10, \infty)$
7) $\left(-\infty, \frac{2}{7}\right] \cup [6, \infty)$
9) $\left(-\infty, -\frac{4}{7}\right] \cup [4, \infty)$
11) $\left(-\infty, -\frac{1}{2}\right] \cup [7, \infty)$
13) $\left(-\infty, -\frac{6}{7}\right] \cup [5, \infty)$
15) $(-\infty, -3] \cup \left[\frac{6}{7}, \infty\right)$
17) $\left[-2 - \frac{\sqrt{21}}{3}, -2 + \frac{\sqrt{21}}{3}\right]$
19) $(-\infty, -2] \cup [0, \infty)$



Chapter 5

Advanced Function Concepts

Identifying Domain Algebraically (L30)

Objective: Identify the domain of a function that is described algebraically.

When trying to identify the domain of a function that has been described algebraically or whose graph is not known, we will often need to consider what is *not* permissible for the function, then exclude any values of x that will make the function undefined from the interval $(-\infty, \infty)$. What is left will be our domain. With virtually every algebraic function, this amounts to avoiding the following situations.

- Negatives under an even radical $\left(\sqrt{}\,,\ \sqrt[4]{}\,,\ \sqrt[6]{}\,,\ \ldots\right)$
- Zero in a denominator

In the previous chapters, we dealt exclusively with linear equations. While equations of the form $y = mx + b$ represent y as a function of x, they are also included in a much larger family of functions known as *polynomials*. Polynomials are functions of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x + a_1 x + a_0,$$

where each of the coefficients a_i represent real numbers (with $a_n \neq 0$) and n represents a nonnegative integer. These functions include quadratics, which are of the form $y = ax^2 + bx + c$. Since polynomials contain no radicals or variables in a denominator, we can immediately conclude that their domain will always be all real numbers, or $(-\infty, \infty)$. We reiterate this with our first example.

Example 228. Find the domain of $f(x) = \frac{1}{3}x^2 - x$.

$f(x) = \dfrac{1}{3}x^2 - x$	No radicals or variables in a denominator
	No values of x need to be excluded
All real numbers or $(-\infty, \infty)$	Our solution

Our next example will be of a *rational function*, which is defined as a ratio of two polynomial functions. We will explore rational functions and their graphs in a later chapter. Since rational functions usually include expressions in a denominator, their domains will often require us to exclude one or more values of x.

Example 229. Find the domain of the function $f(x) = \dfrac{3x-1}{x^2+x-6}$.

$$\begin{aligned} f(x) &= \frac{3x-1}{x^2+x-6} && \text{Cannot have zero in a denominator} \\ x^2+x-6 &\neq 0 && \text{Solve by factoring} \\ (x+3)(x-2) &\neq 0 && \text{Set each factor not equal to zero} \\ x+3\neq 0 \ \text{ and } \ x-2 &\neq 0 && \text{Solve each inequality} \\ x &\neq -3, 2 && \text{Our solution as an inequality} \\ (-\infty,-3)\cup(-3,2)\cup(2,\infty) & && \text{Our solution using interval notation} \end{aligned}$$

The notation in the previous example tells us that x can be any value except for -3 and 2. If x were to equal one of those two values, our expression in the denominator would reduce to zero and the function would consequently be undefined. Furthermore, although one can easily see that $x = \frac{1}{3}$ will make the numerator equal zero, since $x = \frac{1}{3}$ does not coincide with the two values obtained above (either -3 or 2), we should not exclude it from our domain.

This example further illustrates that whenever we are finding the domain of a rational function, we need not be concerned at all with the numerator, and instead must restrict our domain to exclude any value for x that would make the *denominator* equal to zero.

For our final two examples, we will introduce a square root in our function, first in the numerator and later in the denominator.

Example 230. Find the domain of $f(x) = \sqrt{-2x+3}$.

$$\begin{aligned} f(x) &= \sqrt{-2x+3} && \text{Even radical; cannot have negative underneath} \\ -2x+3 &\geq 0 && \text{Set greater than or equal to zero and solve} \\ -2x &\geq -3 && \text{Remember to switch direction of inequality} \\ x \leq \frac{3}{2} \ \text{ or } & \left(-\infty, \frac{3}{2}\right] && \text{Our solution as an inequality or an interval} \end{aligned}$$

The notation in the above example states that our variable can be $\frac{3}{2}$ or any real number less than $\frac{3}{2}$. But any number greater than $\frac{3}{2}$ would make the function undefined.

Example 231. Find the domain of $m(x) = \dfrac{-x}{\sqrt{7x-3}}$.

The even radical tells us that we cannot have a negative value underneath. But also, the denominator cannot equal zero. This results in two inequalities.

$$7x-3 \geq 0 \ \text{ AND } \ 7x-3 \neq 0$$

Solving for x, we get the following.

$$x \geq \frac{3}{7} \text{ AND } x \neq \frac{3}{7}$$

Our final solution is $x > \frac{3}{7}$, or $\left(\frac{3}{7}, \infty\right)$ as an interval. This represents the intersection of both inequalities above.

The previous two examples can be generalized as follows.

- In instances where the given function is a square root (or even radical), to find the domain we may set up and solve an inequality in which the entire expression underneath is set ≥ 0.
- In instances where the numerator of a given function is a polynomial and the denominator is a square root (or even radical), to find the domain we may set up and solve an inequality in which the expression underneath is set > 0 (strictly positive).

Since these two cases certainly do not handle every possible function than we may encounter, one should always be cautious when attempting to find the domain of any function.

Combining Functions

Function Arithmetic (L32)

Objective: Add, subtract, multiply, and divide functions.

In this section, we demonstrate how two (or more) functions can be combined to create new functions. This is accomplished using five common operations: the four basic arithmetic operations of addition, subtraction, multiplication and division, and a fifth operation that we will establish later in the section, known as a *composition*.

The notation for the four basic functions is as follows.

$$\begin{array}{rl}
\text{Addition} & (f+g)(x) = f(x) + g(x) \\
\text{Subtraction} & (f-g)(x) = f(x) - g(x) \\
\text{Multiplication} & (f \cdot g)(x) = f(x)g(x) \\
\text{Division} & \left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}, \text{ where } g(x) \neq 0
\end{array}$$

As we will see in the next few examples, when applying the specified operations, one must be careful to completely simplify, by distributing and combining like terms where it is necessary. We will demonstrate this for each operation, highlighting the most critical steps in the process.

Example 232. Find $f+g$, where $f(x) = x^2 - x - 2$ and $g(x) = x + 1$.

$(f+g)(x)$	Consider the problem
$f(x) + g(x)$	Rewrite as a sum of two functions
$(x^2 - x - 2) + (x + 1)$	Substitute functions, inserting parentheses
$x^2 - x - 2 + x + 1$	Simplify; remove the parentheses
$x^2 - x + x - 2 + 1$	Combine like terms
$(f+g)(x) = x^2 - 1$	Our solution
$= (x-1)(x+1)$	Our solution in factored form

We include the factored form of $f+g$ in the previous example to reinforce the methods of factorization learned in an earlier chapter. Generally, either form (expanded or factored) would be considered acceptable.

Although the parentheses are not entirely necessary in our first example, we have included them nevertheless, to reinforce that each operation is applied to an *entire* function or expression. This will become more apparent in our next example (subtraction), when we will need to distribute a negative sign.

Example 233. Find $g-f$, where $f(x) = x^2 - x - 2$ and $g(x) = x + 1$.

$(g-f)(x)$	Consider the problem
$g(x) - f(x)$	Rewrite as a difference of two functions
$(x + 1) - (x^2 - x - 2)$	Substitute functions, inserting parentheses
$x + 1 - x^2 + x + 2$	Simplify; distribute the negative sign
$-x^2 + x + x + 1 + 2$	Combine like terms
$(g-f)(x) = -x^2 + 2x + 3$	Our solution
$= -(x-3)(x+1)$	Our solution in factored form

Example 234. Find $(h \cdot k)(x)$, where $h(x) = 3x^2 - 4x$ and $k(x) = x - 2$.

$(h \cdot k)(x)$	Consider the problem
$h(x) \cdot k(x)$	Rewrite as a product of two functions
$(3x^2 - 4x)(x - 2)$	Substitute functions, inserting parentheses
$3x^3 - 6x^2 - 4x^2 + 8x$	Expand by distributing
$3x^3 - 10x^2 + 8x$	Combine like terms
$(h \cdot k)(x) = 3x^3 - 10x^2 + 8x$	Our solution
$= x(3x-4)(x-2)$	Our solution in factored form

Example 235. Find $\left(\dfrac{g}{f}\right)(x)$, where $f(x) = x^2 - x - 2$ and $g(x) = x + 1$.

$$\left(\frac{g}{f}\right)(x) \qquad \text{Consider the problem}$$

$$\frac{g(x)}{f(x)} \qquad \text{Rewrite as a quotient of two functions}$$

$$\frac{x+1}{x^2 - x - 2} \qquad \text{Substitute functions, parentheses unnecessary}$$

$$\frac{x+1}{(x+1)(x-2)} \qquad \text{Factor (if possible)}$$

$$x \neq -1 \quad \text{and} \quad x \neq 2 \qquad \text{Restrict denominator: } g(x) \neq 0$$

$$\frac{\cancel{x+1}}{\cancel{(x+1)}(x-2)} \qquad \text{Simplify: reduce } \frac{x+1}{x+1}$$

$$\left(\frac{g}{f}\right)(x) = \frac{1}{x-2}, \quad x \neq -1 \qquad \text{Our solution with added restriction}$$

The previous example presents us with a new precautionary measure that we must be careful not to overlook. This has to do with the simplification of g/f and the requirement that we include the necessary restriction of $x \neq -1$. Although the *domain* of the resulting quotient is still $x \neq -1, 2$, we have included $x \neq -1$ as part of our final answer, since the simplified expression allows us to easily determine that x cannot equal 2, but fails to carry through the additional restriction.

In general, whenever we simplify any function, we must be careful to insure that the domain of the resulting expression will be in agreement with the initial *unsimplified* expression. In the chapter on rational functions, we will see the graphical consequence that arises when the restriction $x \neq -1$ is overlooked.

Thus far, we have sought to create new functions by combining two functions f and g accordingly, keeping the variable x in place throughout. We could, however, just as easily evaluate the functions $f + g$, $f - g$, $f \cdot g$, and f/g at certain values of x. We do this in our next example.

Example 236. Find $(h \cdot k)(5)$, where $h(x) = 2x - 4$ and $k(x) = -3x + 1$.

$$h(x) = 2x - 4 \text{ and } k(x) = -3x + 1 \qquad \text{Evaluate each function at } 5$$

$$h(5) = 2(5) - 4 = 6 \qquad \text{Evaluate } h \text{ at } 5$$

$$k(5) = -3(5) + 1 = -14 \qquad \text{Evaluate } k \text{ at } 5$$

$$\begin{aligned}(h \cdot k)(5) &= (h(5)) \cdot (k(5)) && \text{Multiply the two results}\\ &= (6)(-14) \\ &= -84 && \text{Our solution}\end{aligned}$$

The clear advantage to this process is that the simplification can be substantially easier when the variable has been replaced with a constant. One major disadvantage, however, is that our end result represents only a single value, instead of an entire function. Particularly in situations where the resulting function is not demanded, students will likely find it more efficient to use this approach when evaluating $f+g$, $f-g$, $f \cdot g$ and f/g at a specified value.

Composite Functions (L33)

Objective: Construct, evaluate, and interpret composite functions.

In addition to the four basic arithmetic operations $(+, -, \cdot, \div)$, we will now discuss a fifth operation, known as a *composition* and denoted by $\circ$ (not to be confused with a product, $\cdot$). The result of a composition is called a *composite function* and is defined as follows.

$$\boldsymbol{(f \circ g)(x) = f(g(x))}$$

The notation $(f \circ g)(x)$ above should always be interpreted as "f of g of x". In this situation, we consider g to be the *inner* function, since it is being substituted into f for x. Consequently, we refer to f as the *outer* function.

Similarly, if we reversed the order of the two functions f and g, then the resulting composite function $(g \circ f)(x) = g(f(x))$ will have inner function f and outer function g, and should be interpreted as "g of f of x". As we will see, one should never assume that the two composite functions $f \circ g$ and $g \circ f$ will be equal.

The idea behind a composition, though relatively simple, can often pose a formidable challenge at first. We will begin by evaluating a composite function at a single value. This is accomplished by first evaluating the inner function at the specified value, and then substituting ("plugging in") the corresponding *output* into the outer function.

Example 237. Find $(f \circ g)(3)$, where $f(x) = x^2 - 2x + 1$ and $g(x) = x - 5$.

$$\begin{aligned}(f \circ g)(3) &= f(g(3)) && \text{Rewrite } f \circ g \text{ as inner and outer functions}\\[1em] g(3) &= (3) - 5 = -2 && \text{Evaluate inner function at } x = 3\\ & && \text{Use output of } -2 \text{ as input for } f\\ f(-2) &= (-2)^2 - 2(-2) + 1 && \text{Evaluate outer function at } x = -2\\ &= 4 + 4 + 1 && \text{Simplify}\\ (f \circ g)(3) &= 9 && \text{Our solution}\end{aligned}$$

We can also identify a composite function in terms of the variable. In the next example, we will substitute the inner function into the outer function for every instance of the variable and then simplify. This approach is often referred to as the "inside-out" approach by some instructors.

Example 238. Find $(f \circ g)(x)$, where $f(x) = x^2 - x$ and $g(x) = x + 3$.

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) && \text{Rewrite } f \circ g \text{ as inner and outer functions} \\
& && \text{Our inner function is } g(x) = x + 3 \\
f(x+3) & && \text{Replace each } x \text{ in } f \text{ with } (x+3) \\
& && \text{Make sure to include parentheses!} \\
(x+3)^2 - (x+3) & && \text{Simplify; expand binomial} \\
(x^2 + 6x + 9) - (x+3) & && \text{Distribute negative} \\
x^2 + 6x + 9 - x - 3 & && \text{Combine like terms} \\
(f \circ g)(x) &= x^2 + 5x + 6 && \text{Our solution} \\
&= (x+3)(x+2) && \text{Our solution in factored form}
\end{aligned}$$

It is important to reiterate that $(f \circ g)(x)$ usually will *not* equal $(g \circ f)(x)$ as the next example shows. Again, we will take the "inside-out" approach, where the inner function is now f and the outer function is g.

Example 239. Find $(g \circ f)(x)$, where $f(x) = x^2 - x$ and $g(x) = x + 3$.

$$\begin{aligned}
(g \circ f)(x) &= g(f(x)) && \text{Rewrite } g \circ f \text{ as inner and outer functions} \\
& && \text{Our inner function is } f(x) = x^2 - x \\
g(x^2 - x) & && \text{Replace each } x \text{ in } f \text{ with } (x^2 - x) \\
(x^2 - x) + 3 & && \text{Simplify; remove parentheses} \\
(g \circ f)(x) &= x^2 - x + 3 && \text{Our solution}
\end{aligned}$$

Notice that a simple calculation of the discriminant,

$$b^2 - 4ac = (-1)^2 - 4(1)(3) = -11 < 0,$$

tells us that the resulting composite function is irreducible (not factorable) over the real numbers.

Here is another example, for additional practice.

Example 240. Find $(m \circ n)(x)$, where $m(x) = 5x^2 - x + 1$ and $n(x) = x - 4$.

$$\begin{aligned}
(m \circ n)(x) &= m(n(x)) && \text{Rewrite } m \circ n \text{ as inner and outer functions} \\
& && \text{Our inner function is } n(x) = x - 4 \\
g(x-4) & && \text{Replace each } x \text{ in } m \text{ with } (x-4) \\
& && \text{Make sure to include parentheses!} \\
5(x-4)^2 - (x-4) + 1 & && \text{Simplify; expand binomial} \\
5(x^2 - 8x + 16) - (x-4) + 1 & && \text{Distribute negative and the five} \\
5x^2 - 40x + 80 - x + 4 + 1 & && \text{Combine like terms} \\
(m \circ n)(x) &= 5x^2 - 41x + 85 && \text{Our solution}
\end{aligned}$$

It is also possible to compose a function with itself, as the next example shows.

Example 241. Find $(g \circ g)(x)$, where $g(x) = x^2 - 2x$.

$(g \circ g)(x) = g(g(x))$	Rewrite $g \circ g$ as inner and outer functions
	Our inner function is $g(x) = x^2 - 2x$
$g(x^2 - 2x)$	Replace each x in g with $x^2 - 2x$
	Make sure to include parentheses!
$(x^2 - 2x)^2 - 2(x^2 - 2x)$	Simplify; expand binomial
$(x^4 - 4x^3 + 4x^2) - 2(x^2 - 2x)$	Distribute -2
$x^4 - 4x^3 + 4x^2 - 2x^2 + 4x$	Combine like terms
$(g \circ g)(x) = x^4 - 4x^3 + 2x^2 + 4x$	Our solution

We close this section by demonstrating the "outside-in" approach to finding a composite function $f \circ g$. The idea behind this approach is to *first* rewrite the outer function f by its given expression, replacing each instance of the variable with the general $g(x)$. To see that this will yield the same result as the "inside-out" approach, we will revisit example 238 above.

Example 242. Find $(f \circ g)(x)$, where $f(x) = x^2 - x$ and $g(x) = x + 3$.

$(f \circ g)(x) = f(g(x))$	Rewrite $f \circ g$ as inner and outer functions
	Our outer function is $f(x) = x^2 - x$
$[g(x)]^2 - [g(x)]$	Replace each x in f with $g(x)$
$(x+3)^2 - (x+3)$	Replace each $g(x)$ by $x + 3$
	Make sure to include parentheses!
$(x^2 + 6x + 9) - (x + 3)$	Simplify; expand binomial
$x^2 + 6x + 9 - x - 3$	Distribute negative
$x^2 + 5x + 6$	Combine like terms
$(f \circ g)(x) = x^2 + 5x + 6$	Our solution
$= (x+3)(x+2)$	Our solution in factored form

Inverse Functions

Definition and the Horizontal Line Test (L34)

Objective: Understand the definition of an inverse function and graphical implications. Determine whether a function is invertible.

In this section, we introduce the notion of an inverse function to a function f, and develop an understanding of the relationship (both algebraic and graphical) between a function f and its inverse.

One often considers the operations of addition and subtraction to be "opposites" of one another, and similarly for multiplication and division. The reason for this, naturally, is because each of these operations "undoes" the other. In mathematics, since the term "opposite" can take on different meanings, we instead consider addition and subtraction (or multiplication and division) to be *inverse operations* of one another. This notion of an inverse can be applied to entire functions, which we will now discuss.

We start by analyzing a very basic function which is reversible, a linear function. Consider the function $f(x) = 3x + 4$. Thinking of f as a process, we start with an input x and apply two steps, in order:

1. multiply by 3
2. add 4.

To reverse this process, we seek a function g which will undo each of these steps, by taking the output from f, $3x + 4$, and returning the original input x. If we think of the real-world reversible two-step process of first putting on socks then putting on shoes, to reverse the process, we first take off the shoes, and then we take off the socks. In much the same way, the function g should undo the last step of f first. That is, the function g should:

1. subtract 4, then
2. divide by 3.

Following this procedure, we get $g(x) = \dfrac{x-4}{3}$.

Now we can test our function to see if it conceptually agrees with our "feet, socks, and shoes" analogy. Just as in the first part of the process we began with our bare feet and ended up in shoes, the reverse process brings us back, in the end, to our bare feet. We can see if this holds for f and g by using what we already know about functions.

For example, if $x = 5$, then

$$f(5) = 3(5) + 4 = 15 + 4 = 19.$$

Substituting the output 19 from f as our new input for g, we get our original input for f.

$$g(19) = \frac{19-4}{3} = \frac{15}{3} = 5$$

To check that g does this for all x in the domain of f (not just a single value), we will need to find and simplify the composite function $(g \circ f)(x) = g(f(x))$.

$$g(f(x)) = g(3x+4) = \frac{(3x+\cancel{4}) - \cancel{4}}{3} = \frac{\cancel{3}x}{\cancel{3}} = x$$

If we carefully examine the arithmetic, as we simplify $g(f(x))$, we can actually see g "undoing" the addition of 4 first, followed by the multiplication by 3.

Not only does g "undo" f, but f also undoes g, which we can verify by once again looking at a composite function. This time we will find and simplify $(f \circ g)(x) = f(g(x))$.

$$f(g(x)) = f\left(\frac{x-4}{3}\right) = \cancel{3}\left(\frac{x-4}{\cancel{3}}\right) + 4 = (x - \cancel{4}) + \cancel{4} = x$$

In each composition, we began and ended with the variable x, which can be thought of as the bare feet in our analogy. Two functions f and g which are related in this manner are defined to be *inverse functions*, or simply *inverses*, of each other. More precisely, using the language of function composition, two functions f and g are said to be inverses if both:

- $g(f(x)) = x$ for all x in the domain of f, and
- $f(g(x)) = x$ for all x in the domain of g.

We say that a function f is *invertible* if an inverse function of f exists. If two functions g and f are inverses of each other, then we denote this by $g(x) = f^{-1}(x)$, and similarly $f(x) = g^{-1}(x)$. This notation can be a bit "gnarly" at first, since an inverse function f^{-1} of f must not be confused with the reciprocal function, $1/f$. The primary difference between these two functions is that a reciprocal function satisfies the property that

$$f(x) \cdot (1/f)(x) = 1,$$

whereas for inverses,

$$(f \circ f^{-1})(x) = x \quad \text{and} \quad (f^{-1} \circ f)(x) = x.$$

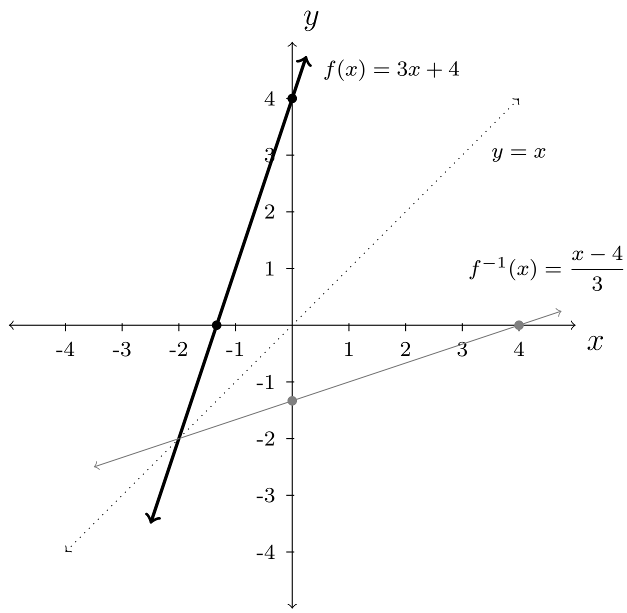
Using our function $f(x) = 3x + 4$, we can see this distinction.

- Original Function: $f(x) = 3x + 4$
- Inverse Function: $f^{-1}(x) = \dfrac{x-4}{3}$
- Reciprocal Function: $\left(\dfrac{1}{f}\right)(x) = \dfrac{1}{3x+4}$

Properties of Inverse Functions:
Let f and f^{-1} be inverse functions of one another.

- The range of f is the domain of f^{-1} and the domain of f is the range of f^{-1}.
- $f(a) = b$ if and only if $f^{-1}(b) = a$.
- The point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}.

As a direct consequence of the third property above, we will see that the graph of f^{-1} may be obtained by reflecting the graph of f about the line $y = x$. Again, we will use our example, by graphing the inverse functions $f(x) = 3x+4$ and $f^{-1}(x) = \dfrac{x-4}{3}$ on the same set of axes.



Again, from the third property, the figure above confirms that the $y-$intercept $(0, b)$ of the graph of f will be an $x-$intercept $(b, 0)$ of the graph of f^{-1}. Similarly, the $x-$intercept of the graph of f will be a $y-$intercept of the graph of f^{-1}.

Let us now turn our attention to the quadratic function $f(x) = x^2$. Is f invertible? If we consider the idea of "undoing" an operation, a likely candidate for the inverse of f is the function $g(x) = \sqrt{x}$. Checking the composition gives us

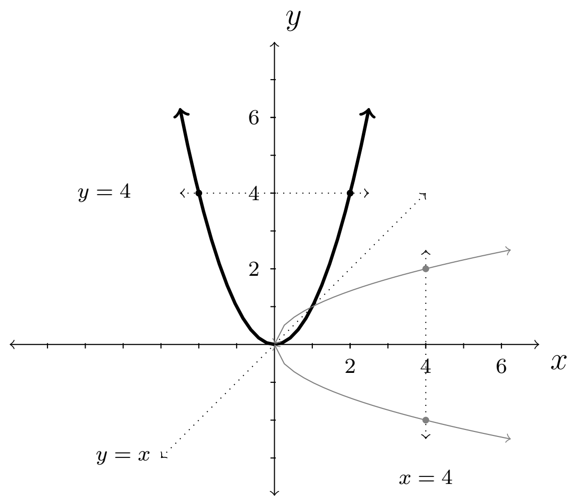
$$(g \circ f)(x) = g(f(x)) = \sqrt{x^2} = |x|,$$

which is not equal to x, for all real numbers in the domain of f, $(-\infty, \infty)$.

This subtle issue arises when we input a negative value for x into the composition above. For example, when $x = -2$, $f(-2) = (-2)^2 = 4$, but $g(4) = \sqrt{4} = 2$. Hence, g fails to return the original input $x = -2$ from its output of 4. What g does, however, is match the output 4 to a *different* input, namely $x = 2$, since $f(2)$ also equals 4.

Since both $f(-2)$ and $f(2)$ equal 4, it will be impossible to construct a function which inputs $x = 4$ and outputs *both* $x = 2$ and $x = -2$. This is due to the fact that, by definition, a function assigns a real number x with exactly one other real number.

Furthermore, we know that if and inverse f^{-1} of $f(x) = x^2$ exists, its graph can be obtained by reflecting the graph of x^2 about the line $y = x$.



In the above graph, we see that the vertical line $x = 4$ intersects the reflection of the parabola $y = x^2$ about the diagonal $y = x$ twice, which fails the Vertical Line Test, and as such, our proposed inverse cannot represent y as a function of x.

The vertical line $x = 4$ corresponds to the *horizontal line* $y = 4$ intersecting the graph of the parabola $y = x^2$. The fact that the horizontal line $y = 4$ intersects the graph of $y = x^2$ twice further confirms that two *different* inputs, namely $x = -2$ and $x = 2$, are paired with the *same* output, 4, which is the cause of all our trouble in attempting to find an inverse function to $f(x) = x^2$.

In general, in order for a function to be invertible, the function must have the property that any two inputs for x can never be paired with the same output, or else we will run into the same problem as with $f(x) = x^2$. We give this property a name.

> A function f is said to be *one-to-one* if f matches different inputs to different outputs. Equivalently, f is one-to-one if and only if whenever $f(c) = f(d)$, then $c = d$.

Graphically, we can identify one-to-one functions using the following test.

> **The Horizontal Line Test (HLT):**
> A function f is one-to-one if and only if no horizontal line intersects the graph of f more than once.

We say that the graph of a function *passes* the Horizontal Line Test if no horizontal line intersects the graph more than once; otherwise, we say the graph of the function *fails* the Horizontal Line Test.

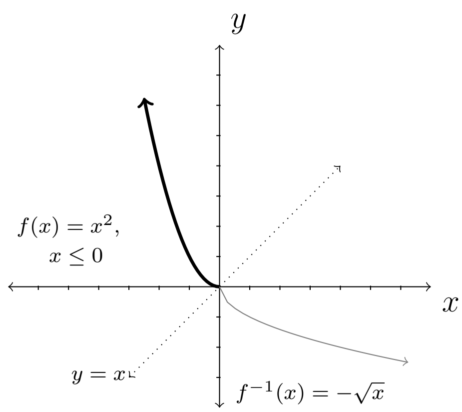
Lastly, we have argued that if f is invertible, then f must be one-to-one, since otherwise the reflection of the graph of $y = f(x)$ about the line $y = x$ will fail the Vertical Line Test. It turns out that being one-to-one is also enough to guarantee invertibility of a function f. To see this, we can think of f as the set of ordered pairs which constitute its graph. If switching the x- and y-coordinates of the points results in a function (i.e., passes the VLT), then f is invertible and we have found the graph of its inverse, f^{-1}. This is precisely what the Horizontal Line Test does for us: it checks to see whether or not a set of points describes x as a function of y.

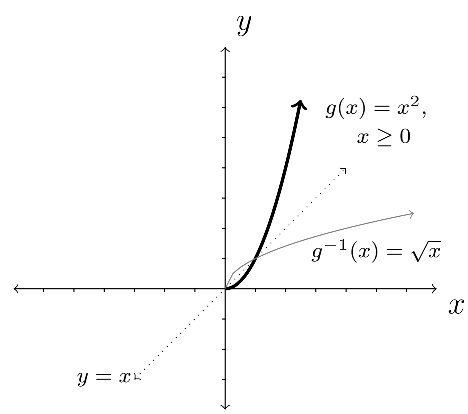
We can now summarize our results.

Equivalent Conditions for Invertibility:
Suppose f is a function. The following statements are equivalent.

- f is invertible (f^{-1} exists).
- f is one-to-one.
- The graph of f passes the Horizontal Line Test.

In the case of $f(x) = x^2$, since the corresponding parabola fails the Horizontal Line Test, f is not invertible. If we were to restrict the domain of our function to either the left half ($x \le 0$) or right half ($x \ge 0$) of the parabola, however, we could produce a function that passes the HLT and consequently has an inverse, as seen in the following two graphs.





In the next subsection, we will outline the process of determining whether or not a function is invertible, and if so, find its inverse function algebraically.

Finding Inverses Algebraically (L35)

Objective: Find the inverse of a given function.

Recall that a function f is one-to-one if and only if whenever $f(c) = f(d)$, then $c = d$. Using this definition, we will now test whether a given function is one-to-one and consequently invertible.

Example 243. Determine if the function $f(x) = \dfrac{1-2x}{5}$ is one-to-one.

Notice that f is a linear function with a nonzero slope. Hence, its graph passes the Horizontal Line Test. To confirm that f is one-to-one algebraically, we begin by assuming $f(c) = f(d)$ and attempt to deduce that $c = d$.

$$\begin{aligned} f(c) &= f(d) \\ \frac{1-2c}{5} &= \frac{1-2d}{5} \\ 1-2c &= 1-2d \\ -2c &= -2d \\ c &= d \checkmark \end{aligned}$$

Hence, f is one-to-one.

Example 244. Determine if the function $g(x) = \dfrac{2x}{1-x}$ is one-to-one.

The function g is known as a rational function, and will be formally discussed in a later chapter. To determine whether or not g is one-to-one, we must use an algebraic approach. Again, we begin with the assumption that $g(c) = g(d)$.

$$\begin{aligned} g(c) &= g(d) \\ \frac{2c}{1-c} &= \frac{2d}{1-d} \\ 2c(1-d) &= 2d(1-c) \\ 2c-2cd &= 2d-2dc \\ 2c &= 2d \\ c &= d \checkmark \end{aligned}$$

Hence, g is one-to-one.

Example 245. Determine if the function $h(x) = x^2 - 2x + 4$ is one-to-one.

Notice that h is a quadratic function, whose graph is a parabola, and consequently fails the Horizontal Line Test. This means that our function should not be one-to-one. We now verify this algebraically.

Let $h(c) = h(d)$. As we work our way through the problem, we encounter a nonlinear equation, which requires us to set the right-hand side equal to zero and factor accordingly.

$$\begin{array}{rcll} h(c) &=& h(d) & \\ c^2-2c+4 &=& d^2-2d+4 & \\ c^2-2c &=& d^2-2d & \\ c^2-d^2-2c+2d &=& 0 & \text{Factor by grouping} \\ (c+d)(c-d)-2(c-d) &=& 0 & \text{Difference of squares} \\ (c-d)((c+d)-2) &=& 0 & \\ c-d=0 &\text{or}& c+d-2=0 & \\ c=d &\text{or}& c=2-d & \end{array}$$

We get $c = d$ as one possibility, but we also get the possibility that $c = 2 - d$. This suggests that h will likely not be one-to-one.

Letting $d = 0$, we get $c = 0$ or $c = 2$. This implies that, $h(0) = 4$ and $h(2) = 4$, and we have produced two different inputs with the same output. Hence, h is not one-to-one, as anticipated.

Once we have established whether a function f is one-to-one, and consequently invertible, our next task is to identify f^{-1} precisely. In the previous part of this section, we noticed that switching each point, (x, y), of the graph of f produced a point (y, x) on the graph of f^{-1}. This is our motivation in the steps for finding an inverse algebraically, as we will be switching the x and y coordinates to do so.

Steps for finding the Inverse of a Function

1. Rewrite $f(x)$ as y.
2. Switch x and y.
3. Solve for y.
4. Rewrite y as $f^{-1}(x)$.

In the next few examples, we find the inverse of each function f, as well as confirm that the domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}. We also check each answer using function composition. We leave it as an exercise to the reader to graph each function (using a graphing utility where necessary), and verify that the two functions are reflections of each other about the line $y = x$.

Example 246. Find the inverse f^{-1} of the function $f(x) = \dfrac{1-2x}{5}$. Verify using compositions that f and f^{-1} are inverses, and that the domain and range of f equal the range and domain of f^{-1}, respectively.

We write $y = f(x)$ and proceed to switch x and y

$$\begin{aligned}
y &= f(x) \\
y &= \frac{1-2x}{5} \\
x &= \frac{1-2y}{5} && \text{Switch } x \text{ and } y \\
5x &= 1-2y && \text{Solve for } y \\
5x-1 &= -2y \\
\frac{5x-1}{-2} &= y \\
y &= -\frac{5}{2}x+\frac{1}{2}
\end{aligned}$$

We have $f^{-1}(x) = -\frac{5}{2}x + \frac{1}{2}$.

To verify this answer, we first check that $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f, which is all real numbers.

$$\begin{aligned}
\left(f^{-1}\circ f\right)(x) &= f^{-1}(f(x)) \\
&= -\frac{5}{2}f(x)+\frac{1}{2} \\
&= -\frac{5}{2}\left(\frac{1-2x}{5}\right)+\frac{1}{2} \\
&= -\frac{1}{2}(1-2x)+\frac{1}{2} \\
&= -\frac{1}{2}+x+\frac{1}{2} \\
&= x \checkmark
\end{aligned}$$

We now check that $\left(f\circ f^{-1}\right)(x)=x$ for all x in the domain of f^{-1} which is also all real numbers.

$$\begin{aligned}
\left(f\circ f^{-1}\right)(x) &= f(f^{-1}(x)) \\
&= \frac{1-2f^{-1}(x)}{5} \\
&= \frac{1-2\left(-\frac{5}{2}x+\frac{1}{2}\right)}{5} \\
&= \frac{1+5x-1}{5} \\
&= \frac{5x}{5} \\
&= x \checkmark
\end{aligned}$$

Since both f and f^{-1} are linear functions with nonzero slopes, their domain and range is all real numbers, $(-\infty,\infty)$.

Example 247. Find the inverse g^{-1} of the function $g(x)=\dfrac{2x}{1-x}$. Verify using compositions that g and g^{-1} are inverses, and that the domain and range of g equal the range and domain of g^{-1}, respectively.

Notice that the domain of g is $(-\infty,1)\cup(1,\infty)$. One can verify graphically, that the range of g is $(-\infty,-2)\cup(-2,\infty)$.

To find $g^{-1}(x)$, we start with $y=g(x)$.

$$\begin{aligned}
y &= g(x) \\
y &= \frac{2x}{1-x} \\
x &= \frac{2y}{1-y} && \text{Switch } x \text{ and } y \\
x(1-y) &= 2y && \text{Solve for } y\text{; clear denominator} \\
x-xy &= 2y && \text{Distribute } x \\
x &= xy+2y && \text{Move } y \text{ terms to one side} \\
x &= y(x+2) && \text{Factor out } y \\
y &= \frac{x}{x+2} && \text{Divide by } x+2
\end{aligned}$$

We have $g^{-1}(x) = \dfrac{x}{x+2}$.

Notice that the domain of g^{-1} matches the range of g from earlier, $(-\infty, -2) \cup (-2, \infty)$. Again, we can use the graph of g^{-1} to verify that the range of g^{-1} also matches the domain of g, $(-\infty, 1) \cup (1, \infty)$.

To check that our inverse is correct, we first check that $\left(g^{-1} \circ g\right)(x) = x$.

$$\begin{aligned}
\left(g^{-1} \circ g\right)(x) &= g^{-1}(g(x)) \\
&= g^{-1}\left(\frac{2x}{1-x}\right) \\
\left(g^{-1} \circ g\right)(x) &= \frac{\left(\dfrac{2x}{1-x}\right)}{\left(\dfrac{2x}{1-x}\right)+2} \\
&= \frac{\left(\dfrac{2x}{1-x}\right)}{\left(\dfrac{2x}{1-x}\right)+2} \cdot \frac{(1-x)}{(1-x)} && \text{Clear denominators} \\
&= \frac{2x}{2x+2(1-x)} \\
&= \frac{2x}{2x+2-2x} \\
&= \frac{2x}{2} \\
&= x \ \checkmark
\end{aligned}$$

Lastly, we check that $\left(g \circ g^{-1}\right)(x) = x$.

$$\begin{aligned}
\left(g \circ g^{-1}\right)(x) &= g\left(g^{-1}(x)\right) \\
&= g\left(\frac{x}{x+2}\right) \\
&= \frac{2\left(\dfrac{x}{x+2}\right)}{1-\left(\dfrac{x}{x+2}\right)} \\
&= \frac{2\left(\dfrac{x}{x+2}\right)}{1-\left(\dfrac{x}{x+2}\right)} \cdot \frac{(x+2)}{(x+2)} && \text{Clear denominators}
\end{aligned}$$

$$\begin{aligned} &= \frac{2x}{(x+2)-x} \\ &= \frac{2x}{2} \\ &= x \checkmark \end{aligned}$$

For our last two examples, we revisit the inverse relationship between quadratics and functions containing square roots.

Example 248. Find the inverse h^{-1} of the function $h(x) = 3\sqrt{x}+4$. Compare the domain and range of h with that of h^{-1}. Verify using compositions that h and h^{-1} are inverses.

Notice that the domain of h is $x \geq 0$, or $[0, \infty)$, and the range is $y \geq 4$, or $[4, \infty)$.

To find $h^{-1}(x)$, we start with $y = h(x)$.

$$\begin{aligned} y &= h(x) \\ y &= 3\sqrt{x}+4 \end{aligned}$$

$$\begin{aligned} x &= 3\sqrt{y}+4 && \text{Switch } x \text{ and } y \\ x-4 &= 3\sqrt{y} && \text{Solve for } y \\ \frac{x-4}{3} &= \sqrt{y} \\ y &= \left(\frac{x-4}{3}\right)^2 && \text{Square both sides} \end{aligned}$$

We have

$$h^{-1}(x) = \left(\frac{x-4}{3}\right)^2 = \frac{1}{9}(x-4)^2,$$

whose graph is a parabola, opening upwards with vertex $(4, 0)$.

Consequently, the range of h^{-1} is $y \geq 0$, or $[0, \infty)$, which coincides with the domain of h. In order for our functions to truly be inverses of one another, however, we must impose a restriction on the domain of h^{-1}, which would otherwise be all real numbers. Instead, we only take the right half of the graph of our parabola, which coincides with a domain of h^{-1} of $x \geq 4$, or $[4, \infty)$. This restriction guarantees that the domain of h^{-1} matches the range of h, and that the graph of h^{-1} passes the Horizontal Line Test, which is a requirement of invertibility. We leave it as an exercise to the reader to show that $(h \circ h^{-1})(x) = x$ and $(h^{-1} \circ h)(x) = x$.

For our last example, we begin with a quadratic function, whose domain has already been restricted, in order to guarantee the existence of an inverse.

Example 249. Find the inverse f^{-1} of the function $f(x) = -2x^2 - 20x - 30$, where $x \geq -5$. Verify using compositions that f and f^{-1} are inverses, and that the domain and range of f equal the range and domain of f^{-1}, respectively.

To find $f^{-1}(x)$, we start with $y = f(x)$.

$$\begin{aligned} y &= f(x) \\ y &= -2x^2 - 20x - 30 \\ x &= -2y^2 - 20y - 30 \quad \text{Switch } x \text{ and } y \end{aligned}$$

Any further attempt to solve for y, however, will lead us to a dead end. This is due in large part to the fact that we cannot combine the terms $-2y^2$ and $-20y$. Instead, we first convert the quadratic $f(x)$ to its vertex form.

$$h = \frac{-b}{2a} = \frac{-(-20)}{2(-2)} = \frac{20}{-4} = -5$$

$$k = f(h) = -2(-5)^2 - 20(-5) - 30 = -50 + 100 - 30 = 20$$

$$\text{Vertex Form: } f(x) = -2(x+5)^2 + 20, \text{ where } x \geq -5$$

We can now use our vertex form to find f^{-1}, as follows.

$$\begin{aligned} y &= f(x) \\ y &= -2(x+5)^2 + 20 \\ x &= -2(y+5)^2 + 20 && \text{Switch } x \text{ and } y \\ x - 20 &= -2(y+5)^2 && \text{Solve for } y \\ \frac{x-20}{-2} &= (y+5)^2 \\ \sqrt{\frac{x-20}{-2}} &= y + 5 && \text{Square root both sides} \\ \sqrt{\frac{x-20}{-2}} - 5 &= y \end{aligned}$$

So,

$$f^{-1}(x) = \sqrt{\frac{x-20}{-2}} - 5 = \sqrt{\frac{20-x}{2}} - 5.$$

Using our standard form for f, we see that the graph of f is the right half of a parabola (since we were given that $x \geq -5$), opening downward with vertex $(-5, 20)$. Thus we can conclude that the range of f is $y \leq 20$. Similarly, if we consider our answer for f^{-1}, we see that our inverse function has a domain of $20 - x \geq 0$, or $x \leq 20$, which agrees with the range of f. Furthermore, since a square root must always be nonnegative, we can conclude that the range of f^{-1} is $y \geq -5$, which agrees with the given domain restriction ($x \geq -5$) of f.

It is important to mention that in our steps for finding f^{-1}, we were required to introduce a square root into the equation. Although this would usually require us to include a $\pm$, our final answer only shows a positive square root. This is not by accident, but is in fact necessary, since including a $\pm$ will produce an expression whose graph fails the Vertical Line Test, and can therefore not be the correct inverse function of f. Furthermore, because we are given that the domain of f is $x \geq -5$, a decision must be made to only include a positive square root for f^{-1}, and disregard the case of a negative square root. If we were instead initially given that $x \leq -5$ for our quadratic f, our answer for f^{-1} would in fact require a negative square root. Interpreted graphically, such a change would correspond to the graph of f as the left half of our parabola ($x \leq -5$), instead of the right half ($x \geq -5$).

To conclude this section, we will check that $(f^{-1} \circ f)(x) = x$. We leave it as an exercise to the reader to confirm that $(f \circ f^{-1})(x) = x$. As when we found f^{-1}, in each case, it will again be beneficial to use the vertex form for f, rather than the standard form.

$$\begin{aligned}
\left(f^{-1} \circ f\right)(x) &= f^{-1}\left(f(x)\right) \\
&= f^{-1}\left(-2(x+5)^2 + 20\right) \\
&= \sqrt{\frac{20 - \left(-2(x+5)^2 + 20\right)}{2}} - 5
\end{aligned}$$

$$\begin{aligned}
\left(f^{-1} \circ f\right)(x) &= \sqrt{\frac{20 + 2(x+5)^2 - 20}{2}} - 5 \\
&= \sqrt{\frac{2(x+5)^2}{2}} - 5 \\
&= \sqrt{(x+5)^2} - 5 \\
&= (x+5) - 5 \\
&= x \checkmark
\end{aligned}$$

Transformations

Introduction

In this section, we will continue to become more comfortable with general function notation and use it to establish a "database" of actions that may be applied to a particular function, each of which resulting in a predictable transformation of the graph of the original function. The three fundamental transformations which we will discuss are:

- translations, or "shifts"
- reflections
- scalings, or "stretches" and "shrinks"

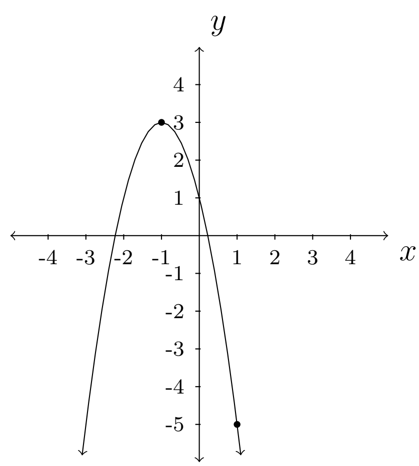
Each of these transformations will not only be identified by their name, but also by whether they have an effect on the original graph vertically or horizontally.

Eventually, once we have described, in detail, each action and its respective transformation, we will be able to consider transformations resulting from two or more actions on the given function. In fact, our first example, taken directly from the familiar chapter on quadratics (see page 162) will demonstrate such a transformation.

Example 250. Sketch a complete graph of $g(x) = -2(x + 1)^2 + 3$.

Recall that the vertex of the graph of g is at $(h, k) = (-1, 3)$. The negative leading coefficient ($a = -2$) reminds us that the graph opens (or points) downward. Although it is not necessary to find the intercepts in order to determine the general shape of the graph, we will include a table of points for the graph of g, which include both the $x-$ and $y-$intercepts, as well as a reference point at $(1, -5)$. We leave it as an exercise to the reader to verify that the values from the table are accurate. For the purposes of this example, we will only identify the vertex and reference point directly on our graph.

x	$g(x)$
-1	3
0	-1
1	-5
$-1 + \sqrt{6}/2$	0
$-1 - \sqrt{6}/2$	0



Although it should be relatively straightforward to deduce the graph of g using the methods from the previous chapter, if we were to instead consider the fundamental quadratic function $f(x) = x^2$ and compare it to g, we would actually notice *four* contributing factors which act on f and transform its graph to the graph of g shown above. We will identify each factor below, using a numbered list to help keep track of the changes. Later, we will see how rearranging the order of each of our actions can produce a different transformation of the original graph.

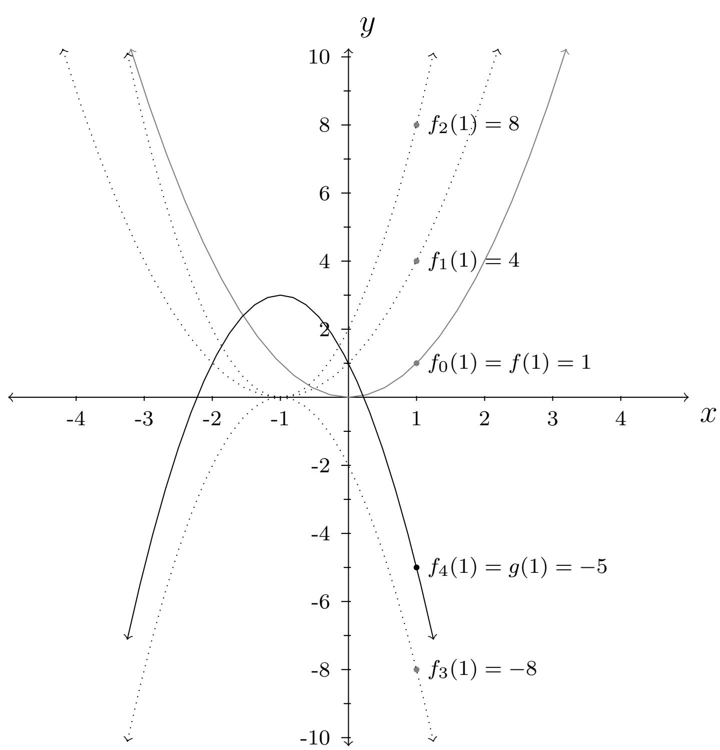
Original Function: $f(x) = x^2$ New Function: $g(x) = -2(x + 1)^2 + 3$

Contributing factors, taken in order:

1. $+1$ inside parentheses results in a horizontal shift 1 unit left.
2. multiplier of 2 outside parentheses results in a vertical stretch by a factor of 2.
3. negative multiplier $(-)$ results in a reflection about the $x-$axis.
4. $+3$ outside of parentheses results in a vertical shift 3 units up.

By considering each individual action separately, we can actually determine a sequence of functions and corresponding graphical transformations that, taken as a whole, result in the graph of g. This sequence is detailed below, along with each of their graphs, which include a reference point when $x = 1$.

Number	Function			Resulting Action
0.	$f_0(x) = f(x)$	$=$	x^2	Original Function
1.	$f_1(x)$	$=$	$(x+1)^2$	Horizontal Shift
2.	$f_2(x)$	$=$	$2(x+1)^2$	Vertical Stretch
3.	$f_3(x)$	$=$	$-2(x+1)^2$	Reflection about $x-$axis
4.	$f_4(x) = g(x)$	$=$	$-2(x+1)^2 + 3$	Vertical Shift



Now that we have seen the result of a combination of multiple actions on a familiar function $f(x) = x^2$, we turn our attention to understanding the effects of each single action on the graph of an arbitrary function.

Translations (L36)

Objective: Graph or identify a function that is represented by either a vertical or horizontal translation of a known function.

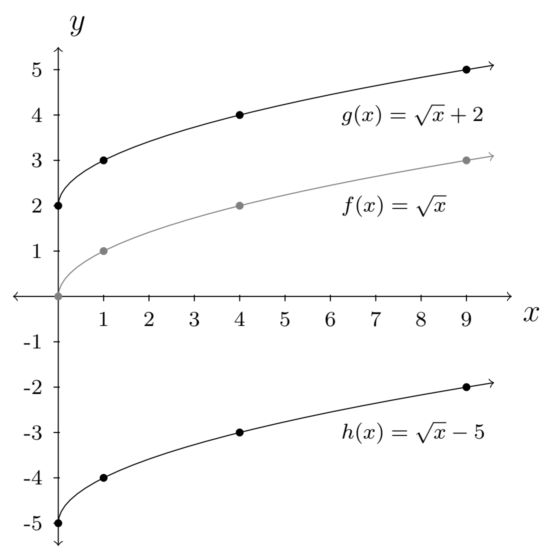
First, we consider the action of adding (or subtracting) a number to a function. In our earlier example, $f(x) = -2(x+1)^2 + 3$, this was done in two different places: once inside of the square (+1) and once outside of it (+3). And, as we have already seen, adding 1 inside of the square resulted in a horizontal shift of the graph, while adding 3 outside of the square resulted in a vertical shift. Such transformations are also referred to as *translations*. Our two cases demonstrate the possible effects that adding a number to a function can have on its graph. What remains to be seen is how the sign of the number can effect the graph.

We will begin by exploring vertical shifts of the function $f(x) = \sqrt{x}$.

Example 251. Graph the functions $g(x) = \sqrt{x} + 2$ and $h(x) = \sqrt{x} - 5$ and describe them as transformations of the graph of $f(x) = \sqrt{x}$.

First, observe that we can rewrite each function in terms of f as $g(x) = f(x) + 2$ and $h(x) = f(x) - 5$. It is also worth noting that the domain of all three functions is $[0, \infty)$. Next, we make a table of values to help sketch the graph of each function on the same set of axes.

x	$f(x)$	$g(x)$	$h(x)$
0	0	2	-5
1	1	3	-4
4	2	4	-3
9	3	5	-2



From our graph, we see that the graph of g represents a vertical shift of the graph of f up 2 units, while the graph of h represents a vertical shift of the graph of f down 5 units.

Our results are generalized as follows.

Vertical Shifts
Let f be a function and k a real number. Consider the function

$$g(x) = f(x) + k.$$

- If $k > 0$, then the graph of g represents a *vertical shift*, or translation, of the graph of f *up* k units.
- If $k < 0$, then the graph of g represents a *vertical shift*, or translation, of the graph of f *down* k units.

Next, we will explore horizontal shifts of the function $f(x) = \dfrac{1}{x}$.

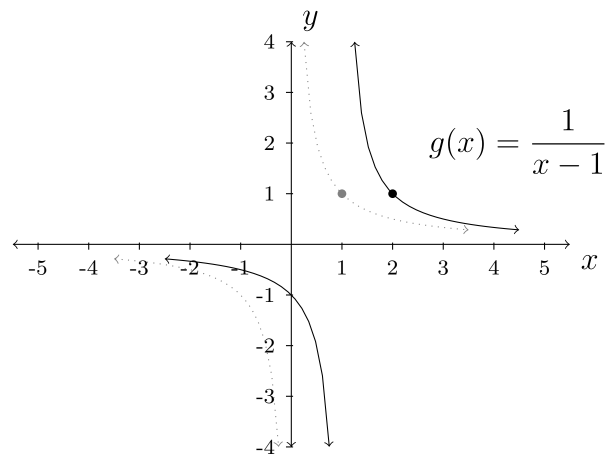
Example 252. Graph the functions $g(x) = \dfrac{1}{x-1}$ and $h(x) = \dfrac{1}{x+2}$ and describe them as transformations of the graph of $f(x) = \dfrac{1}{x}$.

First, observe that we can rewrite each function in terms of f as $g(x) = f(x-1)$ and $h(x) = f(x+2)$. Also notice that the domains of each function exclude a different value for x.

