Abstract Algebra by Pinter, Chapter 18

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Abstract

Chapter 18 on Ideals

Contents

1 A. Examples of Subrings

1.1 Q1

$$\{x + \sqrt{3}y : x, y \in \mathbb{Z}\}$$

Closed wrt subtraction

$$(x + \sqrt{3}y)(v + \sqrt{3}w) = xv + \sqrt{3}(yv + xw) + 3yw \in \mathbb{R}$$

Thus it's a subring.

1.2 Q2

As before, it's closed under subtraction and multiplication.

1.3 Q3

$$\{x2^y : x, y \in \mathbb{Z}\}$$

Closed under multiplication because:

$$x_1 2^{y_1} \cdot x_2 2^{y_2} = (x_1 x_2) 2^{y_1 + y_2}$$

Also contains negatives since $x \in \mathbb{Z}$.

To show closure under addition is trivial for positive powers since

$$x2^y + v2^w = x2^{(y-w)}2^w + v2^w = (x2^{(y-w)} + v)2^w$$

Now for the negative case, assume $y > w$, hence $y - w$ is positive and the formulation still holds.

1.4 Q4

The sum and product of continuous functions are continuous.

1.5 Q5

The sum and product on any interval $[0, 1]$ also remains continuous, and hence also includes $\mathcal{C}$

1.6 Q6

Addition and negatives remain in $\mathcal{M}_2(\mathbb{R})$ as does multiplication

$$\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & xy \end{pmatrix}$$

2 B. Examples of Ideals

2.1 Q1

Identify which of the following are ideals of $\mathbb{Z} \times \mathbb{Z}$

2.1.1 $\{(n, n) : n \in \mathbb{Z}\}$

$$\begin{aligned} (n, n) + (m, m) &= (m + n, m + n) \in I \\ -(n, n) &= (-n, -n) \in I \\ (n, n) \cdot (a, b) &= (na, nb) \notin I \end{aligned}$$

Not an ideal.

2.1.2 $\{(5n, 0) : n \in \mathbb{Z}\}$

$$\begin{aligned} (5m, 0) + (5n, 0) &= (5(m + n), 0) \in I \\ -(5n, 0) &= (5(-n), 0) \in I \\ (5n, 0) \cdot (a, b) = (5(na), 0) \in I \end{aligned}$$

Is an ideal.

2.1.3 $\{(n, m) : n + m$ **is even** $\}$

$$\begin{aligned}(n_1, m_1) + (n_2, m_2) = (n_1 + n_2, m_1 + m_2) \in I \\ -(n, m) \in I \\ (n, m) \cdot (a, b) = (na, mb)\end{aligned}$$

na is even and mb is even, so $na + mb$ is even so $(na, mb) \in I$.

Is an ideal.

2.1.4 $\{(2n, 3m) : n, m \in \mathbb{Z}\}$

$$\begin{aligned}(2n_1, 3m_1) + (2n_2, 3m_2) = (2(n_1 + n_2), 3(m_1 + m_2)) \in I \\ -(2n, 3m) = (2(-n), 3(-m)) \in I \\ (2n, 3m) \cdot (a, b) = (2na, 3mb) \in I\end{aligned}$$

Is an ideal

2.2 Q2

List all the ideals of $\mathbb{Z}_{12}$

$\mathbb{Z}_{12} = \langle 1 \rangle$ and is cyclic. All subgroups are also cyclic.

- $\mathbb{Z}_{12} =< 1 >=< 5 >=< 7 >=< 11 >$ because $\gcd(m, 12) = 1$
- $< 4 >=< 8 >, < 4 >= \{4, 8, 0\}$
- $< 3 >=< 9 >, < 3 >= \{3, 6, 9, 0\}$
- $< 2 >=< 10 >, < 2 >= \{2, 4, 6, 8, 10, 0\}$
- $< 6 >= \{6, 0\}$
- $< 0 >= \{0\}$

Let $m \in \bar{m} =< m >$, then $< m >= \{mj : j \in \mathbb{Z}_{12}\}$

Let $x \in < m >$ and $y \in \mathbb{Z}_{12}$, since $x \in < m >$, then $x = mj$ for some $j \in \mathbb{Z}_{12}$, thus $xy = mjy$, thus $< m >$ is an ideal of $\mathbb{Z}_{12}$.

