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Chapter 16, Problem 6ED Bookmark Show all steps: ON

Problem

Let G be a group. By an *automorphism* of G we mean an isomorphism $f: G \to G$.

Let $h: G \to I(G)$ be the function defined by $h(a) = \phi_a$. Prove that h is a homomorphism from G onto $I(G)$ and that C is its kernel.

Step-by-step solution

Step 1 of 4

Suppose that $I(G)=\{\phi_a : a \in G\}$ is the set of all the inner automorphisms of G. Consider a mapping $h: G \to I(G)$ defined by

$$h(a)=\phi_a.$$

Objective is to prove that h is a homomorphism from G onto $I(G)$. Also show that kernel of this homomorphism will be same as the center C of G.

The center of any group G defined as:

$$C=\{a \in G : ax = xa \text{ for every } x \in G\}.$$

Comment

Step 2 of 4

If G and H are two groups, a homomorphism from G to H is a function $f: G \to H$ such that for any two elements a, b in G,

$$f(ab)=f(a)f(b).$$

Let $x, y \in G$. Then

$$h(xy)=\phi_{xy}.$$

Now by the property $\phi_a \quad \phi_b = \phi_{ab}$, it implies that

$$
\begin{aligned}
h(xy) &= \phi_{xy} \\
&= \phi_x \quad \phi_y \\
&= h(x) \quad h(y).
\end{aligned}
$$

Therefore, *h* is a homomorphism.

Let $\phi_a \in I(G)$. Then correspondingly the element *a* will belong to *G*. That is, for all $\phi_a \in I(G)$ there exists $a \in G$ such that $h(a) = \phi_a$, a onto mapping.

Comment

Step 3 of 4

According to the definition of kernel,

$$\ker h = \{a \in G : h(a) = e\},$$

where *e* is the identity of $I(G)$. Since $h(a) = \phi_a$, so

$$
\begin{aligned}
\ker h &= \{a \in G : \phi_a = e\} \\
&= \{a \in G : \phi_a(x) = e(x)\} \\
&= \{a \in G : axa^{-1} = x\}
\end{aligned}
$$

The last equality is obtained by definition of inner automorphism and identity function. Solve the condition $axa^{-1} = x$ by multiplying both the sides by *a* as:

$$
\begin{aligned}
axa^{-1}a &= xa \\
ax &= xa
\end{aligned}
$$

for all $x \in G$. That is, $a \in \ker h$ if it satisfies the condition that for all *x* in *G*, $ax = xa$. Therefore, $a \in C$ and thus kernel of *h* contains all the center elements.

Comment

Step 4 of 4

Hence, the function *h* is a homomorphism from *G* onto $I(G)$ with $\ker h = C$, a center of *G*.

Comment