Function	Domain
$f(x) = \dfrac{1}{x}$	$x \neq 0$
$g(x) = \dfrac{1}{x-1}$	$x \neq 1$
$h(x) = \dfrac{1}{x+2}$	$x \neq -2$

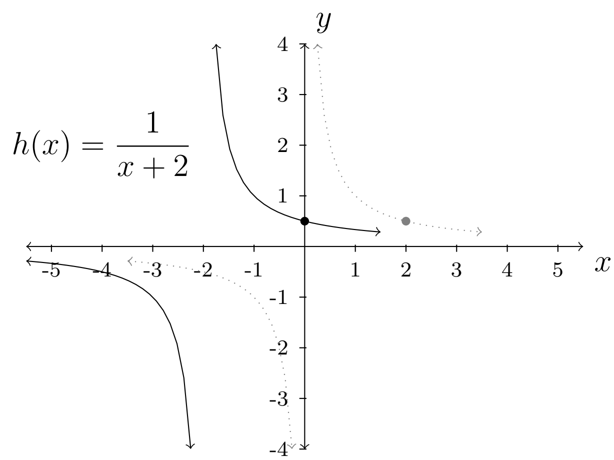
Again, we construct a table of values to help graph each function. In this example, it will be easier to compare each graph to our original graph one at a time. In each figure, the graph of f appears using a dotted curve. A set of two reference points, having the same $y-$coordinate has also been included in each graph.

x	$f(x)$	$g(x)$
-3	$-\frac{1}{3}$	$-\frac{1}{4}$
-2	$-\frac{1}{2}$	$-\frac{1}{3}$
-1	-1	$-\frac{1}{2}$
0	DNE	-1
1	1	DNE
2	$\frac{1}{2}$	1
3	$\frac{1}{3}$	$\frac{1}{2}$



From our graph, we see that the graph of g represents a horizontal shift of the graph of f to the right 1 unit.

x	$f(x)$	$h(x)$
-3	$-\frac{1}{3}$	-1
-2	$-\frac{1}{2}$	DNE
-1	-1	1
0	DNE	$\frac{1}{2}$
1	1	$\frac{1}{3}$
2	$\frac{1}{2}$	$\frac{1}{4}$
3	$\frac{1}{3}$	$\frac{1}{5}$



Similarly, we see that the graph of h represents a horizontal shift of the graph of f to the left 2 units.

It is worth noting that *adding* 2 from x in the case of h above resulted in a horizontal shift of the graph of f to the *left*. Often, this goes against what we would typically expect from adding a positive constant to x, since the left-half of the $x-$axis is considered the negative half (when $x < 0$).

Instead, if we consider plugging values into the expression $x + 2$, then evaluating $h(x) = f(x + 2)$ at $x = -2$ (two units to the *left* of zero) will yield the same $y-$coordinate as evaluating $f(x)$ at $x = 0$. We have also seen this notion at work when identifying the vertex of a parabola using the standard form of a quadratic function. For example, the graph of $h(x) = (x+2)^2$ has a vertex at $(-2, 0)$, which is two units to the left of the vertex $(0, 0)$ associated with the graph of $f(x) = x^2$.

A similar observation is worth mentioning when we consider the resulting graph from *subtracting* a positive constant from x, as in the case of g above. In this case, the resulting transformation is a horizontal shift to the *right*.

Again, we can generalize our findings.

Horizontal Shifts
Let f be a function and h a real number. Consider the function

$$g(x) = f(x - h).$$

- If $h > 0$, then the graph of g represents a *horizontal shift*, or translation, of the graph of f *right* h units.
- If $h < 0$, then the graph of g represents a *horizontal shift*, or translation, of the graph of f *left* h units.

Before moving on to our next type of transformation, it is important to point out the nature of the associated transformation of the graph of a function $f(x)$, when adding (or subtracting) a constant either inside or outside of the function. Specifically, a change to the original function occurring outside, such as $f(x)+4$, results in a *vertical* change of the graph of the original function, whereas a change occurring inside, such as $f(x+4)$, results in a *horizontal* change of the original graph. This will be a recurring theme, as we explore each of our remaining transformation types, and will be helpful as we encounter more advanced functions.

Reflections (L37)

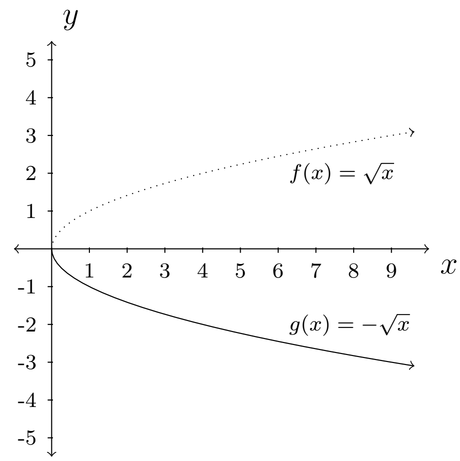
Objective: Graph or identify a function that is represented by either a vertical or horizontal reflection of a known function about the $y-$axis or $x-$axis, respectively.

Next, we consider the action of multiplication by -1. Given a function $f(x)$, we will consider the functions $-f(x)$ and $f(-x)$, whose graphs will represent reflections of the graph of f about either the $x-$axis or $y-$axis. Following along the same theme as in the previous subsection, one can initially guess that the graph of $-f(x)$ will represent a vertical reflection of the graph of f about the $x-$axis, since the negative sign occurs outside of the original function. This guess should also make sense to us, since multiplication of $f(x)$ by a negative sign would change the $y-$coordinate of any point $(x, y) = (x, f(x))$ on the graph of f, from either positive to negative or negative to positive.

Example 253. Graph the function $g(x) = -\sqrt{x}$ and describe it as a transformation of the graph of $f(x) = \sqrt{x}$.

Notice that since $g(x) = -f(x)$, the domain of g will be the same as that of f, $[0, \infty)$, whereas the range of g will be $(-\infty, 0]$. Next, we make a table of values to help sketch the graph of each function on the same set of axes.

x	$f(x)$	$g(x)$
0	0	0
1	1	-1
4	2	-2
9	3	-3

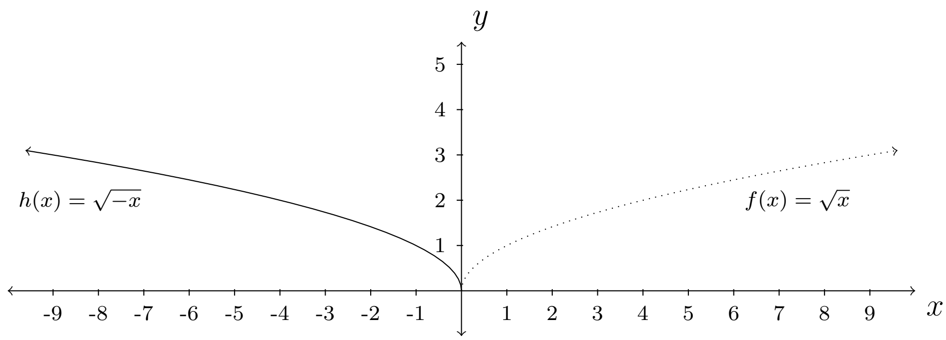


From our graph, we easily see that the graph of g represents a vertical reflection of the graph of f about the $x-$axis, as expected.

Example 254. Graph the function $h(x) = \sqrt{-x}$ and describe it as a transformation of the graph of $f(x) = \sqrt{x}$.

Notice that since $h(x) = f(-x)$, the range of h will be the same as that of f, $[0, \infty)$, whereas the domain of h will be $(-\infty, 0]$. Next, we make a table of values to help sketch the graph of each function on the same set of axes.

x	$f(x)$	$h(x)$
-9	DNE	-3
-4	DNE	-2
-1	DNE	-1
0	0	0
1	1	DNE
4	2	DNE
9	3	DNE



From our graph, we easily see that the graph of h represents a horizontal reflection of the graph of f about the $y-$axis, as expected.

Our two examples above are generalized as follows.

Reflections
Let f be a function.

- The graph of $g(x) = -f(x)$ represents a *reflection*, of the graph of f *about the x-axis* (a vertical change).
- The graph of $g(x) = f(-x)$ represents a *reflection*, of the graph of f *about the y-axis* (a horizontal change).

Scalings (L38)

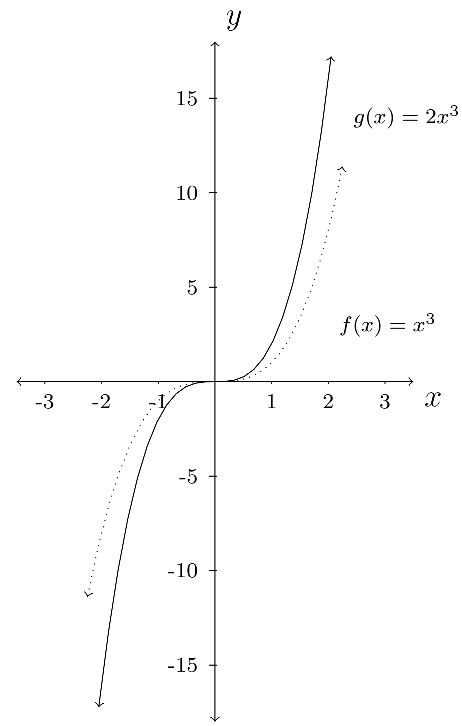
Objective: Graph or identify a function that is represented by either a vertical or horizontal scaling of a known function.

In this last portion of the transformations section, we focus our attention on scalings of the graph of a function f, also known as "stretches" or "shrinks". Again, we will look to classify both vertical and horizontal scalings, and will treat each case separately, beginning with vertical scalings.

Example 255. Graph the function $g(x) = 2x^3$ and describe it as a transformation of the graph of $f(x) = x^3$.

As we have seen in both of the previous subsections, we can anticipate a vertical effect on the graph of f, from the multiplication by 2 *outside*, or after, the cubing of x. In this case, we have that $g(x) = 2f(x)$, which will result in a doubling of every $y-$coordinate from our original graph. Our table and graphs below confirm this.

x	$f(x)$	$g(x)$
-3	-27	-54
-2	-8	-16
-1	-1	-2
0	0	0
1	1	2
2	8	16
3	27	54



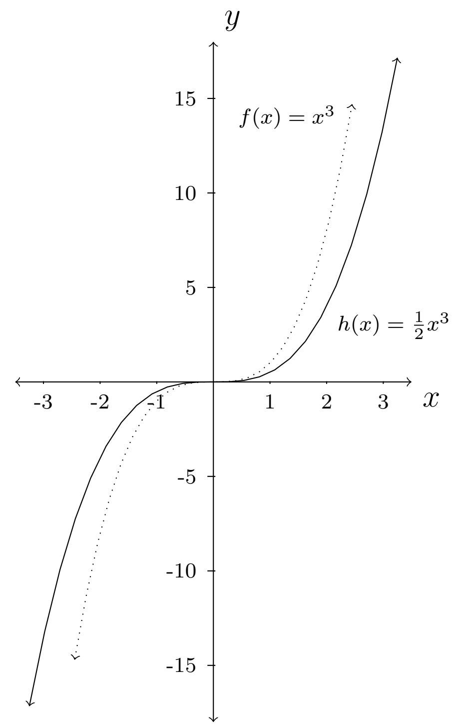
It is worth mentioning here that for the purposes of easily displaying our graphs, we have taken the liberty of adjusting the $y-$axis to easily fit the page. This adjustment should not affect our ability to identify the resulting transformation of the graph of f in any way.

As anticipated, our graph of g represents a vertical stretch of the graph of f. In this case, for a given value x, since every $y-$coordinate for the graph of g equals *twice* the value of the corresponding $y-$coordinate on the graph of f, we say that the resulting transformation is a *vertical stretch* of the graph of f *by a factor of* 2.

Example 256. Graph the function $h(x) = \frac{1}{2}x^3$ and describe it as a transformation of the graph of $f(x) = x^3$.

Again, we can anticipate a vertical effect on the graph of f, from the multiplication by $\frac{1}{2}$ *outside*, or after, the cubing of x. In this case, we have that $h(x) = \frac{1}{2}f(x)$, which will result in a halving of every $y-$coordinate from our original graph. Both the table and graph that follow confirm this.

x	$f(x)$	$h(x)$
-3	-27	$-\dfrac{27}{2}$
-2	-8	-4
-1	-1	$-\dfrac{1}{2}$
0	0	0
1	1	$\dfrac{1}{2}$
2	8	4
3	27	$\dfrac{27}{2}$



As with the previous example, for the purposes of easily displaying our graphs, we have taken the liberty of adjusting the $y-$axis to easily fit the page.

Now, our graph of h represents a vertical shrink of the graph of f. In this case, for a given value x, since every $y-$coordinate for the graph of h equals *half* the value of the corresponding $y-$coordinate on the graph of f, we say that the resulting transformation is a *vertical shrink* of the graph of f *by a factor of* 2.

Notice that despite the change from the last example (stretch to shrink), we still keep the same *factor* of 2. This is because the use of the term "shrink" tells us to *divide* by 2, as opposed to multiplying when the term "stretch" is used. Instead, if we were to mistakenly claim that the transformation for h represented a vertical shrink by a factor of $\frac{1}{2}$, this would actually mean that each $y-$coordinate for the graph of f should be divided by $\frac{1}{2}$, or doubled, which does not match the correct transformation for h.

In each of the previous two examples, we witnessed a vertical stretch when $f(x)$ was multiplied by 2 and a vertical shrink when $f(x)$ was multiplied by $\frac{1}{2}$. The fundamental difference in these two cases depends on the multiplier, and whether it is greater than or less than one. We now summarize each of these cases for vertical scalings.

Vertical Scalings
Let f be a function and a a positive real number. Consider the function

$$g(x) = af(x).$$

- If $a > 1$ the graph of g represents a *vertical stretch*, or expansion, of the graph of f *by a factor of* a.
- If $a < 1$ the graph of g represents a *vertical shrink*, or compression, of the graph of f *by a factor of* $1/a$.

For our last set of examples, we will analyze horizontal scalings.

Example 257. Graph the function $g(x) = (2x)^2$ and describe it as a transformation of the graph of $f(x) = x^2$.

Since $g(x) = f(2x)$, and the action occurs inside of the square, we will anticipate a horizontal effect on the graph of f. It is worth mentioning that the domain and range of g equal the domain and range of f. Again, we make a table to assist in graphing g.

x	$f(x)$	$g(x)$
0	0	0
1	1	4
$\frac{3}{2}$	$\frac{9}{4}$	9
2	4	16
3	9	36
4	16	64

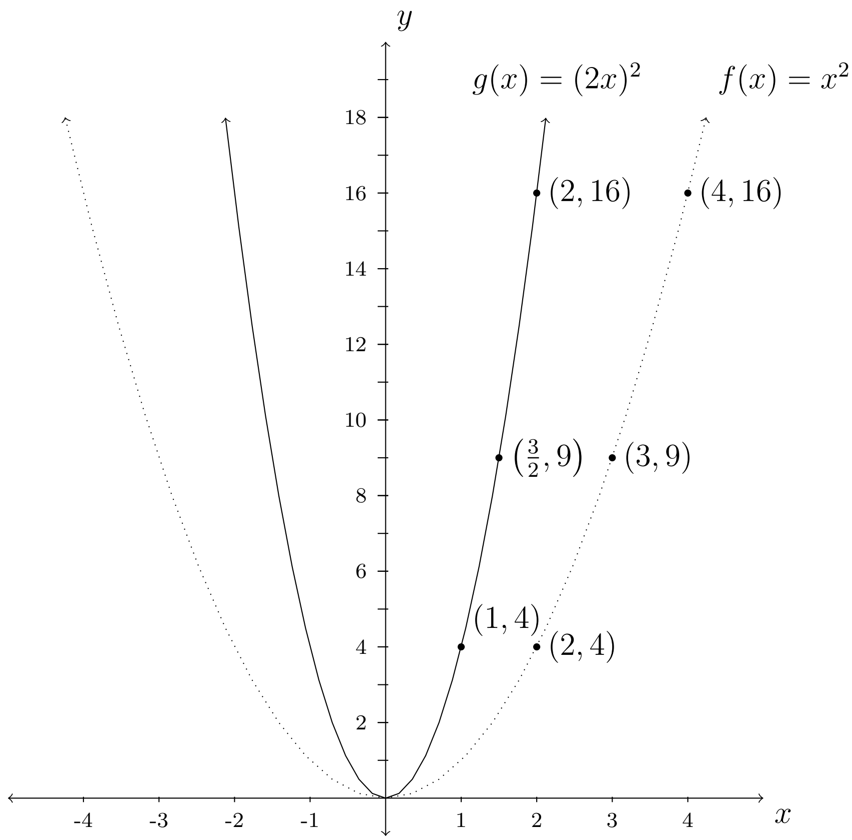
Notice that:

$g(1) = f(2) = 4$,

$g\left(\frac{3}{2}\right) = f(3) = 9$, and

$g(2) = f(4) = 16$.

In this example, we see that the points (x, y) for the graph of f will now correspond to the points $(x/2, y)$ for the graph of g. This results in a horizontal shrink (or compression) of the graph of f by a factor of 2, as shown in the graph below. The point at the origin $(0, 0)$, also the vertex, remains unchanged, since $0/2$ still equals 0. Notice that, as in the previous set of examples, we have adjusted the scaling of the $y-$axis, to easily display the graphs on the page. Again, this should not prevent us from correctly identifying the transformation.



Example 258. Graph the function $h(x) = \left(\frac{x}{2}\right)^2$ and describe it as a transformation of the graph of $f(x) = x^2$.

Since $h(x) = f\left(\frac{x}{2}\right)$, we will anticipate a horizontal effect on the graph of f. Again, we make a table to assist in graphing h.

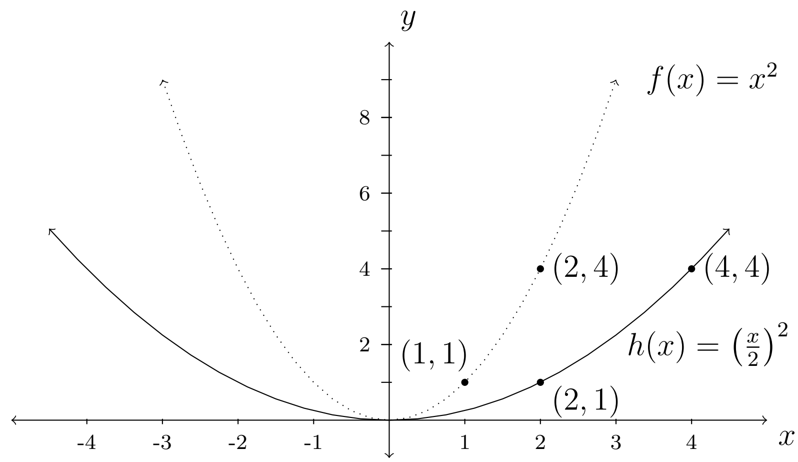
x	$f(x)$	$h(x)$
0	0	0
1	1	$\frac{1}{4}$
2	4	1
3	9	$\frac{9}{4}$
4	16	4

Notice that:

$h(2) = f(1) = 1$, and

$h(4) = f(2) = 4$.

In this example, we see that the points (x, y) for the graph of f will now correspond to the points $(2x, y)$ for the graph of h. This results in a horizontal stretch (or expansion) of the graph of f by a factor of 2, as shown in the graph below. As in the previous example, the origin $(0, 0)$ remains unchanged.



We are now ready to summarize the cases for horizontal scalings.

Horizontal Scalings
Let f be a function and b a positive real number. Consider the function

$$g(x) = f(bx).$$

- If $b > 1$ the graph of g represents a *horizontal shrink*, or compression, of the graph of f *by a factor of* b.
- If $b < 1$ the graph of g represents a *horizontal stretch*, or expansion, of the graph of f *by a factor of* $1/b$.

Transformations Summary (L39)

Objective: Graph or identify a function that is represented by a sequence of transformations of a known function.

Now that we have discussed each of the basic transformations of the graph of a function f, we are ready to consider combining two or more transformations, as demonstrated with our very first example of this section, $f(x) = -2(x+1)^2+3$. It will be critical that we keep track of the order of our actions on the function f, in order to correctly determine the resulting transformation of its graph. To assist in this, we now present the following theorem.

Transformations Summary
Let f be a function. Consider the function

$$g(x) = af(bx+h)+k,$$

where $a \neq 0$ and $b \neq 0$. Then, the graph of g may be obtained from the graph of f by following the sequence of transformations below.

1. **Horizontal Shift**:
   Shift the graph of f by h units to the the left if $h > 0$, or right if $h < 0$.
2. **Horizontal Scale/Horizontal Reflection**:
   Scale the graph from (1.) horizontally, as a shrink by a factor of $|b|$ if $|b| > 1$, or a stretch by a factor of $|1/b|$ if $0 < |b| < 1$.

   If $b < 0$, reflect the graph about the y-axis.
3. **Vertical Scale/Vertical Reflection**:
   Scale the graph from (2.) vertically, as a stretch by a factor of $|a|$ if $|a| > 1$, or a shrink by a factor of $|1/a|$ if $0 < |a| < 1$.

   If $a < 0$, reflect the graph about the x-axis.
4. **Vertical Shift**:
   Shift the graph from (3.) by k units up if $k > 0$, or down if $k < 0$.

In our first example, recall that the order of transformations was as follows.

$$f(x) = -2(x+1)^2 + 3$$

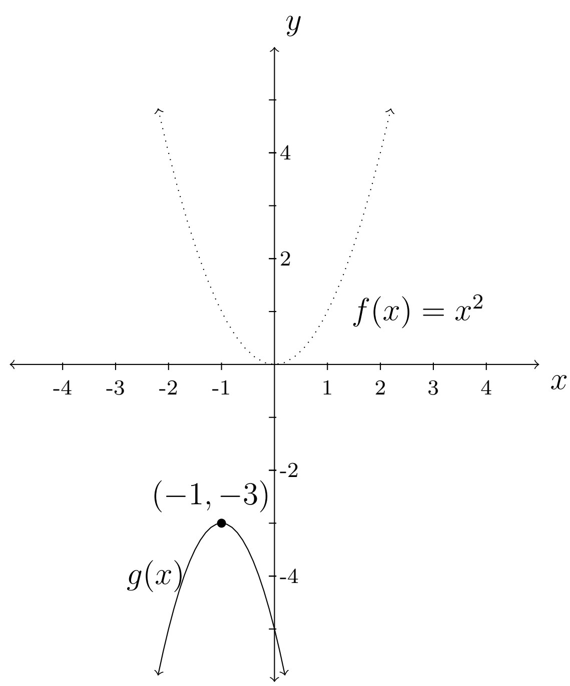
Contributing factors, taken in order:

1. $+1$ inside parentheses results in a horizontal shift 1 unit left.
2. multiplier of 2 outside parentheses results in a vertical stretch by a factor of 2.
3. negative multiplier $(-)$ results in a reflection about the $x-$axis.
4. $+3$ outside of parentheses results in a vertical shift 3 units up.

For our next example, we will rearrange the order of transformations from the previous example and utilize the graph of f to determine the graph of g. We can then work backwards, drawing upon our knowledge of quadratic equations to identify the corresponding function from our graph.

Example 259. Graph the resulting transformation of $f(x) = x^2$ from the sequence of transformations described below. Use the graph to determine the corresponding function $g(x)$. Using the Transformations Summary Theorem, compare your answer with the original sequence.

1. Vertical shift 3 units up
2. Vertical stretch by a factor of 2
3. Horizontal shift 1 unit left
4. Vertical reflection about the $x-$axis



Our resulting function is $g(x) = -2(x+1)^2 - 3$. Using the Transformations Summary Theorem, we see that our original sequence of transformations in this case also corresponds to the following sequence.

1. Horizontal shift 1 unit left
2. Vertical stretch by a factor of 2
3. Vertical reflection about the $x-$axis
4. Vertical shift 3 units down

We conclude this section with one final example.

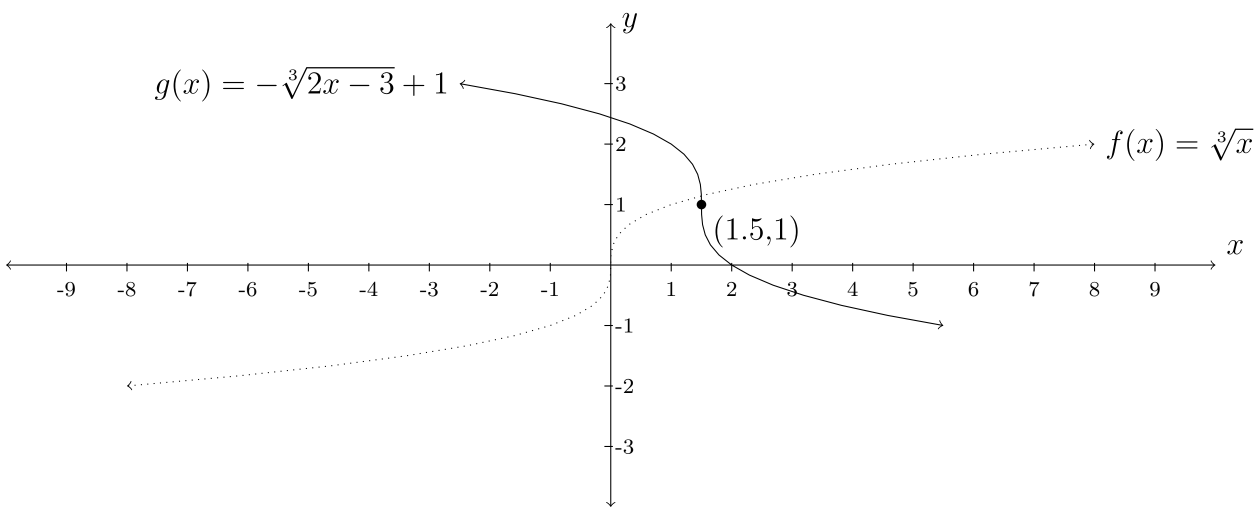
Example 260. Use the Transformations Summary Theorem to sketch a graph of the function below, as a transformation of $f(x) = \sqrt[3]{x}$.

$$g(x) = -\sqrt[3]{2x-3} + 1$$

Using the theorem, we can break down our graph as the following sequence of transformations of the graph of f. Since it will be difficult to keep track of each change in the graph, we use the point $(0,0)$ as a reference, keeping track of how it is affected by each change.

1. Horizontal shift 3 units to the right. $\Rightarrow$ $(3,0)$
2. Horizontal shrink by a factor of 2. $\Rightarrow$ $(1.5,0)$
3. Vertical reflection about the $x-$axis. $\Rightarrow$ $(1.5,0)$
4. Vertical shift 1 unit up. $\Rightarrow$ $(1.5,1)$

Our graph of g is shown below.



This last example demonstrates the significant challenge that comes from having to interpret the graph of a function that results from a sequence of two or more transformations. Although we have only kept track of how the function g affected one reference point in our example, by focusing our attention on just a few points of reference, we can come to a better understanding of the general shape of the graph of g, and how it relates to the graph of the fundamental function f.

Piecewise-Defined and Absolute Value Functions

Piecewise-Defined Functions (L40)

Objective: Define, evaluate, solve, and graph piecewise-defined functions

A *piecewise-defined* (or simply, a *piecewise*) function is a function that is defined in pieces. More precisely, a piecewise-defined function is a function that is presented using one or more expressions, each defined over non-intersecting intervals. An example of a piecewise-defined function is shown below.

$$f(x) = \begin{cases} 2x-1 & \text{if } x>0 \\ x^2-1 & \text{if } x\le 0 \end{cases}$$

To evaluate a piecewise-defined function at a particular value of the variable, we must first compare our value to the various intervals (or domains) applied to each piece, and then substitute our value into the piece that coincides with the correct domain. For example, since $x=1$ is greater than zero, we would use the expression $2x-1$ to evaluate $f(1)$,

$$f(1) = 2(1) - 1 = 2 - 1 = 1.$$

Similarly, since $x=-1$ is less than zero, we would use the expression x^2-1 to evaluate $f(-1)$,

$$f(-1) = (-1)^2 - 1 = 1 - 1 = 0.$$

Below is a table of points obtained from the piecewise-defined function f above.

Example 261.

$$f(x) = \begin{cases} 2x-1 & \text{if } x>0 \\ x^2-1 & \text{if } x\le 0 \end{cases}$$

x	$f(x)$
2	$2(2)-1=3$
1	$2(1)-1=1$
0	$(0)^2-1=-1$
-1	$(-1)^2-1=0$
-2	$(-2)^2-1=3$

We have included an extra line between the values of $x=0$ and $x=1$ in the table above, in order to emphasize the changeover from one piece of our function $(2x-1)$ to another (x^2-1).

The value of $x=0$ is very important, since it is an endpoint for the two domains of our function, $(0,\infty)$ and $(-\infty,0]$.

A common misconception among students is to evaluate $f(0)$ at both $2x-1$ and x^2-1 because it seems to "straddle" both individual domains. And although the values for both pieces are equal at $x=0$,

$$2(0) - 1 = -1 = 0^2 - 1$$

this will often not be the case. Regardless, we must be careful to *always* associate $x=0$ with x^2-1, since it is contained in our second piece's domain $(0\le 0)$ and not in our first. Our next example demonstrates what can happen with a piecewise function, if one mishandles such values of x.

Example 262.

$$g(x) \;=\; \begin{cases} 2x+1 & \text{if } \; x > 0 \\ x^2 - 1 & \text{if } \; x \leq 0 \end{cases}$$

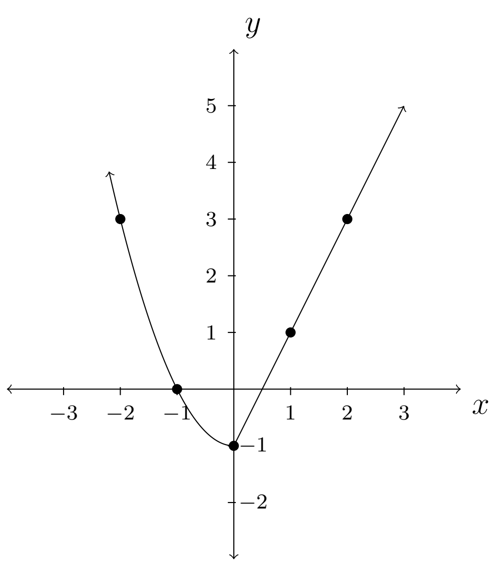
x	$g(x)$
2	$2(2)+1=5$
1	$2(1)+1=3$
0	$(0)^2-1=-1$
-1	$(-1)^2-1=0$
-2	$(-2)^2-1=3$

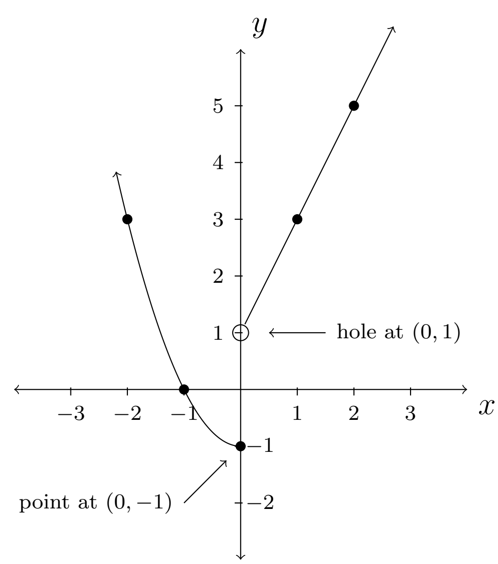
In this example, we see that both pieces for $g(x)$ do not "match up", since the values we obtain for both pieces at $x = 0$ do not agree:

$$g(0) = 0^2 - 1 = -1, \;\text{ but }\; 2(0) + 1 = 1.$$

Remember that when evaluating any function at a value of x in its domain, we should always only ever get a *single value* for $g(x)$, since this is how we defined a function earlier in the chapter. Furthermore, if we were to associate two values ($g(0) = \pm 1$) to $x = 0$, our graph would consequently contain points at $(0, -1)$ and $(0, 1)$, and therefore fail the Vertical Line Test.

When we consider the graphs of both f and g, since both pieces of f seem to "match up" at $x = 0$, we will see that the graph of f will be one *continuous* graph, with no breaks or separations appearing. On the other hand, since both pieces of g do not "match up" at $x = 0$, we will see that the graph of g will contain a break at $x = 0$, known as a *discontinuity* in the graph. The formal definition of a *continuous function* is one that is usually reserved for a follow-up course to Algebra (either Precalculus or Calculus). Both graphs are shown below.





Notice that in order for us to have a *complete* sketch of the graph of g, we have evaluated *both* pieces of g at $x = 0$, so that we can properly identify the *point* at the end of the quadratic piece x^2-1 and the *hole* at the end of the linear piece, $2x+1$. In general, whenever faced with the task of graphing a piecewise-defined function, one should always make sure to identify exactly where each piece of the graph starts and stops, even if a location corresponds to a hole, i.e., a coordinate pair that is *not* actually a point on the graph.

We can also observe, both from how our functions are defined (algebraically) and from their graphs that the domain of both f and g is all real numbers, or $(-\infty, \infty)$. To identify the range of each function, we can project each of our graphs onto the $y-$axis. In doing so, we obtain a range of $[-1, \infty)$ for both f and g. Notice that although both functions produce distinctly different graphs, their range is coincidentally the same, since the quadratic piece x^2-1 begins at the same minimum value ($y=-1$) for each graph.

As we have already discussed evaluating piecewise-defined functions at a value of x, we will now address the issue of solving an equation that involves a piecewise function for all possible values of x. We will do this, once again, using our functions f and g from before.

For some constant k, to find all x such that $f(x)=k$, we will use the strategy outlined below, which will be the same for any piecewise-defined function.

- Set each separate piece equal to k and solve for x.
- Compare your answers for x to the domain applied to each piece. Only keep those solutions that coincide with the specified domain.

We illustrate this approach by finding all possible zeros (or roots) of both f and g.

Example 263. Find the set of all zeros of

$$f(x) \;=\; \begin{cases} 2x-1 & \text{if } x>0 \\ x^2-1 & \text{if } x\le 0 \end{cases}$$

.

$$\begin{aligned}
f(x)&=0 && \text{Apply to each piece separately}\\
\\
2x-1=0,\ x&>0 && \text{First piece; solve for } x\\
x=\frac{1}{2},\ x&>0 && \text{One solution; coincides with domain}\\
\\
x^2-1=0\ ,\quad x&\le 0 && \text{Second piece; solve for } x\\
(x-1)(x+1)=0,\ x&\le 0 && \text{Solve by factoring}\\
x=\pm 1,\ x&\le 0 && \text{Two potential solutions}\\
x=-1,\ x&\le 0 && \text{Exclude } x=1;\text{does not coincide with domain}\\
\\
f(x)=0 \text{ when } x=-1,\ &\frac{1}{2} && \text{Our answer}
\end{aligned}$$

Example 264. Find the set of all zeros of

$$g(x) = \begin{cases} 2x+1 & \text{if } x > 0 \\ x^2 - 1 & \text{if } x \leq 0 \end{cases}$$

.

$$\begin{aligned}
g(x) &= 0 && \text{Apply to each piece separately} \\
\\
2x + 1 = 0,\ x &> 0 && \text{First piece; solve for } x \\
x = -\frac{1}{2},\ x &> 0 && \text{Invalid solution; does not coincide with domain} \\
\\
x^2 - 1 = 0\ ,\quad x &\leq 0 && \text{Second piece; solve for } x \\
(x-1)(x+1) = 0,\ x &\leq 0 && \text{Solve by factoring} \\
x = \pm 1,\ x &\leq 0 && \text{Two potential solutions} \\
x = -1,\ x &\leq 0 && \text{Exclude } x = 1; \text{does not coincide with domain} \\
\\
g(x) = 0 \text{ when } x &= -1 && \text{Our answer}
\end{aligned}$$

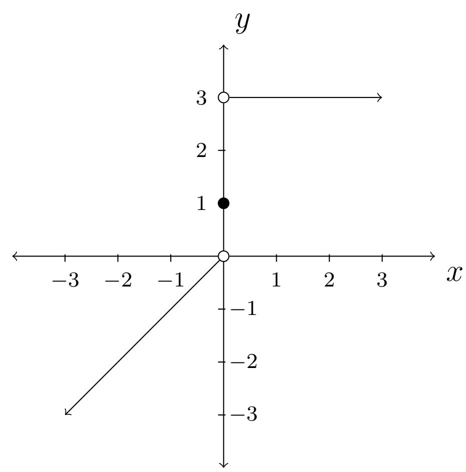
Each of the previous examples can also be confirmed by the graphs that we obtained earlier.

For our next example, we will graph a piecewise function that consists of three pieces.

Example 265.

$$h(x) = \begin{cases} 3 & \text{if } x > 0 \\ 1 & \text{if } x = 0 \\ x & \text{if } x < 0 \end{cases}$$

x	$h(x)$
2	3
1	3
0	1
-1	-1
-2	-2



The graph of h

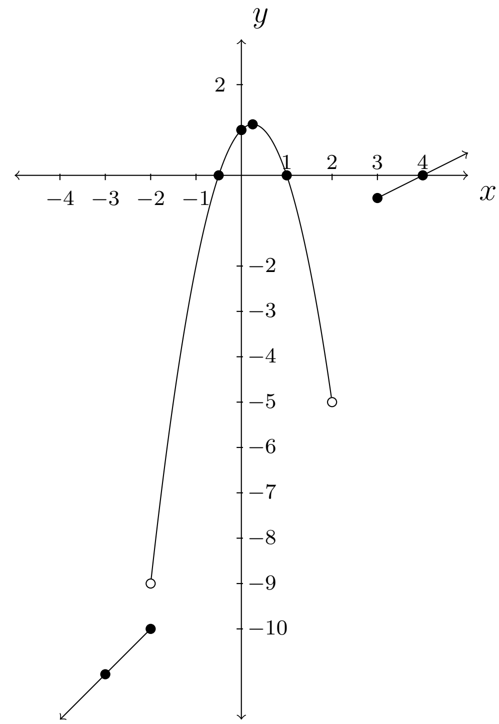
Here, we see that our graph consists of three pieces, one of which is a single point at $(0, 1)$. We can also once again determine both algebraically and graphically that our domain is $(-\infty, \infty)$. Using our graph, we obtain a range of $(-\infty, 0) \cup \{1\} \cup \{3\}$. Our complete graph also contains holes at $(0, 3)$ and $(0, 0)$.

We can easily identify all three of the coordinate pairs associated with $x = 0$ (two holes and one point) by evaluating all three pieces at $x = 0$. To reinforce this concept, we will present another example of a piecewise function that consists of three pieces.

Example 266.

$$f(x) \;=\; \begin{cases} \frac{x}{2} - 2 & \text{if } \; x \geq 3 \\ -2x^2 + x + 1 & \text{if } \; -2 < x < 2 \\ x - 8 & \text{if } \; x \leq -2 \end{cases}$$

x	$f(x)$
4	0
3	$-\frac{1}{2}$
1	0
$\frac{1}{4}$	$\frac{9}{8}$
0	1
$-\frac{1}{2}$	0
-1	-2
-2	-10
-3	-11



The graph of f

In the previous example, we see that there is a "gap" in our domain between the $x-$coordinates of 2 and 3. Hence, our domain is $(-\infty, 2) \cup [3, \infty)$. From our graph, we see that our range also contains a gap between the $y-$coordinates of -10 and -9. Hence, our range is $(-\infty, -10] \cup (-9, \infty)$. In our example we have have also identified several other essential coordinate pairs that should be included in our graph. We will now list each pair below, as well as the piece that is used to obtain it. We include the function f, once again, for reinforcement.

$$f(x) = \begin{cases} \frac{x}{2} - 2 & \text{if } x \geq 3 \\ -2x^2 + x + 1 & \text{if } -2 < x < 2 \\ x - 8 & \text{if } x \leq -2 \end{cases}$$

- A $y-$intercept at $(0, 1)$ from our second piece
- An $x-$intercept at $(4, 0)$ from our first piece
- Two $x-$intercepts at $(1, 0)$ and $\left(-\frac{1}{2}, 0\right)$ from our second piece
- A vertex at $\left(\frac{1}{4}, \frac{9}{8}\right)$ from our second piece
- An endpoint at $\left(3, -\frac{1}{2}\right)$ from our first piece
- An endpoint at $(-2, -10)$ from our third piece
- Two holes at $(-2, -9)$ and $(2, -5)$ from our second piece.

Lastly, we have included the point at $(-3, -11)$, to help identify the slope of the third piece of our graph.

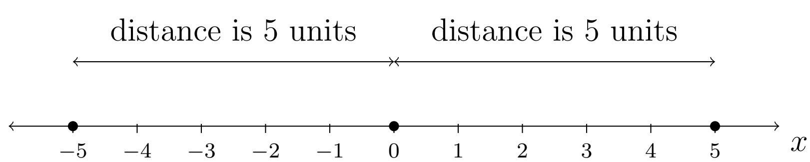
Although this example may first appear to be quite complicated, when considered on the level of each individual piece, we see that our training in the chapters leading up to this section has adequately prepared us to handle these, as well as more challenging piecewise-defined functions that we will eventually encounter.

Functions Containing an Absolute Value (L41)

Graphing Functions Containing an Absolute Value

Objective: Graph a variety of functions that contain an absolute value

There are a few ways to describe what is meant by the absolute value $|x|$ of a real number x. A common description is that $|x|$ represents the distance from the number x to 0 on the real number line. So, for example, $|5| = 5$ and $|-5| = 5$, since each is 5 units away from 0 on the real number line.



Another way to define an absolute value is by the equation $|x| = \sqrt{x^2}$. Using this definition, we have

$$|5| = \sqrt{(5)^2} = \sqrt{25} = 5 \qquad \text{and} \qquad |-5| = \sqrt{(-5)^2} = \sqrt{25} = 5.$$

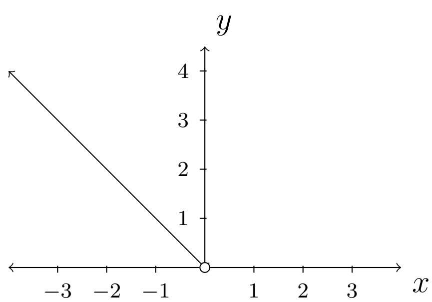
The long and short of both of these descriptions is that $|x|$ takes negative real numbers and assigns them to their positive counterparts, while it leaves positive real numbers (and zero) alone. This last description is the one we shall adopt, and is summarized in the following definition.

> The **absolute value** of a real number x, denoted $|x|$, is given by
>
> $$|x| = \begin{cases} -x, & \text{if } \; x < 0 \\ x, & \text{if } \; x \geq 0 \end{cases}$$

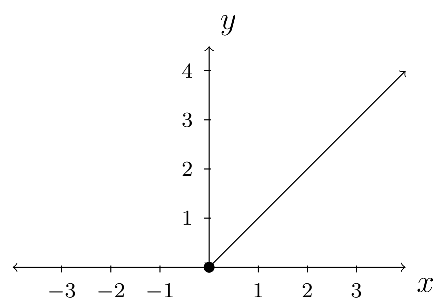
Notice that we have defined $|x|$ using a piecewise-defined function. To check that this definition agrees with what we previously understood to be the absolute value of x, observe that since $5 \geq 0$, to find $|5|$ we use the rule $|x| = x$, so $|5| = 5$. Similarly, since $-5 < 0$, we use the rule $|x| = -x$, so that $|-5| = -(-5) = 5$.

We will now graph some functions that contain an absolute value. Our strategy is to use our knowledge of the absolute value coupled with what we now know about graphing linear functions and piecewise-defined functions.

Example 267. Sketch a complete graph of $f(x) = |x|$.

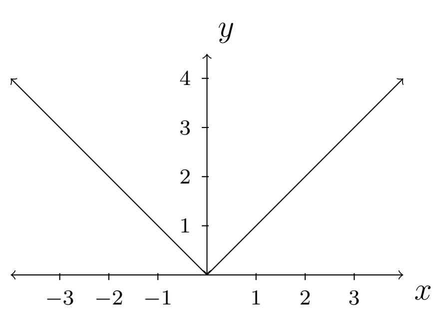


$f(x) = |x|, \; x < 0$



$f(x) = |x|, \; x \geq 0$

Notice that we have a hole at $(0, 0)$ in the graph when $x < 0$. As we have seen before, this is due to the fact that the points on $y = -x$ approach $(0, 0)$ as the x-values approach 0. Since x is required to be strictly less than zero on this interval, we include a hole at the origin. Notice, however, that when $x \geq 0$, we get to include the point at $(0, 0)$, which effectively fills in the hole from our first piece. Our final graph is shown below.

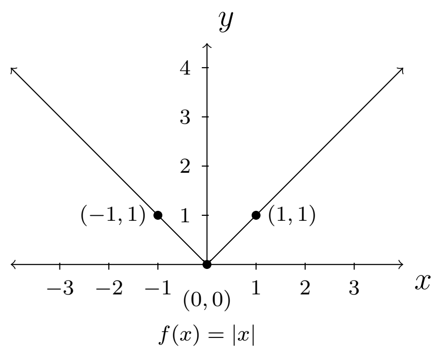


$f(x) = |x|$

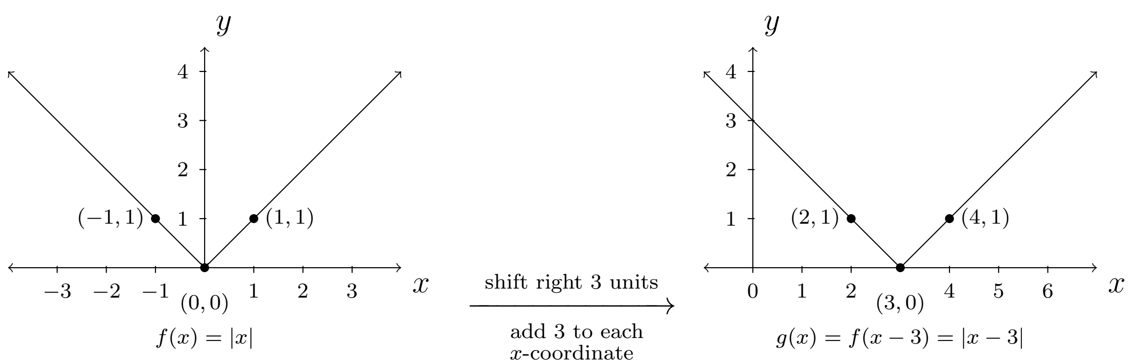
By projecting our graph onto the $x-$axis, we see that the domain of $f(x) = |x|$ is $(-\infty, \infty)$, as expected. Projecting onto the $y-$axis gives us our range of $[0, \infty)$. Our function is also increasing over the interval $[0, \infty)$ and decreasing over the interval $(-\infty, 0]$. We can also say that the graph of f has an absolute minimum at $y = 0$, since this coordinate coincides with the (absolute) lowest point on the graph, which occurs at the origin. From our graph, we can further conclude that there is no absolute maximum value of f, since the y values on the graph extend infinitely upwards.

Example 268. Use the graph of $f(x) = |x|$ to graph the function $g(x) = |x - 3|$.

We begin by graphing $f(x) = |x|$ and labeling three reference points: $(-1, 1)$, $(0, 0)$ and $(1, 1)$.

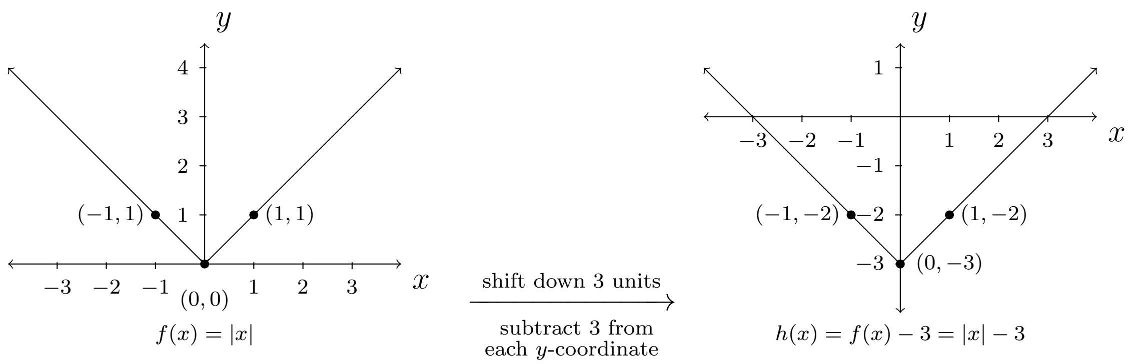


Since $g(x) = |x - 3| = f(x - 3)$, we will add 3 to each of the $x-$coordinates of the points on the graph of $y = f(x)$ to obtain the graph of $y = g(x)$. This shifts the graph of $y = f(x)$ to the *right* by 3 units and moves the points $(-1, 1)$ to $(2, 1)$, $(0, 0)$ to $(3, 0)$ and $(1, 1)$ to $(4, 1)$. Connecting these points in the classic '∨' fashion produces the graph of $y = g(x)$.



Example 269. Use the graph of $f(x) = |x|$ to graph the function $h(x) = |x| - 3$.

Since $h(x) = |x| - 3 = f(x) - 3$, we will subtract 3 from each of the $y-$coordinates of the points on the graph of $y = f(x)$ to obtain the graph of $y = h(x)$. This shifts the graph of $y = f(x)$ *down* by 3 units and moves the points $(-1, 1)$ to $(-1, -2)$, $(0, 0)$ to $(0, -3)$ and $(1, 1)$ to $(1, -2)$. Connecting these points with the '∨' shape produces our graph of $y = h(x)$.



Example 270. Use the graph of $f(x) = |x|$ to graph the function $k(x) = 4 - 2|3x + 1|$.

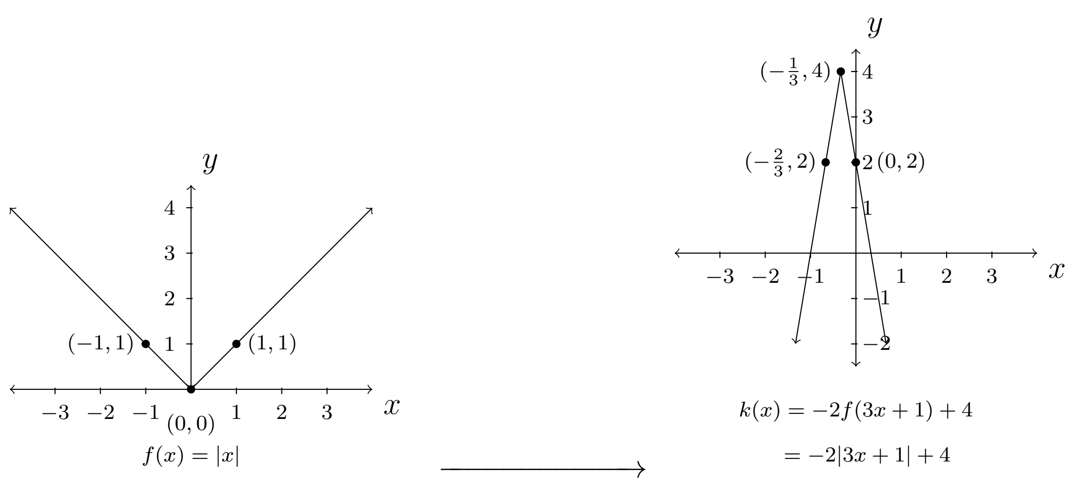
Notice that

$$\begin{aligned} k(x) &= 4 - 2|3x + 1| \\ &= 4 - 2f(3x + 1) \\ &= -2f(3x + 1) + 4. \end{aligned}$$

First, we will determine the corresponding transformations resulting from inside of the absolute value. Instead of $|x|$, we have $|3x + 1|$, so we must first subtract 1 from each of the $x-$coordinates of points on the graph of $y = f(x)$, then divide each of those new values by 3. This corresponds to a horizontal shift left by 1 unit followed by a horizontal shrink by a factor of 3. These transformations move the points $(-1, 1)$ to $\left(-\frac{2}{3}, 1\right)$, $(0, 0)$ to $\left(-\frac{1}{3}, 0\right)$ and $(1, 1)$ to $(0, 1)$.

Next, we will determine the corresponding transformations resulting from what appears outside of the absolute value. We must first multiply each $y-$coordinate of our new points by -2 and then *add* 4. Geometrically, this corresponds to a vertical *stretch* by a factor of 2, a reflection across the $x-$axis and finally, a vertical shift *up* by 4 units.

The resulting transformations move the points $\left(-\frac{2}{3}, 1\right)$ to $\left(-\frac{2}{3}, 2\right)$, $\left(-\frac{1}{3}, 0\right)$ to $\left(-\frac{1}{3}, 4\right)$ and $(0, 1)$ to $(0, 2)$. Connecting our final points with the usual '$\vee$' shape produces the graph of $y = k(x)$, shown below.



Absolute Value as a Piecewise Function (L42)

Objective: Interpret a function containing an absolute value as a piecewise-defined function.

By definition, we know that

$$|x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

If $m \neq 0$ and b is a real number, we may generalize the definition above as follows.

$$\begin{aligned} |mx+b| &= \begin{cases} -(mx+b), & \text{if } mx+b < 0 \\ mx+b\ , & \text{if } mx+b \geq 0 \end{cases} \\ &= \begin{cases} -mx-b, & \text{if } mx+b < 0 \\ mx+b, & \text{if } mx+b \geq 0 \end{cases} \end{aligned}$$

Notice that since we have never specified whether m is positive or negative above, it would not be wise to attempt to simplify either inequality in our new definition. Once we are given a value for m, as in our next example, we will be able to simplify our piecewise representation completely.

Example 271. Express $g(x) = |x-3|$ as a piecewise-defined function.

$$g(x) = |x-3| = \begin{cases} -(x-3), & \text{if } \quad x-3 < 0 \\ (x-3), & \text{if } (x-3) \geq 0 \end{cases}$$

Simplifying, we get

$$g(x) = \begin{cases} -x+3, & \text{if } x < 3 \\ x-3, & \text{if } x \geq 3 \end{cases}$$

Our piecewise answer above should begin to make sense, when one considers the graph of g as a horizontal shift of $y = |x|$ to the right by 3 units.

Example 272. Express $h(x) = |x| - 3$ as a piecewise-defined function.

Since the variable within the absolute value remains unchanged, the domains for each piece in our resulting function will not change. Instead, we need only subtract 3 from each piece of our answer. Thus, we get the following representation.

$$h(x) = \begin{cases} -x-3, & \text{if } x < 0 \\ x-3, & \text{if } x \geq 0 \end{cases}$$

Similarly, this answer again seems reasonable, as the graph of $h(x) = |x| - 3$ represents a vertical shift of $y = |x|$ down by 3 units.

Example 273. Express $k(x) = 4 - 2|3x+1|$ as a piecewise-defined function and identify any $x-$ and $y-$intercepts on its graph. Determine the domain and range of $k(x)$.

