A Book of Abstract Algebra | (2nd Edition)

Problem

By the proof of the basic theorem of field extensions, if *p*(*x*) is an irreducible polynomial of degree *n* in *F*[*x*], then $F[x]$, then $F[x]/\langle p(x)\rangle \cong F(c)$ where c is a root of *p*(*x*). By Theorem 1 in this chapter, *F*(*c*) is of degree *η* over F. Using the paragraph preceding Theorem 1:

Construct a field of eight elements. (It is to be an extension of $\mathbb{Z}_2$).

Step-by-step solution

Step 1 of 2

The objective is to construct a field of eight elements.

Comment

Step 2 of 2

Let $f(x)=x^3+x+1\in \ _2[x]$.

Then $f(0)=1$ and $f(1)=1+1+1 \ =1$.

So , $f(x)$ has no zeros in $_2$ and thus is irreducible over $_2[x]$.

Then there exist an extension field E of $_2$ containing a zero α of $f(x)$.

Since every element β of a simple extension $E=F(\alpha)$ can be uniquely expressed in the form $\beta=b_0+b_1\alpha+\ldots+b_{n-1}\alpha^{n-1}$ where $b_i\in F$ and α is algebraic over F , $_2(\alpha)$ has elements $0,\ 1,\ \alpha,\ \alpha^2,\ 1+\alpha,\ 1+\alpha+\alpha^2,\ 1+\alpha^2,\ \alpha+\alpha^2$.

This gives a field of eight elements.

Comment