CHAPTER 2:

DISCRETE-TIME SIGNALS & SYSTEMS

Lesson #4: DT signals

Lesson #5: DT systems

Lesson #6: DT convolution

Lesson #7: Difference equation models

Lesson #8: Block diagram for DT LTI systems

Duration: 9 hrs

Lecture #4
DT signals

1. Representations of DT signals

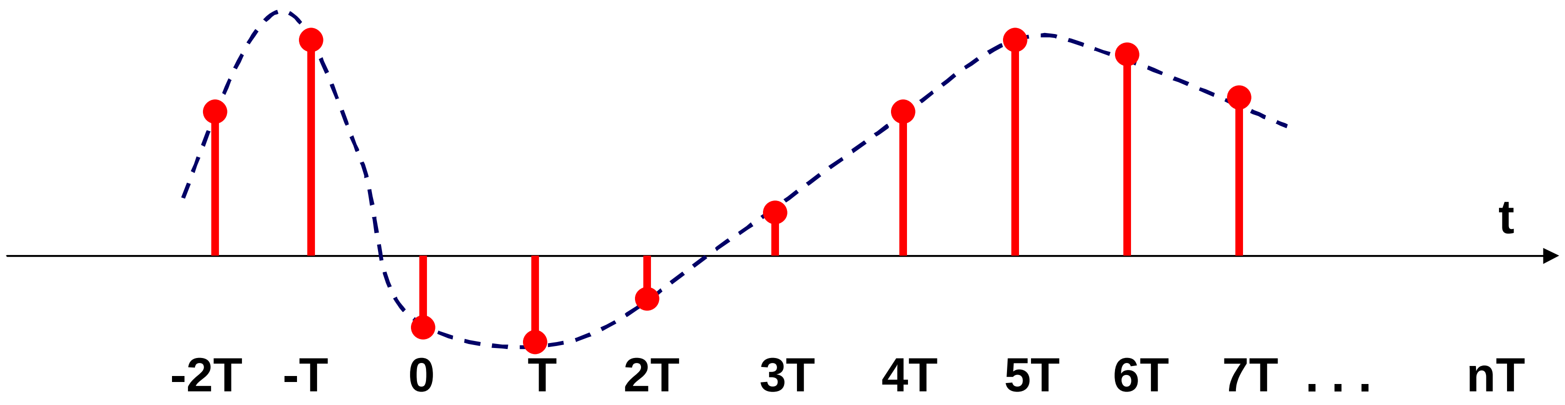
2. Some elementary DT signals

3. Simple manipulations of DT signals

4. Characteristics of DT signals

Sampled signals

Converting a CT signal into a DT signal by sampling: given $x_a(t)$ to be a CT signal, $x_a(nT)$ is the value of $x_a(t)$ at $t = nT$ → DT signal is defined only for n an integer



$$\boxed{x_a(t)\big|_{t=nT} = x_a(nT) \equiv x(n), \quad -\infty < n < \infty}$$

Representations of DT signals

1. Functional representation

$$x[n] = \begin{cases} 1, & n = 1,3 \\ 4, & n = 2 \\ 0, & n \neq \end{cases}$$

2. Tabular representation

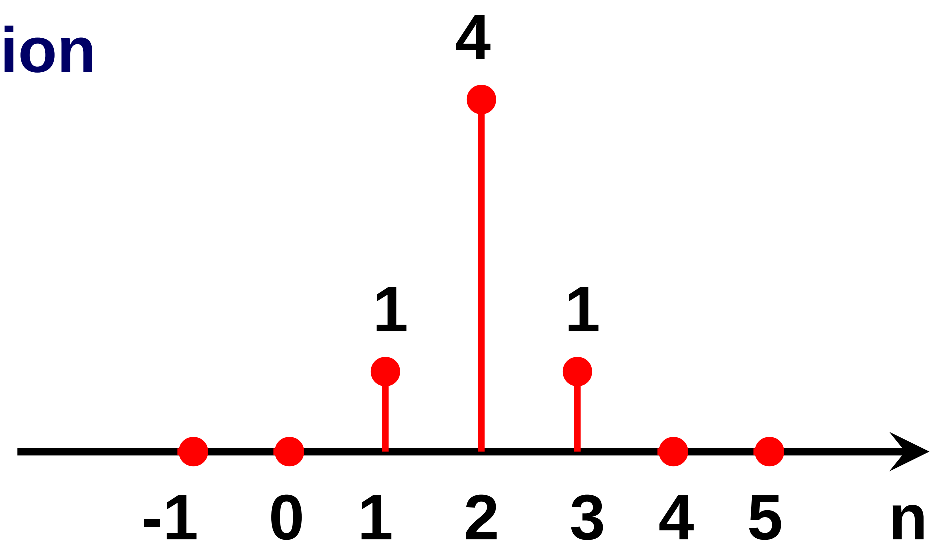
n	… -1	0	1	2	3	4 …
x[n]	… 0	0	1	4	1	0 …

Representations of DT signals

3. Sequence representation

$$x[n] = \{\underset{\uparrow}{0}, 1, 4, 1\}$$

4. Graphical representation



Lecture #4
DT signals

1. Representations of DT signals

2. Some elementary DT signals

3. Simple manipulations of DT signals

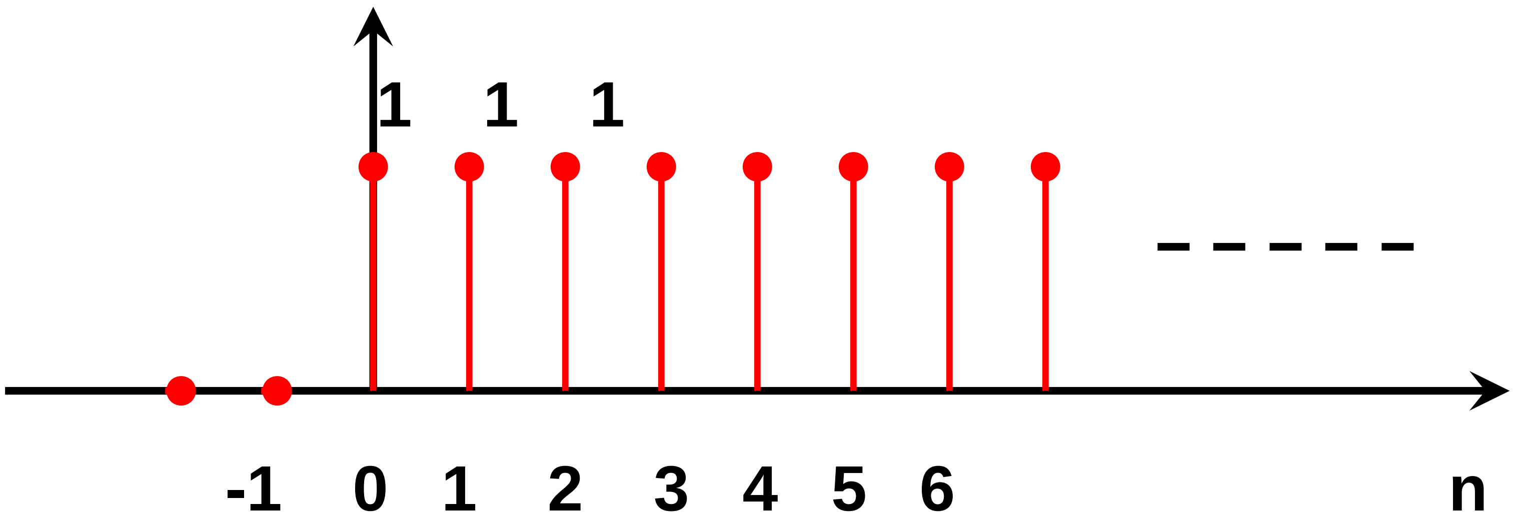
4. Classification of DT signals

Some elementary DT signals

1. Unit step sequence
2. Unit impulse signal
3. Sinusoidal signal
4. Exponential signal

Unit step

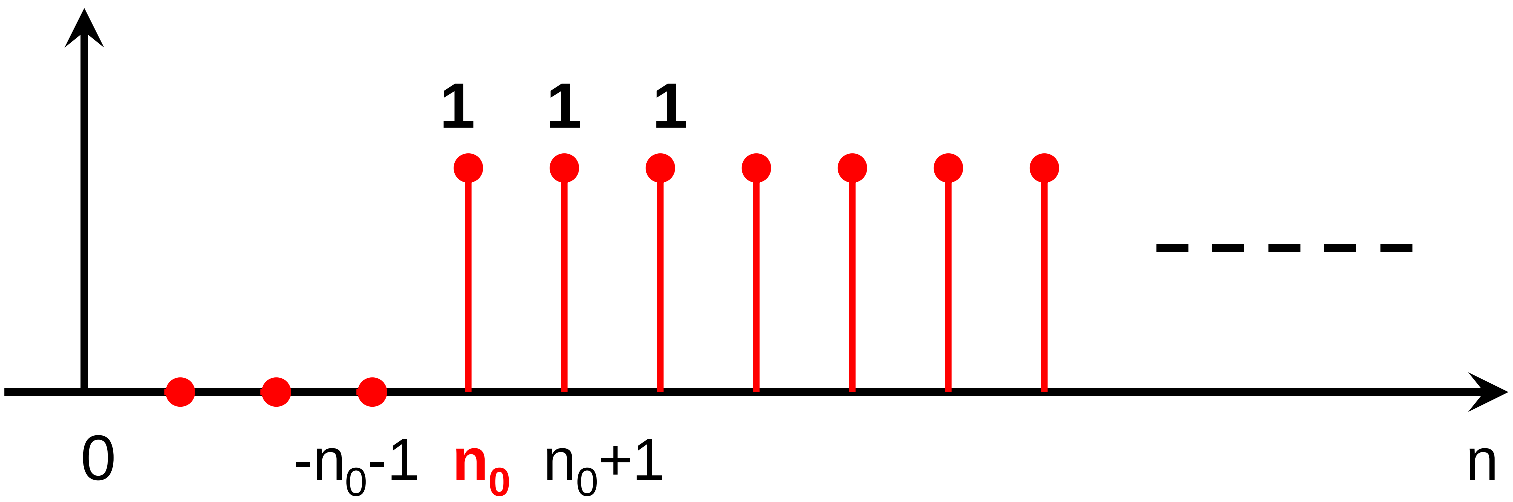
$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



Time-shifted unit step

$$u[n-n_0]=\begin{cases}1, & n \geq n_0 \\ 0, & n < n_0\end{cases}$$

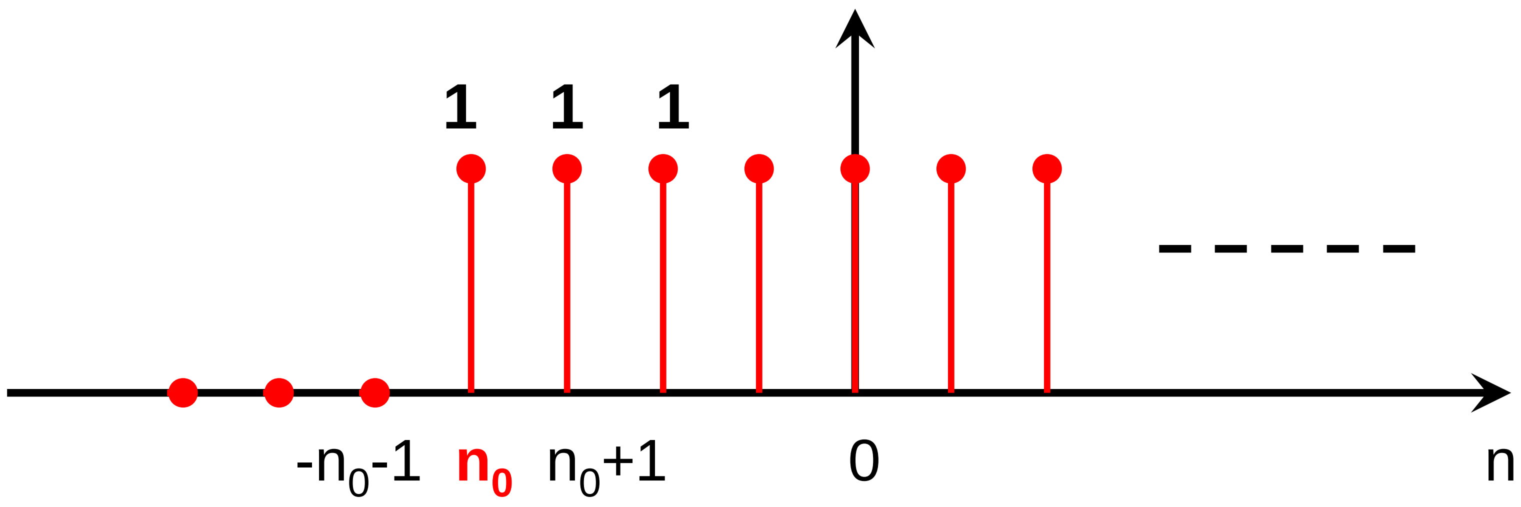
For $n_0 > 0$



Time-shifted unit step

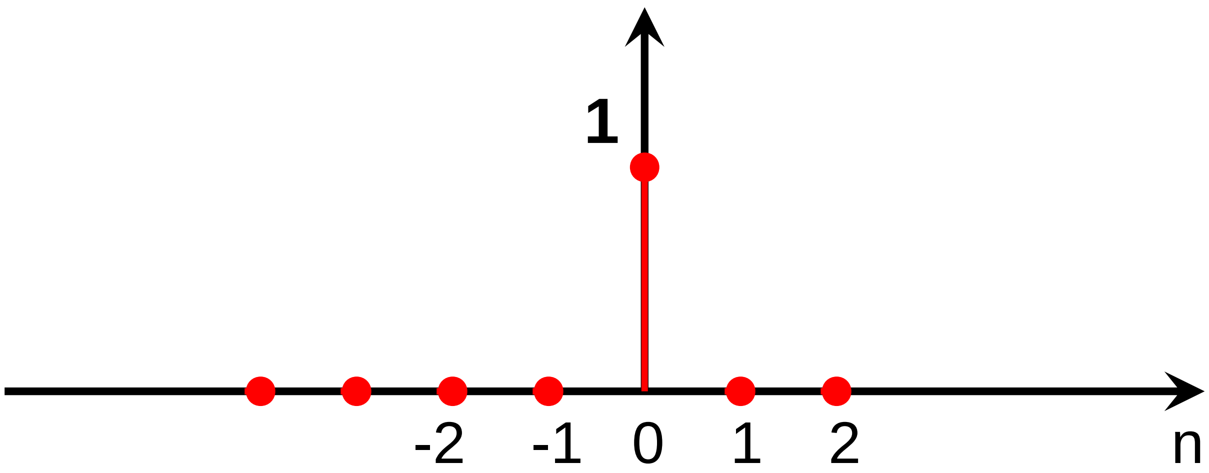
$$u[n-n_0]=\begin{cases}1, & n \geq n_0 \\ 0, & n < n_0\end{cases}$$

For $n_0 < 0$



Unit impulse

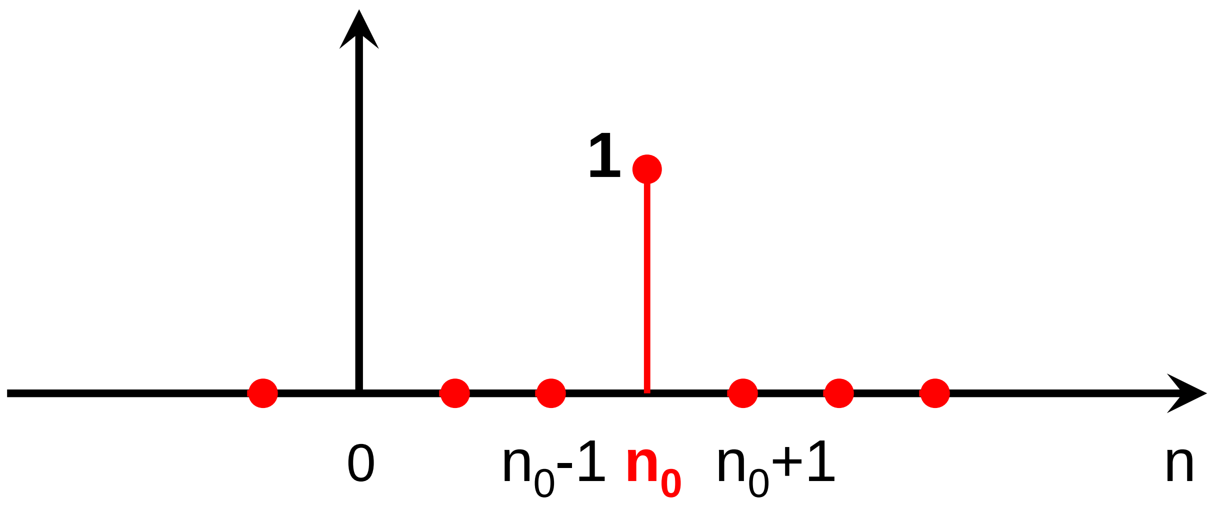
$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



Time-shifted unit impulse

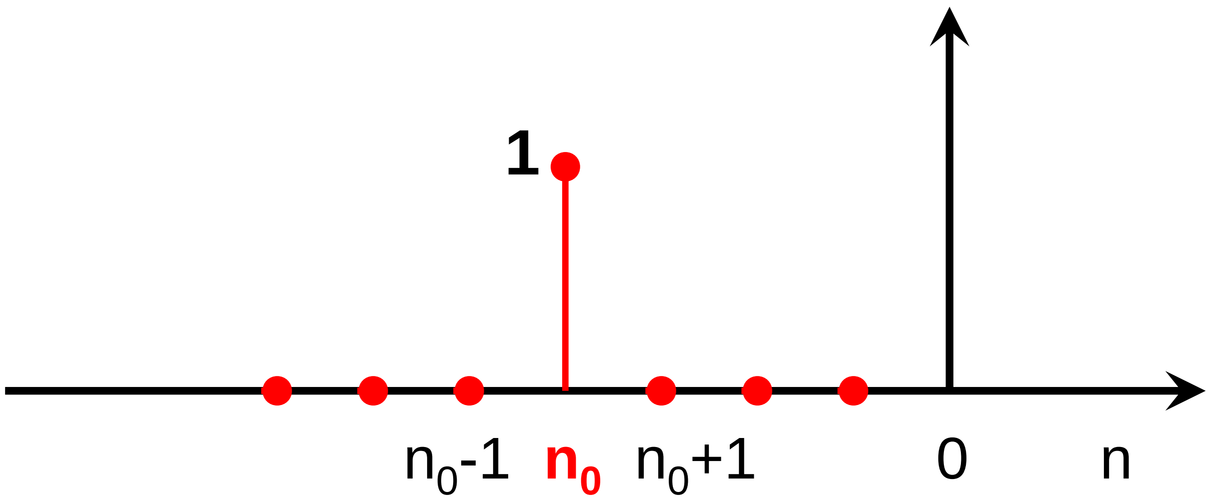
$$\delta[n-n_0]=\begin{cases}1, & n=n_0\\ 0, & n\neq n_0\end{cases}$$

