CHAPTER 3:

THE Z-TRANSFORM & ITS APPLICATION TO THE ANALYSIS OF LTI SYSTEMS

Lesson #9: The Z-transform

Lesson #10: Z-transform properties

Lesson #11: Inversion of Z-transform

Lesson #12: Analysis of LTI systems in the z-domain

Lecture #9

The Z-transform

1. **Definition of the Z-transform**
2. Region of convergence
3. Examples of Z-transform

Introduction to Z-Transform (ZT)

Recall the Laplace Transform:

$$\boxed{F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt}$$

Z-Transform is the discrete-time counterpart of the Laplace Transform:

$$\boxed{F(z) = \sum_{n=-\infty}^{\infty} f[n] z^{-n}}$$

s and z take values in the complex plane

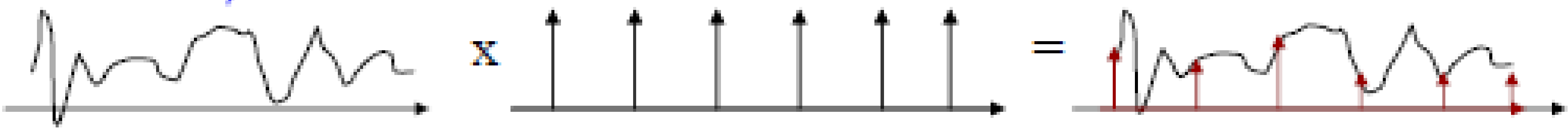
t and *n* are time variables

Infinite integral replaced by infinite sum

e^{-st} replaced by z^{-n}

From Laplace Transform to Z-Transform

Take a CT signal f(t) and sample it:



Sampling = multiplying by an infinite series of time-shifted impulses, and then summing

$$f(t) \xrightarrow{sampling} f_s(t)$$

$$f_s(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - nT) = \sum_{n=-\infty}^{\infty} f(nT)\delta(t - nT)$$

From Laplace Transform to Z-Transform

Take the Laplace Transform of the sampled signal:

$$L[f_s(t)] = \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} f(nT)\delta(t-nT) \right] e^{-st} dt = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(nT)\delta(t-nT)e^{-st} dt$$

$$= \sum_{n=-\infty}^{\infty} f(nT) \int_{-\infty}^{\infty} \delta(t-nT)e^{-st} dt = \sum_{n=-\infty}^{\infty} f(nT)e^{-snT}$$

Let ***f(nT) = f[n]*** and ***e^{sT} = z*** then:

$$\boxed{F(z) = \sum_{n=-\infty}^{\infty} f[n]z^{-n}}$$

Definition of the Z-transform

For a DT signal x[n] = …, x[-2], x[-1], x[0], x[1], x[2], …

The **bilateral Z-transform** of x[n] is defined to be

$$\boxed{X(z) = ZT\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}}$$

Sum goes over all integer values

Assume that values of z exist such that the summation converges. **z takes values in the complex plane**

Poles and Zeros of Z-transform

$$X(z) = ZT\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

Poles $p_k \leftarrow\rightarrow X(p_k) = \infty$

Zeros $z_k \leftarrow\rightarrow X(z_k) = 0$

Poles are roots of denominator polynomial

Zeros are roots of numerator polynomial

Note: find these *after canceling any common factors* – and do this for polynomial *in z (not in z^{-1})*

Lecture #9

The Z-transform

1. Definition of the Z-transform

2. Region of convergence

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Region of convergence (ROC)

ROC: set of all values of z, for which X(z) converges

We will find the ROC for these cases:

1. Right-sided signal ($x[n] = 0, n<n_0$)
2. Left-sided signal ($x[n] = 0, n>n_0$)
3. Two-sided signal ($-\infty<n<+\infty$)
4. Finite-duration signal

Right-sided signal

for right-sided $x[n]$

$$X(z)=\sum_{n=n_0}^{\infty} x[n]z^{-n}$$

$$X(z)=\sum_{n=n_0}^{\infty} x[n]\left(\frac{1}{z}\right)^n$$

Sum goes from n_0 to infinity

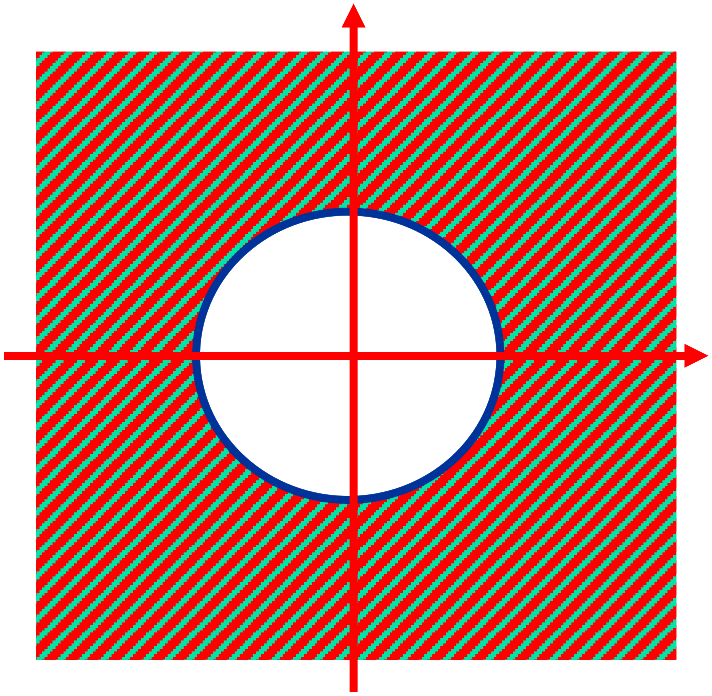
$\boxed{\text{As } n\to\infty\text{, need } (1/z)^n\to 0 \text{ for sum to converge.}}$

Happens for z OUTSIDE the poles $|z|>r_{max}$.

$$\boxed{|z| > r_{max}}$$

Right-sided signal

$$\boxed{|z| > r_{max}}$$



Infinite Egg White!!!

