Recursive Least square

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Least square method

Linear Regression

$$y(k) = \varphi^T(k)\,\theta + e(k)$$

Least square is

$$\theta = \left[\sum_{k=1}^{N} \varphi^T(k)\,\varphi(k)\right]^{-1} \sum_{k=1}^{N} y(k)\,\varphi(k)$$

regressive least square is to find the relation

$$\theta_{N+1} = f\left[\theta_N, y(N), \varphi(N)\right]$$

Let us define $G_N = \sum_{k=1}^{N} \varphi^T(k)\,\varphi(k)$

$$\theta_N = G_N^{-1} \sum_{k=1}^{N} y(k)\,\varphi(k)$$

$$\theta_{N+1} = G_{N+1}^{-1} \sum_{k=1}^{N+1} y(k)\,\varphi(k)$$

Here

$$\sum_{k=1}^{N+1} y(k)\varphi(k) = \sum_{k=1}^{N} y(k)\varphi(k) + y(N+1)\varphi(N+1)$$
$$= G_N\theta_N + y(N+1)\varphi(N+1)$$
$$+\varphi^T(N+1)\varphi(N+1)\theta_N - \varphi^T(N+1)\varphi(N+1)\theta_N$$
$$= \left[G_N + \varphi^T(N+1)\varphi(N+1)\right]\theta_N$$
$$+\varphi(N+1)\left[y(N+1) - \varphi^T(N+1)\theta_N\right]$$
$$= G_{N+1}\theta_N + \varphi(N+1)\left[y(N+1) - \varphi^T(N+1)\theta_N\right]$$

So

$$\theta_{N+1} = G_{N+1}^{-1}\left\{G_{N+1}\theta_N + \varphi(N+1)\left[y(N+1) - \varphi^T(N+1)\theta_N\right]\right.$$
$$= \theta_N + G_{N+1}^{-1}\varphi(N+1)\left[y(N+1) - \varphi^T(N+1)\theta_N\right]$$

We use following matrix inverse lemma

$$\left(A + BC^{-1}D\right)^{-1} = A^{-1} - A^{-1}B\left(C + DA^{-1}B\right)^{-1}DA^{-1}$$

where A, B, C, D are any matrices, A and C are non-singular.
$A = G_N$, $C = 1$, $B = \varphi^T(N+1)$, $D = \varphi(N+1)$

$$G_{N+1}^{-1} = \left[G_N + \varphi^T(N+1)\varphi(N+1)\right]^{-1}$$
$$= G_N^{-1} - G_N^{-1}\varphi^T(N+1)\left[1 + \varphi(N+1)G_N^{-1}\varphi^T(N+1)\right]^{-1}\varphi(N+$$

If we define $P_N = G_N^{-1}$, $P_{N+1} = G_{N+1}^{-1}$, the recursive least square is

$$\theta_{N+1} = \theta_N + P_{N+1}\varphi(N+1)\left[y(N+1) - \varphi^T(N+1)\theta_N\right]$$
$$P_{N+1} = P_N - \frac{P_N\varphi^T(N+1)\varphi(N+1)P_N}{1+\varphi(N+1)P_N\varphi^T(N+1)}$$

Or in other form

$$P_{N+1} = \frac{P_N\left[1+\varphi(N+1)P_N\varphi^T(N+1)\right]-P_N\varphi^T(N+1)\varphi(N+1)P_N}{1+\varphi(N+1)P_N\varphi^T(N+1)}$$
$$= \frac{P_N}{1+\varphi(N+1)P_N\varphi^T(N+1)}$$

So

$$\theta_{N+1} = \theta_N + \frac{P_N\varphi(N+1)}{1+\varphi(N+1)P_N\varphi^T(N+1)}\left[y\left(N+1\right)-\varphi^T\left(N+1\right)\theta_N\right]$$
$$P_{N+1} = P_N - \frac{P_N\varphi^T(N+1)\varphi(N+1)P_N}{1+\varphi(N+1)P_N\varphi^T(N+1)}$$