For $n_0 > 0$



Time-shifted unit impulse

$$\delta[n-n_0]=\begin{cases}1, & n=n_0\\ 0, & n\neq n_0\end{cases}$$



Relation between unit step and unit impulse

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

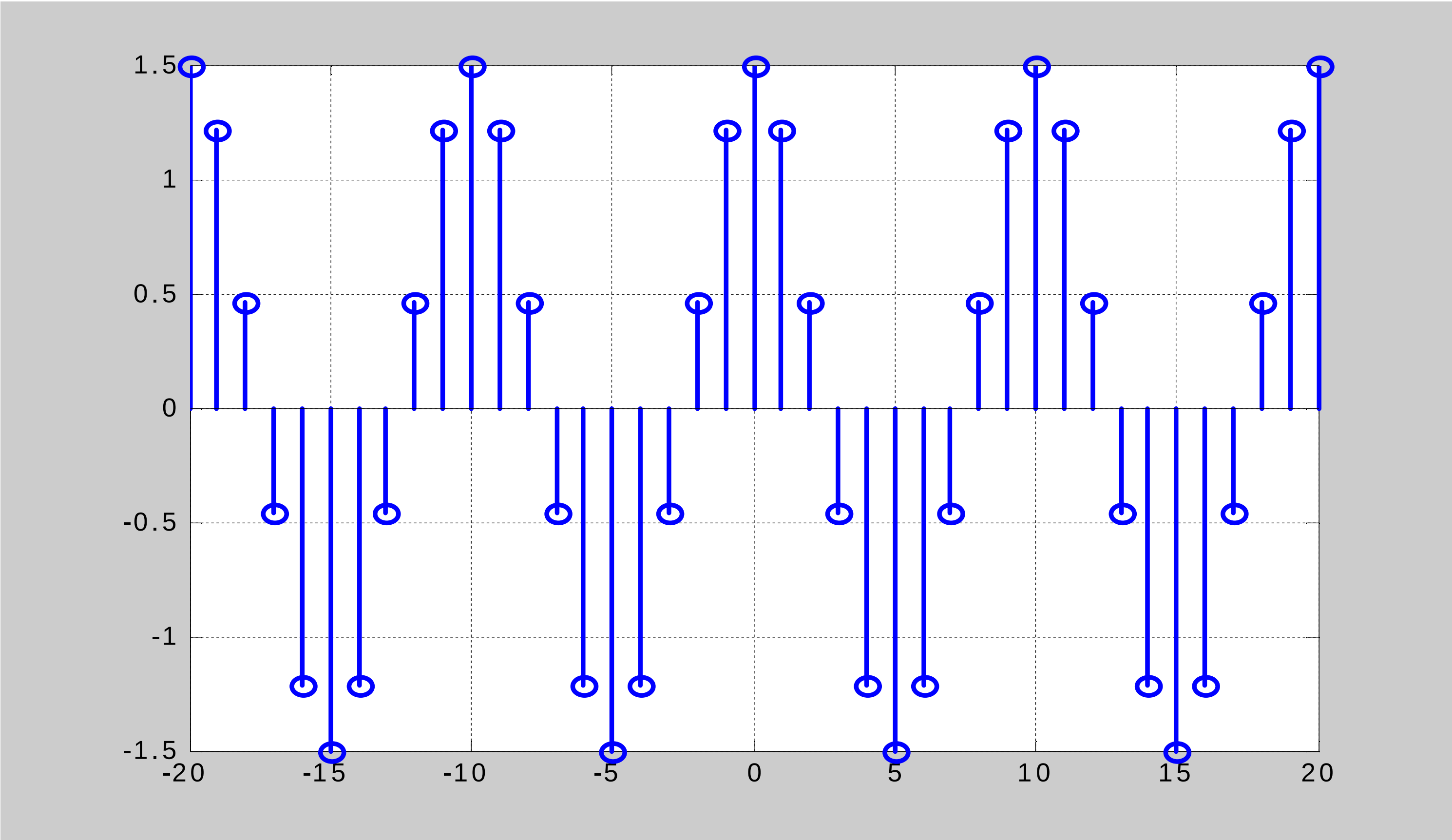
$$\delta[n] = u[n] - u[n-1]$$

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$

$$\sum_{n=-\infty}^{\infty} x[n]\delta[n-n_0] = x[n_0]$$

Sinusoidal signal

$$x(n) = A\cos(\Omega n + \theta), \quad -\infty < n < +\infty$$
$$= A\cos(2\pi F n + \theta), \quad -\infty < n < +\infty$$



Exponential signal

$$x[n] = Ca^n$$

1. If *C* and *a* are real, then *x[n]* is a real exponential

a > 1 → growing exponential

0 < a < 1 → shrinking exponential

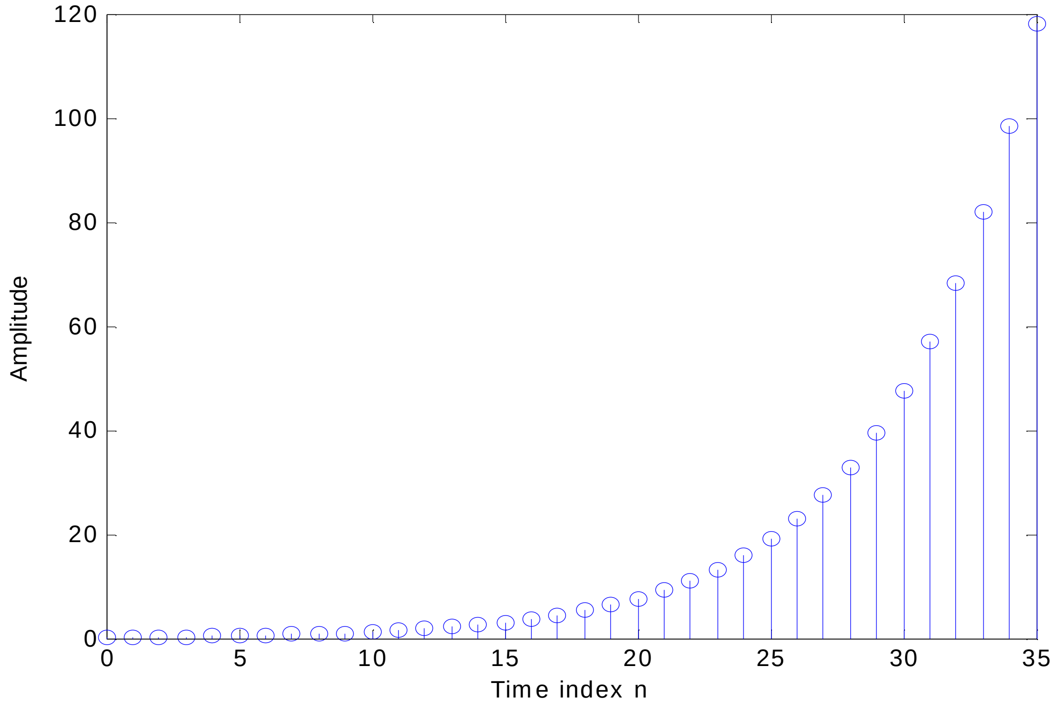
-1 < a < 0 → alternate and decay

a < -1 → alternate and grows

2. If *C* or *a* or both is complex, then *x[n]* is a complex exponential

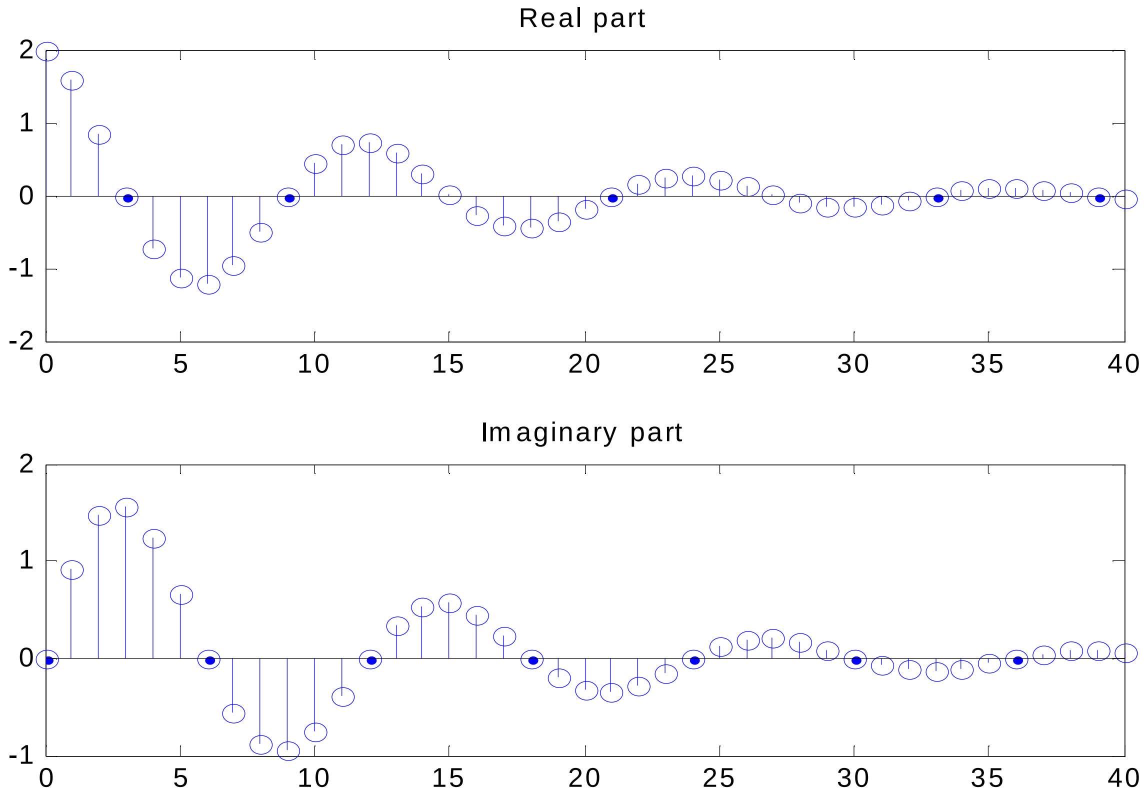
An example of real exponential signal

$$x[n] = (0.2)(1.2)^n$$



An example of complex exponential signal

$$x[n] = 2e^{\left(-\frac{1}{12}+j\frac{\pi}{6}\right)n}$$



Periodic exponential signal

- **Recall:** A DT sinusoidal signal is periodic only if its frequency is a rational number
- Consider complex exponential signal:

$$x[n] = Ce^{j\Omega_0 n} = C\cos(\Omega_0 n) + j\sin(\Omega_0 n)$$

- It is also periodic only if its frequency is a rational number:

$$\boxed{F_0 = \frac{k}{N} \quad or \quad \frac{\Omega_0}{2\pi} = \frac{k}{N}}$$

Fundamental period

- The fundamental period can be found as

$$\boxed{N = \frac{k2\pi}{\Omega_0}}$$

Where k is the smallest integer such that N is an integer

- **Step 1:** Is $\frac{\Omega_0}{2\pi}$ rational?
- **Step 2:** If yes, then periodic; reduce to

$$\frac{\Omega_0}{2\pi} = \frac{k}{N} = \frac{\#\,cycles}{\#\,points}$$

Examples

Determine which of the signals below are periodic. For the ones that are, find the fundamental period and fundamental frequency

$$x_1[n] = e^{j\frac{\pi}{6}n}$$

$$\frac{\Omega_0}{2\pi} = \frac{\pi}{6(2\pi)} = \frac{1}{12} = \frac{k}{N}$$ → One cycle in 12 points

$N = 12$: fundamental period

$$\Omega_0 = \frac{2\pi}{N} = \frac{2\pi}{12} = \frac{\pi}{6}$$: fundamental frequency

Examples

Determine which of the signals below are periodic. For the ones that are, find the fundamental period and fundamental frequency

$$x_2[n] = \sin\left(\frac{3\pi}{5}n + 1\right)$$

Examples

Determine which of the signals below are periodic. For the ones that are, find the fundamental period and fundamental frequency

$$x_3[n] = \cos(2n - \pi)$$

Examples

Determine which of the signals below are periodic. For the ones that are, find the fundamental period and fundamental frequency

$$x_4[n] = \cos(1.2\pi n)$$

Lecture #4
DT signals

1. Representations of DT signals
2. Some elementary DT signals
3. **Simple manipulations of DT signals**
4. Classification of DT signals

Simple manipulations of DT signals

- Adding and subtracting signals
- Transformation of time:
  - Time shifting
  - Time scaling
  - Time reversal
- Transformation of amplitude:
  - Amplitude shifting
  - Amplitude scaling
  - Amplitude reversal

Adding and Subtracting signals

- Do it “point by point”
- Can do using a table, or graphically, or by computer program
- Example: x[n] = u[n] – u[n-4]

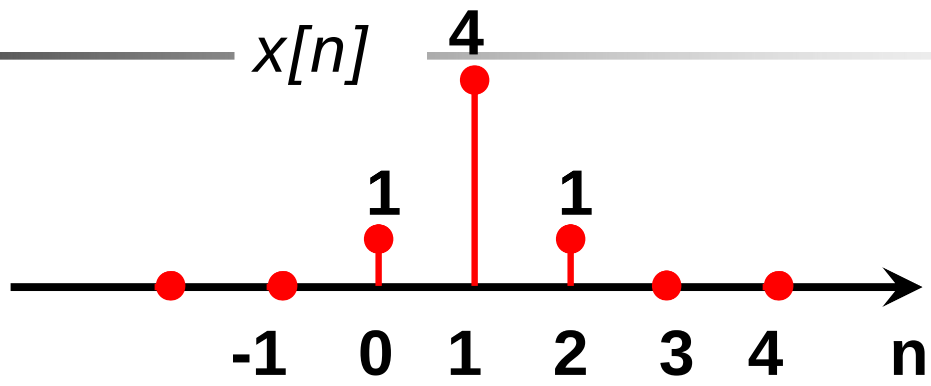
n	<=-1	0	1	2	3	>=4
x[n]	0	1	1	1	1	0

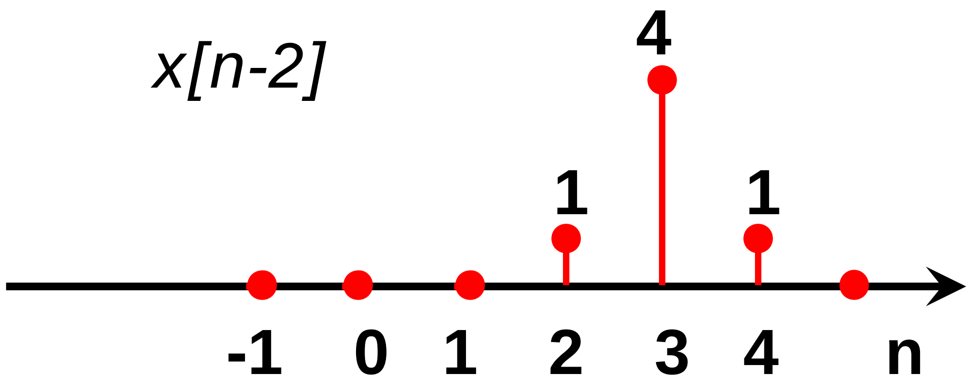
Time shifting a DT signal

$x[n] \rightarrow x[n - k]$; k is an integer

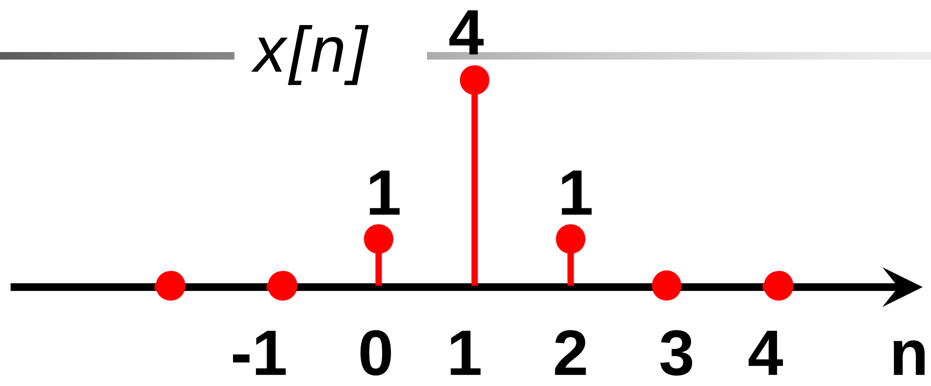
- **<u>k > 0:</u>** right-shift x[n] by |k| samples (delay of signal)
- **<u>k < 0:</u>** left-shift x[n] by |k| samples (advance of signal)

