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Chapter 27, Problem 1EG | Bookmark | Show all steps: ON

Problem

Let F be a field, and let c be transcendental over F. Prove the following:

$\{a(c):a(x) \in F[x]\}$ is an integral domain isomorphic to $F[x]$.

Step-by-step solution

Step 1 of 3

Consider that F is any arbitrary field and let $c \in F$ is transcendental over F. Objective is to prove that the set $\{a(c):a(x)\in F[x]\}$ is an integral domain isomorphic to $F[x]$.

Assume the substitution function $\sigma_c : F[x] \to E$ as follows:

For every polynomial $a(x)\in F[x]$,

$\sigma_c(a(x)) = a(c)$,

where E is a field, F is a subfield of E.

Comment

Step 2 of 3

It is known that substitution function is a homomorphism. The range of this homomorphism is the set of all the elements $a(c)$, for all $a(x)$ in $F[x]$, that is,

$\text{Range}\,\sigma_c = \{a(c):a(x)\in F[x]\}$.

Since $\text{Range}\,\sigma_c$ is a field and every field is an integral domain, therefore $\{a(c):a(x)\in F[x]\}$ is an integral domain.

Consider the following result:

Element c is transcendental over F if and only if kernel of homomorphism σ_c is trivial, that is, $\ker\sigma_c = \{0\}$.

Thus, the mapping from $F[x]$ to $\text{Range}\,\sigma_c$ is bijective homomorphism.

Comment

Step 3 of 3

Hence, $\{a(c):a(x)\in F[x]\}$ is isomorphic to $F[x]$.

Comment