Algorithm: start from initial conditions $\theta_0 = \alpha$, $P_0 = \beta I$, $\beta > 0$at

$$t=1, \quad \begin{array}{l} P_1 = P_0 - \frac{P_0\varphi^T(1)\varphi(1)P_0}{1+\varphi(1)P_0\varphi^T(1)} \\ \theta_1 = \theta_0 + P_1\varphi\left(1\right)\left[y\left(1\right)-\varphi^T\left(1\right)\theta_0\right] \end{array}$$

$$t=2, \quad \begin{array}{l} P_2 = P_1 - \frac{P_1\varphi^T(2)\varphi(2)P_1}{1+\varphi(2)P_1\varphi^T(2)} \\ \theta_2 = \theta_1 + P_2\varphi\left(1\right)\left[y\left(2\right)-\varphi^T\left(2\right)\theta_1\right] \end{array}$$

when $N \to \infty$, $\theta_N \to \theta^*$, $y\left(N\right) = \varphi^T\left(N\right)\theta^* + e\left(N\right)$, $e^2\left(N\right)$ is minimized.

Remark 1 *P_k is positive and decreased.*

$$P_{N+1} = G_{N+1}^{-1} = \left[G_N + \varphi^T\left(N+1\right)\varphi\left(N+1\right)\right]^{-1}$$
$$P_{N+1}^{-1} = P_N^{-1} + \varphi^T\left(N+1\right)\varphi\left(N+1\right)$$

Since $\varphi^T\left(N+1\right)\varphi\left(N+1\right) > 0$, P_k^{-1} is positive and increased. So P_0 should be selected big enough (β should be big positive constant), such that P_{N+1} is not near zero.

Remark 2 *Since*

$$\begin{aligned}\theta_N &= \left[\sum_{k=1}^{N} \varphi^T(k)\,\varphi(k)\right]^{-1} \sum_{k=1}^{N} y(k)\,\varphi(k)\\ &= P_N \sum_{k=1}^{N} y(k)\,\varphi(k)\end{aligned}$$

when $N \to \infty$, $P_N \to P^*$

$$\theta^* = P^* \sum_{k=1}^{\infty} y(k)\,\varphi(k)$$

it has no relation with θ_0, θ_0 *can be any theoretically.*

Matrix case

$$Y(k) = \Phi(k)\,\Theta + E(k)$$

Least square is

$$\Theta = \left[\sum_{k=1}^{l} \Phi^T(k)\,\Phi(k)\right]^{-1} \sum_{k=1}^{l} Y^T(k)\,\Phi(k)$$

Recursive least square is

$$\begin{array}{l}\Theta_{N+1} = \Theta_N + \left[Y(N+1) - \Theta_N \Phi(N+1)\right]\Phi^T(N+1)\,P_{N+1}\\ P_{N+1} = P_N - P_N \Phi(N+1)\\ \left[I + \Phi^T(N+1)\,P_N \Phi(N+1)\right]^{-1} \Phi^T(N+1)\,P_N\end{array}$$

Continuous time **case**

$$\dot{\varphi}(t) = \theta\varphi(t)$$

Least square

$$\theta = Y\Gamma^{-1}$$

where $\Gamma(t) = \int \varphi\varphi^T dt$, $Y(t) = \int \dot{\varphi}\varphi^T dt$.

$$\begin{aligned} \dot{\theta} &= \dot{Y}\Gamma^{-1} + Y\dot{\Gamma}^{-1} \\ &= \dot{\varphi}\varphi^T\Gamma^{-1} + Y\dot{\Gamma}^{-1} \end{aligned}$$

Since $\Gamma\Gamma^{-1} = I$

$$\begin{aligned} &\dot{\Gamma}\Gamma^{-1} + \Gamma\dot{\Gamma}^{-1} = 0 \\ &\dot{\Gamma}^{-1} = -\Gamma^{-1}\dot{\Gamma}\Gamma^{-1} = -\Gamma^{-1}\left(\varphi\varphi^T\right)\Gamma^{-1} \end{aligned}$$

So

$$\begin{aligned} \dot{\theta} &= \dot{\varphi}\varphi^T\Gamma^{-1} - Y\Gamma^{-1}\left(\varphi\varphi^T\right)\Gamma^{-1} \\ &= \dot{\varphi}\varphi^T\Gamma^{-1} - \theta\left(\varphi\varphi^T\right)\Gamma^{-1} \\ &= \left[\dot{\varphi} - \theta\varphi\right]\varphi^T\Gamma^{-1} \end{aligned}$$

