A Book of Abstract Algebra | (2nd Edition)

Problem

Let F be a field, and let $a(x)$, $b(x) \in F[x]$. Prove the following:

If $a(x)$ and $b(x)$ are relatively prime in $F[x]$, they are relatively prime in $K[x]$, for any extension K of F. Conversely, if they are relatively prime in $K[x]$, then they are relatively prime in $F[x]$.

Step-by-step solution

Step 1 of 3

Consider that F is any field, and K is some extension field of F. Let $a(x), b(x) \in F[x]$.

Objective is to prove that $a(x)$ and $b(x)$ are relatively prime in $F[x]$ if and only they are relatively prime in $K[x]$.

Let $a(x)$ and $b(x)$ are relatively prime in $F[x]$. Then their greatest common divisor will be 1. So, there are polynomials $f(x), g(x) \in F[x]$ such that

$$a(x)f(x)+b(x)g(x)=1.$$

Comment

Step 2 of 3

If c is some common root of $a(x)$ and $b(x)$, then the substitution of c for x yields

$$\begin{aligned} 0\cdot f(x)+0\cdot g(x)&=1 \\ 0&=1, \end{aligned}$$

a contradiction. Thus, $a(x)$ and $b(x)$ have no common root in any extension K of F. Hence, both are relatively prime in $K[x]$.

Conversely, assume that $a(x)$ and $b(x)$ are relatively prime in $K[x]$. Let both the polynomials have a non-constant greatest common divisor $d(x)$ in F and c is the root of $d(x)$ in K. Since d divides both $a(x)$ and $b(x)$, therefore c is a common root of $a(x)$ and $b(x)$ in $K[x]$. This is a contradiction to the hypothesis.

Comment

Step 3 of 3

Hence, $a(x)$ and $b(x)$ are relatively prime in $F[x]$ if and only they are relatively prime in $K[x]$.

Comment