We set $k(x) = 0$ to find any zeros: $4 - 2|3x+1| = 0$.

Isolating the absolute value gives us $|3x+1| = 2$, so either

$$3x + 1 = 2 \qquad \text{or} \qquad 3x + 1 = -2.$$

This results in $x = \frac{1}{3}$ or $x = -1$, so our $x-$intercepts are $\left(\frac{1}{3}, 0\right)$ and $(-1, 0)$.

For our $y-$intercept, substituting $x = 0$ into $k(x)$ gives us

$$y = k(0) = 4 - 2|3(0) + 1| = 2.$$

So our y-intercept is at $(0, 2)$. Rewriting the expression for k as a piecewise function gives us the following.

$$\begin{aligned}
k(x) &= \begin{cases} 4 - 2[-(3x+1)], & \text{if } \; 3x + 1 < 0 \\ 4 - 2(3x+1), & \text{if } \; 3x + 1 \geq 0 \end{cases} \\
&= \begin{cases} 4 + 6x + 2, & \text{if } \; 3x < -1 \\ 4 - 6x - 2, & \text{if } \; 3x \geq -1 \end{cases} \\
&= \begin{cases} 6x + 6, & \text{if } \; x < -\frac{1}{3} \\ -6x + 2, & \text{if } \; x \geq -\frac{1}{3} \end{cases}
\end{aligned}$$

Either algebraically, or using the graph of k from page 243, we see that the domain of k is $(-\infty, \infty)$ while the range is $(-\infty, 4]$.

Practice Problems

Identifying Domain Algebraically

Find the domain of each of the following functions. Express your answers using interval notation.

1. $f(x) = |3x - 2|$
2. $g(x) = (3x-2)^2$
3. $h(x) = \dfrac{1}{3x-2}$
4. $k(x) = \sqrt{3x-2}$
5. $k(x) = \sqrt[3]{3x-2}$
6. $\ell(x) = \sqrt[4]{2-3x}$
7. $m(x) = \dfrac{x-2}{\sqrt{3x-2}}$
8. $n(x) = \dfrac{\sqrt{3x-2}}{x-2}$

Find the domain of each of the following functions. Express your answers using interval notation.

9. $g(x) - 4x^2$
10. $f(x) = x^4 - 13x^3 + 56x^2 - 19$
11. $g(x) = x^2 - 4$
12. $k(x) = \dfrac{x}{x-8}$
13. $h(x) = \dfrac{x-5}{x+4}$

14. $h(x) = \dfrac{x-2}{x+1}$
15. $k(x) = \dfrac{x-2}{x-2}$
16. $k(x) = \dfrac{3x}{x^2+x-2}$
17. $g(x) = \dfrac{2x}{x^2-9}$
18. $f(x) = \dfrac{2x}{x^2+9}$
19. $h(x) = \dfrac{x+4}{x^2-36}$
20. $f(x) = \sqrt{3-x}$
21. $g(x) = \sqrt{2x+5}$
22. $f(x) = 5\sqrt{x-1}$
23. $h(x) = 9x\sqrt{x+3}$
24. $k(x) = \dfrac{\sqrt{7-x}}{x^2+1}$
25. $f(x) = \sqrt{6x-2}$
26. $g(x) = \dfrac{6}{\sqrt{6x-2}}$
27. $k(x) = \dfrac{4}{\sqrt{x-3}}$
28. $g(x) = \dfrac{x}{\sqrt{x-8}}$
29. $h(x) = \sqrt[3]{6x-2}$
30. $k(x) = \dfrac{6}{4-\sqrt{6x-2}}$
31. $f(x) = \dfrac{\sqrt{6x-2}}{x^2-36}$
32. $g(x) = \dfrac{\sqrt[3]{6x-2}}{x^2+36}$
33. $h(x) = \sqrt{x-7} + \sqrt{9-x}$
34. $h(t) = \dfrac{\sqrt{t}-8}{5-t}$
35. $f(r) = \dfrac{\sqrt{r}}{r-8}$
36. $k(v) = \dfrac{1}{4-\dfrac{1}{v^2}}$
37. $f(y) = \sqrt[3]{\dfrac{y}{y-8}}$
38. $k(w) = \dfrac{w-8}{5-\sqrt{w}}$

Combining Functions

Function Arithmetic

In each of the following exercises, use the pair of functions f and g to find the following values if they exist.

- $(f+g)(2)$
- $(f-g)(-1)$
- $(g-f)(1)$
- $(fg)\left(\frac{1}{2}\right)$
- $\left(\frac{f}{g}\right)(0)$
- $\left(\frac{g}{f}\right)(-2)$

1. $f(x) = 3x+1 \quad g(x) = 4-x$
2. $f(x) = x^2 \quad g(x) = -2x+1$
3. $f(x) = x^2-x \quad g(x) = 12-x^2$
4. $f(x) = 2x^3 \quad g(x) = -x^2-2x-3$
5. $f(x) = \sqrt{x+3} \quad g(x) = 2x-1$
6. $f(x) = \sqrt{4-x} \quad g(x) = \sqrt{x+2}$

7. $f(x) = 2x \qquad g(x) = \dfrac{1}{2x+1}$

8. $f(x) = x^2 \qquad g(x) = \dfrac{3}{2x-3}$

9. $f(x) = x^2 \qquad g(x) = \dfrac{1}{x^2}$

10. $f(x) = x^2 + 1 \qquad g(x) = \dfrac{1}{x^2+1}$

In each of the following exercises, use the pair of functions f and g to find the domain of the indicated function then find and simplify an expression for it.

- $(f+g)(x)$
- $(f-g)(x)$
- $(fg)(x)$
- $\left(\frac{f}{g}\right)(x)$

11. $f(x) = 2x + 1 \qquad g(x) = x - 2$

12. $f(x) = 1 - 4x \qquad g(x) = 2x - 1$

13. $f(x) = x^2 \qquad g(x) = 3x - 1$

14. $f(x) = x^2 - x \qquad g(x) = 7x$

15. $f(x) = x^2 - 4 \qquad g(x) = 3x + 6$

16. $f(x) = -x^2 + x + 6 \quad g(x) = x^2 - 9$

17. $f(x) = \dfrac{x}{2} \qquad g(x) = \dfrac{2}{x}$

18. $f(x) = x - 1 \qquad g(x) = \dfrac{1}{x-1}$

19. $f(x) = x \qquad g(x) = \sqrt{x+1}$

20. $f(x) = g(x) = \sqrt{x-5}$

For each of the following exercises, let f be the function defined by

$$f = \{(-3,4), (-2,2), (-1,0), (0,1), (1,3), (2,4), (3,-1)\}$$

and let g be the function defined

$$g = \{(-3,-2), (-2,0), (-1,-4), (0,0), (1,-3), (2,1), (3,2)\}.$$

Use f and g to compute each of the indicated values if they exist.

21. $(f+g)(-3)$

22. $(f-g)(2)$

23. $(fg)(-1)$

24. $(g+f)(1)$

25. $(g-f)(3)$

26. $(gf)(-3)$

27. $\left(\frac{f}{g}\right)(-2)$

28. $\left(\frac{f}{g}\right)(-1)$

29. $\left(\frac{f}{g}\right)(2)$

30. $\left(\frac{g}{f}\right)(-1)$

31. $\left(\frac{g}{f}\right)(3)$

32. $\left(\frac{g}{f}\right)(-3)$

Composite Functions

In each of the following exercises, use the given pair of functions to find the following values if they exist.

- $(g \circ f)(0)$
- $(f \circ g)(-1)$
- $(f \circ f)(2)$
- $(g \circ f)(-3)$
- $(f \circ g)\left(\frac{1}{2}\right)$
- $(f \circ f)(-2)$

1. $f(x) = x^2$, $g(x) = 2x + 1$
2. $f(x) = 4 - x$, $g(x) = 1 - x^2$
3. $f(x) = 4 - 3x$, $g(x) = |x|$
4. $f(x) = |x - 1|$, $g(x) = x^2 - 5$
5. $f(x) = 4x + 5$, $g(x) = \sqrt{x}$
6. $f(x) = \sqrt{3 - x}$, $g(x) = x^2 + 1$
7. $f(x) = \dfrac{3}{1 - x}$, $g(x) = \dfrac{4x}{x^2 + 1}$
8. $f(x) = \dfrac{x}{x + 5}$, $g(x) = \dfrac{2}{7 - x^2}$

In each of the following exercises, use the given pair of functions to find and simplify expressions for the following functions and state the domain of each using interval notation.

- $(g \circ f)(x)$
- $(f \circ g)(x)$
- $(f \circ f)(x)$

9. $f(x) = 2x + 3$, $g(x) = x^2 - 9$
10. $f(x) = x^2 - x + 1$, $g(x) = 3x - 5$
11. $f(x) = x^2 - 4$, $g(x) = |x|$
12. $f(x) = 3x - 5$, $g(x) = \sqrt{x}$
13. $f(x) = |x + 1|$, $g(x) = \sqrt{x}$
14. $f(x) = 3 - x^2$, $g(x) = \sqrt{x + 1}$
15. $f(x) = |x|$, $g(x) = \sqrt{4 - x}$
16. $f(x) = x^2 - x - 1$, $g(x) = \sqrt{x - 5}$
17. $f(x) = 3x - 1$, $g(x) = \dfrac{1}{x + 3}$
18. $f(x) = \dfrac{3x}{x - 1}$, $g(x) = \dfrac{x}{x - 3}$
19. $f(x) = \dfrac{x}{2x + 1}$, $g(x) = \dfrac{2x + 1}{x}$
20. $f(x) = \dfrac{2x}{x^2 - 4}$, $g(x) = \sqrt{1 - x}$

In each of the following exercises, use $f(x) = -2x$, $g(x) = \sqrt{x}$ and $h(x) = |x|$ to find and simplify expressions for the following functions and state the domain of each using interval notation.

21. $(h \circ g \circ f)(x)$
22. $(h \circ f \circ g)(x)$
23. $(g \circ f \circ h)(x)$
24. $(g \circ h \circ f)(x)$
25. $(f \circ h \circ g)(x)$
26. $(f \circ g \circ h)(x)$

In each of the following exercises, write the given function as a composition of two or more non-identity functions. (There are several correct answers, so check your answer using function composition.)

27. $p(x) = (2x+3)^3$
28. $P(x) = \left(x^2 - x + 1\right)^5$
29. $h(x) = \sqrt{2x-1}$
30. $H(x) = |7-3x|$
31. $r(x) = \dfrac{2}{5x+1}$
32. $R(x) = \dfrac{7}{x^2-1}$
33. $q(x) = \dfrac{|x|+1}{|x|-1}$
34. $Q(x) = \dfrac{2x^3+1}{x^3-1}$
35. $v(x) = \dfrac{2x+1}{3-4x}$
36. $w(x) = \dfrac{x^2}{x^4+1}$
37. Let $g(x) = -x$, $h(x) = x+2$, $j(x) = 3x$ and $k(x) = x-4$. In what order must these functions be composed with $f(x) = \sqrt{x}$ to create $F(x) = 3\sqrt{-x+2} - 4$?
38. What linear functions could be used to transform $f(x) = x^3$ into $F(x) = -\frac{1}{2}(2x-7)^3 + 1$? What is the proper order of composition?

For each of the following exercises, let f be the function defined by

$$f = \{(-3,4), (-2,2), (-1,0), (0,1), (1,3), (2,4), (3,-1)\}$$

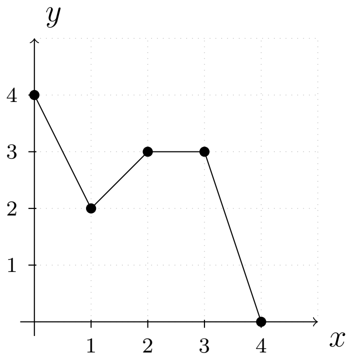
and let g be the function defined

$$g = \{(-3,-2), (-2,0), (-1,-4), (0,0), (1,-3), (2,1), (3,2)\}.$$

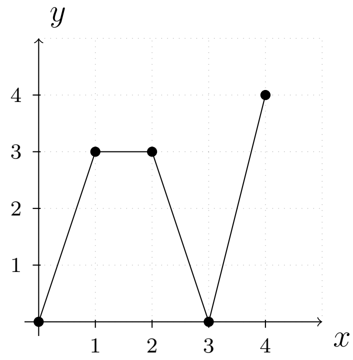
Use f and g to compute each of the indicated values if they exist.

39. $(f \circ g)(3)$
40. $f(g(-1))$
41. $(f \circ f)(0)$
42. $(f \circ g)(-3)$
43. $(g \circ f)(3)$
44. $g(f(-3))$
45. $(g \circ g)(-2)$
46. $(g \circ f)(-2)$
47. $g(f(g(0)))$

In each of the following exercises, use the graphs of $y = f(x)$ and $y = g(x)$ below to find the function value.



$y = f(x)$



$y = g(x)$

48. $(g \circ f)(1)$
49. $(f \circ g)(3)$
50. $(g \circ f)(2)$
51. $(f \circ g)(0)$
52. $(f \circ f)(1)$
53. $(g \circ g)(1)$

Inverse Functions

In each of the following exercises, show that the given function is one-to-one and find its inverse. Check your answers algebraically and graphically. Verify that the range of f is the domain of f^{-1} and vice-versa.

1. $f(x) = 6x - 2$
2. $f(x) = 42 - x$
3. $f(x) = \dfrac{x-2}{3} + 4$
4. $f(x) = 1 - \dfrac{4+3x}{5}$
5. $f(x) = \sqrt{3x-1} + 5$
6. $f(x) = 2 - \sqrt{x-5}$
7. $f(x) = 3\sqrt{x-1} - 4$
8. $f(x) = 1 - 2\sqrt{2x+5}$
9. $f(x) = \sqrt[5]{3x-1}$
10. $f(x) = 3 - \sqrt[3]{x-2}$
11. $f(x) = x^2 - 10x$, $x \geq 5$
12. $f(x) = 3(x+4)^2 - 5$, $x \leq -4$
13. $f(x) = x^2 - 6x + 5$, $x \leq 3$
14. $f(x) = 4x^2 + 4x + 1$, $x < -1$
15. $f(x) = \dfrac{3}{4-x}$
16. $f(x) = \dfrac{x}{1-3x}$
17. $f(x) = \dfrac{2x-1}{3x+4}$
18. $f(x) = \dfrac{4x+2}{3x-6}$

19. $f(x) = \dfrac{-3x-2}{x+3}$

20. $f(x) = \dfrac{x-2}{2x-1}$

Find the inverses of each of the following functions.

21. $f(x) = ax + b,\ a \neq 0$

22. $f(x) = a\sqrt{x-h} + k,\ a \neq 0, x \geq h$

23. $f(x) = ax^2 + bx + c$ where $a \neq 0,\ x \geq -\dfrac{b}{2a}$.

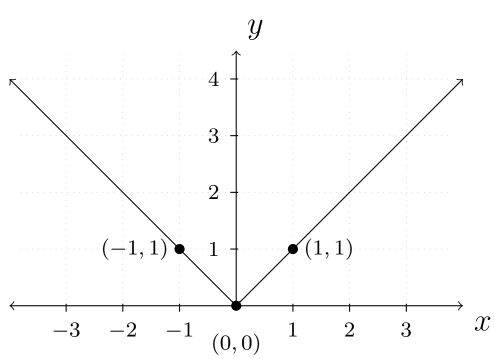
24. $f(x) = \dfrac{ax+b}{cx+d}$ where c and d are not both zero.

Transformations

Suppose $(2, -3)$ is on the graph of $y = f(x)$. In each of the following exercises, use the given point to find a point on the graph of the given transformed function.

1. $g(x) = f(x) + 3$
2. $g(x) = f(x+3)$
3. $g(x) = f(x) - 1$
4. $g(x) = f(x-1)$
5. $g(x) = 3f(x)$
6. $g(x) = f(3x)$
7. $g(x) = -f(x)$
8. $g(x) = f(-x)$
9. $g(x) = f(x-3) + 1$
10. $g(x) = 2f(x+1)$
11. $g(x) = 10 - f(x)$
12. $g(x) = 3f(2x) - 1$
13. $g(x) = \frac{1}{2}f(4-x)$
14. $g(x) = 5f(2x) + 3$
15. $g(x) = 2f(1-x) - 1$
16. $g(x) = \dfrac{f(3x)-1}{2}$
17. $g(x) = \dfrac{4-f(3x-1)}{7}$

The complete graph of $f(x) = |x|$ is given below. In each of the following exercises, use it to graph the given transformed function.

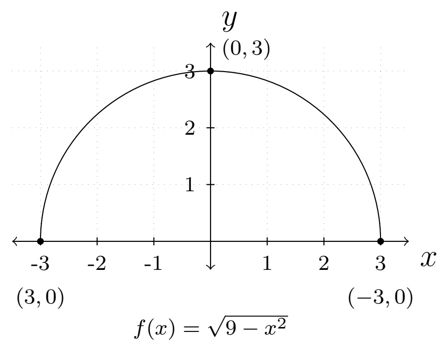


$f(x) = |x|$

18. $g(x) = f(x) + 1$
19. $g(x) = f(x) - 2$
20. $g(x) = f(x + 1)$
21. $g(x) = f(x - 2)$
22. $g(x) = 2f(x)$
23. $g(x) = f(2x)$
24. $g(x) = 2 - f(x)$
25. $g(x) = f(2 - x)$
26. $g(x) = 2 - f(2 - x)$

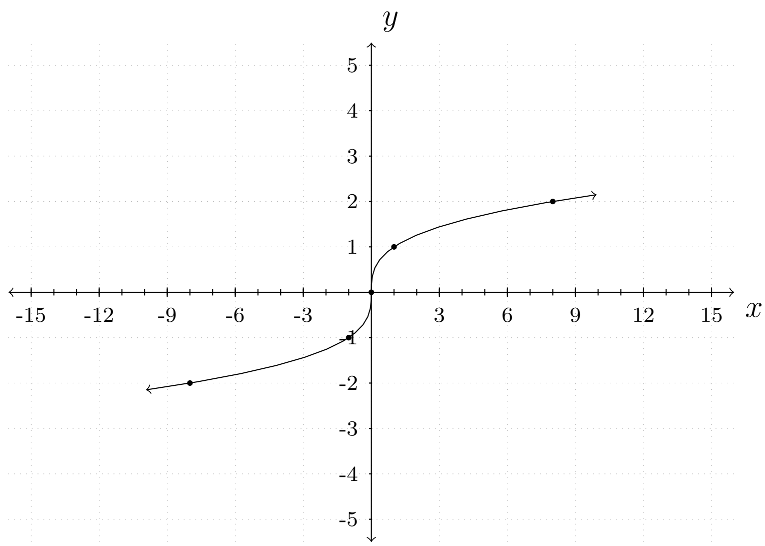
27. Some of the answers to the previous nine exercises should be equal. Which ones are equal? What properties of the graph of $y = f(x)$ contribute to this?

The complete graph of $f(x) = \sqrt{9 - x^2}$ is given below. Use the graph of f to graph the each of the given transformations.

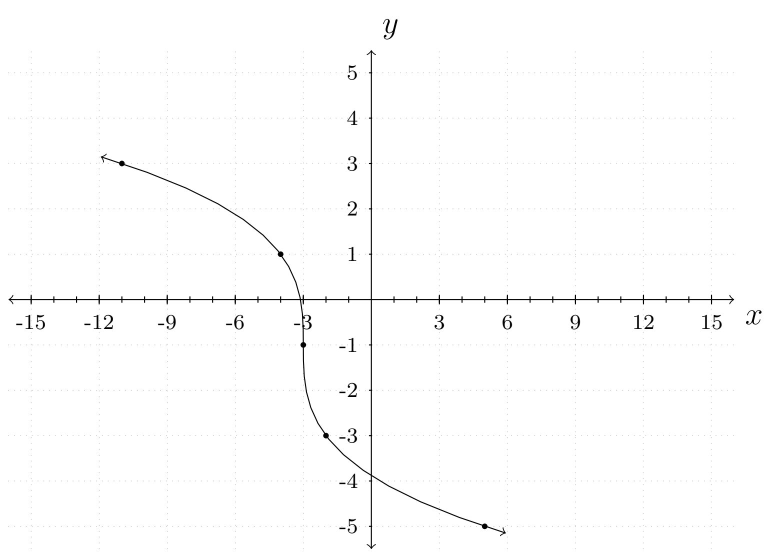


28. $g(x) = f(x) + 3$
29. $h(x) = f(x) - \frac{1}{2}$
30. $j(x) = f\left(x - \frac{2}{3}\right)$
31. $a(x) = f(x + 4)$
32. $b(x) = f(x + 1) - 1$
33. $c(x) = \frac{3}{5}f(x)$
34. $d(x) = -2f(x)$
35. $k(x) = f\left(\frac{2}{3}x\right)$
36. $m(x) = -\frac{1}{4}f(3x)$
37. $n(x) = 4f(x - 3) - 6$
38. $p(x) = 4 + f(1 - 2x)$

39. The graphs of $y = f(x) = \sqrt[3]{x}$ and $y = g(x)$ are shown below. Find a formula for g based on transformations of the graph of f. Check your answer by confirming that the points shown on the graph of g satisfy the equation $y = g(x)$.



$f(x) = \sqrt[3]{x}$



$g(x)$

40. A function f is said to be *even* if $f(x) = f(-x)$. The graph of an even function will be symmetric about the $y-$axis, since $f(-x)$ represents a reflection of the graph of f about the $y-$axis. Determine both algebraically (using compositions) and graphically (using transformations) whether each of the following fundamental functions is even.

    (a) $g(x) = x^2$
    (b) $k(x) = \sqrt{x}$
    (c) $\ell(x) = |x|$
    (d) $m(x) = x^3$
    (e) $n(x) = \sqrt[3]{x}$
    (f) $p(x) = \dfrac{1}{x}$
    (g) $q(x) = \sqrt{9 - x^2}$

41. A function f is said to be *odd* if $-f(x) = f(-x)$. Since $f(-x)$ represents a reflection of the graph of f about the $y-$axis and $-f(x)$ represents a reflection of the graph of f about the $x-$axis, whenever these two reflections produce the same graph, the corresponding function will be odd. In this case, the graph of an odd function is said to be *symmetric about the origin*. Determine both algebraically (using compositions) and graphically (using transformations) whether each of the following fundamental functions is odd.

    (a) $g(x) = x^2$
    (b) $k(x) = \sqrt{x}$
    (c) $\ell(x) = |x|$
    (d) $m(x) = x^3$
    (e) $n(x) = \sqrt[3]{x}$
    (f) $p(x) = \dfrac{1}{x}$
    (g) $q(x) = \sqrt{9 - x^2}$

Let $f(x) = \sqrt{x}$. Find a formula for a function g whose graph is obtained from f from the given sequence of transformations.

42. (1) shift right 2 units; (2) shift down 3 units
43. (1) shift down 3 units; (2) shift right 2 units
44. (1) reflect across the x-axis; (2) shift up 1 unit
45. (1) shift up 1 unit; (2) reflect across the x-axis
46. (1) shift left 1 unit; (2) reflect across the y-axis; (3) shift up 2 units
47. (1) reflect across the y-axis; (2) shift left 1 unit; (3) shift up 2 units
48. (1) shift left 3 units; (2) vertical stretch by a factor of 2; (3) shift down 4 units
49. (1) shift left 3 units; (2) shift down 4 units; (3) vertical stretch by a factor of 2
50. (1) shift right 3 units; (2) horizontal shrink by a factor of 2; (3) shift up 1 unit
51. (1) horizontal shrink by a factor of 2; (2) shift right 3 units; (3) shift up 1 unit

Piecewise-Defined and Absolute Value Functions

Piecewise-Defined Functions

1. Let $f(x) = \begin{cases} x+5 & \text{if} \quad x \le -3 \\ \sqrt{9-x^2} & \text{if} \quad -3 < x \le 3 \\ -x+5 & \text{if} \quad x > 3 \end{cases}$

   Compute the following function values.

   (a) $f(-4)$ (b) $f(-3)$ (c) $f(3)$

   (d) $f(3.1)$ (e) $f(-3.01)$ (f) $f(2)$

2. Let $f(x) = \begin{cases} x^2 & \text{if} \quad x \le -1 \\ \sqrt{1-x^2} & \text{if} \quad -1 < x \le 1 \\ x & \text{if} \quad x > 1 \end{cases}$

   Compute the following function values.

   (a) $f(4)$ (b) $f(-3)$ (c) $f(1)$

   (d) $f(0)$ (e) $f(-1)$ (f) $f(-0.99)$

In each of the following exercises, find all possible x such that $f(x) = 0$. Then sketch the graph of the given piecewise-defined function. Use your graph to identify the domain and range of each function.

3. $f(x) = \begin{cases} 4-x & \text{if} \quad x \le 3 \\ 2 & \text{if} \quad x > 3 \end{cases}$

4. $f(x) = \begin{cases} x^2 & \text{if} \quad x \le 0 \\ 2x & \text{if} \quad x > 0 \end{cases}$

5. $f(x) = \begin{cases} -3 & \text{if} \quad x < 0 \\ 2x-3 & \text{if} \quad 0 \le x \le 3 \\ 3 & \text{if} \quad x > 3 \end{cases}$

6. $f(x) = \begin{cases} x^2-4 & \text{if} \quad x \le -2 \\ 4-x^2 & \text{if} \quad -2 < x < 2 \\ x^2-4 & \text{if} \quad x \ge 2 \end{cases}$

7. $f(x) = \begin{cases} -2x-4 & \text{if} \quad x < 0 \\ 3x & \text{if} \quad x \ge 0 \end{cases}$

8. $f(x) = \begin{cases} x^2 & \text{if} \quad x \le -2 \\ 3-x & \text{if} \quad -2 < x < 2 \\ 4 & \text{if} \quad x \ge 2 \end{cases}$

9. $f(x) = \begin{cases} \dfrac{1}{x} & \text{if} \quad -6 < x < -1 \\ x & \text{if} \quad -1 < x < 1 \\ \sqrt{x} & \text{if} \quad 1 < x < 9 \end{cases}$

Functions Containing an Absolute Value

In each of the following exercises, find the zeros of each function and the x- and y-intercepts of each graph, if any exist. Then graph the given absolute value function and express it as a piecewise-defined function. Use the graph to determine the domain and range of each function.

1. $f(x) = |x+4|$
2. $f(x) = |x| + 4$
3. $f(x) = |4x|$
4. $f(x) = -3|x|$
5. $f(x) = |2x-5|$
6. $f(x) = |-2x+5|$
7. $f(x) = 2|x - \frac{5}{2}|$
8. $f(x) = \dfrac{1}{3}|2x-1|$
9. $f(x) = 3|x+4| - 4$

Selected Answers

Identifying Domain Algebraically

1. $(-\infty, \infty)$
3. $(-\infty, \frac{2}{3}) \cup (\frac{2}{3}, \infty)$
5. $(-\infty, \infty)$
7. $(\frac{2}{3}, \infty)$
9. $(-\infty, \infty)$
11. $(-\infty, \infty)$
13. $(-\infty, -4) \cup (-4, \infty)$
15. $(-\infty, 2) \cup (2, \infty)$
17. $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
19. $(-\infty, -6) \cup (-6, 6) \cup (6, \infty)$
21. $[-\frac{5}{2}, \infty)$
23. $[-3, \infty)$
25. $[3, \infty)$
27. $(3, \infty)$
29. $(-\infty, \infty)$
31. $(-\infty, -6) \cup (-6, 6) \cup (6, \infty)$
33. $[7, 9]$
35. $(-\infty, 8) \cup (8, \infty)$
37. $(-\infty, \infty)$

Combining Functions

Function Arithmetic

1. $f(x) = 3x + 1, \quad g(x) = 4 - x$
   - $(f+g)(2) = 9$
   - $(f-g)(-1) = -7$
   - $(g-f)(1) = -1$
   - $(fg)(\frac{1}{2}) = \frac{35}{4}$
   - $\left(\frac{f}{g}\right)(0) = \frac{1}{4}$
   - $\left(\frac{g}{f}\right)(-2) = -\frac{6}{5}$

3. $f(x) = x^2 - x, \quad g(x) = 12 - x^2$
   - $(f+g)(2) = 10$
   - $(f-g)(-1) = -9$
   - $(g-f)(1) = 11$
   - $(fg)(\frac{1}{2}) = -\frac{47}{16}$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = \frac{4}{3}$

5. $f(x) = \sqrt{x+3}, \quad g(x) = 2x - 1$
   - $(f+g)(2) = 3 + \sqrt{5}$
   - $(f-g)(-1) = 3 + \sqrt{2}$
   - $(g-f)(1) = -1$
   - $(fg)(\frac{1}{2}) = 0$
   - $\left(\frac{f}{g}\right)(0) = -\sqrt{3}$
   - $\left(\frac{g}{f}\right)(-2) = -5$

7. $f(x) = 2x, \quad g(x) = \dfrac{1}{2x+1}$
   - $(f+g)(2) = \frac{21}{5}$
   - $(f-g)(-1) = -1$
   - $(g-f)(1) = -\frac{5}{3}$
   - $(fg)(\frac{1}{2}) = \frac{1}{2}$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = \frac{1}{12}$

9. $f(x) = x^2, \quad g(x) = \dfrac{1}{x^2}$
   - $(f+g)(2) = \frac{17}{4}$
   - $(f-g)(-1) = 0$
   - $(g-f)(1) = 0$
   - $(fg)(\frac{1}{2}) = 1$
   - $\left(\frac{f}{g}\right)(0) =$DNE
   - $\left(\frac{g}{f}\right)(-2) = \frac{1}{16}$

11. $f(x) = 2x + 1, \quad g(x) = x - 2$
    - $(f+g)(x) = 3x - 1,$ all reals
    - $(f-g)(x) = x + 3,$ all reals
    - $(fg)(x) = 2x^2 - 3x - 2,$ all reals
    - $\left(\frac{f}{g}\right)(x) = \frac{2x+1}{x-2}, \ x \neq 2$

13. $f(x) = x^2, \quad g(x) = 3x - 1$
    - $(f+g)(x) = x^2 + 3x - 1,$ all reals
    - $(f-g)(x) = x^2 - 3x + 1,$ all reals
    - $(fg)(x) = 3x^3 - x^2,$ all reals
    - $\left(\frac{f}{g}\right)(x) = \frac{x^2}{3x-1}, \ x \neq \frac{1}{3}$

15. $f(x) = x^2 - 4, \quad g(x) = 3x + 6$
    - $(f+g)(x) = x^2 + 3x + 2,$ all reals
    - $(f-g)(x) = x^2 - 3x - 10,$ all reals
    - $(fg)(x) = 3x^3 + 6x^2 - 12x - 24,$ all reals
    - $\left(\frac{f}{g}\right)(x) = \frac{x^2-4}{3x+6}, \ x \neq -2$

17. $f(x) = \dfrac{x}{2}, \quad g(x) = \dfrac{2}{x}$

- $(f+g)(x) = \frac{x^2+4}{2x},\ x \neq 0$
- $(f-g)(x) = \frac{x^2-4}{2x},\ x \neq 0$
- $(fg)(x) = 1,\ x \neq 0$
- $\left(\frac{f}{g}\right)(x) = \frac{x^2}{4},\ x \neq 0$

19. $f(x) = x, \quad g(x) = \sqrt{x+1}$

- $(f+g)(x) = x + \sqrt{x+1},\ x \geq -1$
- $(f-g)(x) = x - \sqrt{x+1},\ x \geq -1$
- $(fg)(x) = x\sqrt{x+1},\ x \geq -1$
- $\left(\frac{f}{g}\right)(x) = \frac{x}{\sqrt{x+1}},\ x > -1$

21. 2
23. 0
25. 3
27. DNE
29. 4
31. -2

Composite Functions

1. $f(x) = x^2, \quad g(x) = 2x+1$

- $(g \circ f)(0) = 1$
- $(f \circ g)(-1) = 1$
- $(f \circ f)(2) = 16$
- $(g \circ f)(-3) = 19$
- $(f \circ g)(\frac{1}{2}) = 4$
- $(f \circ f)(-2) = 16$

3. $f(x) = 4 - 3x, \quad g(x) = |x|$

- $(g \circ f)(0) = 4$
- $(f \circ g)(-1) = 1$
- $(f \circ f)(2) = 10$
- $(g \circ f)(-3) = 13$
- $(f \circ g)(\frac{1}{2}) = \frac{5}{2}$
- $(f \circ f)(-2) = -26$

5. $f(x) = 4x + 5, \quad g(x) = \sqrt{x}$

- $(g \circ f)(0) = \sqrt{5}$
- $(f \circ g)(-1) =$DNE
- $(f \circ f)(2) = 57$
- $(g \circ f)(-3) =$DNE
- $(f \circ g)(\frac{1}{2}) = 4\sqrt{\frac{1}{2}} + 5$
- $(f \circ f)(-2) = -7$

7. $f(x) = \dfrac{3}{1-x}, \quad g(x) = \dfrac{4x}{x^2+1}$

- $(g \circ f)(0) = \frac{6}{5}$
- $(f \circ g)(-1) = 1$
- $(f \circ f)(2) = \frac{3}{4}$
- $(g \circ f)(-3) = \frac{48}{25}$
- $(f \circ g)(\frac{1}{2}) = -5$
- $(f \circ f)(-2) =$DNE

9. $f(x) = 2x + 3, \quad g(x) = x^2 - 9$
   - $(g \circ f)(x) = 4x^2 + 12x$
   - $(f \circ g)(x) = 2x^2 - 15$
   - $(f \circ f)(x) = 4x = 9$

11. $f(x) = x^2 - 4, \quad g(x) = |x|$
   - $(g \circ f)(x) = |x^2 - 4|$
   - $(f \circ g)(x) = x^2 - 4$
   - $(f \circ f)(x) = x^4 - 8x^2 + 12$

13. $f(x) = |x + 1|, \quad g(x) = \sqrt{x}$
   - $(g \circ f)(x) = \sqrt{|x + 1|}$
   - $(f \circ g)(x) = |\sqrt{x} + 1|$
   - $(f \circ f)(x) = |x + 1| + 1$

15. $f(x) = |x|, \quad g(x) = \sqrt{4 - x}$
   - $(g \circ f)(x) = \sqrt{4 - |x|}$
   - $(f \circ g)(x) = \sqrt{4 - x}$
   - $(f \circ f)(x) = |x|$

17. $f(x) = 3x - 1, \quad g(x) = \dfrac{1}{x + 3}$
   - $(g \circ f)(x) = \frac{1}{3x+2}$
   - $(f \circ g)(x) = \frac{x}{x+3}$
   - $(f \circ f)(x) = 9x - 4$

19. $f(x) = \dfrac{x}{2x + 1}, \quad g(x) = \dfrac{2x + 1}{x}$
   - $(g \circ f)(x) = \frac{4x+1}{x}$
   - $(f \circ g)(x) = \frac{2x+1}{5x+2}$
   - $(f \circ f)(x) = \frac{x}{4x+1}$

21. $h(g(f(x))) = |\sqrt{-2x}|$
23. $g(f(h(x))) = \sqrt{-2|x|}$
25. $f(h(g(x))) = -2|\sqrt{x}|$

27. $f(x) = x^3, \quad g(x) = 2x + 3$
29. $f(x) = \sqrt{x}, \quad g(x) = 2x - 1$
31. $f(x) = \dfrac{2}{x}, \quad g(x) = 5x + 1$
33. $f(x) = \dfrac{x + 1}{x - 1}, \quad g(x) = |x|$
35. $f(x) = \dfrac{x + 1}{3 - 2x}, \quad g(x) = 2x$
37. $k \circ j \circ f \circ h \circ g$

39. 4
41. 3
43. -4
45. 0
47. -3
49. 4
51. 4
53. 0

Inverse Functions

1. $f^{-1}(x) = \dfrac{x+2}{6}$
3. $f^{-1}(x) = 3x - 10$
5. $f^{-1}(x) = \frac{1}{3}(x-5)^2 + \frac{1}{3},\ x \geq 5$
7. $f^{-1}(x) = \frac{1}{9}(x+4)^2 + 1,\ x \geq -4$
9. $f^{-1}(x) = \frac{1}{3}x^5 + \frac{1}{3}$
11. $f^{-1}(x) = 5 + \sqrt{x+25}$
13. $f^{-1}(x) = 3 - \sqrt{x+4}$
15. $f^{-1}(x) = \dfrac{4x-3}{x}$
17. $f^{-1}(x) = \dfrac{4x+1}{2-3x}$
19. $f^{-1}(x) = \dfrac{-3x-2}{x+3}$
21. $f^{-1}(x) = \dfrac{x-b}{a}$
23. $f^{-1}(x) = \dfrac{-b + \sqrt{b^2 - 4a(c-x)}}{2a}$

Transformations

1. $(2, 0)$
3. $(2, -4)$
5. $(2, -9)$
7. $(2, 3)$
9. $(5, -2)$
11. $(2, 13)$
13. $(2, -\frac{3}{2})$
15. $(-1, -7)$
17. $(1, 1)$

Each answer below describes the resulting transformation of the graph of $f(x) = |x|$.

19. Shift down 2 units
21. Shift right 2 units
23. Vertical stretch (or horizontal shrink) by a factor of 2
25. Shift right 2 units
27. Exercises 22. and 23. agree; 21. and 25. agree. $|kx| = |k| \cdot |x|$, where $k \in \mathbb{R}$

Each answer below describes the resulting transformation of the graph of $f(x) = \sqrt{9 - x^2}$.

29. Shift down 1/2 units
31. Shift left 4 units
33. Vertical shrink by a factor of 5/3
35. Horizontal stretch by a factor of 3/2
37. Shift right 3 units, vertical stretch by a factor of 4, shift down 6 units

39. $g(x) = -2\sqrt[3]{x+3} - 1$
41. (d), (e), and (f)
43. $g(x) = \sqrt{x-2} - 3$
45. $g(x) = -\sqrt{x} - 1$
47. $g(x) = \sqrt{-x-1} + 2$
49. $g(x) = 2\sqrt{x+3} - 8$
51. $g(x) = \sqrt{2x-6} + 1$

Piecewise-Defined and Absolute Value Functions

Piecewise-Defined Functions

1. $f(x) = \begin{cases} x+5 & \text{if} \quad x \leq -3 \\ \sqrt{9-x^2} & \text{if} \quad -3 < x \leq 3 \\ -x+5 & \text{if} \quad x > 3 \end{cases}$

   (a) $f(-4) = 1$ (b) $f(-3) = 2$ (c) $f(3) = 0$

   (d) $f(3.1) = 1.9$ (e) $f(-3.01) = 1.99$ (f) $f(2) = \sqrt{5}$

3. D: $(-\infty, \infty)$; R: $[1, \infty)$; No zeros
5. D: $(-\infty, \infty)$; R: $[-3, 3]$; $x = 3/2$
7. D: $(-\infty, \infty)$; R: $(-4, \infty)$; $x = -2, 0$
9. D: $(-6, -1) \cup (-1, 1) \cup (1, 9)$; R: $(-1, 1) \cup (1, 3)$; $x = 0$

Functions Containing an Absolute Value

1. No zeros; $y-$int at $(0, 4)$; D: $(-\infty, \infty)$; R: $[4, \infty)$

$$f(x) = \begin{cases} x+4 & \text{if} \quad x \geq 0 \\ -x+4 & \text{if} \quad x < 0 \end{cases}$$

3. Zero at $x = 0$; $y-$int at $(0, 0)$; D: $(-\infty, \infty)$; R: $[0, \infty)$

$$f(x) = \begin{cases} 4x & \text{if} \quad x \geq 0 \\ -4x & \text{if} \quad x < 0 \end{cases}$$

5. Zero at $x = \frac{5}{2}$; $y-$int at $(0, 5)$; D: $(-\infty, \infty)$; R: $[0, \infty)$

$$f(x) = \begin{cases} 2x-5 & \text{if} \quad x \geq \frac{5}{2} \\ -2x+5 & \text{if} \quad x < \frac{5}{2} \end{cases}$$

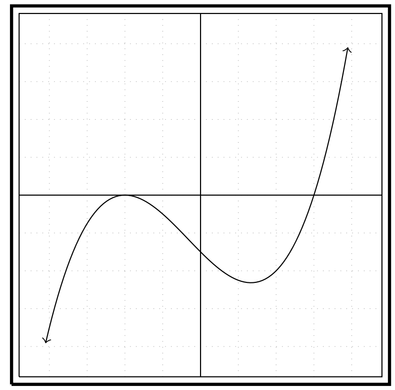
7. Zero at $x = \frac{5}{2}$; $y-$int at $(0, 5)$; D: $(-\infty, \infty)$; R: $[0, \infty)$

$$f(x) = \begin{cases} 2x-5 & \text{if} \quad x \geq \frac{5}{2} \\ -2x+5 & \text{if} \quad x < \frac{5}{2} \end{cases}$$

9. Zeros at $x = -\frac{16}{3}, \; -\frac{8}{3}$; $y-$int at $(0, 8)$; D: $(-\infty, \infty)$; R: $[-4, \infty)$

$$f(x) = \begin{cases} 3x+8 & \text{if} \quad x \geq -4 \\ -3x-16 & \text{if} \quad x < -4 \end{cases}$$

Chapter 6



Polynomials

Introduction and Terminology (L43)

Objective: Identify key features of and classify a polynomial by degree and number of nonzero terms.

A *polynomial* in terms of a variable x is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0,$$

where each *coefficient*, a_i, is a real number ($a_n \neq 0$) and the exponent, or *degree* of the polynomial, n, is a nonnegative integer.

Examples of polynomials include: $f(x) = x^2 + 5$, $f(x) = x$ and $f(x) = -3x^7 + 4x^3 - 5x$. Before classifying polynomials, we will take a moment to establish some key terminology. For our general polynomial above, the:

degree	is	n
coefficients	are	$a_n, a_{n-1}, \ldots, a_1, a_0$
leading coefficient	is	a_n
leading term	is	$a_n x^n$
constant term	is	$a_0 x^0 = a_0$.

A concrete example will help to clarify each of these terms.

Example 274. Identify the degree, leading coefficient, leading term and constant term for the polynomial

$$f(x) = -19x^5 + 4x^4 - 6x + 21.$$

The degree of this polynomial is $n = 5$, since five is the greatest exponent.

The leading term, which is the term that contains the greatest exponent (degree), is $a_n x^n = -19x^5$.

The leading coefficient is the real number being multiplied by x^n in the leading term, namely $a_n = -19$.

The constant term is $a_0 = 21$, which also represents the $y-$intercept for the graph of the given polynomial, just as it did in the chapter on quadratics.

The complete set of coefficients for the given polynomial is

$$\{a_5 = -19,\ a_4 = 4,\ a_3 = 0,\ a_2 = 0,\ a_1 = -6,\ a_0 = 21\}.$$

It is important to point out the fact that the previous example contains no *cubic* or *quadratic* terms, since the respective coefficients are both zero. This example demonstrates that not every polynomial will contain a nonzero coefficient for every term. As another example, the *power function* $f(x) = x^{10}$ is also characterized as a polynomial having degree $n = 10$, leading coefficient $a_{10} = 1$, and trailing coefficients $a_i = 0$ for $i = 9, 8, \ldots, 1, 0$.

Before we can identify and begin to classify a polynomial, we may need to simplify the given expression for x, by distributing and combining all like terms. The general form of a polynomial should be reminiscent of the standard form of a quadratic, with possibly more terms. Hence the name “polynomial”, meaning “many terms”.

The following example shows how to identify a polynomial after the necessary simplification has taken place.

Example 275. Identify the degree, leading coefficient, leading term and constant term for the given polynomial function.

$$\begin{aligned} f(x) &= 3(x+1)(x-1) + 4x^3 + 2x + 3 \\ &= 3(x^2 - 1) + 4x^3 + 2x + 3 \\ &= 3x^2 - 3 + 4x^3 + 2x + 3 \\ &= 4x^3 + 3x^2 + 2x \end{aligned}$$

Upon simplifying, we see that f has degree $n = 3$, since three is the greatest exponent.

The leading term is $4x^3$ with a leading coefficient of $a_n = 4$.

Since no constant term is shown, $a_0 = 0$ is our constant term.

Now that we can identify the essential components of a polynomial, we will categorize polynomials based upon their degree, as well as the number of terms, after all necessary simplification.

Types of Polynomials

Degree	Type	Example
0	Constant	-1
1	Linear	$2x + \sqrt{5}$
2	Quadratic	$5x^2 - 32x + 2$
3	Cubic	$(-1/2)x^3$
4	Quartic	$-3x^4 + 2x^2 + 3x + 1$
5	Quintic	$-2x^5$
6 or more	n^{th} Degree	$-2x^7 + 52x^6 + 12$

One point of note in the table above is the appearance of both rational and irrational coefficients $\left(-1/2 \text{ and } \sqrt{5}\right)$. The appearance of such coefficients is permissible in polynomials, since our coefficients a_i are only required to be real numbers. A coefficient containing the imaginary number $i = \sqrt{-1}$, on the other hand, is not permitted.

Polynomial Characterizations by Number of Nonzero Terms

Number of Terms	Name	Example
1	Monomial	$4x^5$
2	Binomial	$2x^3 + 1$
3	Trinomial	$-23x^{18} + 4x^2 + 3x$
4	Tetranomial	$-23x^{18} + 4x^2 + 3x + 1$
5 or more	Polynomial	$-2x^4 + x^3 + 15x^2 - 41x + 12$

Example 276. Describe the type and characterization (number of terms) of the polynomial function shown below.

$$f(x) = -19x^5 + 4x^4 - 6x + 21$$

Polynomials are typically named by their degree first and then their number of terms. The polynomial above is a *quintic tetranomial*; quintic because it is degree five and tetranomial because it contains four terms.

Example 277. Describe the type and characterization (number of terms) of the polynomial function shown below.

$$f(x) = x^3 + x^2$$

The polynomial above is a *cubic binomial*, since it has degree three and contains two terms.

Example 278. Describe the type and characterization (number of terms) of the polynomial function shown below.

$$f(x) = 21x^4 + 12x^2 - 3x^2 - 9x^2 - 22x^4$$

Upon simplifying, we see that the given polynomial reduces to $f(x) = -x^4$. As a result, our polynomial is a quartic (degree four) monomial (one term).

This section “sets the table” for the basic terminology that will be used throughout the chapter. In the next section, we will review some additional prerequisite factoring techniques which will be necessary for working with certain polynomials, and provide a brief summary of all factoring methods that have been discussed up to this point. Once we have finished our review of factoring, we will be ready to begin the natural (albeit lengthy) method of analyzing and graphing a polynomial function.

Sign Diagrams (L44)

Objective: Construct a sign diagram for a given polynomial expression.

If a polynomial function or expression is completely factored, it will be beneficial to us to construct a sign diagram for the polynomial, in order to answer questions about its graph and confirm any other findings. Therefore, we devote this section to the construction of a sign diagram for a factored polynomial. Note that expanded polynomials first require us to find a complete factorization prior to constructing a sign diagram. This will require us to first employ factoring techniques and possibly polynomial division, which we reserve for a later section.

Recall that the roots of a quadratic expression represent the dividers in its corresponding sign diagram. This carries over directly to a polynomial expression.

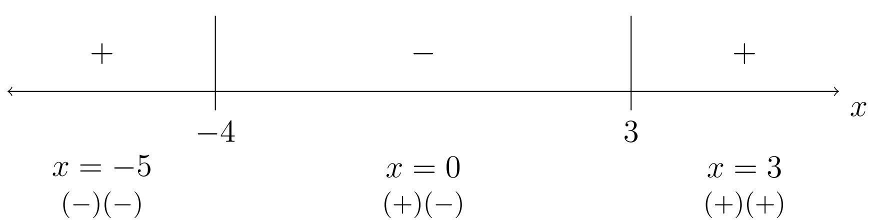
We begin with an example for quadratics.

Example 279. Construct a sign diagram for the polynomial function $f(x) = 2x^2 + 3x - 20$.

Although our first example is not factored, we can apply the *ac*-method to quickly factor our function.

$$\begin{aligned} f(x) &= 2x^2 + 3x - 20 \\ &= 2x^2 + 8x - 5x - 20 \\ &= 2x(x+4) - 5(x+4) \\ &= (x+4)(2x-5) \end{aligned}$$

This gives us two roots, $x = -4$ and $x = \frac{5}{2}$, which serve as the dividers in our accompanying diagram. For our three test values, we will use $x = -5, 0,$ and 3.

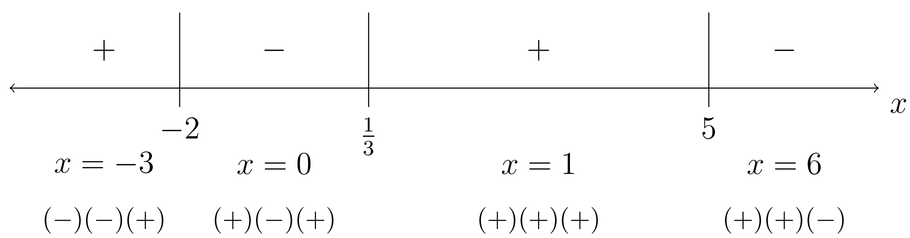


The previous example should be a familiar one, and one that we are comfortable with, since it ties in directly with the chapter on quadratics (degree-2 polynomials). For polynomials with a degree of $n \geq 3$, our diagram should look similar. The primary exceptions will be number of factors in our expression, and the number of dividers in our diagram. Again, we will focus primarily on polynomials which are already factored for our examples.

Example 280. Construct a sign diagram for the factored polynomial function

$$g(x) = (x+2)(3x-1)(5-x).$$

Our roots are $x = -2, \frac{1}{3}$, and 5. Consequently, the following diagram shows three dividers.

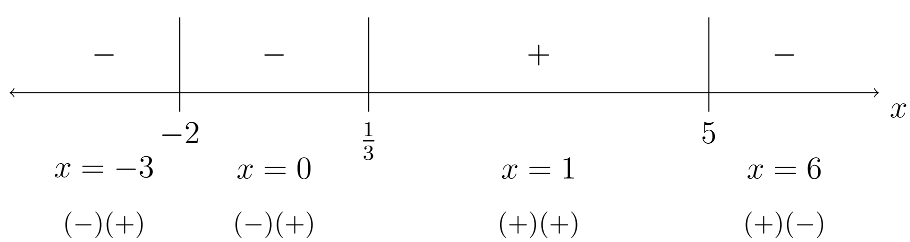


For our next example, we will make a slight change to the function g from the previous example, by including an extra factor of $x+2$.

Example 281. Construct a sign diagram for the factored polynomial function

$$h(x) = (x+2)^2(3x-1)(5-x).$$

Since the roots of h equal those from g, our diagram will have the same dividers and test values.



In the previous diagram, we see that each of our sign calculations have excluded the $(x+2)^2$ factor, since it will always contribute a positive sign and therefore has no impact on the end result. For example, for the test value $x = -3$, we get

$$(-)^2(-)(+) = \cancel{(-)^2}(-)(+),$$

which reduces to a negative sign. This simplification in our sign calculation can be employed for any factor that appears in our function with an *even* exponent.

Additionally, our last two diagrams look almost identical, with the lone exception being the sign associated with our first interval, $(-\infty, -2)$. This should make some sense, however, since we only changed the factor associated with the root $x = -2$ from one example to the next. The reason behind the change in diagram will become more clear to us as we explore polynomials further.

For our last example, we will present both the sign diagram and the accompanying graph for the given polynomial. Although the techniques to graphing a polynomial have not yet been discussed, for any function it is often helpful to utilize a graphing utility such as Desmos, in order to better understand the makeup of the function and how its graph is related.

Example 282. Construct a sign diagram for the factored polynomial function

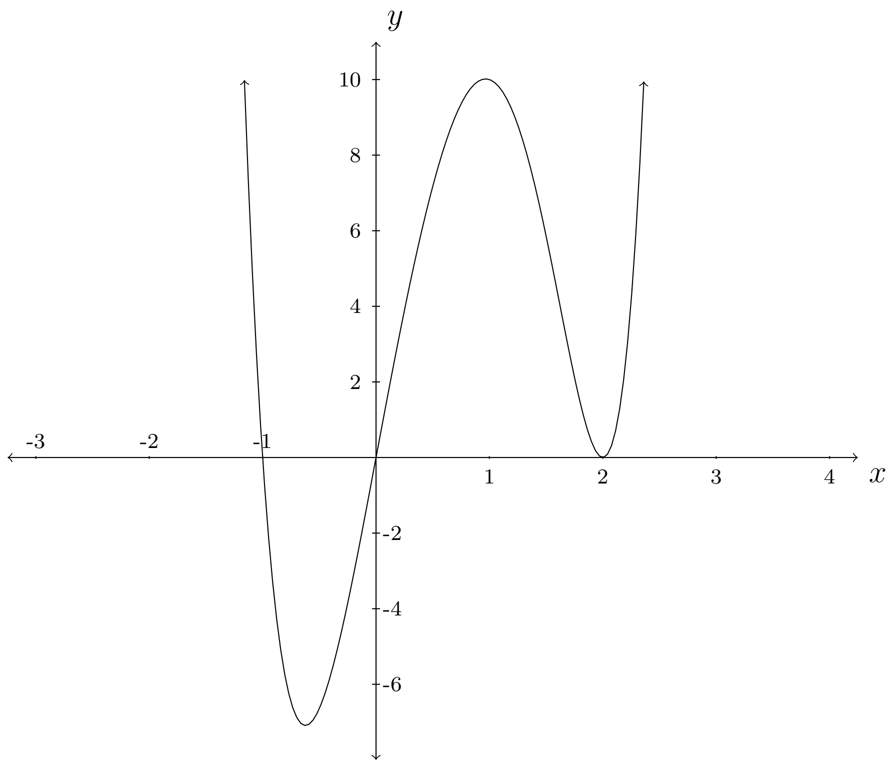
$$f(x) = x(x+1)(x-2)^2(x^2+4).$$

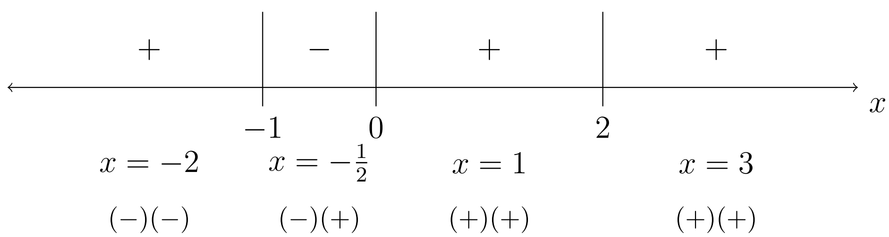
Use Desmos or a similar graphing utility to construct a graph of f.

