CHAPTER 4:

FREQUENCY ANALYSIS OF SIGNALS AND SYSTEMS

Lesson #13: Review of CTFT and Sampling

Lesson #14: Discrete-Time Fourier Transform (DTFT) & Inverse DTFT

Lesson #15: DTFT properties

Lesson #16: Frequency spectrum of DT signals

Lesson #17: Frequency-domain characteristics of LTI systems

Lesson #18: Digital filters

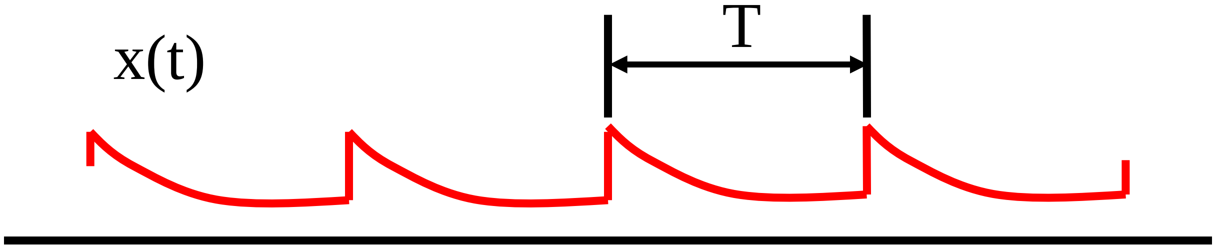
Lecture #13

Review of CTFT and Sampling

1. **The Fourier series for CT periodic signals**
2. The Fourier Transform for CT aperiodic signals (CTFT)
3. Sampling of CT signals
4. Aliasing

The Fourier series of CT periodic signal

x(t) is periodic and satisfies the Dirichlet conditions



Fourier series: $$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{j\frac{2\pi}{T}nt} \quad \forall t$$

Fourier coefficients: $$a_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-j\frac{2\pi}{T}nt} dt$$

Power density spectrum of periodic signal

A periodic signal has infinite energy and finite average power, which is

$$P_x = \frac{1}{T}\int_{-T/2}^{T/2} |\, x(t)\,|^2 dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t).x^*(t)dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\left(\sum_{n=-\infty}^{\infty} a_n^* e^{-j\frac{2\pi}{T}nt}\right)dt$$

$$= \sum_{n=-\infty}^{\infty} a_n^*\left[\frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j\frac{2\pi}{T}nt}dt\right] = \sum_{n=-\infty}^{\infty} |\, a_n\,|^2$$

Parseval's relation:

$$\boxed{P_x = \frac{1}{T}\int_{-T/2}^{T/2} |\, x(t)\,|^2 dt = \sum_{n=-\infty}^{\infty} |\, a_n\,|^2}$$

Lecture #13

Review of CTFT and Sampling

1. The Fourier series for CT periodic signals
2. **The Fourier Transform for CT aperiodic signals (CTFT)**
3. Sampling of CT signals
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The Fourier transform of CT aperiodic signal

- ***x(t)*** is aperiodic and satisfies the Dirichlet conditions

Fourier transform pair:

$$\boxed{\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \equiv FT\{x(t)\} \\ x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \equiv FT^{-1}\{X(\omega)\} \end{aligned}}$$

Fourier transform properties

Linearity: $\mathrm{ax}(\mathrm{t})+\mathrm{by}(\mathrm{t}) \overset{\mathrm{F}}{\leftrightarrow} \mathrm{aX}(\omega)+bY(\omega)$

Time shift: $x(t-\tau) \overset{\mathrm{F}}{\leftrightarrow} \mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{X}(\omega)$

Time reverse: $x(-t) \overset{\mathrm{F}}{\leftrightarrow} \mathrm{X}(-\omega)$

Convolution in time domain: $x(t) * y(t) \overset{\mathrm{F}}{\leftrightarrow} \mathrm{X}(\omega).Y(\omega)$

Multiplication in time domain: $x(t).y(t) \overset{\mathrm{F}}{\leftrightarrow} \dfrac{1}{2\pi}\mathrm{X}(\omega) * Y(\omega)$

Energy density spectrum of aperiodic signal

Let x(t) be finite energy signal, which is

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t).x^*(t) dt$$

$$= \int_{-\infty}^{\infty} x(t) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t} d\omega \right) dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \left(\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

Parseval's relation:

$$\boxed{E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega}$$

Lecture #13

Review of CTFT and Sampling

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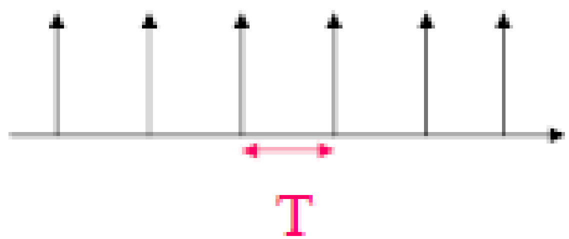
3. **Sampling of CT signals**
4. Aliasing

Sampling of CT signals

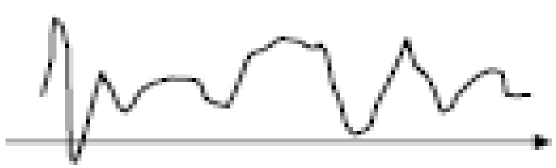
Define the CT impulse train as:

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$



A periodic signal !

x(t) is CT signal we want to sample:



To sample *x(t)*, we will multiply *x(t)* by *p(t)*

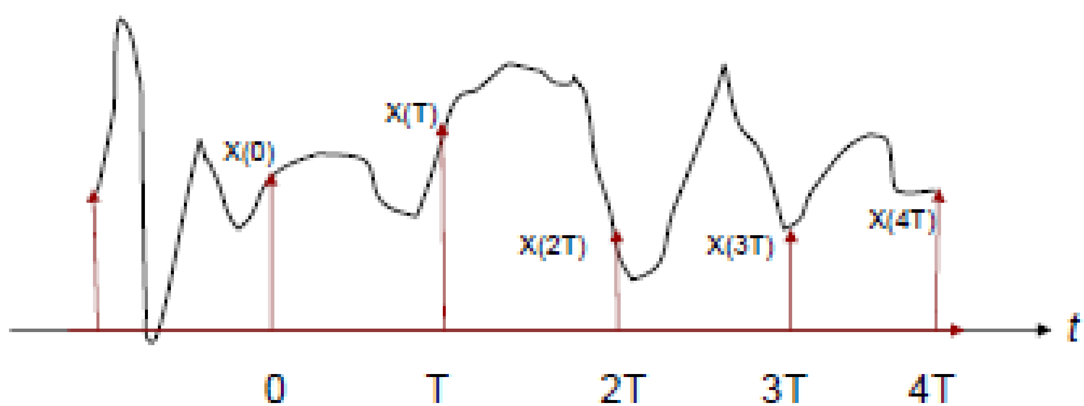
Sampling of CT signals

Let $x_s(t)$ be the sampled signal. Then,

$$x(t) \xrightarrow{\quad sampling \quad} x_s(t)$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$$

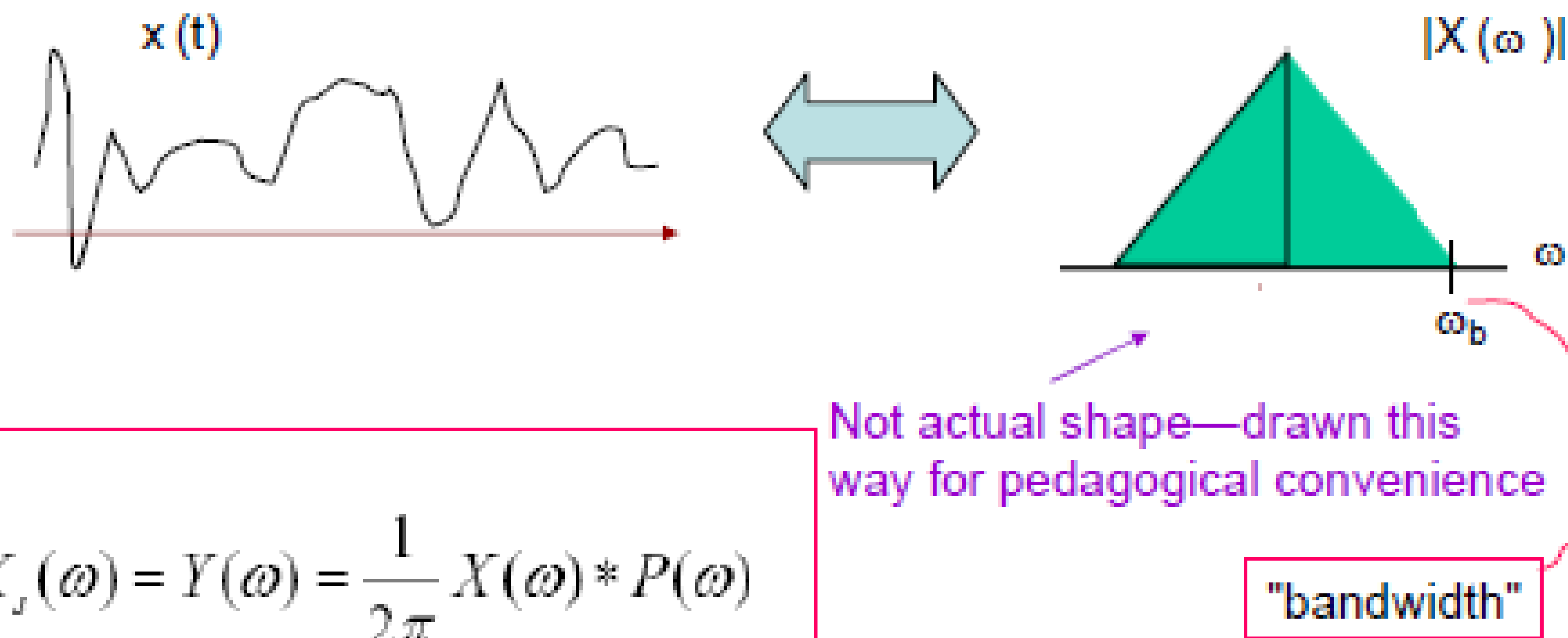
Signal ***$x_s(t)$*** consists of a train of CT impulses – take off the arrow heads to get ***x(n)*** – a DT signal



Spectrum of sampled signals

Consider ***$y(t) = x_s(t) = x(t).p(t)$*** in the frequency domain

Take Fourier transform of *$x(t)$*



$$X_s(\omega) = Y(\omega) = \frac{1}{2\pi} X(\omega) * P(\omega)$$

$X(\omega) = FT\{x(t)\}$ and $P(\omega) = FT\{p(t)\}$

Spectrum of sampled signals

Finding FT{p(t)}, using the continuous-time FT of periodic signals

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad \xleftrightarrow{FT} \quad P(\omega) = \sum_{n=-\infty}^{\infty} 2\pi a_n \delta(\omega - n\omega_s)$$

$$a_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-j\omega t} dt = \frac{1}{T} \quad \Rightarrow \quad \boxed{P(\omega) = \sum_{n=-\infty}^{\infty} \omega_s \delta(\omega - n\omega_s)}$$

A CT impulse train has a FT that is an impulse train in frequency

Spacing between pulses in time is T

Spacing between pulses in frequency is 2π/T

Increasing period in time domain decreases it in frequency domain

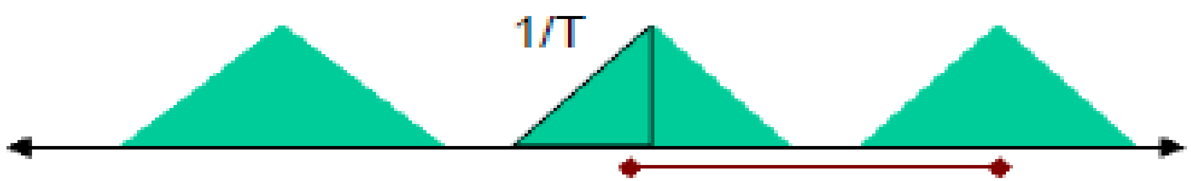
Spectrum of sampled signals

Back to $X_s(\omega)$; with ω_s: **the sampling frequency**

$$X_s(\omega)=\frac{1}{2\pi}X(\omega)*\left[\sum_{n=-\infty}^{\infty}\omega_s\delta(\omega-n\omega_s)\right]$$

$$=\frac{\omega_s}{2\pi}\sum_{n=-\infty}^{\infty}X(\omega-n\omega_s)=\frac{1}{T}\sum_{n=-\infty}^{\infty}X(\omega-n\omega_s)$$

The effect of sampling is an infinite sum of scaled, shifted copies of the continuous time signal's Fourier Transform



$$w_s=\frac{2\pi}{T}$$

Lecture #13

Review of CTFT and Sampling

1. The Fourier series for CT periodic signals
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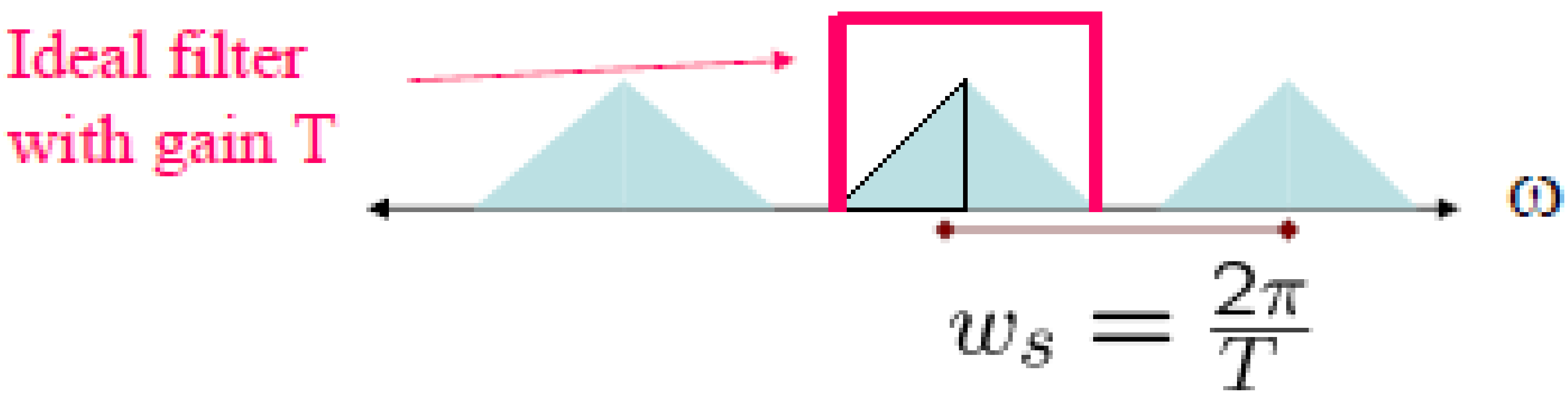
4. **Aliasing**