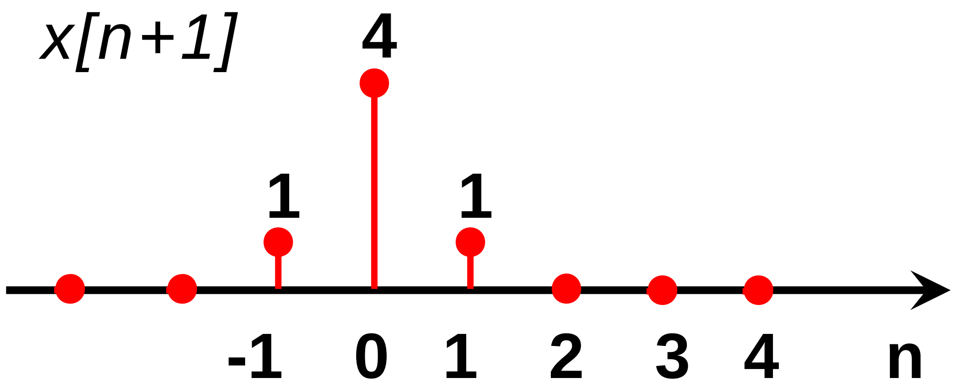
Examples of time shifting





Examples of time shifting





Time scaling a DT signal

$$x[n] \rightarrow y[n] = x[an]$$

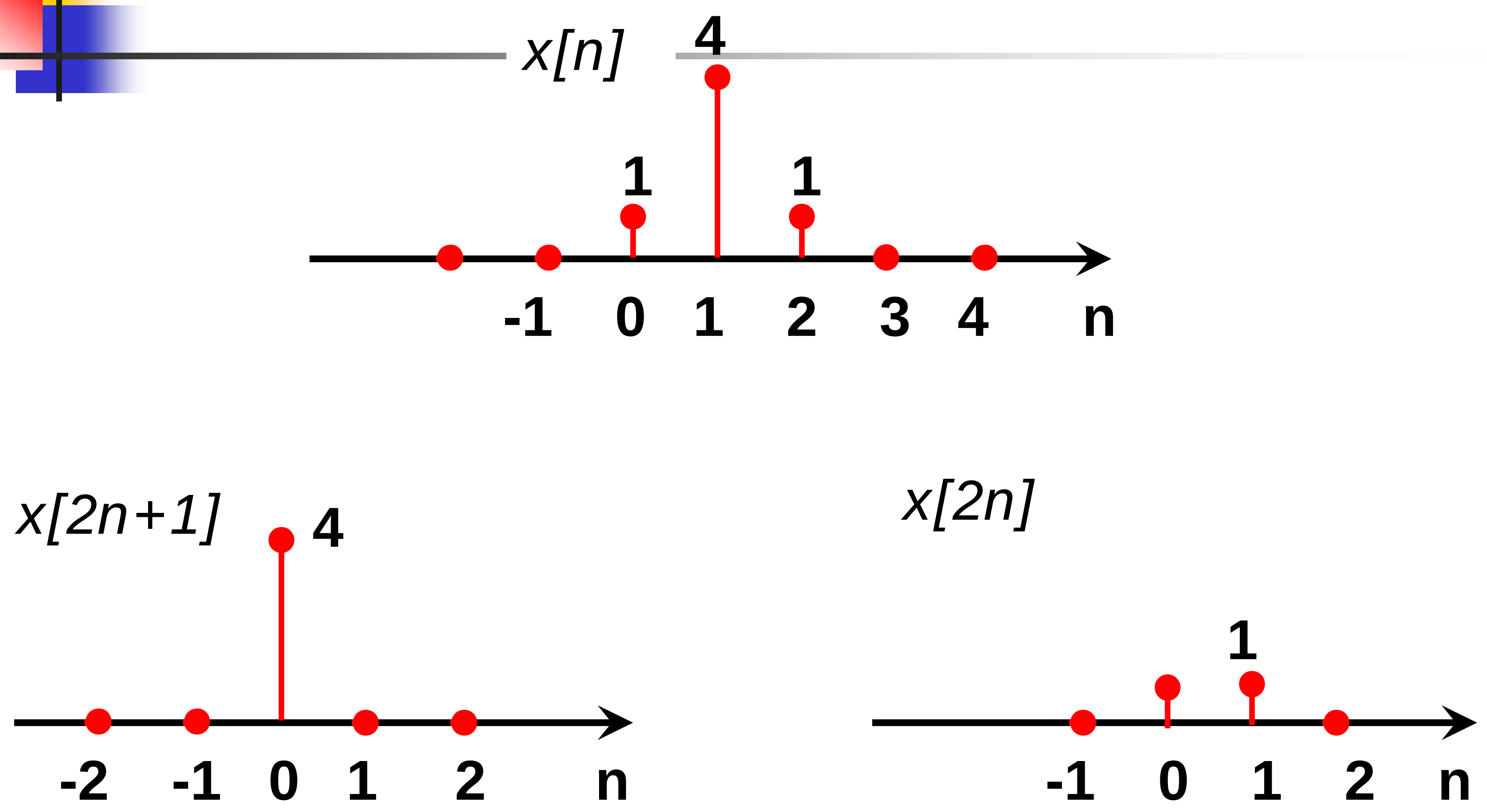
- **$|a| > 1$:** speed up by a factor of a

  a must be an integer

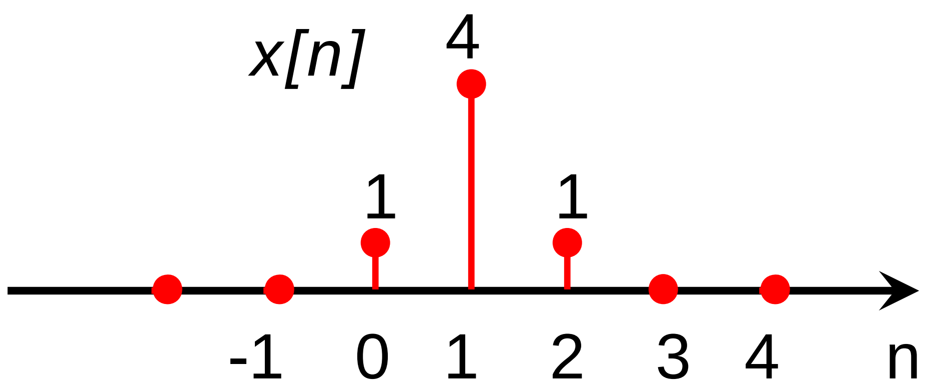
- **$|a| < 1$:** slow down by a factor of a

  a = 1/K; K must be an integer

Examples of time scaling



Examples of time scaling



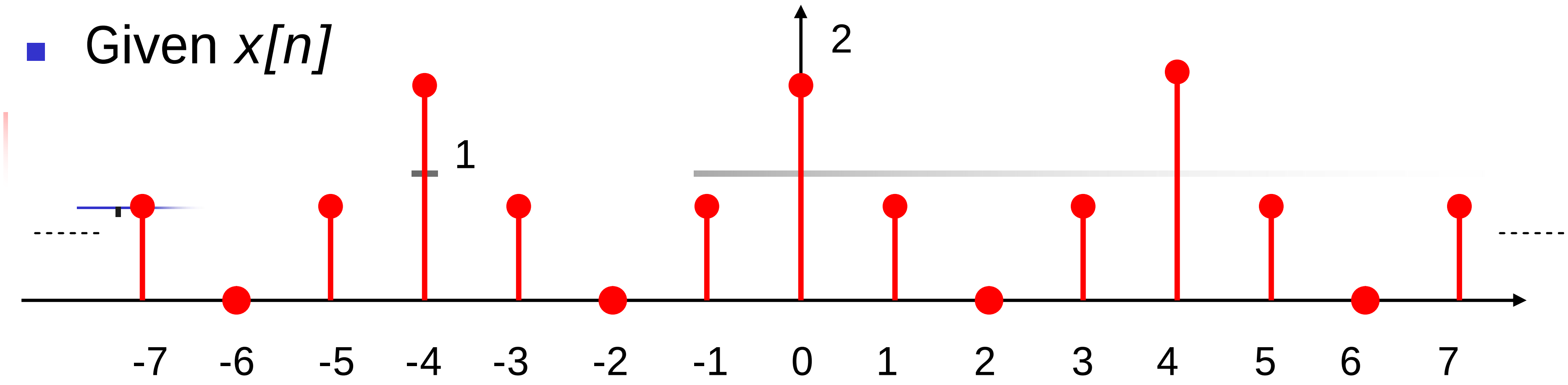
n	x[n]	y[n]=x[n/2]
0	1	1
1	4	?
2	1	4
3	0	?

How to find y[1] and y[3]??

One solution is <u>linear interpolation</u> used in a simple compression scheme

Examples of time scaling

- Given *x[n]*

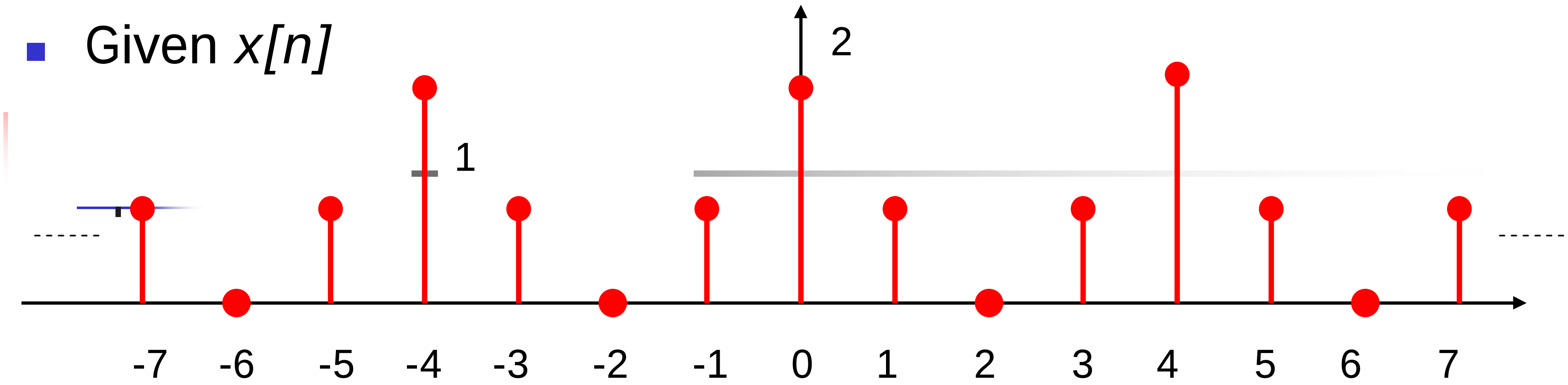


- $w_1[n] = x[2n]$

- What does $w_1[n/2]$ look like? **Just look like x[n]!**

Examples of time scaling

- Given *x[n]*



- Find $w_2[n] = x[2n+1]$

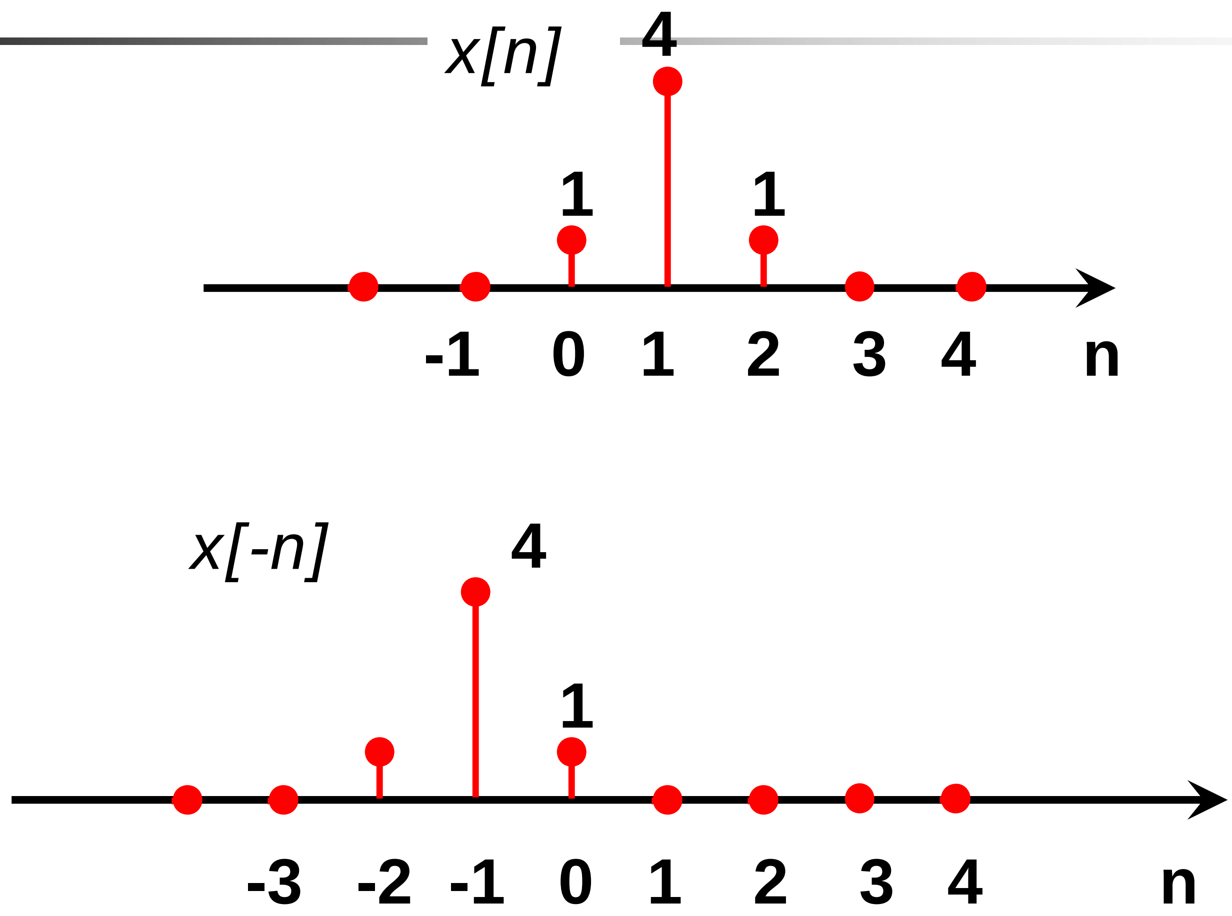
- What does $w_2[n/2]$ look like? **Just look like $w_2[n]$!**

Time reversal a DT signal

$$x[n] \rightarrow x[-n]$$

Flip a signal about the vertical axis

Examples of time reversal



Combining time reversal and time shifting

$$x[n] \rightarrow y[n] = x[-n-k]$$

Method 1: Flip first, then shift

Method 2: Shift first, then flip

Example – Method 1

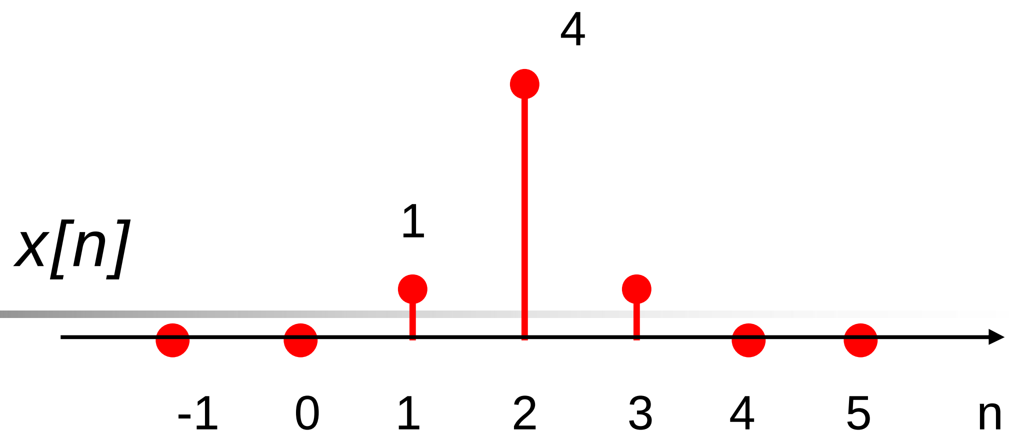
$$x[n] \rightarrow y[n] = x[-n-k]$$

Method 1: Flip first, then shift

Ex. Find $y[n] = x[-n-2] = x[-(n+2)]$

$$x[n] \xrightarrow{\text{Flip}} x[-n] = w[n] \xrightarrow{\text{Advance by 2 (left shift)}} w[n+2] = x[-(n+2)]$$

Example
Method 1



Example – Method 2

$$\boxed{x[n] \rightarrow y[n] = x[-n-k]}$$

Method 2: Shift first, then flip

Ex. Find y[n] = x[-n-2] =

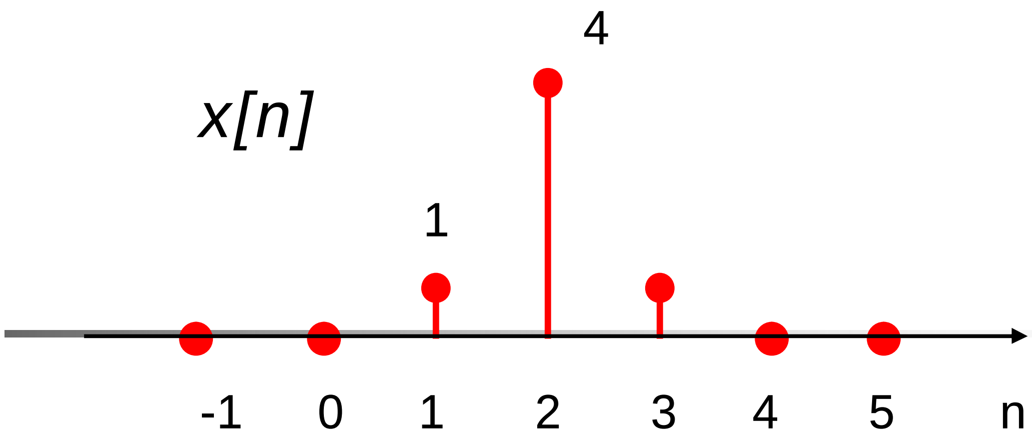
$x[n] \quad \rightarrow \quad x[n-2] = w[n] \quad \rightarrow \quad w[-n] = x[-n-2]$

Delay by 2

(right shift)

Flip w[n]

Example Method 2



Combining time shifting and time scaling

$$\boxed{x[n] \rightarrow y[n] = x[an-b]}$$

Method 1: time scale then shift

Method 2: shift then time scale

Be careful!!! For some cases, method 1 or 2 doesn't work.

