

A Book of Abstract Algebra | (2nd Edition)

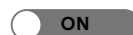


Chapter 29, Problem 5EA



Bookmark

Show all steps:



Problem

Find a basis of $\mathbb{Q}(\sqrt{5}, \sqrt{7})$ over \mathbb{Q} and describe the elements of $\mathbb{Q}(\sqrt{5}, \sqrt{7})$. (See the example at the end of this chapter.)

Step-by-step solution

Step 1 of 2

The objective is to find a basis of $(\sqrt{5}, \sqrt{7})$ over \mathbb{Q} and describe the elements of $\mathbb{Q}(\sqrt{5}, \sqrt{7})$.

[Comment](#)

Step 2 of 2

The minimal polynomial of $\sqrt{5}$ over \mathbb{Q} is $x^2 - 5$ (It is monic and irreducible (5 – Eisenstein) with $\sqrt{5}$ as a root.)

Hence $\left[\left[\sqrt{5} \right] : \right] = 2$ and a basis is $\{1, \sqrt{5}\}$.

Claim: $\sqrt{7} \notin \left(\sqrt{5} \right)$.

Suppose on the contrary that $\sqrt{7} \in \left(\sqrt{5} \right)$.

Therefore there exists $a, b \in \mathbb{Q}$ with $\sqrt{7} = a + b\sqrt{5}$.

Squaring gives $49 = a^2 + 2ab\sqrt{5} + 5b^2$ from which it follows that $\sqrt{5} = \frac{49 - a^2 - 5b^2}{2ab} \in \mathbb{Q}$,

a contradiction.

Hence $x^2 - 7$ is irreducible over $\left(\sqrt{5} \right)$; it is the minimal polynomial over $\left(\sqrt{5} \right)$.

So $\left[\left(\sqrt{5}, \sqrt{7} \right) : \left(\sqrt{5} \right) \right] = 2$ and that $\{1, \sqrt{7}\}$ is a basis for $\left(\sqrt{5}, \sqrt{7} \right)$ over $\left(\sqrt{5} \right)$.

$$\begin{aligned} \left[\left(\sqrt{5}, \sqrt{7} \right) : \right] &= \left[\left(\sqrt{5}, \sqrt{7} \right) : \left(\sqrt{5} \right) \right] \left[\left(\sqrt{5} \right) : \right] \\ &= 2 \cdot 2 \\ &= 4 \end{aligned}$$

The extension has basis $\{1, \sqrt{5}, \sqrt{7}, \sqrt{35}\}$ over \mathbb{Q} .

Any element of $\left(\sqrt{5}, \sqrt{7} \right)$ is of the form :

$$a + b\sqrt{5} + c\sqrt{7} + d\sqrt{35} : a, b, c, d \in \mathbb{Q}.$$

[Comment](#)

