A Book of Abstract Algebra | (2nd Edition)

Chapter 30, Problem 1EE

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Problem

We will show that $2\pi/5$ is a constructible angle, and it will follow that the regular pentagon is constructible.

If $r = \cos k + i \sin k$ is a complex number, prove that $1/r = \cos k - i \sin k$. Conclude that $r + 1/r = 2 \cos k$.

By de Moivre's theorem,

$$\omega = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$$

is a complex fifth root of unity. Since

$$x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1)$$

ω is a root of $p(x) = x^4 + x^3 + x^2 + x + 1$.

Step-by-step solution

Step 1 of 3

Here, objective is to prove that $\frac{1}{r} = \cos k - i \sin k$ and $r + \frac{1}{r} = 2\cos k$.

Comment

Step 2 of 3

Consider $r = \cos k + i\sin k$

$$\frac{1}{r} = \frac{1}{\cos k + i\sin k}$$

Multiply and divide with $\cos k - i\sin k$.

$$\frac{1}{r} = \frac{(\cos k - i\sin k)}{(\cos k + i\sin k)(\cos k - i\sin k)}$$

$$\frac{1}{r} = \frac{(\cos k - i\sin k)}{\cos^2 k - i^2 \sin^2 k}$$

$$\frac{1}{r} = \frac{(\cos k - i\sin k)}{\cos^2 k + \sin^2 k} \qquad (\because i^2 = -1)$$

$$\frac{1}{r} = \frac{(\cos k - i\sin k)}{1} \qquad (\because \cos^2 k + \sin^2 k = 1)$$

Therefore, $\frac{1}{r} = \cos k - i\sin k$

Comment

Step 3 of 3

Consider $r+\frac{1}{r}$

$$r+\frac{1}{r}=\cos k+i\sin k+\cos k-i\sin k$$
$$=2\cos k$$

Hence, $\frac{1}{r}=\cos k-i\sin k$ and $r+\frac{1}{r}=2\cos k$.

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