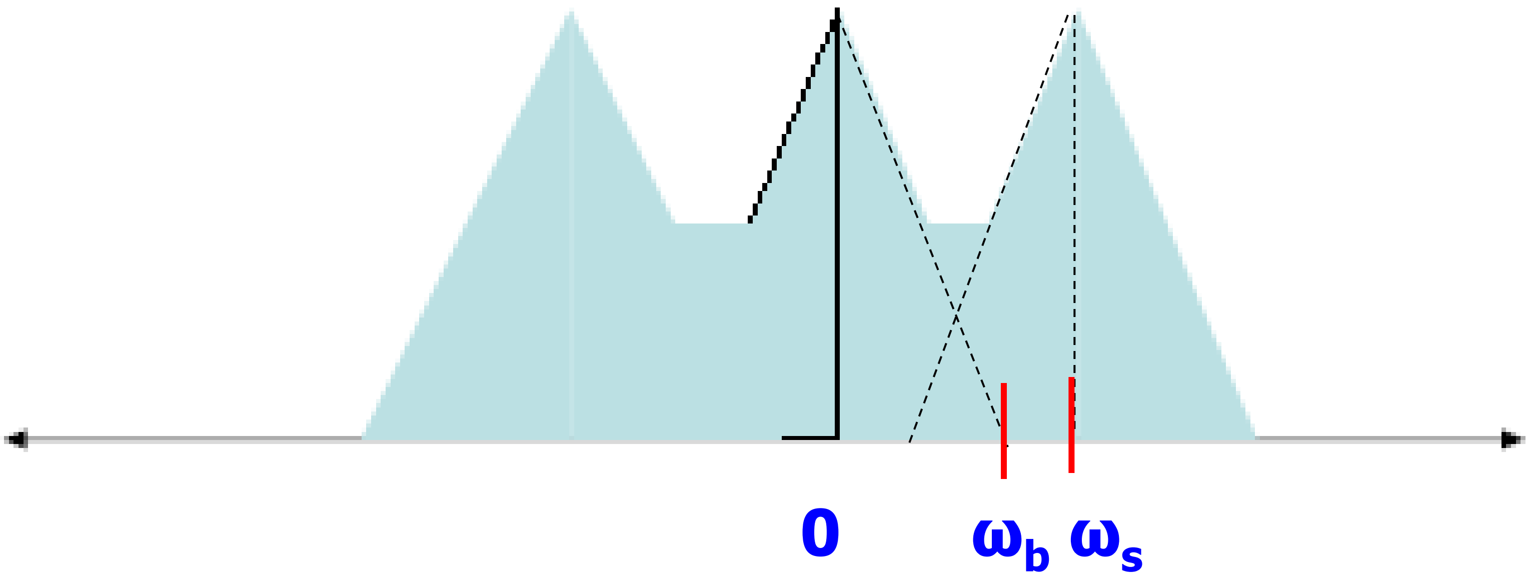
Aliasing

Note that the triangles don't overlap – so with an ideal low pass filter with cut-off frequency $\omega=\pi/T$, we could filter $x_s(t)$ to perfectly recover $x(t)$

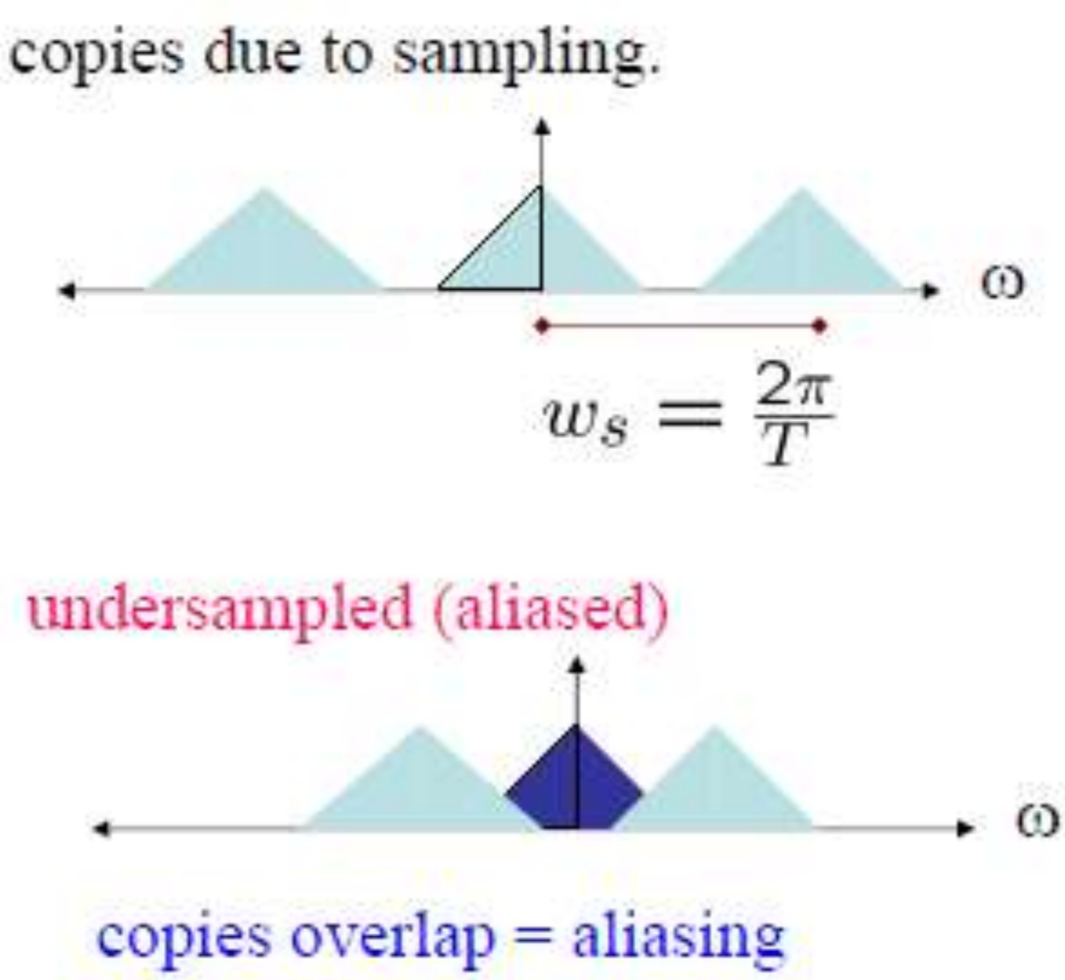


Aliasing (cont)

- The triangles overlap → can't recover x(t)
- Happens when $\omega_s = 2\pi/T < 2\omega_b$



Aliasing (cont)



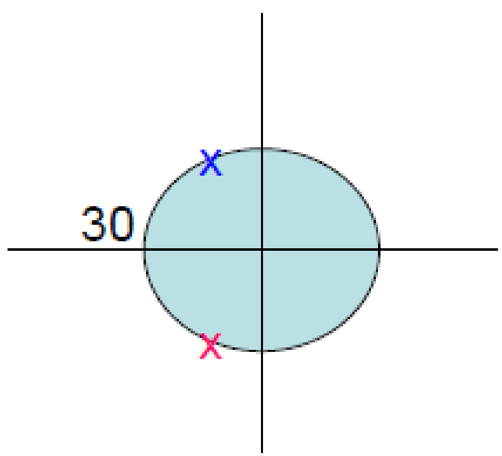
- **Avoid aliasing:** sampling faster than twice the highest frequency component – The Nyquist-Shannon sampling theorem

Examples

Ex: You sample a 40 Hz sinusoid at a sampling frequency of 60 Hz. You reconstruct a sinusoid from its samples. What frequency will the reconstructed sinusoid have?

60 Hz $\Rightarrow$ 2π

So 40 Hz $\Rightarrow$ $4\pi/3$ $\Rightarrow$ $2\pi/3$



So it looks like 20 Hz

(due to aliasing)

Examples

Ex: You sample a 40 Hz sinusoid at a sampling frequency of 120 Hz. You reconstruct a sinusoid from its samples. What frequency will the reconstructed sinusoid have?

40 < 60 No aliasing
Reconstructed sinusoid will have frequency of 40 Hz, like original signal

Examples

- Ex: You sample a 30 Hz sinusoid at a sampling frequency of 40 Hz. You reconstruct a sinusoid from its samples. What frequency will the reconstructed sinusoid have?

Examples

Ex: You sample a 149 Hz sinusoid at a sampling frequency of 150 Hz. You reconstruct a sinusoid from its samples. What frequency will the reconstructed sinusoid have?

Lecture #14

Discrete-Time Fourier Transform (DTFT)

1. From CTFT to DTFT

2. Convergence of the DTFT

3. The relation between DTFT and ZT

4. DTFT of signals with poles on the unit circle

5. Inverse DTFT

From CTFT to DTFT

Take a CT signal **x(t)** and sample it:

$$x(t) \xrightarrow{sampling} x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$$

The CTFT of the sampled signal is:

$$\mathrm{FT}\{x_s(t)\} = \sum_{n=-\infty}^{\infty} x(nT)\,\mathrm{FT}\{\delta(t-nT)\}$$

$$= \sum_{n=-\infty}^{\infty} x(nT) \int_{-\infty}^{\infty} \delta(t-nT) e^{-j\omega t} dt$$

$$= \sum_{n=-\infty}^{\infty} x(nT) e^{-j\omega nT} \underset{\omega T=\Omega}{\overset{T=1}{=}} \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} = X(\Omega)$$

DTFT formula

$$X(\Omega) = \mathrm{DTFT}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

- Discrete in time, but continuous in frequency and periodic with period of 2π
- Gives the complex frequency spectrum of DT signal
- Not all DTFT is converge

Lecture #14

Discrete-Time Fourier Transform (DTFT)

1. From CTFT to DTFT
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Convergence of the DTFT

We always have:

$$\left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}\right| \le \sum_{n=-\infty}^{\infty} \left|x[n]e^{-j\Omega n}\right|$$

$$\le \sum_{n=-\infty}^{\infty} \left|x[n]\right|\left|e^{-j\Omega n}\right|$$

$$\le \sum_{n=-\infty}^{\infty} \left|x[n]\right|$$

→ DTFT exists when:

$$\boxed{\sum_{n=-\infty}^{\infty} \left|x[n]\right| < \infty}$$

absolutely summable

not all discrete time signals have a DTFT

Examples

1) Find DTFT of x(n) where $x[n] = a^n u[n]$

$$X(\Omega) =$$

if |a| < 1

$\boxed{\text{If } |a| \geq 1\text{, DTFT does not exist}}$

Examples

2) Find DTFT of x(n) where $y[n] = a^n u[-n]$

$\Upsilon(\Omega) =$

if |a| > 1

$\boxed{\text{If } |a| \leq 1\text{, DTFT does not exist}}$

Examples

3) Find DTFT of p(n) where $p[n]=u[n]-u[n-N]$

Show that this DTFT has a linear phase term

$$P(\Omega)=\sum_{n=0}^{N-1}1.e^{-j\Omega n}=\frac{1-e^{-j\Omega N}}{1-e^{-j\Omega}}$$

Phase: -Ω(N-1)/2 → linear in phase

Examples

4) Find DTFT of h(n) where $h[n] = \delta[n] + 2\delta[n-1] + 2\delta[n-2] + \delta[n-3]$

Show that this DTFT has a linear phase term

$$\mathrm{H}(\Omega) = \sum_{n=0}^{3} h[n] e^{-j\Omega n} = 1 + 2e^{-j\Omega} + 2e^{-j2\Omega} + e^{-j3\Omega}$$

Phase: -3Ω/2 → linear in phase

Lecture #14

Discrete-Time Fourier Transform (DTFT)

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From ZT to DTFT

Recall ZT of **x(t)**:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Evaluating X(z) on the unit circle, if the unit circle is in the ROC of X(z)

$$X(z)\Big|_{z=e^{j\Omega}} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} = X(\Omega)$$

From ZT to DTFT

DTFT is the Z-transform of x(n) evaluated on the unit circle

$$\boxed{X(\Omega) = X(z)\Big|_{z=e^{j\Omega}}}$$

If the ROC of the ZT contains the unit circle, we can get the DTFT from the ZT by substitution **$z = e^{j\Omega}$**

Example

Find DTFT of x(n) where $x[n] = a^n u[n]$

$$X(z) = \frac{z}{z-a} \quad ROC: |z| > |a|$$

→ DTFT exists when ROC includes the unit circle, which means **|a| < 1:**

$$X(\Omega) = X(z)\Big|_{z=e^{j\Omega}} = \frac{e^{j\Omega}}{e^{j\Omega} - a}$$

If |a|≥1, DTFT does not exist

Lecture #14

Discrete-Time Fourier Transform (DTFT)

1. From CTFT to DTFT
2. Convergence of the DTFT
3. The relation between DTFT and ZT
4. **DTFT of signals with poles on the unit circle**
5. Inverse DTFT

DTFT of signal with poles on the unit circle

DTFT of *x[n]* can be determined by evaluating its ZT *X(z)* on the unit circle, provided that the unit circle lies within the ROC, or none of poles is on the unit circle

Rescue: just extend the DTFT!!!

Allowing the DTFT to contain impulses at certain frequencies corresponding to the location of the poles

Examples

Ex1: Consider this signal $x[n] = u[n]$

$$X(z) = \frac{z}{z-1} \quad ROC: |z| > 1$$

→ DTFT does not exist

Evaluate $X(z)$ on the unit circle, except at $z = 1 = e^{jk2\pi}$:

$$X(\Omega) = \frac{e^{j\Omega}}{e^{j\Omega}-1} = \frac{e^{j\Omega/2}.e^{j\Omega/2}}{e^{j\Omega/2}(e^{j\Omega/2}-e^{-j\Omega/2})} = \frac{e^{j\Omega/2}}{2j\sin(\Omega/2)} \qquad \Omega \neq k2\pi$$

$$\Rightarrow \quad |X(\Omega)| = \frac{1}{2|\sin(\Omega/2)|} \qquad \Omega \neq k2\pi$$

At $\Omega = k2\pi$, $X(\Omega)$ contains impulses

Examples

Ex2: Consider this signal *x[n] = (-1)ⁿu[n]*

$$X(z) = \frac{z}{z+1} \quad ROC: |z| > 1$$

→ DTFT does not exist

Evaluate *X(z)* on the unit circle, except at ***z = -1 = e^{j(2k+1)π}***

$$X(\Omega) = \frac{e^{j\Omega}}{e^{j\Omega}+1} = \frac{e^{j\Omega/2}.e^{j\Omega/2}}{e^{j\Omega/2}(e^{j\Omega/2}+e^{-j\Omega/2})} = \frac{e^{j\Omega/2}}{2\cos(\Omega/2)} \quad \Omega \neq (2k+1)\pi$$

$$\Rightarrow \quad |X(\Omega)| = \frac{1}{2|\cos(\Omega/2)|} \quad \Omega \neq (2k+1)\pi$$

At *Ω = (2k+1)π*, *X(Ω)* contains impulses

Examples

Ex3: Consider this signal $x[n] = (\cos\Omega_0 n)u[n]$

$$X(z) = \frac{z - \cos\Omega_0}{z^2 - 2z\cos\Omega_0 + 1} \quad ROC: |z| > 1$$ → DTFT does not exist

Evaluate $X(z)$ on the unit circle, except at

Lecture #14

Discrete-Time Fourier Transform (DTFT)

1. From CTFT to DTFT
2. Convergence of the DTFT
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4. DTFT of signals with poles on the unit circle
5. **Inverse DTFT**

The inverse of DTFT

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega) e^{j\Omega l} d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}\right] e^{j\Omega l} d\Omega$$

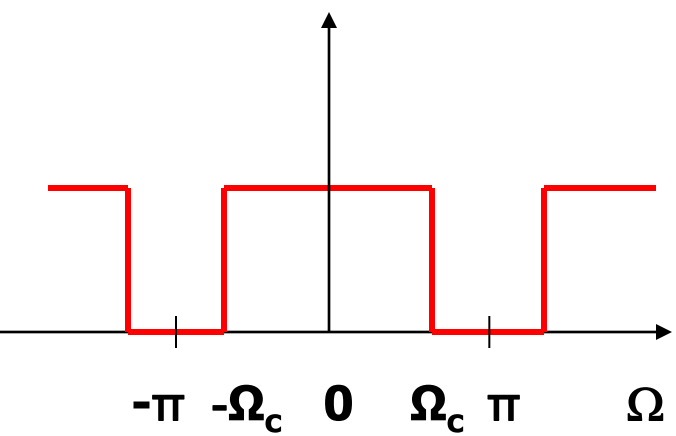
$$= \sum_{n=-\infty}^{\infty} x[n]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega(l-n)} d\Omega\right] = x[l]$$