To make sure, plug values into the table to check

Example – Method 1

$$x[n] \rightarrow y[n] = x[an-b]$$

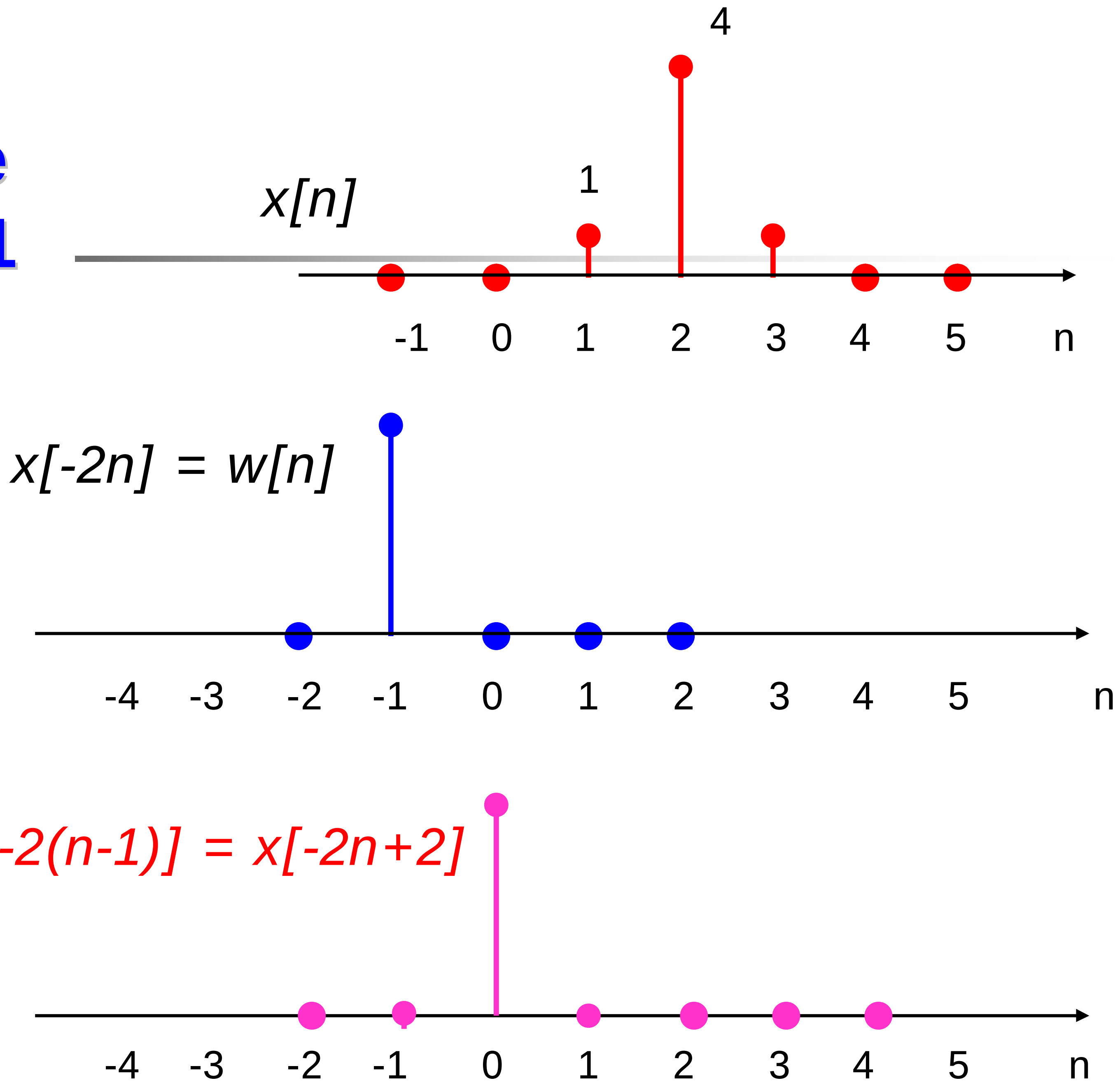
Method 1: time scale then shift

Ex. Find $y[n] = x[2-2n]$

$y[n] = x[-2(n-1)]$

$$x[n] \xrightarrow[\text{by } -2]{\text{Time scale}} x[-2n] = w[n] \xrightarrow[\text{by } 1]{\text{Delay}} w[n-1] = x[-2(n-1)]$$

Example Method 1



Example – Method 2

$$x[n] \rightarrow y[n] = x[an-b]$$

Method 2: shift then time scale

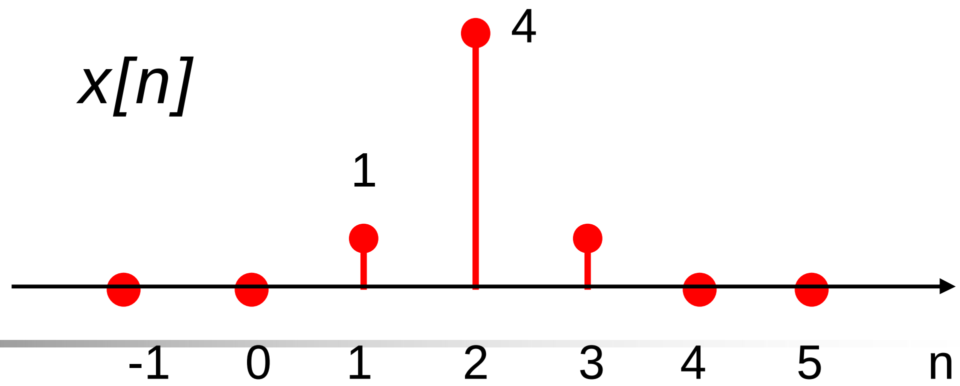
Ex. Find $y[n] = x[2-2n]$

$$x[n] \quad \rightarrow \quad x[n+2] = w[n] \quad \rightarrow \quad w[-2n] = x[2-2n]$$

advance by 2

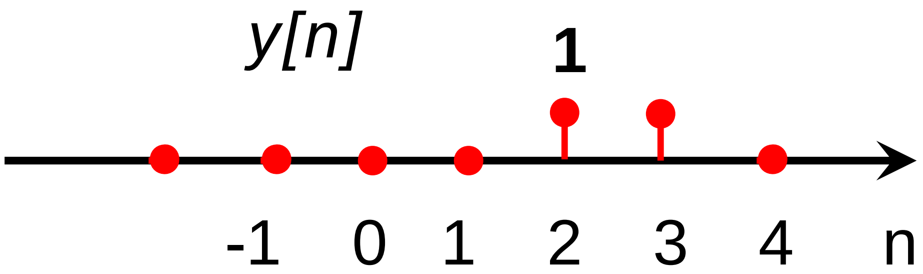
Time scale by -2

Example Method 2



Example

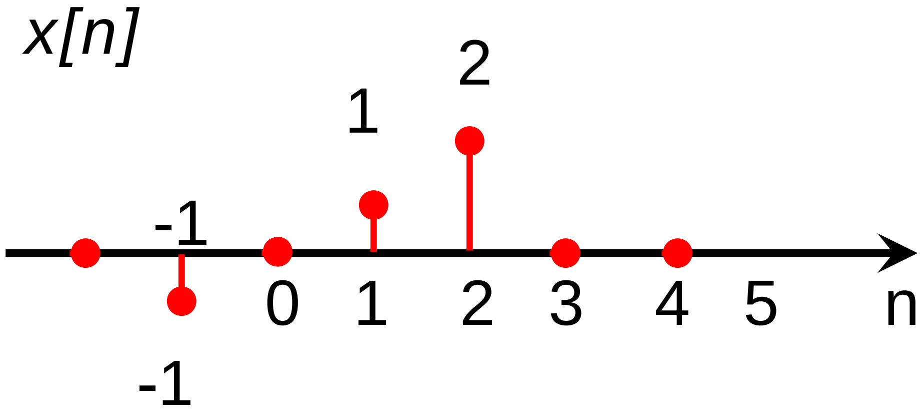
y[n] = x[2n-3]??



n	x[n]	y[n]
-1	0	0
0	0	0
1	1	0
2	4	1
3	1	1
4	0	0

Exercise

- Find $x[n]=(u[n+1]-u[n-5])(nu[2-n])$



Lecture #4
DT signals

1. Representations of DT signals
2. Some elementary DT signals
3. Simple manipulations of DT signals
4. **Characteristics of DT signals**

Characteristics of DT signals

- Symmetric (even) and anti-symmetric (odd) signals
- Energy and power signals

Even and odd signals

- A DT signal $x_e[n]$ is *even* if

$$\boxed{\text{Even}: x_e[n] = x_e[-n]}$$

And the signal $x_o[n]$ is *odd* if

$$\boxed{\text{Odd}: x_o[n] = -x_o[-n]}$$

- Any DT signal can be expressed as the sum of an even signal and an odd signal:

$$\left\{\begin{aligned} x_e[n] &= \tfrac{1}{2}(x[n] + x[-n]) \\ x_o[n] &= \tfrac{1}{2}(x[n] - x[-n]) \end{aligned}\right.$$

$$\boxed{x[n] = x_e[n] + x_o[n]}$$

Even and odd signals

- How to find $x_e[n]$ and $x_o[n]$ from a given $x[n]$?
- Step 1: find $x[-n]$
- Step 2: find

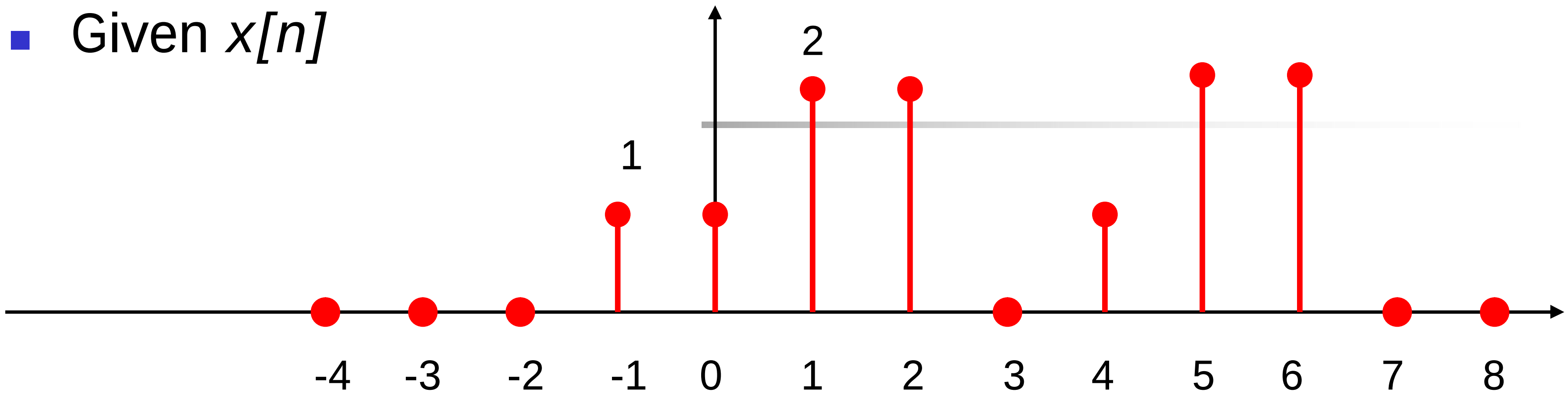
$$x_e[n] = \tfrac{1}{2}(x[n] + x[-n])$$

- Step 3: find

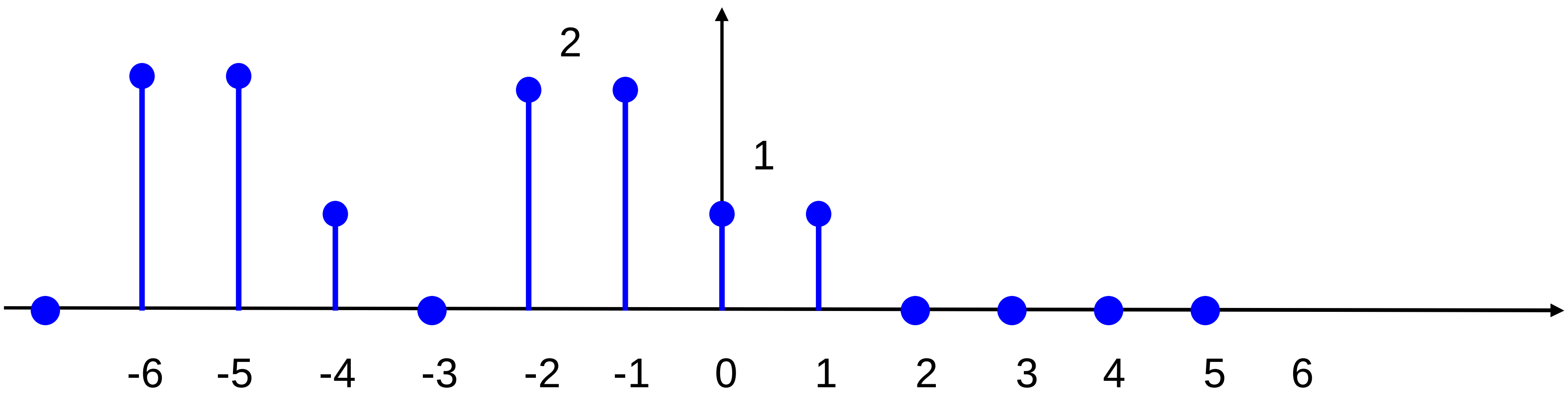
$$x_o[n] = \tfrac{1}{2}(x[n] - x[-n])$$

Example

- Given *x[n]*

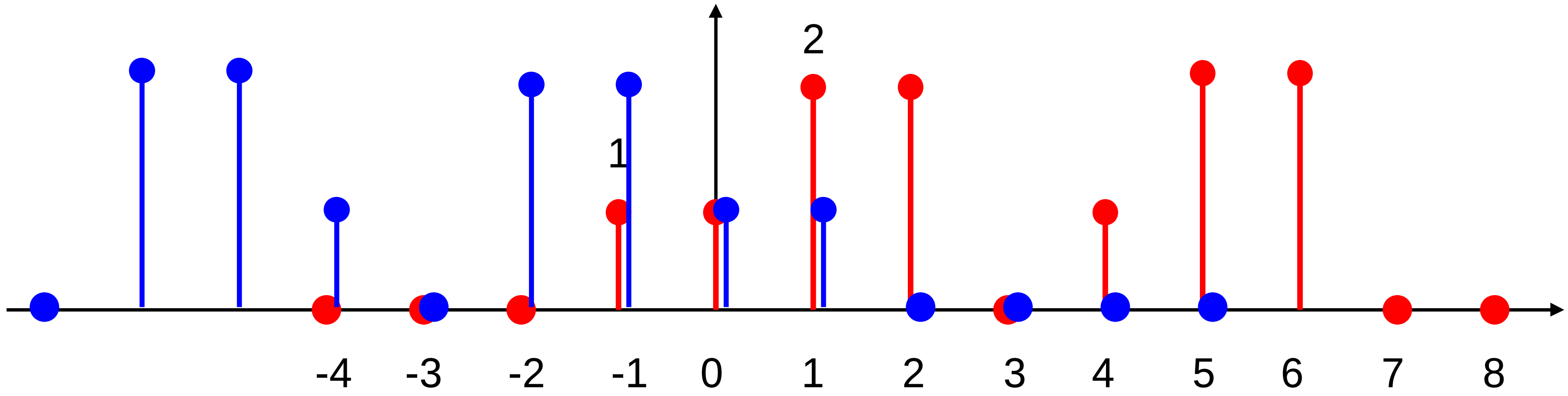


- Find *x[-n]*



Example

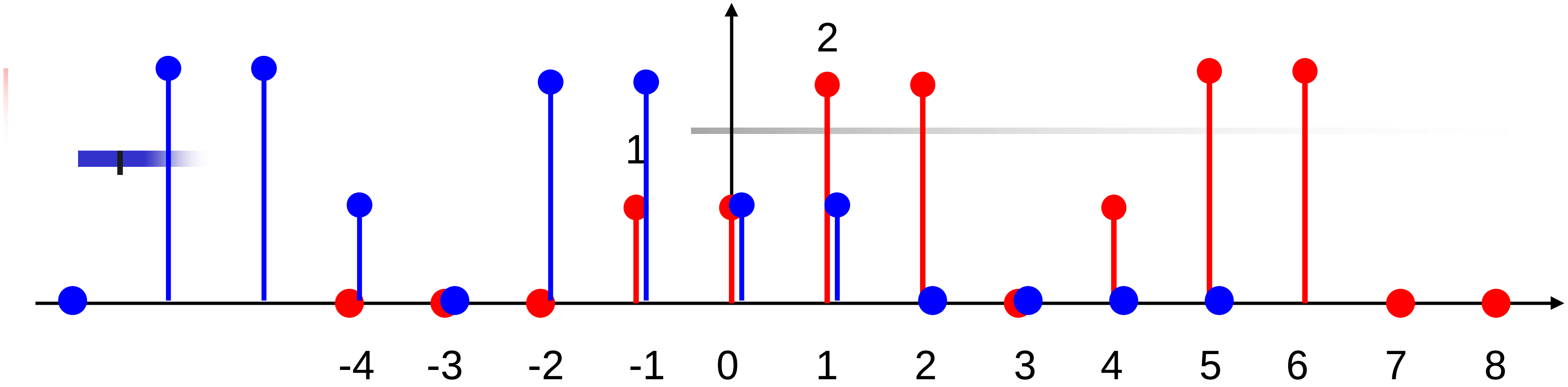
- Find $x[n] + x[-n]$



- Find $x_e[n]$

Example

- Find *x[n] - x[-n]*



- Find $x_0[n]$

Energy and power signals

- Define the signal energy: $$\boxed{E = \sum_{n=-\infty}^{\infty} |x[n]|^2}$$
- Define the signal power: $$\boxed{P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2}$$
- E is finite → x[n] is called an **energy signal**
- E is finite → P = 0
- E is infinite → P maybe finite or infinite. If P is finite and nonzero → x[n] is called **power signal**

Examples

- Determine which of the signals below are energy signals? Which are power signals?

(a) Unit step

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=0}^{\infty} 1^2 = \infty$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} 1^2 = \lim_{N\to\infty} \frac{N+1}{2N+1} = 1/2 < 0$$

Unit step is a power signal

Examples

- Determine which of the signals below are energy signals? Which are power signals?

(b) $x[n] = \begin{cases} (1/2)^n, & n \geq 0 \\ (2)^n, & n < 0 \end{cases}$

$$\sum_{n=n_0}^{\infty} a^n = \begin{cases} \dfrac{a^{n_0}}{1-a} & if \quad |a| < 1 \\ \infty & if \quad |a| \geq 1 \end{cases}$$

Examples

- Determine which of the signals below are energy signals? Which are power signals?

