A Book of Abstract Algebra | (2nd Edition)

Chapter 17, Problem 2EC

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Problem

Let $\mathcal{M}_2(\mathbb{R})$ designate the set of all 2 × 2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

whose entries are real numbers *a*, *b*, *c*, and *d*, with the following addition and multiplication:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} a+r & b+s \\ c+t & d+u \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} ar+bt & as+bu \\ cr+dt & cs+du \end{pmatrix}$$

Show that $\mathcal{M}_2(\mathbb{R})$ is not commutative and has a unity.

Step-by-step solution

Step 1 of 3

Consider that $M_2(R)$ is the set of all 2×2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a,b,c,d\in R$ (real number), with the following addition and multiplication:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}+\begin{pmatrix} r & s \\ t & u \end{pmatrix}=\begin{pmatrix} a+r & b+s \\ c+t & d+u \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} r & s \\ t & u \end{pmatrix}=\begin{pmatrix} ar+bt & as+bu \\ cr+dt & cs+du \end{pmatrix}.$$

Objective is to show that $M_2(R)$ is not commutative but contain a unity.

Comment

Step 2 of 3

The $M_2(R)$ will be commutative if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\begin{pmatrix} r & s \\ t & u \end{pmatrix}=\begin{pmatrix} r & s \\ t & u \end{pmatrix}\cdot\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Consider the following example: let $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}\in M_2(R)$. Then

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}=\begin{pmatrix} 4 & 7 \\ 2 & 3 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}\cdot\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ 2 & 7 \end{pmatrix}$$

Since $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}\neq\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}\cdot\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Therefore, $M_2(R)$ is not commutative.

Comment

Step 3 of 3

Consider the 2×2 matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\in M_2(R)$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Thus, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ will be the unity of $M_2(R)$.

Comment