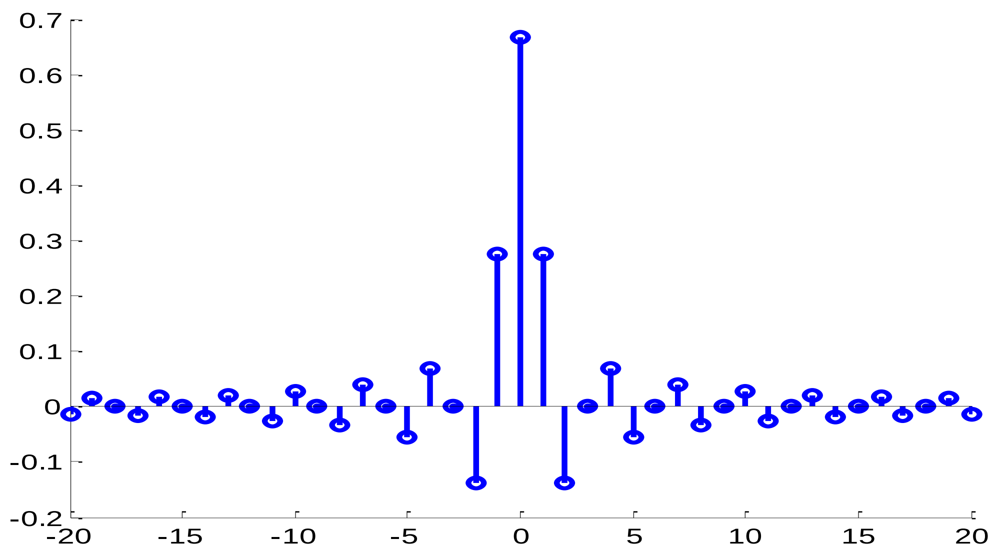
$$\boxed{x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)\, e^{j\Omega n} d\Omega}$$

Examples

Ex1. Find x(n) from its DTFT X(Ω): $X(\Omega) = \begin{cases} 1, & |\Omega| \le \Omega_c \\ 0, & \Omega_c < |\Omega| < \pi \end{cases}$

$$x[n] =$$

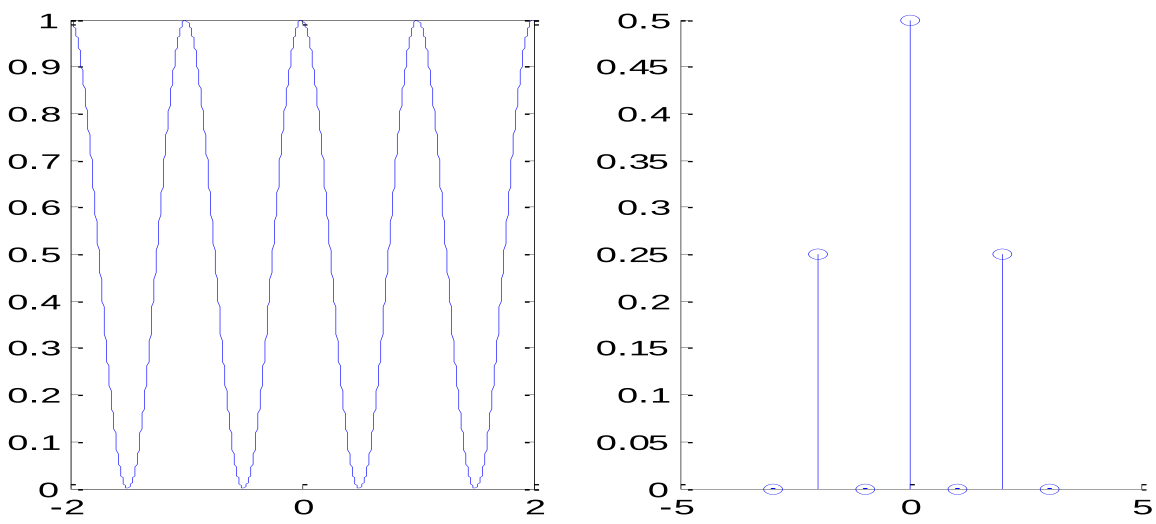




Examples

Ex2. Find x(n) from its DTFT X(Ω): $X(\Omega) = \cos^2 \Omega$

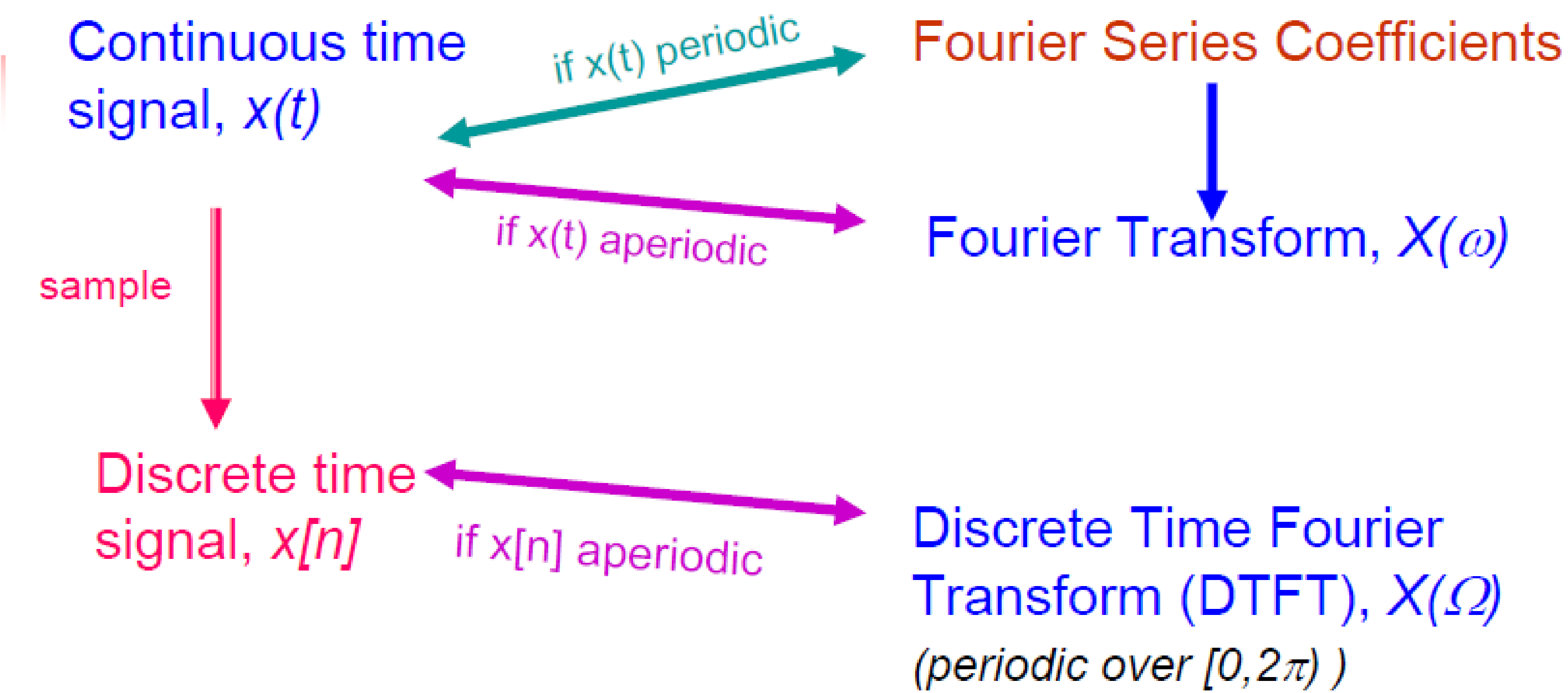
$$X(\Omega) =$$



Examples

Ex3. Find x(n) from its DTFT X(Ω): $X(\Omega) = \dfrac{e^{j\Omega}}{e^{j\Omega} - 2}$

SUMMARY



$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega \longleftrightarrow X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

Lecture #15

DTFT properties

1. **Linearity**
2. Time shifting
3. Frequency shifting and modulation
4. Differentiation in the frequency domain
5. Convolution in time domain
6. Convolution in frequency domain
7. Symmetry

Linearity

$$\boxed{ax[n]+by[n] \overset{\text{DTFT}}{\leftrightarrow} aX(\Omega)+bY(\Omega)}$$

The DTFT of a linear combination of two or more signals is equal to the same linear combination of the DTFT of the individual signals.

Example

Determine the DTFT of the signal $x[n] = a^{|n|}$ if |a| < 1

$$x[n] = \begin{cases} a^n & n \ge 0 \\ a^{-n} & n < 0 \end{cases} = a^n u[n] - \left[-\left(\frac{1}{a}\right)^n \right] u[-n-1]$$

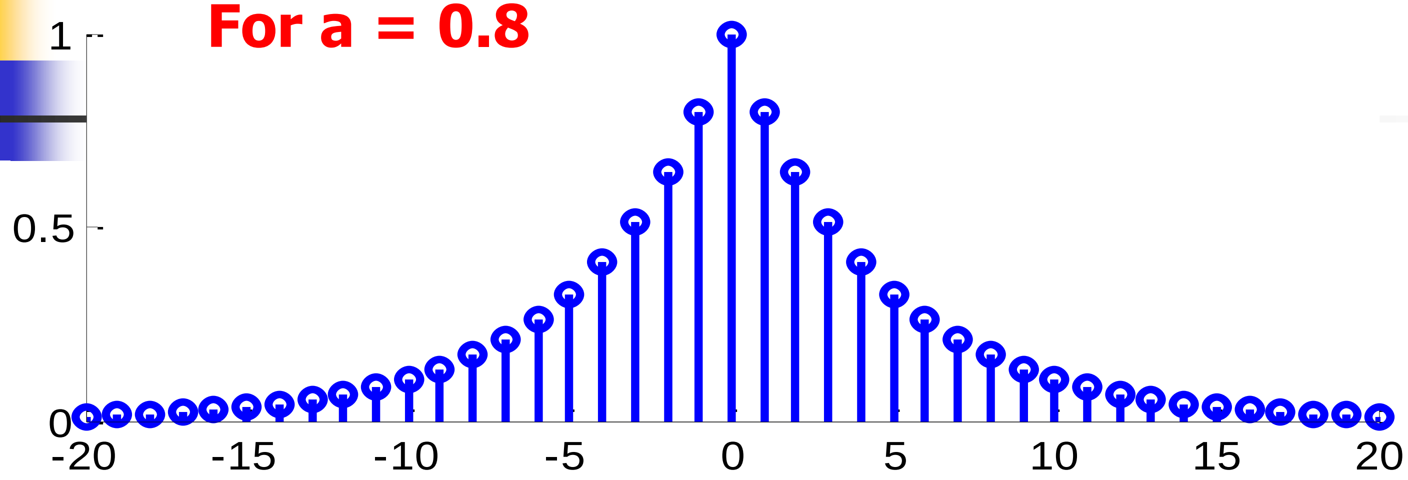
$$X(z) = \frac{z}{z-a} - \frac{z}{z-1/a} \quad ROC: |a| < |z| < \frac{1}{|a|}$$

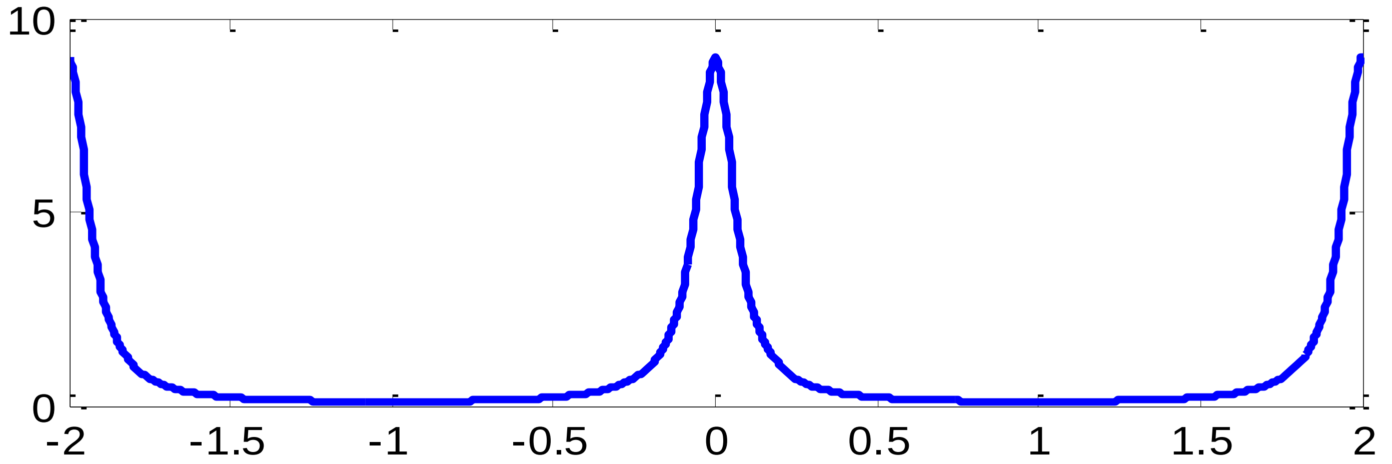
$$\Rightarrow X(\Omega) = \frac{e^{j\Omega}}{e^{j\Omega}-a} - \frac{e^{j\Omega}}{e^{j\Omega}-1/a} = \frac{e^{j\Omega}}{e^{j\Omega}-a} - \frac{ae^{j\Omega}}{ae^{j\Omega}-1}$$

$$= \frac{1-a^2}{1-ae^{j\Omega}-ae^{-j\Omega}+a^2} = \frac{1-a^2}{1-2a\cos\Omega+a^2}$$

Example (cont)

For a = 0.8





Lecture #15

DTFT properties

1. Linearity

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4. Differentiation in the frequency domain

5. Convolution in time domain

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Time shifting

$$\boxed{x[n-n_0] \overset{\text{DTFT}}{\longleftrightarrow} e^{-j\Omega n_0} X(\Omega)}$$

Proof: infer from the shifting property of ZT, then evaluate ZT on the unit circle

$$x[n-n_0] \overset{\text{ZT}}{\longleftrightarrow} z^{-n_0} X(z)$$

→ A shift in time causes **a linear phase shift** in frequency – **no change in DTFT magnitude**

Lecture #15

DTFT properties

1. Linearity
2. Time shifting
3. **Frequency shifting and modulation**
4. Differentiation in the frequency domain
5. Convolution in time domain
6. Convolution in frequency domain
7. Symmetry

Frequency shifting and modulation

$$e^{j\Omega_0 n}x[n] \stackrel{\text{DTFT}}{\longleftrightarrow} X(\Omega-\Omega_0)$$

$$\cos(\Omega_0 n)x[n] \stackrel{\text{DTFT}}{\longleftrightarrow} \frac{1}{2}X(\Omega-\Omega_0)+\frac{1}{2}X(\Omega+\Omega_0)$$

Proof:

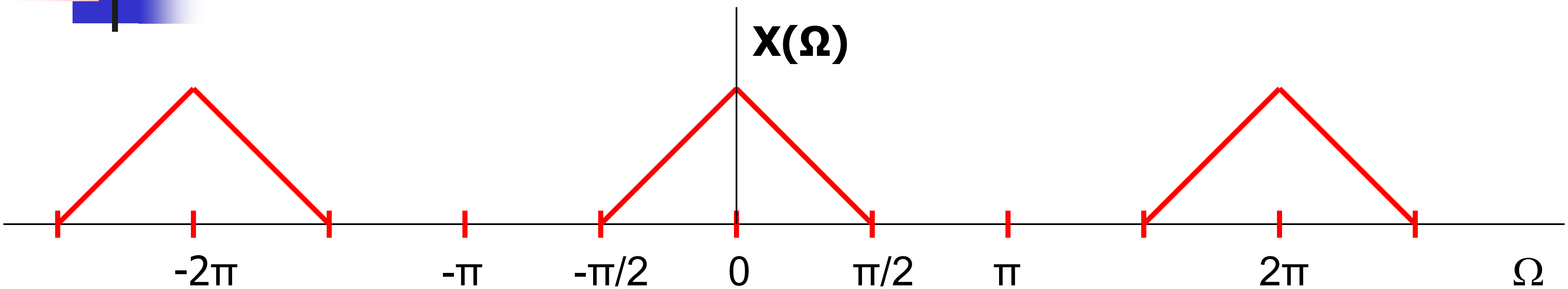
$$e^{j\Omega_0 n}x[n] \stackrel{DTFT}{\longleftrightarrow} \sum_{n=-\infty}^{\infty}(e^{j\Omega_0 n}x[n])e^{-j\Omega n} = \sum_{n=-\infty}^{\infty}x[n]e^{-j(\Omega-\Omega_0)n} = X(\Omega-\Omega_0)$$

$$\cos(\Omega_0 n) = \frac{1}{2}e^{j\Omega_0 n}+\frac{1}{2}e^{-j\Omega_0 n} \Rightarrow \ldots$$

Modulation causes **a shift** in frequency

Example

$$y_1[n] = x[n]\cos 0.5\pi n \quad and \quad y_2[n] = x[n]\cos \pi n$$



Lecture #15

DTFT properties

1. Linearity
2. Time shifting
3. Frequency shifting and modulation
4. **Differentiation in the frequency domain**
5. Convolution in time domain
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7. Symmetry

Differentiation in the frequency domain

$$\boxed{nx[n] \overset{DTFT}{\leftrightarrow} j\frac{dX(\Omega)}{d\Omega}}$$

Proof: infer from the differentiation-in-the-z-domain property of ZT, then evaluate ZT on the unit circle

$$nx[n] \overset{Z}{\leftrightarrow} -z\frac{dX(z)}{dz}$$

$$nx[n] \overset{DTFT}{\leftrightarrow} -e^{j\Omega}\frac{dX(\Omega)}{d(e^{j\Omega})} = -e^{j\Omega}\frac{dX(\Omega)}{je^{j\Omega}d\Omega} = j\frac{dX(\Omega)}{d\Omega}$$

Lecture #15

DTFT properties

1. Linearity
2. Time shifting
3. Frequency shifting and modulation
4. Differentiation in the frequency domain
5. **Convolution in time domain**
6. Convolution in frequency domain
7. Symmetry

Convolution in time domain

$$x_1[n] * x_2[n] \overset{\text{DTFT}}{\leftrightarrow} X_1(\Omega).X_2(\Omega)$$

Proof:

Infer from the convolution-in-time-domain property of ZT, then evaluate ZT on the unit circle

Convolution in time ←→ Multiplication in frequency

Example

Given $h[n] = a^n u[n]$, $|a| < 1$

Find its inverse system $h_i[n]$ but don't use ZT

$$H(\Omega) = \frac{e^{j\Omega}}{e^{j\Omega} - a}$$

$$\Rightarrow H_i(\Omega) = \frac{e^{j\Omega} - a}{e^{j\Omega}} = 1 - ae^{-j\Omega}$$

$$\Rightarrow h_i[n] = \delta[n] - a\delta[n-1]$$

Lecture #15

DTFT properties

1. Linearity
2. Time shifting
3. Frequency shifting and modulation
4. Differentiation in the frequency domain
5. Convolution in time domain
6. **Convolution in frequency domain**
7. Symmetry

Convolution in frequency domain

$$x_1[n].x_2[n] \overset{\text{DTFT}}{\longleftrightarrow} \frac{1}{2\pi}\int_{2\pi} X_1(\lambda)X_2(\Omega-\lambda)d\lambda = \frac{1}{2\pi}X_1(\Omega) * X_2(\Omega)$$

$$x_1[n].x_2[n] \overset{\text{DTFT}}{\longleftrightarrow} \sum_{n=-\infty}^{\infty}(x_1[n].x_2[n])e^{-j\Omega n}$$

Multiplication in time

$$= \sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{2\pi} X_1(\lambda)e^{j\lambda n}d\lambda\right)x_2[n]e^{-j\Omega n}$$

$$= \frac{1}{2\pi}\int_{2\pi} X_1(\lambda)\left(\sum_{n=-\infty}^{\infty}x_2[n]e^{-j(\Omega-\lambda)n}\right)d\lambda$$

$$= \frac{1}{2\pi}\int_{2\pi} X_1(\lambda)X_2(\Omega-\lambda)d\lambda$$

Convolution in frequency

Lecture #15

DTFT properties

1. Linearity
2. Time shifting
3. Frequency shifting and modulation
4. Differentiation in the frequency domain
5. Convolution in time domain
6. Convolution in frequency domain
7. **Symmetry**

Symmetry properties

Consider x[n] and X(Ω) are complex-valued functions

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} (x_R[n] + jx_I[n])(\cos\Omega n - j\sin\Omega n)$$

$$= X_R(\Omega) + jX_I(\Omega)$$

$$\Rightarrow X_R(\Omega) = \sum_{n=-\infty}^{\infty} (x_R[n]\cos\Omega n + x_I[n]\sin\Omega n)$$

$$\Rightarrow X_I(\Omega) = -\sum_{n=-\infty}^{\infty} (x_R[n]\sin\Omega n - x_I[n]\cos\Omega n)$$

Symmetry properties

Consider x[n] and X(Ω) are complex-valued functions

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}[X_R(\Omega) + jX_I(\Omega)](\cos\Omega n + j\sin\Omega n)d\Omega$$

$$= x_R[n] + jx_I[n]$$

$$\Rightarrow x_R[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}[X_R(\Omega)\cos\Omega n - X_I(\Omega)\sin\Omega n]d\Omega$$

$$\Rightarrow x_I[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}[X_R(\Omega)\sin\Omega n + X_I(\Omega)\cos\Omega n]d\Omega$$

Real signal

$$x[n] = x_R[n] \quad and \quad x_I[n] = 0$$

$$\Rightarrow X_R(\Omega) = \sum_{n=-\infty}^{\infty} x[n] \cos \Omega n = X_R(-\Omega) : even$$

$$\Rightarrow X_I(\Omega) = -\sum_{n=-\infty}^{\infty} x[n] \sin \Omega n = -X_I(-\Omega) : odd$$

$$\Rightarrow X^*(\Omega) = X(-\Omega)$$

$$\Rightarrow \begin{cases} |X(\Omega)| = \sqrt{X_R^2(\Omega) + X_i^2(\Omega)} = |X(-\Omega)| : even \\ \angle X(\Omega) = arctg \dfrac{X_I(\Omega)}{X_R(\Omega)} = -\angle X(-\Omega) : odd \end{cases}$$

Real signal (cont)

$$x[n]\ even:$$

$$X_R(\Omega) = x[0] + 2\sum_{n=1}^{\infty} x[n]\cos\Omega n;\ X_I(\Omega) = 0$$

$$x[n] = \frac{1}{\pi}\int_0^{\pi} X_R(\Omega)\cos\Omega n\, d\Omega$$

$$x[n]\ odd:$$

$$X_I(\Omega) = -2\sum_{n=1}^{\infty} x[n]\sin\Omega n;\ X_R(\Omega) = 0$$

$$x[n] = -\frac{1}{\pi}\int_0^{\pi} X_I(\Omega)\sin\Omega n\, d\Omega$$

Symmetry properties

$x[n]$	$X(\Omega)$
$x^*[n]$	$X^*(-\Omega)$
$x^*[-n]$	$X^*(\Omega)$
$x_R[n]$	$X_e(\Omega)$
$jx_I[n]$	$X_o(\Omega)$
$x_e[n]$	$X_R(\Omega)$
$x_o[n]$	$jX_I(\Omega)$

$$\text{any real signal}\begin{cases} X(\Omega) = X^*(-\Omega) \\ |X(\Omega)| = |X(-\Omega)| \\ \angle X(\Omega) = -\angle X(-\Omega) \end{cases}$$

real and even	*real and even*
real and odd	*imaginary and odd*

Lecture #16

Frequency spectrum of DT signals

1. **Frequency spectrum**
2. Amplitude spectrum and phase spectrum
3. Energy spectral density (ESD)
4. Bandwidth

Frequency spectrum

- Representation of the signal in the frequency domain
- Being generated via frequency analysis tools:
  - CTFT for aperiodic CT signal
  - CT Fourier series for periodic CT signal
  - DTFT for aperiodic DT signal
  - DT Fourier series for periodic DT signal

Frequency spectrum analysis

- The technical process of decomposing a complex signal into simpler parts.
- The process of quantifying the various amounts (e.g. amplitudes, powers, intensities, or phases), versus frequency.
- **Summary:**

- Aperiodic CT signals have aperiodic continuous-frequency spectra
- Periodic CT signals have aperiodic discrete-frequency spectra
- Aperiodic DT signals have periodic continuous-frequency spectra
- Periodic DT signals have periodic discrete-frequency spectra

Frequency range of some biological signals

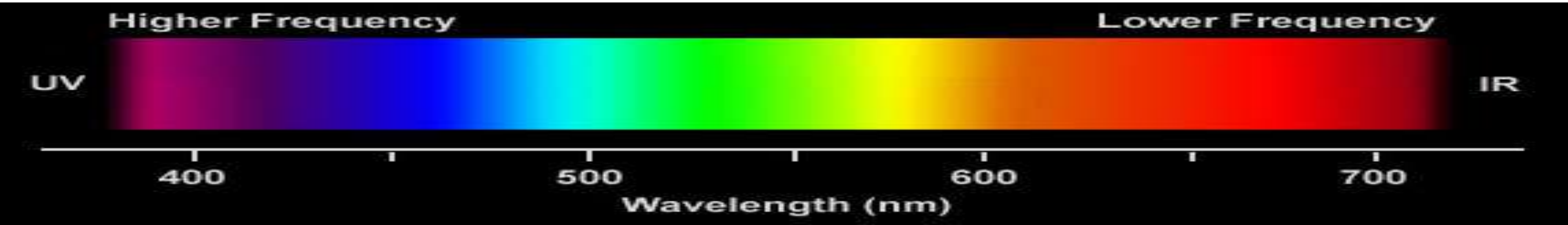
Type of signal	Frequency range (Hz)
Electroretinogram	0-20
Pneumogram	0-40
Electrocardiogram (ECG)	0-100
Electroencephalogram (EEG)	0-100
Electromyogram	10-200
Sphygmomanogram	0-200

Frequency range of some biological signals

Type of signal	Frequency range (Hz)
Wind noise	100-1000
Seismic exploration signals	10-100
Earthquake and nuclear explosion signals	0.01-10
Seismic signals	0.1-1

Frequency range of some EM signals

Type of signal	Frequency range (Hz)
Radio broadcast	$3x10^{-4} – 3x10^{6}$
Shortwave radio signals	$3x10^{6} – 3x10^{10}$
Radar, satellite communications…	$3x10^{8} – 3x10^{10}$
Infrared	$3x10^{11} – 3x10^{14}$
Visible light	$3.7x10^{14} – 7.7x10^{14}$
Ultraviolet	$3x10^{15} – 3x10^{16}$
X-rays	$3x10^{17} – 3x10^{18}$



Frequency spectrum representation

- Real and imaginary parts:

$$X(\Omega) = X_R(\Omega) + jX_I(\Omega)$$

- Absolute and argument:

$$X(\Omega) = |X(\Omega)|e^{j\varphi(\Omega)} = \sqrt{[X_R(\Omega)]^2 + [X_I(\Omega)]^2}e^{jarctg\left(\frac{X_I(\Omega)}{X_R(\Omega)}\right)}$$

- Magnitude and phase:

$$X(\Omega) = A(\Omega)e^{j\theta(\Omega)}$$

$$A(\Omega)\, can\, be\, positive, negative\, or\, zero$$

$$\varphi(\Omega) = \begin{cases} \theta(\Omega) + 2k\pi \; if \;\; A(\Omega) \geq 0 \\ \theta(\Omega) + (2k+1)\pi \; if \;\; A(\Omega) < 0 \end{cases}$$

Example

- Consider this even signal:

$$x[n] = \begin{cases} 1, & -2 \le n \le 2 \\ 0, & elsewhere \end{cases}$$

- Its frequency spectrum:

$$X(\Omega) = X_R(\Omega) = 1 + 2\sum_{n=1}^{2} \cos \Omega n$$

$$= 1 + 2\cos \Omega n + 2\cos 2\Omega n$$

- Magnitude and phase: $A(\Omega) = 1 + 2\cos \Omega n + 2\cos 2\Omega n$

$$\theta(\Omega) = 0$$

- Absolute and argument: $|X(\Omega)| = |1 + 2\cos \Omega n + 2\cos 2\Omega n|$

$$\varphi(\Omega) = \begin{cases} 0 \; if \;\; X(\Omega) \ge 0 \\ \pi \; if \;\; X(\Omega) < 0 \end{cases}$$

Lecture #16

Frequency spectrum of DT signals

1. Frequency spectrum
2. **Amplitude spectrum and phase spectrum**
3. Energy spectral density (ESD)
4. Bandwidth

Amplitude spectrum and phase spectrum

$$\boxed{X(\Omega) = \left|X(\Omega)\right| e^{j\varphi(\Omega)}}$$

Amplitude spectrum **Phase spectrum**

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \quad ; \quad X(-\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{j\Omega n}$$

$$\Rightarrow \quad X(\Omega) = X^*(-\Omega)$$

$$\Rightarrow \quad |X(\Omega)| = |X(-\Omega)| \quad \text{and} \quad \angle X(\Omega) = -\angle X(-\Omega)$$

Example

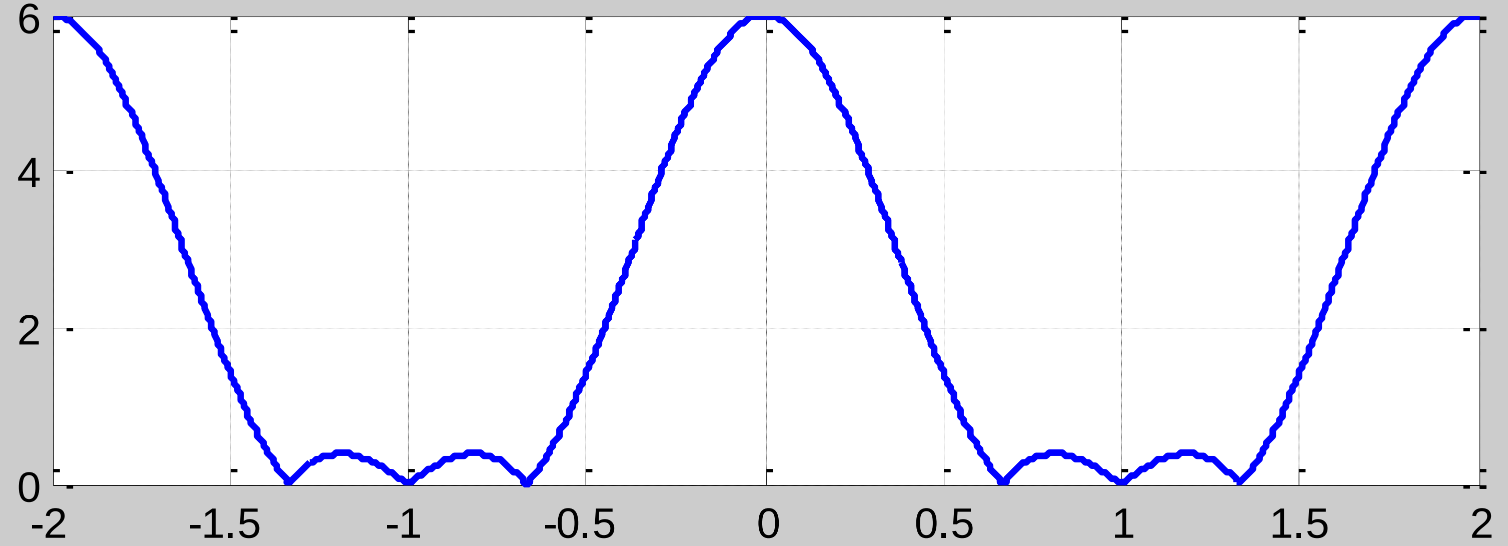
Find and plot amplitude spectrum and phase spectrum:

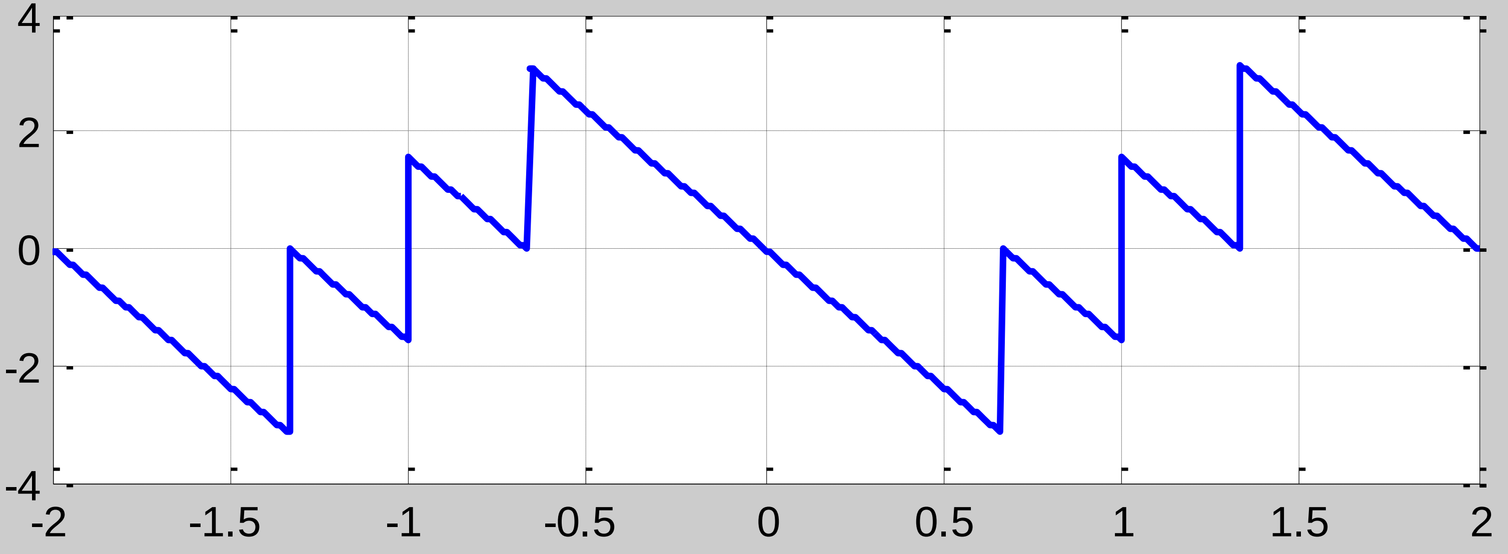
$$x[n] = u[n] - u[n - 4]$$

Using Matlab to plot amplitude spectrum and phase spectrum

```
w = -2*pi:pi/255:2*pi; % freq. -2π → 2π, resolution of π/255

X =4*sinc(2*w/pi)./sinc(w/(2*pi)).*exp(-j*1.5*w);

subplot(2,1,1);

plot(w/pi,abs(X)); % plot amplitude spectrum

subplot(2,1,2);

plot(w/pi,phase(X)); % plot phase spectrum
```

Amplitude spectrum





Phase spectrum

Lecture #16

Frequency spectrum of DT signals

1. Frequency spectrum
2. Amplitude spectrum and phase spectrum
3. **Energy spectral density (ESD)**
4. Bandwidth

Energy spectral density (ESD)

$$E=\sum_{n=-\infty}^{\infty}|x[n]|^2=\sum_{n=-\infty}^{\infty}x[n]x^*[n]=\sum_{n=-\infty}^{\infty}x[n]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}X^*(\Omega)e^{-j\Omega n}d\Omega\right]$$

Changing the order of summation & intergral:

$$\mathrm{E}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{X}^*(\Omega)\left[\sum_{\mathrm{n}=-\infty}^{\infty}\mathrm{x}[\mathrm{n}]\mathrm{e}^{-\mathrm{j}\Omega\mathrm{n}}\right]\mathrm{d}\Omega=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\mathrm{X}(\Omega)\right|^2\mathrm{d}\Omega$$

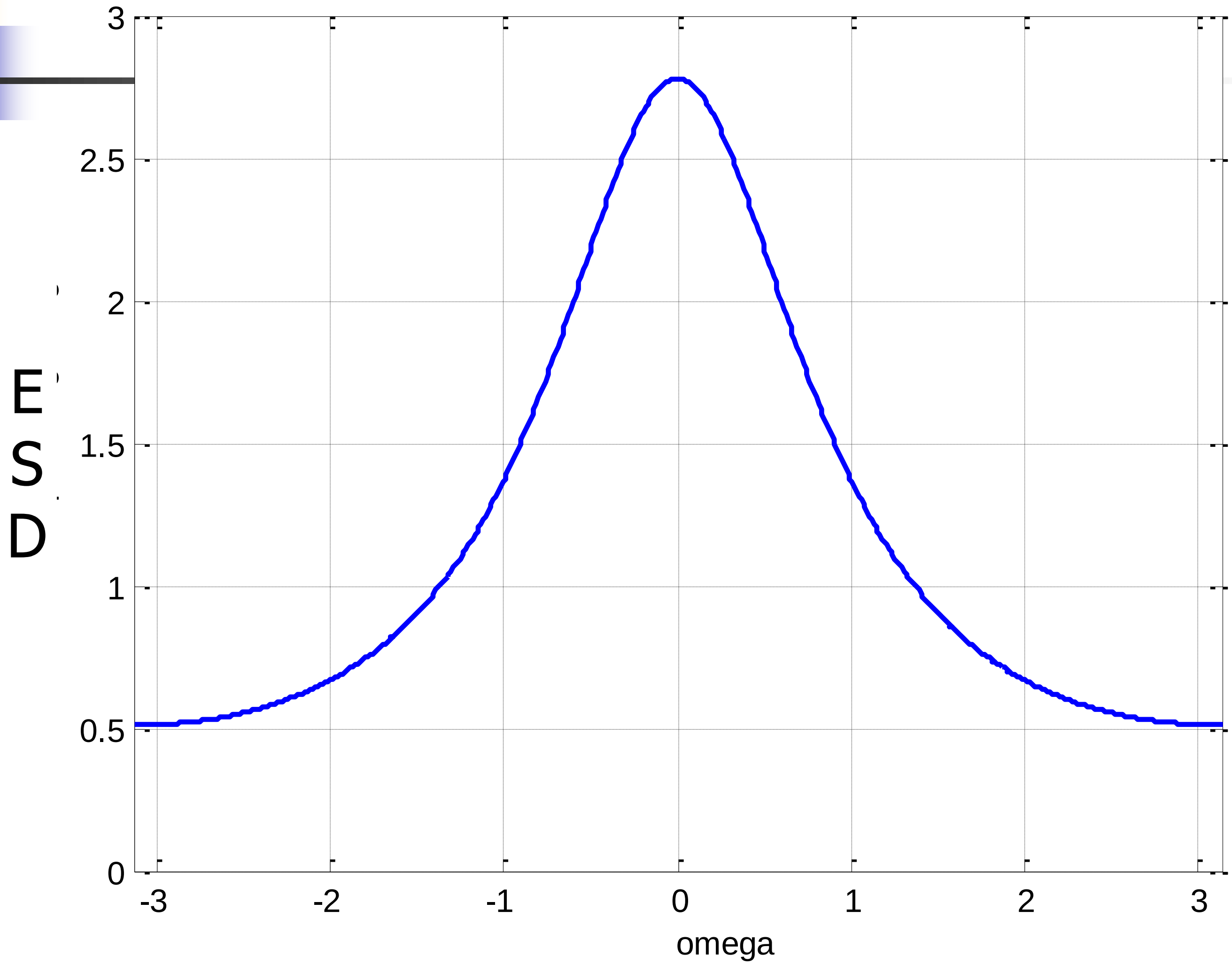
Energy spectral density

Example

Given $x[n] = a^n u[n]$, $-1<a<1$

$$X(\Omega) = \frac{1}{1 - ae^{-j\Omega}}$$

Example



Lecture #16

Frequency spectrum of DT signals

1. Frequency spectrum
2. Amplitude spectrum and phase spectrum
3. Energy spectral density (ESD)
4. **Bandwidth**

Bandwidth of a signal

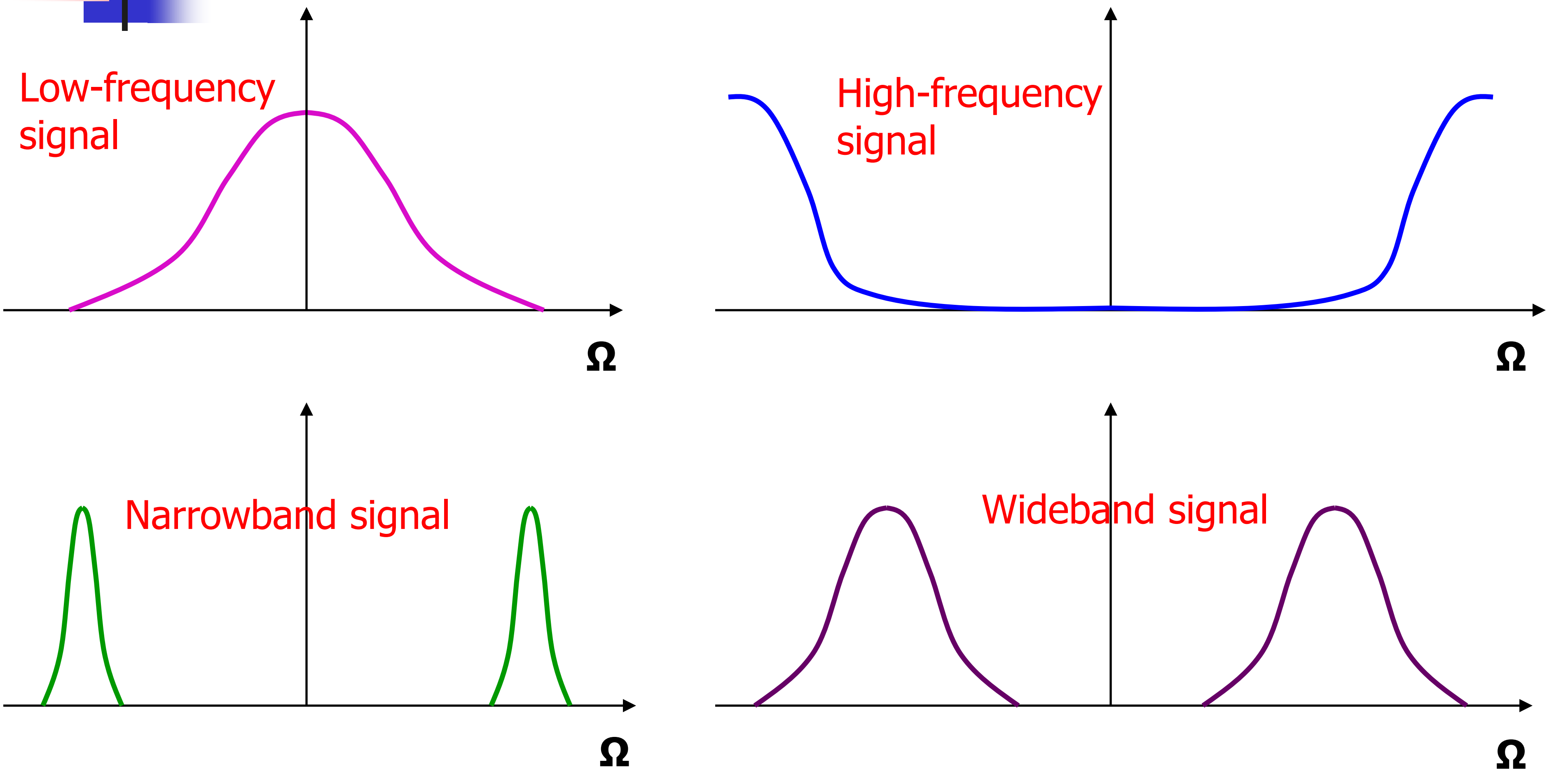
- **Bandwidth:** the range of frequencies over which the power or energy density spectrum is concentrated.
- **Ex:** a signal has *95%* of its ESD or PSD concentrated in the range from F_1 to F_2 then the 95% bandwidth of the signal is (F_2-F_1).
- Similarly, we may define *75%* or *90%* or *99%* bandwidth of the signal

Frequency-domain classification of signals

- Low-frequency signal: PSD or ESD concentrates about zero frequency
- High-frequency signal: PSD or ESD concentrates at high frequencies
- Bandpass signal: PSD or ESD concentrates somewhere in the broad frequency range between low frequency and high frequency

- Narrowband signal: the bandwidth $(F_2\text{-}F_1)$ is much smaller (by a factor of $10+$) than $(F_1+F_2)/2$
- Wideband signal

Frequency-domain classification of signals



Lecture #17

Frequency-domain characteristics of LTI systems

1. **Frequency response function**
2. Response to complex exponential and sinusoidal signals
3. Steady-state and transient response
4. Response to aperiodic input signals

Frequency response

- For impulse response, **h(n),** its DTFT is often called **frequency response H(Ω)**
- **H(Ω)** completely characterizes a LTI system in the frequency domain
- **H(Ω)** allows us to determine the steady-state response of the system to any arbitrary weighted linear combination of sinusoids or complex exponential
- **H(Ω)** exists if the system is BIBO stable

$$H(\Omega) = \underbrace{\left|H(\Omega)\right|}_{\text{Amplitude response}} \underbrace{e^{j\varphi(\Omega)}}_{\text{Phase response}}$$

Amplitude response **Phase response**

Determination of frequency response

1. From impulse response: Just take DTFT of $h[n]$

2. From difference equation:

- Take DTFT for both side
- Put the $Y(\Omega)$ on left side
- Divide both side by $X(\Omega)$

3. From block diagram:

(1) Find the difference equation, then find $H(\Omega)$ from equation

(2) Put $X(\Omega)=1$ as input, then directly find $Y(\Omega)=H(\Omega)$

4. From transfer function:

Evaluate $H(z)$ on the unit circle

Example of frequency response

- A LTI causal system is described by the following equation:

$$y[n]+0.1y[n-1]+0.85y[n-2]=x[n]-0.3x[n-1]$$

- First, checking the stability of the system (by using Matlab):

```
b = [1 -0.3];
a = [1 0.1 0.85];
zplane(b,a) % plot zeros and poles to check if all poles are inside the unit circle
```