(c) $\mathrm{x[n]} = \cos\left(\frac{\pi}{4}\mathrm{n}\right)(\mathrm{u[n]} - \mathrm{u[n-4]})$

$$x[n] = \begin{cases} \cos\left(\frac{\pi}{4}n\right) & 0 \le n \le 3 \\ 0 & otherwise \end{cases} = \delta[n] + \frac{\sqrt{2}}{2}\delta[n-1] - \frac{\sqrt{2}}{2}\delta[n-3]$$

Lecture #5

DT systems

1. DT system

2. DT system properties

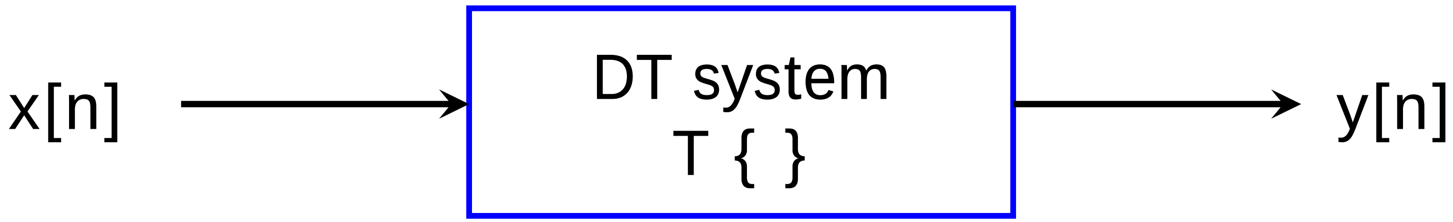
Input-output description of DT systems

Think of a DT system as an operator on DT signals:

- It processes DT input signals, to produce DT output signals

- **Notation: y[n] = T{x[n]} ←→** y[n] is the response of the system T to the excitation x[n]

- Systems are assumed to be a “black box” to the user

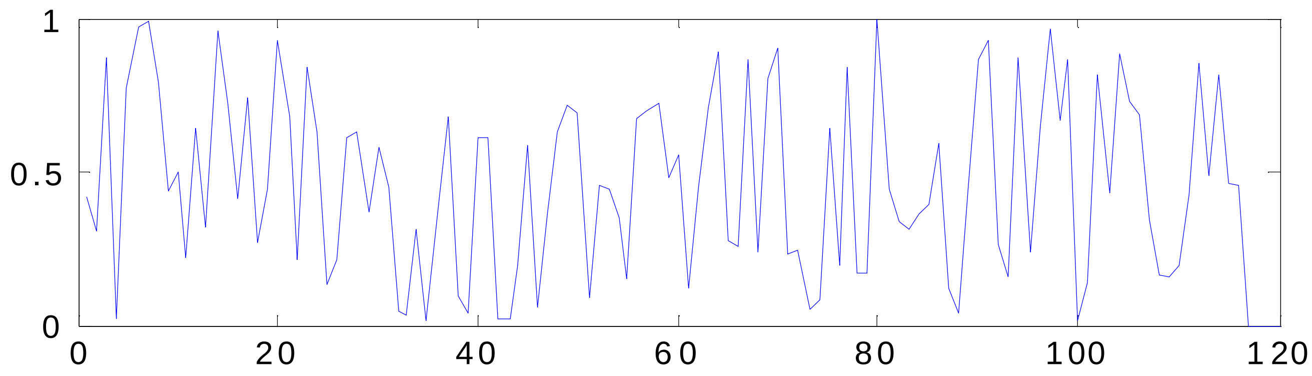


DT system example

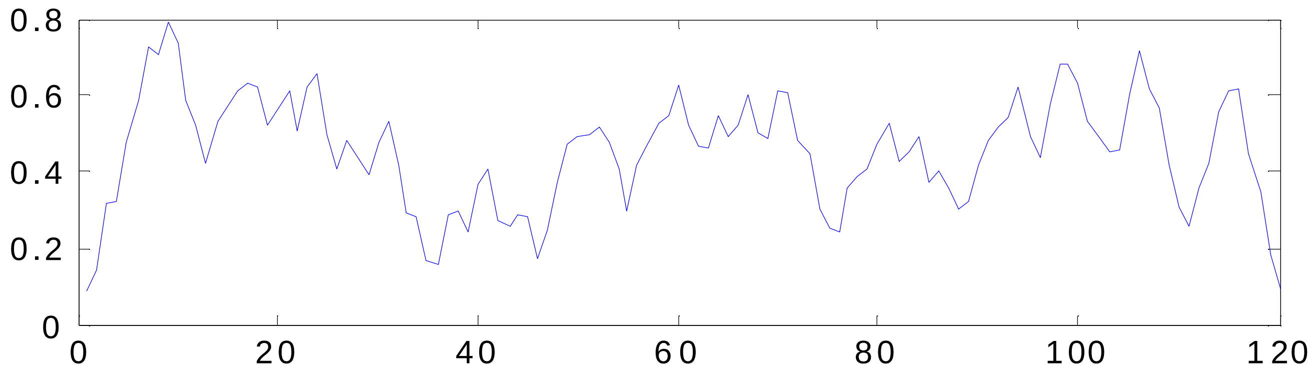
A digital low pass filter:

$$y[n] = 1/5\{x[n]+x[n-1]+x[n-2]+x[n-3]+x[n-4]\}$$

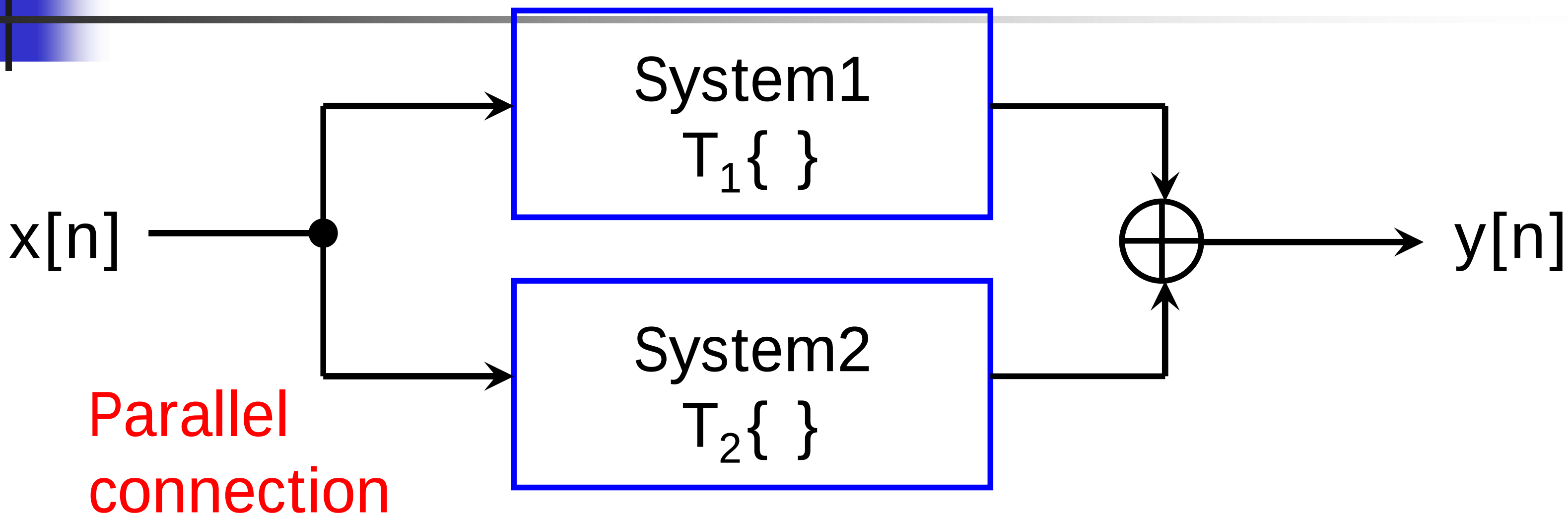
Before filtering



After filtering



Interconnection of DT systems

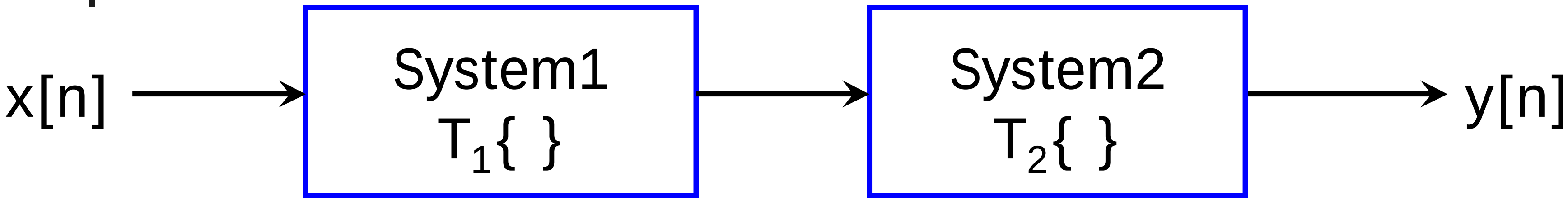


$$y[n] = y_1[n] + y_2[n] = T_1\{x[n]\} + T_2\{x[n]\} = (T_1 + T_2)\{x[n]\}$$

$$= T\{x[n]\}$$

$y[n] = T\{x[n]\}$: notation for the total system

Interconnection of DT systems

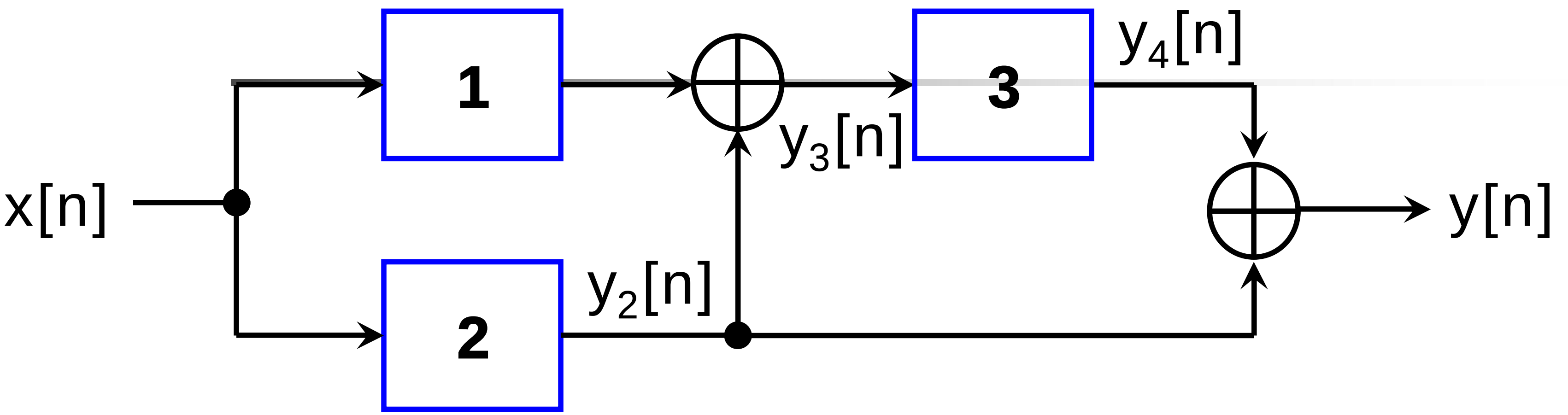


Cascade connection

$$\mathbf{y[n] = T_2\{y_1[n]\} = T_2\{T_1\{x[n]\}\} = T\{x[n]\}}$$

$y[n] = T\{x[n]\}$: notation for the total system

Example



$$y_3[n] = T_1\{x[n]\} + T_2\{x[n]\}$$

$$y_4[n] = T_3\{y_3[n]\} = T_3\{T_1\{x[n]\} + T_2\{x[n]\}\}$$

$$y[n] = y_2[n] + y_4[n] = T_2\{x[n]\} + T_3\{T_1\{x[n]\} + T_2\{x[n]\}\}$$

Lecture #5

DT systems

1. DT system

2. DT system properties

DT system properties

- Memory
- Invertibility
- Causality
- Stability
- Linearity
- Time-invariance

Memory

- **$y[n_0] = f(x[n_0])$** → system is memoryless (static)
- Otherwise, system has memory (dynamic), meaning that its output depends on inputs rather than just at the time of the output
- <u>Ex:</u>

a) $y[n] = x[n] + 5$:

b) $y[n]=(n+5)x[n]$:

c) $y[n]=x[n+5]$:

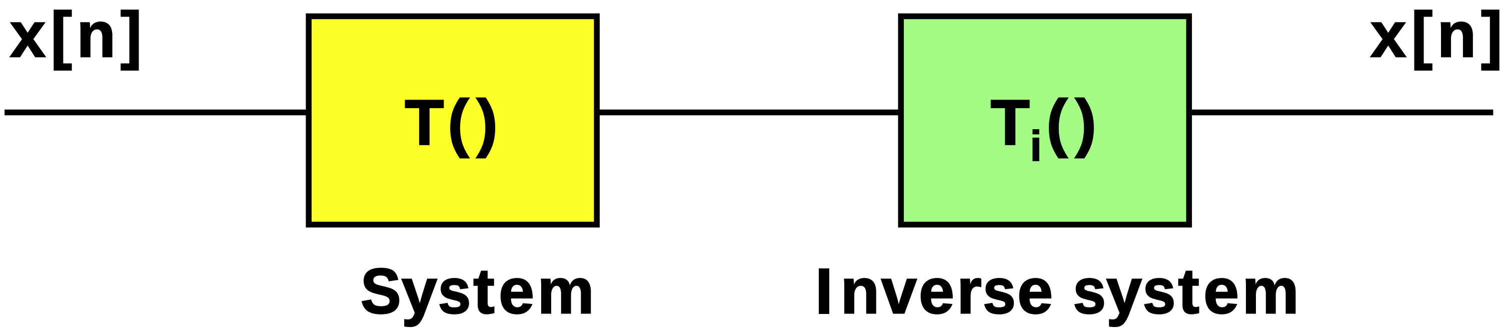
Invertibility

A system is said to be invertible if distinct inputs result in distinct outputs

Ex.: $y[n] = |x[n]|$ is ...

Invertibility

A system is said to be invertible if distinct inputs result in distinct outputs



$$\boxed{T_i[T(x[n])] = x[n]}$$

Examples for invertibility

Determine which of the systems below are invertible

a) Unit advance $y[n] = x[n+1]$ is invertible

$\rightarrow$ $y[n-1] = x[n]$ *is inverse system*

b) Accumulator $y[n] = \sum_{k=-\infty}^{n} x[k]$

c) Rectifier $y[n] = |x[n]|$

Causality

- The output of a causal system (at each time) **does not depend on** future inputs
- All memoryless systems are causal
- All causal systems can have memory or not

Examples for causality

- Determine which of the systems below are causal:

a) $y[n] = x[-n]$

b) $y[n] = (n+1)x[n-1]$

c) $y[n] = x[(n-1)^2]$

d) $y[n] = \cos(w_0 n + x[n])$

e) $y[n] = 0.5y[n-1] + x[n-1]$

Stability

- If a system "blow up" it is **not stable**

  In particular, if a "well-behavior" signal (all values have finite amplitude) results in infinite magnitude outputs, the system is **unstable**

- BIBO stability: "**bounded input – bounded output**" – if you put finite signals in, you will get finite signals out

Examples for stability

- Determine which of the systems below are BIBO stable:

a) A unit delay system

b) An accumulator

c) $y[n] = \cos(x[n])$

d) $y[n] = \ln(x[n])$

e) $y[n] = \exp(x[n])$

Linearity

Scaling signals and adding them, then processing through the system

same as

Processing signals through system, then scaling and adding them

If $T(x_1[n]) = y_1[n]$ and $T(x_2[n]) = y_2[n]$

$\rightarrow T(ax_1[n] + bx_2[n]) = ay_1[n] + by_2[n]$

Time-invariance

- If you time shift the input, get the **same** output, but with the **same** time shift
- The behavior of the system **doesn't change** with time

$$\textbf{If } T(x[n]) = y[n]$$

$$\textbf{then } T(x[n-n_0]) = y[n-n_0]$$

Examples for linearity and time-invariance

Determine which of the systems below are linear, which ones are time-invariant

a) $y[n] = nx[n]$

Linear

Not time-invariant

Examples for linearity and time-invariance

Determine which of the systems below are linear, wich ones are time-invariant

b) $y[n] = x^2[n]$

Examples for linearity and time-invariance

Determine which of the systems below are linear, wich ones are time-invariant

c) $$y[n]=\sum_{r=0}^{M} b_r x[n-r]$$

Example for DT system properties

Given the system below: $$y[n]=\left[\frac{n+2.5}{n+1.5}\right]^2 x[n]$$

a) Memoryless ?

b) Invertible ?

c) Causal ?

Example for DT system properties

Given the system below: $$y[n] = \left[\frac{n+2.5}{n+1.5}\right]^2 x[n]$$

c) Stable ?

Example for DT system properties

Given the system below: $$y[n] = \left[\frac{n+2.5}{n+1.5}\right]^2 x[n]$$

d) time-invariant ?

e) Linear ?

Lecture #6

DT convolution

1. **DT convolution formula**
2. DT convolution properties
3. Computing the convolution sum
4. DT LTI properties from impulse response

Computing the response of DT LTI systems to arbitrary inputs

Method 1: based on the direct solution of the input-output equation for the system

Method 2:

- Decompose the input signal into a sum of elementary signals
- Find the response of system to each elementary signal
- Add those responses to obtain the total response of the system to the given input signal

$$x[n] = \sum_k c_k x_k[n]$$

$$x_k[n] \quad \rightarrow \quad y_k[n]$$

$$x[n] \quad \rightarrow \quad y[n] = \sum_k c_k y_k[n]$$

DT convolution formula

Convolution: an operation between the input signal to a system and its impulse response, resulting in the output signal

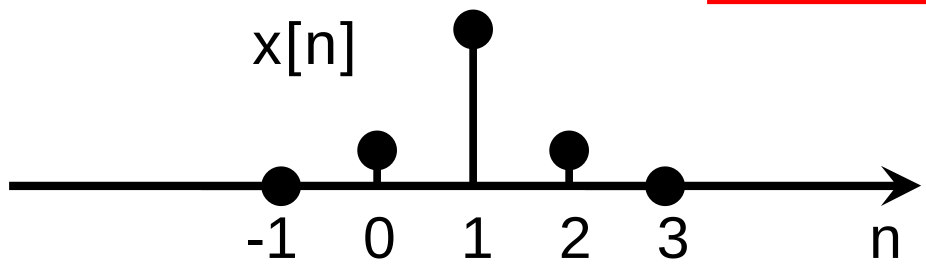
CT systems: convolution of 2 signals involves integrating the product of the 2 signals – where one of signals is flipped and shifted

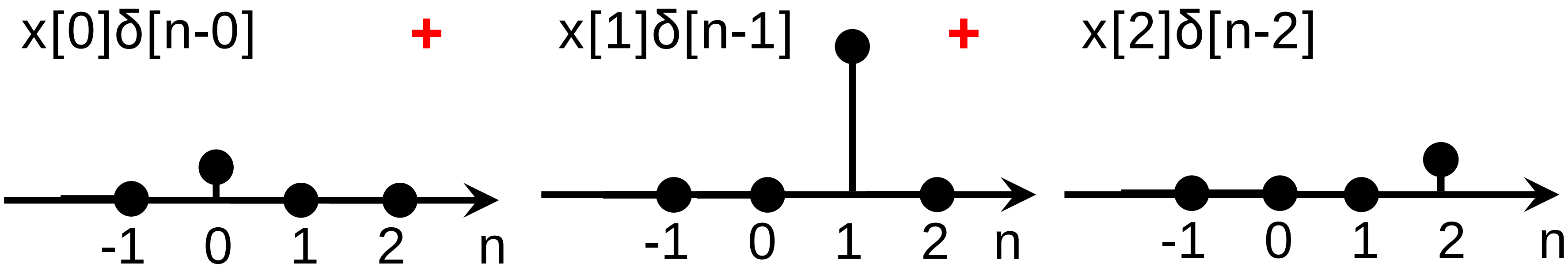
DT systems: convolution of 2 signals involves summing the product of the 2 signals – where one of signals is flipped and shifted

Impulse representation of DT signals

We can describe any DT signal x[n] as: $$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]$$

Example:

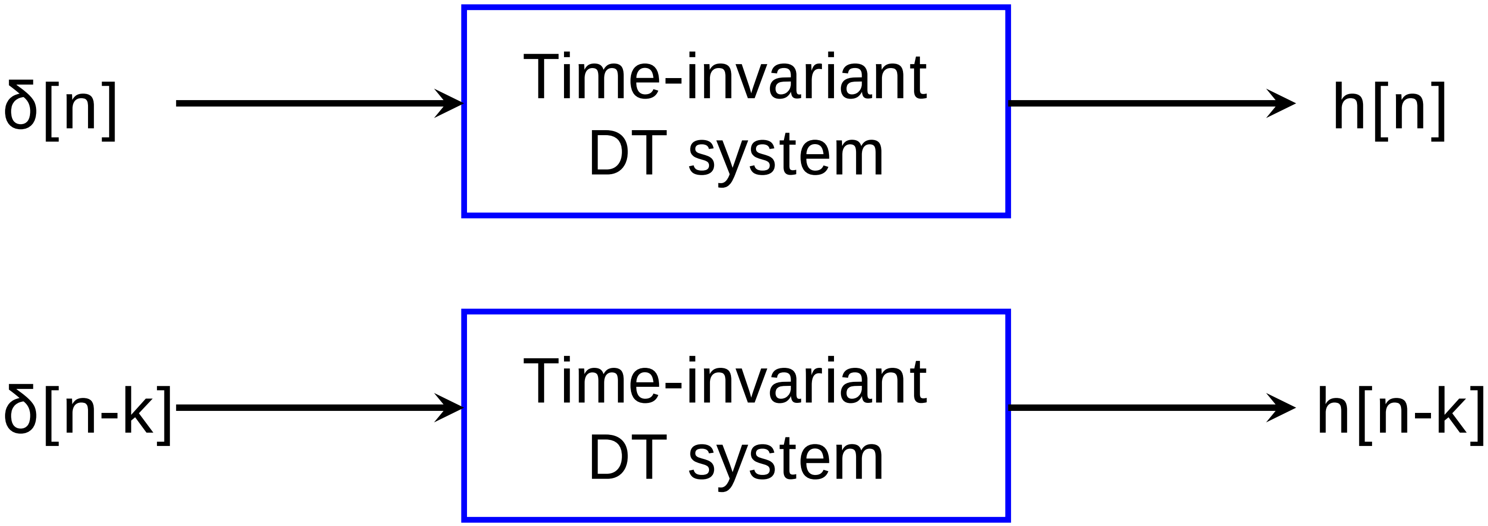




Impulse response of DT systems

Impulse response: the output results, in response to a unit impulse

Denotation: **$h_k[n]$:** impulse response of a system, to an impulse at time *k*



Remember: the impulse response is a sequence of values that may go on forever!!!