Ideals are $< 0 >, < 1 >, < 2 >, < 3 >, < 4 >, < 6 >$

2.3 Q3

See previous exercise

2.4 Q4

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & x \end{pmatrix} \notin \mathcal{M}_2(\mathbb{R})$$

2.5 Q5

The product of a continuous and non-continuous function are non-continuous, hence $\mathcal{C}(\mathbb{R})$ is not an ideal of $\mathcal{F}(\mathbb{R})$

2.6 Q6

2.6.1 a

Assume he means multiplication here.

$$f(x) \cdot g(x) = 0 \quad \forall x \in \mathbb{Q}$$

Thus $f \cdot g \in I$

2.6.2 b

Likewise $f(0)g(0) = 0g(0) = 0$, so $f \cdot g \in I$

2.7 Q7

Ideals of P_3 such that $AB = A \cap B \in I$. See also 17D5

$$P_3 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$

Any subgroup must contain $\emptyset$.

$A + A = \emptyset$ so A is its own negative.

$\{\emptyset, \{a\}\}$, $\{\emptyset, \{b\}\}$, $\{\emptyset, \{c\}\}$ are all ideals since $\{a\}\{a,c\} = \{a\}$ and $\{a\}\{b,c\} = \emptyset$.

Likewise $\{\emptyset, \{a\}, \{c\}, \{a,c\}\}$, $\{\emptyset, \{a\}, \{b\}, \{a,b\}\}$, $\{\emptyset, \{b\}, \{c\}, \{b,c\}\}$ since $\{a,c\}\{b,c\} = \{c\}$

Lastly we have P_3 itself

2.8 Q8

Example of a non-ideal subring is $\{\emptyset, \{a,c\}\}$ which is closed under addition, negatives and multiplication.

2.9 Q9

$$A = \langle(1,1)\rangle = \{(0,0), (1,1), (2,2)\}$$

3 C. Elementary Properties of Subrings

3.1 Q1

Let $x \in B$ and since B is a ring then $0 \in B$, thus $0 - x = -x \in B$.

So B is closed wrt negatives and hence addition since $x - (-y) = x + y \in B$

3.2 Q2

As per part 1

3.3 Q3

A ring is a group under addition. Hence order of a subring divides ring by Lagrange.

3.4 Q4

A has no zero divisors, hence neither does $B \implies B$ is an integral domain.

3.5 Q5

B is a subring of field F. Let $b \in B, b \neq 0$, then $b^{-1} \in F$ (because F is a field and contains inverses). Every field is an integral domain, hence so is B.

3.6 Q6

F is a commutative ring with inverses and unity.

Since B is a subring, it also is commutative.

Since B also contains inverses and is closed wrt multiplication, it must contain 1_F.

Thus B is a field.

3.7 Q7

3.7.1 a

$$B = \langle 2 \rangle = \{0, 2, 4, \ldots, 16\}$$

3.7.2 b

$$B = \langle 9 \rangle = \{0, 9\}$$

3.8 Q8

$$f(e) = e$$
$$f(x_1 + x_2) = f(x_1) + f(x_2)$$
$$f(x_1 x_2) = f(x_1) f(x_2)$$

But $\forall x \in B \quad f(x) = x$

$$x_1, x_2 \in B$$
$$x_1 + x_2 = f(x_1) + f(x_2) = f(x_1 + x_2)$$

Likewise for multiplication.

Since A is a ring $\forall -x \in A$ st $x + (-x) = e$ but $x \in B$

$$f(x) + f(-x) = f(e) = f(x + (-x)) = x + (-x)$$

Hence $-x \in B$ also.

3.9 Q9

$$ax = xa \qquad bx = xb$$
$$(a + b)x = x(a + b)$$

So $a + b$ also is in the center.

$$(ab)x = axb = x(ab)$$

Finally $0x = 0 = x0$

$$-a \in A$$
$$-ax = -(ax) = -(xa) = -xa$$

By associativity.