Right-sided signal

If x[n] is not causal then X(z) will not converge at z = ∞

→ we can't include ∞ in the ROC

Ex: x[n] = u[n+1]

$$X(z) = \sum_{n=-1}^{\infty} z^{-n} = z + \sum_{n=0}^{\infty} z^{-n}$$

ROC: $r_{max} < |z| < \infty$

Left-sided signal

for left-sided $x[n]$

$$X(z) = \sum_{n=-\infty}^{n_0} x[n] z^{-n}$$

Sum goes from minus infinity to n_o

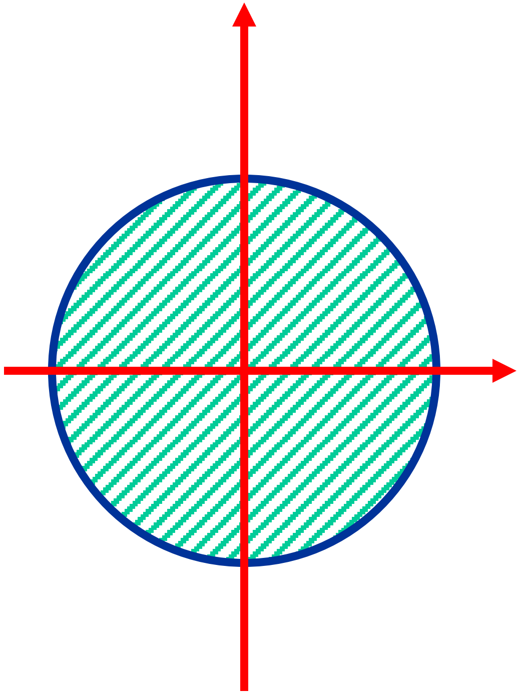
$$\boxed{\text{As } n \to -\infty \text{, need } (1/z)^n \to 0 \text{ or } z^\infty \to 0}$$

Happens for z INSIDE the poles (rather than outside)

$$\boxed{|z| < r_{min}}$$

Left-sided signal

$$\boxed{|z| < r_{min}}$$



Egg Yolk!!!

Left-sided signal

If x[n] has values at some times > 0, then X(z) does not converge at z = 0 → we can't include 0 in the ROC

Ex: x[n] = u[-n+1]

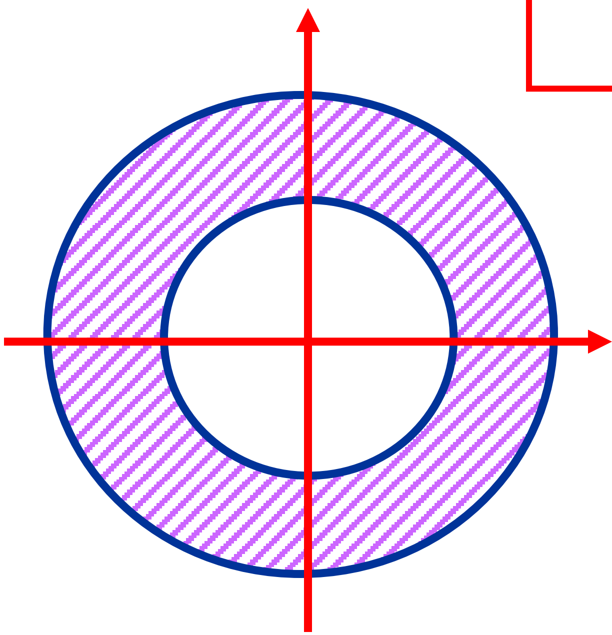
$$X(z) \equiv \sum_{n=-\infty}^{1} z^{-n} = z^{-1} + \sum_{n=0}^{\infty} z^{n}$$

ROC: $0 < |z| < r_{min}$

Two-sided signal

- Two-sided signal = Left-sided signal + right-sided signal

$$|a| < |z| < |b|$$



Donut!!!

Note: if $|a| \geq |b|$ then X(z) does not exist

Finite-duration signal

For $x(n) = \delta(n-m) \Rightarrow X(z) = \sum_{n=-\infty}^{\infty} \delta(n-m) z^{-n} = z^{-m}$

$$\boxed{x[n] = \sum_{k=n_1}^{n_2} x[k]\delta[n-k] \Rightarrow X(z) = \sum_{k=n_1}^{n_2} x[k] z^{-k}}$$

ROC: all of z, except:

$z = 0$ if $k > 0$

$z = \infty$ if $k < 0$

Lecture #9

The Z-transform

1. Definition of the Z-transform

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Examples of Z-transform

Find the ZT of the right-sided and the left-sided sequences

$$x_1[n] = a^n u[n] \quad \text{and} \quad x_2[n] = -(a^n)u[-n-1]$$

$$X_1(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \begin{cases} \dfrac{1}{1-az^{-1}} & |az^{-1}| < 1 \\ \infty & |az^{-1}| > 1 \end{cases}$$

$$= \frac{z}{z-a}, \quad |z| > |a|$$

Examples of Z-transform

- Find the ZT of the right-sided and the left-sided sequences

$$x_1[n] = a^n u[n] \quad \text{and} \quad x_2[n] = -(a^n)u[-n-1]$$

$$X_2(z) = -\sum_{n=-1}^{-\infty} a^n z^{-n} = -\sum_{n=1}^{\infty} (a^{-1}z)^n = \begin{cases} -\dfrac{a^{-1}z}{1-a^{-1}z} & |a^{-1}z| < 1 \\ -\infty & |a^{-1}z| > 1 \end{cases}$$