Before we get started, it is important to spend some time discussing the factorization of f. Specifically, the factor of x will contribute a root of $x = 0$. This is the only instance in which our diagram requires a divider at $x = 0$.

Additionally, the factor of $x^2 + 4$ is often misinterpreted. By setting the expression equal to zero and solving for x, we see that the factor contributes two *imaginary* roots at $x = \pm 2i$. Furthermore, if we look more closely at this factor, we see that for any real number x, $x^2 + 4$ will always be positive. Hence, this factor will have no impact on our sign diagram calculations, and will be omitted. One should caution, however, that this factor does have an impact on the graph of f.

We can now conclude that the set of roots for f are $x = -1, 0,$ and 2. The accompanying diagram and graph are shown below. As before, we have also omitted the factor of $(x-2)^2$, since the squared factor will not impact our signs.





By looking at the graph of our last example, one should begin to notice the relationship that the graph of a polynomial has with its precise makeup and, consequently, its sign diagram. In particular, close attention should be paid to the nature of the graph of f near its real roots. In the case of $x = -1$ and $x = 0$ in our last example, we see that the graph *crosses over* the $x-$axis. Alternatively, our graph *turns around* or "bounces off" at the root $x = 2$. This difference in the local behavior of the graph of f at its roots is not just a coincidence, but rather a consequence of the makeup of the function f, as we will see when we discuss the *multiplicity* of the root of a polynomial in a later section.

Factoring

Some Special Cases (L45)

Objective: Factor a general polynomial expression using one or more factorization methods.

When factoring polynomials there are a few special products that, if we can recognize them, can be easily broken down. The first is one we have seen before, when factoring some quadratics in which there is no linear term.

When expanding, we know that the product of a sum and difference of the same two terms results in a difference of two squares.

$$\boxed{\text{Difference of Two Squares: } a^2 - b^2 = (a+b)(a-b)}$$

Consequently, if faced with the difference of two squares, one can conclude that such an expression will always factor as a product of the sum and difference of their square roots. Our first four examples demonstrate this fact.

Example 283. Factor each of the given binomial expressions completely over the real numbers.

1. $x^2 - 16$
2. $9x^2 - 25y^2$
3. $x^2 - 24$
4. $2x^2 - 5$

1. In this first example, we see that $a = x$ and $b = 4$ for our difference of two squares.

$$\begin{aligned} x^2 - 16 &= (x)^2 - (4)^2 \\ &= (x+4)(x-4) \end{aligned}$$

2. Taking the square roots of $9x^2$ and $25y^2$ gives us $a = 3x$ and $b = 5y$ for our second expression.

$$\begin{aligned} 9x^2 - 25y^2 &= (3x)^2 - (5y)^2 \\ &= (3x+5y)(3x-5y) \end{aligned}$$

3. Our third expression poses a bit of a challenge, since it is the first which does not present us with the difference of two *perfect* squares. In this case, $a = x$, but $b = \sqrt{24} = \sqrt{4 \cdot 6} = 2\sqrt{6}$.

$$\begin{aligned} x^2 - 24 &= (x)^2 - \left(2\sqrt{6}\right)^2 \\ &= \left(x + 2\sqrt{6}\right)\left(x - 2\sqrt{6}\right) \end{aligned}$$

4. Similarly, our final expression presents us with two terms, neither of which are perfect squares. In this case, $a = \sqrt{2x^2} = \sqrt{2}x$ and $b = \sqrt{5}$.

$$\begin{aligned} 2x^2 - 5 &= \left(\sqrt{2}x\right)^2 - \left(\sqrt{5}\right)^2 \\ &= \left(\sqrt{2}x + \sqrt{5}\right)\left(\sqrt{2}x - \sqrt{5}\right) \end{aligned}$$

   Note that in this last case, we have $\sqrt{2}x$ (or $x\sqrt{2}$) in our factorization, and not $\sqrt{2x}$.

It is important to note that, unlike differences, a *sum* of squares will never factor over the real numbers. Such expressions only factor over the complex numbers. Hence, we say that they are *irreducible* over the reals. This can be seen in our next example, where we will attempt to employ the $ac-$method to factor.

Example 284. Factor the expression $x^2 + 36$ completely over the real numbers and over the complex numbers.

For the expression $x^2 + 36$, $ac = 36$ and $b = 0$, as we have no linear term. So we need to identify two integers, m and n, such that $m + n = 0$ and $m \cdot n = 36$. Our choices for $m \cdot n$ are $1 \cdot 36$, $2 \cdot 18$, $3 \cdot 12$, $4 \cdot 9$ and $6 \cdot 6$. But, since there are no combinations from these that will both multiply to 36 *and* add to 0, we conclude that the given expression is irreducible over the reals.

Notice that $x^2 + 36$ does, however, factor over the complex numbers.

$$\begin{aligned} x^2 + 36 &= x^2 - (-36) \\ &= x^2 - \left(\sqrt{-36}\right)^2 \\ &= x^2 - \left(\sqrt{36}\sqrt{-1}\right)^2 \\ &= x^2 - (6i)^2 \\ &= (x - 6i)(x + 6i) \end{aligned}$$

We can further make sense of this result by recalling the methods from the chapter on quadratics. Since the discriminant of $x^2 + 36$ is

$$\begin{aligned} b^2 - 4ac &= 0^2 - 4(1)(36) \\ &= -144 \\ &< 0, \end{aligned}$$

we know that the given expression has no real roots. Hence, any factorization must contain imaginary numbers. By setting the expression equal to zero and extracting square roots, we get $x = \pm 6i$, which further supports our factorization.

We present the general factorization for the sum of two squares over the complex numbers below.

Sum of Two Squares: $a^2 + b^2 = (a + bi)(a - bi)$

For graphing purposes, we will primarily be concerned with factorization over the real numbers.

In many cases, we can also recognize an entire expression as a perfect square (or a squared binomial).

$$\boxed{\text{Perfect Square: } a^2 + 2ab + b^2 = (a+b)^2}$$

While it might seem difficult to recognize a perfect square at first glance, by employing the $ac-$method, we can see that in the case where $m = n$, the resulting factorization will be a perfect square. In this case, we can factor by identifying the square roots of the first and last terms and using the sign from the middle term. This is demonstrated in the following example.

Example 285. Factor each of the given trinomial expressions completely over the real numbers.

1. $x^2 - 6x + 9$
2. $4x^2 + 20xy + 25y^2$

1. For our first expression, $a = 1,\ b = -6$, and $c = 9$. So we must find two integers m and n such that $m + n = -6$ and $mn = ac = 9$. In this case, the numbers we need are -3 and -3. Consequently, we will have a perfect square.

   Using the square roots of $a = 1$ and $c = 9$ and the negative sign from the linear term, our factorization is
   $$x^2 - 6x + 9 = (x-3)^2 .$$

2. For our second expression, $a = 4,\ b = 20$, and $c = 25$. So we are looking for an m and n such that $m + n = 20$ and $mn = ac = 100$. Quickly, we see that $m = n = 10$, and again, we have a perfect square.

   In this case, our factorization is
   $$4x^2 + 20xy + 25y^2 = (2x + 5y)^2 .$$

Another factoring shortcut involves sums and differences of cubes. Both sums and differences of cubes have very similar factorizations.

$$\boxed{\begin{array}{c}\text{Sum of Cubes: } a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)\\ \text{Difference of Cubes: } a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)\end{array}}$$

As with all of the formulas in this section, we can verify those for a sum and difference of cubes by expanding the right-hand side. For example, for the difference of cubes,

$$\begin{aligned}(a-b)\left(a^2+ab+b^2\right) &= a\left(a^2+ab+b^2\right) - b\left(a^2+ab+b^2\right)\\ &= a^3 + \cancel{a^2b} + \cancel{ab^2} - \cancel{a^2b} - \cancel{ab^2} - b^3\\ &= a^3 - b^3\end{aligned}$$

Comparing the formulas one may notice that the only difference resides in the signs between the terms. One way to remember these two formulas is to think of "**SOAP**":

S	The first sign in our factorization is the **Same** sign as the given expression.
O	The second sign in our factorization is the **Opposite** sign as the given expression.
AP	The last sign in our factorization is **Always Positive**.

Example 286. Factor each of the given binomial expressions completely over the real numbers.

1. $m^3 - 27$
2. $125p^3 + 8r^3$

1. In our first expression, our desired cube roots for each term are $a = m$ and $b = 3$. Using the "Same, Opposite, Always Positive" acronym, we have the following factorization.

$$\begin{aligned} m^3 - 27 &= (m-3)\left((m)^2 + 3m + (3)^2\right) \\ &= (m-3)\left(m^2 + 3m + 9\right) \end{aligned}$$

2. In second expression, our desired cube roots for each term are

$$\begin{aligned} a &= \sqrt[3]{125p^3} \\ &= \sqrt[3]{125}\sqrt[3]{p^3} \\ &= 5p \end{aligned} \qquad\qquad \begin{aligned} b &= \sqrt[3]{8r^3} \\ &= \sqrt[3]{8}\sqrt[3]{r^3} \\ &= 2r \end{aligned}$$

Using the "Same, Opposite, Always Positive" acronym, we have the following factorization.

$$\begin{aligned} 125p^3 + 8r^3 &= (5p+2r)\left((5p)^2 - (5p)(2r) + (2r)^2\right) \\ &= (5p+2r)\left(25p^2 - 10pr + 4r^2\right) \end{aligned}$$

The second expression in our last example illustrates an important point. When we identify the first and last terms of the trinomial in our factorization, we must square each cube root in its entirety. In this case, both the coefficients and variables are squared, so that $(5p)^2$ becomes $25p^2$, and $(2r)^2$ becomes $4r^2$.

After factoring a sum or difference of cubes, it should be natural to attempt to factor the resulting trinomial expression (our second factor). As a general rule, however, this factor should always be irreducible over the reals, with the main exception being that of a GCF in the given expression that might have been initially overlooked.

Our last special case comes up frequently enough that we will devote the next subsection to it.

Quadratic Type (L46)

Objective: Recognize and factor a polynomial expression of quadratic type.

Recall that a quadratic expression in terms of a variable x is an expression of the form

$$ax^2 + bx + c.$$

If y is any algebraic expression, we say that the expression

$$ay^2 + by + c$$

is an expression of *quadratic type.*

In just about every case we will see, we will consider y as a power of x, $y = x^n$, so that our expression of quadratic type will appear as follows.

$$\boxed{\begin{gathered}\text{Quadratic Type:}\\ ax^{2n} + bx^n + c \;=\; a\left[x^n\right]^2 + b\left[x^n\right] + c\end{gathered}}$$

If $y = x^3$, then the expression

$$ay^2 + by + c = ax^6 + bx^3 + c$$

would be an expression of quadratic type.

Similarly, if $y = x^4$, then the expression

$$ay^2 + by + c = ax^8 + bx^4 + c$$

would be an expression of quadratic type.

In each of these last two examples, notice the exponential pattern, where the middle term has an exponent that is half that of the leading term's. This will always be apparent, as long as the middle coefficient b is nonzero.

By viewing certain expressions as quadratic type, we can often apply more familiar methods, such as the $ac-$method, to obtain a complete factorization.

For example, if we let $y = x^2$, then the difference of fourth powers $x^4 - 16$ can be rewritten as a difference of squares, $y^2 - 4^2$, leading us to the complete factorization over the real numbers shown below.

$$\begin{aligned} x^4 - 16 &= \left(x^2\right)^2 - 4^2 \\ &= y^2 - 4^2, \quad y = x^2 \\ &= (y+4)(y-4) \\ &= \left(x^2+4\right)\left(x^2-4\right) \\ &= \left(x^2+4\right)(x+2)(x-2) \end{aligned}$$

Example 287. Factor the trinomial expression $x^4 + 2x^2 - 24$ completely over the real numbers.

Notice that the given trinomial exhibits quadratic type characteristics, since the degree of four is double the exponent appearing in the middle term. Consequently, we will let $y = x^2$ and rewrite the expression in terms of y.

$$y^2 + 2y - 24$$

Applying the $ac-$method, we see the following.

$$\begin{aligned} y^2 + 2y - 24 &= y^2 + 6y - 4y - 24 \\ &= y\,(y+6) - 4\,(y+6) \\ &= (y+6)\,(y-4) \end{aligned}$$

Substituting back for x, we have $(x^2+6)\,(x^2-4)$. The first factor is a sum of squares, which is irreducible over the reals. The second factor of $x^2 - 4$ is a difference of perfect squares, which we know is factorable as $(x+2)\,(x-2)$.

Our final factorization is

$$x^4 + 2x^2 - 24 = \left(x^2+6\right)(x+2)\,(x-2)\,.$$

Example 288. Find all real roots of the polynomial expression $x^4 - 12x^2 + 27$.

In this example, we are not asked to factor the given expression, but instead to solve for when the expression equals zero. Still, we can start by finding a complete factorization, again substituting $y = x^2$ and employing the $ac-$method.

$$\begin{aligned} x^4 - 12x^2 + 27 &= y^2 - 12y + 27 \\ &= y^2 - 3y - 9y + 27 \\ &= y\,(y-3) - 9\,(y-3) \\ &= (y-3)\,(y-9) \\ &= \left(x^2-3\right)\left(x^2-9\right) \\ &= \left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)(x-3)\,(x+3) \end{aligned}$$

Here, we see that after using the $ac-$method and substituting back for x, we end up with two quadratic factors which can *both* be factored as a difference of squares. In the case of the first factor, $x^2 - 3$, our factorization requires a square root, since 3 is not a perfect square.

Setting each of the four factors equal to zero gives us our set of real roots, $\{\pm\sqrt{3}, \pm 3\}$.

In each of our last two examples, we have seen a degree-four polynomial having two and four real roots, respectively. We can also easily identify degree-four polynomials having no, one, or three real roots. The expression $x^4 + x^2 + 1$, for example, factors as $\left(x^2+1\right)^2$, which has only complex roots at $\pm i$. In general, a degree-n polynomial can have as few as zero and as many as n unique real roots. This is a fact which we will more formally state in a later section, once we have discussed the *multiplicity* of a root.

The following example should look familiar.

Example 289. Factor each of the following polynomial expressions completely over the real numbers.

1. $x^8 + 2x^4 - 24$
2. $x^6 + 2x^3 - 24$

Before we begin, notice that the coefficients for each of the given expressions match those in Example 287, with the only difference being the exponents appearing in each expression.

1. Despite the fact that the first expression has a higher degree, its factorization will be simpler than the second expression's. In this case, we will let $y = x^4$, and apply the $ac-$method as before.

$$\begin{aligned} x^8 + 2x^4 - 24 &= \left(x^4\right)^2 + 2\left(x^4\right) - 24 \\ &= y^2 + 2y - 24 \\ &= (y+6)(y-4) \\ &= \left(x^4+6\right)\left(x^4-4\right) \end{aligned}$$

   Though it might not be obvious, our first factor $x^4 + 6$ is in fact irreducible over the real numbers. One way we can realize this is to think of $x^4 + 6$ as a vertical shift of the graph of x^4 up six units. The resulting graph will lie entirely in the upper-half of the $xy-$plane, and therefore will not intersect the $x-$axis. Hence, the factor of $x^4 + 6$ will have no real roots, and consequently any factorization will involve the introduction of imaginary numbers. Alternatively, one might also notice that raising any real number to the fourth power and adding six will never produce an output of zero, leading us to again conclude that the expression has no real roots. Lastly, we could also recognize $x^4 + 6$ as a sum of squares, namely $(x^2)^2 + \left(\sqrt{6}\right)^2$, which we have already discussed as one expression type that is irreducible over the real numbers.

   On the other hand, we can view our second factor $x^4 - 4$ as a difference of two squares, and factor it as follows.

$$\begin{aligned} x^4 - 4 &= \left(x^2\right)^2 - 2^2 \\ &= \left(x^2+2\right)\left(x^2-2\right) \\ &= \left(x^2+2\right)\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right) \end{aligned}$$

   Our complete factorization over the reals is then

$$x^8 + 2x^4 - 24 = \left(x^4+6\right)\left(x^2+2\right)\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right).$$

2. In the case of the second expression, if we let $y = x^3$, we start out with the same two factors for y, which we rewrite as

$$\left(x^3+6\right)\left(x^3-4\right).$$

Although neither 6 nor 4 are perfect cubes, we can still break down each of the factors above by using the formulas for the sum and difference of cubes from earlier in the section. For our first factor, letting $a = x$ and $b = \sqrt[3]{6}$, we can write $x^3 + 6$ as

$$(a+b)\left(a^2 - ab + b^2\right) = \left(x + \sqrt[3]{6}\right)\left(x^2 - \sqrt[3]{6}x + \left(\sqrt[3]{6}\right)^2\right).$$

Similarly, for the second factor, if $a = x$ and $b = \sqrt[3]{4}$, we can write $x^3 - 4$ as

$$(a-b)\left(a^2 + ab + b^2\right) = \left(x - \sqrt[3]{4}\right)\left(x^2 + \sqrt[3]{4}x + \left(\sqrt[3]{4}\right)^2\right).$$

Our complete factorization over the reals is then

$$\begin{aligned} x^6 + 2x^3 - 24 &= \left(x^3 + 6\right)\left(x^3 - 4\right) \\ &= \left(x + \sqrt[3]{6}\right)\left(x^2 - \sqrt[3]{6}x + \left(\sqrt[3]{6}\right)^2\right)\left(x - \sqrt[3]{4}\right)\left(x^2 + \sqrt[3]{4}x + \left(\sqrt[3]{4}\right)^2\right). \end{aligned}$$

We end the subsection on quadratic type with one final example.

Example 290. Factor each polynomial expression completely over the reals and find its set of real roots.

1. $x^4 - 49$
2. $x^6 - 4x^3 - 5$

1. Setting $y = x^2$, we can quickly factor our first expression as a difference of squares, breaking down one of its factors in a similar manner.

$$\begin{aligned} x^4 - 49 &= \left(x^2\right)^2 - 49 \\ &= y^2 - (7)^2 \\ &= (y+7)(y-7) \\ &= \left(x^2 + 7\right)\left(x^2 - 7\right) \\ &= \left(x^2 + 7\right)\left(x + \sqrt{7}\right)\left(x - \sqrt{7}\right) \end{aligned}$$

From our two linear factors, we obtain $x = \pm 7$ as our two real roots.

2. For our second expression, setting $y = x^3$, we apply the $ac-$method.

$$\begin{aligned} x^6 - 4x^3 - 5 &= \left(x^3\right)^2 - 4\left(x^3\right) - 5 \\ &= y^2 - 4y - 5 \\ &= (y+1)(y-5) \\ &= \left(x^2 + 1\right)\left(x^2 - 5\right) \\ &= \left(x^2 + 1\right)\left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right) \end{aligned}$$

Our two linear factors give us $x = \pm\sqrt{5}$ as the real roots for our given expression.

Division

Polynomial (Long) Division (L47)

Objective: Apply polynomial division to a rational expression.

Up until this point, every polynomial expression that we have encountered has either already been provided in its factored form or is easily factorable using one or more of the many techniques that we have learned. We must be careful, however, to consider the very likely possibility that a given polynomial is not factorable using elementary methods (GCF, grouping, $ac-$method, etc.). In many cases, obtaining a complete factorization can prove almost impossible without the aid of mathematical computating software. Although, there is still one powerful tool that can help us to dissect certain factorable, yet formidable, polynomials. This tool is known as the *Rational Root Theorem*, and we will see its use in a later section.

In order to successfully employ the Rational Root Theorem, we first must understand polynomial division. As we will see, dividing polynomials is a process very similar to long division of whole numbers.

Before we begin with our first example, let's recall the terminology and format associated with division.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

Alternatively, multiplying both sides of the above equation by the divisor, we have the following.

$$\frac{\text{dividend}}{\cancel{\text{divisor}}} \cdot \cancel{\text{divisor}} = \text{quotient} \cdot \text{divisor} + \frac{\text{remainder}}{\cancel{\text{divisor}}} \cdot \cancel{\text{divisor}}$$
$$\text{dividend} = \text{quotient} \cdot \text{divisor} + \text{remainder}$$

We begin with dividing a polynomial by a monomial, which simply utilizes the distributive property. In the following example, we can think of the stated division as a distribution of the divisor (denominator) to each term in the dividend (numerator).

Example 291. Divide and simplify the following expressions.

1. $\dfrac{9x^5 + 6x^4 - 18x^3 - 24x^2}{3x^2}$
2. $\dfrac{8x^3 + 4x^2 - 2x + 6}{4x^2}$

1. By distributing the expression $\dfrac{1}{3x^2}$ to each of the four terms in the numerator, our first expression becomes

$$\frac{9x^5 + 6x^4 - 18x^3 - 24x^2}{3x^2} = \frac{9x^5}{3x^2} + \frac{6x^4}{3x^2} - \frac{18x^3}{3x^2} - \frac{24x^2}{3x^2}.$$

We can then reduce each individual quotient to produce the following expression.

$$3x^3 + 2x^2 - 6x - 8$$

In this case, our answer is $3x^3 + 2x^2 - 6x - 8$, and we can summarize our results as follows.

$$\frac{9x^5 + 6x^4 - 18x^3 - 24x^2}{3x^2} = 3x^3 + 2x^2 - 6x - 8$$

In this first example, our expression reduced completely, producing our quotient polynomial expression and a remainder of zero.

2. Again, we will begin with the second expression by splitting it up, or distributing the denominator to each of the three terms in the numerator. Reducing each individual quotient gives us our answer.

$$\begin{aligned}\frac{8x^3 + 4x^2 - 2x + 6}{4x^2} &= \frac{8x^3}{4x^2} + \frac{4x^2}{4x^2} - \frac{2x}{4x^2} + \frac{6}{4x^2} \\ &= 2x + 1 - \frac{1}{2x} + \frac{3}{2x^2}\end{aligned}$$

Unlike our first expression, here our expression does not reduce completely, i.e., it contains a nonzero remainder. Because of this, since our answer includes two rational (or fractional) expressions, one could also combine them to form a single rational expression.

$$\begin{aligned}\frac{8x^3 + 4x^2 - 2x + 6}{4x^2} &= 2x + 1 - \frac{1}{2x} \cdot \frac{x}{x} + \frac{3}{2x^2} \\ &= 2x + 1 - \frac{x}{2x^2} + \frac{3}{2x^2} \\ &= 2x + 1 + \frac{3 - x}{2x^2}\end{aligned}$$

Furthermore, if we wanted to identify the remainder in this example, we could multiply both sides of the equation by the divisor, $4x^2$.

$$\begin{aligned}\frac{8x^3 + 4x^2 - 2x + 6}{\cancel{4x^2}} \cdot \cancel{4x^2} &= (2x + 1) \cdot 4x^2 + \left(\frac{3 - x}{2\cancel{x^2}}\right) \cdot 4\cancel{x^2} \\ \underbrace{8x^3 + 4x^2 - 2x + 6}_{\text{dividend}} &= \underbrace{(2x + 1)}_{\text{quotient}}\ \underbrace{4x^2}_{\text{divisor}} + \underbrace{6 - 2x}_{\text{remainder}}\end{aligned}$$

Lastly, one final observation to point out from this particular answer is the initial reduction of the second term, $\frac{4x^2}{4x^2}$, which equals one (not zero), and therefore does not simply disappear from the expression altogether.

For division by polynomial expressions that contain more than a single term, long division is usually required. To illustrate the relationship between polynomial division and standard numerical long division, an example with whole numbers is provided in order to review the (general) steps that will also be used for polynomial long division.

Example 292. Divide 631 by 4.

$$
\begin{array}{r}
157 \\
4\,\overline{)\,631} \\
\underline{400} \\
231 \\
\underline{200} \\
31 \\
\underline{28} \\
3
\end{array}
$$

Recalling the process associated with long division, we begin to construct our quotient (157) by comparing the divisor (4) with the largest placeholder of our dividend (631), then subtracting, and repeating this process until we have worked our way down to the ones digit. We know we have finished, once we are left with a remainder (3) that is less than our divisor.

Expressing our answer in the form

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}},$$

we have $\dfrac{631}{4} = 157 + \dfrac{3}{4}$. Or, in the alternate form

$$\text{dividend} = \text{quotient} \cdot \text{divisor} + \text{remainder},$$

we have $631 = 145 \cdot 4 + 3$.

The general process for division of polynomials follows closely with that for dividing integers. The only real difference is in the terminology that we use: *term* in place of *number/digit*, and *degree* instead of *value*.

General Steps for Polynomial (Long) Division

Let $D(x)$ and $d(x)$ represent two nonzero polynomial functions. The steps for simplifying the rational expression $\dfrac{D(x)}{d(x)}$ are as follows.

1. Divide the leading term of the dividend D by the leading term of the divisor d. Label the resulting term $a_n x^n$, and write it above the dividend. This will be the leading term of the quotient, $q(x)$.
2. Multiply $a_n x^n$ by the divisor, distribute, and simplify. Label this as $d_1(x)$ and write it directly below the dividend, D, making sure to align terms according to exponents.
3. Subtract the resulting terms from the dividend. Label the new expression D_1.
4. Repeat steps (1)-(3) for the divisor d and the new expression D_i until the degree of D_i is *less than* the degree of the divisor. Relabel the final new dividend as the remainder, $r(x)$. The entire polynomial expression appearing above the original dividend is the quotient, $q(x)$.

$$
\begin{array}{rl}
 & \underline{\quad q(x) \quad} \\
d(x) \,\big) & D(x) \\
- & \underline{d_1(x)} \\
 & \quad D_1(x) \\
 & - \underline{d_2(x)} \\
 & \qquad \ddots \\
 & \qquad - \underline{d_i(x)} \\
 & \qquad\quad D_i(x) = r(x)
\end{array}
$$

Step (3) above often tends to pose the greatest challenge for students. It is important to keep in mind that we are are always subtracting the top term from the bottom term, which is why we must change the signs of the term(s) on the bottom. In most cases, we will need to utilize the distributive property.

A basic example should clear up any confusion, and we begin by revisiting Example 291.

Example 293. Divide $9x^5 + 6x^4 - 18x^3 - 24x^2$ by $3x^2$. Simplify and express your answer in the form

$$\frac{D(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

$$3x^2 \overline{\big) \quad 9x^5 + 6x^4 - 18x^3 - 24x^2}$$

We set up our division process by first writing the dividend and the divisor in the appropriate locations.

$$
\begin{array}{rl}
 & 3x^3 \\
3x^2 \big) & \overline{\;9x^5 + 6x^4 - 18x^3 - 24x^2}
\end{array}
$$

Next, we identify the leading term for our quotient, $3x^3$.

$$
\begin{array}{rl}
 & 3x^3 \\
3x^2 \big) & \overline{\;9x^5 + 6x^4 - 18x^3 - 24x^2} \\
 & \underline{-9x^5} \\
 & 6x^4
\end{array}
$$

Multiplying and subtracting produces our new expression, $D_1(x) = 6x^4$.

Notice that in this example we need only carry down the remaining terms from the dividend when our new expression contains terms which are alike to them. In this case, for example, while it is perfectly fine to carry down $-18x^3 - 24x^2$, it is not necessary until these terms play a role in the subtraction step.

$$
\begin{array}{rl}
 & 3x^3 + 2x^2 \\
3x^2 \big) & \overline{\;9x^5 + 6x^4 - 18x^3 - 24x^2} \\
 & \underline{-9x^5} \\
 & 6x^4 \\
 & \underline{-6x^4} \\
 & -18x^3
\end{array}
$$

Repeating our division steps gives us the second term in our quotient, $2x^2$.

Multiplying this term by the divisor, subtracting, and carrying down the next term in the dividend finishes up the second round of our steps for division. Our new expression is $D_2(x) = -18x^3$.

$$
\begin{array}{rl}
 & 3x^3 + 2x^2 - 6x \\
3x^2 \big) & \overline{\;9x^5 + 6x^4 - 18x^3 - 24x^2} \\
 & \underline{-9x^5} \\
 & 6x^4 \\
 & \underline{-6x^4} \\
 & -18x^3 \\
 & \underline{18x^3} \\
 & -24x^2
\end{array}
$$

Again, we repeat our division steps to produce the third term in our quotient, $-6x$.

Multiplying and subtracting produces another new expression, $D_3(x) = -24x^2$. Since the degree of D_3 equals that of our divisor, $d(x) = 3x^2$, we will need to apply our steps for division one final time.

$$
\begin{array}{rl}
 & 3x^3 + 2x^2 - 6x - 8 \\
3x^2 \big) & \overline{\;9x^5 + 6x^4 - 18x^3 - 24x^2} \\
 & \underline{-9x^5} \\
 & 6x^4 \\
 & \underline{-6x^4} \\
 & -18x^3 \\
 & \underline{18x^3} \\
 & -24x^2 \\
 & \underline{24x^2} \\
 & 0
\end{array}
$$

After our fourth and final round of steps, our new expression produces a remainder of $r(x) = 0$. This should come as no real surprise, based upon our earlier calculations from Example 291. Many examples that we will encounter after this first one will not work out as nicely.

We express the results of our division in the required form as follows.

$$\frac{9x^5 + 6x^4 - 18x^3 - 24x^2}{3x^2} = 3x^3 + 2x^2 - 6x - 8 + \frac{0}{3x^2}$$

Because our divisor in this last example is a monomial, our steps do not require the distributive property. The next example includes a binomial divisor, which will emphasize the importance of the distributive property in the division process.

Example 294. Divide $3x^3 - 5x^2 - 32x + 7$ by $x - 4$. Simplify and express your answer in the form

$$\frac{D(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

$$\begin{array}{r|rrrr}
 & 3x^2 & +7x & -4 & \\ \hline
x-4 & 3x^3 & -5x^2 & -32x & +7 \\
 & -3x^3 & +12x^2 & & \\ \hline
 & & 7x^2 & -32x & \\
 & & -7x^2 & +28x & \\ \hline
 & & & -4x & +7 \\
 & & & 4x & -16 \\ \hline
 & & & & -9
\end{array}$$

For this second example, we have opted to show the entire division process all at once. Since our quotient,

$$3x^2 + 7x - 4,$$

is a trinomial, we must apply the steps for division three times.

In this second example, our remainder is the constant term $r(x) = -9$, which has one degree less than our linear divisor, $d(x) = x - 4$.

Our answer is $\dfrac{3x^3 - 5x^2 - 32x + 7}{x - 4} = 3x^2 + 7x - 4 + \dfrac{-9}{x - 4}$.

If we look at the first round of long division steps, we see a leading term in our quotient of $3x^2$. Multiplying by our divisor produces the following binomial expression.

$$3x^2(x - 4) = 3x^3 - 12x^2$$

Since we must subtract *each* term of this expression from our dividend, however, we see that the distributive property has already been applied. Specifically,

$$-(3x^3 - 12x^2) = -3x^3 + 12x^2.$$

What remains now is to simply add each term to the respective term in the dividend, which, if employed correctly, should *always* eliminate the leading term in each round of steps.

Failure to distribute a negative across *all* terms in this critical step will always lead to an incorrect quotient and remainder. This is the most common mistake in the polynomial division process, and is one that often takes a long amount of time to correct. Consequently, checking one's work is critical throughout the division process. We emphasize these points in this first example where distribution of a negative is required, in the hopes that it is not overlooked.

We continue with another example.

Example 295. Divide $6x^3 - 8x^2 + 10x + 103$ by $4 + 2x$. Simplify and express your answer in the form

$$\frac{D(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

Before we begin, we wish to point out the important prerequisite for polynomial division that all expressions be written in *descending power order*. In this case, we will start out by rewriting our divisor as $2x + 4$.

$$
\begin{array}{rrrrr}
 & & 3x^2 & -10x & +25 \\
\cline{2-5}
2x+4 \big) & 6x^3 & -8x^2 & +10x & +103 \\
 & -6x^3 & -12x^2 & & \\
\cline{2-3}
 & & -20x^2 & +10x & \\
 & & 20x^2 & +40x & \\
\cline{3-4}
 & & & 50x & +103 \\
 & & & -50x & -100 \\
\cline{4-5}
 & & & & 3
\end{array}
$$

This example is similar to the previous one. Specifically, the divisor is a linear binomial, and the dividend has a degree of 3.

Consequently, the steps for polynomial division are employed three times, yielding a constant remainder.

Our answer should be expressed as follows.

$$\frac{6x^3 - 8x^2 + 10x + 103}{2x + 4} = 3x^2 - 10x + 25 + \frac{3}{2x + 4}$$

In each of the previous two examples the dividend contained only nonzero coefficients. In other words, no term was "skipped over" in the expression for $D(x)$. Our last example will address the importance of keeping track of polynomial *place holders*, in the event that a specific term carries with it a zero coefficient, and is therefore omitted from the original expression for the dividend, $D(x)$.

Our last example demonstrates the importance of these preliminary steps.

Example 296. Divide and simplify the given expression.

$$\frac{2x^4 + 42x - 4x^2}{x^2 + 3x} \qquad \text{Reorder dividend; need } x^3 \text{ term, add } 0x^3$$

$$x^2 + 3x \,\big)\overline{\,2x^4 + \mathbf{0x^3} - 4x^2 + 42x} \qquad \text{Divide the leading terms : } \frac{2x^4}{x^2} = 2x^2$$

$$
\begin{array}{rrrrrl}
 & \mathbf{2x^2} & & & & \\
\cline{2-5}
x^2+3x \big) & 2x^4 & +0x^3 & -4x^2 & +42x & \quad \text{Multiply this term by divisor : } 2x^2(x^2+3x) = 2x^4 + 6x^3 \\
 & \mathbf{-2x^4} & \mathbf{-6x^3} & & & \quad \text{Subtract, changing terms} \\
\cline{2-3}
 & & \mathbf{-6x^3} & \mathbf{-4x^2} & & \quad \text{Bring down the next term, } -4x^2
\end{array}
$$

$$
\begin{array}{rrrrrl}
 & 2x^2 & \mathbf{-6x} & & & \\
\cline{2-5}
x^2+3x \big) & 2x^4 & +0x^3 & -4x^2 & +42x & \quad \text{Repeat, divide new leading term by } x^2 : \frac{-6x^3}{x^2} = -6x \\
 & -2x^4 & -6x^3 & & & \\
\cline{2-3}
 & & -6x^3 & -4x^2 & & \quad \text{Multiply this term by divisor : } -6x(x^2+3x) = -6x^3 - 18x^2 \\
 & & \mathbf{+6x^3} & \mathbf{+18x^2} & & \quad \text{Subtract, changing signs} \\
\cline{3-4}
 & & & \mathbf{14x^2} & \mathbf{+42x} & \quad \text{Bring down the next term, } 42x
\end{array}
$$

$$
\begin{array}{rl}
\begin{array}{r}
\quad 2x^2 - 6x \ \mathbf{+14} \\
x^2+3x \,\overline{)\, 2x^4 + 0x^3 - 4x^2 + 42x} \\
\underline{-2x^4 - 6x^3} \qquad\qquad\quad \\
-6x^3 - 4x^2 \qquad\quad \\
\underline{+6x^3 + 18x^2} \qquad\quad \\
14x^2 + 42x \\
\underline{\mathbf{-14x^2 - 42x}} \\
0
\end{array} &
\begin{array}{l}
\text{Repeat, divide new leading term by } x^2 : \dfrac{14x^2}{x^2} = 14 \\
\\ \\ \\
\text{Multiply this term by the divisor} : 14(x^2+3x) = 14x^2 + 42x \\
\text{Subtract, changing signs} \\
\text{Zero remainder}
\end{array}
\end{array}
$$

$2x^2 - 6x + 14$ Our solution

So we have,

$$\frac{2x^4 - 4x^2 + 42x}{x^2 + 3x} = 2x^2 - 6x + 14$$

It is important to take a moment to check each problem, to verify that the exponents decrease incrementally and that none are skipped.

This final example also illustrates that, just as with classic numerical long division, sometimes our remainder will be zero.

Synthetic Division (L48)

Objective: Apply synthetic division to a rational expression.

Next, we will introduce a method of division that can be used to streamline the polynomial division process and is often preferred over the more traditional long division method. This method, known as *synthetic division*, although usually quicker than traditional polynomial division, this alternative method can only be implemented when the given divisor is *linear*. Specifically, we will require $d(x)$ to be of the form $x - c$.

For our first example, we will divide $x^3 + 4x^2 - 5x - 14$ by $x - 2$, which one can check will produce a quotient of $x^2 + 6x + 7$ and a remainder of zero using polynomial long division.

$$\frac{x^3 + 4x^2 - 5x - 14}{x - 2} = x^2 + 6x + 7$$

The method of synthetic division focuses primarily on the coefficients of both the divisor and dividend. We must still pay careful attention, however, to the powers of our exponents, which will serve as placeholders throughout the process. To start the process, we will write our coefficients in what we will refer to as a *synthetic division tableau* prior to dividing.

To divide $x^3 + 4x^2 - 5x - 14$ by $x - 2$, we first write 2 in the place of the divisor since 2 is zero of the factor $x - 2$ and we write the coefficients of $x^3 + 4x^2 - 5x - 14$ in for the dividend.

As our next step, we 'bring down' the first coefficient of the dividend. We will then multiply and add repeatedly.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & & & \\ \hline & & & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & & & \\ \hline & 1 & & & \end{array}$$

Next, take the 2 from the divisor and multiply by the 1 that was brought down to get 2. Write this underneath the 4, then add to get 6.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & & \\ \hline & 1 & & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & & \\ \hline & 1 & 6 & & \end{array}$$

Now multiply the 2 from the divisor by the 6 to get 12, and add it to the -5 to get 7.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & 12 & \\ \hline & 1 & 6 & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & 12 & \\ \hline & 1 & 6 & 7 & \end{array}$$

Finally, multiply the 2 in the divisor by the 7 to get 14, and add it to the -14 to get 0.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \mathbf{0} \end{array}$$

The first three numbers in the last row of our tableau will be the coefficients of the desired quotient polynomial. Remember, we started with a third degree polynomial and divided by a first degree polynomial, so the quotient will be a second degree polynomial. Hence the quotient is $x^2 + 6x + 7$. The number in bold represents the remainder, which is zero in this case.

Due in large part to its speed, synthetic division is often a 'tool of choice' for dividing polynomials by divisors of the form $x - c$. It is important to reiterate that synthetic division will *only* work for these kinds of divisors (linear divisors with leading coefficient 1), and we will need to use polynomial long division for divisors having degree larger than 1.

Another observation worth mentioning is that when a polynomial (of degree at least 1) is divided by $x - c$, the result will be a quotient polynomial of exactly one less degree than the original polynomial. This is a direct result of the divisor being a linear expression.

For a more complete understanding of the relationship between long and synthetic division, students are encouraged to trace each step in synthetic division back to its corresponding step in long division.

We conclude this section with three examples using synthetic division. We will summarize each example using the form below.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

Example 297. Use synthetic division to perform the following polynomial division. Find the quotient and the remainder polynomials.

$$\frac{5x^3 - 2x^2 + 1}{x - 3}$$

When setting up the synthetic division tableau, we need to enter 0 for the coefficient of x in the dividend as a placeholder, just like in polynomial division.

Setting up and working through the tableau gives us the following result.

$$\begin{array}{r|rrrr} 3 & 5 & -2 & 0 & 1 \\ & & 15 & 39 & 117 \\ \hline & 5 & 13 & 39 & \mathbf{118} \end{array}$$

Since the dividend was a third degree polynomial, the quotient is a quadratic polynomial with coefficients 5, 13 and 39. Our quotient is then $q(x) = 5x^2 + 13x + 39$ and the remainder is $r(x) = 118$.

Putting this all together, we have the following equation.

$$\frac{5x^3 - 2x^2 + 1}{x - 3} = 5x^2 + 13x + 39 + \frac{118}{x - 3}$$

Example 298. Use synthetic division to perform the following polynomial division. Find the quotient and the remainder polynomials.

$$\frac{x^3 + 8}{x + 2}$$

For this division, since we have a factor of $x + 2$, we must use the zero of $x = -2$ to begin. Here, we will once again stress that it is critical to take the time in order to ensure we have set the synthetic division tableau up correctly at the onset of the problem. Failure to do so will result in an incorrect answer, as well as a considerable amount time spent re-doing the problem.

$$\begin{array}{r|rrrr} -2 & 1 & 0 & 0 & 8 \\ & & -2 & 4 & -8 \\ \hline & 1 & -2 & 4 & \mathbf{0} \end{array}$$

We then obtain a quotient of $q(x) = x^2 - 2x + 4$ and remainder of $r(x) = 0$. This gives us the following equation.

$$\frac{x^3 + 8}{x + 2} = x^2 - 2x + 4$$

This answer is a great reminder of the factoring rules for cubic polynomials that we outlined earlier in the chapter.

Example 299. Use synthetic division to perform the following polynomial division. Find the quotient and the remainder polynomials.

$$\frac{4-8x-12x^2}{2x-3}$$

To divide $4-8x-12x^2$ by $2x-3$, two things must be done. First, we write the dividend in descending powers of x as $-12x^2-8x+4$. Second, since synthetic division works only for factors of the form $x-c$, we factor $2x-3$ as $2\left(x-\frac{3}{2}\right)$. Our strategy is to first divide $-12x^2-8x+4$ by 2, to get $-6x^2-4x+2$. Next, we divide by $x-\frac{3}{2}$. The tableau becomes

$$\begin{array}{c|ccc} \frac{3}{2} & -6 & -4 & 2 \\ & & -9 & -\frac{39}{2} \\ \hline & -6 & -13 & -\mathbf{\frac{35}{2}} \end{array}$$

From this, we get a quotient of $q(x)=-6x-13$ and a remainder of $r(x)=-\frac{35}{2}$. This gives us the following equation.

$$\frac{-6x^2-4x+2}{x-\frac{3}{2}} = -6x-13+\frac{-\frac{35}{2}}{x-\frac{3}{2}}$$

Multiplying both sides by of our equation by $\frac{2}{2}$ and distributing gives us our desired answer.

$$\frac{-12x^2-8x+4}{2x-3} = -6x-13+\frac{-35}{2x-3}$$

Note that we could also multiply both sides of our last equation by $2x-3$ to obtain the following equation.

$$-12x^2-8x+4=(2x-3)(-6x-13)-35$$

While both of the forms above are certainly equivalent, the previous one may remind us of the familiar classic division algorithm for integers, shown below.

$$\text{dividend} = (\text{divisor})\cdot(\text{quotient}) + \text{remainder}$$

The first form, however, will be particularly useful when we graph more complicated rational functions in the next chapter.

End Behavior (L49)

Objective: Identify and describe the end behavior of the graph of a polynomial function.

The *end behavior* of any function refers to what happens near the extreme ends of its graph. We also often refer to these as the "tails" of the graph. The ends of the graph of a function correspond to points having large positive or negative $x-$coordinates. Because of this, we can associate the expressions

$$x \to \infty \qquad \text{and} \qquad x \to -\infty$$

to the end behavior of a function. For example, the sentence

$$\text{As } x \to \infty, \;\; f(x) \to \infty.$$

describes a function for which the right-hand side of its graph, i.e. when $x \to \infty$, points upward. Alternatively, the sentence

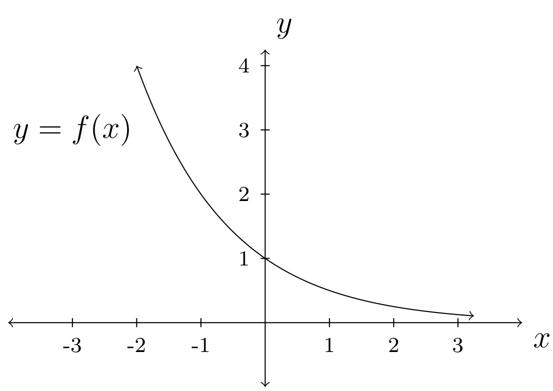
$$\text{As } x \to \infty, \;\; f(x) \to -\infty.$$

describes a function for which the right-hand side of its graph points downward.

In each of the above mathematical statements, we are identifying both a horizontal direction and a vertical direction:

1. the independent variable x getting large (either positively or negatively),
2. and the effect this has on the values of $f(x)$.

Example 300. Describe the end behavior of the function f whose graph is shown below.



Although this graph is one that we typically see in a precalculus setting (known as an exponential function), we can still discuss its end behavior. In this case, as the values of x increase, we see that the points on the graph approach the $x-$axis. This translates to the following statement.

$$\text{As } x \to \infty, \;\; f(x) \to 0.$$

On the other hand, as the values of x tend towards $-\infty$, we see that the $y-$coordinates for their respective points continue to increase. Hence, we can say the following.

$$\text{As } x \to -\infty, \;\; f(x) \to \infty.$$

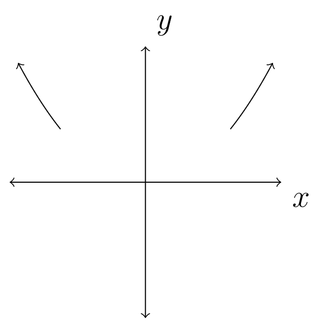
Prior to this chapter, we have not had much need to discuss end behavior at great length, since most of the functions which we have been exposed to have been relatively easily diagnosed and graphed. A quadratic function, $f(x) = ax^2 + bx + c$, for example, will either open up or down, depending on the sign of the leading coefficient, a. As we begin to graph polynomials, however, we will see our graphs take more than a few turns, which will require us to have a better understanding about the nature of their tails.

For each algebraic function, the corresponding graph will describe two such statements: one for the left-hand side of the graph ($x \to -\infty$) and one for the right-hand side of the graph ($x \to \infty$). In the case of polynomials, there are only four cases for these two statements, summarized as follows.

Let
$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$$
be a polynomial function with degree n and nonzero leading coefficient a_n.

The end behavior of f is described by one of the following four cases.

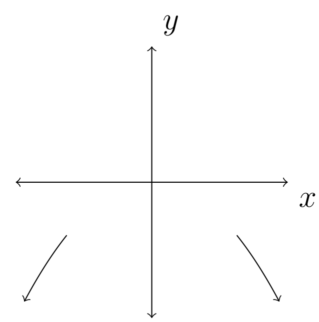
I. n even, $a_n > 0$



As $x \to \infty,\ f(x) \to \infty$

As $x \to -\infty,\ f(x) \to \infty$

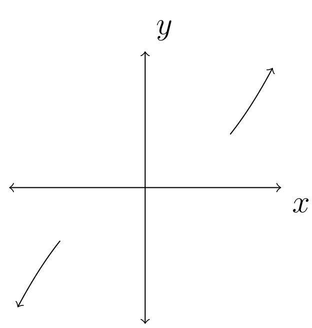
II. n even, $a_n < 0$



As $x \to \infty,\ f(x) \to -\infty$

As $x \to -\infty,\ f(x) \to -\infty$

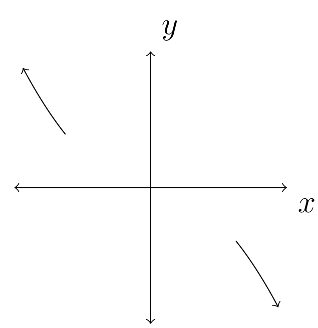
III. n odd, $a_n > 0$



As $x \to \infty,\ f(x) \to \infty$

As $x \to -\infty,\ f(x) \to -\infty$

IV. n odd, $a_n < 0$



As $x \to \infty,\ f(x) \to -\infty$

As $x \to -\infty,\ f(x) \to \infty$

An important initial observation of the previous figure is that the cases for the end behavior of a polynomial only depend on its leading term, $a_n x^n$. More specifically, the end behavior of a polynomial depends only on the parity of its degree n (even or odd) and the sign of its leading coefficient a_n (positive or negative). Additionally, we can see that cases I and II also include all quadratic functions (when $n = 2$).

Identifying the end behavior for an expanded polynomial, is much more straightforward than for a factored polynomial, as we will see in our next example.

Example 301. Determine the end behavior of each of the following functions.

1. $f(x) = 1 - 3x^4$
2. $g(x) = -2x^3 + 10000x^2 + 1000$
3. $h(x) = x(2x-1)(x-5)^2$
4. $k(x) = -2(1-3x)^2(x+1)(x-1)(x^2+1)$

1. The polynomial $f(x) = 1-3x^4$ is in expanded form, though not written in descending-power order. We can easily re-write f as $f(x) = -3x^4 + 1$. In this case, the degree $n = 4$ is even, and the leading coefficient $a_n = -3$ is negative. Hence, we are in case II:

   $$\text{As } x \to -\infty,\ f(x) \to -\infty. \qquad \text{As } x \to \infty,\ f(x) \to -\infty.$$

2. The polynomial g is also in expanded form, with odd degree $n = 3$ and negative leading coefficient, $a_n = -2$. The "large" quadratic and constant terms will not affect the end behavior of g, and so we are in case IV:

   $$\text{As } x \to -\infty,\ g(x) \to \infty. \qquad \text{As } x \to \infty,\ g(x) \to -\infty.$$

3. The polynomial h is written in factored form, which is helpful for identifying roots/$x-$intercepts, but not necessarily for describing the tails of the graph of h. Although we could, with enough time, expand h completely to describe the function's end behavior, this will quickly prove to be an inefficient strategy. Recall, however, that for end behavior we need only focus on finding the leading term, $a_n x^n$. We do this by identifying any parts of h that will contribute to the leading term. In this case, we identify in boldface font all contributing components to the leading term of h below.

   $$h(x) = \mathbf{x}(\mathbf{2x} - 1)(\mathbf{x} - 5)^{\mathbf{2}}$$

   So, when we expand, the leading term of h will be

   $$a_n x^n = x(2x)(x)^2 = 2x^4.$$

   We can now see that h has even degree $n = 4$, and positive leading coefficient, $a_n = 2$. Hence, we are in case I:

   $$\text{As } x \to -\infty,\ h(x) \to \infty. \qquad \text{As } x \to \infty,\ h(x) \to \infty.$$

4. Similarly, the polynomial k is written in factored form, and will require us to find the leading term, $a_n x^n$. Again, we identify all contributing components to the leading term of k in boldface font below.

$$k(x) = \mathbf{-2}(1\mathbf{-3x})^{\mathbf{2}}(\mathbf{x}+1)(\mathbf{x}-1)(\mathbf{x^2}+1)$$

So, when we expand, the leading term of k will be

$$\begin{aligned} a_n x^n &= -2(-3x)^2(x)(x)(x^2) \\ &= -2(9x^2)(x^4) \\ &= -18x^6. \end{aligned}$$

Through careful analysis, we see that k has an even degree, $n = 6$, and a negative leading coefficient, $a_n = -18$. Hence, we are in case II:

As $x \to -\infty,\ k(x) \to -\infty.$ As $x \to \infty,\ k(x) \to -\infty.$

In the previous example, we witnessed a new technique to quickly identify the end behavior of a polynomial that is given in factored form. The idea behind this technique is to only focus on the contributing components to a polynomial's leading term, $a_n x^n$, ignoring all others. This essentially boils down to focusing on three things:

- any constant multiplier,
- the leading term of each factor,
- and the power associated with each factor.

In general, if we suppose that a polynomial f has the factorization

$$f(x) = c \cdot (\text{Factor 1})^{k_1} \cdot (\text{Factor 2})^{k_2} \cdot \ldots \cdot (\text{Factor m})^{k_m},$$

then the leading term for f will equal

$$a_n x^n = c \cdot (\text{Leading Term 1})^{k_1} \cdot (\text{Leading Term 2})^{k_2} \cdot \ldots \cdot (\text{Leading Term m})^{k_m}.$$

Note that "Leading Term 1" refers to the leading term of Factor 1, and so on for the other factors.

This approach is similar to one that we have likely seen for identifying the constant term for a factored polynomial. To identify the constant term, a_0 of f, we would have

$$a_n x^n = c \cdot (\text{Constant Term 1})^{k_1} \cdot (\text{Constant Term 2})^{k_2} \cdot \ldots \cdot (\text{Constant Term m})^{k_m}.$$

Example 302. Find the leading and constant terms for the given function, and use them to identify the end behavior and $y-$intercept of its graph.

$$f(x) = 3(-2x+1)^2(x-2)^2(x-5)$$

First, we boldface the contributors for the leading term.

$$f(x) = \mathbf{3}(-\mathbf{2x}+1)^{\mathbf{2}}(\mathbf{x}-2)^{\mathbf{2}}(\mathbf{x}-5)$$

This gives us the following.

$$\begin{aligned} a_n x^n &= 3(-2x)^2(x)^2(x) \\ &= 3(4x^2)x^3 \\ &= 12x^5 \end{aligned}$$

Next, we boldface the contributors for the constant term.

$$f(x) = \mathbf{3}(-2x+\mathbf{1})^{\mathbf{2}}(x-\mathbf{2})^{\mathbf{2}}(x-\mathbf{5})$$

This gives us the following.

$$\begin{aligned} a_0 &= 3(1)^2(-2)^2(-5) \\ &= 3(1)(4)(-5) \\ &= -60 \end{aligned}$$

Hence, we have that

$$f(x) = 12x^5 + \ldots + (-60),$$

with middle terms unknown.

Since our degree, $n = 5$, is odd, and our leading coefficient, $a_n = 12$, is positive, we are in case III for end behavior.

As $x \to -\infty$, $f(x) \to -\infty$. As $x \to \infty$, $f(x) \to +\infty$.

Our constant term also tells us that the graph of f has a $y-$intercept at $(0, -60)$.

It is natural to ask why the additional terms of a polynomial have no impact on its end behavior. To address this, let us consider factoring out the leading term from f, which will give us the following.