- Second, take DTFT for two sides:

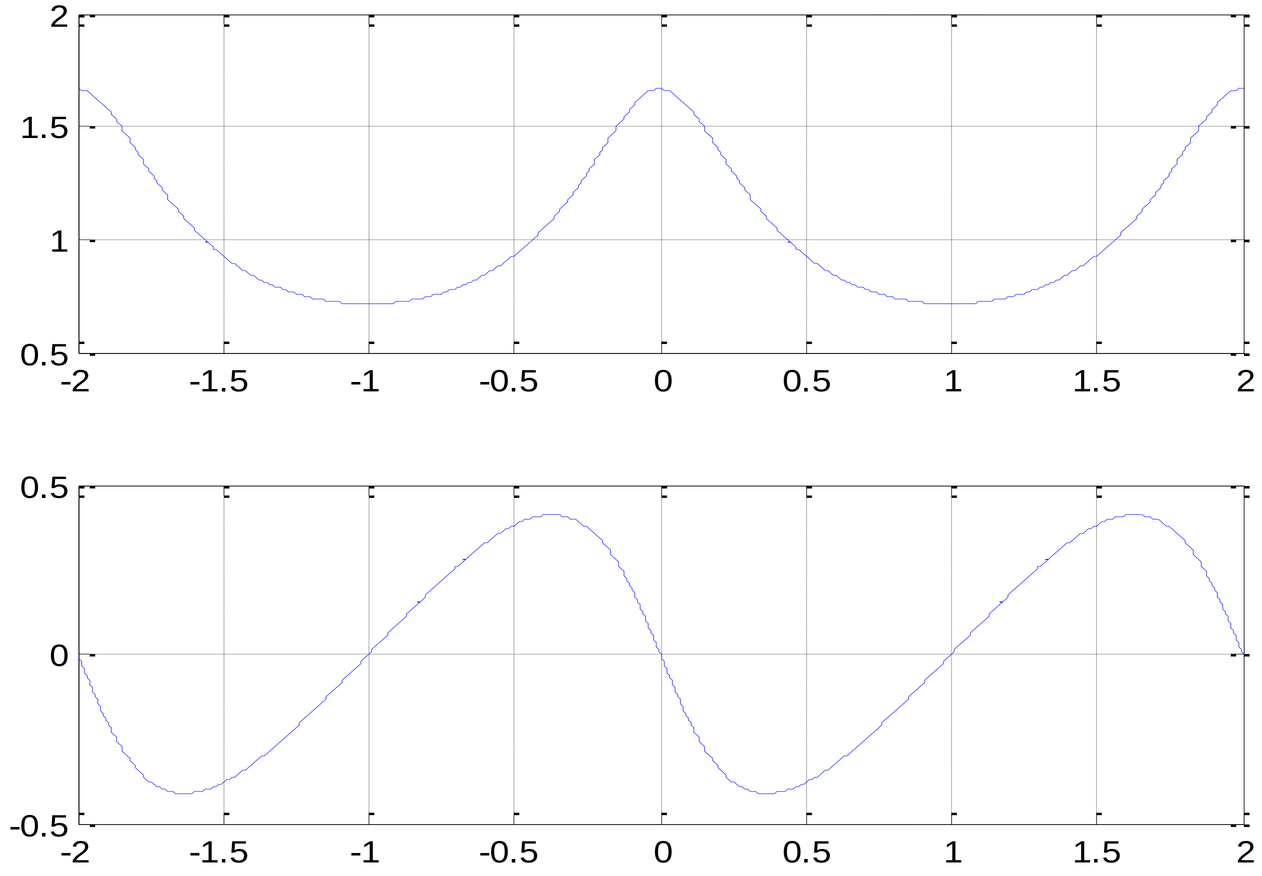
$$H(\Omega)=\frac{1-0.3e^{-j\Omega}}{1+0.1e^{-j\Omega}+0.85e^{-j2\Omega}}$$

Example of amplitude and phase responses

$$H(\Omega) = \frac{1}{1 - 0.4e^{-j\Omega}}$$

Example of amplitude and phase responses

$$H(\Omega) = \frac{1}{1 - 0.4e^{-j\Omega}}$$



Example of moving average filter

$$y[n] = \frac{1}{3}(x[n+1\} + x[n] = x[n-1])$$

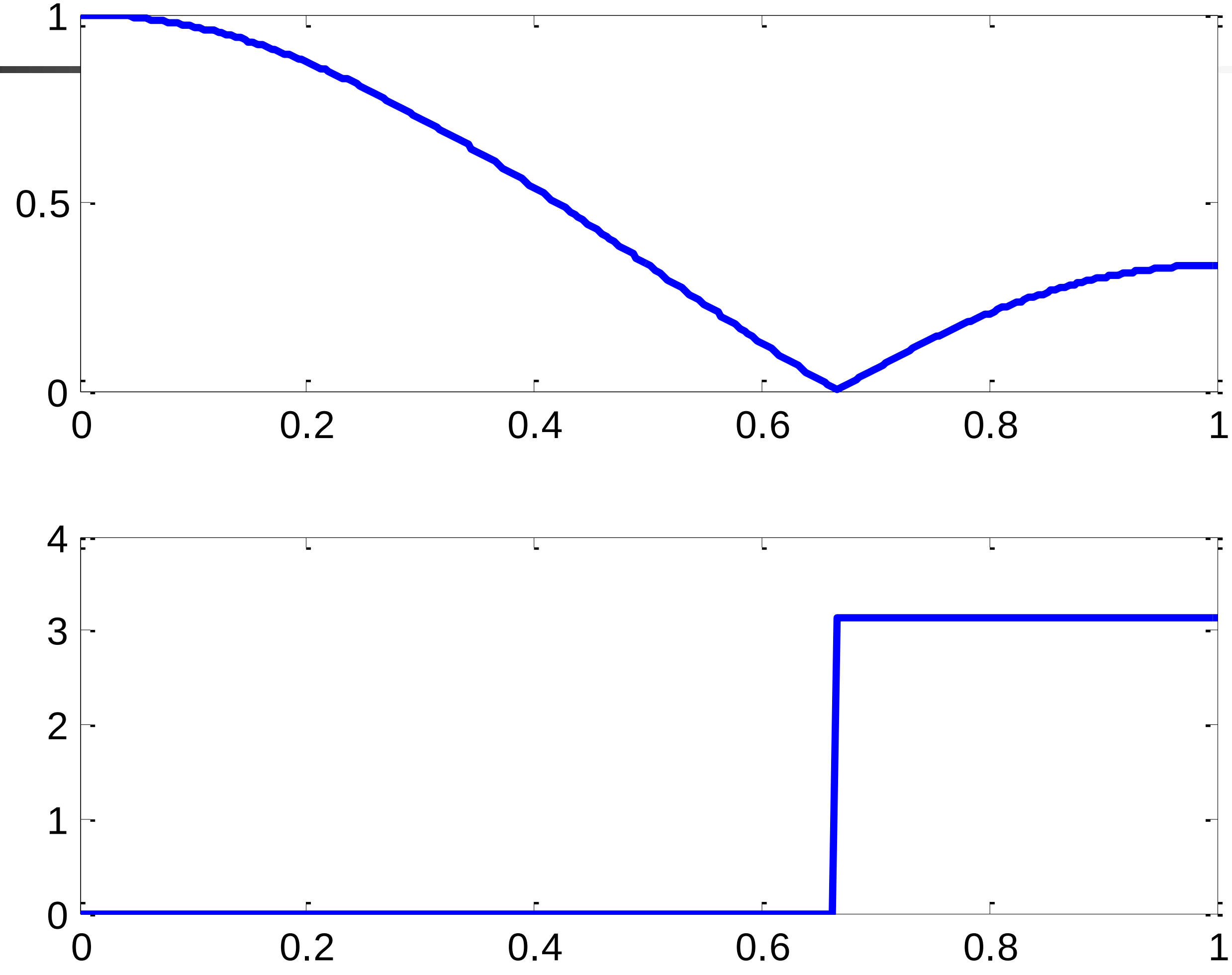
$$H(\Omega) = \frac{1}{3}(e^{j\Omega} + 1 + e^{-j\Omega}) = \frac{1}{3}(1 + 2\cos\Omega)$$

Hence

$$|H(\Omega)| = \frac{1}{3}|1 + 2\cos\Omega|$$

$$\angle H(\Omega) = \begin{cases} 0, & 0 \le \Omega \le 2\pi/3 \\ \pi, & 2\pi/3 < \Omega \le \pi \end{cases}$$

Example of moving average filter



Lecture #17

Frequency-domain characteristics of LTI systems

1. Frequency response function

2. Response to complex exponential and sinusoidal signals

3. Steady-state and transient response

4. Response to aperiodic input signals

Response to complex exponential signals

$$x[n] = Ae^{j\Omega_0 n}, \quad -\infty < n < \infty \Rightarrow y[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\left(Ae^{j\Omega_0 (n-k)}\right)$$

$$= A\left[\sum_{k=-\infty}^{\infty} h[k]\left(e^{-j\Omega_0 k}\right)\right]e^{j\Omega_0 n}$$

$$= (Ae^{j\Omega_0 n})H(\Omega_0) = x[n]H(\Omega_0)$$

$$\boxed{y[n] = x[n]H(\Omega_0) = \underbrace{Ae^{j\Omega_0 n}}_{\text{eigenfunction}}\underbrace{H(\Omega_0)}_{\text{eigenvalue}}}$$

Example

Determine the output signal of system $h[n] = (1/2)^n u[n]$

to this input signal $x[n] = Ae^{j\frac{\pi}{2}n}, \quad -\infty < n < \infty$

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\Omega n} = \frac{1}{1 - \frac{1}{2} e^{-j\Omega}}$$

$$At\ n = \frac{\pi}{2} : H\left(\frac{\pi}{2}\right) = \frac{1}{1 + j\frac{1}{2}} = \frac{2}{\sqrt{5}} e^{-j26.6^0}$$

$y[n]$

Response to sinusoidal signals

$$x[n] = A\cos(\Omega_0 n) = \frac{A}{2}e^{j\Omega_0 n} + \frac{A}{2}e^{-j\Omega_0 n}, \quad -\infty < n < \infty$$

$$y[n] = \frac{A}{2}e^{j\Omega_0 n}H(\Omega_0) + \frac{A}{2}e^{-j\Omega_0 n}H(-\Omega_0)$$

$$= \frac{A}{2}e^{j\Omega_0 n}\,|H(\Omega_0)|\,e^{j\angle H(\Omega_0)} + \frac{A}{2}e^{-j\Omega_0 n}\,|H(\Omega_0)|\,e^{-j\angle H(\Omega_0)}$$

$$= \frac{A}{2}|H(\Omega_0)|\left(e^{j\Omega_0 n}e^{j\angle H(\Omega_0)} + e^{-j\Omega_0 n}e^{-j\angle H(\Omega_0)}\right)$$

$$\boxed{y[n] = A\,|H(\Omega_0)|\cos\left(\Omega_0 n + \angle H(\Omega_0)\right)}$$

Example

- Determine the response of the system $h[n] = (1/2)^n u[n]$ to the input signal

$$x[n] = 10 - 5\sin\frac{\pi}{2}n + 20\cos\pi n, \quad -\infty < n < \infty$$

One more example

- A LTI causal system is described by the following equation:

$$y[n] = ay[n-1] + bx[n],\ a \text{ real and } 0<a<1$$

a) Determine the amplitude and phase responses

b) Choose b so that the maximum value of $|H(\Omega)|$ is unity, and sketch $|H(\Omega)|$ and $\angle H(\Omega)$ for $a = 0.9$

c) Determine the output of the system to the input:

$$x[n] = 5 + 12\sin\frac{\pi}{2}n - 20\cos\left(\pi n + \frac{\pi}{4}\right), \quad -\infty < n < \infty$$

One more example

a) The frequency response is:

$$H(\Omega) = \frac{b}{1 - ae^{-j\Omega}}$$

The amplitude response and phase response are:

One more example

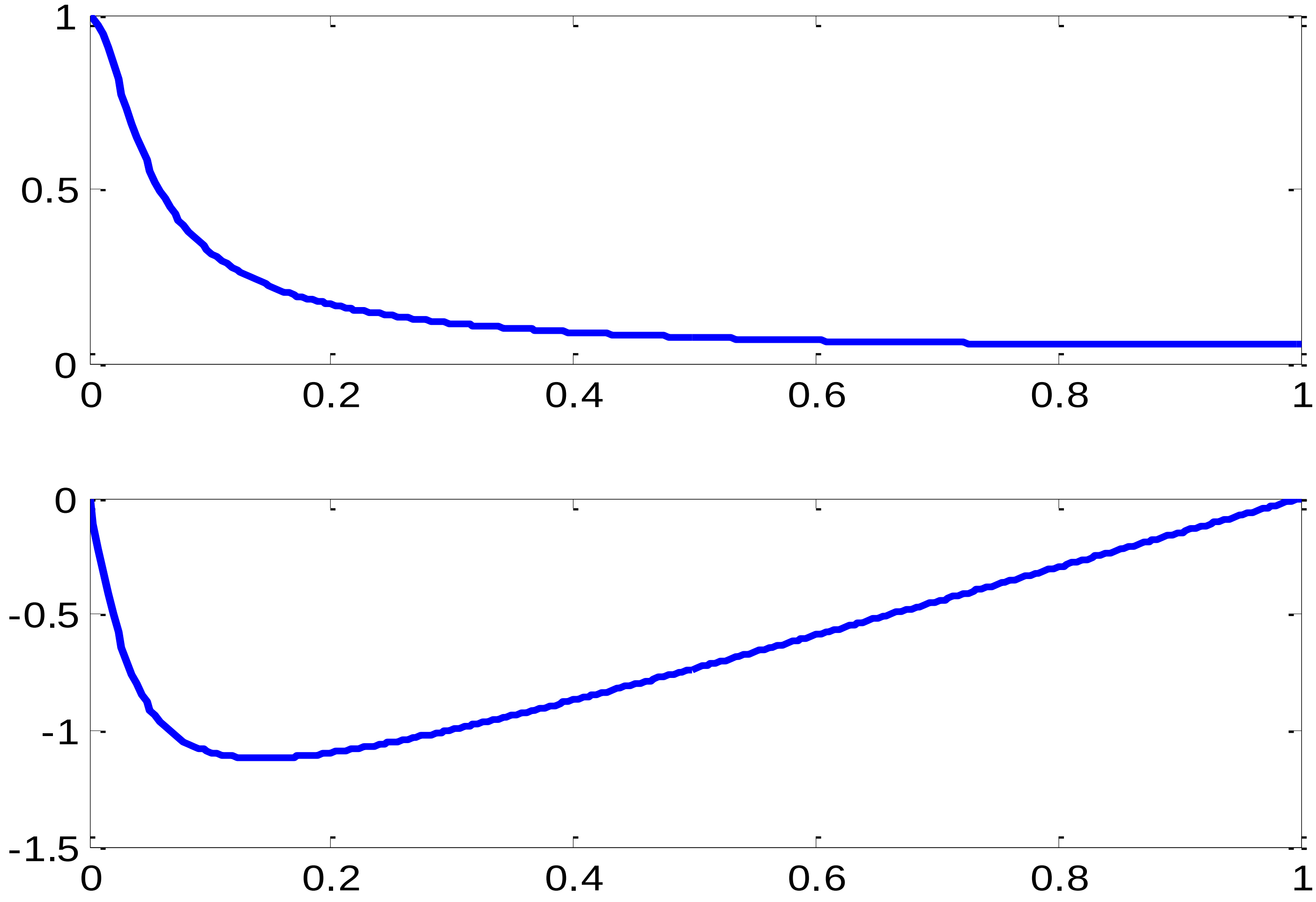
Choose *a = 0.9* and *b = 1-a* :

$$|H(\Omega)| = \frac{1-a}{\sqrt{1+a^2-2a\cos\Omega}} = \frac{0.1}{\sqrt{1.81-1.8\cos\Omega}}$$

$$\angle H(\Omega) = 0 - arctg\,\frac{0.9\sin\Omega}{1-0.9\cos\Omega}$$

One more example

b)