Response of LTI DT systems to arbitrary inputs

δ[n-k] → LTI DT system → h[n-k]

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \rightarrow \text{LTI DT system} \rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Convolution sum

Notation: y[n] = x[n] * h[n]

Convolution sum in more details

$y[0] = x[0] * h[0]$

$= \dots + x[-2]h[2] + x[-1]h[1] + x[0]h[0]$

$+ x[1]h[-1] + x[2]h[-2] + \dots +$

The general output:

$y[n] = \dots + x[-2]h[n+2] + x[-1]h[n+1] + x[0]h[n]$

$+ x[1]h[n-1] + x[2]h[n-2] + \dots + x[n-1]h[1]$

$+ x[n]h[0] + x[n+1]h[-1] + x[n+2]h[-2]$

Note: the sum of the arguments in each term is always *n*

Lecture #6

DT convolution

1. DT convolution formula

2. DT convolution properties

3. Computing the convolution sum

4. DT LTI properties from impulse response

Convolution sum properties

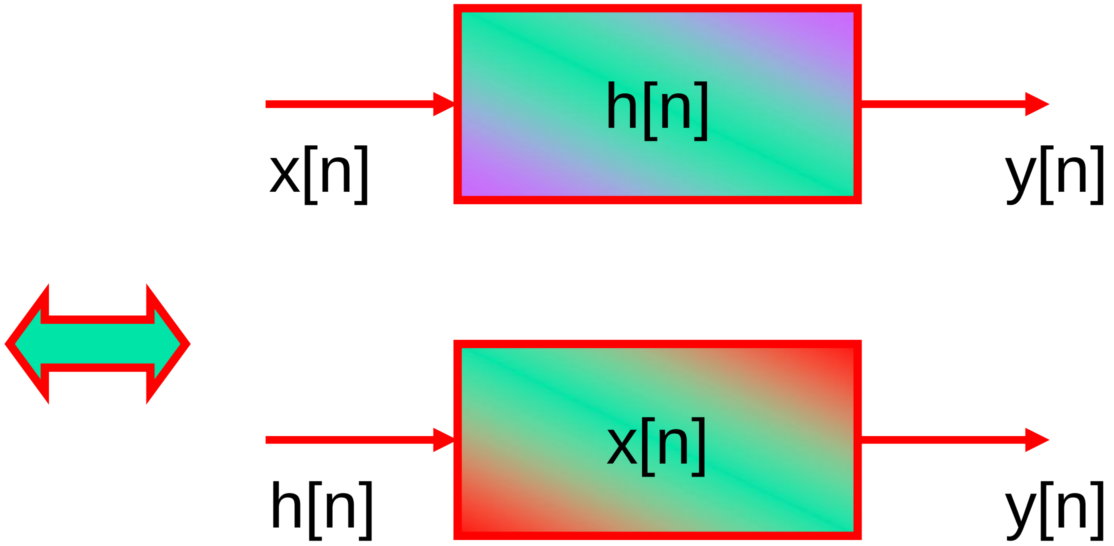
- $\delta[n] * x[n] = x[n]$

  $\delta[n-m] * x[n] = x[n-m]$

  $\delta[n] * x[n-m] = x[n-m]$
- Commutative law
- Associative law
- Distributive law

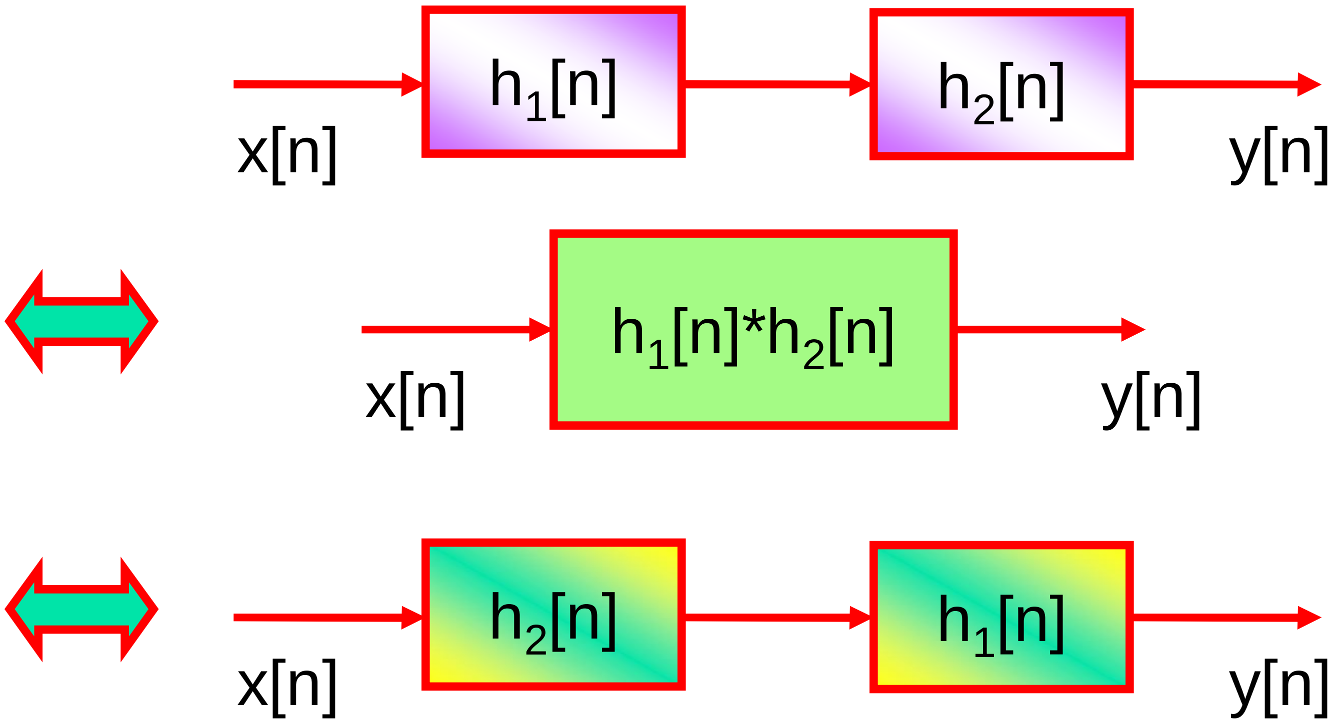
Commutative law

$$\boxed{x[n] * h[n] = h[n] * x[n]}$$



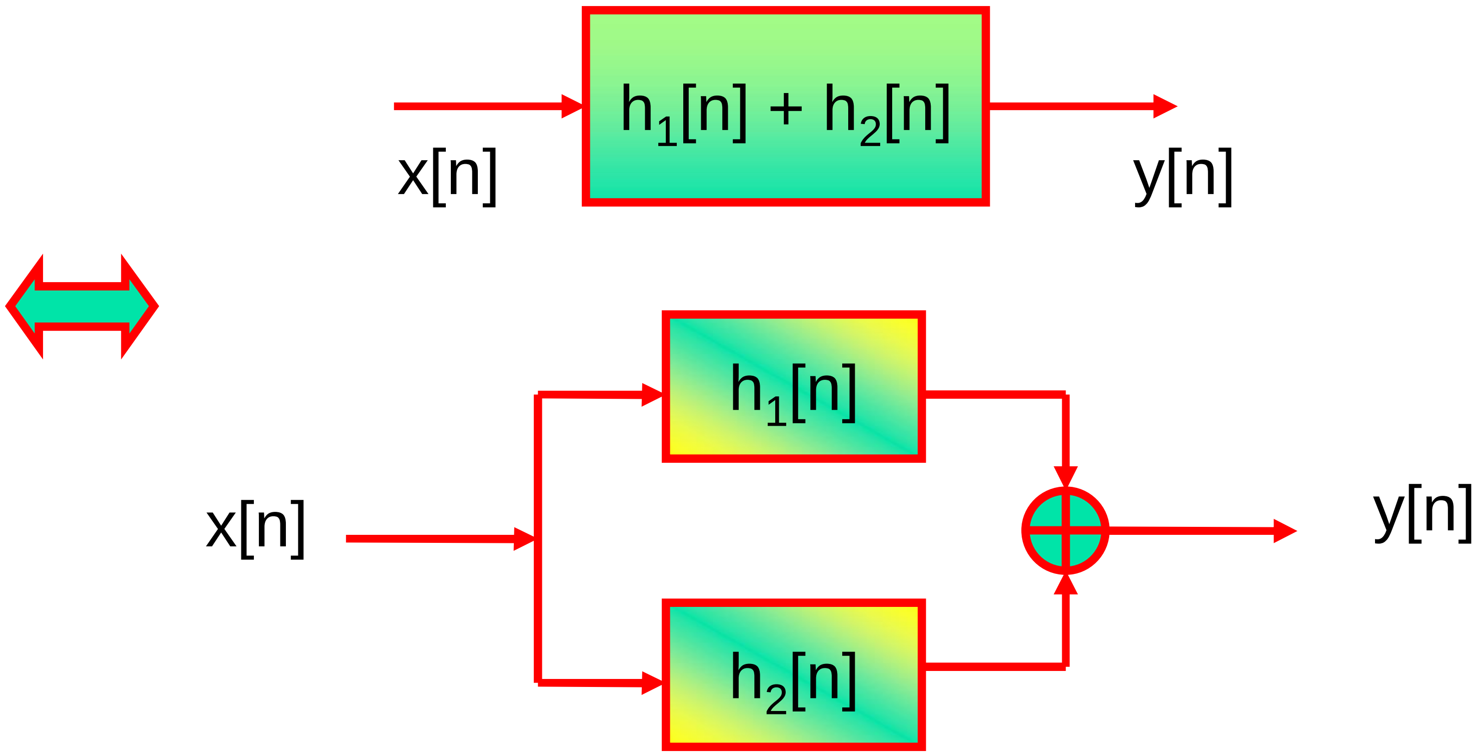
Associative law

$$(x[n]*h_1[n])*h_2[n] = x[n]*(h_1[n]*h_2[n])$$



Distributive law

$$\boxed{x[n]*(h_1[n]+h_2[n]) = (x[n]*h_1[n]) + (x[n]*h_2[n])}$$



Lecture #6

DT convolution

1. DT convolution formula
2. DT convolution properties
3. **Computing the convolution sum**
4. DT LTI properties from impulse response

Computing the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \quad \Rightarrow \quad y[n_0] = \sum_{k=-\infty}^{\infty} x[k]h[n_0-k]$$

1. **Fold** *h[k]* about $k = 0$, to obtain *h[-k]*

2. **Shift** *h[-k]* by n_0 to the right (left) if n_0 is positive (negative), to obtain $h[n_0-k]$

3. **Multiply** *x[k]* and $h[n_0-k]$ for all *k*, to obtain the product

$$x[k].h[n_0-k]$$

4. **Sum** up the product for all *k*, to obtain $y[n_0]$

Repeat from 2-4 fof all of n

The length of the convolution sum result

$$\boxed{y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]}$$

Suppose:

Length of x[k] is N_x → $N_1 \le k \le N_1 + N_x - 1$

Length of h[n-k] is N_h → $N_2 \le n\text{-}k \le N_2 + N_h - 1$

→ $N_1 + N_2 \le n \le N_1 + N_2 + N_x + N_h - 2$

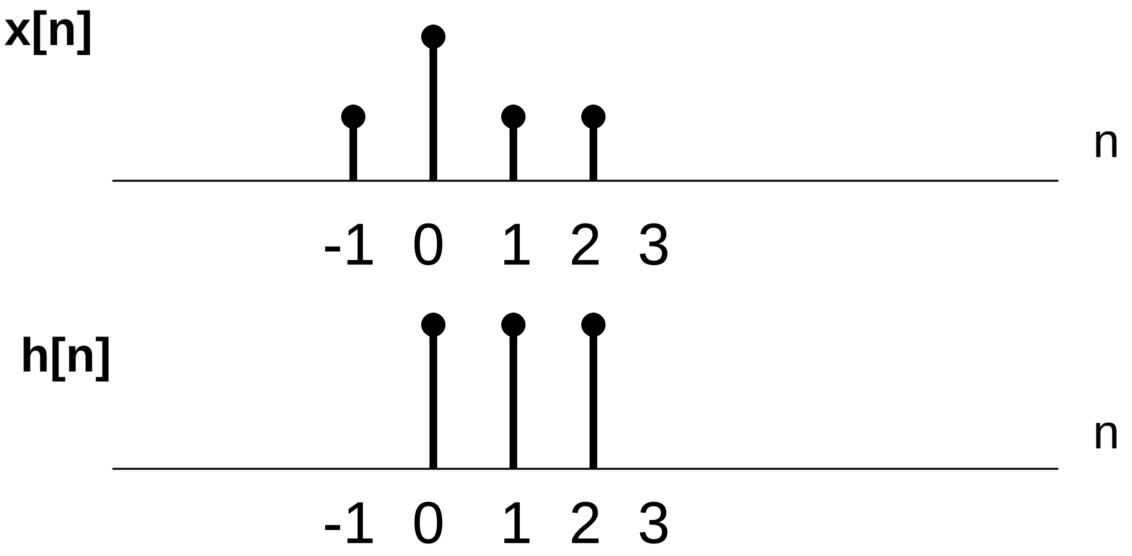
Length of y[n]:

$$\boxed{\mathbf{N_y = N_x + N_h - 1}}$$

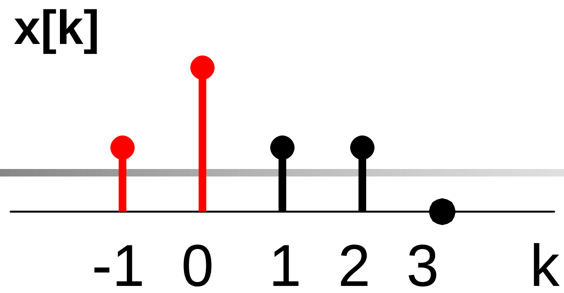
Example 1

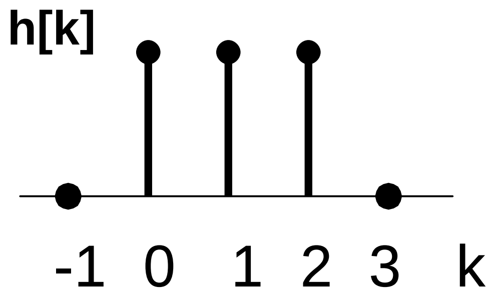
Find y[n] = x[n]*h[n] where

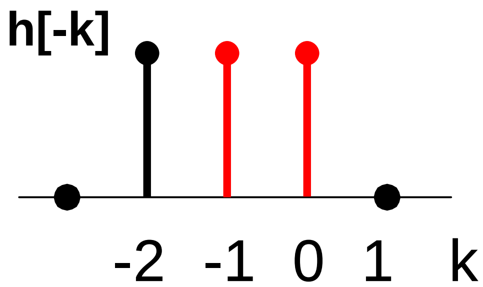
$$x[n] = u[n+1] - u[n-3] + \delta[n] \qquad h[n] = 2\left(u[n] - u[n-3]\right)$$



Ex1 (cont)

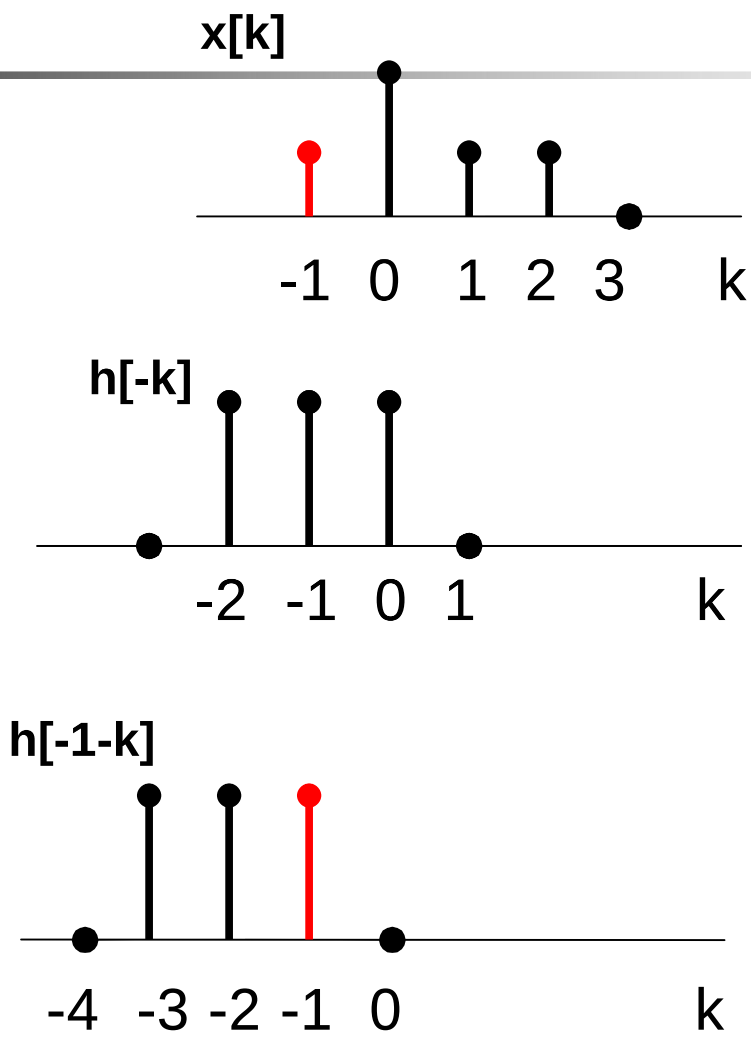






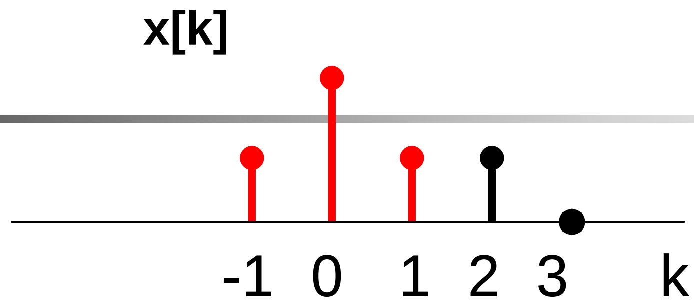
y[0] = 6;