4 D. Elementary Properties of Ideals

4.1 Q1

Explain why J is an ideal of A iff J is closed with respect to subtraction and J absorbs products in A.

$$0 - x = -x \in J$$
$$x - (-y) = x + y \in J$$

So J is closed wrt negatives and addition from the statement about subtraction.

4.2 Q2

If A is a ring with unity, prove that J is an ideal of A iff J is closed with respect to addition and J absorbs products in A.

Note that A is a ring with unity, and by definition must include -1.

Then note that since J absorbs products, that $(-1) \cdot a = -a \in J$.

4.3 Q3

Prove that the intersection of any two ideals of A is an ideal of A.

1. Since $x, y \in I_j$, and I_j is an ideal, $x - y \in I_j, \forall j \in J$. Therefore $x - y \in \bigcap_{j \in J} I_j = I$.
2. Since $x \in I_j, rx \in I_j, \forall j \in J$. Therefore $rx \in I$.

4.4 Q4

Prove that J is an ideal of A and $1 \in J$, then $J = A$.

Since ideals absorb products, then if $1 \in J$, then since $a \cdot 1 = a \in J$, then $J = A$.

4.5 Q5

Prove that if J is an ideal of A and J contains an invertible element a of A, then $J = A$.

$$a \cdot a^{-1} = 1 \in J$$

By previous exercise $J = A$.

4.6 Q6

Explain why a field F can have no nontrivial ideals.

Every nonzero element of a field is invertible. Hence the only ideals are $\{0\}$ or F itself.

5 E. Examples of Homomorphisms

5.1 Q1

Let $f, g \in \mathcal{F}(\mathbb{R})$

$$\phi(f+g) = (f+g)(0) = f(0) + g(0) = \phi(f) + \phi(g)$$
$$\phi(f \cdot g) = (f \cdot g)(0) = f(0)g(0) = \phi(f)\phi(g)$$

$$K = \{f \in \mathcal{F}(\mathbb{R}) : f(0) = 0\}$$

Range is $[-\infty, \infty]$.

5.2 Q2

$$h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$
$$h(x, y) = x$$
$$h(x_1 + x_2, y_1 + y_2) = x_1 + x_2 = h(x_1, y_1) + h(x_2, y_2)$$
$$h(x_1 x_2, y_1 y_2) = x_1 x_2 = h(x_1, y_1)h(x_2 y_2)$$

$$\begin{aligned} K &= \{x, y \in \mathbb{R} \times \mathbb{R} : h(x, y) = 0\} \\ &= \{(0, y) : y \in \mathbb{R}\} \end{aligned}$$

Range is $[-\infty, \infty]$

5.3 Q3

$$h : \mathbb{R} \to \mathcal{M}_2(\mathbb{R})$$
$$h(x) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$$
$$h(x+y) = \begin{pmatrix} x+y & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} = h(x) + h(y)$$
$$h(xy) = \begin{pmatrix} xy & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} = h(x)h(y)$$
$$K = \{0\}$$

Range is

$$\begin{pmatrix} \pm\infty & 0 \\ 0 & 0 \end{pmatrix}$$

5.4 Q4

$$h : \mathbb{R} \times \mathbb{R} \to \mathcal{M}_2(\mathbb{R})$$
$$h(x, y) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

$$\begin{aligned} h(x_1 + x_2, y_1 + y_2) &= \begin{pmatrix} x_1 + x_2 & 0 \\ 0 & y_2 + y_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 & 0 \\ 0 & y_1 \end{pmatrix} + \begin{pmatrix} x_2 & 0 \\ 0 & y_2 \end{pmatrix} \\ &= h(x_1, y_1) + h(x_2, y_2) \end{aligned}$$

$$K = \{(0, 0)\}$$

Range is

$$\begin{pmatrix} \pm\infty & 0 \\ 0 & \pm\infty \end{pmatrix}$$

5.5 Q5

$$f : \mathbb{R} \times \mathbb{R} \to \mathcal{M}_2(\mathbb{R})$$
$$f(x, y) = \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix}$$