Recursive least square is

$$\begin{aligned} \dot{\theta} &= \left[\dot{\varphi} - \theta\varphi\right]\varphi^T\Gamma^{-1} \\ \dot{\Gamma}^{-1} &= -\Gamma^{-1}\left(\varphi\varphi^T\right)\Gamma^{-1} \end{aligned}$$

Other forms of least square

1) Forgetting factor method

$$y(k) = \varphi^T(k)\theta + e(k)$$

criterion $\lambda^2(k) > 1$

$$J = \sum_{k=1}^{l} e^2(k) = \sum_{k=1}^{l} \lambda^2(k)\left[y(k) - \varphi^T(k)\theta\right]^2$$
$$= \sum_{k=1}^{l}\left[\lambda(k)y(k) - \lambda(k)\varphi^T(k)\theta\right]^2$$
$$= \sum_{k=1}^{l}\left[y_1(k) - \varphi_1^T(k)\theta\right]^2$$

Least square is

$$\theta_{N+1} = \theta_N + \frac{P_N\varphi_1(N+1)}{1+\varphi_1(N+1)P_N\varphi_1^T(N+1)}\left[y_1(N+1) - \varphi_1^T(N+1)\theta_N\right]$$
$$\theta_{N+1} = \theta_N + \frac{P_N\varphi^T(N+1)\varphi(N+1)P_N}{\frac{1}{\lambda^2(k)}+\varphi(N+1)P_N\varphi^T(N+1)}\left[y(N+1) - \varphi^T(N+1)\theta_N\right]$$
$$P_{N+1} = P_N - \frac{P_N\varphi^T(N+1)\varphi(N+1)P_N}{\frac{1}{\lambda^2(k)}+\varphi(N+1)P_N\varphi^T(N+1)}$$

The forgetting factor $0 < \delta = \frac{1}{\lambda^2(k)} < 1$. It is used for slow time-varying system, for example the system is changed after $t = 15$, we can select

$$\lambda^2(k) = \begin{cases} 1 & K < 15 \\ 1.1 & K > 15 \end{cases}$$

So the data before $t = 15$ does not effect a lot on LS.

2) Reset P_k.

since

$$P_{N+1} = P_N - \frac{P_N \varphi^T(N+1)\varphi(N+1) P_N}{1 + \varphi(N+1) P_N \varphi^T(N+1)}$$

$P_N \to 0$, $\theta_{N+1} \approx \theta_N$. LS is stopped when P_N is small. When $t = t_1$, we let $P_{t_1} = P_0$

Homework 2

(1) Identify ARX model

$$A(q)\, y(k) = B(q)\, u(k)$$

where $A(q) = 1 + a_1 q^{-1} + a_2 q^{-2} + a_3 q^{-3}$, $B(q) = b_1 q^{-1} + b_2 q^{-2} + b_3 q^{-3}$, $A(q)$ is stable.

(2) Use matrix least square identify following linear system

$$x(k+1) = Ax(k) + Cu(k-1)$$

where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, $x(0) = [0,0]^T$ .A is stable

(3) Identify following nonlinear system

$$\begin{aligned} y_1(k) &= \tfrac{c_1}{1+c_2 \cos k} sgn(\sin k) + b_1 y_2(k-1)\, u_1(k) \\ y_2(k) &= \tfrac{c_3}{1+c_4 \sin k} y_1(k-1) \ln(1+k) + b_2 y_2(k-1)\, u_2(k) \end{aligned}$$

where $0 < c_i < 1$ ($i = 1, \cdots 4$), $1 < b_i < 3$ ($i = 1, 2$)

(4) Use least square to identify continuous time system (chemical

kinetic system)

$$\dot{c}_1(t) = -\frac{k_1}{V_l} c_1(t) Q(t)$$
$$\dot{c}_2(t) = -\frac{k_2}{V_l} c_2(t) Q(t)$$
$$\dot{Q}(t) = K_s [Q_m - Q(t)] - Q(t) [k_1 c_1(t) + k_2 c_2(t)]$$

where $Q_m = 1.68 \times 10^{-6}$, $V_l = 0.008$, $K_s = 0.2$,
$c_1(0) = c_2(0) = 2 \times 10^{-5}$, $Q(0) = 10^{-8}$
the parameters $10^4 < k_i < 10^6$, $i = 1, 2$

- *choose (1) parameters of plants; (2) input*
- *identify the parameter via LS*
- *compare plant and model*
- *comment*