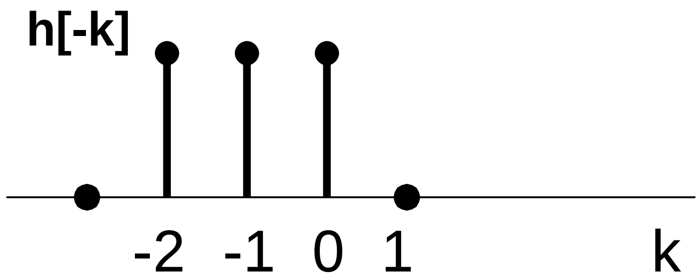
Ex1 (cont)



y[-1] = 2;

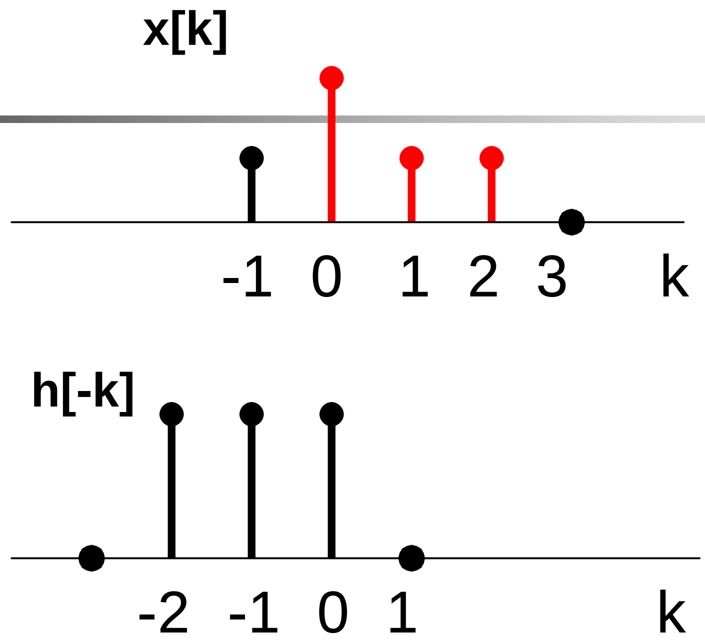
Ex1 (cont)





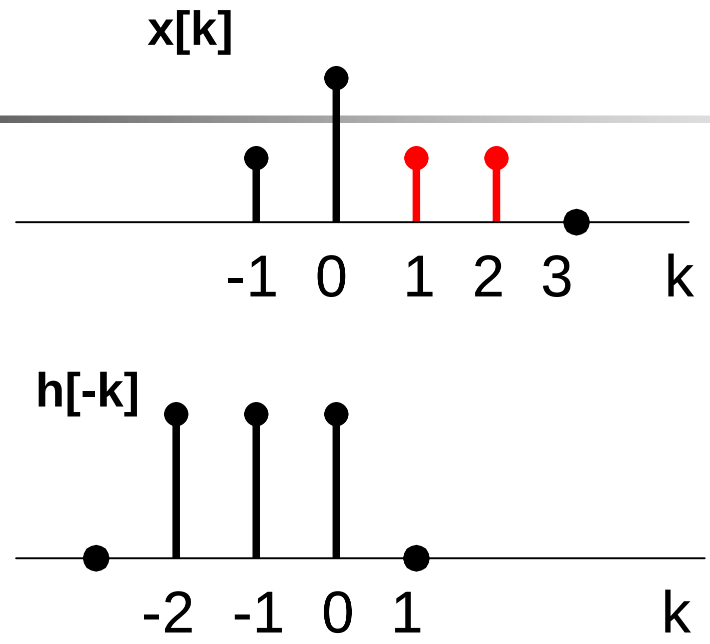
y[1] = ?

Ex1 (cont)



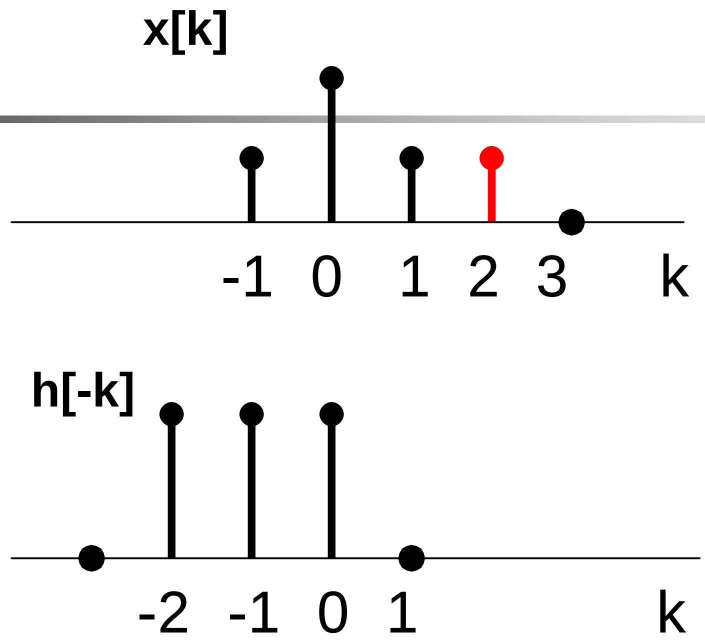
y[2] = ?

Ex1 (cont)



y[3] = ?

Ex1 (cont)



y[4] = ?

Example 2

Find y[n] = x[n]*h[n] where $x[n] = a^n u[n]$ $h[n] = u[n]$

Try it both ways (first flip x[n] and do the convolution and then flip h[n] and do the convolution). Which method do you prefer?

Remember!

$$\sum_{n=n_0}^{\infty} a^n = \begin{cases} \dfrac{a^{n_0}}{1-a} & if \quad |a|<1 \\ \infty & if \quad |a| \geq 1 \end{cases}$$

$$\sum_{n=n_0}^{n_1} a^n = a^{n_0} \frac{1-a^{(n_1-n_0+1)}}{1-a}$$

Example 2

First flip x[n] and do the convolution

Factor out a^n to
get form you know

Therefore

$$\boxed{y[n] = \left(\frac{a^{n+1} - 1}{a - 1} \right) u[n]}$$

Example 2

First flip h[n] and do the convolution

you know this form

Therefore
$$\boxed{y[n]=\left(\frac{a^{n+1}-1}{a-1}\right)u[n]}$$

Tip: first flip the simpler signal!!!

Example 3

- Find $y[n] = x[n]*h[n]$ where $x[n] = b^n u[n]$ and $h[n] = a^n u[n+2]$

  $|a| < 1$, $|b| < 1$, $a \neq b$

Example 4

Compute output of a system with impulse response

$h[n] = a^n u[n-2]$, $|a| < 1$ when the input is $x[n] = u[-n]$

Flipping x[n] because it is simpler

Lecture #6

DT convolution

1. DT convolution formula
2. DT convolution properties
3. Computing the convolution sum
4. **DT LTI properties from impulse response**

Recall impulse response

- System output, in response to the unit impulse input:

$$h[n] = T\{\delta[n]\}$$

- May go on forever → there are 2 kinds of LTI systems:

1. Finite Impulse Response (FIR) system
2. Infinite Impulse Response (IIR) system

- Be one of system representations
- Be convolved with the input to result in the output

Calculation of the impulse response

- Applying the unit impulse function to the input-output equation
- **Ex.:** $$y[n]=\frac{1}{4}y[n-1]+\frac{1}{2}x[n]+\frac{1}{2}x[n-1]$$

Suppose this system is causal

$$h[0]=\frac{1}{4}h[-1]+\frac{1}{2}\delta[0]+\frac{1}{2}\delta[-1]=\frac{1}{2}$$

$$h[1]=\frac{1}{4}h[0]+\frac{1}{2}\delta[1]+\frac{1}{2}\delta[0]=\frac{1}{4}x\frac{1}{2}+\frac{1}{2}=\frac{5}{8}=\frac{5}{8}\left(\frac{1}{4}\right)^0$$

$$h[2]=\frac{1}{4}h[1]+\frac{1}{2}\delta[2]+\frac{1}{2}\delta[1]=\frac{5}{8}\left(\frac{1}{4}\right)^1$$

$$h[3]=\frac{1}{4}h[2]+\frac{1}{2}\delta[3]+\frac{1}{2}\delta[2]=\frac{1}{4}x\frac{1}{4}x\frac{5}{8}=\frac{5}{8}\left(\frac{1}{4}\right)^2$$

$$y[n]=(1/2)\delta[n]+(1/4)^{n-1}(5/8)u[n-1]$$

DT LTI properties from impulse response

- Memoryless system: impulse response must have the form below

$$h[n] = K\delta[n]$$

- Invertible system: system with $h[n]$ is invertible if there exists another impulse response $h_i[n]$ such that

$$h[n]*h_i[n] = \delta[n]$$

DT LTI properties from impulse response

- Causal system: *h[n]* is zero for all time *n<0*

The system is causal → output does not depend on future inputs →

$$y[n_0] = \sum_{k=-\infty}^{\infty} h[k]x[n_0-k] = \sum_{k=0}^{\infty} h[k]x[n_0-k] + \sum_{k=-\infty}^{-1} h[k]x[n_0-k]$$
$$= \underbrace{\{h[0]x[n_0] + h[1]x[n_0-1] + \ldots\}}_{\text{Present and past inputs}} + \underbrace{\{h[-1]x[n_0+1] + h[-2]x[n_0+2] + \ldots\}}_{\text{Future inputs}}$$

Thus, the second term should be zero ←→ h[n] = 0   n < 0

DT LTI properties from impulse response

- Causal system: *h[n]* is zero for all time *n<0*

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$\downarrow \qquad\qquad \downarrow$$

$$= \sum_{k=0}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{n} x[k]h[n-k]$$

If at least one value of *h[k]* for a negative *k* is not zero, the system is not causal

DT LTI properties from impulse response

- BIBO stable system: finite input, finite output
- If x[n] is bounded then:

$$|x[n]| \leq M_x < \infty$$

Take the absolute:

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \Rightarrow |y[n]| \leq \sum_{k=-\infty}^{\infty} |h[k]x[n-k]|$$

$$|y[n]| \leq M_x \sum_{k=-\infty}^{\infty} |h[k]|$$

The output is bounded if the impulse response satisfies:

$$\boxed{\sum_{n=-\infty}^{+\infty} |h[n]| < \infty}$$

Examples

1. Is $h[n] = 0.5^n u[n]$ BIBO stable? Causal?

Stable Causal

2. Is $h[n] = 3^n u[n]$ BIBO stable? Causal?

3. Is $h[n] = 3^n u[-n]$ BIBO stable? Causal?

Lecture #7

Difference equation model

1. LTI systems characterized by linear constant coefficient difference equations

2. Recursive solution of difference equations

3. Closed form solution of difference equations

Linear constant coefficient difference equations

General form:

$$y[n]+a_1 y[n-1]+\ldots+a_N y[n-N]=b_0 x[n]+b_1 x[n-1]+\ldots+b_M x[n-M]$$

$$\boxed{\Leftrightarrow \sum_{k=0}^{N} a_k y[n-k]=\sum_{r=0}^{M} b_r x[n-r], \quad a_0=1}$$

N, M: non-negative integers

N: order of equation

a_k, b_r: real constant coefficients

Linear constant coefficient difference equations

Two common ways to write:

$$\boxed{\begin{gathered}\sum_{k=0}^{N} a_k y[n-k] = \sum_{r=0}^{M} b_r x[n-r], \quad a_0 = 1 \\ y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{r=0}^{M} b_r x[n-r]\end{gathered}}$$

Solving the equation in 2 ways:

1. Solving the equation recursively, one value at a time. Need to start the iteration with initial conditions

2. Finding a formula for *y[n]* (a closed form)

Recursive solution of difference equations

1) Put y[n] on the left hand side by itself

$$y[n] = -a_1 y[n-1] - \dots - a_N y[n-N] + b_0 x[n] + \dots + b_M x[n-M]$$

2) To calculate a given output at time $n = n_0$, that is $y[n_0]$, we add the weighted M+1 inputs $b_0 x[n_0] + \dots + b_M x[n_0-M]$ to the weighted N past outputs $-a_1 y[n_0-1] - \dots - a_N y[n_0-N]$

3) Increase the time index to $n = n_0+1$ and recursively calculate the next output. This can continue forever.

To start this recursion somewhere, for example at $n_0 = 0$, we need to know the N initial conditions $y[n_0-1], y[n_0-2], \dots, y[n_0-N]$

Example 1

Solve iteratively to find the 1st 3 terms of

$y[n] - 2y[n-1] = x[n-1]$

with initial condition $y[-1] = 10$, and with the input $x[n] = 2u[n]$

n	**x[n]**	**y[n]**
-1	0	**10** (initial condition)
0	2	y[0] = x[-1]+2y[-1] = **20**
1	2	y[1] = x[0]+2y[0] = 2+2(20) = **42**
2	2	y[2] = x[1]+2y[1] = 2+2(42) = **86**

Example 2

Find the 1st 3 terms of $y[n+2] - y[n+1] + 0.24y[n] = x[n+2] - 2x[n+1]$

with initial condition $y[-1] = 2$, $y[-2] = 1$, and input $x[n] = nu[n]$

This system is time invariant, so it is equivalent to

$y[n] - y[n-1] + 0.24y[n-2] = x[n] - 2x[n-1]$

n	x[n]	y[n]
-2	0	**1** (initial condition)
-1	0	**2** (initial condition)
0	0	y[n] = x[n]-2x[n-1]+y[n-1]-0.24y[n-2] y[0] = **1.76**
1	1	y[1] = **2.28**
2	2	y[2] = **1.8576**

Closed form solutions of difference equations

Total response = zero-input component + zero-state component
= natural response + forced response
= complementary response + particular response

1. Find the complementary response, assume input = 0.
2. Find the particular response, assume all initial conditions = 0. Choose the form of the particular response same as the form of input
3. Total response = complementary + particular. Use initial conditions to find *N* constants from the complementary response

Example

Given $y[n] - 0.3y[n-1] = x[n]$ with $y[-1] = 0$ and $x[n] = (0.6)^n$

Example (cont)

Combining particular and complementary solutions:

Lecture #8

Block diagram for DT LTI systems

1. **Components for block diagram**

2. Direct form I realization of a system

3. Direct form II realization of a system

Components for block diagram

Components: branch points, summation, delays, gains, LTI systems

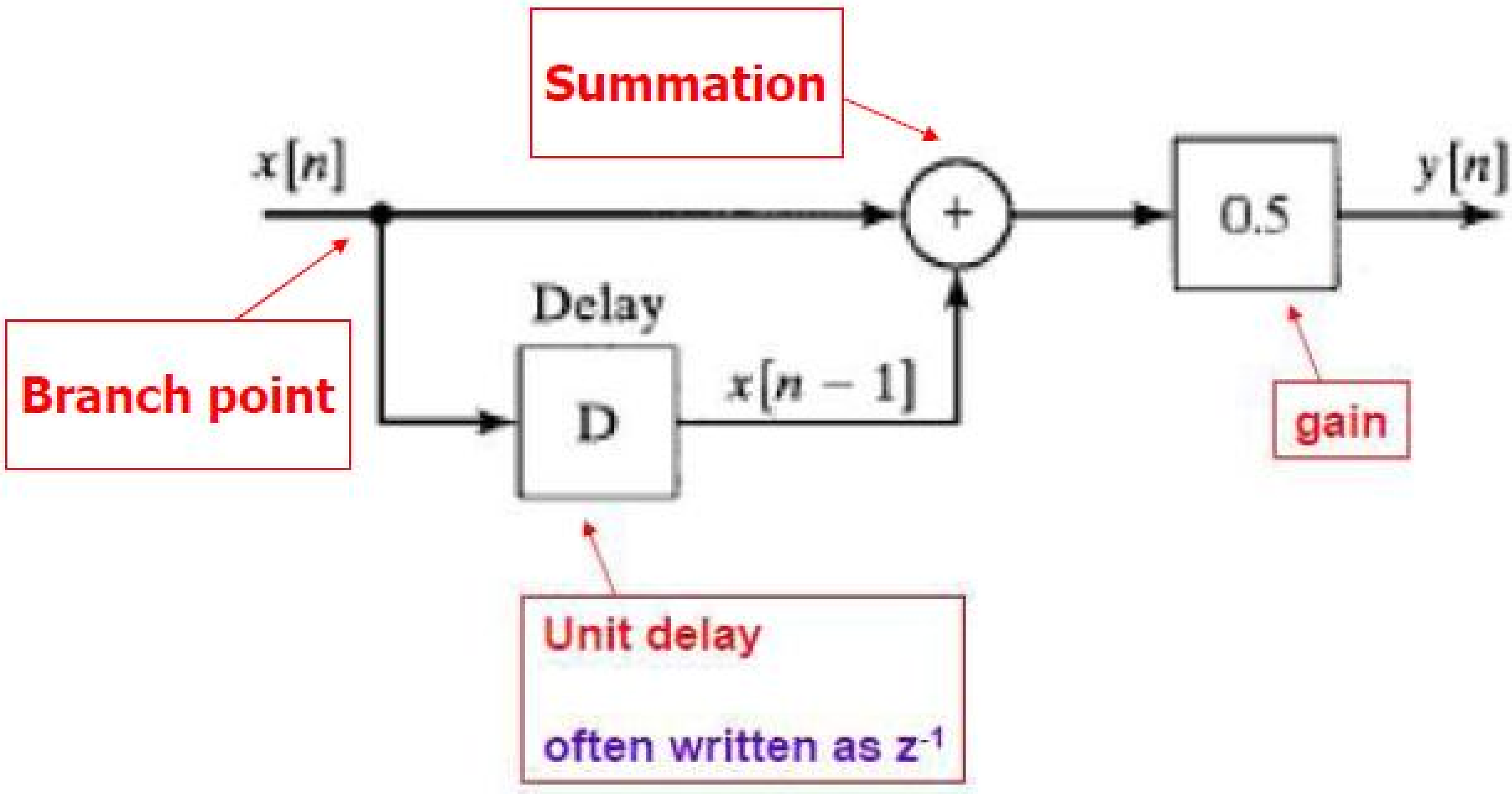
• Signals travel along lines, arrows point direction of inputs and outputs