One more example

c) The amplitude and phase response at $\Omega=0$, $\pi/2$, and π respectively:

The output:

Lecture #17

Frequency-domain characteristics of LTI systems

1. Frequency response function

2. Response to complex exponential and sinusoidal signals

3. Steady-state and transient response

4. Response to aperiodic input signals

Steady-state and transient response

The system response can be considered as a sum of 2 terms:

$$\boxed{y[n] = y_{ss}[n] + y_{tr}[n]}$$

$y_{ss}[n]$: steady-state response

$y_{tr}[n]$: transient response, decays toward zero as **$n \rightarrow \infty$**

In many practical applications, the transient response is unimportant and therefore, it is usually ignored

Example

The LTI system described by first-order equation:

$$y[n] - ay[n-1] = x[n] \qquad |a| < 1$$

Its response to the input: $x[n] = Ae^{j\Omega n},\ n \geq 0$

is determined as

$$y[n] = a^{n+1} y[-1] - \frac{Aa^{n+1} e^{-j\Omega(n+1)}}{1 - ae^{-j\Omega}} e^{j\Omega n} + \frac{A}{1 - ae^{-j\Omega}} e^{j\Omega n}, \quad n \geq 0$$

where *y[-1]* is the initial condition

Example

$$y[n] = a^{n+1}y[-1] - \frac{Aa^{n+1}e^{-j\Omega(n+1)}}{1-ae^{-j\Omega}}e^{j\Omega n} + \frac{A}{1-ae^{-j\Omega}}e^{j\Omega n}$$

Transient response Steady-state response

$$\Rightarrow \quad y_{ss}[n] = AH(\Omega)e^{j\Omega n} \equiv AH(e^{j\Omega})e^{j\Omega n}$$

$$\boxed{x[n] = \sum_{k=1}^{M} A_k z_k^n \quad \Rightarrow \quad y_{ss}[n] = \sum_{k=1}^{M} A_k H(z_k) z_k^n}$$

Lecture #17

Frequency-domain characteristics of LTI systems

1. Frequency response function

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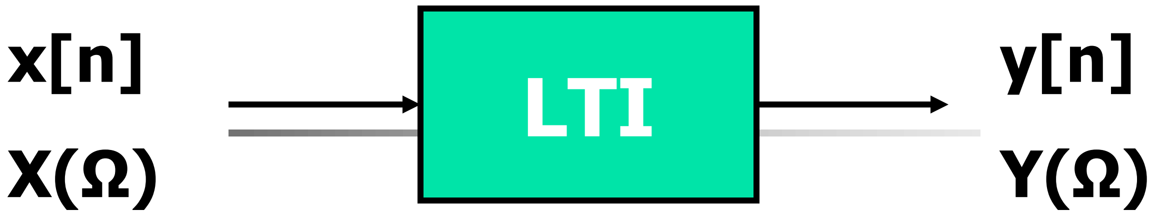
Response to aperiodic input signals



- From the convolution property, we have:

$$\boxed{\begin{aligned} &Y(\Omega) = X(\Omega)H(\Omega) \\ &|Y(\Omega)| = |X(\Omega)||H(\Omega)| \\ &\angle Y(\Omega) = \angle X(\Omega) + \angle H(\Omega) \end{aligned}}$$

$$For\ \Omega = \Omega_0 :$$

$$|Y(\Omega_0)| = |X(\Omega_0)||H(\Omega_0)|$$

$$\angle Y(\Omega_0) = \angle X(\Omega_0) + \angle H(\Omega_0)$$

- The LTI system *attenuates* some frequency components and *amplifies* other frequency components of the input

- The output can't contain frequency components that are not contained in the input

Example

- A LTI causal system is described by the following equation:

$$y[n] - 0.5y[n-1] = x[n]$$

Determine the spectrum and the ESD of the output when the input signal is: $x[n] = \left(\frac{1}{4}\right)^n u[n]$

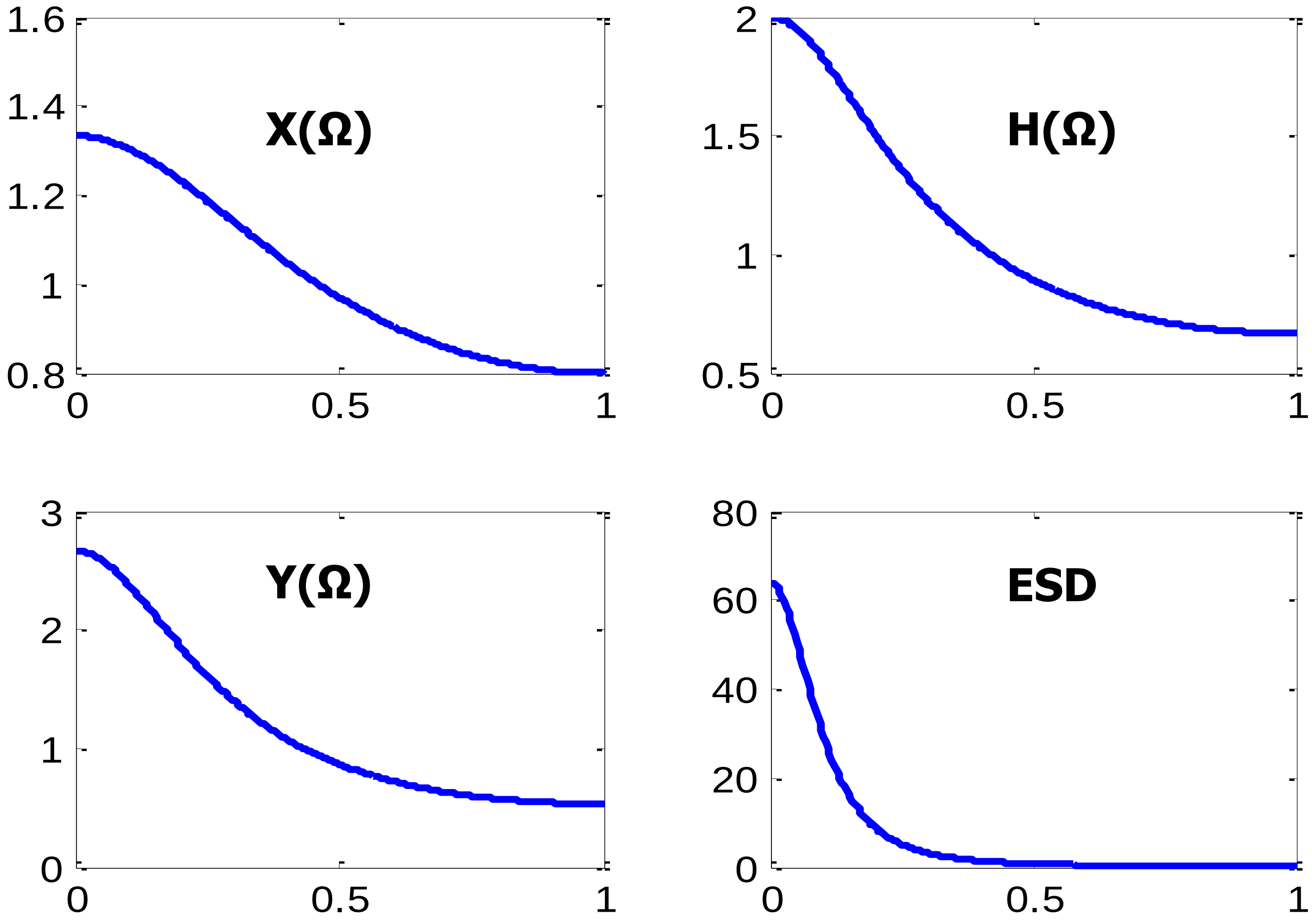
- The frequency response: $H(\Omega) = \dfrac{1}{1 - \frac{1}{2}e^{-j\Omega}}$

- The input spectrum: $X(\Omega) = \dfrac{1}{1 - \frac{1}{4}e^{-j\Omega}}$

- The output spectrum: $Y(\Omega) = X(\Omega).H(\Omega) = \dfrac{1}{\left(1 - \frac{1}{4}e^{-j\Omega}\right)\left(1 - \frac{1}{2}e^{-j\Omega}\right)}$

$$ESD = |Y(\Omega)|^2 = \frac{1}{\left(\frac{5}{4} - \cos\Omega\right)\left(\frac{17}{16} - \frac{1}{2}\cos\Omega\right)}$$

Example



Lecture #18

Digital filters

1. Digital filters

2. Ideal filters

Filters

- Provide a convenient means to change the nature of a signal.

- Change the frequency characteristics of a signal in a specific way, letting some frequencies in the signal pass while blocking others

- **4 basic types:**

- Low pass filter (LPF): lets low frequencies pass while blocking high frequencies

- High pass filter (HPF): does the opposite

- Band pass filter (BPF): allows a band of frequencies to pass

- Band stop filter (BSF): block all frequencies inside a band

Digital filters

- DT systems that perform mathematical operations on a DT signal to reduce or enhance certain aspects of that signal.
- Digital filters are difference equations defined by a list of filter coefficients
- **Ex:** Moving average filter:

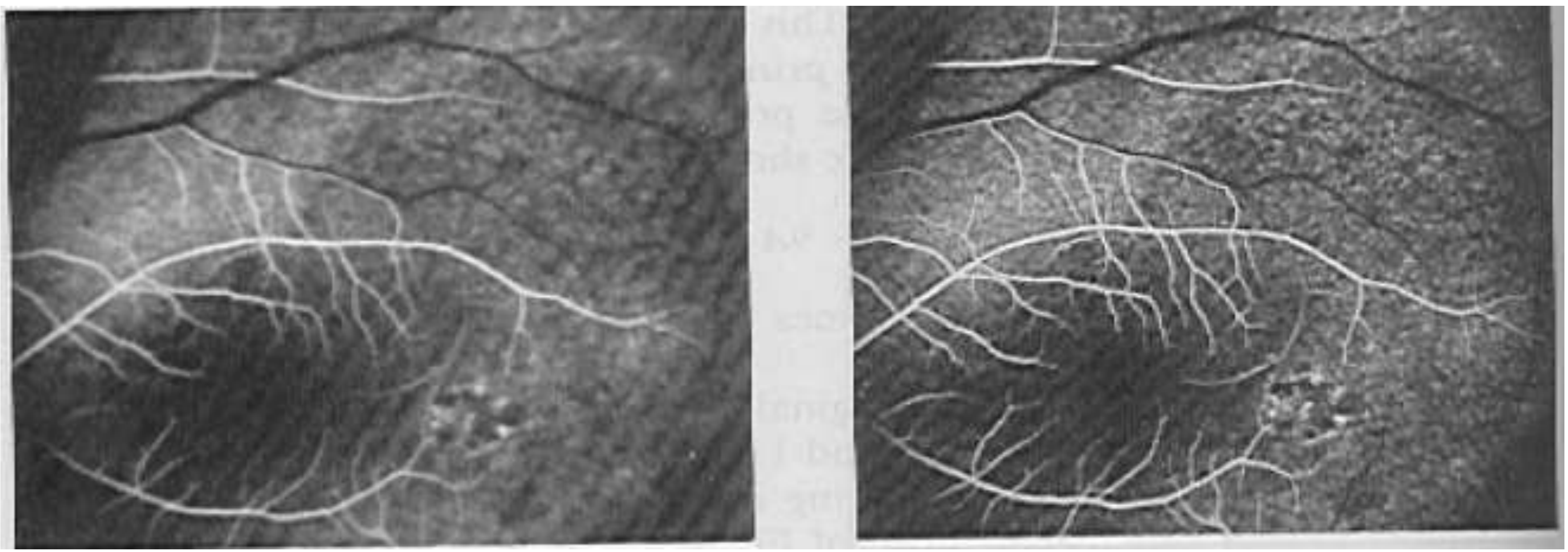
$$y[n] = 1/3\{x[n+1]+x[n]+x[n-1]\}$$

Typical applications of digital filters

- Noise suppression: radio signal, biomedical signal, analog media signal...
- Enhancement of selected frequency range: treble/bass control, equalizers in audio, image edge enhancement…
- Bandwidth limiting: aliasing prevention, interference avoidance…
- Removal of specific frequencies: DC removal, 60 Hz signal removal, notch filter…
- Special operations: differentiation, integration, phase shift…

Example of signal after lowpass filtering

Example of signal after highpass filtering



Lecture #18

Digital filters

1. Digital filters

2. Ideal filters

Ideal digital filters

$$y[n] = \begin{cases} Cx[n-n_0], & \Omega_1 < \Omega < \Omega_2 \\ 0, & \Omega \neq \end{cases}$$

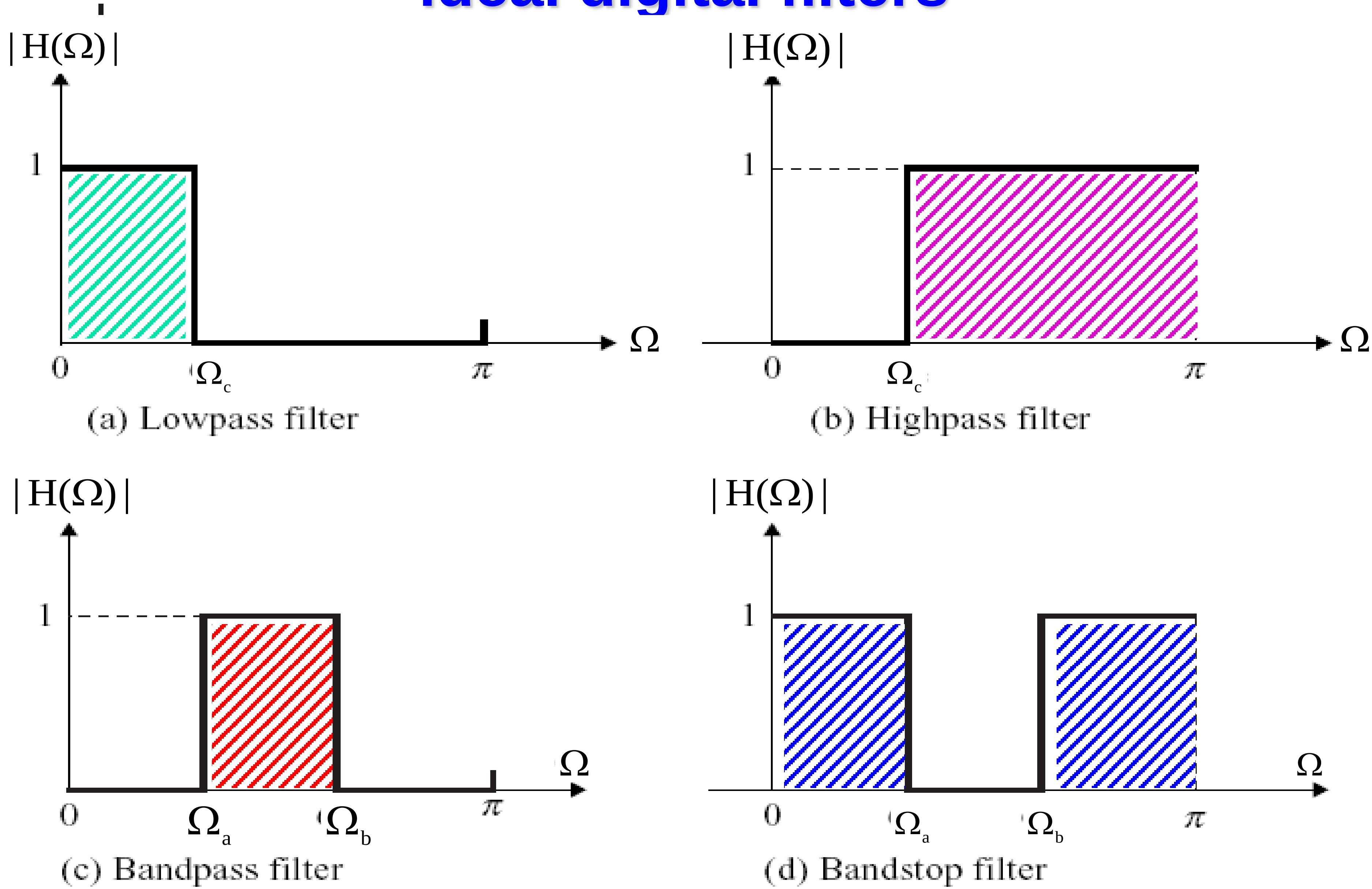
$$\Rightarrow \quad Y(\Omega) = CX(\Omega)e^{-j\Omega n_0} = X(\Omega)H(\Omega), \quad \Omega_1 < \Omega < \Omega_2$$

$$\Rightarrow \quad H(\Omega) = \begin{cases} Ce^{-j\Omega n_0}, \Omega_1 < \Omega < \Omega_2 \\ 0, \quad \Omega \neq \end{cases}$$

$$\Rightarrow \quad \boxed{\begin{aligned} &|H(\Omega)| = C, \quad \Omega_1 < \Omega < \Omega_2 \\ &\theta(\Omega) = -\Omega n_0, \quad \Omega_1 < \Omega < \Omega_2 \end{aligned}}$$