$$= \quad \frac{z}{z-a}, \quad |z| < |a|$$

The ROC must be specified for the bilateral Z-Transform to be unique

Examples of Z-transform

Find the ZT of the left-sided signal

$$x[n] = 3^n u[-n-1] + 4^n u[-n-1].$$

Examples of Z-transform

Find the ZT of the two-sided signal $x[n] = a^{|n|}$

Examples of Z-transform

Find the ZT of the two-sided signal

$$h[n] = (.5)^n u[n-1] + 3^n u[-n-1].$$

Examples of Z-transform

Find the ZT of the signal

$$x[n] = \frac{1}{2}\delta[n-1] + 3\delta[n+1]$$

Examples of Z-transform

- Find the ZT of the right-sided-sided signal

$$x[n] = r^n \sin(bn) u[n]$$

$$x(n) = \frac{1}{2j} r^n e^{jbn} u(n) - \frac{1}{2j} r^n e^{-jbn} u(n)$$

$$= \frac{1}{2j} (r\ e^{jb})^n u(n) - \frac{1}{2j} (re^{-jb})^n u(n)$$

$$\Rightarrow X(z) = \frac{1}{2j} \left(\frac{z}{z - r\ e^{jb}} - \frac{z}{z - r\ e^{-jb}} \right)$$

$$\boxed{r^n \sin(bn) u(n) \quad \overset{z}{\longleftrightarrow} \quad \frac{rz \sin b}{z^2 - 2rz \cos b + r^2}, \quad |z| > |r|}$$

Lecture #9

Z-transform properties

1. **Linearity**
2. Time shifting
3. Frequency scaling
4. Multiplication by *n*
5. Convolution in time
6. Initial value
7. Final value

Linearity

$$\boxed{ax[n]+by[n] \overset{z}{\longleftrightarrow} aX(z)+bY(z)}$$

The new ROC is the intersection of ROC{X(z)} and ROC{Y(z)}

If aX(z) + bY(z) cancels pole then the new ROC is bigger

Lecture #9

Z-transform properties

1. Linearity

2. Time shifting

3. Frequency scaling

4. Multiplication by *n*

5. Convolution in time

8. Initial value

9. Final value

Time shifting

$$x[n-n_0] \overset{Z}{\longleftrightarrow} z^{-n_0}X(z)$$

The new ROC is the same as ROC{X(z)} except for z = 0 if $n_0>0$ and z = ∞ if $n_0<0$

Proof:

$$Z\{x[n-n_o]\} = \sum_{n=-\infty}^{\infty} x[n-n_o]z^{-n}$$

$$let \quad l = n - n_o \qquad \sum_{l=-\infty}^{\infty} x[l]z^{-(l+n_o)} = z^{-n_o}\sum_{l=-\infty}^{\infty} x[l]z^{-l} = z^{-n_o}X(z)$$

Delay of k means that the Z-transform is multiplied by z^{-k}

Example of applying the time-shifting property

Determine the ZT of the signal:

$$w[n] = \frac{1}{4}\left\{(-1)^n + (3)^{n-5}\right\}u[n-5]$$

Lecture #9

Z-transform properties

1. Linearity
2. Time shifting
3. **Frequency scaling**
4. Multiplication by n
5. Convolution in time
6. Initial value
7. Final value

Frequency scaling

$$\boxed{a^n x[n] \overset{Z}{\leftrightarrow} X\left(\frac{z}{a}\right)}$$

The new ROC is the scaled ROC{X(z)} with factor */a/* (bigger or smaller)

Proof:

$$ZT\{a^n x[n]\} = \sum_{n=-\infty}^{\infty} a^n x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\left(\frac{z}{a}\right)^{-n} = X\left(\frac{z}{a}\right)$$

Multiplication by a^n results in a complex scaling in the z-domain

Example of applying the frequency-scaling property

Determine the ZT of the signal:

$$x[n] = a^n u[n]$$

Lecture #9

Z-transform properties

1. Linearity
2. Time shifting
3. Frequency scaling
4. **Multiplication by *n***
5. Convolution in time
6. Initial value
7. Final value

Multiplication by n

$$\boxed{nx[n] \overset{z}{\leftrightarrow} -z\frac{dX(z)}{dz}}$$

The new ROC is the same ROC{X(z)}

Proof:

$$X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n} \Rightarrow \frac{dX(z)}{dz}=-\sum_{n=-\infty}^{\infty}nx[n]z^{-n-1}=-\frac{1}{z}\sum_{n=-\infty}^{\infty}nx[n]z^{-n}$$

$$\Rightarrow ZT\{nx[n]\}=\sum_{n=-\infty}^{\infty}nx[n]z^{-n}=-z\frac{dX(z)}{dz}$$

Example of applying the multiplication-by-n property

Determine the ZT of the signal:

$$x[n] = na^n u[n]$$

Lecture #9

Z-transform properties

1. Linearity
2. Time shifting
3. Frequency scaling
4. Multiplication by n
5. **Convolution in time**
6. Initial value
7. Final value

Convolution in time

$$\boxed{y[n] = x[n] * h[n] \overset{Z}{\longleftrightarrow} X(z)H(z)}$$

The new ROC is the intersection of ROC{X(z)} and ROC{Y(z)}

If poles cancel zeros then the new ROC is bigger

Proof:

$$y[n] = x[n] * h[n] \overset{Z}{\longleftrightarrow} \sum_{n=-\infty}^{\infty} [\sum_{k=-\infty}^{\infty} x[k]h[n-k]]z^{-n}$$

Switching the order of the summation:

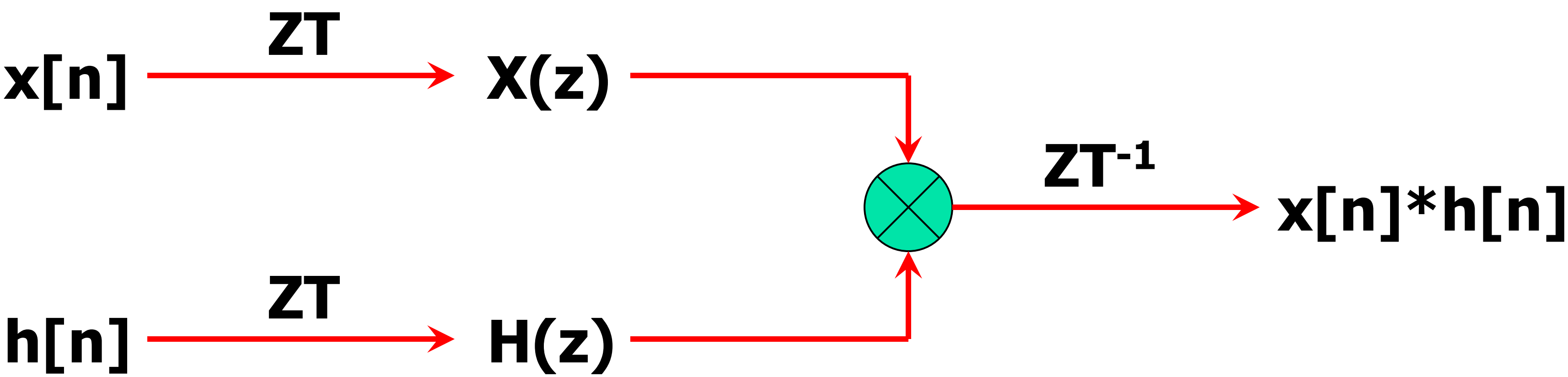
$$Y(z) = \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} h[n-k]z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x[k]z^{-k} \sum_{n-k=-\infty}^{\infty} h[n-k]z^{-(n-k)}$$

$$= X(z).H(z)$$

Application of the convolution property

Convolution in time domain ⟹ multiplication in Z domain



Example

Compute the convolution of the signals:

$x_1[n] = \delta[n] - 2\delta[n-1] + \delta[n-2]$

$x_2[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5]$

Lecture #9

Z-transform properties

1. Linearity
2. Time shifting
3. Frequency scaling
4. Multiplication by n
5. Convolution in time
6. **Initial value**
7. Final value

Initial value theorem

If x[n] is causal then x[0] is the initial value of the function x[n]

$$\boxed{x[0] = \lim_{z \to \infty} X(z)}$$

Proof:

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + \ldots$$

$$Obviously, \quad as\ z \to \infty, \quad z^{-n} \to 0$$

Initial value theorem

If $x[n] = 0$ with $n < n_0$ then $x[n_0]$ is the initial value, and

$$\boxed{x[n_0] = \lim_{z \to \infty} [z^{n_0} X(z)]}$$

Proof:

$$X(z) = \sum_{n=n_0}^{\infty} x[n] z^{-n} = x[n_0] z^{-n_0} + x[n_0+1] z^{-(n_0+1)} + x[n_0+2] z^{-(n_0+2)} + \ldots$$

$$z^{n_0} X(z) = x[n_0] + x[n_0+1] z^{-1} + x[n_0+2] z^{-2} + \ldots$$

$$As\ z \to \infty, \quad z^{n_0} X(z) = x[n_0]$$

Initial value theorem

Ex 1: Find the initial value of $x[n]$ where $X(z) = z/(z-0.6)$

Ex 2: Find the initial value of $x[n]$ where $X(z) = (z^3)/(z-0.6)$

Lecture #9

Z-transform properties

1. Linearity
2. Time shifting
3. Frequency scaling
4. Multiplication by n
5. Convolution in time
6. Initial value
7. **Final value**

Final value theorem

If this limit exists then x[n] has a final value (steady-state value)

$$\boxed{\lim_{n\to\infty} x[n] = x[\infty] = \lim_{z\to 1}[(z-1)X(z)]}$$

Proof: Exercise

Examples

Ex1: unit step *u[n]*

$$ZT\{u[n]\} = \frac{z}{z-1}$$

$\Rightarrow \lim_{z \to 1}(z-1).\frac{z}{z-1} = 1$ => Ok !

Ex2: sinusoidal signal

$$ZT\left\{sin\left[\frac{\pi n}{2}\right]u[n]\right\} = \cdots$$

Examples

Ex3: the impulse response of an LTI system is $h[n] = a^n u[n]$, $|a|<1$.

Determine the value of the step response of the system as $n \to \infty$

The step response of the system is:

$y[n] = h[n]*u[n] \to Y(z) = X(z).H(z)$

Lecture #10

Inversion of Z-transform

1. Inverse Z-transform formula

2. Using partial fraction expansion to invert the ZT

3. Using power series expansion to invert ZT

FT formula building

Using the Cauchy theorem:

$$\boxed{I = \frac{1}{2\pi j}\oint_C z^{n-1}dz = \begin{cases} 1 & if \;\; z = 0 \\ 0 & if \;\; z \neq 0 \end{cases}}$$

Z: complex variable

Counterclockwise contour integral is along a closed path in the z plane

FT formula building

Multiplying two sides of ZT definition formula by $\frac{1}{2\pi j} z^{l-1}$

$$\frac{1}{2\pi j} X(z) z^{l-1} = \frac{1}{2\pi j} \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} z^{l-1}$$

Taking the integral:

$$\frac{1}{2\pi j} \oint_C X(z) z^{l-1} dz = \frac{1}{2\pi j} \oint_C \left(\sum_{n=-\infty}^{\infty} x[n]\, z^{-n} z^{l-1} \right) dz$$

Applying the Cauchy theorem

$$= \sum_{n=-\infty}^{\infty} x[n] \frac{1}{2\pi j} \oint_C z^{(l-n)-1} dz$$

$$= x[l]$$

Inverse ZT formula

The inverse ZT is defined by

$$\boxed{x[n] = \frac{1}{2\pi j}\oint_{C} X(z) z^{n-1} dz}$$

Counterclockwise contour integral is along a closed path in the z plane.