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0 \\ &= a_n x^n \left(1 + \frac{a_{n-1}}{a_n x} + \ldots + \frac{a_2}{a_n x^{n-2}} + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n}\right) \end{aligned}$$

If we use $g(x)$ to denote the expression in parentheses,

$$g(x) = 1 + \frac{a_{n-1}}{a_n x} + \ldots + \frac{a_2}{a_n x^{n-2}} + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n},$$

then

$$\begin{aligned} f(x) &= a_n x^n \underbrace{\left(1 + \frac{a_{n-1}}{a_n x} + \ldots + \frac{a_2}{a_n x^{n-2}} + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n}\right)}_{g(x)} \\ &= a_n x^n \cdot g(x). \end{aligned}$$

But recall that the end behavior of a polynomial is determined when $x \to \pm\infty$. So, as x gets large (either positively or negatively), with the exception of the first term, all subsequent terms in the expression for g will approach zero.

$$\text{As } x \to \pm\infty, \;\; g(x) = 1 + \cancelto{0}{\frac{a_{n-1}}{a_n x}} + \ldots + \cancelto{0}{\frac{a_2}{a_n x^{n-2}}} + \cancelto{0}{\frac{a_1}{a_n x^{n-1}}} + \cancelto{0}{\frac{a_0}{a_n x^n}} \to 1.$$

Therefore, since $g(x)$ approaches 1, $f(x) = a_n x^n \cdot g(x)$ will approach its leading term, $a_n x^n$. Hence, we conclude that the end behavior of a polynomial f will coincide with the end behavior of its leading term.

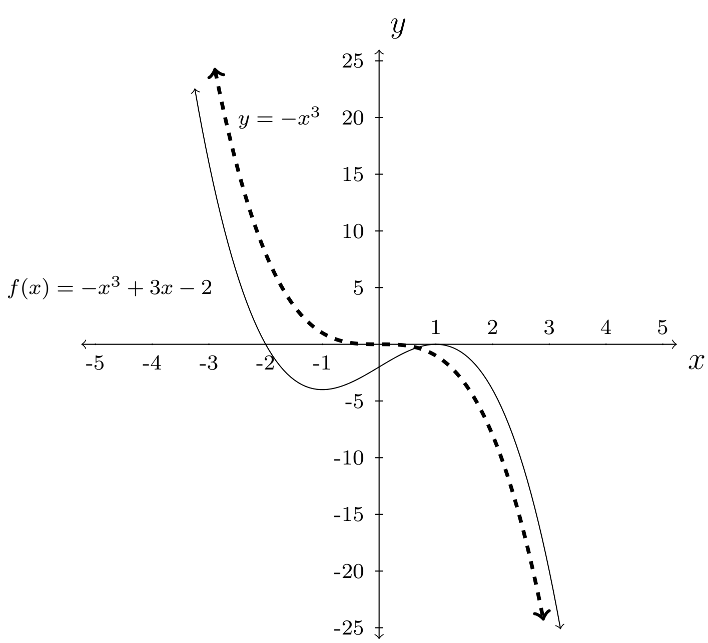
Furthermore, for any polynomial $f(x)$, if we were to graph the two curves $y = f(x)$ and $y = a_n x^n$ using Desmos or another graphing utility, and continue to 'zoom out', the two graphs would become virtually indistinguishable from one another. We demonstrate this in our next example.

Example 303. Determine the end behavior of the polynomial function below, and graph both the function and its leading term on a single set of axes.

$$f(x) = -x^3 + 3x - 2$$

The leading term of f is $-x^3$, with odd degree, $n = 3$, and negative leading coefficient, $a_n = -1$. Hence, we are in case IV:

As $x \to -\infty$, $f(x) \to \infty$. As $x \to \infty$, $f(x) \to -\infty$.



Local Behavior (L50)

Objective: Identify all real roots and their corresponding multiplicities for a polynomial function that is easily factorable.

In College Algebra and Precalculus, when we refer to the *local behavior* of a function f, we will be concerned with anything of interest in the interior of the graph of f, and not its end behavior. For polynomials, this is the $x-$ and $y-$intercepts of the graph. These points coincide with when $f(x) = 0$ for any $x-$intercepts, and when $x = 0$ in the case of the $y-$intercept. In Calculus, local behavior will also include points where the graph changes inflection or achieves a local maximum or minimum value.

Since we should be very familiar with finding a $y-$intercept at this point, we will start with a simple example.

Example 304. Find the $y-$intercept for each of the following polynomials.

1. $f(x) = 5x^3 - \frac{1}{2}x^2 + 6x - 18$
2. $g(x) = \frac{1}{2}(x-2)^2(x+5)(x-3)$

1. $f(0) = -18$. Hence, the graph of f has a $y-$intercept at $(0, -18)$.
2. In the case of g, we have to identify the constant term a_0 in the expanded form of the polynomial. Recalling Example 302 from our last section, we can easily obtain this value without the need to expand g in its entirety.

$$\begin{aligned} g(0) &= \frac{1}{2}(0-2)^2(0+5)(0-3) \\ &= \frac{1}{2}(-2)^2(5)(-3) \\ &= \frac{1}{2}(4)(-15) \\ &= 2(-15) \\ &= -30 \end{aligned}$$

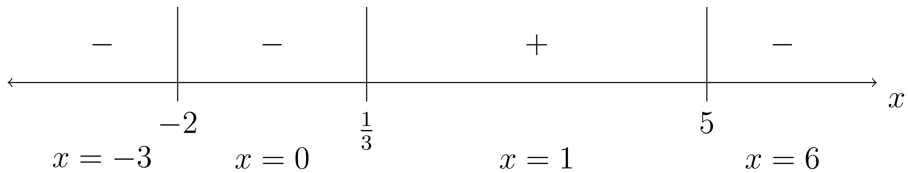
Hence, our $y-$intercept for the graph of g is $(0, -30)$.

Although it is certainly important to identify the $y-$intercept of any polynomial, the primary objective of this section will be finding the roots of a polynomial and classifying the respective $x-$intercepts of its graph. Since roots/$x-$intercepts coincide with when a function equals zero, $f(x) = 0$, this section will depend heavily on working with a polynomial that is either in factored form or for which a complete factorization is easily obtainable. In a subsequent section of this chapter, we will see more a more advanced technique for finding a complete factorization of a polynomial, using polynomial division and the Rational Root Theorem.

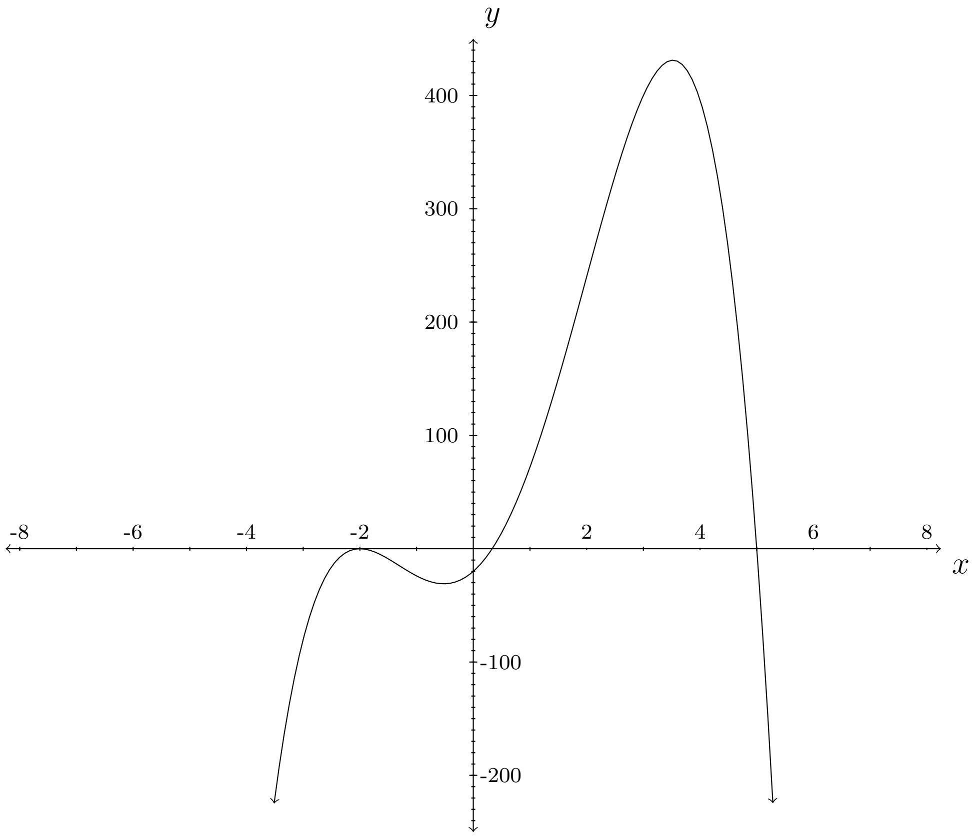
We begin our exploration of $x-$intercepts by revisiting a past example.

Example 305. Find all roots of the polynomial function $h(x) = (x+2)^2(3x-1)(5-x)$, and graph h using Desmos or another graphing utility. For each root, identify whether the graph of h crosses over or turns around at the corresponding $x-$intercept.

For this example, we will first recall the work done Example 281, where we identified the roots of h to be $x = -2, \frac{1}{3}$, and 5, as well as the following sign diagram.



Our graph of h is shown below.



Based upon our picture, we see that the graph of h crosses over the $x-$axis at $x = \frac{1}{3}$ and $x = 5$. The graph turns around at $x = -2$.

In our last example, we have included our sign diagram to point out a connection. Our diagram confirms the nature of each $x-$intercept without the need to graph h, since both sides of our *turnaround point* $x = -2$ show the *same* sign (either $+|+$ or $-|-$). Similarly, the signs *change* from either positive to negative ($+|-$) or negative to positive ($-|+$) for each of our *crossover points*.

In fact, this idea of turnaround and crossover points can be parsed down to one basic concept, known as the *multiplicity* of a root. We define the multiplicity of a root below, followed immediately by an example for clarification.

Suppose f is a polynomial function with real root $x = c$. For some positive integer k, if $(x-c)^k$ is a factor of f but $(x-c)^{k+1}$ is not, then we say $x = c$ is a root of f having associated multiplicity k.

Example 306. Determine the set of roots and corresponding multiplicities for the following functions.

1. $f(x) = x^6 - 2x^5 - 15x^4$
2. $g(x) = (x-6)^5(x+2)^2(x^2+1)$

1. Factoring f gives us the following.
$$\begin{aligned} f(x) &= x^6 - 2x^5 - 15x^4 \\ &= x^4(x^2 - 2x - 15) \\ &= x^4(x-5)(x+3) \end{aligned}$$
We then can easily see that f has a root at $x = 0$ with multiplicity four, and roots at $x = 5$ and $x = -3$, each with multiplicity one.
2. Since g is already factored, we see that $x = 6$ is a root having multiplicity five, and $x = -2$ is a root having multiplicity two. The factor of $x^2 + 1$ is meant to throw us off, since its roots are the imaginary numbers $\pm i$.

Another way of describing the multiplicity k of a root $x = c$ is that k represents the maximum number of factors of $(x - c)$ that divide the polynomial f (with a remainder of 0). That is,
$$f(x) = (x-c)^k \cdot q(x),$$
where $(x - c)$ is *not* a factor of the quotient $q(x)$.

If we apply this idea to g in our last example, we see that although $(x-6)^4$ divides our polynomial,
$$g(x) = (x-6)^4 \cdot \underbrace{(x-6)(x+2)^2(x^2+1)}_{q(x)},$$
the value of four does not represent the *maximum* number of factors of $(x-6)$ that divide g:
$$g(x) = (x-6)^5 \cdot \underbrace{(x+2)^2(x^2+1)}_{q(x)}.$$

At this point, we are ready to highlight the importance of multiplicities in graphing polynomials.

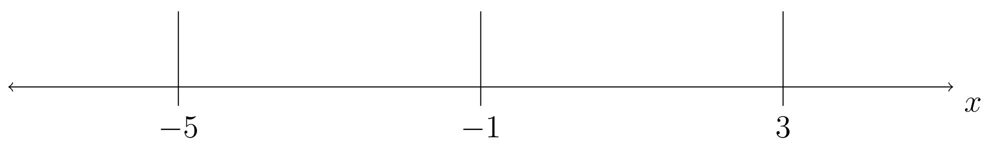
Let f be a polynomial function with a real root at $x = c$ having multiplicity k.

- If k is *even*, the corresponding $x-$intercept $(c, 0)$ is a *turnaround point.* In other words, the graph of f touches and rebounds from the x-axis at $(c, 0)$, leaving the $y-$values to maintain the same sign on either side of the root $x = c$.
- If k is *odd*, the corresponding $x-$intercept $(c, 0)$ is a *crossover point.* In other words, the graph of f crosses through the x-axis at $(c, 0)$, leaving the $y-$values to change signs on either side of the root $x = c$.

Combining this new notion about multiplicities of roots with all that we have already learned about polynomials will enable us to quickly identify all important aspects of a particular polynomial function, culminating in a sketch of its graph. We capitalize on this in our next example.

Example 307. Construct a sign diagram for the factored polynomial $f(x) = -(x-3)^2(x+1)(x+5)^2$.

The dividers for our sign diagram come from the set of roots of f, namely $\{-5, -1, 3\}$.



Instead of assigning test values, however, we will use both multiplicities and end behavior to determine our various signs.

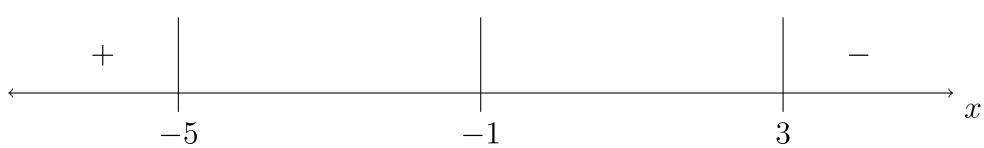
First, we identify the leading term of f.

$$a_n x^n = -(x)^2(x)(x)^2 = -x^5$$

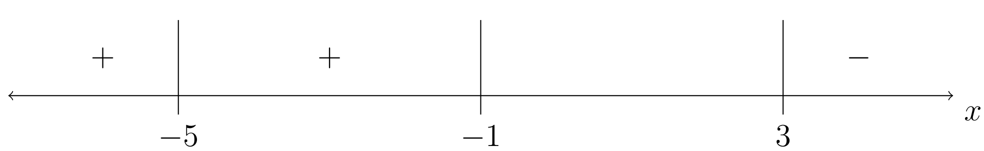
Since $a_n < 0$ and $n = 5$ is odd, our end behavior follows case IV:

$$\text{As } x \to -\infty,\ f(x) \to \infty. \qquad \text{As } x \to \infty,\ f(x) \to -\infty.$$

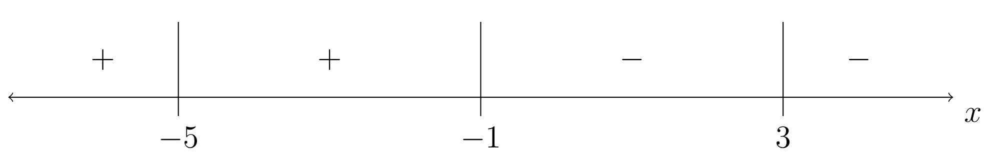
This tells us that our diagram will begin with a positive sign and end with a negative sign.



Furthermore, the multiplicity of the root $x = -5$ is two, which is even. So, our diagram must contain the same signs on either side of $x = -5$, namely two positive signs.



Applying this same idea to both of our remaining roots, we get the following diagram, and are done!



In fact, we can take this last example further, and easily sketch a graph of our polynomial. All that remains is to identify the $y-$intercept.

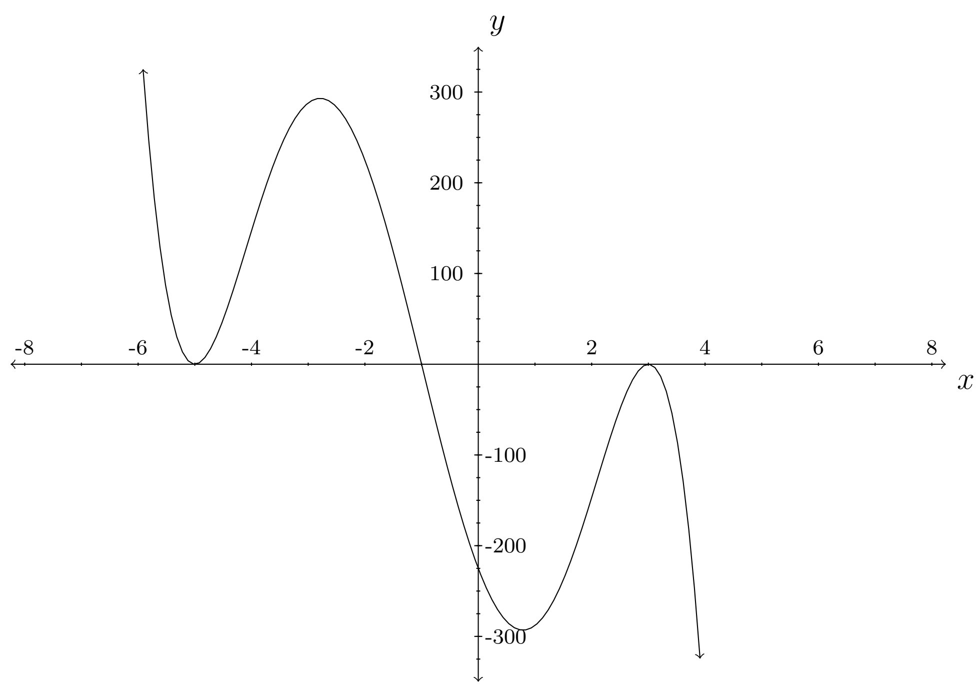
Example 308. Sketch a graph of the factored polynomial $f(x) = -(x-3)^2(x+1)(x+5)^2$, making sure to identify a clearly defined scale and any $x-$ and $y-$intercepts.

Since the roots, $x = 3$ and $x = -5$ have even multiplicities, their corresponding $x-$intercepts will be turnaround points. Our sign diagram confirms this, and further shows that the intercept at $x = -5$ will be a local minimum $(+|+)$, whereas the intercept at $x = 3$ will be a local maximum $(-|-)$. On the other hand, the root at $x = -1$ has an odd multiplicity, and the $x-$intercept at $x = -1$ will be a crossover point.

For a $y-$intercept, we evaluate the function at $x = 0$.

$$\begin{aligned} f(0) &= -(0-3)^2(0+1)(0+5)^2 \\ &= -(9)(1)(25) \\ &= -225 \end{aligned}$$

Since our $y-$intercept is a large negative value, we will have to shrink our scale for the $y-$axis accordingly.



This last example achieves what we have sought after since beginning the chapter. In it, we have identified the end behavior and all intercepts of a completely factored polynomial. We have further used both a sign diagram and a multiplicity argument in order to graph the given function.

What remains is to take a closer look at more challenging polynomials, for which a factorization may not be given or even easily identifiable.

The Rational Root Theorem (L51)

Objective: Apply the Rational Root Theorem to determine a set of possible rational roots for and a factorization of a given polynomial.

The Rational Root Theorem is used to identify a list of all possible rational roots for a given polynomial.

> **Rational Root Theorem:** Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial of degree n with $n \geq 1$, and a_0, a_1, $\ldots a_n$ are integers. If r is a rational root of f, then r is of the form $\pm\frac{p}{q}$, where p is a factor of the constant term a_0, and q is a factor of the leading coefficient a_n.

The Rational Root Theorem gives us a list of numbers to test as roots of a given polynomial using synthetic division, which is a nicer approach than simply guessing at possible roots. If none of the numbers in the list turn out to be roots, then either the polynomial has no real roots at all, or all of the real roots will be irrational numbers.

Example 309. Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. Use the Rational Root Theorem to list all of the possible rational roots of f.

To generate a complete list of rational roots, we need to take each of the factors of the constant term, $a_0 = -3$, and divide them by each of the factors of the leading coefficient $a_4 = 2$.

The factors of -3 are ± 1 and ± 3. Since the Rational Root Theorem tacks on a $\pm$ anyway, for the moment, we consider only the positive factors 1 and 3. The factors of 2 are 1 and 2, so the Rational Root Theorem gives the list $\left\{\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{3}{1}, \pm\frac{3}{2}\right\}$ or $\left\{\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 3\right\}$.

Additionally, we can evaluate f at each of the eight potential rational roots in our list, to see if any of them are indeed roots. Starting with ± 1, we see that

$$f(1) = 2 + 4 - 1 - 6 - 3 = -4 \neq 0 \qquad \text{and} \qquad f(-1) = 2 - 4 - 1 + 6 - 3 = 0.$$

Hence, we can conclude that $x = -1$ is a root of f and $x = 1$ is not. Using synthetic division, we can then divide f by the linear factor $x + 1$ as follows.

$$\begin{array}{r|rrrrr} -1 & 2 & 4 & -1 & -6 & -3 \\ & & -2 & -2 & 3 & 3 \\ \hline & 2 & 2 & -3 & -3 & \mathbf{0} \end{array}$$

We can then begin to factor f,

$$2x^4 + 4x^3 - x^2 - 6x - 3 = (x + 1)(2x^3 + 2x^2 - 3x - 3)$$

The resulting quotient polynomial is then factorable by grouping,

$$2x^3 + 2x^2 - 3x - 3 = (2x^2 - 3)(x + 1).$$

Factoring out a 2 from the expression $2x^2 - 3$, allows us to factor it as the difference of two squares,

$$\begin{aligned} 2x^2 - 3 &= 2\left(x^2 - \frac{3}{2}\right) \\ &= 2\left(x - \sqrt{\frac{3}{2}}\right)\left(x + \sqrt{\frac{3}{2}}\right) \\ &= 2\left(x - \frac{\sqrt{6}}{2}\right)\left(x + \frac{\sqrt{6}}{2}\right) \end{aligned}$$

So, a complete factorization for f would be

$$2x^4 + 4x^3 - x^2 - 6x - 3 = 2\left(x - \frac{\sqrt{6}}{2}\right)\left(x + \frac{\sqrt{6}}{2}\right)(x+1)^2,$$

and the set of real roots for f is $\left\{-1, \pm\frac{\sqrt{6}}{2}\right\}$.

Graphing Summary (L52)

Objective: Graph a polynomial function in its entirety.

At this point, we have addressed all key features of polynomials individually. This section pulls each of these aspects together, for a detailed analysis of a polynomial, culminating in a complete sketch of its graph. Along the way, we will need to address each of the following aspects for our polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$. It is important to note that there is no universally accepted order to this checklist.

- Find the $y-$intercept of the graph of f, $(0, f(0)) = (0, a_0)$.
- Use the degree n and leading coefficient a_n to determine the end behavior of the graph of f.
- Identify a complete factorization of f, and use it to find any $x-$intercepts of the graph of f. Using multiplicities, classify each $x-$intercept as a crossover or turnaround ("bounce") point.
- Using the $x-$intercepts, construct a sign diagram for f.

In each polynomial we encounter, we will carefully examine the function, making sure not to omit any of the checklist items above and to compare each item to those that precede it along the way for accuracy. Although the process will take some time, if we are thorough, our end result should be a complete, accurate sketch of the given polynomial.

Example 310. Sketch a complete graph of $f(x) = 14x^4 - 17x^3 - 6x^2 + 7x + 2$.

We will start with the $y-$intercept, which is $(0, 2)$.

Next, we see that f has even degree and positive leading coefficient. So, the tails of the graph of f both point upwards. In other words, as $x \to \pm\infty$, $f(x) \to \infty$.

Since f is degree-4, contains more than four terms, and is not of quadratic type, we will apply the Rational Root Theorem. In this case, our set of possible rational roots is

$$\left\{\pm 1, \pm 2, \pm\frac{1}{2}, \pm\frac{1}{7}, \pm\frac{1}{14}, \pm\frac{2}{7}\right\}$$

Fortunately, we see that $f(1) = 14 - 17 - 6 + 7 + 2 = 0$. So, $x - 1$ is a factor of f. Dividing, we get:

$$\begin{array}{r|r}
 & 14x^3 - 3x^2 - 9x - 2 \\ \hline
x-1 & 14x^4 - 17x^3 - 6x^2 + 7x + 2 \\
 & -14x^4 + 14x^3 \\ \hline
 & -3x^3 - 6x^2 \\
 & 3x^3 - 3x^2 \\ \hline
 & -9x^2 + 7x \\
 & 9x^2 - 9x \\ \hline
 & -2x + 2 \\
 & 2x - 2 \\ \hline
 & 0
\end{array}
\qquad
\begin{array}{r|rrrrr}
1 & 14 & -17 & -6 & 7 & 2 \\
 & & 14 & -3 & -9 & -2 \\ \hline
 & 14 & -3 & -9 & -2 & \mathbf{0}
\end{array}$$

So, $f(x) = (x - 1)(14x^3 - 3x^2 - 9x - 2)$. Applying the Rational Root Theorem a second time, we can see that $x = 1$ is also a root of the cubic factor of f, since $14 - 3 - 9 - 2 = 0$. Again, we can divide to factor f further.

$$\begin{array}{r|r}
 & 14x^2 + 11x + 2 \\ \hline
x-1 & 14x^3 - 3x^2 - 9x - 2 \\
 & -14x^3 + 14x^2 \\ \hline
 & 11x^2 - 9x \\
 & -11x^2 + 11x \\ \hline
 & 2x - 2 \\
 & -2x + 2 \\ \hline
 & 0
\end{array}
\qquad
\begin{array}{r|rrrr}
1 & 14 & -3 & -9 & -2 \\
 & & 14 & 11 & 2 \\ \hline
 & 14 & 11 & 2 & \mathbf{0}
\end{array}$$

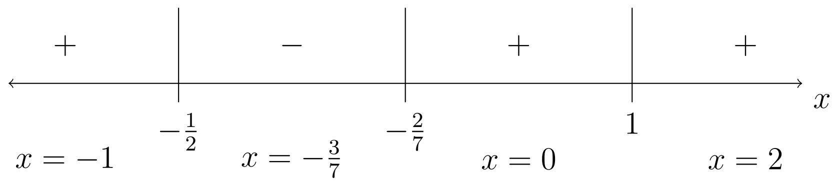
So, $f(x) = (x - 1)^2(14x^2 + 11x + 2)$. Factoring the remaining quadratic, we have

$$f(x) = (x - 1)^2(7x + 2)(2x + 1),$$

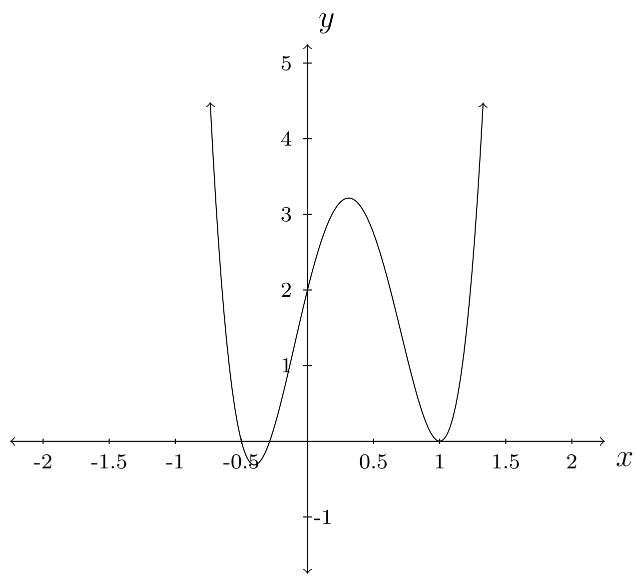
with accompanying set of roots $\{1, -\frac{1}{2}, -\frac{2}{7}\}$.

Using multiplicities, we conclude that the $x-$intercept $(1, 0)$ is a turnaround point, and the intercepts $\left(-\frac{1}{2}, 0\right)$ and $\left(-\frac{2}{7}, 0\right)$ are crossover points.

Though not necessary for graphing, a sign diagram confirms our end and local behavior findings.



Putting all of this information together results in the following graph.



Polynomial Inequalities (L53)

Objective: Solve a polynomial inequality by constructing a sign diagram.

Example 311. Solve the polynomial inequality

$$x^4 + 6x^2 - 15x \leq x^4 + 2x^3 - 7x^2.$$

Just as with quadratic inequalities, we begin by setting one side equal to zero. This gives us

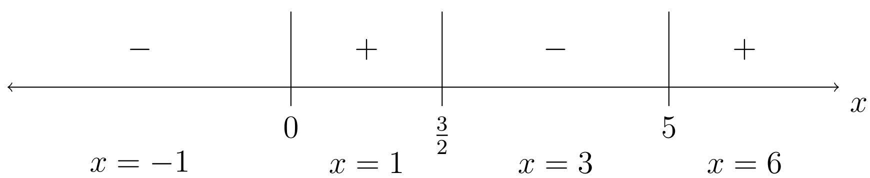
$$2x^3 - 13x^2 + 15x \geq 0.$$

In order to construct a sign diagram, we must find a factorization and identify the roots of the left-hand side of our inequality.

$$2x^3 - 13x^2 + 15x = 2x\left(x - \frac{3}{2}\right)(x - 5)$$

So the dividers in our diagram will be the roots $x = 0, \frac{3}{2}$, and 5. Below is a chart for testing the intervals in our sign diagram, as well as the end result.

Interval	Test Value	Signs	Result
$(-\infty, 0)$	$x = -1$	$(-)(-)(-)$	$-$
$(0, \frac{3}{2})$	$x = 1$	$(+)(-)(-)$	$+$
$(\frac{3}{2}, 5)$	$x = 3$	$(+)(+)(-)$	$-$
$(5, \infty)$	$x = 6$	$(+)(+)(+)$	$+$



So, using our diagram as an aide, we see that the solution to the inequality

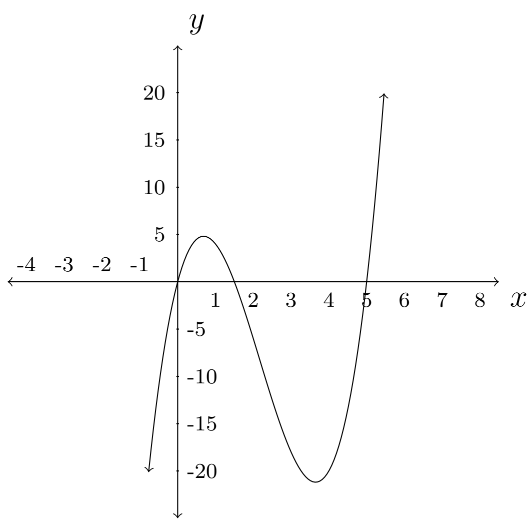
$$2x^3 - 13x^2 + 15x \geq 0,$$

as well as our original inequality

$$x^4 + 6x^2 - 15x \leq x^4 + 2x^3 - 7x^2,$$

will be

$$\left[0, \frac{3}{2}\right] \cup [5, \infty).$$



Since our given inequality was inclusive ($\leq$ or $\geq$), we include the corresponding endpoints in our answer.

We can verify that our answer is correct by comparing it to the graph of the function

$$f(x) = 2x^3 - 13x^2 + 15x,$$

which lies above (or on) the $x-$axis over the intervals in our answer.

Practice Problems

Introduction and Terminology

Identify the degree, set of coefficients, leading coefficient, leading term and constant term for each of the polynomials listed. Classify each polynomial by both degree and number of nonzero terms. If it is not already provided, write the polynomial in descending-power order.

1. $f(x) = -2x^3 - 1$
2. $f(x) = -2x^4 + 4x + 1$
3. $f(x) = 40 - x^3$
4. $f(x) = (x-1)^2$
5. $f(x) = 32x^5 + x^2 + x$
6. $f(x) = 4x^2 - 3x^4$
7. $f(x) = -2x^4 - 4x^2 - 6x - 8$
8. $f(x) = 5x + 3x^2 + x^3 + \sqrt{3}$
9. $f(x) = \frac{1}{2}x^4 - 5x^2 - \frac{1}{2}$
10. $f(x) = 12 - 6x + 3x^2 - 2x^3 - x^6$
11. $f(x) = -3x^4 - 12x^3 + x - 13$

Sign Diagrams

Construct a sign diagram for the factored polynomial functions below. Use Desmos to graph each function and check the accuracy of your diagram. Identify the interval(s) where the function is positive and where it is negative.

1. $f(x) = x^3(x-2)(x+2)$
2. $g(x) = (x^2+1)(1-x)$
3. $h(x) = x(x-3)^2(x+3)$
4. $k(x) = (3x-4)^3$
5. $\ell(x) = (x^2+2)(x^2+3)$
6. $m(x) = -2(x+7)^2(1-2x)^2$
7. $f(x) = (x^2-1)(x+4)$
8. $g(x) = (x^2-1)(x^2-16)$
9. $h(x) = -2x^3(3x-1)(2-x)$
10. $k(x) = (x^2-4x+1)(x+2)^2$

Factoring

Some Special Cases

Completely factor each of the following polynomial expressions.

1. $2x^2 - 11x + 15$
2. $5n^3 + 7n^2 - 6n$
3. $54u^3 - 16$
4. $54 - 128x^3$
5. $n^2 - n$
6. $2x^4 - 21x^2 - 11$
7. $24az - 18ah + 60yz - 45yh$
8. $5u^2 - 9uv + 4v^2$
9. $16x^2 + 48xy + 36y^2$
10. $-2x^3 + 128y^3$
11. $20uv - 60u^3 - 5xv + 15xu^2$
12. $2x^3 + 5x^2y + 3y^2x$

Quadratic Type

Completely factor each of the following polynomials over the real numbers and identify the set of all real roots.

1. $x^4 + 13x^2 + 40$
2. $x^4 - 5x^2 + 4$
3. $x^4 - 17x^2 + 16$
4. $x^4 - 3x^2 - 40$
5. $3x^4 - 32x^2 + 45$
6. $x^4 + x^2 - 12$
7. $x^4 - 3x^2 - 10$
8. $x^6 - 82x^3 + 81$
9. $8x^4 + 2x^2 - 3$
10. $2x^4 - 19x^2 + 9$

Division

Polynomial (Long) Division

Use polynomial long division to divide and simplify each of the given expressions. Express each answer in the form below.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

1. $\dfrac{20x^4 + x^3 + 2x^2}{4x^3}$
2. $\dfrac{5x^4 + 45x^3 + 4x^2}{9x}$
3. $\dfrac{20x^4 + x^3 + 40x^2}{10x}$
4. $\dfrac{3x^3 + 4x^2 + 2x}{8x}$
5. $\dfrac{12x^4 + 24x^3 + 3x^2}{6x}$
6. $\dfrac{5x^4 + 16x^3 + 16x^2}{4x}$
7. $\dfrac{10x^4 + 50x^3 + 2x^2}{10x^2}$
8. $\dfrac{3x^4 + 18x^3 + 27x^2}{9x^2}$
9. $\dfrac{x^2 - 2x - 71}{x + 8}$
10. $\dfrac{x^2 - 3x - 53}{x - 9}$
11. $\dfrac{x^2 + 13x + 32}{x + 5}$
12. $\dfrac{x^2 - 10x + 16}{x - 7}$
13. $\dfrac{x^2 - 2x - 89}{x - 10}$
14. $\dfrac{x^2 + 4x - 26}{x + 7}$
15. $\dfrac{x^2 - 4x - 38}{x - 8}$
16. $\dfrac{x^2 - 4}{x - 2}$
17. $\dfrac{x^3 + 15x^2 + 49x - 55}{x + 7}$
18. $\dfrac{x^3 - 26x - 41}{x + 4}$
19. $\dfrac{3x^3 + 9x^2 - 64x - 68}{x + 6}$
20. $\dfrac{9x^3 + 45x^2 + 27x - 5}{9x + 9}$
21. $\dfrac{x^3 - x^2 - 16x + 8}{x - 4}$
22. $\dfrac{x^2 - 10x + 22}{x - 4}$
23. $\dfrac{x^3 - 16x^2 + 71x - 56}{x - 8}$
24. $\dfrac{x^3 - 4x^2 - 6x + 4}{x - 1}$
25. $\dfrac{8x^3 - 66x^2 + 12x + 37}{x - 8}$
26. $\dfrac{3x^2 + 9x - 9}{3x - 3}$
27. $\dfrac{2x^2 - 5x - 8}{2x + 3}$
28. $\dfrac{3x^2 - 32}{3x - 9}$
29. $\dfrac{4x^2 - 23x - 38}{4x + 5}$
30. $\dfrac{2x^3 + 21x^2 + 25x}{2x + 3}$

31. $\dfrac{4x^3 - 21x^2 + 6x + 19}{4x + 3}$

32. $\dfrac{8x^3 - 57x^2 + 42}{8x + 7}$

33. $\dfrac{2x^3 + 12x^2 + 4x - 37}{2x + 6}$

34. $\dfrac{45x^2 + 56x + 19}{9x + 4}$

35. $\dfrac{10x^2 - 32x + 9}{10x - 2}$

36. $\dfrac{4x^2 - x - 1}{4x + 3}$

37. $\dfrac{27x^2 + 87x + 35}{3x + 8}$

38. $\dfrac{4x^2 - 33x + 28}{4x - 5}$

39. $\dfrac{48x^2 - 70x + 16}{6x - 2}$

40. $\dfrac{12x^3 + 12x^2 - 15x - 4}{2x + 3}$

41. $\dfrac{24x^3 - 38x^2 + 29x - 60}{4x - 7}$

Synthetic Division

Use synthetic division to divide and simplify each of the given expressions. Express each answer in the form below.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

1. $\dfrac{x^4 - 4x^3 + 2x^2 - x + 1}{x + 2}$
2. $\dfrac{x^4 - 2x^3 + 7x^2 - 6x + 3}{x - 2}$
3. $\dfrac{2x^4 - 2x^3 - 10x^2 + 1}{x + 2}$
4. $\dfrac{5x^4 - 2x^3 + 4x^2 - 5x}{x - 1}$
5. $\dfrac{-x^4 - x^3 + x^2 + x + 1}{x + 5}$
6. $\dfrac{x^4 - 3x^3 + 2x^2 - x + 1}{x - 4}$
7. $\dfrac{12x^4 - x^3 + x^2 - 3x + 1}{x + 2}$
8. $\dfrac{3x^4 + 3x^3 + 13x^2 - 4x + 14}{x + 1}$
9. $\dfrac{1x^4 - 3x^3 + 5x^2 - 14x + 2}{x - 2}$
10. $\dfrac{2x^4 - 2x + 1}{x + 3}$
11. $\dfrac{x^4 - 3x - 4}{x - 3}$
12. $\dfrac{x^4 - 4x^3 + 13x^2 - 5x + 7}{x - 4}$

Use synthetic division to divide and simplify each of the given expression from Exercises 9-41.

End Behavior

Determine the end behavior of each of the following functions. Write your answers as mathematical sentences. Graph each function on Desmos to check your answers.

1. $f(x) = -2x^3 + 4x + 1$
2. $g(x) = 32x^5 + x^2 + 15$
3. $h(x) = -3x^4 + 4x^2$
4. $k(x) = 15x^4 - 32x^2 - x - 14$
5. $\ell(x) = x^5 + 40$
6. $m(x) = 5x^5 + 3x^2 + x + 14$
7. $n(x) = 123x^4 - 7x^3 - 5x^2 - 3x + 1$
8. $p(x) = x^3 - 1$
9. $q(x) = -23x^6 + x^3 + x^2 + x + 1$

Identify the degree, leading coefficient, and constant term of each polynomial function below. Use the degree and leading coefficient to identify the end behavior of the graph of each function. Write your answers as mathematical sentences. Graph each function on Desmos to check your answers.

10. $f(x) = x^3(x-2)(x+2)$
11. $g(x) = (x^2+1)(1-x)$
12. $h(x) = x(x-3)^2(x+3)$
13. $k(x) = (3x-4)^3$
14. $\ell(x) = (x^2+2)(x^2+3)$
15. $m(x) = -2(x+7)^2(1-2x)^2$
16. $f(x) = (x^2-1)(x+4)$
17. $g(x) = (x^2-1)(x^2-16)$
18. $h(x) = -2x^3(3x-1)(2-x)$
19. $k(x) = (x^2-4x+1)(x+2)^2$

Local Behavior

Determine the set of roots and corresponding multiplicities for the following functions. In each case, classify the corresponding $x-$intercept as either a turnaround or crossover point. Use Desmos to check your answers.

1. $f(x) = x^3(x-2)(x+2)$
2. $g(x) = (x^2+1)(1-x)$
3. $h(x) = x(x-3)^2(x+3)$
4. $k(x) = (3x-4)^3$
5. $\ell(x) = (x^2+2)(x^2+3)$
6. $m(x) = -2(x+7)^2(1-2x)^2$
7. $f(x) = (x^2-1)(x+4)$
8. $g(x) = (x^2-1)(x^2-16)$
9. $h(x) = -2x^3(3x-1)(2-x)$
10. $k(x) = (x^2-4x+1)(x+2)^2$
11. $f(x) = \frac{1}{2}(x-2)^2(x+5)(x-3)$
12. $g(x) = (x+2)^2(3x-1)(5-x)$

The Rational Root Theorem

Use the Rational Root Theorem to identify a set of possible rational roots for each of the polynomial functions below. Evaluate the function at $x = 1$. If $x = 1$ is a real root, divide the polynomial by $x-1$ and factor the resulting quotient. If $x = 1$ is not a real root, evaluate the function at at least one of your remaining possible roots, in order to determine if they are actual roots of the polynomial. If successful, divide your polynomial by the respective factor and factor the remaining quotient. Use Desmos to help determine the actual set of real roots.

1. $f(x) = x^3 - 2x^2 - 5x + 6$
2. $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$
3. $f(x) = x^5 - x^4 - 37x^3 + 37x^2 + 36x - 36$
4. $f(x) = 3x^3 + 3x^2 - 11x - 10$

Use the Rational Root Theorem to identify a set of possible rational roots for each of the polynomial functions below. Evaluate the function at at least two of your possible roots, in order to determine if they are actual roots of the polynomial. If successful, divide your polynomial by the respective factor. Use Desmos to help determine the actual set of real roots.

5. $f(x) = x^4 - 2x^3 + 5x^2 - 8x + 4$
6. $f(x) = x^3 + 4x^2 - 11x + 6$
7. $f(x) = -2x^3 + 19x^2 - 49x + 20$
8. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
9. $f(x) = x^4 - 9x^2 - 4x + 12$
10. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
11. $f(x) = 6x^3 + 19x^2 - 6x - 40$

Graphing Summary

Factor each polynomial below, and sketch a complete graph of the function, making sure to have a clearly defined scale and label any intercepts. Use Desmos to compare your results.

1. $f(x) = -17x^3 + 5x^2 + 34x - 10$
2. $f(x) = x^4 - 9x^2 + 14$
3. $f(x) = 3x^4 - 14x^2 - 5$
4. $f(x) = 2x^4 - 7x^2 + 6$
5. $f(x) = x^5 - 2x^4 - x + 2$
6. $f(x) = 2x^5 + 3x^4 - 32x - 48$
7. $f(x) = x^6 - 6x^3 - 16$
8. $f(x) = 2x^6 - 7x^3 + 5$
9. $f(x) = -x^3 + 7x^2 - x + 7$
10. $f(x) = 3x^4 - 5x^3 - 12x^2$
11. $f(x) = 2x^3 - 5x^2 - x$
12. $f(x) = -x^4 - 2x^2 + 15$

Get a complete factorization of each polynomial below by first dividing the function by $x-1$. Then sketch a graph of the function, making sure to have a clearly defined scale and label any intercepts. Use Desmos to compare your results.

13. $f(x) = x^3 - 2x^2 - 5x + 6$
14. $f(x) = x^3 + 4x^2 - 11x + 6$
15. $f(x) = x^5 - x^4 - 37x^3 + 37x^2 + 36x - 36$
16. $f(x) = x^4 - 2x^3 + 5x^2 - 8x + 4$

Use the Rational Root Theorem and polynomial division to get a complete factorization of each polynomial function below. Then sketch a graph of the function, making sure to have a clearly defined scale and label any intercepts. Use Desmos to compare your results.

17. $f(x) = x^4 - 9x^2 - 4x + 12$
18. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
19. $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$
20. $f(x) = 3x^3 + 3x^2 - 11x - 10$
21. $f(x) = 6x^3 + 19x^2 - 34x - 40$
22. $f(x) = -2x^3 + 19x^2 - 49x + 20$
23. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
24. $f(x) = x^4 + 4x^3 - x - 4$
25. $f(x) = 2x^3 - 5x^2 - 52x + 60$
26. $f(x) = -x^3 - x^2 + 39x + 45$
27. $f(x) = -2x^4 + 7x^3 + 17x^2 - 28x - 36$
28. $f(x) = x^7 - 5x^6 - 24x^5 + 120x^4 - 25x^3 + 125x^2$

Polynomial Inequalities

Solve each polynomial inequality below, expressing your answers using interval notation. Use Desmos to help confirm that each answer is correct.

1. $x^4 + x^2 \geq 6$
2. $x^4 - 9x^2 \leq 4x - 12$
3. $4x^3 \geq 3x + 1$
4. $x^4 \leq 16 + 4x - x^3$
5. $3x^2 + 2x < x^4$
6. $\dfrac{x^3 + 2x^2}{2} < x + 2$
7. $\dfrac{x^3 + 20x}{8} \geq x^2 + 2$
8. $19x^2 + 20 > 2x^3 + 49x$
9. $x^3 < 4x^2$
10. $x^3 - 7x^2 \leq 12x - 84$
11. $(x - 1)^2 \geq 4$
12. $2x^4 > 5x^2 + 3$

Selected Answers

Introduction and Terminology

1. $n = 3, \quad a_n = -2, \quad a_n x^n = -2x^3, \quad a_0 = -1, \quad \{-2, 0, 0, -1\}$
3. $n = 3, \quad a_n = -1, \quad a_n x^n = -1x^3, \quad a_0 = 40, \quad \{-1, 0, 0, 40\}$
5. $n = 5, \quad a_n = 32, \quad a_n x^n = 32x^5, \quad a_0 = 0, \quad \{32, 0, 0, 1, 1, 0\}$
7. $n = 4, \quad a_n = -2, \quad a_n x^n = -2x^4, \quad a_0 = -8, \quad \{-2, 0, -4, -6, -8\}$
9. $n = 4, \quad a_n = \frac{1}{2}, \quad a_n x^n = \frac{1}{2}x^4, \quad a_0 = -\frac{1}{2}, \quad \{\frac{1}{2}, 0, -5, 0, -\frac{1}{2}\}$
11. $n = 4, \quad a_n = -3, \quad a_n x^n = -3x^4, \quad a_0 = -13, \quad \{-3, -12, 0, 1, -13\}$

Sign Diagrams

1. $$\begin{array}{ccccccc} - & | & + & | & - & | & + \\ \hline & -2 & & 0 & & 2 & \end{array} \; x$$

3. $$\begin{array}{ccccccc} + & | & - & | & + & | & + \\ \hline & -3 & & 0 & & 3 & \end{array} \; x$$

5. $$\begin{array}{c} + \\ \hline \end{array} \; x$$

7. $$\begin{array}{ccccccc} - & | & + & | & - & | & + \\ \hline & -4 & & -1 & & 1 & \end{array} \; x$$

9. $$\begin{array}{ccccccc} - & | & + & | & - & | & + \\ \hline & 0 & & \frac{1}{3} & & 2 & \end{array} \; x$$

Factoring

Some Special Cases

1. $(2x-5)(x-3)$
3. $2(3x-2)(9x^2+6x+4)$
5. $n(n-1)$
7. $3(2a+15y)(4z-3h)$
9. $4(2x+3y)^2$
11. $-5(4u-x)(3u^2-v)$

Quadratic Type

1. $(x^2+8)(x^2+5)$
3. $(x-1)(x+1)(x-4)(x+4)$
5. $(3x^2-5)(x^2-9) = 3(x-\frac{\sqrt{15}}{3})(x-\frac{\sqrt{15}}{3})(x-3)(x+3)$
7. $(x^2-5)(x^2+2) = (x-\sqrt{5})(x+\sqrt{5})(x^2+2)$
9. $(2x^2-\frac{1}{2})(4x^2+3) = 2(x-\frac{\sqrt{2}}{2})(x+\frac{\sqrt{2}}{2})(4x^2+3)$

Division

Polynomial (Long) Division

1. $$\begin{array}{r|l} & \quad\quad\quad\quad\quad 5x+\frac{1}{4} \\ \hline 4x^3 & 20x^4+x^3+2x^2 \\ & \underline{-\,20x^4} \\ & \quad\quad\quad x^3 \\ & \quad\quad \underline{-\,x^3} \\ & \quad\quad\quad\quad\quad 2x^2 \end{array}$$

3.
$$
\begin{array}{rr}
 & 2x^3 + \frac{1}{10}x^2 + 4x \\
\cline{2-2}
10x\big) & 20x^4 + x^3 + 40x^2 \\
 & -20x^4 \\
\cline{2-2}
 & x^3 \\
 & -x^3 \\
\cline{2-2}
 & 40x^2 \\
 & -40x^2 \\
\cline{2-2}
 & 0
\end{array}
$$

5.
$$
\begin{array}{rr}
 & 2x^3 + 4x^2 + \frac{1}{2}x \\
\cline{2-2}
6x\big) & 12x^4 + 24x^3 + 3x^2 \\
 & -12x^4 \\
\cline{2-2}
 & 24x^3 \\
 & -24x^3 \\
\cline{2-2}
 & 3x^2 \\
 & -3x^2 \\
\cline{2-2}
 & 0
\end{array}
$$

7.
$$
\begin{array}{rr}
 & x^2 + 5x + \frac{1}{5} \\
\cline{2-2}
10x^2\big) & 10x^4 + 50x^3 + 2x^2 \\
 & -10x^4 \\
\cline{2-2}
 & 50x^3 \\
 & -50x^3 \\
\cline{2-2}
 & 2x^2 \\
 & -2x^2 \\
\cline{2-2}
 & 0
\end{array}
$$

9.
$$
\begin{array}{rr}
 & x - 10 \\
\cline{2-2}
x+8\big) & x^2 - 2x - 71 \\
 & -x^2 - 8x \\
\cline{2-2}
 & -10x - 71 \\
 & 10x + 80 \\
\cline{2-2}
 & 9
\end{array}
$$

11.
$$
\begin{array}{rr}
 & x + 8 \\
\cline{2-2}
x+5\big) & x^2 + 13x + 32 \\
 & -x^2 - 5x \\
\cline{2-2}
 & 8x + 32 \\
 & -8x - 40 \\
\cline{2-2}
 & -8
\end{array}
$$

13.
$$
\begin{array}{rr}
 & x + 8 \\
\cline{2-2}
x-10\big) & x^2 - 2x - 89 \\
 & -x^2 + 10x \\
\cline{2-2}
 & 8x - 89 \\
 & -8x + 80 \\
\cline{2-2}
 & -9
\end{array}
$$

15.
$$
\begin{array}{rr}
 & x + 4 \\
\cline{2-2}
x-8\big) & x^2 - 4x - 38 \\
 & -x^2 + 8x \\
\cline{2-2}
 & 4x - 38 \\
 & -4x + 32 \\
\cline{2-2}
 & -6
\end{array}
$$

17.
$$
\begin{array}{rr}
 & x^2 + 8x - 7 \\
\cline{2-2}
x+7\big) & x^3 + 15x^2 + 49x - 55 \\
 & -x^3 - 7x^2 \\
\cline{2-2}
 & 8x^2 + 49x \\
 & -8x^2 - 56x \\
\cline{2-2}
 & -7x - 55 \\
 & 7x + 49 \\
\cline{2-2}
 & -6
\end{array}
$$

19.
$$
\begin{array}{rr}
 & 3x^2 - 9x - 10 \\
\cline{2-2}
x+6\big) & 3x^3 + 9x^2 - 64x - 68 \\
 & -3x^3 - 18x^2 \\
\cline{2-2}
 & -9x^2 - 64x \\
 & 9x^2 + 54x \\
\cline{2-2}
 & -10x - 68 \\
 & 10x + 60 \\
\cline{2-2}
 & -8
\end{array}
$$

21.
$$
\begin{array}{rr}
 & x^2 + 3x - 4 \\
\cline{2-2}
x-4\big) & x^3 - x^2 - 16x + 8 \\
 & -x^3 + 4x^2 \\
\cline{2-2}
 & 3x^2 - 16x \\
 & -3x^2 + 12x \\
\cline{2-2}
 & -4x + 8 \\
 & 4x - 16 \\
\cline{2-2}
 & -8
\end{array}
$$

23.
$$
\begin{array}{rr}
 & x^2 - 8x + 7 \\ \cline{2-2}
x-8\big) & x^3 - 16x^2 + 71x - 56 \\
 & -x^3 + 8x^2 \phantom{{}+71x-56} \\ \cline{2-2}
 & -8x^2 + 71x \phantom{{}-56} \\
 & 8x^2 - 64x \phantom{{}-56} \\ \cline{2-2}
 & 7x - 56 \\
 & -7x + 56 \\ \cline{2-2}
 & 0
\end{array}
$$

25.
$$
\begin{array}{rr}
 & 8x^2 - 2x - 4 \\ \cline{2-2}
x-8\big) & 8x^3 - 66x^2 + 12x + 37 \\
 & -8x^3 + 64x^2 \phantom{{}+12x+37} \\ \cline{2-2}
 & -2x^2 + 12x \phantom{{}+37} \\
 & 2x^2 - 16x \phantom{{}+37} \\ \cline{2-2}
 & -4x + 37 \\
 & 4x - 32 \\ \cline{2-2}
 & 5
\end{array}
$$

27.
$$
\begin{array}{rr}
 & x - 4 \\ \cline{2-2}
2x+3\big) & 2x^2 - 5x - 8 \\
 & -2x^2 - 3x \phantom{{}-8} \\ \cline{2-2}
 & -8x - 8 \\
 & 8x + 12 \\ \cline{2-2}
 & 4
\end{array}
$$