$$\begin{aligned} f(x_1 + x_2, y_1 + y_2) &= \begin{pmatrix} x_1 + x_2 & 0 \\ y_2 + y_2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} x_1 & 0 \\ y_1 & 0 \end{pmatrix} + \begin{pmatrix} x_2 & 0 \\ y_2 & 0 \end{pmatrix} \\ &= f(x_1, y_1) + f(x_2, y_2) \end{aligned}$$

$$f((x_1, y_1) \otimes (x_2, y_2)) = f(x_1x_2, y_1x_2) = \begin{pmatrix} x_1x_2 & 0 \\ y_1x_2 & 0 \end{pmatrix}$$
$$f(x_1, y_1)f(x_2y_2) = \begin{pmatrix} x_1 & 0 \\ y_1 & 0 \end{pmatrix} \begin{pmatrix} x_2 & 0 \\ y_2 & 0 \end{pmatrix} = \begin{pmatrix} x_1x_2 & 0 \\ y_1x_2 & 0 \end{pmatrix}$$
$$K = \{(0, 0)\}$$

5.6 Q6

$$h : P_C \to P_C$$
$$h(A) = A \cap D$$
$$D \subseteq C$$

$$\begin{aligned} h(A + B) &= h((A - B) \cup (B - A)) \\ &= [(A - B) \cup (B - A)] \cap D \\ &= [(A - B) \cap D] \cup [(B - A) \cap D] \\ &= [A \cap D - B \cap D] \cup [B \cap D - A \cap D] \\ &= h(A) + h(B) \\ h(AB) &= h(A \cap B) = A \cap B \cap D \\ &= (A \cap D) \cap (B \cap D) \\ &= h(A)h(B) \end{aligned}$$

$$K = \{A \in P_C : A \cap D = \emptyset\}$$

Range is every subset of D.

5.7 Q7

Rules for ring homomorphisms:

$$f(a+b) = f(a) + f(b) \qquad f(ab) = f(a)f(b)$$

$$f(0) = 0 \qquad f(1_A) = 1_B$$

$$f(n) = f(1 + \cdots + 1) = f(1) + \cdots + f(1) = nf(1)$$

$$mf(1) = 0 \qquad f(1)^2 = f(1)$$

Homomorphisms for $\phi_i : \mathbb{Z}_2 \to \mathbb{Z}_4$

$$\phi_e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The other mappings do not work:

- $1 \to 1$ then $2f(1) \neq 0$
- $1 \to 2$ then $f(1)^2 \neq f(1)$
- $1 \to 3$ then $f(1)^2 \neq f(1)$

Homomorphisms for $\phi_i : \mathbb{Z}_2 \to \mathbb{Z}_4$

$$\phi_e = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\phi_a = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 4 & 2 \end{pmatrix}$$

The other mappings do not work:

- $1 \to 1$ then $f(1)^2 \neq f(1)$
- $1 \to 2$ then $f(1)^2 \neq f(1)$
- $1 \to 3$ then $3f(1) \neq 0$
- $1 \to 5$ then $3f(1) \neq 0$

6 F. Elementary Properties of Homomorphisms

6.1 Q1

Prove $f(A) = \{f(x) : x \in A\}$ is a subring of B.

Since f is a homomorphism, ring operations are obeyed in the homomorphism. For negatives we note that $f(0_A) = 0_B = 1_B + (-1_B)$ and every negative is expressible as $(-1_B) \cdot a$ where $a \in B$.

6.2 Q2

Prove the kernel of f is an ideal of A.

$$K = \{x \in A : f(x) = 0_B\}$$

From f being a homomorphism, we conclude K is a subring of A.

To show it's an ideal, for any $a \in A$ and $x \in K$, then $f(ax) = 0 = f(x)$. So K absorbs the product ax.

Thus the kernel of a homomorphism is an ideal of the input ring.