LTI case: can avoid integration – use easier methods

Lecture #10

Inversion of Z-transform

1. Inverse Z-transform formula

2. Using partial fraction expansion to invert the ZT

3. Using power series expansion to invert ZT

Partial fraction expansion

Similar to what you saw for Laplace transforms

$$Y(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}=\frac{N(z)}{D(z)}=\sum_{k=1}^{N}\frac{r_k z}{z-p_k}$$

$$p_k=\text{pole} \quad r_k=\text{residue}$$

Distinct poles: $$r_k=\frac{Y(z)}{z}(z-p_k)\Big|_{z=p_k}$$

Partial fraction expansion

Repeated poles:

$$F(z) = \frac{N(z)}{(z-p_1)^r} = \frac{k_1}{z-p_1} + \frac{k_2}{(z-p_1)^2} + \cdots + \frac{k_r}{(z-p_1)^r}$$

$$k_r = F(z)(z-p_1)^r \Big|_{z=p_1}$$

$$k_i = \frac{1}{(r-i)!} \frac{d^{r-i}}{dz^{r-i}} [F(z)(z-p_1)^r] \Bigg|_{z=p_1}$$

$$i = 1, \ldots, r-1$$

Z-Transform table

1 $\delta(n) \leftrightarrow 1$

2 $\delta(n-m) \leftrightarrow z^{-m}$

3 $a^{n}u[n] \leftrightarrow \dfrac{z}{z-a}$

4 $na^{n}u[n] \leftrightarrow \dfrac{az}{(z-a)^{2}}$

5 $n^{2}a^{n}u[n] \leftrightarrow \dfrac{az(z+a)}{(z-a)^{3}}$

6 $a^{n}\cos(\Omega n)u[n] \leftrightarrow \dfrac{z(z-a\cos\Omega)}{z^{2}-2az\cos\Omega+a^{2}}$

7 $a^{n}\sin(\Omega n)u[n] \leftrightarrow \dfrac{az\sin\Omega}{z^{2}-2az\cos\Omega+a^{2}}$

Examples of partial fraction expansion

- Distinct poles
  - $X(z) = \dfrac{2z^2 - 5z}{(z-2)(z-3)}, |z| > 3$

Divide X(z) by z, to "save" z for later

$$\frac{X(z)}{z} = \frac{2z-5}{(z-2)(z-3)} = \frac{(z-2)+(z-3)}{(z-2)(z-3)}$$

$$= \frac{1}{z-3} + \frac{1}{z-2}, |z| > 3$$

$$\Rightarrow x(n) = (3^n + 2^n)u(n)$$

Examples (cont)

Repeated poles

$$X(z) = \frac{2z}{(z-2)(z-1)^2}, \quad |z| > 2$$

$$\frac{X(z)}{2z} = \frac{1}{(z-2)(z-1)^2} = \frac{A}{z-2} + \frac{B}{z-1} + \frac{C}{(z-1)^2}, |z| > 2$$

$$A(z-1)^2 + B(z-1)(z-2) + C(z-2) = 1$$

$$A = 1; B = -1; C = -1;$$

$$X(z) = 2\left[\frac{z}{z-2} - \frac{z}{z-1} - \frac{z}{(z-1)^2}\right], |z| > 2$$

$$\boxed{x(n) = 2(2^n - 1 - n)u(n)}$$

Examples (cont)

Complex poles

$$X(z) = \frac{z}{z^2 - 0.5z + 0.25} \quad |z| > 0.5$$

$$a^n \sin(\Omega n)u[n] \leftrightarrow \frac{az\sin\Omega}{z^2 - 2az\cos\Omega + a^2}$$

$$X(z) = \frac{4}{\sqrt{3}} \text{ x } \frac{(0.5)z\sin\frac{\pi}{3}}{z^2 - 2(0.5)z\cos\frac{\pi}{3} + (0.5)^2} \, |z| > 0.5$$

$$\boxed{x(n) = \frac{4}{\sqrt{3}}(0.5)^n \sin\left(\frac{\pi}{3}n\right)u(n)}$$

Examples (cont)

Time-shift property

$$W(z) = \frac{z^{-4}}{z^2 - 2z - 3}, |z| > 3$$

Divide W(z) by z, to "save" z for later

$$\frac{W(z)}{z} = \frac{z^{-5}}{z^2 - 2z - 3} = \frac{z^{-5}}{(z+1)(z-3)} = \left(\frac{-\frac{1}{4}}{z+1} + \frac{\frac{1}{4}}{z-3} \right) z^{-5}$$

$$w[n] = -\frac{1}{4}(-1)^{n-5} u[n-5] + \frac{1}{4}(3)^{n-5} u[n-5]$$

Examples (cont)

- Given $h(n) = a^n u(n)$ ($|a|<1$) and $x(n) = u(n)$.