29.
$$
\begin{array}{rr}
 & x - 7 \\ \cline{2-2}
4x+5\big) & 4x^2 - 23x - 38 \\
 & -4x^2 - 5x \phantom{{}-38} \\ \cline{2-2}
 & -28x - 38 \\
 & 28x + 35 \\ \cline{2-2}
 & -3
\end{array}
$$

31.
$$
\begin{array}{rr}
 & x^2 - 6x + 6 \\ \cline{2-2}
4x+3\big) & 4x^3 - 21x^2 + 6x + 19 \\
 & -4x^3 - 3x^2 \phantom{{}+6x+19} \\ \cline{2-2}
 & -24x^2 + 6x \phantom{{}+19} \\
 & 24x^2 + 18x \phantom{{}+19} \\ \cline{2-2}
 & 24x + 19 \\
 & -24x - 18 \\ \cline{2-2}
 & 1
\end{array}
$$

33.
$$
\begin{array}{rr}
 & x^2 + 3x - 7 \\ \cline{2-2}
2x+6\big) & 2x^3 + 12x^2 + 4x - 37 \\
 & -2x^3 - 6x^2 \phantom{{}+4x-37} \\ \cline{2-2}
 & 6x^2 + 4x \phantom{{}-37} \\
 & -6x^2 - 18x \phantom{{}-37} \\ \cline{2-2}
 & -14x - 37 \\
 & 14x + 42 \\ \cline{2-2}
 & 5
\end{array}
$$

35.
$$
\begin{array}{rr}
 & x - 3 \\ \cline{2-2}
10x-2\big) & 10x^2 - 32x + 9 \\
 & -10x^2 + 2x \phantom{{}+9} \\ \cline{2-2}
 & -30x + 9 \\
 & 30x - 6 \\ \cline{2-2}
 & 3
\end{array}
$$

37.
$$
\begin{array}{rr}
 & 9x + 5 \\ \cline{2-2}
3x+8\big) & 27x^2 + 87x + 35 \\
 & -27x^2 - 72x \phantom{{}+35} \\ \cline{2-2}
 & 15x + 35 \\
 & -15x - 40 \\ \cline{2-2}
 & -5
\end{array}
$$

39.
$$
\begin{array}{rr}
 & 8x - 9 \\ \cline{2-2}
6x-2\big) & 48x^2 - 70x + 16 \\
 & -48x^2 + 16x \phantom{{}+16} \\ \cline{2-2}
 & -54x + 16 \\
 & 54x - 18 \\ \cline{2-2}
 & -2
\end{array}
$$

41.
$$
\begin{array}{rr}
 & 6x^2 + x + 9 \\ \cline{2-2}
4x-7\big) & 24x^3 - 38x^2 + 29x - 60 \\
 & -24x^3 + 42x^2 \phantom{{}+29x-60} \\ \cline{2-2}
 & 4x^2 + 29x \phantom{{}-60} \\
 & -4x^2 + 7x \phantom{{}-60} \\ \cline{2-2}
 & 36x - 60 \\
 & -36x + 63 \\ \cline{2-2}
 & 3
\end{array}
$$

Synthetic Division

1. $$\begin{array}{r|rrrrr} -2 & 1 & -4 & 2 & -1 & 1 \\ & & -2 & 12 & -28 & 58 \\ \hline & 1 & -6 & 14 & -29 & \mathbf{59} \end{array}$$

3. $$\begin{array}{r|rrrrr} -2 & 2 & -2 & -10 & 0 & 1 \\ & & -4 & 12 & -4 & 8 \\ \hline & 2 & -6 & 2 & -4 & \mathbf{9} \end{array}$$

5. $$\begin{array}{r|rrrrr} -5 & -1 & -1 & 1 & 1 & 1 \\ & & 5 & -20 & 95 & -480 \\ \hline & -1 & 4 & -19 & 96 & -\mathbf{479} \end{array}$$

7. $$\begin{array}{r|rrrrr} -2 & 12 & -1 & 1 & -3 & 1 \\ & & -24 & 50 & -102 & 210 \\ \hline & 12 & -25 & 51 & -105 & \mathbf{211} \end{array}$$

9. $$\begin{array}{r|rrrrr} 2 & 1 & -3 & 5 & -14 & 2 \\ & & 2 & -2 & 6 & -16 \\ \hline & 1 & -1 & 3 & -8 & -\mathbf{14} \end{array}$$

11. $$\begin{array}{r|rrrrr} 3 & 1 & 0 & 0 & -3 & -4 \\ & & 3 & 9 & 27 & 72 \\ \hline & 1 & 3 & 9 & 24 & \mathbf{68} \end{array}$$

End Behavior

1. As $x \to -\infty,\ f(x) \to \infty.$ As $x \to \infty,\ f(x) \to -\infty.$
3. As $x \to -\infty,\ h(x) \to -\infty.$ As $x \to \infty,\ h(x) \to -\infty.$
5. As $x \to -\infty,\ \ell(x) \to -\infty.$ As $x \to \infty,\ \ell(x) \to \infty.$
7. As $x \to -\infty,\ n(x) \to \infty.$ As $x \to \infty,\ n(x) \to \infty.$
9. As $x \to -\infty,\ q(x) \to -\infty.$ As $x \to \infty,\ q(x) \to -\infty.$
11. $a_n x^n = -1x^3$, $a_0 = 1$, As $x \to -\infty,\ f(x) \to \infty.$ As $x \to \infty,\ f(x) \to -\infty.$
13. $a_n x^n = 27x^3$, $a_0 = -64$, As $x \to -\infty,\ k(x) \to -\infty.$ As $x \to \infty,\ k(x) \to \infty.$
15. $a_n x^n = -8x^4$, $a_0 = -98$, As $x \to -\infty,\ m(x) \to -\infty.$ As $x \to \infty,\ m(x) \to -\infty.$
17. $a_n x^n = 1x^4$, $a_0 = 16$, As $x \to -\infty,\ g(x) \to \infty.$ As $x \to \infty,\ g(x) \to \infty.$
19. $a_n x^n = 1x^4$, $a_0 = 2$, As $x \to -\infty,\ k(x) \to \infty.$ As $x \to \infty,\ k(x) \to \infty.$

Local Behavior

1. $x_1 = 0, k_1 = 3$, Crossover; $x_2 = 2, k_2 = 1$, Crossover; $x_3 = -2, k_3 = 1$, Crossover
3. $x_1 = 0, k_1 = 1$, Crossover; $x_2 = 3, k_2 = 2$, Turnaround; $x_3 = -3, k_3 = 1$, Crossover
5. No real roots
7. $x_1 = 1, k_1 = 1$, Crossover; $x_2 = -1, k_2 = 1$, Crossover; $x_3 = -4, k_3 = 1$, Crossover
9. $x_1 = 0, k_1 = 3$, Crossover; $x_2 = \frac{1}{3}, k_2 = 1$, Crossover; $x_3 = 2, k_3 = 1$, Crossover
11. $x_1 = 2, k_1 = 2$, Turnaround; $x_2 = -5, k_2 = 1$, Crossover; $x_3 = 3, k_3 = 1$, Crossover

The Rational Root Theorem

1. $f(x) = \big(x-1\big)\big(x+2\big)\big(x-3\big)$
   List of possible rational roots: $\{\pm 6, \pm 3, \pm 2, \pm 1\}$
3. $f(x) = \big(x-1\big)^2\big(x+1\big)\big(x-6\big)\big(x+6\big)$
   List of possible rational roots: $\{\pm 36, \pm 18, \pm 12, \pm 9, \pm 6, \pm 4, \pm 3, \pm 2, \pm 1\}$
5. $f(x) = \big(x^2+4\big)\big(x-1\big)^2$
   List of possible rational roots: $\{\pm 4, \pm 2, \pm 1\}$
7. $f(x) = \, -2\big(x-\frac{1}{2}\big)\big(x-4\big)\big(x-5\big)$
   List of possible rational roots: $\{\pm 20, \pm 10, \pm 5, \pm 4, \pm\frac{5}{2}, \pm 2, \pm 1, \pm\frac{1}{2}\}$
9. $f(x) = \big(x-1\big)\big(x+2\big)^2\big(x-3\big)$
   List of possible rational roots: $\{\pm 12, \pm 6, \pm 4, \pm 3, \pm 2, \pm 1\}$
11. $f(x) = 6\big(x+2\big)\big(x-\frac{4}{3}\big)\big(x+\frac{5}{2}\big)$
    List of possible rational roots:
    $\{\pm 40, \pm 20, \pm\frac{40}{3}, \pm 10, \pm 8, \pm\frac{20}{3}, \pm 5, \pm 4, \pm\frac{10}{3}, \pm\frac{8}{3}, \pm\frac{5}{2}, \pm 2, \pm\frac{5}{3}, \pm\frac{4}{3}, \pm 1, \pm\frac{5}{6}, \pm\frac{2}{3}, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}\}$

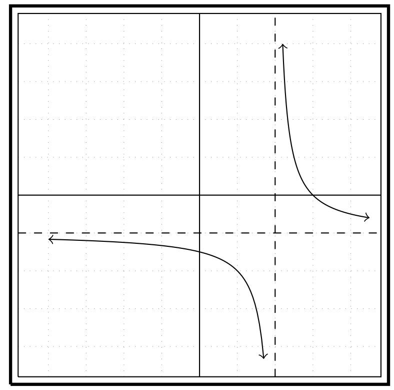
Graphing Summary

1. $f(x) = \, -17\big(x-\frac{5}{17}\big)\big(x+\sqrt{2}\big)\big(x-\sqrt{2}\big)$
3. $f(x) = (3x^2+1)\,\big(x+\sqrt{5}\big)\big(x-\sqrt{5}\big)$
5. $f(x) = \big(x^2+1\big)\big(x-1\big)\big(x+1\big)\big(x-2\big)$
7. $f(x) = (x^3-8)(x^3+2) = (x-2)(x^2+2x+4)(x+\sqrt[3]{2})(x^2-\sqrt[3]{2}x+\sqrt[3]{4})$
9. $f(x) = -(x^2+1)\,\big(x-7\big)$
11. $f(x) = 2x\left(x-\frac{5}{4}+\frac{\sqrt{33}}{4}\right)\left(x-\frac{5}{4}-\frac{\sqrt{33}}{4}\right)$
13. $f(x) = \big(x-1\big)\big(x+2\big)\big(x-3\big)$
15. $f(x) = \big(x-1\big)^2\big(x+1\big)\big(x-6\big)\big(x+6\big)$
17. $f(x) = \big(x-1\big)\big(x+2\big)^2\big(x-3\big)$

19. $f(x) = 2\left(x+1\right)\left(x-\frac{1}{2}\right)\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)$
21. $f(x) = 6\left(x+2\right)\left(x-\frac{4}{3}\right)\left(x+\frac{5}{2}\right)$
23. $f(x) = 36\left(x-\frac{1}{2}\right)^2\left(x+\frac{1}{3}\right)^2$
25. $f(x) = 2\left(x-6\right)\left(x+\frac{7}{4}+\frac{\sqrt{129}}{4}\right)\left(x+\frac{7}{4}-\frac{\sqrt{129}}{4}\right)$
27. $f(x) = -2\left(x+1\right)\left(x-2\right)\left(x+2\right)\left(x-\frac{9}{2}\right)$

Polynomial Inequalities

1. $(-\infty, \sqrt{2}] \cup [\sqrt{2}, \infty)$
3. $[1, \infty)$
5. $(-\infty, 0) \cup (2, \infty)$
7. $[4, \infty)$
9. $(-\infty, 4)$
11. $(-\infty, -1] \cup [3, \infty)$



Chapter 7

Rational Functions

Introduction and Terminology (L54)

Objective: Define and identify key features of rational functions

A *rational function* is a function that can be represented as a ratio (or fraction) of two polynomials p and q. The general form of a rational function f is

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0},$$

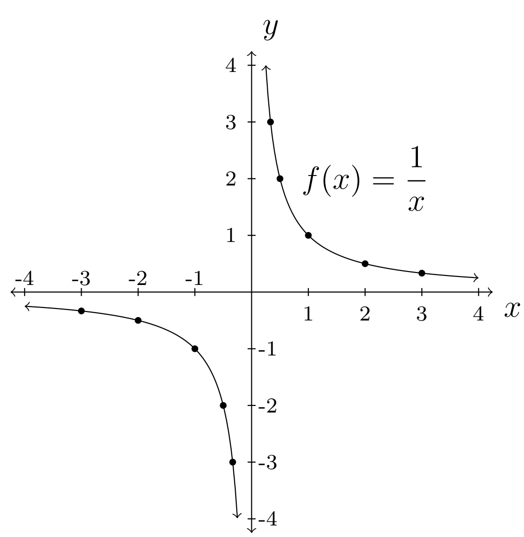
where each of the a_i and b_j are real numbers (for $i, j = 0, 1, 2, \ldots$), with a_n and $b_m \neq 0$, and both m and n are nonnegative integers.

We have already encountered at least one example of a rational function, namely $f(x) = \dfrac{1}{x}$, whose graph we should also be familiar with.

Example 312. $f(x) = \dfrac{1}{x}$

Domain: $x \neq 0$ or $(-\infty, 0) \cup (0, \infty)$

Range: $y \neq 0$ or $(-\infty, 0) \cup (0, \infty)$



As with any function, we can easily identify the $y-$intercept of the graph of f by evaluating the function at $x = 0$.

$$f(0) = \frac{p(0)}{q(0)} = \frac{\cancel{a_n \cdot 0^n} + \cancel{a_{n-1} \cdot 0^{n-1}} + \ldots + \cancel{a_1 \cdot 0} + a_0}{\cancel{b_m \cdot 0^m} + \cancel{b_{m-1} \cdot 0^{m-1}} + \ldots + \cancel{b_1 \cdot 0} + b_0} = \frac{a_0}{b_0}$$

Hence, the graph of f will have a $y-$intercept at the point $\left(0, \dfrac{a_0}{b_0}\right)$.

For rational functions that are already factored, finding the $y-$intercept just requires some simplification after substituting zero for x. The following example demonstrates this.

Example 313. $f(x) = \dfrac{x^3 - x^2 - 8x + 12}{-2x^2 + 14x - 12} \qquad g(x) = \dfrac{(x-2)^2(x+3)}{2(x-6)(1-x)}$

The $y-$intercept of the graph of f is $(0, \frac{12}{-12}) = (0, -1)$.

To find the $y-$intercept of the graph of g, we evaluate $g(0)$ below.

$$g(0) = \frac{(0-2)^2(0+3)}{2(0-6)(1-0)} = \frac{(-2)^2(3)}{2(-6)(1)} = \frac{12}{-12} = -1$$

Hence, as was the case with f, the $y-$intercept of the graph of g is $(0, -1)$. In fact, it is left as an exercise to the reader to show that f and g are the same function.

To identify the domain of a rational function $f(x) = \frac{p(x)}{q(x)}$, we must eliminate all real numbers x which make the denominator equal to zero. In other words, the domain of f is the set of all x such that $q(x) \neq 0$. Identifying the domain of a rational function that is given in factored form is relatively straightforward, whereas rational functions that are given in expanded form must first be factored. Again, we provide an example for each case.

Example 314. $f(x) = \dfrac{3(x+4)(x-2)^2}{(x+3)^2(2x-3)} \qquad g(x) = \dfrac{-x^2 - 4x + 45}{2x^3 - 5x^2 - 18x + 45}$

The domain of f is $x \neq -3, \frac{3}{2}$, or $(-\infty, -3) \cup (-3, \frac{3}{2}) \cup (\frac{3}{2}, \infty)$.

To find the domain of g, one must first use grouping and the $ac-$method to obtain the following factorization of g.

$$\begin{aligned} g(x) &= \frac{-x^2 - 4x + 45}{2x^3 - 5x^2 - 18x + 45} \\ &= \frac{-(x+9)(x-5)}{(x^2-9)(2x-5)} \\ &= \frac{-(x+9)(x-5)}{(x-3)(x+3)(2x-5)} \end{aligned}$$

We now can identify three zeros for the denominator of g that must be excluded from our domain.

The domain of g is $x \neq 3, -3, \frac{5}{2}$, or $(-\infty, -3) \cup (-3, \frac{5}{2}) \cup (\frac{5}{2}, 3) \cup (3, \infty)$.

Notice that in our previous example, the interval notation for each domain always contains one more interval than the number of values that are excluded from the domain. This will always be the case, since we can think of our excluded values as partitioning dividers of the $x-$axis, $(-\infty, \infty)$. In other words, three dividers partition the real number line into four intervals. The multiplicity of each excluded zero of the denominator $q(x)$ will also play a role in the nature of the graph of f, as we will see in a later section.

To find all possible $x-$intercepts for the graph of $f(x) = \frac{p(x)}{q(x)}$, we set the function equal to zero and solve for all possible x, keeping *only* those values that are also in our domain. Since $f(x)$ can only equal zero if its numerator is zero, this amounts to finding all roots of the polynomial p.

$$\begin{aligned} f(x) &= 0 \\ \frac{p(x)}{q(x)} &= 0 \\ \cancel{q(x)} \cdot \frac{p(x)}{\cancel{q(x)}} &= 0 \cdot q(x) \end{aligned}$$

$$p(x) = 0, \ q(x) \neq 0$$

Furthermore, referring to the results from the chapter on polynomials, we can again use the multiplicity of each zero of p to determine whether the corresponding $x-$intercept will represent a crossover or turnaround point, as in the following example.

Example 315. $f(x) = \dfrac{(x+3)^2(x-1)(x-4)}{(x-1)^2(x^2+2)}$

The numerator has three zeros ($x = -3, 1,$ and 4), but the corresponding graph only has two $x-$intercepts, since $x = 1$ is also a zero of the denominator, and therefore not in the domain of f. The $x-$intercept at $(-3, 0)$ is a turnaround point, since the multiplicity of the zero $x = -3$ is even. The $x-$intercept at $(4, 0)$ is a crossover point, since the multiplicity of the zero $x = 4$ is odd.

The graph of the reciprocal function $f(x) = \frac{1}{x}$ in our first example of this chapter also has two interesting characteristics, known as *asymptotes*. Asymptotes do not appear in the graph of a polynomial, but often show up when analyzing rational functions and more advanced functions such as exponentials and logarithms. An asymptote usually appears in the form of a line (horizontal, vertical, or slanted) that the graph of a function f approaches. The graph of $f(x) = \frac{1}{x}$ has a horizontal asymptote at $y = 0$, since the end behavior of the graph (as $x \to \pm\infty$) approaches zero, as well as a vertical asymptote at $x = 0$, since the local behavior of the graph near $x = 0$ (as $x \to 0$) tends towards $\pm\infty$.

Later on, we will outline the procedures for finding both horizontal and vertical asymptotes for a rational function, as well as the case where the graph of f has a slant (or oblique) asymptote. We will also see an example of the special case of a curvilinear asymptote, in which the graph of f approaches a nonlinear curve, as x approaches $\pm\infty$. Although all polynomials are, by definition, rational functions (with denominator $q(x) = 1$), the concept of an asymptote (horizontal, vertical, or otherwise) demonstrates a critical aspect that separates most rational functions from polynomials. We close this section with a few more examples of rational functions and their graphs. While each function shares a common domain ($x \neq 5$), the corresponding graphs exhibit some clear differences. Specifically, close attention should be paid to the existence, location, and nature of the graph of each function near the $y-$int and $x-$int(s), as well as the horizontal and vertical asymptotes.

Example 316.

$$f(x) = \frac{-2x+4}{x-5} = \frac{-2(x-2)}{x-5}$$

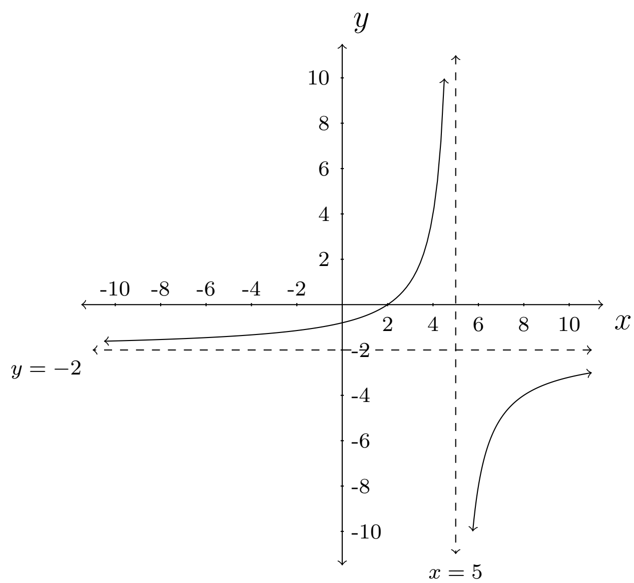
$y-$intercept at $(0, -\frac{4}{5})$

Domain: $x \neq 5$ or $(-\infty, 5) \cup (5, \infty)$

$x-$intercept at $(2, 0)$

Horizontal asymptote at $y = -2$

Vertical asymptote at $x = 5$



Example 317.

$$g(x) = \frac{x^2-4x+4}{x-5} = \frac{(x-2)^2}{x-5}$$

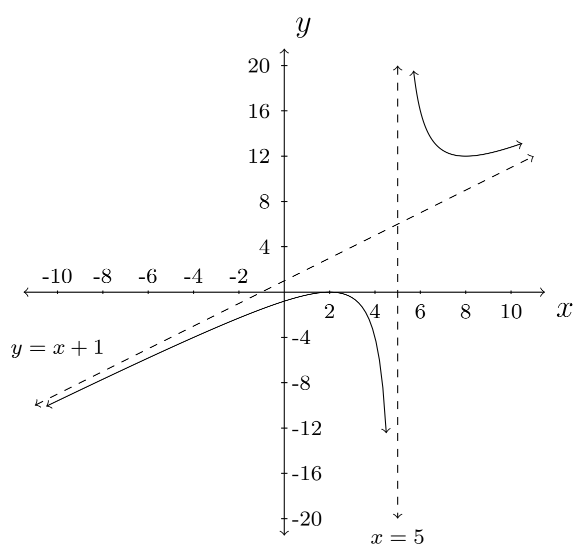
$y-$intercept at $(0, -\frac{4}{5})$

Domain: $x \neq 5$ or $(-\infty, 5) \cup (5, \infty)$

$x-$intercept at $(2, 0)$

Slant (oblique) asymptote at $y = x + 1$

Vertical asymptote at $x = 5$



Example 318.

$$h(x) = \frac{x^2+25}{x^2-10x+25} = \frac{x^2+25}{(x-5)^2}$$

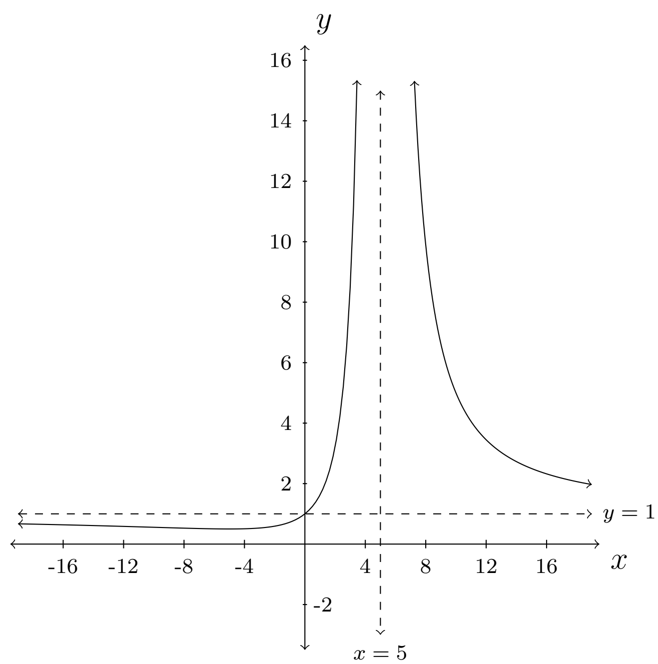
$y-$intercept at $(0, 1)$

Domain: $x \neq 5$ or $(-\infty, 5) \cup (5, \infty)$

No $x-$intercepts

Horizontal asymptote at $y = 1$

Vertical asymptote at $x = 5$



Example 319.

$$k(x) = \frac{x^3-5x^2}{10x-50} = \frac{x^2(x-5)}{10(x-5)}$$

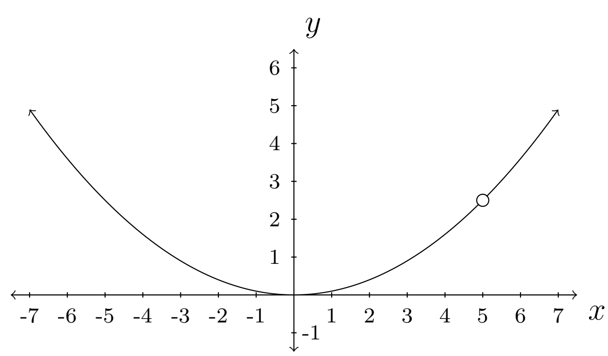
$y-$intercept at $(0, 0)$

Domain: $x \neq 5$ or $(-\infty, 5) \cup (5, \infty)$

$x-$intercept at $(0, 0)$

No horizontal or vertical asymptotes

Hole at $(5, \frac{5}{2})$



Sign Diagrams (L55)

Objective: Construct a sign diagram for a given rational function.

As with polynomial functions, throughout this chapter we will periodically reference the sign diagram of a rational function or expression, to both answer questions about particular functions and verify our work. As before, there is a reliance on factorization that is needed for construction of a sign diagram, since the roots of a given expression will be used as the dividers in the corresponding sign diagram.

We begin with an example for polynomial functions.

Example 320. Construct a sign diagram for the polynomial function $f(x) = x^3-5x^2+3x+9$. Use the fact that $f(-1) = 0$.

Since f is a polynomial that has four terms, we could first try to factor f by grouping. But we quickly see that this method will fail to yield a factorization.

$$\begin{aligned} f(x) &= x^3 - 5x^2 + 3x + 9 \\ &= x^2(x-5) + 3(x+3) \end{aligned}$$

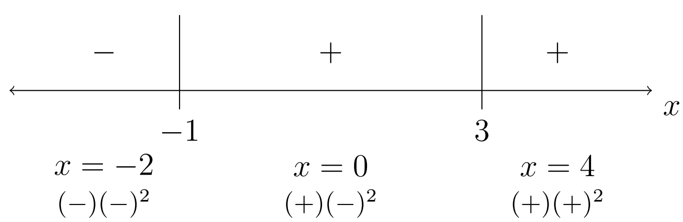
$$x - 5 \neq x - 3$$

Instead, if we use the fact that $f(-1) = 0$, we can employ polynomial division to factor f completely.

$$\begin{array}{r|l} & x^2 - 6x + 9 \\ \hline x+1 & x^3 - 5x^2 + 3x + 9 \\ & -x^3 - x^2 \\ \cline{2-2} & - 6x^2 + 3x \\ & 6x^2 + 6x \\ \cline{2-2} & 9x + 9 \\ & - 9x - 9 \\ \cline{2-2} & 0 \end{array}$$

$$\begin{aligned} f(x) &= x^3 - 5x^2 + 3x + 9 \\ &= (x+1)(x^2 - 6x + 9) \\ &= (x+1)(x-3)^2 \end{aligned}$$

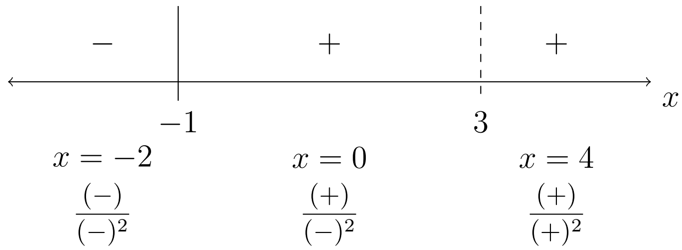
Roots of f: $x = -1, 3$



Since a rational function includes an expression in a denominator, the only additional consideration that we need to make to construct a sign diagram is to identify the roots of both the numerator *and* the denominator as dividers in our diagram. In other words, the dividers of our diagram for a rational function f will consist of all roots and all values not in the domain of f.

Example 321. Construct a sign diagram for the rational function $g(x) = \dfrac{x+1}{(x-3)^2}$.

Since $x = -1$ is a root of g, and $g(3)$ is undefined (the domain of g is $x \neq 3$), we will again place dividers at -1 and 3.



In Example 321, since $x = 3$ is not in the domain of g, we have used a dashed divider to signify this fundamental difference from Example 320. As we will see later, this results in different graphical implications. In other words, the graph of f behaves differently when $x = c$ is a root of f versus when it is excluded from the domain.

Example 322. Construct a sign diagram for the rational function $h(x) = \dfrac{x^3 - 5x^2 + 3x + 9}{x - 3}$.

Since the numerator of h matches f from Example 320, we can factor h as follows.

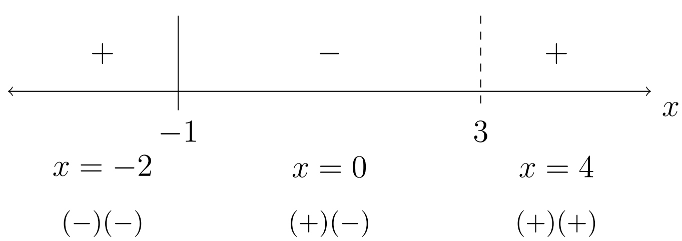
$$h(x) = \frac{(x+1)(x-3)^2}{x-3}$$

Although we might be tempted to cancel out the denominator of h completely, it still remains that $h(3)$ is undefined, i.e., the domain of h is $x \neq 3$. Hence, we will include a dashed divider in our sign diagram at $x = 3$.

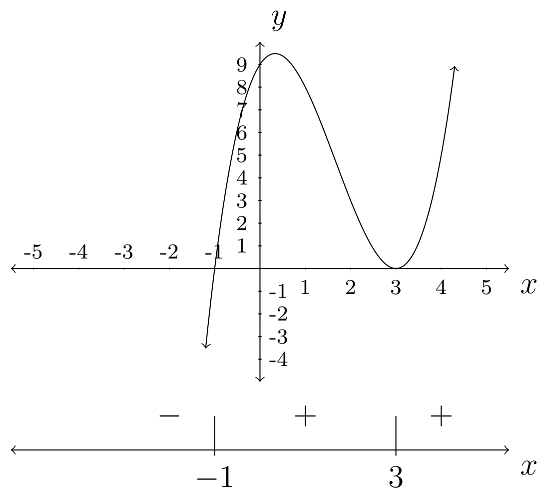
When determining the various signs for our diagram, however, we can consider working with the simplified expression

$$\frac{(x+1)(x-3)^{\cancel{2}}}{\cancel{x-3}} = (x+1)(x-3),$$

since none of our test values equal 3.

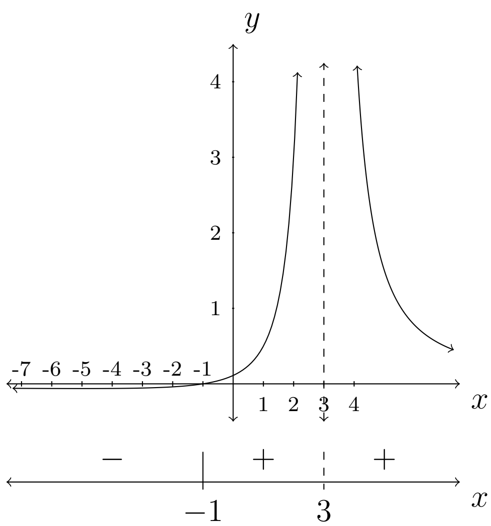


Although we are not yet ready to completely graph a rational function, let's look at the graphs of each of our last three examples, in order to see the differences at $x = 3$.

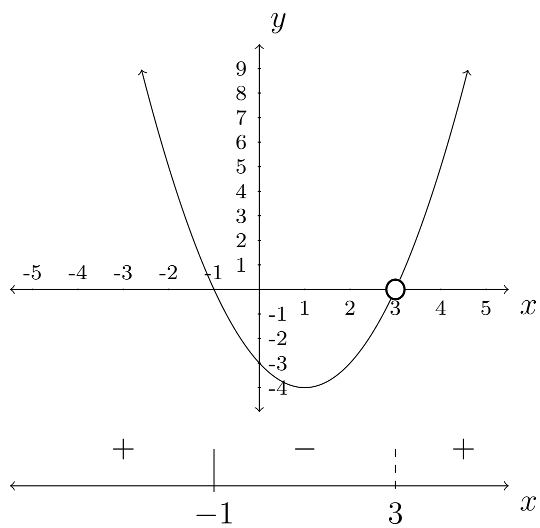


$f(x) = (x+1)(x-3)^2$

In the case of Example 320 above, there should be no real surprises, since we are given a cubic polynomial with positive leading coefficient. The multiplicity of $x = -1$ being odd produces a sign change in our diagram, which results in a "crossover point" at our corresponding $x-$intercept. Alternatively, the even multiplicity of $x = 3$ creates a "turnaround point" (or bounce off) at the corresponding $x-$intercept.



$g(x) = \dfrac{x+1}{(x-3)^2}$



$h(x) = \dfrac{(x+1)(x-3)^2}{x-3}$

In Examples 321 and 322 above, we see the same $x-$intercept at $x = -1$ as in Example 320. The fundamental difference between polynomials and rational functions centers around what happens at $x = 3$. Unlike our first graph, both rational functions exhibit a break in the graph at this value of x, since it is not in the domain of either function.

Still, the difference in the expressions for both g and h results in two distinctly different breaks in each of our graphs (a vertical asymptote in the first graph and a hole in the second graph). We will more closely examine these differences in a later section. For the purposes of this section, we simply wish to stress the importance of identifying the corresponding divider in each of our sign diagrams.

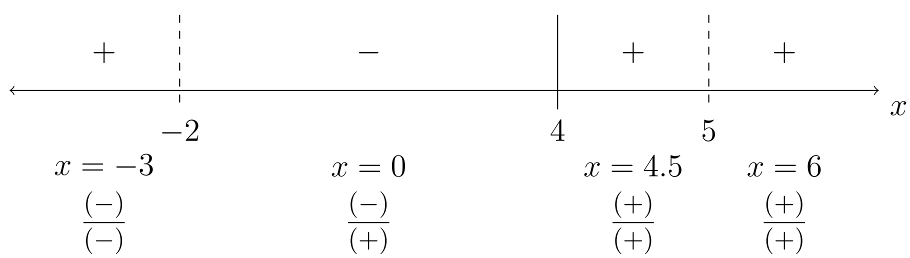
Example 323. Construct a sign diagram for the rational function $f(x) = \dfrac{x^2 - 9x + 20}{x^2 - 3x - 10}$.

A factorization of f gives us the following.

$$f(x) = \frac{(x-4)(x-5)}{(x+2)(x-5)}$$

Hence, we need dividers for $x = -2, 4$, and 5. Since our domain is $x \neq -2, 5$ we will use dashed dividers for these values, and a solid divider for the $x-$intercept at $x = 4$.

As in Example 322, we will use the simplified expression $\dfrac{x-4}{x+2}$ for each of our test values, since we are not testing $x = 5$.



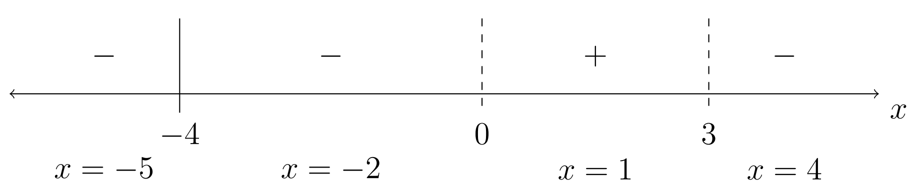
We leave it as an exercise to the reader to graph

$$f(x) = \frac{x^2 - 9x + 20}{x^2 - 3x - 10} = \frac{(x-4)(x-5)}{(x+2)(x-5)}$$

using Desmos, in order to see the differences near the two values excluded from our domain, $x = -2$ and $x = 5$.

We conclude this section with one final example

Example 324. Identify a rational function f having the following sign diagram.



We know that at each divider value c, our function must have a corresponding factor of $x - c$. Furthermore, solid dividers will see our factor in the numerator, whereas dashed dividers signify that c is not in our domain, and so our factor must lie in the denominator. This gives us the following candidate for f.

$$\frac{x+4}{x(x-3)}$$

But also, the lack of a sign change at $x = -4$ (negative to negative) tells us that the corresponding $x-$intercept will be a turnaround point. This further tells us that the multiplicity at $x = -4$ must be even. So we refine our candidate to the following.

$$\frac{(x+4)^2}{x(x-3)}$$

Since our signs change at $x = 0$ (negative to positive) and $x = 3$ (positive to negative), we will keep an odd multiplicity of 1 for each of their respective factors in the denominator.

A simple check of one of our test values will tell us whether our answer is correct.

When $x = -5$, we get $\dfrac{(-)^2}{(-)(-)} = +$, which is not what we want. This suggests that each of our signs will be the opposite of what we are looking for. By multiplying our candidate by a negative, we obtain our final answer.

$$f(x) = \frac{-(x+4)^2}{x(x-3)}$$

We leave it as an exercise for the reader to check that our function does indeed match the given diagram.

End Behavior

Horizontal Asymptotes (L56)

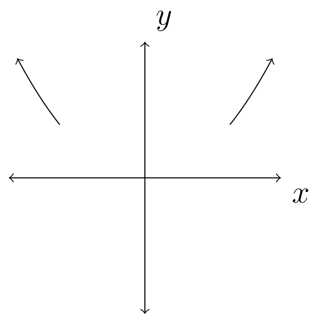
Objective: Identify a horizontal asymptote in the graph of a rational function.

Next, we will look at the end (or long run) behavior of the graph of a rational function f, as $x \to \pm\infty$. For clarity, we will first state the main result of this subsection.

Let $f(x) = \dfrac{p(x)}{q(x)}$ be a rational function with leading terms $a_n x^n$ and $b_m x^m$ of $p(x)$ and $q(x)$, respectively.

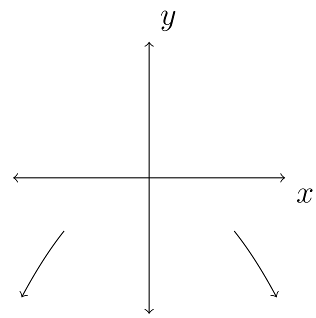
- If $n = m$, the graph of f will have a horizontal asymptote at $y = \dfrac{a_n}{b_m}$.
- If $n < m$, the graph of f will have a horizontal asymptote at $y = 0$.
- If $n > m$, the graph of f will not have a horizontal asymptote.

Since any polynomial is, by definition, also a rational function, we will begin by including the possibilities that $f(x) \to \infty$ or $f(x) \to -\infty$ for either the left (as $x \to -\infty$) or right (as $x \to \infty$) end behavior of the graph of a rational function f.



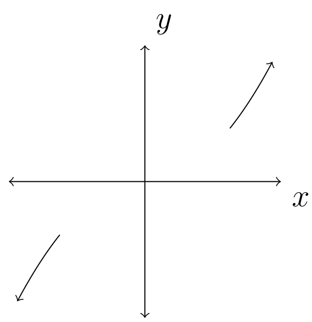
As $x \to \infty, \; f(x) \to \infty$

As $x \to -\infty, \; f(x) \to \infty$



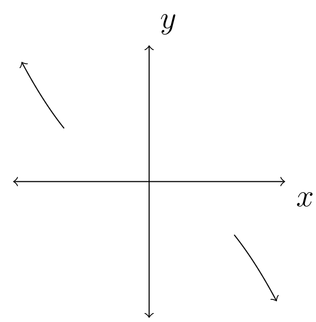
As $x \to \infty, \; f(x) \to -\infty$

As $x \to -\infty, \; f(x) \to -\infty$



As $x \to \infty, \; f(x) \to \infty$

As $x \to -\infty, \; f(x) \to -\infty$



As $x \to \infty, \; f(x) \to -\infty$

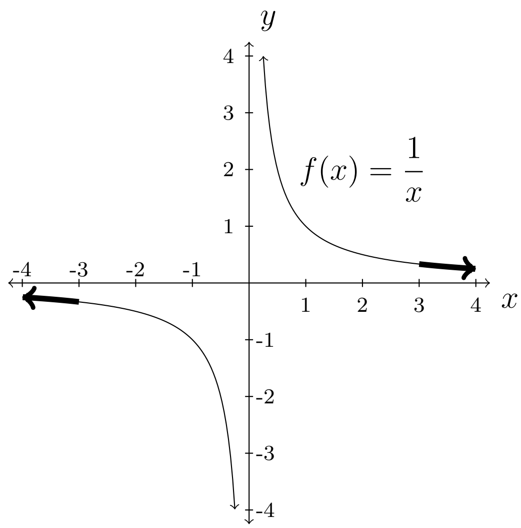
As $x \to -\infty, \; f(x) \to \infty$

Recall that we used two aspects of a polynomial to identify the end behavior of its graph:

1. the parity of the degree (even or odd), and
2. the sign of the leading coefficient (positive or negative).

As with polynomials, we will use the degree and leading coefficient of both the numerator and denominator of a rational function f, to identify the end behavior of its graph.

To start, let us again consider the graph of the reciprocal function $f(x) = \dfrac{1}{x}$.



This example presents us with the first instance in which a graph does not tend towards either ∞ or $-\infty$, but instead "levels off" as the values of x grow in either the positive (right) or negative (left) direction.

$$\text{As } x \to \infty,\ f(x) \to 0^+.$$

$$\text{As } x \to -\infty,\ f(x) \to 0^-.$$

Here, we use a $+$ or $-$ in the exponent to further describe how the tails of the graph approach 0, either from *above* $(+)$ or from *below* $(-)$. These identifiers can just as easily be omitted entirely, but provide a bit more insight into the graph of the function f. The tails of the graph are thickened for additional emphasis of this concept.

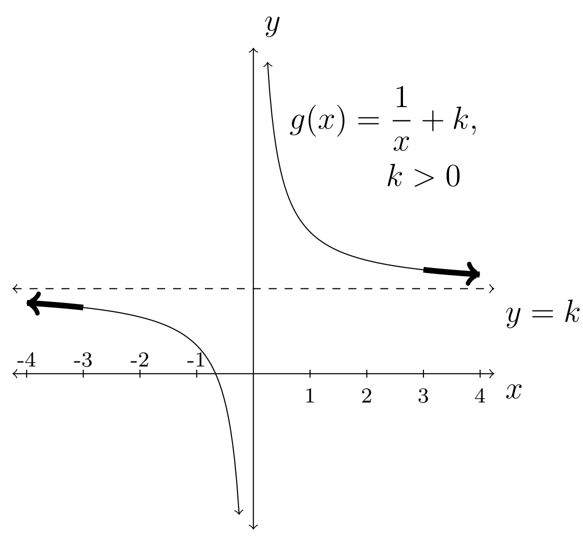
In fact, for any real number k, we can transform the graph above, by simply adding k to the function, to produce a new rational function whose graph levels off at k. The resulting graph represents a vertical shift of the graph of $\frac{1}{x}$ by k units. The shift is up when $k > 0$ and down when $k < 0$.

Example 325.

$$\begin{aligned} g(x) &= f(x) + k \\ &= \frac{1}{x} + k \cdot \frac{x}{x} \\ &= \frac{kx+1}{x} \end{aligned}$$

$$\text{As } x \to \infty,\ g(x) \to k^+.$$

$$\text{As } x \to -\infty,\ g(x) \to k^-.$$



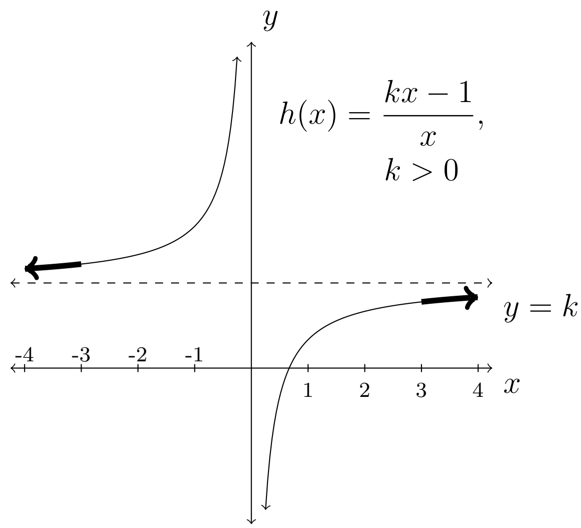
Furthermore, if we first replace x by $-x$ in f, and then add k, this will reflect the graph of f about the $y-$axis and shift it vertically by k units, producing a slightly different end behavior as in our previous two examples.

Example 326.

$$\begin{aligned} h(x) &= f(-x) + k \\ &= \frac{1}{-x} + k \\ &= -\frac{1}{x} + k \cdot \frac{x}{x} \\ &= \frac{kx - 1}{x} \end{aligned}$$

As $x \to \infty,\ h(x) \to k^-$.

As $x \to -\infty,\ h(x) \to k^+$.



One of the most important takeaways from each of the above examples is that, unlike a polynomial, a rational function can possibly "level off" along a horizontal line $y = k$, for any real number k, as x approaches either ∞ or $-\infty$. In this case, we say that the corresponding graph has a *horizontal asymptote* at $y = k$.

Notice that in each of the previous two examples, when adding $k \neq 0$, we have increased the degree of the numerator (from 0 to 1), which matches the degree of the denominator.

$$g(x) = \frac{kx^1 + 1}{1x^1} \qquad\qquad h(x) = \frac{kx^1 - 1}{1x^1}$$

In fact, whenever the numerator and denominator of a rational function f have the *same* degree, the corresponding graph will have a horizontal asymptote along the line $y = \dfrac{a_n}{b_m}$, where a_n and b_m represent the leading coefficients of the numerator and denominator of f, respectively.

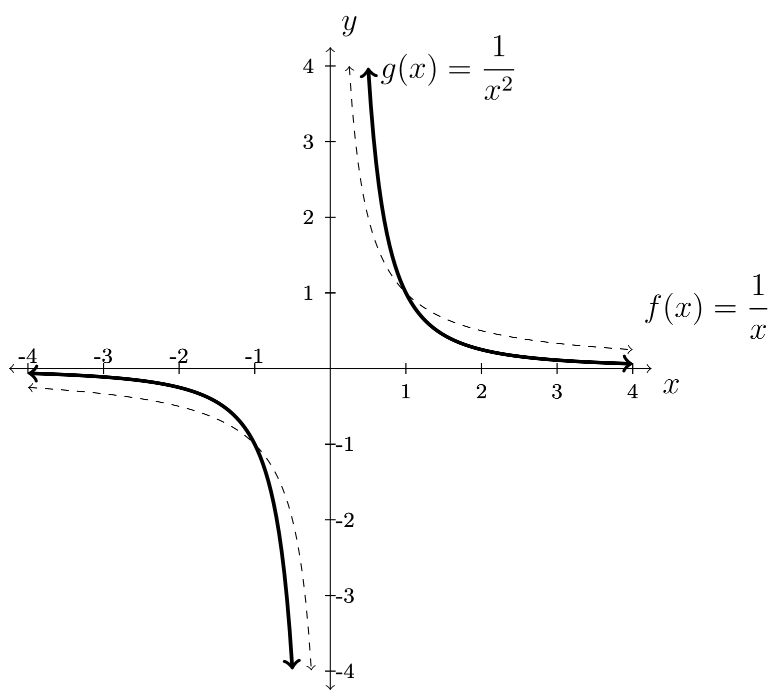
Stated more formally:

> Let $f(x) = \dfrac{p(x)}{q(x)}$, with leading terms $a_n x^n$ and $b_m x^m$ of $p(x)$ and $q(x)$, respectively. If $n = m$, the graph of f will have a horizontal asymptote at $y = \dfrac{a_n}{b_m}$. In other words, when the degrees of p and q are equal, as $x \to \pm\infty$, $f(x) \to \dfrac{a_n}{b_m}$.

We see this at work in Example 316, where the graph of $f(x) = \dfrac{-2x+4}{x-5}$ has a horizontal asymptote at $y = \frac{-2}{1} = -2$, and again in Example 318, where the graph of $h(x) = \dfrac{x^2+25}{x^2-10x+25}$ has a horizontal asymptote at $y = \frac{1}{1} = 1$. In each of these cases, we can now easily confirm that, for example, as $x \to \pm\infty$, $h(x) \to 1$. To determine the precise nature of the tails or ends of the graph ($h(x) \to 1^+$ versus $h(x) \to 1^-$), however, requires further analysis. This is aided by polynomial division and often a sign diagram, which we will see in a subsequent section.

We have now seen that the graph of a rational function will "level off" along a horizontal line $y = \dfrac{a_n}{b_m}$ when n and m are equal. What remains is to determine the end behavior when $n \neq m$. This gives us two additional cases to consider: 1) $n < m$ and 2) $n > m$.

In the case when $n < m$, we may again look to $f(x) = \dfrac{1}{x}$, which has a horizontal asymptote along the $x-$axis, i.e., the line $y = 0$. In fact, this will generally be the case when $n < m$, since we can think of the denominator, $q(x)$, as growing much faster than the numerator, $p(x)$, whenever x approaches either ∞ or $-\infty$. If we look at the difference in degrees of p and q, we can further see that the nature with which our graph approaches the $x-$axis changes. Specifically, When $m - n$ is larger (2,3,...), the graph will approach the $x-$axis more quickly. We will use $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x^2}$ to illustrate this point. Recall that both f and g have numerators which have a degree of zero.



As x gets large, the act of squaring causes the denominator to grow more quickly, which in turn, makes the values of $g(x) = \dfrac{1}{x^2}$ approach zero more quickly than the values of $f(x) = \dfrac{1}{x}$. Regardless, in both cases, the horizontal asymptote is the same:

Let $f(x) = \dfrac{p(x)}{q(x)}$, with degrees n and m of $p(x)$ and $q(x)$, respectively. If $n < m$, the graph of f will have a horizontal asymptote at $y = 0$. In other words, as $x \to \pm\infty$, $f(x) \to 0$.

Revisiting Example 314, the graph of $g(x) = \dfrac{-x^2 - 4x + 45}{2x^3 - 5x^2 - 18x + 45}$ will again have a horizontal asymptote at $y = 0$, since $m > n$. In this case, we can further reason the nature of the graph of g (whether it approaches 0 above or below the $x-$axis) by looking at the leading terms for both the numerator and denominator of g.

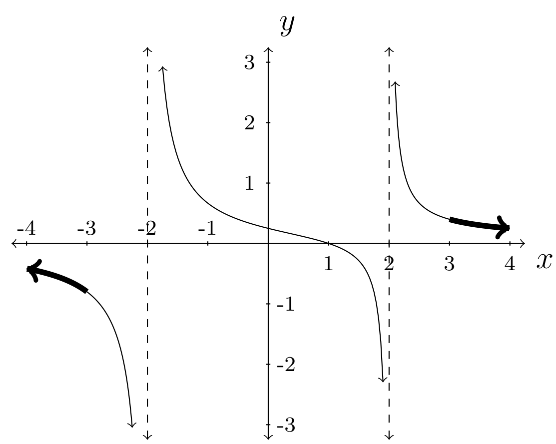
$$g(x) = \frac{-x^2 - 4x + 45}{2x^3 - 5x^2 - 18x + 45}$$

Since the numerator of g has a negative leading coefficient and an even degree, as $x \to \infty$, the numerator of g will approach $-\infty$. The denominator, however, has a positive leading coefficient and an odd degree. Consequently, as $x \to \infty$, the denominator of g will approach ∞. But since $m > n$ and the numerator and denominator differ in sign, as $x \to \infty$, $g(x) \to 0^-$. All of this means that the graph of g will approach the $x-$axis from *below* on the right.

Alternatively, as $x \to -\infty$, we can see that the graph of g will approach the $x-$axis from *above* on the left, since both the numerator and denominator of g will approach $-\infty$.

Example 327. In this example, we see the graph of

$$f(x) = \frac{x-1}{x^2-4}.$$



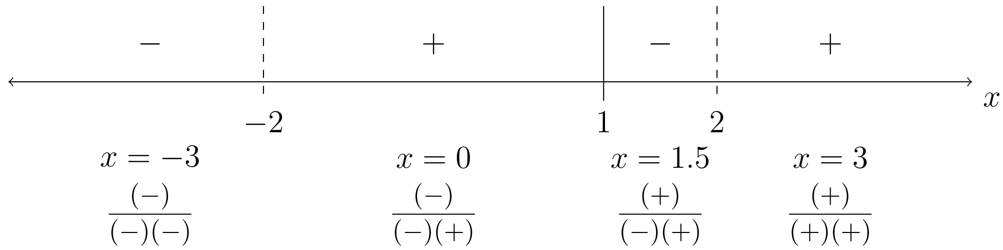
Since the degree of the denominator is greater than the degree of the numerator, we see that the graph of f (again) levels off along the $x-$axis, i.e., the line $y = 0$.

We can also readily determine whether the graph approaches the $x-$axis above or below without too much difficulty.

$$\text{As } x \to \infty,\ f(x) \to 0^+.$$
$$\text{As } x \to -\infty,\ f(x) \to 0^-.$$

In the case when $m > n$, a sign diagram can also verify whether $f(x) \to 0$ from above or below, and we have included one here. The signs on the ends of the diagram will correspond to the nature of the tails of the graph above.

$$f(x) = \frac{x-1}{x^2-4} = \frac{x-1}{(x-2)(x+2)}$$

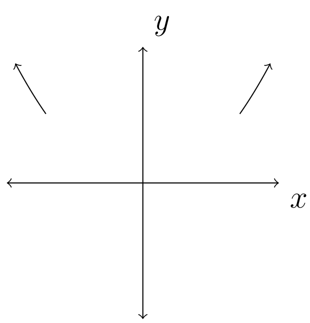


Our last case to consider for a rational function $f(x)$, is when the numerator has a greater degree than the denominator, $n > m$. This case includes all polynomial functions, which represent rational functions, in which the denominator equals the constant (degree-0) function $q(x) = 1$. In this case, as $x \to \pm\infty$, we can think of the numerator as growing faster than the denominator. The result is a graph that has no horizontal asymptote.

> Let $f(x) = \dfrac{p(x)}{q(x)}$, with degrees n and m of $p(x)$ and $q(x)$, respectively. If $n > m$, the graph of f will have no horizontal asymptote.

