A Book of Abstract Algebra | (2nd Edition)

Chapter 17, Problem 1EG

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Problem

If A and B are rings, their *direct product* is a new ring, denoted by $A \times B$, and defined as follows: $A \times B$ consists of all the ordered pairs (x, y) where x is in A and y is in B. Addition in $A \times B$ consists of adding corresponding components:

$(x_1, y_1) + (x_2, y_2) = (x_1+x_2, y_1+y_2)$

Multiplication in $A \times B$ consists of multiplying corresponding components:

$$(x_1, y_1) + (x_2, y_2) = (x_1+x_2, y_1+y_2)$$

If A and B are rings, verify that $A \times B$ is a ring.

Step-by-step solution

Step 1 of 5

Consider the direct product $A \times B$ of two rings A and B with the following addition and multiplication:

$$(x_1, y_1)+(x_2, y_2)=(x_1+x_2, y_1+y_2),$$
$$(x_1, y_1)\cdot(x_2, y_2)=(x_1x_2, y_1y_2),$$

where $x_1, x_2 \in A,\ y_1, y_2 \in B$.

Objective is to show that $A \times B$ satisfies all the axioms to be a ring.

Comment

Step 2 of 5

First show that $((A\times B), +)$ is an abelian group.

(1) The $A \times B$ is closed under addition because $x_1 + x_2 \in A,\ y_1 + y_2 \in B$ so:

$$(x_1, y_1)+(x_2, y_2)=(x_1+x_2, y_1+y_2)\in A\times B.$$

(2) Associative: Let $(a, b), (c, d), (e, f) \in A \times B$. Then

$$\begin{aligned}((a,b)+(c,d))+(e,f)&=(a,b)+((c,d)+(e,f))\\ (a+c,b+d)+(e,f)&=(a,b)+(c+e,d+f)\\ (a+c+e,b+d+f)&=(a+c+e,b+d+f).\end{aligned}$$

Since both the sides are equals, so addition is associative in $A \times B$.

(3) The identity on addition is $(0, 0)$ where first 0 is the identity of ring *A* and second 0 is the identity of ring *B*:

$$\begin{aligned}(a,b)+(0,0)&=(a+0,b+0)\\ &=(a,b)\\ (0,0)+(a,b)&=(a,b).\end{aligned}$$

(4) For every $(a, b) \in A \times B$, the negative of it will be $(-a, -b)$ where $-a \in A$ is the negative of *a* and $-b \in B$ is negative of *b*:

$$\begin{aligned}(a,b)+(-a,-b)&=(a-a,b-b)\\ &=(0,0)\\ (-a,-b)+(a,b)&=(0,0).\end{aligned}$$

(5) Also,

$$\begin{aligned}(a,b)+(c,d)&=(a+c,b+d)\\ &=(c+a,d+b)\\ &=(c,d)+(a,b)\end{aligned}$$

So, addition is commutative.

And from here it conclude that, $A \times B$ is an abelian group.

Comment

Step 3 of 5

Now, show that product is associative. So,

$$\begin{aligned}((a,b)\cdot(c,d))\cdot(e,f)&=(a,b)\cdot((c,d)\cdot(e,f))\\ (ac,bd)\cdot(e,f)&=(a,b)\cdot(ce,df)\\ (ace,bdf)&=(ace,bdf).\end{aligned}$$

Since both the sides are equals, so multiplication is associative in $A \times B$.

Comment

Step 4 of 5

Next is distributive law:

$$
\begin{aligned}
(a,b)\cdot((c,d)+(e,f)) &= (a,b)\cdot(c+e,d+f) \\
&= (ac+ae, bd+bf) \\
(a,b)\cdot(c,d)+(a,b)\cdot(e,f) &= (ac,bd)+(ae,bf) \\
&= (ac+ae, bd+bf).
\end{aligned}
$$

Thus, $(a,b)\cdot((c,d)+(p,q)) = ((a,b)\cdot(c,d)) + ((a,b)\cdot(p,q))$. Similarly, $((c,d)+(p,q))\cdot(a,b) = ((c,d)\cdot(a,b)) + ((p,q)\cdot(a,b))$.

Comment

Step 5 of 5

Hence, $(A\times B, +, \cdot)$ satisfies all the axioms to be a ring.

Comment