  Find $y(n) = x(n)*h(n)$

$$\boxed{y[n] = \frac{1-a^{n+1}}{1-a} u[n]}$$

Examples (cont)

Find the output y(n) to an input x(n) = u(n) and an LTI system with impulse response h(n) = $-3^n u(-n-1)$

$$\boxed{y[n] = -\frac{1}{2}u[n] - \frac{3}{2}(3)^n u[-n-1]}$$

Lecture #10

Inversion of Z-transform

1. Inverse Z-transform formula

2. Using partial fraction expansion to invert the ZT

3. Using power series expansion to invert ZT

Power series expansion

If you can expand $X(z)$ as a series in z^{-1}

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \ldots + x[-2]z^{2} + x[-1]z^{1} + x[0]z^{0} + x[1]z^{-1} + x[2]z^{-2} + \ldots$$

$$X(z) = \ldots + a_{-2}z^{2} + a_{-1}z^{1} + a_{0}z^{0} + a_{1}z^{-1} + a_{2}z^{-2} + \ldots$$

you can pick off $x(n)$ as the coefficients of the series

$$\boxed{a_n \equiv x[n]}$$

Examples of power series expansion

Find the inverse Z-transform of

$$X(z) = 1 + 2z^{-1} + 3z^{-2} \quad \mathrm{ROC:} \quad |z| > 0$$

$$x(0) = 1$$

$$x(1) = 2$$

$$x(2) = 3$$

$$x(n)(n \neq 0,1,2) = 0$$

Examples (cont)

$$X(z)=\frac{1}{1-az^{-1}}, \quad ROC: |z|>|a|$$

Long division

$$X(z)=1+az^{-1}+a^2z^{-2}+a^3z^{-3}+\ldots$$

$$x(0)=1$$

$$x(1)=a$$

$$x(2)=a^2$$

$$x(3)=a^3$$

$$\ldots$$

$$x(n)(n<0)=0$$

Lecture #11

Analysis of LTI systems in the Z-domain

1. Transfer function

2. LTI system properties from transfer function

3. Unilateral Z-transform

4. Using unilateral Z-Transform to solve the difference equations

Transfer function

For system impulse response **h(n),** its ZT is often called **Transfer Function H(z)**

Consider **H(z) = N(z)/D(z)**

The roots of N(z): system zeros

The roots of D(z): system poles

D(z) = 0: characteristic equation

Determination of transfer function

1. From impulse response h(n): Just take Z-Transform

2. From difference equation:

- Take Z-Transform for both side
- Put the Y(z) on left side
- Divide both side by X(z)

3. From block diagram:

(1) Find the difference equation, then find H(z) from equation

(2) Put X(z) as input, then directly find Y(z) from block diagram

Transfer function from difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{r=0}^{M} b_r x[n-r]$$

ZT

$$\Leftrightarrow \quad \sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{r=0}^{M} b_r z^{-r} X(z)$$

-Linear

-Time-shift

Y(z) = X(z) . H(z) $\Rightarrow$

$$\boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{r=0}^{M} b_r z^{-r}}{\sum_{k=0}^{N} a_k z^{-k}}}$$

Transfer function from difference equation

Suppose M = N

Multiply num. and den. by z^N:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{r=0}^{M} b_r z^{-r}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{\sum_{r=0}^{N} b_r z^{r}}{\sum_{k=0}^{N} a_k z^{k}}$$

Example

For the α-filter:

$$y(n)-(1-\alpha)y(n-1)=\alpha\ x(n)$$

Its transfer function:

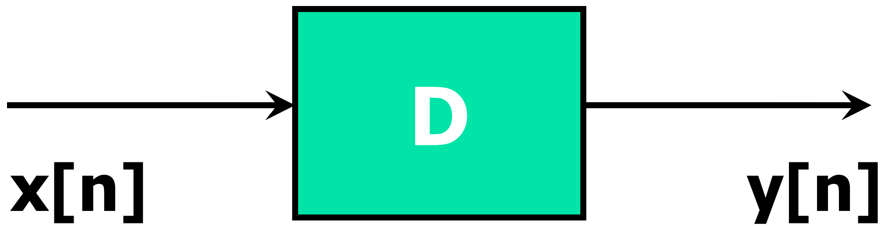
$$H(z)=\frac{\sum_{r=0}^{M} b_r z^{-r}}{\sum_{k=0}^{N} a_k z^{-k}}=\frac{\alpha}{1-(1-\alpha)z^{-1}}$$