- Constant gain
- Linear phase

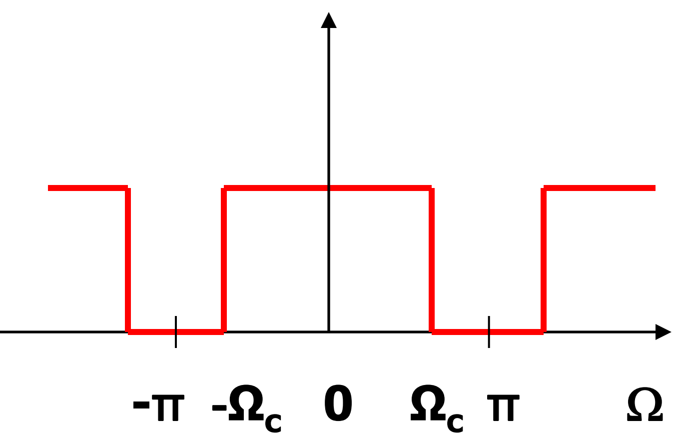
Amplitude responses for some ideal digital filters

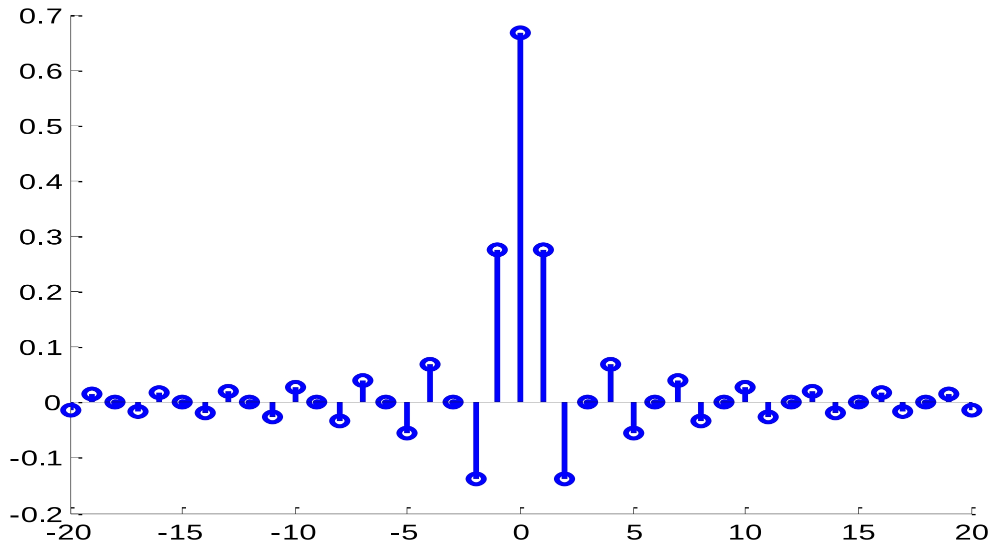


Ideal digital lowpass filters

Frequency response: $$H_L(\Omega) = \begin{cases} 1, & |\Omega| \le \Omega_c \\ 0, & \Omega_c < |\Omega| < \pi \end{cases}$$

$$\boxed{h_L[n] = \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} 1.e^{j\Omega n}\, d\Omega = \frac{\Omega_c}{\pi}\frac{\sin \Omega_c n}{\Omega_c n}}$$



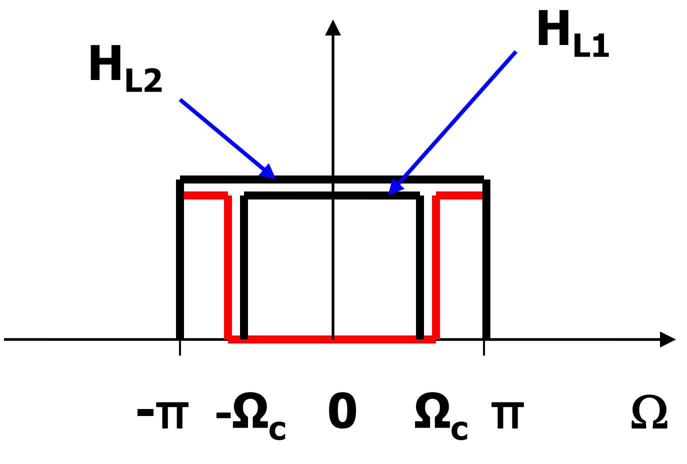


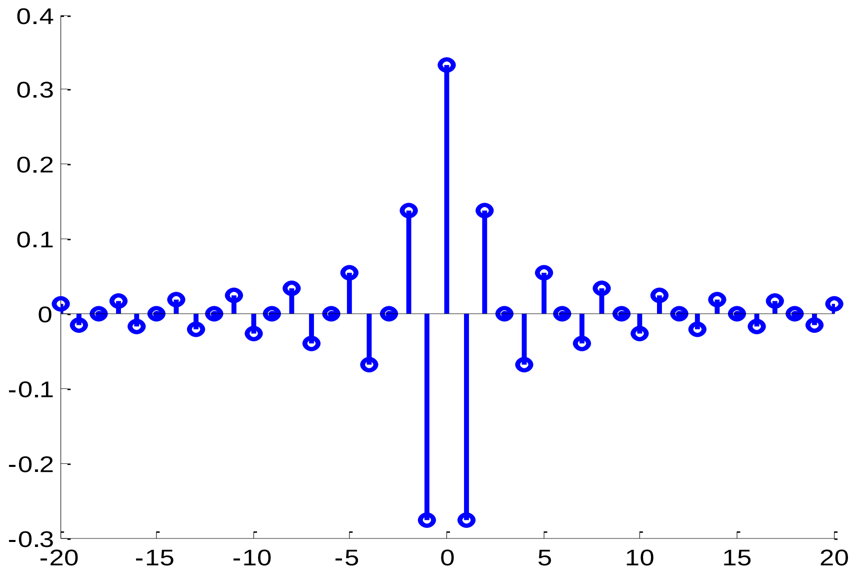
Ideal digital highpass filters

- Frequency response:

$$H_H(\Omega) = \begin{cases} 1, & \Omega_c < |\Omega| < \pi \\ 0, & |\Omega| \le \Omega_c \end{cases}$$

$$\boxed{H_H(\Omega) = H_{L2}(\Omega) - H_{L1}(\Omega) \Rightarrow h[n] = \frac{\sin \pi n}{\pi n} - \frac{\Omega_c}{\pi}\frac{\sin \Omega_c n}{\Omega_c n}}$$



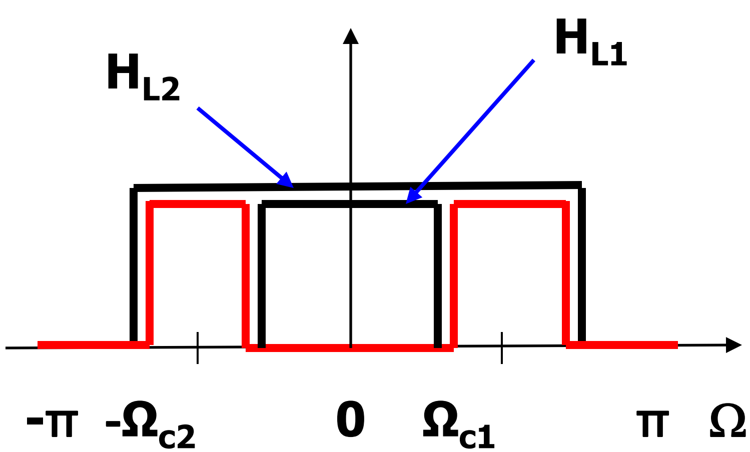


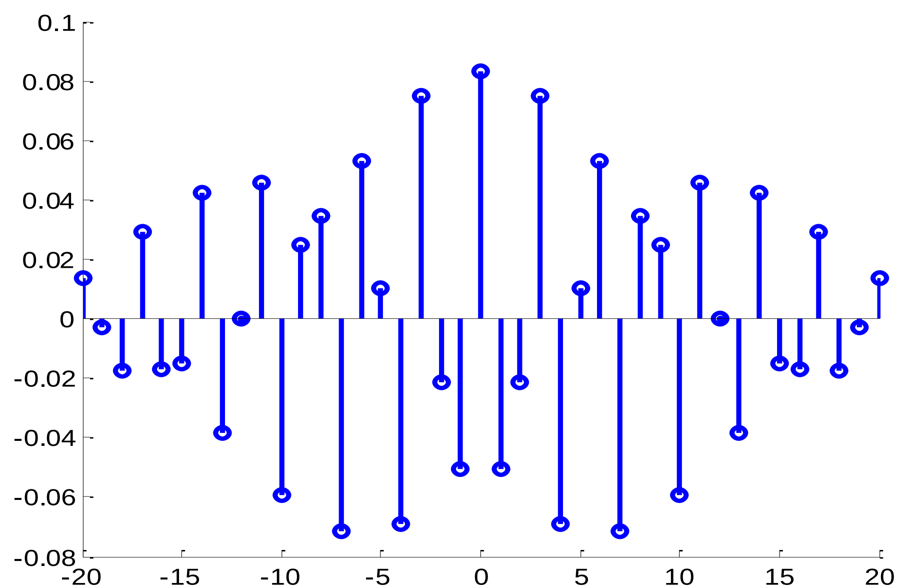
Ideal digital bandpass filters

- Frequency response:

$$H_B(\Omega) = \begin{cases} 1, & \Omega_{c1} \le |\Omega| \le \Omega_{c2} < \pi \\ 0, & elsewhere \end{cases}$$

$$\boxed{H_H(\Omega) = H_{L2}(\Omega) - H_{L1}(\Omega) \Rightarrow h[n] = \frac{\Omega_{c2}}{\pi}\frac{\sin\Omega_{c2}n}{\Omega_{c2}n} - \frac{\Omega_{c1}}{\pi}\frac{\sin\Omega_{c1}n}{\Omega_{c1}n}}$$



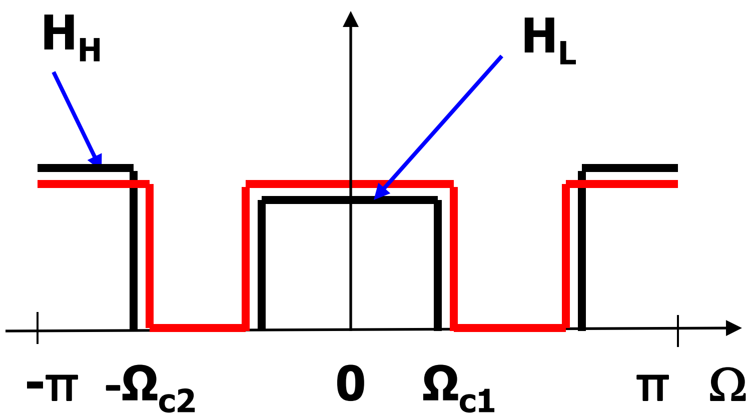


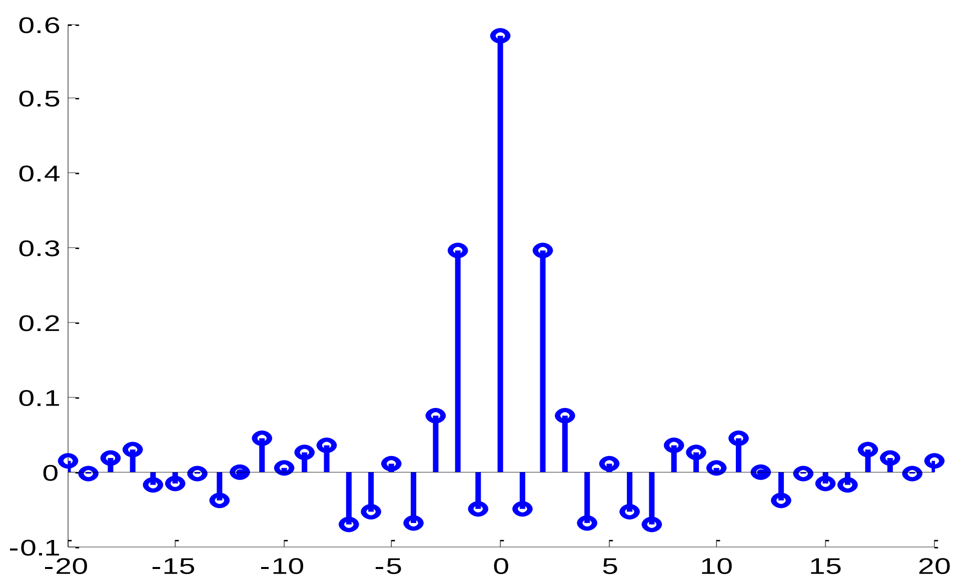
Ideal digital bandstop filters

- Frequency response:

$$H_B(\Omega) = \begin{cases} 1, & |\Omega| \le \Omega_{c1} \\ 0, & \Omega_{c1} < |\Omega| < \Omega_{c2} \\ 1, & \Omega_{c2} \le |\Omega| < \pi \end{cases}$$

$$H_H(\Omega) = H_L(\Omega) + H_H(\Omega) \Rightarrow h[n] = \frac{\Omega_{c1}}{\pi}\frac{\sin\Omega_{c1}n}{\Omega_{c1}n} + \delta[n] - \frac{\Omega_{c2}}{\pi}\frac{\sin\Omega_{c2}n}{\Omega_{c2}n}$$





Amplitude responses for some actual digital filters