• Each component acts the same way, no matter how many components are connected to it

• For linear system, the order of operation does not matter responses to obtain

Components for block diagram

EX: an averaging system y[n] = 0.5(x[n] + x[n-1])



Lecture #8

Block diagram for DT LTI systems

1. Components for block diagram

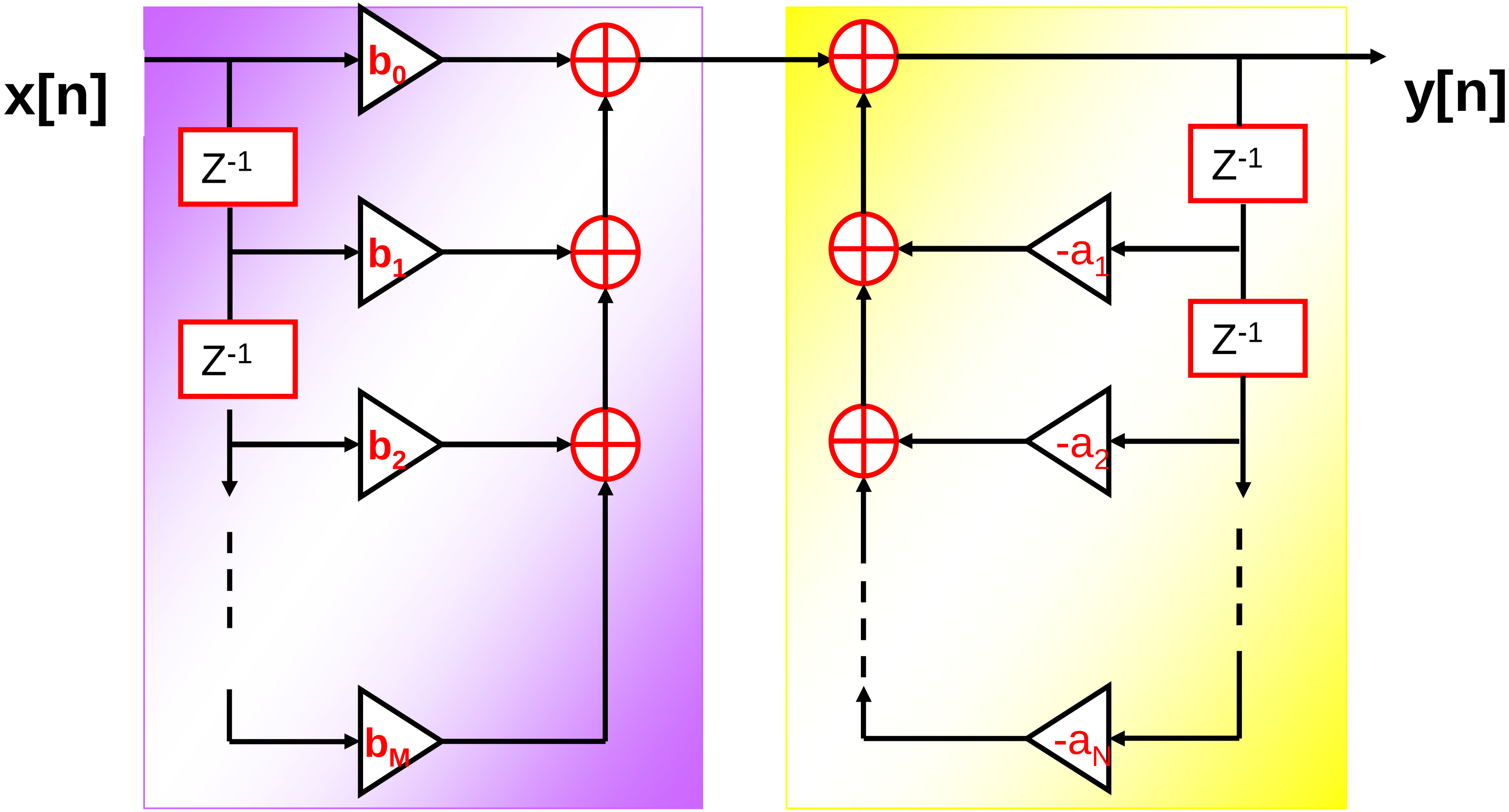
2. Direct form I realization of a system

3. Direct form II realization of a system

Direct form I realization of a system

$$y[n]+a_1 y[n-1]+\ldots+a_N y[n-N]=b_0 x[n]+b_1 x[n-1]+\ldots+b_M x[n-M]$$

$$\Leftrightarrow y[n]=\boxed{b_0 x[n]+b_1 x[n-1]+\ldots+b_M x[n-M]}+\boxed{(-a_1)y[n-1]+\ldots+(-a_N)y[n-N]}$$



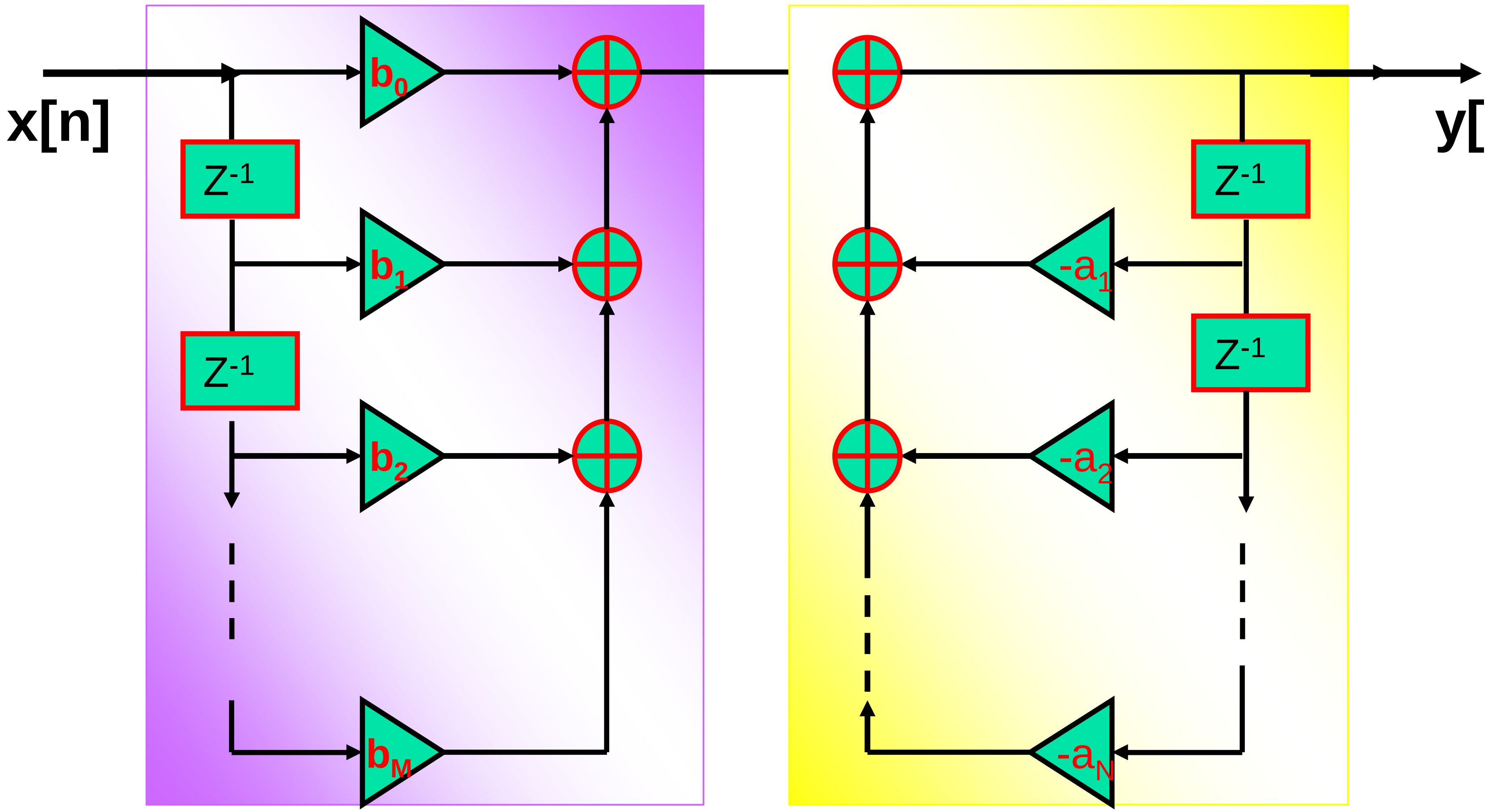
Lecture #8

Block diagram for DT LTI systems

1. Components for block diagram
2. Direct form I realization of a system
3. **Direct form II realization of a system**

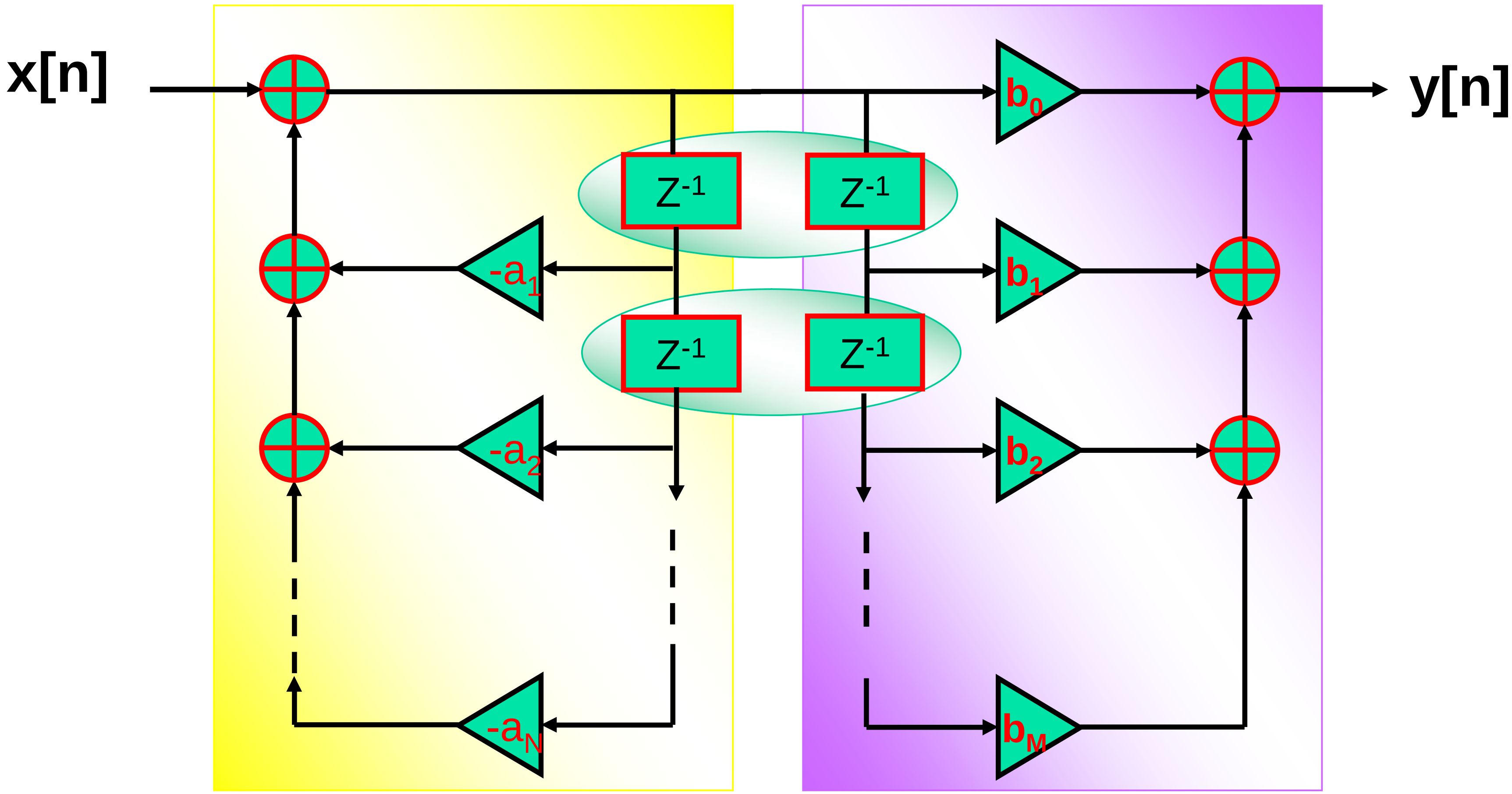
Direct form II realization of a system

To get another realization, we reverse the order of these two systems "violet" and "yellow" without altering the input-output relation

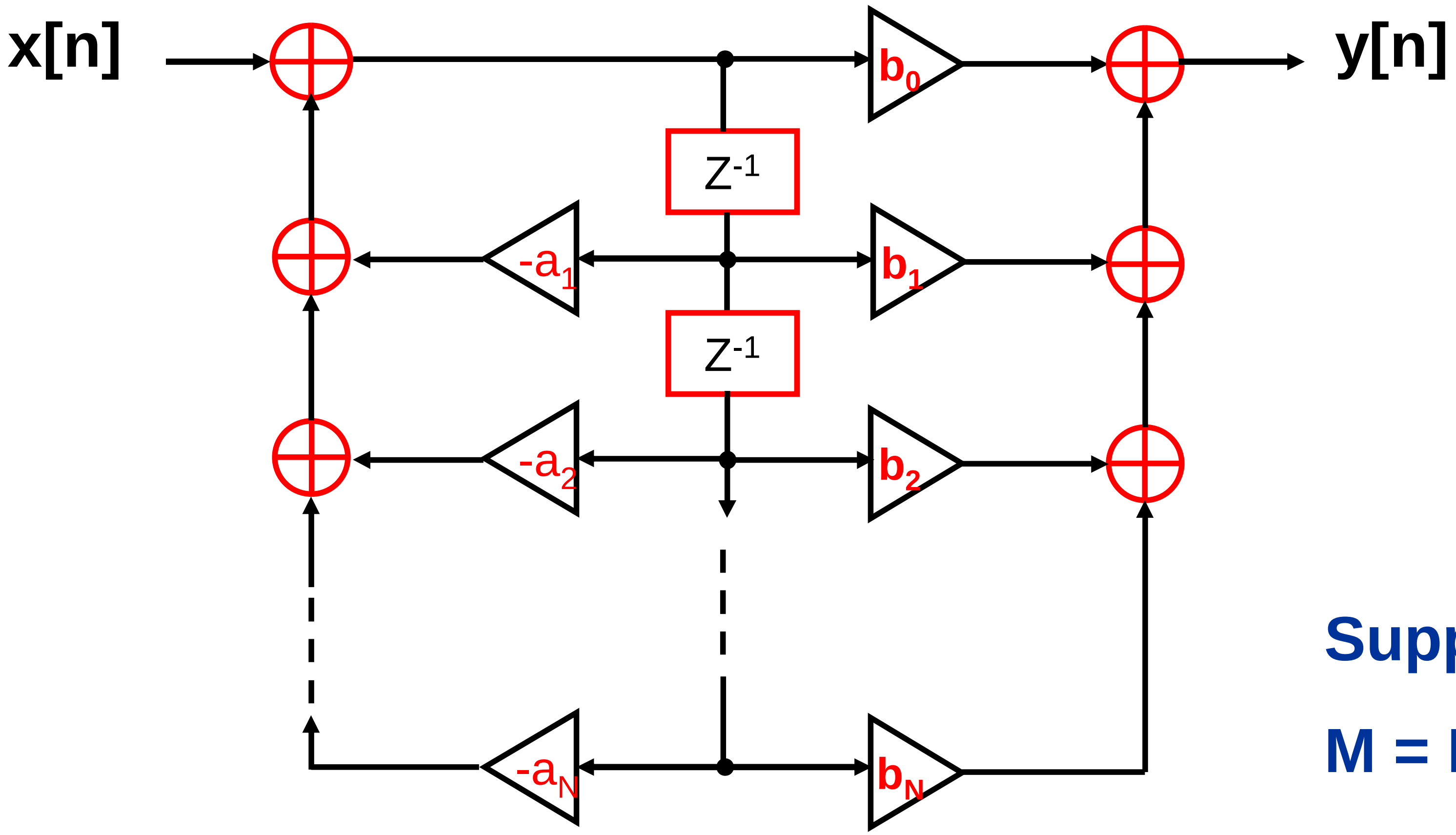


Direct form II realization of a system

We observe that: these two delays contain the same input and hence the same output → these two delays can be merged into one delay



Direct form II realization of a system



Suppose

M = N

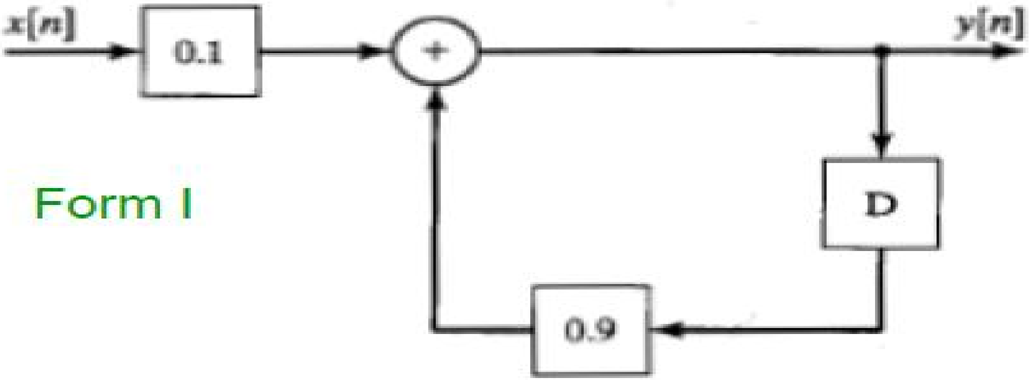
Example of realization LTI system

Example

$$y[n]-(1-\alpha)y[n-1]=\alpha x[n]$$

$\alpha = 0.1$

Form I



Form II