For example: *y[n] – 0.9y[n-1] = 0.1x[n]* → *α = 0.1*

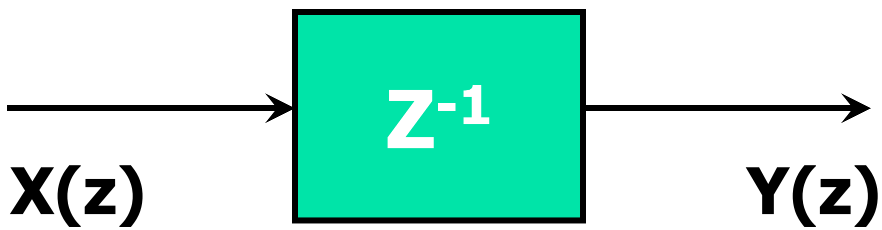
$$H(z)=\frac{0.1z}{z-0.9}$$

Transfer function from block diagram

Find the transfer function of the delay unit

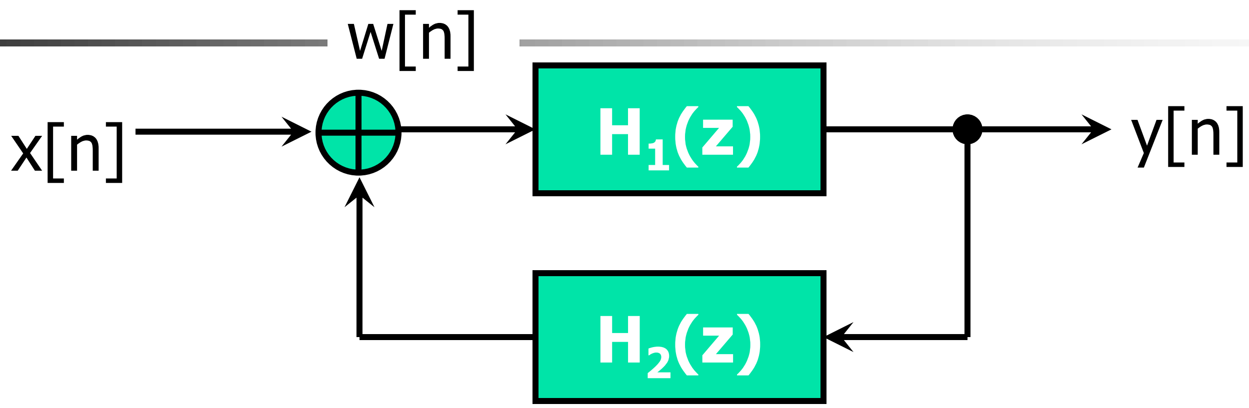


$$y[n] = x[n-1] \rightarrow Y(z) = z^{-1}X(z) \rightarrow H(z) = z^{-1}$$



Transfer function from block diagram

Find the transfer function of this feedback system



$$y[n] = (\, x[n]+y[n]*h_2[n] \,) * h_1[n]$$

$$\rightarrow Y(z) = [X(z)+Y(z).H_2(z)].H_1(z)$$

$$\boxed{H(z)=\frac{Y(z)}{X(z)}=\frac{H_1(z)}{1-H_1(z).H_2(z)}}$$

Lecture #11

Analysis of LTI systems in the Z-domain

1. Transfer function

2. LTI system properties from transfer function

3. Unilateral Z-transform

4. Using unilateral Z-Transform to solve the difference equations

Causality

- **Recall:**

Causal system ←→ $\boxed{h[n] = 0 \quad \forall n < 0}$

- **h(n)** is right-sided signal ←→ ROC of transfer function is

$$\boxed{|z| > r_{max}}$$

- An LTI system is causal if and only if the ROC of the transfer function is the exterior of a circle of radius $r_{max} < \infty$ including the point $z = \infty$

Causality (cont)

- Consider the system:

$$H(z) = \frac{z^2 + 0.4z + 0.9}{z - 0.6} = z + \frac{z + 0.9}{z - 0.6}$$

Unit advance y[n]=x[n+1]
→ noncausal

- For the causal system, the numerator of H(z) **can not be of higher order** than the denominator
- If the numerator order is less equal to the denominator order then the system is causal???

Example

- Consider this system:

$$h[n] = -u[-n-1]$$

$$H(z) = \frac{z}{z-1} \quad ROC: |z| < 1$$

- Numerator order is *1*; denominator order is *1*; but it is non-causal!!!

Stability

- **Recall:**

Stable system $\leftrightarrow$ $\boxed{\sum_{n=-\infty}^{+\infty} |h[n]| \ < \ \infty}$

- **Its transfer function:**

$$\mathrm{H(z)} = \sum_{\mathrm{n}=-\infty}^{\infty} \mathrm{h[n]z^{-n}} \Rightarrow |\mathrm{H(z)}| \le \sum_{\mathrm{n}=-\infty}^{\infty} |\mathrm{h[n]z^{-n}}| = \sum_{\mathrm{n}=-\infty}^{\infty} |\mathrm{h[n]}||\mathrm{z^{-n}}|$$

Unit circle |z| = 1 → $|\mathrm{H(z)}| \le \sum_{\mathrm{n}=-\infty}^{\infty} |\mathrm{h[n]}|$ The ROC includes |z| = 1

- An LTI system is BIBO stable if and only if the ROC of the transfer function includes the unit circle.

Causality and Stability

- The conditions for causality and stability are different and one does not imply the other
- **4 cases:**

- Causal and stable system
- Non-causal and stable system
- Causal and unstable system
- Non-causal and unstable system

- A causal system is BIBO stable, provided that all poles of H(z) lie inside the unit circle.

Examples

Ex1: Given an LTI system:

$$H(z) = \frac{2z^2 - 1.6z - 0.9}{z^3 - 2.5z^2 + 1.96z - 0.48}$$

The poles of H(z):

den = [1 -2.5 1.96 -0.48];

p = roots(den)

→ **p = 1.2 0.8 0.5**

1. |z|>1.2: causal, unstable

2. 0.8<|z|<1.2: non-causal, stable

3. 0.5≠|z|<0.8: non-causal, unstable

Examples

Ex2: A LTI system is characterized by:

$$H(z) = \frac{3-4z^{-1}}{1-3.5z^{-1}+1.5z^{-2}} = \frac{z(3z-4)}{z^2-3.5z+1.5} = \frac{z}{z-0.5} + \frac{2z}{z-3}$$

Specify the ROC of H(z) and find h[n] for the following conditions:

1. The system is stable
2. The system is causal
3. The system is anticausal

Examples

Ex2: A LTI system is characterized by:

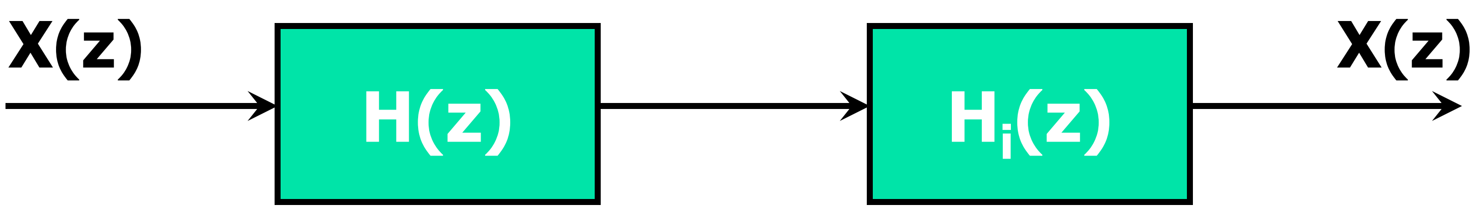
$$H(z) = \frac{3-4z^{-1}}{1-3.5z^{-1}+1.5z^{-2}} = \frac{z(3z-4)}{z^2-3.5z+1.5} = \frac{z}{z-0.5} + \frac{2z}{z-3}$$

Invertibility

- The inverse of a system H(z) is a second system $H_i(z)$ that, when cascaded with H(z), yields the identity system

$$H(z).H_i(z) = 1$$

$$H_i(z) = 1/H(z)$$



Invertibility

$H(z) = N(z)/D(z) \rightarrow H_i(z) = D(z)/N(z)$

H(z) is causal → $M \le N$ and if $H_i(z)$ is causal → $N \le M$.

Hence, for both systems to be causal → **$M = N$**

H(z) is stable: all poles lie inside the unit circle.

$H_i(z)$ is stable: all poles (zeros of H(z)) lie inside the unit circle.

Hence, for both systems to be stable, **both the poles and zeros of H(z) must lie inside the unit circle**

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Unilateral Z-Transform

- In control applications, causal systems are interested in:

$$h[n] = 0 \text{ with all } n < 0$$

→ When considering CT causal systems, the Unilateral Laplace Transform is used

Similarly, when considering DT causal systems, the Unilateral Z-transform is used

- In difference equations, the initial conditions are non-zero

→ the Unilateral Z-Transform is used

Unilateral ZT

$$X^+(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

$$Notation: \quad x[n] \overset{Z^+}{\leftrightarrow} X^+(z)$$

1. It does not contain information about the signal *x[n]* for *n<0*

2. Unilateral $X^+(z)$ is identical to the bilateral *X(z)* of the signal *x[n].u[n]*

3. ROC of $X^+(z)$ is always outside a circle → no need to refer to ROC

Time shifting property of unilateral ZT

$$If \quad x[n] \overset{z^+}{\leftrightarrow} X^+(z)$$

$$then \quad x(n-k) \overset{z^+}{\leftrightarrow} z^{-k}X^+(z) + z^{-k}\sum_{i=-k}^{-1} x[i]z^{-i}, k > 0$$

Ex.: $x(n-1) \leftarrow\rightarrow x(-1) + z^{-1}X^+(z)$

$x(n-2) \leftarrow\rightarrow x(-2) + z^{-1}x(-1) + z^{-2}X^+(z)$

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Using Z-transform to solve the difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{r=0}^{M} b_r x[n-r]$$

1. Take the unilateral ZT of equation

2. Find the output in the Z-domain, Y(z)

3. Use inverse ZT to get the output y(n) from Y(z)

Example

Find the output, $y[n], \quad n \geq 0$

of the system

$$y[n] = 3y[n-1] - 2y[n-2] + x[n]$$

in two cases:

a)

$$x[n] = 3^{n-2}u[n], \quad y[-2] = 0, \quad y[-1] = 0$$

b)

$$x[n] = 3^{n-2}u[n], \quad y[-2] = -\frac{4}{9}, \quad y[-1] = -\frac{1}{3}$$

Example

- a) Take the bilateral ZT of the equation:

$$y[n] = 3y[n-1] - 2y[n-2] + x[n]$$

Example

- b) Take the unilateral ZT of the equation:

$$y[n] = 3y[n-1] - 2y[n-2] + x[n]$$

Homework 3

E1. Find x(n)*h(n) using the Z-transform

$$a)\quad x[n]=\left(\tfrac{1}{4}\right)^n u[n-1],\quad h[n]=[1+(\tfrac{1}{2})^n]u[n]$$

$$b)\quad x[n]=u[n],\quad h[n]=\delta[n]+(\tfrac{1}{2})^n u[n]$$

$$c)\quad x[n]=nu[n],\quad h[n]=2^n u[n-1]$$

Homework 3

E2

We want to design a causal LTI system with the property that if the input is $\mathbf{x[n] = (0.5)^n u[n] - 0.25(0.5)^{n-1} u[n-1]}$

then the output is $\mathbf{y[n] = (1/3)^n u[n]}$

a) Find H(z) and then h(n) of a system that satisfies the foregoing conditions

b) Find the difference equation that characterizes this system

c) Determine a realization of the system that requires the minimum possible amount of memory

d) Determine if the system is stable

Homework 3

E3. Use the Z-transform to evaluate the following series:

$a)\, x = \sum_{n=0}^{\infty} 0.2^{n}$

$b)\, x = \sum_{n=4}^{\infty} 0.2^{n}$

$c)\, x = \sum_{n=0}^{\infty} 0.2^{n} \cos(0.2\mathrm{n})$

Homework 3

E4. A causal linear, time-invariant system is described by this difference equation:

$$y[n] = x[n] + y[n-1]$$

a) Find the system impulse response

b) Find y[n] for an input $x[n] = \left(\frac{1}{2}\right)^n u[n]$. Label the natural and forced responses in y[n]

c) Is the system BIBO stable ?

Homework 3

E5. Find the inverse of the bilateral Z-transform:

$$F(z) = \frac{1.5z^2 - 1.3}{(z-1)(z-9)}$$

for the following regions of convergence.

a) $|z| < 0.9$

b) $|z| > 1$

c) $0.9 < |z| < 1$

d) Find the final values of the functions of parts (a) through (c)