The possibilities for the tails of the graph of f ($f(x) \to \infty$ or $f(x) \to -\infty$) are, again, determined by the leading terms for both the numerator and denominator of f.

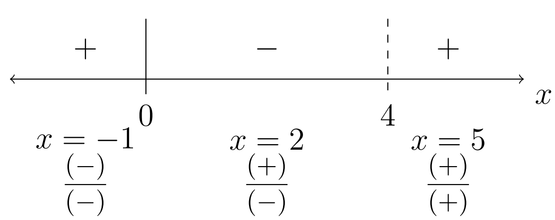
Example 328. Consider the function $f(x) = \dfrac{x^3}{x-4}$. The numerator has a degree of 3, and the denominator has a degree of 1. Hence, the graph of f has no horizontal asymptotes. As $x \to \infty$, we can further see that both the numerator and denominator will approach $+\infty$. Hence, we can conclude that $f(x) \to \infty$. Alternatively, as $x \to -\infty$, we see that both the numerator and denominator will approach $-\infty$, and so $f(x) \to \infty$. This tells us that the tails of f will both point upwards. As in the previous example, a sign diagram confirms this.



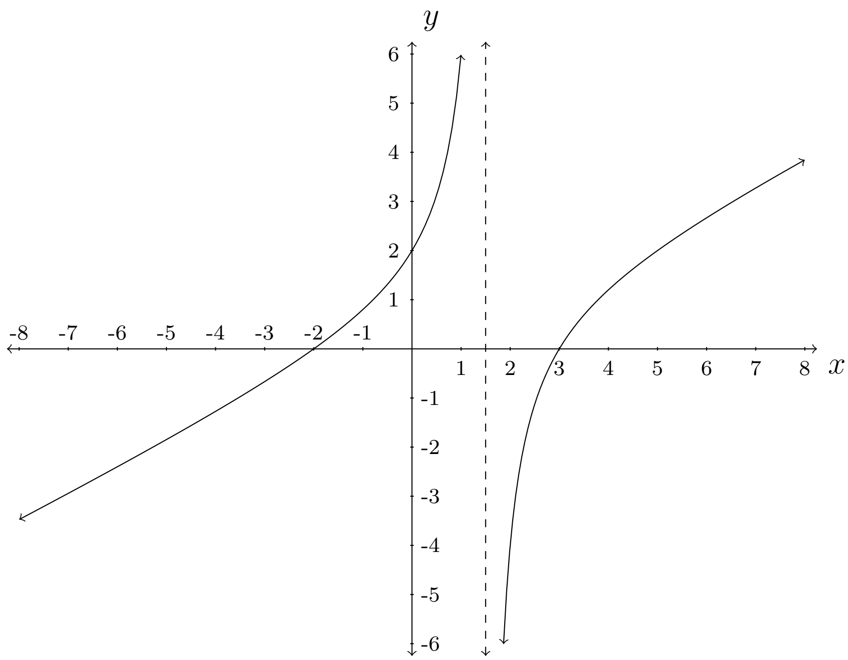
As $x \to \infty$, $f(x) \to \infty$

As $x \to -\infty$, $f(x) \to \infty$

$$f(x) = \frac{x^3}{x-4}$$



Example 329. Let $g(x) = \dfrac{x^2 - x - 6}{2x - 3}$. Again, the graph of g will not have a horizontal asymptote, since the degree of the numerator is greater than the degree of the denominator. We include the graph of g below.



Examples 317 and 319 present us with two additional rational functions whose graphs do not include a horizontal asymptote. In each case, as $x \to \infty$, the $y-$coordinate also approaches ∞. Still, the nature in which the $y-$coordinates grow as x grows is distinctly different for each function and its graph. The same can be said for our last two examples. This has to do with the difference between the degrees of the numerator and denominator, and we will address this next.

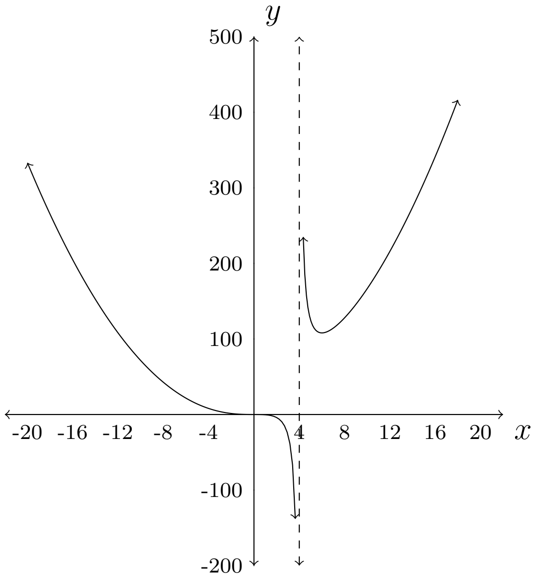
Slant Asymptotes (L57)

Objective: Identify a slant or curvilinear asymptote in the graph of a rational function.

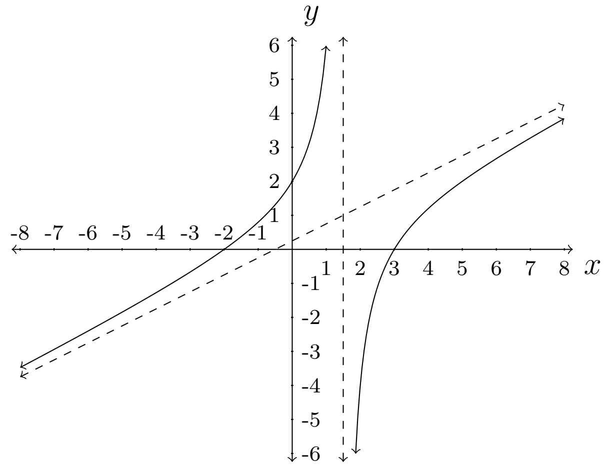
For a rational function

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0},$$

when $n > m$, we know that the graph of f will have no horizontal asymptotes. Depending upon the difference between n and m, however, there is more to discover about the nature of the graph of f, as $x \to \pm\infty$. For example, below are the graphs of Examples 328 and 329.



$$f(x) = \frac{x^3}{x-4}$$



$$g(x) = \frac{x^2 - x - 6}{2x - 3}$$

In the case of $g(x) = \dfrac{x^2 - x - 6}{2x - 3}$, we see that as $x \to \pm\infty$, the graph of g actually approaches a linear asymptote. Whereas horizontal asymptotes are horizontal lines, having a slope of zero, this new type of linear asymptote has a non-zero slope and is consequently *slanted*. Hence, we say that the graph of g contains a *slant* or *oblique asymptote*.

On the other hand, the graph of $f(x) = \dfrac{x^3}{x-4}$ does not appear to contain a slant asymptote. In fact, as $x \to \pm\infty$, the graph of f resembles a parabola. In cases such as these, we could say that the graph of f contains a *curvilinear asymptote*. In other words, the graph of f approaches some identifiable non-linear curve, as x approaches $\pm\infty$.

The central idea behind slant and curvilinear asymptotes is the same, and only requires an understanding of polynomial division. Nevertheless, we will focus almost entirely on slant asymptotes in this section, leaving the topic of curvilinear asymptotes for our last example.

So what causes the graph of a rational function to have such asymptotes? We ponder this question, keeping in mind that we have already imposed the requirement that the degree of the numerator be greater than the denominator, $n > m$. Simply stated, it is the difference between the degrees, $n - m$, which will determine whether the corresponding graph possesses a slant or curvilinear asymptote.

Let's look more closely at $g(x) = \dfrac{x^2 - x - 6}{2x - 3}$. In this case, the degree of the numerator is equal one more than the degree of the denominator, $n = m + 1$ (or $n - m = 1$). This is precisely the case in which the corresponding graph will always contain a slant asymptote!

Let $f(x) = \dfrac{p(x)}{q(x)}$, with degrees n and m of $p(x)$ and $q(x)$, respectively. If $n = m + 1$, the graph of f will have a slant (or oblique) asymptote. In other words, when $n = m + 1$, as $x \to \pm\infty$, the graph of f will approach some line $y = cx + d$, where $c \neq 0$.

Next comes the question of how to find the equation of our slant asymptote. The answer comes to us by employing polynomial division. Recall that when we divide two polynomials, we have the following result.

$$\frac{D(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)} \qquad \text{or} \qquad \frac{\text{Dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

Here, we use the labels d, D, q, r to represent the *divisor, Dividend, quotient,* and *remainder*, respectively. Relabeling in terms of our rational function, $f(x) = \dfrac{p(x)}{q(x)}$, we have the following.

$$\frac{p(x)}{q(x)} = h(x) + \frac{r(x)}{q(x)}$$

But in the case where the degree of p is one more than the degree of q, the resulting quotient function, $h(x)$, will be the desired linear equation for our slant asymptote.

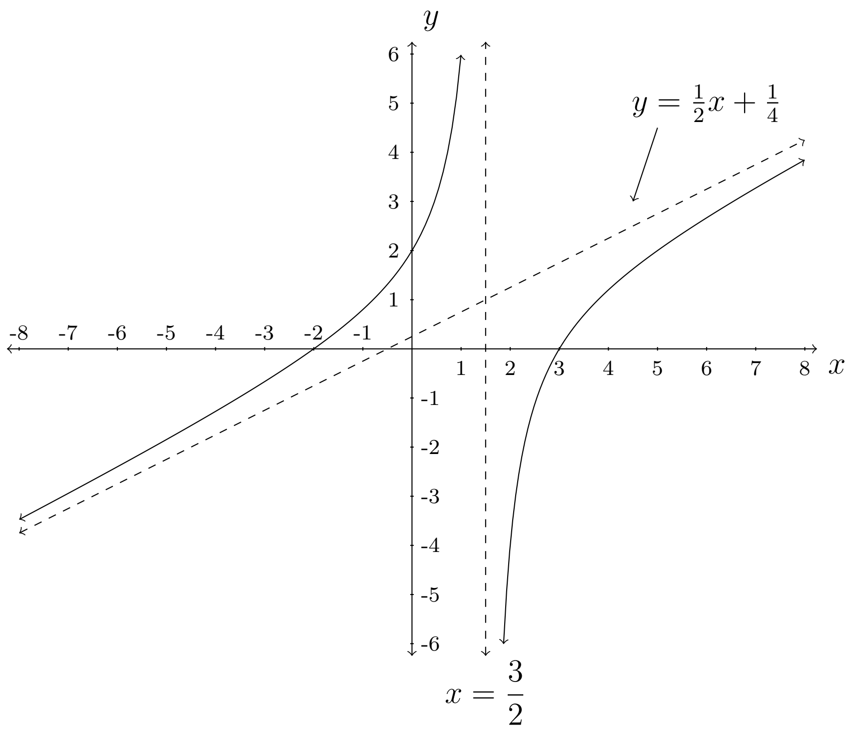
$$\frac{p(x)}{q(x)} = \underbrace{cx + d}_{\substack{\text{slant}\\ \text{asymptote}}} + \frac{r(x)}{q(x)}$$

Example 330. Find the equation of the slant asymptote in the graph of $g(x) = \dfrac{x^2 - x - 6}{2x - 3}$.

$$\begin{array}{r|rrr}
 & \frac{1}{2}x & +\frac{1}{4} & \\
\hline
2x-3 & x^2 & -x & -6 \\
 & -x^2 & +\frac{3}{2}x & \\
\hline
 & & \frac{1}{2}x & -6 \\
 & & -\frac{1}{2}x & +\frac{3}{4} \\
\hline
 & & & -\frac{21}{4}
\end{array}$$

Hence, $\dfrac{x^2 - x - 6}{2x - 3} = \frac{1}{2}x + \frac{1}{4} + \dfrac{-\frac{21}{4}}{2x - 3}$.

So, the graph of g has a slant asymptote at $y = \frac{1}{2}x + \frac{1}{4}$.



$$g(x) = \frac{x^2 - x - 6}{2x - 3} \;=\; \tfrac{1}{2}x + \tfrac{1}{4} + \frac{-\frac{21}{4}}{2x - 3}$$

It is worth noting in this last example, that the trailing expression $\dfrac{-\frac{21}{4}}{2x-3}$, involving the remainder in our polynomial division, can be extremely helpful in determine whether or not the graph of g sits above or below the slant asymptote, as $x \to \pm\infty$. In this case, when x is a large positive value, our trailing expression will be negative. Hence, as $x \to \infty$ the values of $g(x)$ on the right-side tail of our graph will lie slightly *below* our slant asymptote, since

$$\frac{x^2 - x - 6}{2x - 3} = \tfrac{1}{2}x + \tfrac{1}{4} + (\text{a small negative value}).$$

On the other hand, as $x \to -\infty$, the trailing expression will be positive, and so

$$\frac{x^2 - x - 6}{2x - 3} = \tfrac{1}{2}x + \tfrac{1}{4} + (\text{a small positive value}).$$

Hence, the left-side tail of the graph of g will lie slightly *above* our slant asymptote.

Example 331. Find the equation of the slant asymptote in the graph of $f(x) = \dfrac{-2x^3 + x^2 - 2x + 3}{x^2 + 1}$.

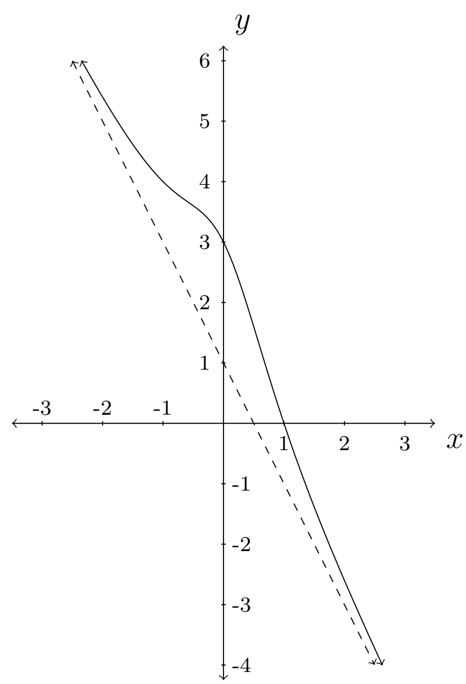
In this example, again, we know that the graph of f will include a slant asymptote, since the degree of the numerator is one more than the degree of the denominator, $3 = 2 + 1$.

$$
\begin{array}{r|l}
 & -2x + 1 \\
\hline
x^2 + 1 & -2x^3 + x^2 - 2x + 3 \\
 & \underline{2x^3 \qquad + 2x} \\
 & \qquad x^2 \qquad + 3 \\
 & \qquad \underline{-x^2 \qquad - 1} \\
 & \qquad\qquad\qquad 2
\end{array}
$$

Hence, $\dfrac{-2x^3 + x^2 - 2x + 3}{x^2 + 1} = -2x + 1 + \dfrac{2}{x^2 + 1}$.

So, the graph of f has a slant asymptote at $y = -2x + 1$.

Notice that the trailing expression above, $\dfrac{2}{x^2 + 1}$ is *positive for all* x. Hence, as $x \to \pm\infty$, the graph of f will lie *above* the slant asymptote $y = -2x + 1$. We include the graph of f below for completeness.



$$f(x) = \frac{-2x^3 + x^2 - 2x + 3}{x^2 + 1} = -2x + 1 + \frac{2}{x^2 + 1}$$

Example 332. Construct a rational function $f(x) = \dfrac{p(x)}{q(x)}$ that has a domain of $x \neq 2$ and a slant asymptote along the line $y = x - 3$.

In this example, we can work backwards from what we just learned to construct the desired rational function f. We will start by filling in the necessary information to the right hand side of the expression below.

$$\frac{p(x)}{q(x)} = \underbrace{cx + d}_{\substack{\text{slant}\\\text{asymptote}}} + \frac{r(x)}{q(x)}$$

Since $x \neq 2$, we may use $q(x) = x - 2$ for our denominator. Similarly, we can replace $cx + d$ with our given asymptote, $x - 3$. Since there are no other restrictions for our function, we are free to choose any polynomial expression for $r(x)$. Since $q(x) = x - 2$, $r(x)$ will be a constant function. For this example, we will use the identity polynomial, $r(x) = 1$.

$$\frac{p(x)}{x-2} = x - 3 + \frac{1}{x-2}$$

All that remains is to obtain a common denominator on the right-hand side, in order to identify the numerator, $p(x)$.

$$\begin{aligned} f(x) &= \frac{x-3}{1} \cdot \frac{x-2}{x-2} + \frac{1}{x-2} \\ &= \frac{x^2 - 5x + 6}{x-2} + \frac{1}{x-2} \\ &= \frac{x^2 - 5x + 7}{x-2} \end{aligned}$$

Our desired polynomial is $f(x) = \dfrac{x^2 - 5x + 7}{x-2}$.

Notice that our answer fits the criteria for a slant asymptote, since $n = m + 1$. Also, we can easily identify many other functions that satisfy this particular problem by changing the expression for $r(x)$ to another constant.

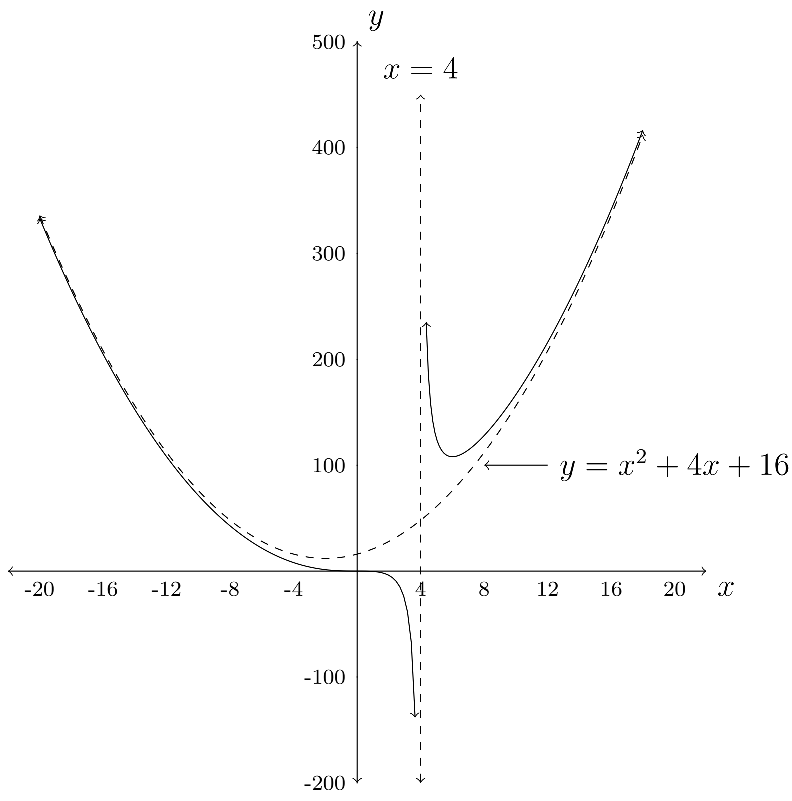
We leave it as an exercise to the reader to determine whether there might be other possibilities for our denominator $q(x)$.

Example 333. In the case of Example 328, we see that $f(x) = \dfrac{x^3}{x-4}$ does not satisfy our criteria for the existence of a slant asymptote, since $n - m \neq 1$. This should now make perfect sense, however, since polynomial division will not produce a linear quotient expression, but rather a *quadratic*, as shown below.

$$
\begin{array}{r l}
 & \quad\;\; x^2 \;\; + 4x + 16 \\ \cline{2-2}
x - 4 \big) & \; x^3 \\
 & \underline{-\,x^3 + 4x^2} \\
 & \qquad 4x^2 \\
 & \qquad \underline{-\,4x^2 + 16x} \\
 & \qquad\qquad\quad 16x \\
 & \qquad\qquad\quad \underline{-\,16x + 64} \\
 & \qquad\qquad\qquad\quad\;\; 64
\end{array}
$$

$$
\begin{aligned}
f(x) &= \frac{x^3}{x-4} \\
&= \underbrace{x^2 + 4x + 16}_{\text{curvilinear asymptote}} + \frac{64}{x-4}
\end{aligned}
$$

Consequently, as $x \to \pm\infty$, we can indeed see that the graph of f approaches the *curvilinear asymptote* $y = x^2 + 4x + 16$.



$$f(x) = \frac{x^3}{x-4} \;=\; x^2 + 4x + 16 \;+\; \frac{64}{x-4}$$

In general, for a given rational function, $f(x) = \dfrac{p(x)}{q(x)}$, with degrees n and m of $p(x)$ and $q(x)$, respectively, it is the difference in degrees, $n - m$, that will dictate the nature of the associated end behavior asymptote for the graph of f. Specifically, if $n - m = 1$, the tails of the graph will resemble a linear graph (having non-zero slope), if $n - m = 2$, the tails will resemble a parabola, if $n - m = 3$, the tails will resemble a cubic graph, and so on. We close the section on end behavior with a table that summarizes the key takeaways.

Applying polynomial division:

$$f(x) = \frac{p(x)}{q(x)} = h(x) + \frac{r(x)}{q(x)}$$

$$n = \text{degree of } p$$
$$m = \text{degree of } q$$

$n - m$	End Behavior of the Graph of f
< 0	Horizontal Asymptote at $y = 0$*
0	Horizontal Asymptote at $y = a_n/b_m$*
1	Slant Asymptote at $y = h(x)$
> 1	Curvilinear Asymptote at $y = h(x)$

*Note that in this case, our asymptote will actually still equal $h(x)$.

Local Behavior

Recall that the domain of a rational function $f(x) = \dfrac{p(x)}{q(x)}$ is the set of all x such that $q(x) \neq 0$. We call those values not in the domain of f *discontinuities*, since they will correspond to a break in the graph of f. In this section we will explore what happens to the graph of a rational function near a given discontinuity.

This boils down to two cases:

1. vertical asymptotes, known as *infinite discontinuities*, and
2. holes, known as *removable discontinuities*.

In order to discuss either of the aforementioned cases, we will need to consider a “simplified expression” for a rational function f.

For example, in Example 322, we looked at the function $h(x) = \dfrac{(x+1)(x-3)^2}{x-3}$.

As long as x does not equal 3, we can think of $(x+1)(x-3)$ as a simplified expression for h, since the two expressions will always produce the same value for $x \neq 3$.

For example, when $x = 5$, we get

$$h(5) = \frac{(5+1)(5-3)^2}{5-3} = \frac{(5+1)(5-3)^{\cancel{2}}}{\cancel{5-3}} = (5+1)(5-3) = 6 \cdot 2 = 12$$

Similarly, if we look at Example 323, a simplified expression for

$$f(x) = \frac{(x-4)(x-5)}{(x+2)(x-5)}$$

would be $\frac{x-4}{x+2}$.

It is important to note that one should never assume that a simplified expression for a rational function f will equal f for *all* values of x. This is evident in each of our examples, since the domain of our simplified expression does not equal the domain of the given function. Nevertheless, the two expressions are closely related, and are needed in order to identify the various discontinuities in the graph of a given rational function.

Vertical Asymptotes (L58)

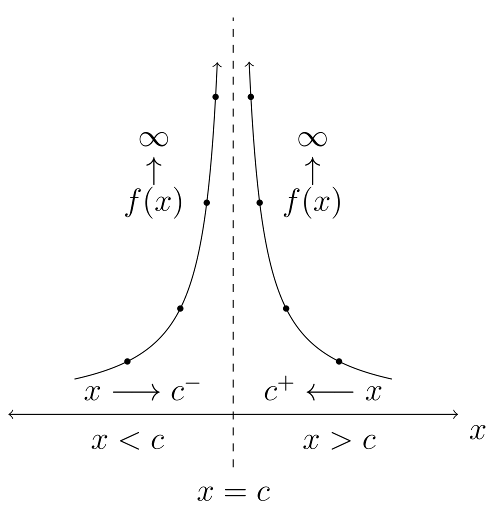
Objective: Identify one or more vertical asymptotes in the graph of a rational function.

The central idea around a vertical asymptote, say $x = c$, is that as x approaches the value of c, either from the left or the right, the values for the corresponding function $f(x)$ will approach either ∞ or $-\infty$.

Approaching from the right: As $x \to c^+$, $f(x) \to \pm\infty$.

Approaching from the left: As $x \to c^-$, $f(x) \to \pm\infty$.

We should be clear here, in that when we say x approaches c *from the right,* what is meant is that we are evaluating the function at values of x that are getting arbitrarily close go c, but are all *greater* than c, i.e., $x > c$. This is precisely why we can write $x \to c^+$ in the statement above. The + in the exponent signifies that $x > c$. The same can be said for when x approaches c from the left. The following graph further illustrates this point.



Our previous graph shows that as x approaches c from either direction, the values for $f(x)$ approach $+\infty$. If, instead, we reflected the right-hand side of the graph across the $x-$axis, we would say that as $x \to c^+$, $f(x) \to -\infty$, since the right-hand side would now point downwards.

Up until this point, we have seen several examples of graphs of rational functions that contain vertical asymptotes. We are now ready to formally state the condition for the existence of a vertical asymptote.

> Let $f(x)$ be a rational function and let $g(x)$ represent the simplified expression for f. If $x = c$ is not in the domain of *both* f and g, then the graph of f will have a vertical asymptote at $x = c$.

In the case where f cannot be simplified, any value not in the domain will correspond to a vertical asymptote in the graph. Examples 316 and 318 are a good place to start. We revisit each of them now.

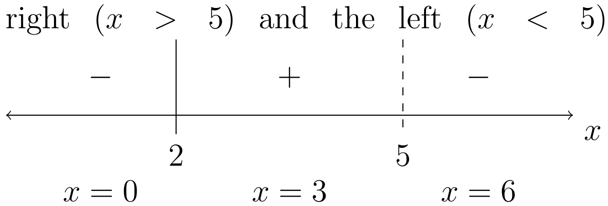
Example 334. $f(x) = \dfrac{-2x+4}{x-5} = \dfrac{-2(x-2)}{x-5}$

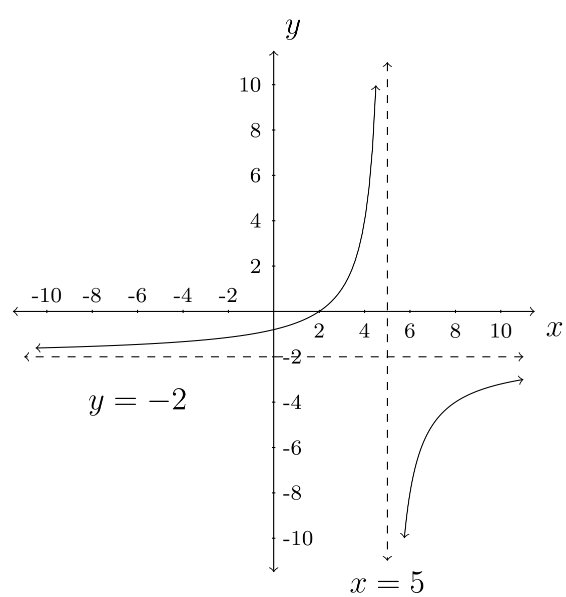
We quickly see that the expression for f is already simplified. In this case,

$$\text{As } x \to 5^+,\ f(x) \to -\infty.$$

$$\text{As } x \to 5^-,\ f(x) \to \infty.$$

A sign diagram for f will also help us to confirm whether $f(x)$ will approach ∞ or $-\infty$, as x approaches 5 from both the right $(x > 5)$ and the left $(x < 5)$.





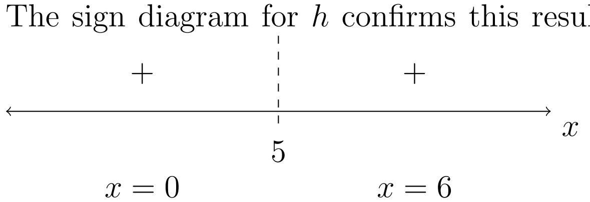
Example 335. $h(x) = \dfrac{x^2+25}{x^2-10x+25} = \dfrac{x^2+25}{(x-5)^2}$

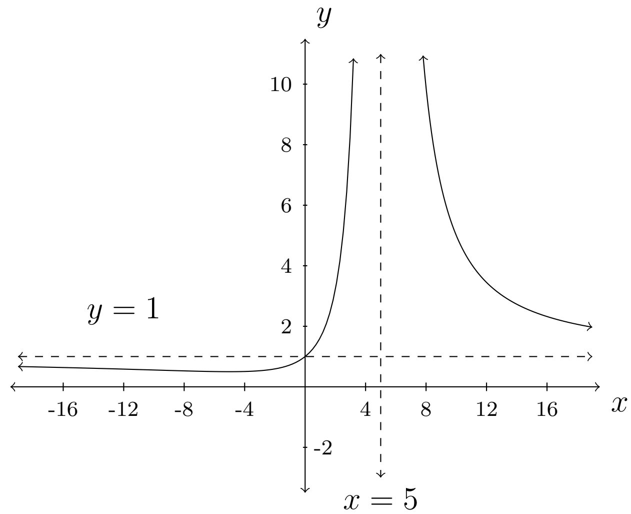
Again, the expression for h is already simplified. In this case,

$$\text{As } x \to 5^+,\ f(x) \to \infty.$$

$$\text{As } x \to 5^-,\ f(x) \to \infty.$$

The sign diagram for h confirms this result.





If, unlike the previous two examples, we were faced with a rational function whose simplified expression did *not* equal the original function, we would still have a vertical asymptote at $x = c$, as long as a factor of $(x - c)$ remained in the simplified expression. This is precisely the same as saying that $x = c$ is not in the domain of both the original function and the simplified expression. A good example of this is Example 323, which we revisit now.

Example 336. $f(x) = \dfrac{x^2-9x+20}{x^2-3x-10} = \dfrac{(x-4)(x-5)}{(x+2)(x-5)}$

Domain of f: $x \neq -2, 5$

Simplified Expression for f: $\dfrac{x-4}{x+2}$

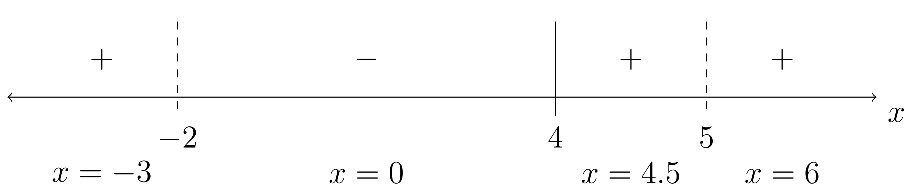
Domain of Simplified Expression: $x \neq -2$

It follows that the graph of f has a vertical asymptote at $x = -2$. Notice, however, that f does *not* have a vertical asymptote at $x = 5$, since $x = 5$ is in the domain of the simplified expression. We will revisit this example, yet again, in the very near future, to discuss the nature of the graph of f near $x = 5$.

The sign diagram for f (shown below) confirms the following statements.

$$\text{As } x \to -2^+,\ f(x) \to -\infty.$$

$$\text{As } x \to -2^-,\ f(x) \to \infty.$$



If we look more closely at each of our last three examples, we can begin to make sense of why a particular function approaches $+\infty$ or $-\infty$, as x approaches our particular discontinuity. The explanation it tied to the notion of *multiplicity* from the chapter on polynomial functions.

Recall that the multiplicity of a root c for a polynomial function $p(x)$ is the maximum number of times, k, that the factor of $x - c$ appears in the polynomial's complete factorization. For example, if we consider the polynomial

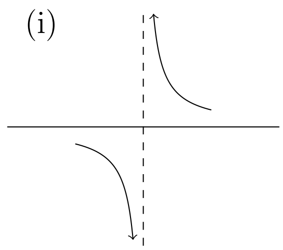
$$p(x) = 2x^3(x-4)^2(x-5),$$

the roots $x_1 = 0$ $x_2 = 4$, and $x_3 = 5$ would have respective multiplicities $k_1 = 3$, $k_2 = 2$, and $k_3 = 1$.

The multiplicity k of a root c helped us to determine whether the corresponding $x-$intercept was a *crossover* point (when k is odd) or a *turnaround* point (when k is even). It turns out that we can use this same idea for visualizing vertical asymptotes, when $x = c$ is a root of the denominator $q(x)$ of a rational function $f(x) = \dfrac{p(x)}{q(x)}$.

Let f be a rational function with vertical asymptote at $x = c$, and let k be the multiplicity of c in the denominator of the *simplified expression* for f.
If k is odd, then the graph of f near $x = c$ will resemble one of the following:

(i)

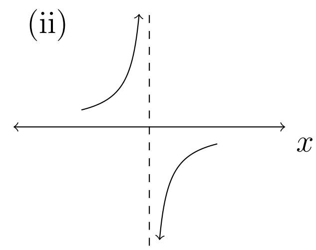


$x = c$

As $x \to c^+,\ f(x) \to \infty$

As $x \to c^-,\ f(x) \to -\infty$

(ii)



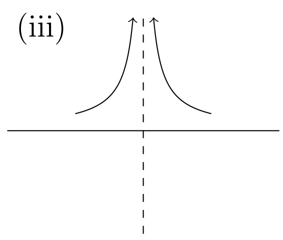
$x = c$

As $x \to c^+,\ f(x) \to -\infty$

As $x \to c^-,\ f(x) \to \infty$

If k is even, then the graph of f near $x = c$ will resemble one of the following:

(iii)

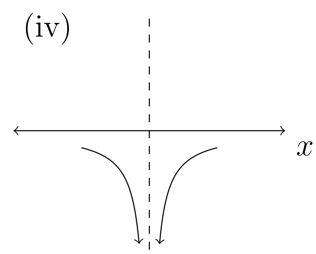


$x = c$

As $x \to c^+,\ f(x) \to \infty$

As $x \to c^-,\ f(x) \to \infty$

(iv)



$x = c$

As $x \to c^+,\ f(x) \to -\infty$

As $x \to c^-,\ f(x) \to -\infty$

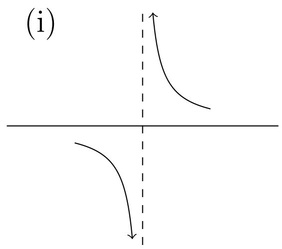
We clearly see this idea at work for $f(x) = \dfrac{-2(x-2)}{x-5}$ (Example 334) and $h(x) = \dfrac{x^2+25}{(x-5)^2}$ (Example 335), whose graphs both show a vertical asymptote at $x = 5$. In the case of f, the multiplicity of $x = 5$ in our denominator ($k = 1$) is odd. Consequently, the two sides of our graph near $x = 5$ approach the vertical asymptote on opposite sides of the $x-$axis. Alternatively, the multiplicity of $x = 5$ in the denominator of h is even ($k = 2$). Consequently, the two sides of our graph near $x = 5$ approach the vertical asymptote on the same side of the $x-$axis.

Furthermore, for a given rational function with vertical asymptote at $x = c$, once we have identified the associated multiplicity k, we can use a sign diagram to determine precisely which case matches our graph: (i) or (ii) when k is odd, and (iii) or (iv) when k is even.

Example 337. We already know that $f(x) = \dfrac{(x-4)(x-5)}{(x+2)(x-5)}$ has a vertical asymptote at $x = -2$ from Example 336.

Since the multiplicity of $x = -2$ in the denominator of our simplified expression $\dfrac{x-4}{x+2}$ is odd ($k = 1$), we know that the nature of the graph near $x = -2$ will resemble one of the following cases.

(i)

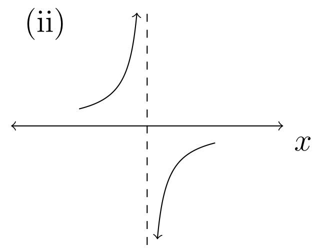


$x = -2$

As $x \to -2^+,\ f(x) \to \infty$

As $x \to -2^-,\ f(x) \to -\infty$

(ii)

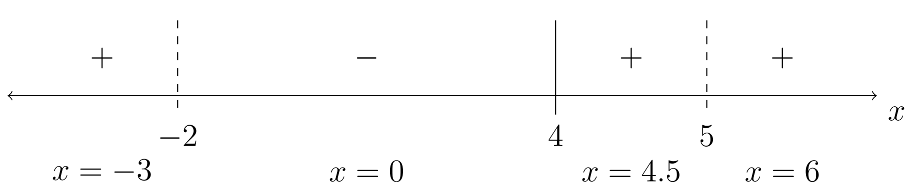


$x = -2$

As $x \to -2^+,\ f(x) \to -\infty$

As $x \to -2^-,\ f(x) \to \infty$

But recall that our sign diagram for f was as follows.



This tells us that our graph will point *upwards* to the left of $x = -2$ and *downwards* to the right of $x = -2$. Hence, case (ii) is the correct graph for our function near $x = -2$. And our corresponding statements match those from before.

$$\text{As } x \to -2^+,\ f(x) \to -\infty.$$

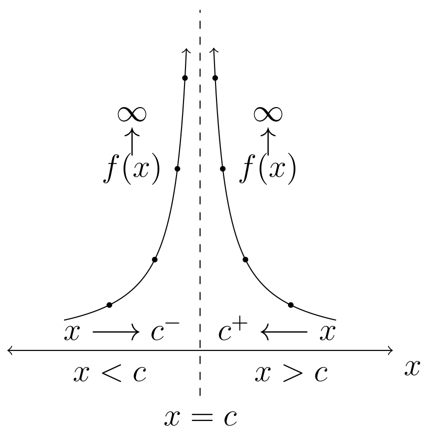
$$\text{As } x \to -2^-,\ f(x) \to \infty.$$

Holes (L59)

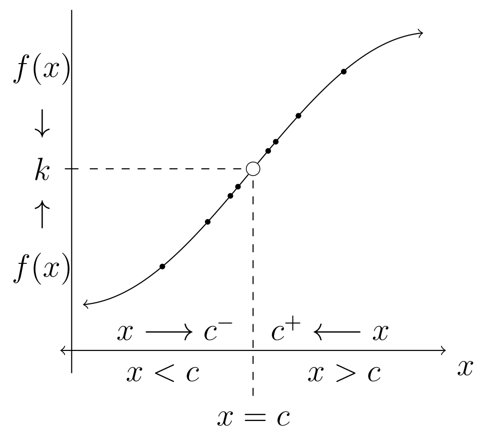
Objective: Identify the precise location of one or more holes in the graph of a rational function.

While Vertical Asymptotes correspond to infinite discontinuities, a hole corresponds to a *removable discontinuity*, since the removal of a single point along a continuous curve creates the hole.

Suppose that the rational function $f(x)$ has a discontinuity at $x = c$, i.e., c is not in the domain of f. If $x = c$ is a vertical asympote of the graph of f, we just saw that as $x \to c$, $f(x) \to \pm\infty$. If $x = c$ represents a hole in the graph of f, however, we will see that as $x \to c$, $f(x) \to k$, for some real number k. This is the fundamental difference between infinite and removable discontinuities.



Infinite Discontinuity



Removable Discontinuity

In the case of the graph of the left, recall that we have the following statements.

$$\text{As } x \to c^+,\ f(x) \to \infty. \qquad \text{As } x \to c^-,\ f(x) \to \infty.$$

Similarly, in the case of the graph on the right, we employ the same idea, using k^+ and k^- in order to identify whether or not the graph of f approaches k from *above* if $f(x) > k$ and *below* if $f(x) < k$.

$$\text{As } x \to c^+,\ f(x) \to k^+. \qquad \text{As } x \to c^-,\ f(x) \to k^-.$$

In virtually all cases, however, it will be sufficient enough to simply state that as $x \to c,\ f(x) \to k$, since further analysis will often prove difficult.

We now state the requirement for a hole, which, as with vertical asymptotes, depends on both the rational function f and its simplified expression.

> Let $f(x)$ be a rational function and let $g(x)$ represent the simplified expression for f. If $x = c$ is not in the domain of f, but *is* in the domain of g, then the graph of f will have a hole at $(c, g(c))$.

For the existence of a vertical asymptote, the identified discontinuity had to be excluded from the domain of *both* f and its simplified expression. This is not the case for a hole, however, as the simplified expression g is, in fact, defined at $x = c$. Furthermore, the value $g(c)$ tells us the precise location of our hole along the $y-$axis.

Example 319 is our first example with a hole, and we revisit it now.

Example 338. Original Function:

$$k(x) = \frac{x^3 - 5x^2}{10x - 50} = \frac{x^2(x-5)}{10(x-5)}$$

Domain of k: $x \neq 5$ or $(-\infty, 5) \cup (5, \infty)$

The original function k is undefined at $x = 5$

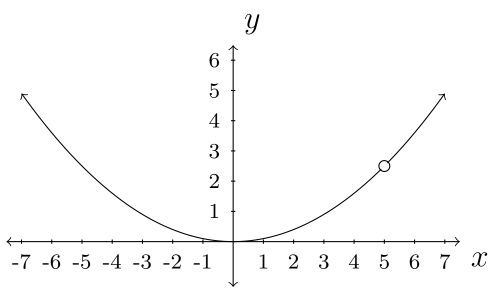
Simplified Expression:

$$g(x) = \frac{x^2\cancel{x-5}}{10\cancel{(x-5)}} = \frac{x^2}{10}$$

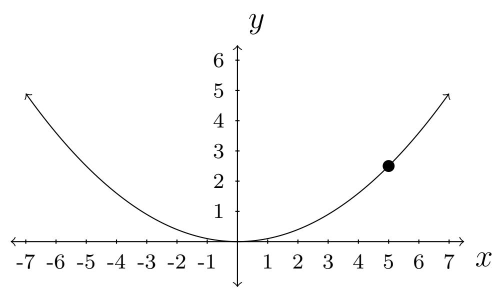
Domain of g: $(-\infty, \infty)$

$$g(5) = \frac{25}{10} = \frac{5}{2}$$

Conclusion: The graph of k has a hole at $(5, g(5)) = (5, \frac{5}{2})$.



Graph of $k(x) = \dfrac{x^3 - 5x^2}{10x - 50}$



Graph of $g(x) = \dfrac{x^2}{10}$

In this first example, we observe a somewhat obvious fact that we have neglected to state until now:

> The graph of a rational function f will always equal the graph of its simplified expression for any x *in the domain* of f.

In this case, the graphs of $k(x) = \dfrac{x^3 - 5x^2}{10x - 50}$ and the familiar quadratic function $g(x) = \dfrac{x^2}{10}$ only differ in their behavior at $x = 5$.

We now take one last look at a familiar example (336), and include its graph for completeness

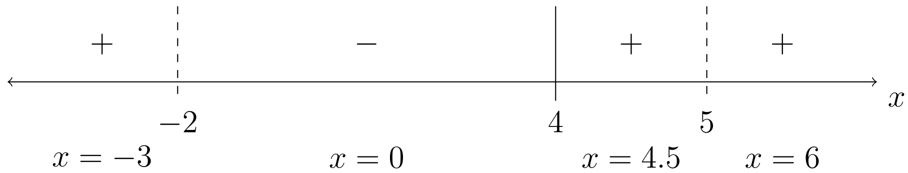
Example 339. $f(x) = \dfrac{x^2 - 9x + 20}{x^2 - 3x - 10} = \dfrac{(x-4)(x-5)}{(x+2)(x-5)}$

The simplified expression for f is $g(x) = \dfrac{x-4}{x+2}$.

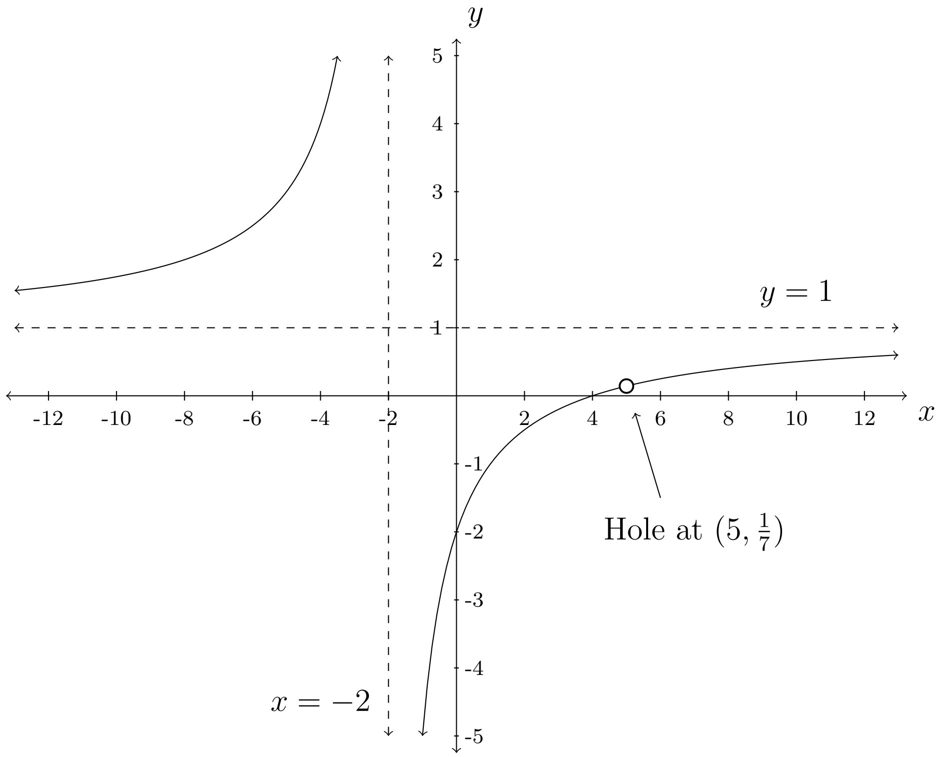
Notice that $x = 5$ is in the domain of g, with $g(5) = \frac{5-4}{5+2} = \frac{1}{7}$.

Since $x = 5$ is not in the domain of f, the graph of f will have a hole at $(5, \frac{1}{7})$.

Recall that our sign diagram for f is as follows.



We now are ready to make sense of the complete graph of f, presented below.



$$f(x) = \frac{x^2 - 9x + 20}{x^2 - 3x - 10} = \frac{(x-4)(x-5)}{(x+2)(x-5)} = \frac{x-4}{x+2},\ x \neq 5$$

In each of our last two examples, the simplified expression for our given rational function has seen a complete elimination of the "offending factor" $(x-c)$ from both the numerator and the denominator. Based upon our criteria for identifying holes, this is certainly a requirement for the denominator. As we will see with our next example, however, it is possible that not all offending factors will completely disappear from the numerator. In this situation, our hole will simply reside on the $x-$axis, since $g(c)$ will equal zero in our simplified expression.

Example 340. Find the domain of $f(x) = \dfrac{4x^2}{x^3 + 3x^2 - 4x}$, and identify all discontinuities. In each case, determine whether the discontinuity is infinite (vertical asymptote) or removable (hole). Once again, we begin by obtaining a complete factorization of f in order to identify its domain and simplified expression.

$$\begin{aligned} f(x) &= \frac{4x^2}{x^3 + 3x^2 - 4x} \\ &= \frac{4x^2}{x(x^2 + 3x - 4)} \\ &= \frac{4x^2}{x(x+4)(x-1)} \end{aligned}$$

The domain of f is $x \neq -4, 0, 1$ or

$$(-\infty, -4) \cup (-4, 0) \cup (0, 1).$$

The simplified expression for f is

$$g(x) = \frac{4x}{(x+4)(x-1)}.$$

Here, we see that the graph of f has three discontinuities, occurring at those values not in the domain, $x = -4, 0$, and 1. Since the factors of $x + 4$ and $x - 1$ still appear in the denominator of the simplified expression, it follows that the discontinuities at $x = -4$ and $x = 1$ will be infinite, and the graph of f will have vertical asymptotes along these lines.

Our third discontinuity at $x = 0$ will be removable, since the simplified expression does not contain a factor of x in its denominator. Hence, the graph of f will have a hole at the point $(0, g(0)) = (0, 0)$.

We leave it as an exercise for the reader to verify our answer by graphing f using Desmos.

Graphing Summary (L60)

Objective: Graph a rational function in its entirety.

At this point, we have addressed all key features of rational functions individually. This section pulls each of these aspects together, for a detailed analysis of a rational function, culminating in a complete sketch of its graph. Along the way, we will need to address each of the following aspects for our rational function $f(x) = \dfrac{p(x)}{q(x)}$. It is important to note that there is no universally accepted order to this checklist.

- Find the $y-$intercept of the graph of f, $(0, f(0))$, if it exists.
- Use the degrees and leading coefficients of p and q to determine whether the graph of f has a horizontal asymptote. If the graph of f has a slant asymptote, use polynomial division to find where it is located.
- Identify a complete factorization of f, and use it to find the domain of the function. This is the set of all x, such that $q(x) \neq 0$.

- Find any $x-$intercepts of the graph of f. This is the set of all x in the domain of f, such that $p(x) = 0$. Using multiplicities, classify each $x-$intercept as a crossover or turnaround ("bounce") point.
- Find the simplified expression g for the given function f, and use it to identify any vertical asymptotes or holes in the graph of f. Use multiplicities to help visualize the nature of the graph of f near its vertical asymptotes. If f has a hole at $x = c$, use g to help plot the hole's precise location at $(c, g(c))$.
- Using both the $x-$intercepts and the discontinuities (those x not in the domain), construct a sign diagram for f.

In each example that follows, we will carefully examine the given function, making sure not to omit any of the checklist items above and to compare each item to those that precede it along the way for accuracy. Although the process will take some time, if we are thorough, our end result should be a complete, accurate sketch of the given rational function. We will start by revisiting our last example.

Example 341. Sketch a complete graph of the rational function below, making sure to have a clearly defined scale and label all key features of your graph (intercepts, asymptotes, and holes).

$$f(x) = \frac{4x^2}{x^3 + 3x^2 - 4x}$$

In this first example, we see that the graph of f will not have a $y-$intercept, since $f(0) = \frac{0}{0}$, which is undefined.

Since the degree of the numerator is less than the degree of the denominator, we conclude that the graph of f has a horizontal asymptote along the $x-$axis, $y = 0$.

Our graph also has no $x-$intercepts, since our numerator only equals zero when $x = 0$, which we know is not in our domain of f.

Furthermore, from the work in our last example, we know that f has a complete factorization of

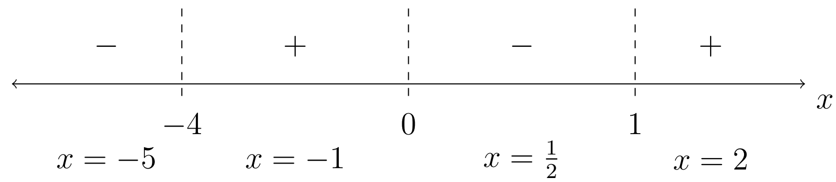
$$f(x) = \frac{4x^2}{x(x + 4)(x - 1)},$$

with corresponding domain $x \neq -4, 0, 1$, and simplified expression

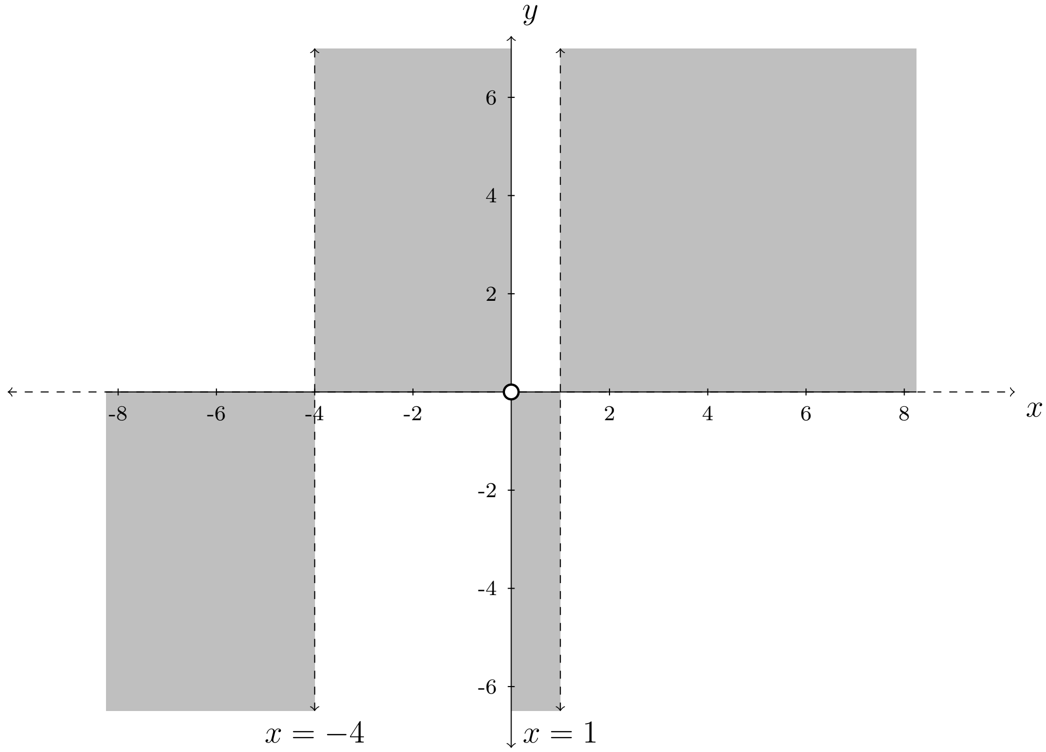
$$g(x) = \frac{4x}{(x + 4)(x - 1)}.$$

Consequently, the graph of f has vertical asymptotes at $x = -4$ and $x = 1$ and a hole at $(0, g(0)) = (0, 0)$.

