A Book of Abstract Algebra | (2nd Edition)

Problem

Prove the following identities involving cartesian products:

$A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Step-by-step solution

Step 1 of 2

Objective:-

The objective is to prove $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Comment

Step 2 of 2

Proof:-

Let A and B are two sets.

The Cartesian product of two sets A and B is:-

$$A \times B = \{(x, y) : x \in A, y \in B\}$$

The intersection of two sets A and B is:-

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

Let $(x, y) \in A \times (B \cap C)$.

$$(x, y) \in A \times (B \cap C)$$
$$\Rightarrow x \in A, y \in B \cap C$$
$$\Rightarrow x \in A, y \in B \text{ and } y \in C$$
$$\Rightarrow (x \in A, y \in B) \text{ and } (x \in A, y \in C)$$

$\Rightarrow (x,y) \in (A \times B)\ and\ (x,y) \in (A \times C)$

$\Rightarrow (x,y) \in (A \times B) \cap (A \times C)$

So,

$$A \times (B \cap C) \subseteq (A \times B) \cap (A \times C) \qquad \ldots\ldots(1)$$

Let $(x,y) \in (A \times B) \cap (A \times C)$.

$(x,y) \in (A \times B) \cap (A \times C)$

$\Rightarrow (x,y) \in (A \times B) \cap (A \times C)$

$\Rightarrow (x,y) \in (A \times B)\ and\ (x,y) \in (A \times C)$

$\Rightarrow (x \in A, y \in B)\ \ and (x \in A, y \in C)$

$\Rightarrow x \in A, y \in B\ \ and\ \ y \in C$

$\Rightarrow x \in A, y \in B \cap C$

$\Rightarrow (x,y) \in A \times (B \cap C)$

So,

$$(A \times B) \cap (A \times C) \subseteq A \times (B \cap C) \qquad \ldots\ldots(2)$$

Let us consider the equation (1) and (2).

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

Comment