6.3 Q3

Prove $f(0) = 0$, and for every $a \in A, f(-a) = -f(a)$.

$$f(0) = f(0+0) = f(0) + f(0) \implies f(0) = 0$$

$$f(a + (-a)) = f(a) + f(-a) = f(0) = 0 = f(a) - f(a)$$

$$\implies f(-a) = -f(a)$$

6.4 Q4

Prove f is injective iff its kernel is equal to $\{0\}$.

$$f(x) = f(y) \iff f(y-x) = 0 \iff y - x \in K$$

Let $x \in K$

$\implies f(x) = 0$

$\implies f(x) = f(0) \qquad$ [since $f(0) = 0$]

$\implies x = 0 \qquad$ [since f is injective]

It follow $K = \{0\}$

Thus f is injective $\implies K = \{0\}$

Now suppose $K = \{0\}$. Then

$f(x) = f(y)$

$\implies f(x) - f(y) = 0$

$\implies f(x-y) = 0$

$\implies x - y \in K$

$\implies x - y = 0 \qquad$ [since $K = \{0\}$]

$\implies x = y$

Hence f is injective.

Thus $K = \{0\} \implies f$ is injective.

Hence f is injective $\iff K = \{0\}$

6.5 Q5

If B is an integral domain, then either $f(1) = 1$ or $f(1) = 0$. If $f(1) = 0$ then $f(x) = 0$ for every $x \in A$. If $f(1) = 1$, the image of every invertible element of A is an invertible element of B.

Integral domain has the cancellation property such that $ab = ac \implies b = c$.

$$f(1) = f(1 \cdot 1) = f(1)f(1)$$
$$f(1) = f(1)f(1)$$

$f(1) = 0$ or 1$

If $f(1) = 0$ then $\forall a \in A$, $f(a) = f(1 \cdot a) = f(1)f(a) = 0$

If $f(1) = 1$ and $\exists x, y \in A$ such that $xy = 1$

$$f(xy) = f(x)f(y) \text{ where } f(y) = (f(x))^{-1}$$

6.6 Q6

Any homomorphic image of a commutative ring is a commutative ring. Any homomorphic image of a field is a field.

Let $a, b \in A$, then $f(a)f(b) = f(b)f(a)$ because $f(ab) = f(ba)$.

If A is a field, then $\forall x \in A$, $\exists x^{-1} \in A$. So by the last exercise, $f(x^{-1}) = (f(x))^{-1}$ and so the inverse of $f(x)$ is a member of B.

$$(f(x))^{-1} \in B$$

6.7 Q7

If the domain A of the homomorphism f is a field, and if the range of f has more than one element, then f is injective.

Since A is a field, the kernel of A is either $\{0\}$ or A itself.

But the range of f is more than one element, so the kernel of A cannot be A and must be $\{0\}$.

Since the kernel of f is $\{0\}$, then f is injective.

7 G. Examples of Isomorphisms

7.1 Q1

$$a \oplus b = a + b + 1$$
$$a \otimes b = ab + a + b$$

7.1.1 Addition

$$\begin{aligned} f(a+b) &= a + b - 1 \\ f(a) \oplus f(b) &= (a-1) + (b-1) - 1 \\ &= a + b - 1 \end{aligned}$$

7.1.2 Multiplication

$$\begin{aligned} f(ab) &= ab - 1 \\ f(a) \otimes f(b) &= (a-1)(b-1) + (a-1) + (b-1) \\ &= ab - b - a + 1 + a + b - 1 - 1 \\ &= ab - 1 \end{aligned}$$

7.2 Q2

$$\mathcal{J} = \{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \}$$
$$f : \mathbb{C} \to \mathcal{J}$$
$$f(a+bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$
$$a + bi = c + di \implies f(a+bi) = f(c+di)$$

7.2.1 Addition

$$f((a+bi) + (c+di)) = \begin{pmatrix} a+c & b+d \\ -(b+d) & a+c \end{pmatrix}$$
$$f(a+bi) + f(c+di) = \begin{pmatrix} a+c & b+d \\ -(b+d) & a+c \end{pmatrix}$$

7.2.2 Multiplication

$$\begin{aligned} f((a+bi)(c+di)) &= f((ac-bd) + (ad+bc)i) \\ &= \begin{pmatrix} a+c & b+d \\ -(b+d) & a+c \end{pmatrix} \\ f(a+bi)f(c+di) &= \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \\ &= \begin{pmatrix} a+c & b+d \\ -(b+d) & a+c \end{pmatrix} \end{aligned}$$