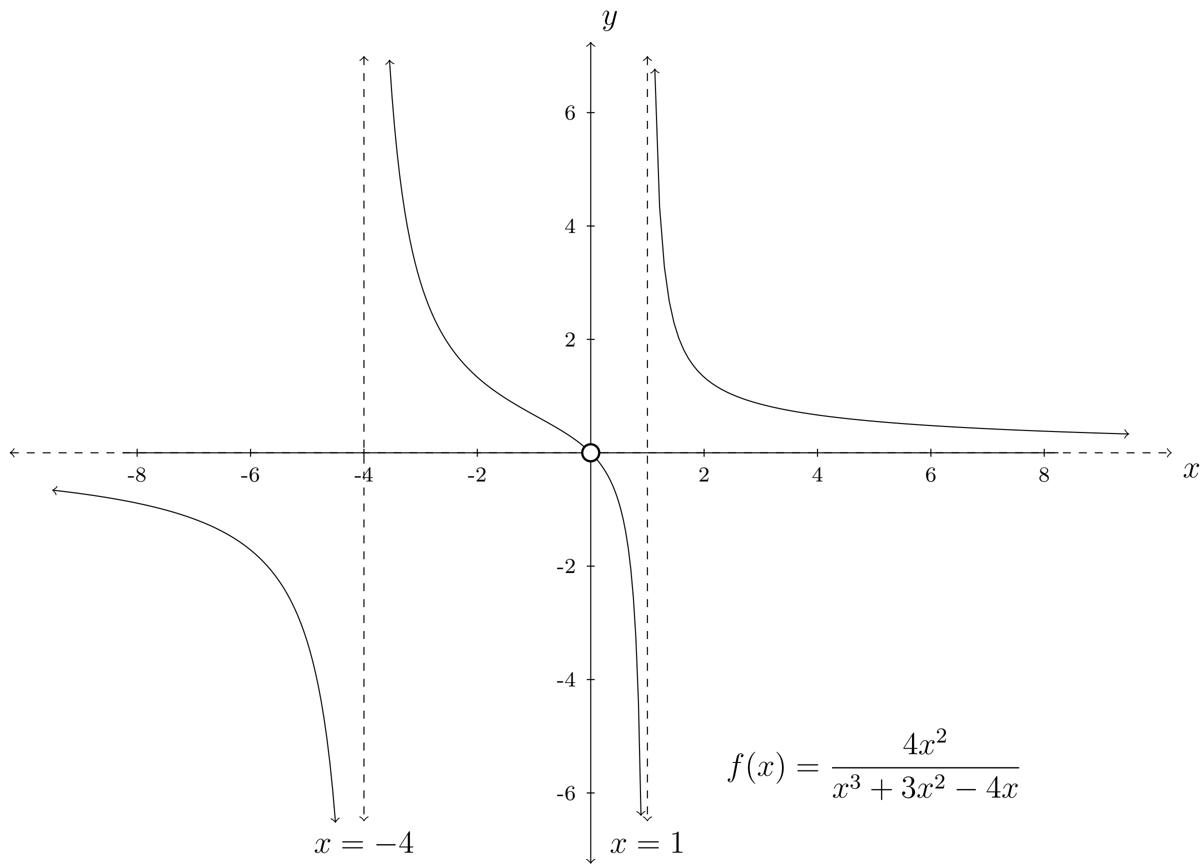
Since the multiplicities of both $x = -4$ and $x = 1$ in the denominator of f are both one (odd), we know that the graph of f will approach each vertical asymptote from opposite sides of the $x-$axis. The following sign diagram confirms this observation.



We are now ready to try our hand at graphing f, and begin our graph by defining a scale for both the $x-$ and $y-$axes, and identifying all intercepts, and asymptotes. This should always be our first step to successfully sketching a decent-looking graph. To emphasize this point, we first show an initial graph that identifies each of these features, and further shades those areas of the $xy-$plane that correspond to our sign diagram above.



We now carefully sketch the graph of f based upon our findings.



Example 342. Sketch a complete graph of the rational function below, making sure to have a clearly defined scale and label all key features of your graph (intercepts, asymptotes, and holes).

$$f(x) = \frac{x^2 - 6x + 9}{x^2 + x - 6}$$

Again, we start by evaluating f at $x = 0$ to identify the $y-$intercept. We get $(0, \frac{9}{-6}) = (0, -\frac{3}{2})$.

Since the degrees of the numerator and denominator are equal and the leading coefficients are also equal, we know that the graph of f will have a horizontal asymptote at the line $y = 1$.

A complete factorization of f gives us

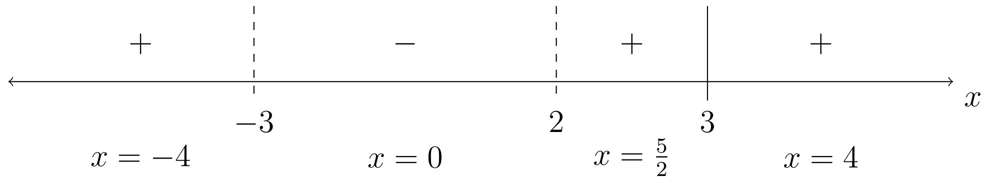
$$f(x) = \frac{(x-3)^2}{(x+3)(x-2)},$$

which is also our simplified expression.

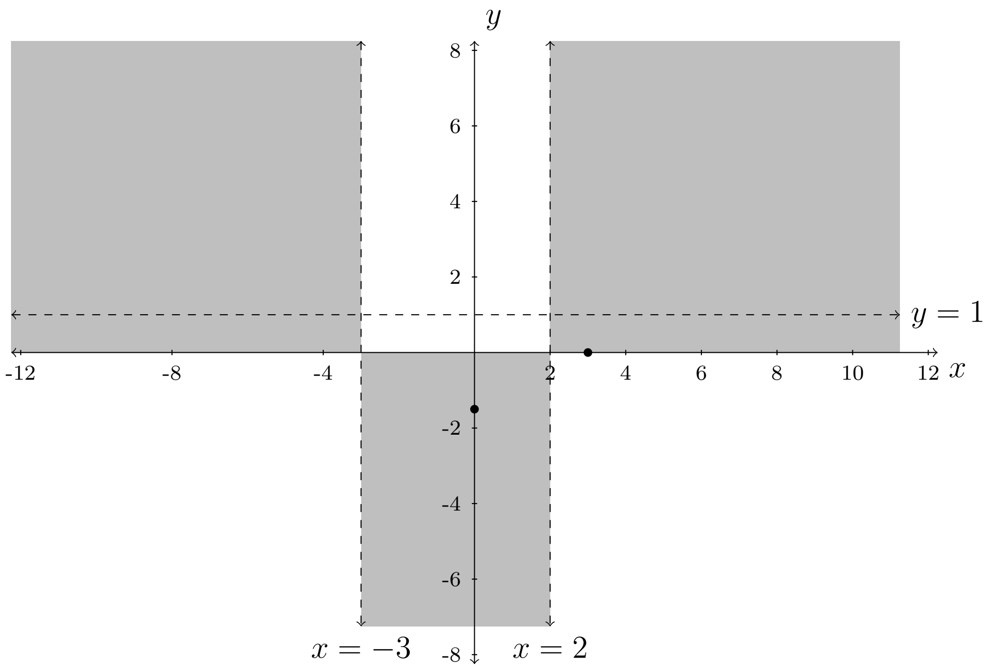
Using our factorization, we see that the graph of f will have an $x-$intercept at $(3, 0)$. This will be a turnaround point, due to the even multiplicity of the root $x = 3$ in the numerator.

Our domain for f is $x \neq -3, 2$, and the corresponding graph will have vertical asymptotes at $x = -3$ and $x = 2$.

Our sign diagram is as follows.



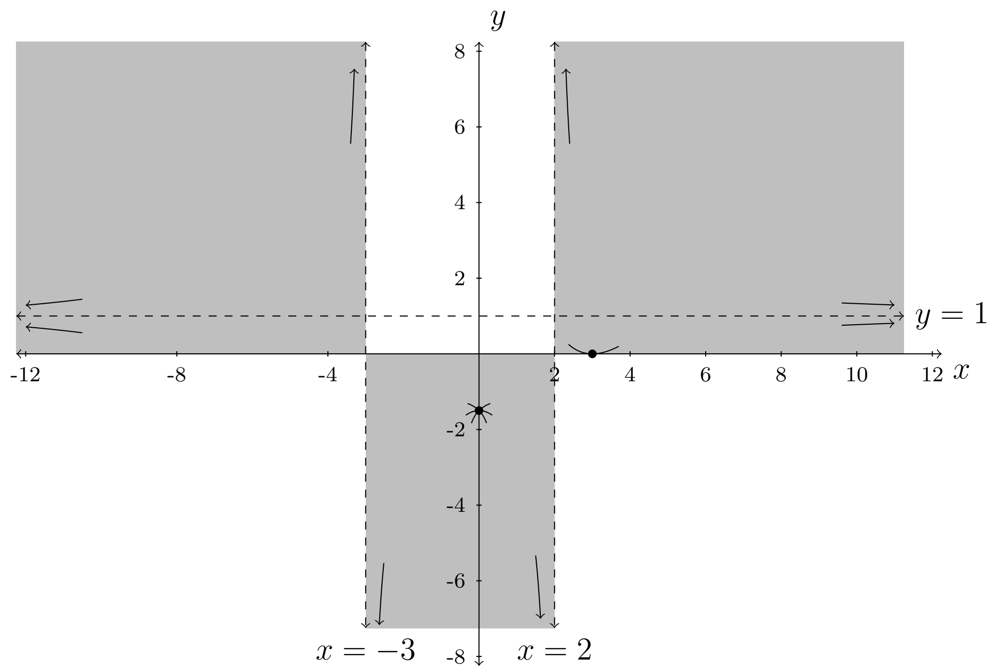
As with our last example, we will build up to the graph of f by first defining a scale, identifying our intercepts and asymptotes, and shading the corresponding areas that should include our graph.



In this particular example, without further algebraic analysis of our function, we will have to make a choice as to how the graph of f approaches the horizontal asymptote, $y = 1$. For example, does it approach from above or below? Does the graph ever cross the horizontal asymptote? In order to answer this last question, one could attempt to solve the equation $f(x) = 1$ for all possible x in order to see exactly where our graph crosses the line $y = 1$.

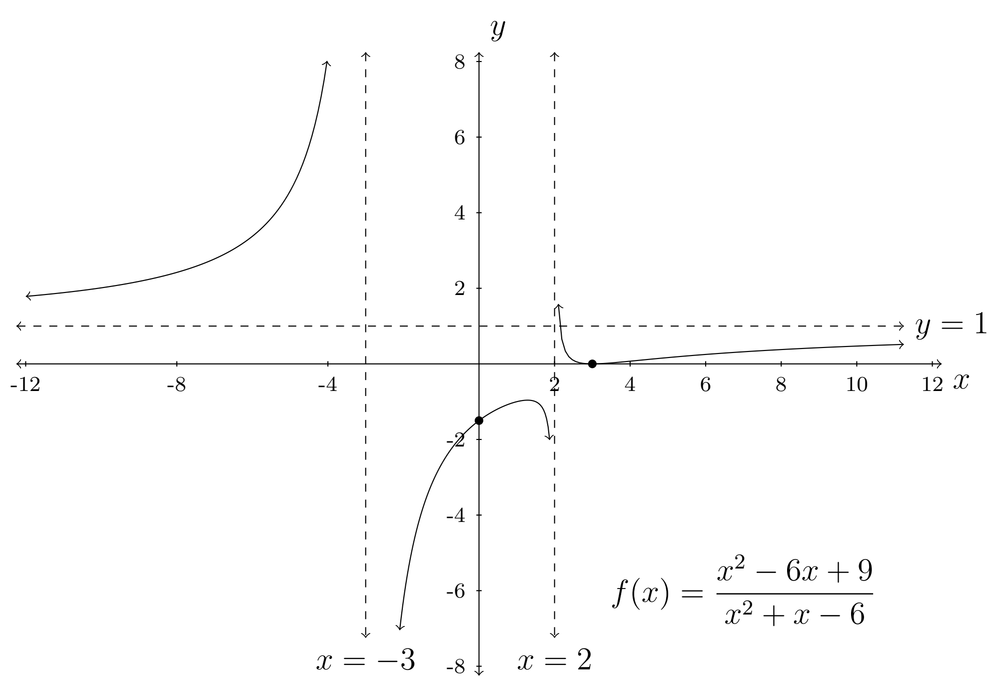
Similarly, we cannot know if our $y-$intercept is also a local maximum value for our graph or how exactly the graph intersects this point without using advanced methods that are typically covered in a calculus course.

On account of these subtleties, before we see the complete graph of f, we will take one more step, and begin to sketch the possibilities for our graph near the asymptotes and our intercepts.



At this point it is important to reinforce the fact that our graph is meant to be a *rough sketch* of the actual graph of f. So, we have to make some choices about how to connect everything up properly. A more complete sketch of f requires advanced techniques and concepts that one would likely see in precalculus or calculus. Nevertheless, as long as there is a logical, smooth connection to each aspect of our graph, we can rest assured that our analysis of the function is sufficient.

We can now conclude this example with the actual graph of f.



As a result, we learn that as $x \to \infty$, $f(x) \to 1^-$, and consequently, our graph does not cross over the horizontal asymptote to the right of our $x-$intercept. Furthermore, our $y-$intercept is not a local maximum for the graph of f, as is often the case. Lastly, as $x \to 2$, we see that the graph of f seems to hold more tightly to the vertical asymptote than when $x \to -3$.

Example 343. Sketch a complete graph of the rational function below, making sure to have a clearly defined scale and label all key features of your graph (intercepts, asymptotes, and holes).

$$f(x) = \frac{x^3 - 16x}{3x^2 - 6x - 24}$$

We will begin by obtaining a complete factorization of f and a simplified expression.

$$\begin{aligned} f(x) &= \frac{x(x^2-16)}{3(x^2-2x-24)} \\ &= \frac{x(x+4)(x-4)}{3(x+4)(x-6)} \end{aligned}$$

The simplified expression for f is

$$\begin{aligned} g(x) &= \frac{x(x-4)}{3(x-6)} \\ &= \frac{x^2-4x}{3x-18} \end{aligned}$$

We quickly see that the graph of f has both an $x-$ and $y-$intercept at the origin, since $f(0) = 0$.

Since the degree of the numerator is one greater than the degree of the denominator, we know that the graph of f will have a slant asymptote, and we can use polynomial division to find its precise location. It is worth noting, however, that we can apply polynomial division to *either* our original function or its simplified expression in order to find our slant asymptote, as demonstrated below.

Original Function:

$$\begin{array}{r|l} & \frac{1}{3}x + \frac{2}{3} \\ \hline 3x^2-6x-72 & x^3 \qquad\quad - 16x \\ & -x^3 + 2x^2 + 24x \\ \hline & \quad 2x^2 + 8x \\ & \quad -2x^2 + 4x + 48 \\ \hline & \qquad\quad 12x + 48 \end{array}$$

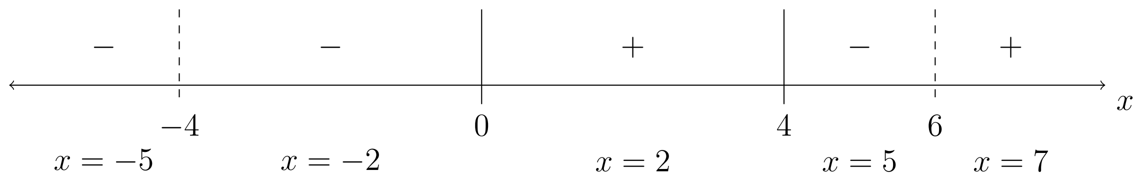
Simplified Expression:

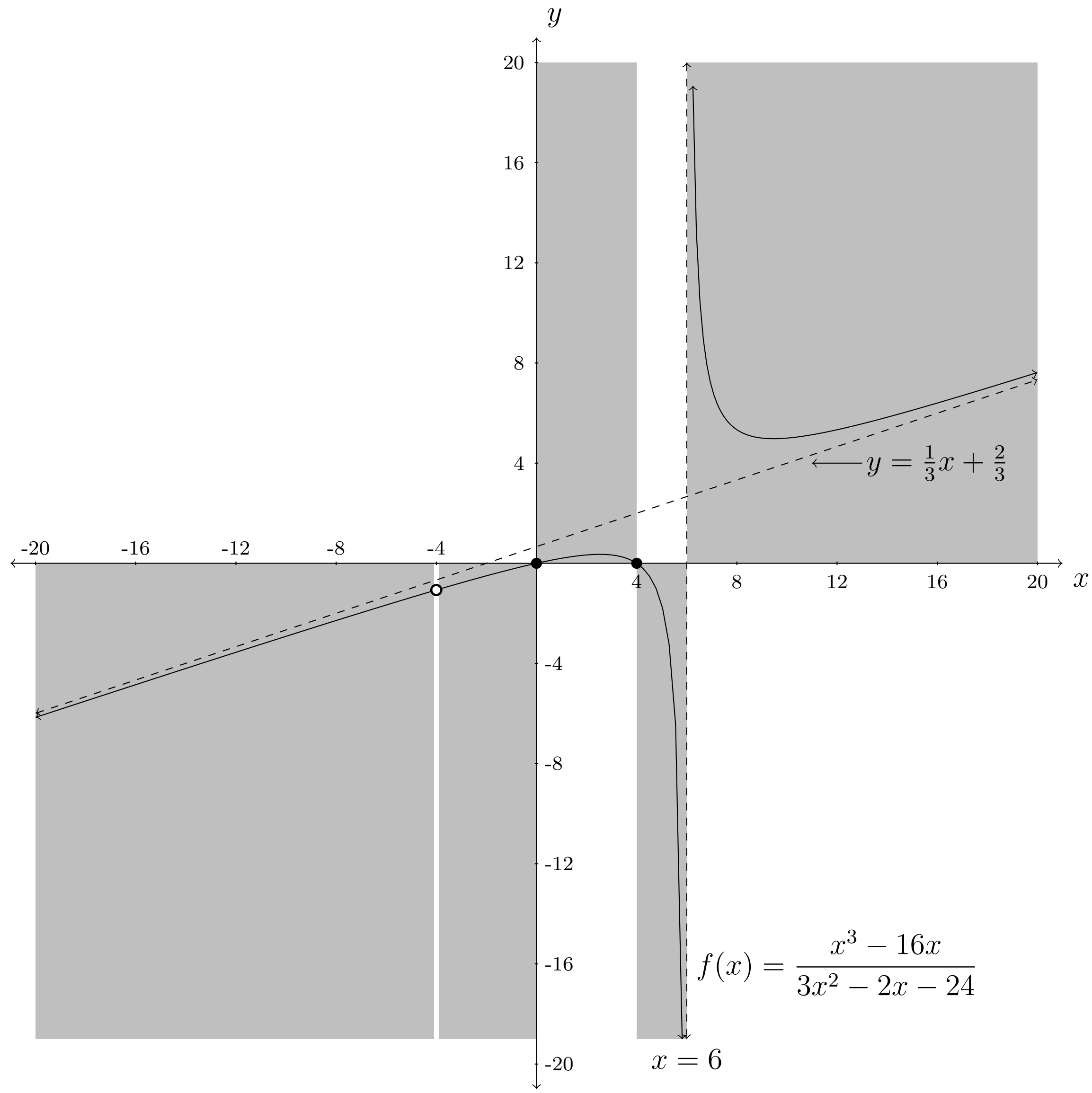
$$\begin{array}{r|l} & \frac{1}{3}x + \frac{2}{3} \\ \hline 3x-18 & x^2 - 4x \\ & -x^2 + 6x \\ \hline & \quad 2x \\ & \quad -2x + 12 \\ \hline & \qquad 12 \end{array}$$

Each calculation carries the same number of steps and yields the same equation for our slant asymptote, $y = \frac{1}{3}x + \frac{2}{3}$. Since dividing the simplified expression for our function should always be a simpler process, it makes sense to use this approach when we encounter similar problems.

Observe that the domain of f is $x \neq -4, 6$. Since the factor of $x + 4$ appears in both the numerator and denominator, the graph of f will contain a hole at $(-4, g(-4)) = (-4, -\frac{16}{15})$ and a vertical asymptote at $x = 6$. Our graph will also include crossover $x-$intercepts at $x = 0$ and $x = 4$, as their multiplicities in the numerator are both odd.

The sign diagram for f is shown below, and we conclude with the graph of f. Though not necessary, we have again included a shading of the regions of our graph that correspond to our sign diagram.

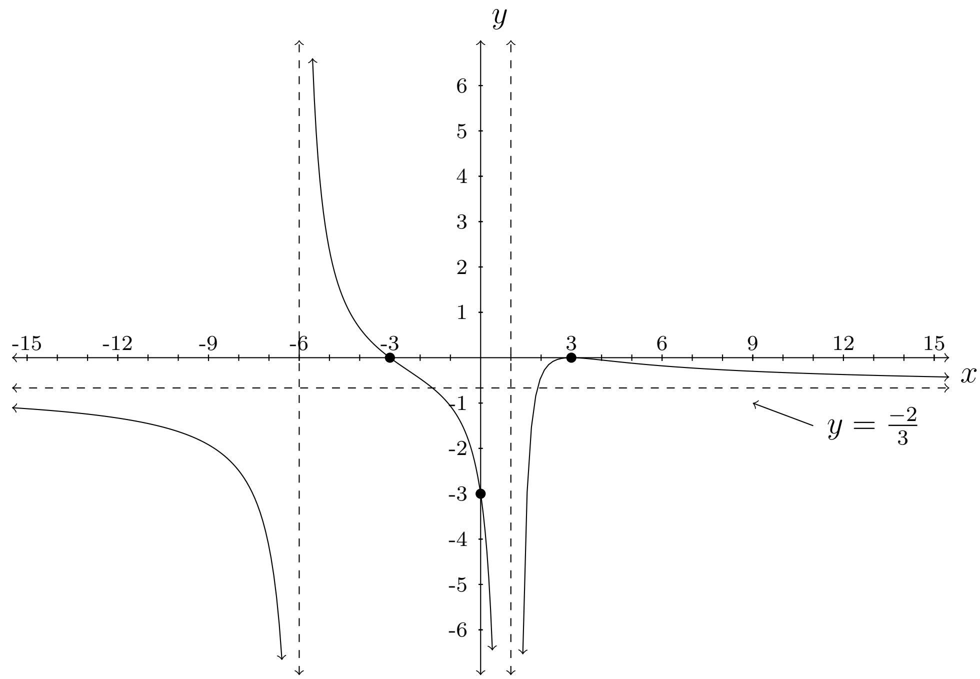




Our graph and, more importantly, its accompanying sign diagram are essential in determining when f is either positive or negative. For example, one could say that $f(x) > 0$ for all x in the set $(0,4)\cup(6,\infty)$, and $f(x) < 0$ when x is in the set $(-\infty,-4)\cup(-4,0)\cup(4,6)$. We will revisit this idea at the beginning of our next section, when we are asked to solve a rational inequality, rather than graph a particular function.

While it is essential that we are able to analyze and graph a rational function, it is equally important that we can work backwards, and correctly interpret a graph to identify the key aspects of an otherwise unknown function. Our last example does just that.

Example 344. Find an explicit form for the rational function whose graph is shown below.



Although it may be easy to feel overwhelmed at what this problem is asking us to do, at this point we can rest assured that the skills outlined throughout the chapter will enable us to systematically analyze each aspect of the graph above, in order to construct the rational function whose graph matches our graph.

We begin by first making the following observations, in no particular order, along with the implications for our function $f(x)$.

$y-$intercept at $(0, -3)$	$\Longrightarrow$	$f(0) = -3$
Crossover $x-$intercept at $x = -3$	$\Longrightarrow$	$(x+3)^1$ appearing in numerator
Turnaround $x-$intercept at $x = 3$	$\Longrightarrow$	$(x-3)^2$ appearing in numerator
Vertical asymptote at $x = -6$, with ends pointing in opposite directions	$\Longrightarrow$	$(x+6)^1$ appearing in denominator
Vertical asymptote at $x = 1$, with ends pointing in same direction	$\Longrightarrow$	$(x-1)^2$ appearing in denominator
Horizontal asymptote at $y = \frac{-2}{3}$	$\Longrightarrow$	Numerator and denominator have same degree, n $\frac{a_n}{b_n} = \frac{-2}{3}$

Since many of our observations involve a *factor* of some kind, we will begin construction of f in its factored form. An expanded form can then be easily obtained by multiplying, if it is required.

We will start by focusing on the implications from our $x-$intercepts and those values not in our domain (corresponding to any vertical asymptotes and holes). Consequently, an initial candidate for f could be

$$f(x) = \frac{(x+3)(x-3)^2}{(x+6)(x-1)^2}.$$

At this point, we will need to check our candidate against all additional aspects of the graph outlined above, making adjustments as necessary. But, we soon see that

$$f(0) = \frac{(0+3)(0-3)^2}{(0+6)(0-1)^2} = \frac{27}{6} \neq -3,$$

as our graph shows. Additionally, though the degrees of both the numerator and denominator are equal (in this case, three), it is not hard to see that our numerator and denominator will both have a leading coefficient of 1, which will not produce the necessary horizontal asymptote at $y = \frac{-2}{3}$.

The solution to this problem is found by making an adjustment to our initial guess, by introducing a *scalar multiplier* k.

$$f(x) = \frac{k(x+3)(x-3)^2}{(x+6)(x-1)^2}$$

All that remains is to determine what k equals. Remember that we know it must be negative, in order to produce a horizontal asymptote at $y = -1$. In order to find the precise multiple that is needed, we must substitute our $y-$intercept into the equation above and solve for k.

$$\begin{aligned} f(0) &= \frac{k(0+3)(0-3)^2}{(0+6)(0-1)^2} \\ &= \frac{27k}{6} \\ &= \frac{9k}{2} \end{aligned}$$

But we know that $f(0) = -3$. So we are left with having to solve $\dfrac{9k}{2} = -3$.

In this case, we get $k = \dfrac{-2}{3}$. Our final answer for f is therefore

$$f(x) = \frac{-2(x+3)(x-3)^2}{3(x+6)(x-1)^2},$$

which further satisfies the requirement for our horizontal asymptote at $y = \frac{-2}{3}$.

Expanding our answer gives us $f(x) = \dfrac{-2x^3 + 6x^2 + 18x - 54}{3x^3 + 12x^2 - 33x + 18}$.

Rational Inequalities

Objective: Solve a rational inequality by constructing a sign diagram.

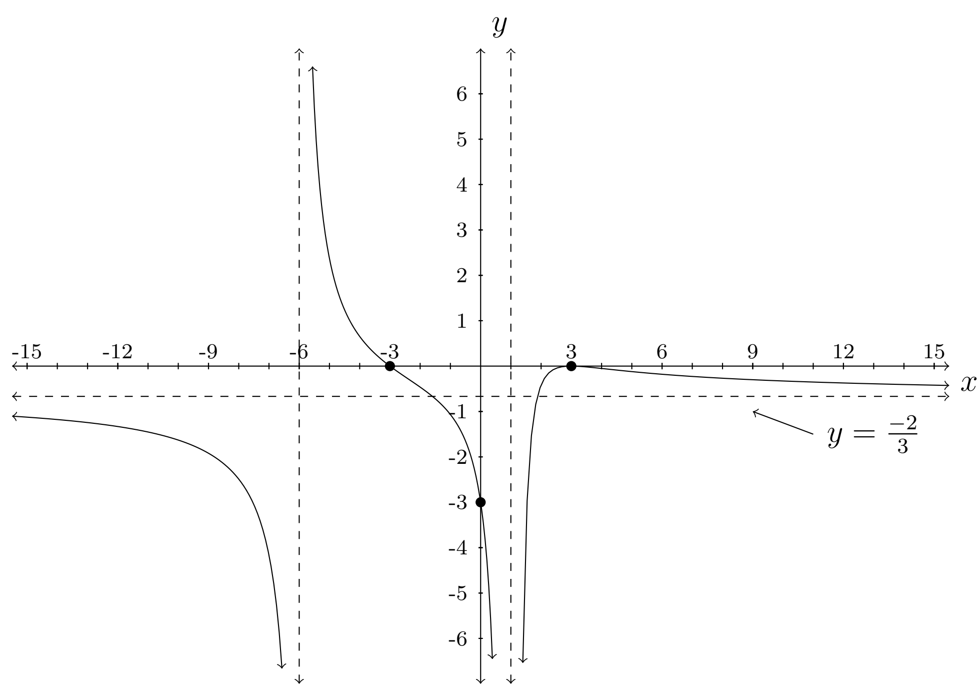
Identifying the solution of a rational inequality is one very practical application of the sign diagram and graph of a rational function. For example, if we are asked to identify when a function $f(x)$ is greater (or less) than zero, we know that this answer will correspond to those values of x such that the point $(x, f(x))$ is above (or below) the $x-$axis. We see this at work at the end of Example 343, where we concluded that the function

$$f(x) = \frac{x^3 - 16x}{3x^2 - 6x - 24} = \frac{x(x+4)(x-4)}{3(x+4)(x-6)}$$

is positive for all x in the set $(0,4) \cup (6,\infty)$ and negative for all x in the set $(-\infty,-4) \cup (-4,0) \cup (4,6)$.

Similarly, we can use the graph for Example 344 to determine when the function $f(x) > 0$. This example is a good starting place, since we were not initially given the expression for f, and had to find it.

Example 345. Determine when the given graph is positive, i.e., when $f(x) > 0$. Express your answer using interval notation.



To answer this question, we need only identify those values of x that correspond to points lying *above* the $x-$axis. Our answer is the single interval $(-6,-3)$.

Note that if we were asked to find when $f(x) \geq 0$ in the previous example, we would need to include the two $x-$intercepts at $x = \pm 3$. Hence, $f(x) \geq 0$ for all x in the set $(-6,-3] \cup \{3\}$.

Recall that since $x = 3$ is a single value, rather than an entire interval of values, we use braces to denote its inclusion in our answer.

Looking at our last example from another angle, let's suppose that we were starting out with a *function*, rather than a graph. We seek to answer the question of when $f(x) > 0$ without the benefit of this visual aide, and will do this using a sign diagram.

Example 346. Solve the inequality $f(x) > 0$ for the function

$$f(x) = \frac{-2x^3 + 6x^2 + 18x - 54}{3x^3 + 12x^2 - 33x + 18}$$

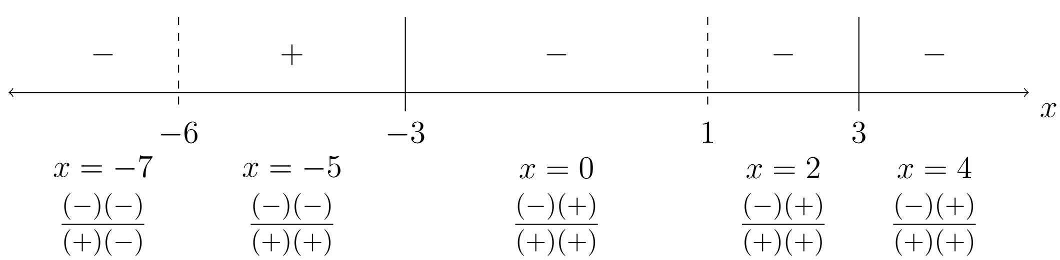
Recall that the given function is that obtained from our graph in the previous example, so we know that our answer should be $(-6, -3)$.

In every problem from here on out, we will need to find a factored form for f so that we can construct a sign diagram. Since this function is a carry-over from a previous example, we know its factorization,

$$f(x) = \frac{-2(x+3)(x-3)^2}{3(x+6)(x-1)^2}.$$

Had we not known this information, factoring f could take a considerable amount of work, since, for example, our denominator $3x^3 + 12x^2 - 33x + 18$ is not easily factorable.

To find the sign diagram for f, we need to identify our $x-$intercepts, as well as those x not in the domain. From our factorization, we see that this is the set $x = \{-6, -3, 1, 3\}$, with $x \pm 3$ being our intercepts. Our diagram is shown below.



One important observation in our diagram is in the calculation of each sign. For each test value, we have excluded the *squared* factors in the numerator and denominator, since both $(x-3)^2$ and $(x-1)^2$ will always contribute a positive sign and not affect the end result. For example, when $x = 0$, we get

$$\frac{(-)(+)(-)^2}{(+)(+)(-)^2},$$

which reduces to the result that we see above. Similarly, we could have excluded the $(+)$ that appears in the denominator of each test value's sign calculation, since the constant multiplier of 3 will have no impact on sign.

At this point we are essentially done, since the factorization and construction of our diagram has done the bulk of the work for us. Since we are asked to find all x such that $f(x) > 0$, we see that this equals to the union of all intervals that correspond to a + sign. This gives us our anticipated answer of $(-6, -3)$.

Recalling our discussion of the last example, if we wished to answer the follow-up question of when $f(x) \geq 0$, we would just need to include all boundary values in our diagram that correspond to $x-$intercepts (when $f(x) = 0$). From our diagram, this would be any value of x that has a *solid* divider, remembering that dashed dividers correspond to values not in our domain. In this case, the function $f(x) \geq 0$ for all x in the set $(-6, -3] \cup \{3\}$.

In each example where we are asked to find when a rational expression or function f is positive, negative, ≥ 0, or ≤ 0, we can take this same approach:

1. Identify a complete factorization of the expression.
2. Construct a sign diagram.
3. Find all intervals that correspond to the desired inequality.
4. In the case of $\geq$ or $\leq$, make sure to include any $x-$intercepts.

Example 347. Solve the inequality

$$\frac{4x^2 - 4x + 1}{x^3 - x^2 - 17x - 15} \leq 0.$$

As with our last example, we will require some additional information for factoring our denominator. In this case, we will include the fact that $x = -1$ is a root of the denominator, and apply polynomial division below. Note that if we did not know this information, we would need to use the Rational Root Theorem to factor the denominator completely.

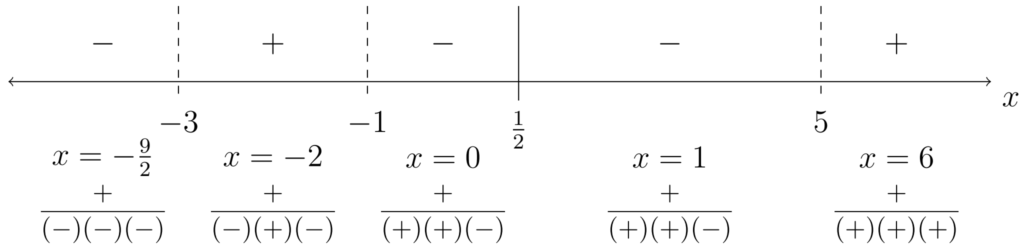
$$\begin{array}{r|rrrr}
 & x^2 & -2x & -15 & \\ \hline
x+1 & x^3 & -x^2 & -17x & -15 \\
 & -x^3 & -x^2 & & \\ \hline
 & & -2x^2 & -17x & \\
 & & 2x^2 & +2x & \\ \hline
 & & & -15x & -15 \\
 & & & 15x & +15 \\ \hline
 & & & & 0
\end{array}$$

Our denominator factors as follows.

$$\begin{aligned} & x^3 - x^2 - 17x - 15 \\ &= (x+1)(x^2 - 2x - 15) \\ &= (x+1)(x+3)(x-5) \end{aligned}$$

A complete factorization of our expression is $\dfrac{(2x-1)^2}{(x+1)(x+3)(x-5)}$, with critical values at $x = -3, -1, \frac{1}{2}$, and 5.

This corresponds to the following diagram.



Our answer will correspond to all "negative" intervals in the diagram above (intervals with a −). Since the inequality we are asked to solve is not strict (it includes when the expression equals zero), we can combine the two intervals with endpoints at $x = \frac{1}{2}$ into one.

$$\text{We conclude that } \frac{4x^2 - 4x + 1}{x^3 - x^2 - 17x - 15} \leq 0 \text{ for all } x \text{ in the set } (-\infty, -3) \cup (-1, 5).$$

Next, how might we handle solving an equation or inequality that does not compare an expression to zero, but some other number or another expression? For example,

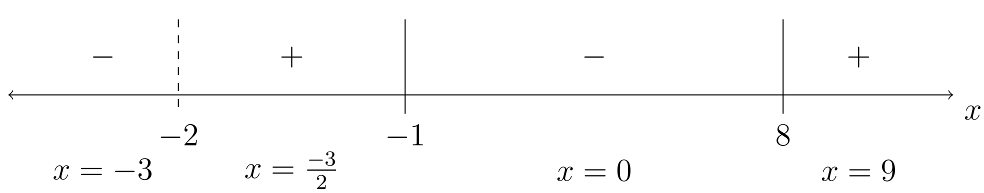
$$\frac{x(x-3)}{x+2} < 4.$$

Since each of our previous answers required us to use a sign diagram in order to determine whether an expression was positive or negative, it should seem logical to set one side of the given equation or inequality equal to zero, and proceed as before. All that remains is to obtain a common denominator so that we have one rational expression on the non-zero side. Our next example demonstrates this.

Example 348. Solve the inequality $\dfrac{x(x-3)}{x+2} < 4$.

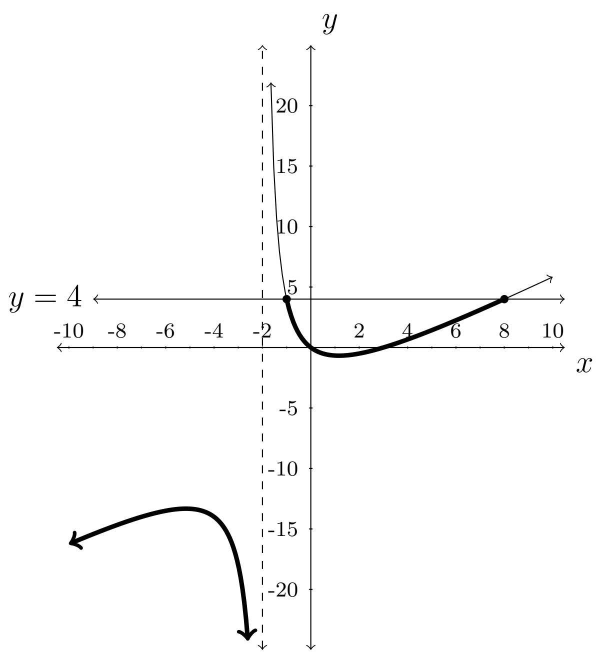
$$\begin{aligned}
\frac{x(x-3)}{x+2} &< 4 \\
\frac{x(x-3)}{x+2} - 4 &< \cancel{4} - \cancel{4} \\
\frac{x(x-3)}{x+2} - 4 \cdot \frac{x+2}{x+2} &< 0
\end{aligned}
\qquad
\begin{aligned}
\frac{x^2 - 3x - 4x - 8}{x+2} &< 0 \\
\frac{x^2 - 7x - 8}{x+2} &< 0 \\
\frac{(x-8)(x+1)}{x+2} &< 0
\end{aligned}$$

A sign diagram for the expression $\dfrac{(x-8)(x+1)}{x+2}$ is shown below.

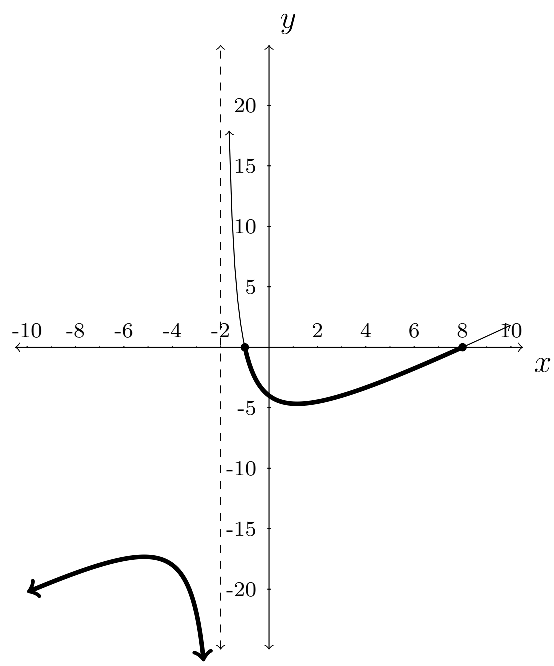


Thus, our inequality holds for all x in the set $(-\infty, -2) \cup (-1, 8)$.

Below, we show the graph of two functions, $f(x) = \dfrac{x(x-3)}{x+2}$ and $g(x) = \dfrac{(x-8)(x+1)}{x+2}$. In the case of the first graph, we see that our solution set coincides with those points that lie below the line $y = 4$, whereas in the case of the second graph, our solution set coincides with those points lying below the $x-$axis (the line $y = 0$). In both graphs, the points coinciding with our solution set appear in bold.



$$f(x) = \frac{x(x-3)}{x+2}$$



$$g(x) = \frac{(x-8)(x+1)}{x+2}$$

This same approach of setting one side of an inequality equal to zero and constructing a sign diagram should be taken with any rational inequality.

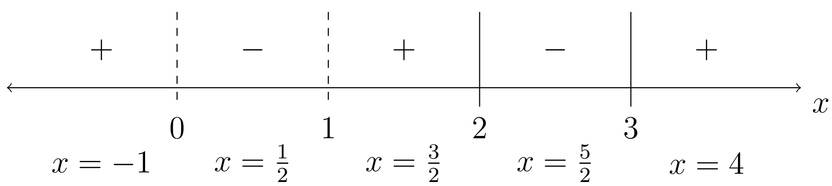
Our final example compares two basic rational expressions.

Example 349. Solve the following inequality

$$\frac{x-6}{x} \geq \frac{-2}{x-1}$$

$$\begin{aligned}
\frac{x-6}{x} &\geq \frac{-2}{x-1} \\
\frac{x-6}{x} + \frac{2}{x-1} &\geq \cancel{\frac{-2}{x-1}} + \cancel{\frac{2}{x-1}} \\
\frac{x-6}{x} + \frac{2}{x-1} &\geq 0 \\
\frac{(x-6)}{x} \cdot \frac{(x-1)}{(x-1)} + \frac{2}{(x-1)} \cdot \frac{x}{x} &\geq 0 \\
\frac{x^2-7x+6+2x}{x(x-1)} &\geq 0 \\
\frac{x^2-5x+6}{x(x-1)} &\geq 0 \\
\frac{(x-3)(x-2)}{x(x-1)} &\geq 0
\end{aligned}$$

This corresponds to the following sign diagram.



From our diagram, we see that $\frac{(x-3)(x-2)}{x(x-1)} \geq 0$, and consequently, $\frac{x-6}{x} \geq \frac{-2}{x-1}$ for all x in the set $(-\infty, 0) \cup (1, 2] \cup [3, \infty)$.

We leave it as an exercise to the reader to confirm our findings using Desmos.

Practice Problems

Perform the indicated tasks below for each of the following rational functions.

- Find the $y-$intercept of the corresponding graph.
- Find the domain of the function.
- Find all $x-$intercepts.
- Graph the function using Desmos, and identify the existence of any horizontal or vertical asymptotes.

1. $f(x) = \dfrac{x+4}{x+2}$
2. $f(x) = \dfrac{x-8}{x-2}$
3. $f(x) = \dfrac{2x-3}{x+3}$
4. $f(x) = \dfrac{2x}{x-1}$
5. $f(x) = \dfrac{x+5}{5-x}$
6. $f(x) = \dfrac{x+4}{2x-1}$
7. $f(x) = \dfrac{x^2}{x-2}$
8. $f(x) = \dfrac{3x+5}{3-x}$
9. $f(x) = \dfrac{x+1}{x^2+1}$

Perform the indicated tasks below for each of the following rational functions.

- Find the $y-$intercept of the corresponding graph.
- Find the domain of the function.
- Find all $x-$intercepts.
- Graph the function using Desmos, and identify the existence of any horizontal, vertical, or slant asymptotes.

10. $f(x) = \dfrac{(x-3)(x+4)}{x-6}$
11. $f(x) = \dfrac{(x-4)^2}{x+2}$
12. $f(x) = \dfrac{x-10}{(x+5)^2}$
13. $f(x) = \dfrac{(x+5)(3x+1)}{(x-1)^2}$
14. $f(x) = \dfrac{(x+3)(x-2)}{(x-4)(x+2)}$
15. $f(x) = \dfrac{2(x-1)^2(x+2)}{(2x-1)^2(x^2+1)}$

For each of exercises 16 through 19, there are many acceptable answers.

16. Find a rational function having a domain of $(-\infty, -3) \cup (-3, \infty)$ and whose graph has an $x-$intercept at $(6, 0)$.
17. Find a rational function whose graph has vertical asymptotes at $x = -2$ and $x = -1$, an $x-$intercept at $(6, 0)$, and a $y-$intercept at $(0, 3)$.
18. A student is asked to construct a rational function whose graph has a vertical asymptote at $x = \frac{1}{2}$, an $x-$intercept at $(2, 0)$, and a $y-$intercept at $(3, 0)$. The student's answer is shown below.
$$f(x) = \frac{x-2}{2x-1} + 3$$
Explain why the student's answer is incorrect, and make the necessary changes to find the desired function.
19. Find a rational function having a domain of $(-\infty, -\sqrt{3}) \cup (-\sqrt{3}, \sqrt{3}) \cup (\sqrt{3}, \infty)$, and whose graph has $x-$intercepts at $(-3, 0)$ and $(3, 0)$, and a $y-$intercept at $(0, 12)$.

Perform the indicated tasks below for each of the following rational functions.

- Find the $y-$intercept of the corresponding graph.
- Factor the function completely over the real numbers.
Note: Functions labeled $r(x)$ will require the Rational Root Theorem.

- Find the domain of the function.
- Find all $x-$intercepts.
- Graph the function using Desmos, and identify the existence of any horizontal, vertical, or slant asymptotes.

20. $f(x) = \dfrac{x^2-4}{x^2-x-12}$
21. $f(x) = \dfrac{2x^2-14x+24}{x^2-6}$
22. $f(x) = \dfrac{6x^2-13x-5}{x^3-3x^2-10x}$
23. $f(x) = \dfrac{x^2+10x+25}{x^3-5x^2+x-5}$
24. $f(x) = \dfrac{x^2-2x-2}{2x^2-7x-4}$
25. $f(x) = \dfrac{x^4-20x^2+64}{x^3+8x^2-x-8}$
26. $r(x) = \dfrac{2x^3+15x^2+16x-12}{x^2+6x+9}$
27. $r(x) = \dfrac{x^2-3x-10}{x^3+12x^2+3x+10}$
28. $r(x) = \dfrac{16-x^4}{x^4+16x^3-3x^2-46x+32}$

"Take the problem further" – Choose any of exercises 1 through 9 above, and multiply either the numerator or denominator (or both) by the factor $(x+1)$. Label your new function $g(x)$. Graph both f (the old function) and g on Desmos. Describe any similarities and differences between the two graphs.

Next, change the newly included factor from $(x+1)$ to $(x+1)^2$. Again, describe how this has impacted the new graph. We will look more closely at the impacts that such changes have on rational functions in the next few sections.

Selected Answers

$y-$intercept; Domain; $x-$intercept; Horizontal, Vertical, and Slant Asymptotes

1. $(0,2)$; $(-\infty,-2)\cup(-2,\infty)$; $(-4,0)$; HA at $y=1$, VA at $x=-2$
2. $(0,4)$; $(-\infty,2)\cup(2,\infty)$; $(8,0)$; HA at $y=1$, VA at $x=2$
3. $(0,-1)$; $(-\infty,-3)\cup(-3,\infty)$; $(\frac{3}{2},0)$; HA at $y=2$, VA at $x=-3$
4. $(0,0)$; $(-\infty,1)\cup(1,\infty)$; $(0,0)$; HA at $y=2$, VA at $x=1$
5. $(0,1)$; $(-\infty,5)\cup(5,\infty)$; $(-5,0)$; HA at $y=-1$, VA at $x=5$
6. $(0,-4)$; $(-\infty,\frac{1}{2})\cup(\frac{1}{2},\infty)$; $(-4,0)$; HA at $y=\frac{1}{2}$, VA at $x=\frac{1}{2}$
7. $(0,0)$; $(-\infty,2)\cup(2,\infty)$; $(0,0)$; HA at $y=0$, VA at $x=2$
8. $(0,\frac{5}{3})$; $(-\infty,3)\cup(3,\infty)$; $(-\frac{5}{3},0)$; HA at $y=-3$, VA at $x=3$
9. $(0,1)$; $(-\infty,\infty)$; $(-1,0)$; HA at $y=0$, No VA

10. $(0,2)$; $(-\infty,6)\cup(6,\infty)$; $(-4,0),(3,0)$; No HA, VA at $x=6$, SA at $y=x+7$

11. $(0,8)$; $(-\infty,-2)\cup(-2,\infty)$; $(4,0)$; No HA, VA at $x=-2$, SA at $y=x-10$
12. $(0,-\frac{2}{5})$; $(-\infty,-5)\cup(-5,\infty)$; $(10,0)$; HA at $y=0$, VA at $x=-5$
13. $(0,5)$; $(-\infty,1)\cup(1,\infty)$; $(-5,0),(-\frac{1}{3},0)$; HA at $y=3$, VA at $x=1$
14. $(0,\frac{3}{4})$; $(-\infty,-2)\cup(-2,4)\cup(4,\infty)$; $(-3,0),(2,0)$; HA at $y=1$, VA at $x=-2$ and $x=4$
15. $(0,4)$; $(-\infty,\frac{1}{2})\cup(\frac{1}{2},\infty)$; $(-2,0),(1,0)$; HA at $y=0$, VA at $x=\frac{1}{2}$
16. $f(x)=\dfrac{x-6}{x+3}$
17. $f(x)=\dfrac{-(x-6)}{(x+2)(x+1)}$
18. The student's graph has a vertical asymptote at $x=\frac{1}{2}$, but adding three to $\dfrac{x-2}{2x+1}$ shifts all points up three units, so the desired $x-$intercept will no longer be at $(2,0)$, since
$$\frac{x-2}{2x-1}+3=\frac{x-2+3(2x-1)}{2x-1}=\frac{7x-5}{2x-1}.$$
Instead, the student needs to rescale the original function by the appropriate factor. In this case, $f(x)=\dfrac{3(x-2)}{2(2x-1)}$.
19. $f(x)=\dfrac{4(x^2-9)}{x^2-3}$
20. $f(x)=\dfrac{x^2-4}{x^2-x-12}=\dfrac{(x+2)(x-2)}{(x+3)(x-4)}$

    $y-$int at $(0,\frac{1}{3})$
    Domain: $(-\infty,-3)\cup(-3,4)\cup(4,\infty)$
    $x-$int(s) at $(-2,0),(2,0)$
    HA at $y=1$, VA at $x=-3$ and $x=4$
21. $f(x)=\dfrac{2x^2-14x+24}{x^2-6}=\dfrac{2(x-3)(x-4)}{(x+\sqrt{6})(x-\sqrt{6})}$

    $y-$int at $(0,-4)$
    Domain: $(-\infty,-\sqrt{6})\cup(-\sqrt{6},\sqrt{6})\cup(\sqrt{6},\infty)$
    $x-$int(s) at $(3,0),(4,0)$
    HA at $y=2$, VA at $x=-\sqrt{6}$ and $x=\sqrt{6}$
22. $f(x)=\dfrac{6x^2-13x-5}{x^3-3x^2-10x}=\dfrac{(3x+1)(2x-5)}{x(x+2)(x-5)}$

    No $y-$int
    Domain: $(-\infty,-2)\cup(-2,0)\cup(0,5)\cup(5,\infty)$
    $x-$int(s) at $(-\frac{1}{3},0),(\frac{5}{2},0)$
    HA at $y=0$, VA at $x=-2$, $x=0$, and $x=5$

23. $f(x) = \dfrac{x^2 + 10x + 25}{x^3 - 5x^2 + x - 5} = \dfrac{(x+5)^2}{(x-5)(x^2+1)}$

    $y-$int at $(0, -5)$
    Domain: $(-\infty, 5) \cup (5, \infty)$
    $x-$int(s) at $(-5, 0)$
    HA at $y = 0$, VA at $x = 5$

24. $f(x) = \dfrac{x^2 - 2x - 2}{2x^2 - 7x - 4} = \dfrac{(x - (1 - \sqrt{3}))(x - (1 + \sqrt{3}))}{(2x+1)(x-4)}$

    $y-$int at $(0, \frac{1}{2})$
    Domain: $(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, 4) \cup (4, \infty)$
    $x-$int(s) at $(1 - \sqrt{3}, 0), (1 + \sqrt{3}, 0)$
    HA at $y = \frac{1}{2}$, VA at $x = -\frac{1}{2}$ and $x = 4$

25. $f(x) = \dfrac{x^4 - 20x^2 + 64}{x^3 + 8x^2 - x - 8} = \dfrac{(x+2)(x-2)(x+4)(x-4)}{(x+8)(x+1)(x-1)}$

    $y-$int at $(0, -8)$
    Domain: $(-\infty, -8) \cup (-8, -1) \cup (-1, 1) \cup (1, \infty)$
    $x-$int(s) at $(-4, 0), (-2, 0), (2, 0)$, and $(4, 0)$
    No HA, VA at $x = -8$, $x = -1$, and $x = 1$, SA at $y = x - 8$

26. $r(x) = \dfrac{2x^3 + 15x^2 + 16x - 12}{x^2 + 6x + 9} = \dfrac{(x+6)(x+2)(2x-1)}{(x+3)^2}$

    $y-$int at $(0, -\frac{4}{3})$
    Domain: $(-\infty, -3) \cup (-3, \infty)$
    $x-$int(s) at $(-6, 0), (-2, 0)$, and $(\frac{1}{2}, 0)$
    No HA, VA at $x = -3$, SA at $y = 2x + 3$

27. $r(x) = \dfrac{x^2 - 3x - 10}{x^3 + 12x^2 + 3x + 10} = \dfrac{(x+2)(x-5)}{(x+10)(x+1)^2}$

    $y-$int at $(0, -1)$
    Domain: $(-\infty, -10) \cup (-10, -1) \cup (-1, \infty)$
    $x-$int(s) at $(-2, 0), (5, 0)$
    HA at $y = 0$, VA at $x = -10$ and $x = -1$

28. $r(x) = \dfrac{16 - x^4}{x^4 + 16x^3 - 3x^2 - 46x + 32} = \dfrac{-(x+2)(x-2)(x^2+4)}{(x+16)(x+2)(x-1)^2}$

    $y-$int at $(0, \frac{1}{2})$
    Domain: $(-\infty, -16) \cup (-16, -2) \cup (-2, 1) \cup (1, \infty)$
    $x-$int(s) at $(2, 0)$ (Hole at $(-2, 0)$)
    HA at $y = -1$, VA at $x = -16$ and $x = 1$