7.3 Q3

$$A = \{(x, x) : x \in \mathbb{Z}\}$$

$\forall x, y \in \mathbb{Z}, (x, x) \in A, (y, y) \in A, (x + y, x + y) \in A$ and $(xy, xy) \in A$

Thus A is a subring of $\mathbb{Z} \times \mathbb{Z}$

The homomorphism $f : \mathbb{Z} \to A$ by $f(x) = (x, x)$ is isomorphic because it is one to one

$$f(x) = f(y) \implies x = y$$

and onto

$$\forall (x, x) \in A, \exists x \in \mathbb{Z} \text{ such that } f(x) = (x, x)$$

Thus

$$\{(x, x) : x \in \mathbb{Z}\} \cong \mathbb{Z}$$

7.4 Q4

Addition and negatives trivially remain inside the set.

$$\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & xy \end{pmatrix}$$

Hence the set is a subring.

$$A = \{ \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} : x \in \mathbb{R}\}$$

Define $f : \mathbb{R} \to A$ by $f(x) = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$, then f is an homomorphism from $\mathbb{R}$ to A.

Hence $A \cong \mathbb{R}$

7.5 Q5

$$f : k\mathbb{Z} \to l\mathbb{Z}$$

$$f(k) = ln \text{ for some } n \neq 0$$

$$\begin{aligned} f(k^2) &= l^2 n^2 \\ &= f(k \cdot k) = f(k + \cdots + k) = kf(k) \\ &= kln \\ k &= ln \end{aligned}$$

But $k \neq l$, so ln does not generate $l\mathbb{Z}$ and f is not an isomorphism.

8 H. Further Properties of Ideals

8.1 Q1

If $J \cap K = \{0\}$, then $jk = 0$ for every $j \in J$ and $k \in K$.

J and K are ideals, so for every $j \in J$ and $k \in K$, then $jk \in J$ and $jk \in K$, so $jk \in J \cap K$.

8.2 Q2

For any $a \in A$, $I_a = \{ax + j + k : x \in A, j \in J, k \in K\}$ is an ideal of A.

$\forall i \in I_a$, and $b \in A$, then $bi = b(ax + j + k) = a(bx) + bj + bk \in I_a$ since J and K are ideals and $bx \in A$.

8.3 Q3

The radical of J is the set $\operatorname{rad} J = \{a \in A : a^n \in J$ *for some* $n \in \mathbb{Z}\}$. *For any ideal J,* $\operatorname{rad} J$ *is an ideal of A.*

$$a^n \in J \qquad b^m \in J$$
$$(a+b)^{n+m} \in J \qquad \text{[see 17m3]}$$

$x \in A$ and $a \in \operatorname{rad} J$ then $(xa)^n = x^n a^n \in J$, so $xa \in \operatorname{rad} J$.

$a, b \in \operatorname{rad} J$, then $(a+b)^{m+n} \in J$ and so $a + b \in \operatorname{rad} J$.

8.4 Q4

For any $a \in A$, $\{x \in A : ax = 0\}$ is an ideal (called the annihilator of a).

Furthermore, $\{x \in A : ax = 0$ for every $a \in A\}$ is an ideal (called the annihilating ideal of A). If A is a ring with unity, its annihilating ideal is equal to $\{0\}$.

Let $b \in A$, then $bx \in Ann(a)$ because $ax = 0$ so $b(ax) = bxa = 0$.

Let $x, y \in Ann(a)$ then $a(x + y) = 0$ so $x + y \in Ann(a)$.

$$I = \{x \in A : ax = 0 \text{ for every } a \in A\}$$

If A is a ring with unity then $a = 1 \implies x = 0$ so $I = \{0\}$.

8.5 Q5

Show that $\{0\}$ and A are ideals of A. (They are trivial ideals; every other ideal of A is a proper ideal.) A proper J of A is called maximal if it is not strictly contained in any strictly larger proper ideal: that is if $J \subseteq K$, where K is an ideal containing some element not in J, then necessarily $K = A$.

Show the following is an example of a maximal ideal: in $\mathcal{F}(\mathbb{R})$, the ideal $J = \{f : f(0) = 0\}$.

$$g \in K \qquad g(0) \neq 0 \qquad g \notin J$$

$$h(x) = g(x) - g(0) \in J$$
$$h(x) - g(x) \in K$$

Continuous function with a nonzero value is invertible.

$h(x) - g(x) = -g(0) \in K$ but $g(0) \neq 0$ so $-1/g(0) \in A$.

But since K is an ideal, that is

$$g(0) \cdot 1/g(0) \in K$$

but this equals 1, and $1 \in K$ so $K = A$ and is maximal.

9 I. Further Properties of Homomorphisms

9.1 Q1

If $f : A \to B$ is a homomorphism from A onto B with kernel K, and J is an ideal of A such that $K \subseteq J$, then $f(J)$ is an ideal of B\$.

f is onto $\exists x : f(x) = y$ so it's an ideal. Closed under addition and negatives and absorbs products.

See also here

9.2 Q2

If $f : A \to B$ is a homomorphism from A onto B, and B is a field, then the kernel of f is a maximal ideal.

The kernel K is a subset of the ideal for A. As shown above $f(J)$ is an ideal of B, which by D6 can only be $\{0\}$ or B itself. Since the homomorphism is onto, then $f(A)$ maps to B, but A is a trivial ideal of A. Thus K, the kernel of f is the proper ideal for A which maps to $\{0\}$ in B.

9.3 Q3

There are no nontrivial homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}$.

$$f(1) = f(1 \cdot 1) = f(1) \cdot f(1)$$
$$f(1) = 1 \text{ or } f(1) = 0$$
$$f(n) = f(1 + \cdots + 1) = f(1) + \cdots + f(1) = nf(1)$$

So $f(n) = n$ or $f(n) = 0$

See also here and here

9.4 Q4

If n is a multiple of m, then $\mathbb{Z}_m$ is a homomorphic image of $\mathbb{Z}_n$.

$f : \mathbb{Z}_n \to \mathbb{Z}_m$\$ by $f(a) = a (\bmod m)$ obeys the homomorphic properties.

See also here

9.5 Q5

If n is odd, there is an injective homomorphism from $\mathbb{Z}_2$ into $\mathbb{Z}_{2n}$.

$$f(x) = nx$$

Above homomorphism is injective since $f(0) = 0$ and $f(1) = n$.

10 J. A Ring of Endomorphisms

10.1 Q1

$$\pi_a(x) = ax$$
$$\pi_a(x + y) = a(x + y) = ax + ay = \pi_a(x) + \pi_a(y)$$

10.2 Q2

$$\pi_a(x) = \pi_a(y) \implies x = y$$

a is not a divisor of zero $\implies \forall x \in A, ax \neq 0$, thus ring A has cancellation property

$$\pi_a(x) = \pi_a(y) = ax = ay \implies x = y$$

10.3 Q3

If a is invertible then $\forall y \in A$, $y = a(a^{-1}y)$ so $x = a^{-1}y$, $f(x) = y$, thus π_a is surjective.

10.4 Q4

$$\mathcal{A} = \{\pi_a : a \in A\}$$
$$[\pi_a + \pi_b](x) = \pi_a(x) + \pi_b(x)$$
$$\pi_a \pi_b = \pi_a \cdot \pi_b$$

1. Addition is abelian
2. Multiplication is associative: $(\pi_a \cdot \pi_b \cdot \pi_c)(x) = (abc)x = a(bcx) = \pi_a((\pi_b \cdot \pi_c)(x))$
3. Distributive over addition

10.5 Q5

$$\phi : A \to \mathcal{A} \text{ given by } \phi(a) = \pi_a$$

As shown above this is homomorphic.

10.6 Q6

$$\phi(a) = \phi(b) \implies \pi_a = \pi_b$$
$$\pi_a(1) = \pi_b(1) \implies a = b$$

$\forall \pi_a \in \mathcal{A}, \exists a \in A : \pi_a = \phi(a)$ by definition.

If a has no divisors of zero, then to show injective property, note that

$$ax = bx \implies a = b$$

$$\pi_a = pi_b \implies \pi_a(x) = ax = \pi_b(x) = bx \implies a = b$$

From the cancellation property since it has no divisors of zero.