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Chapter 29, Problem 4ED

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Problem

Let F be a field, and *K* a field extension of *F*. Prove the following:

Suppose *a, b* ∈ *K* are algebraic over *F* with degrees *m* and η, where *m* and « are relatively prime. Then:

(*a*) *F*(*a, b*)is degree *mn* over *F*.

(*b*) *F*(*a*) ∩ *F*(*b*)= *F*.

Step-by-step solution

Step 1 of 3

Consider a field F and an extension K of F. Suppose $a,b\in K$ are algebraic over F with degrees m and n ,where m and n are relatively prime. The objective is to prove that:

(a) $F(a,b)$ is of degree mn over F.

(b) $F(a)\cap F(b)=F$.

Comment

(a)

Suppose $a,b \in K$ are algebraic over F with minimal polynomials $p(x)$ and $q(x)$ of degrees m and n, respectively.

Now, $[F(a,b):F]=[F(a,b):F(a)][F(a):F]=m[F(a,b):F(a)]$.

Hence, $m \mid [F(a,b):F]$.

Similarly, $n \mid [F(a,b):F]$.

So, $mn \mid [F(a,b):F]$ because m and n are relatively prime.

On the other hand, $[F(a,b):F(a)] \le n$ since the minimal polynomial for b over $F(a)$ can have no greater degree than the minimal polynomial $q(x)$ over F.

Therefore, $[F(a,b):F]=[F(a,b):F(a)][F(a):F] \le mn$.

Since $mn \mid [F(a,b):F]$, $[F(a,b):F]=mn$.

Comment

(b)

Let $L=F(a)\cap F(b)$.

Then,

$[F(a,b):F]=[F(a):F][F(b):F]$, since $F(a)$ and $F(b)$ are intermediate fields

$=[F(a):L][L:F][F(b):L][L:F]$, since $F \subseteq F(a)\cap F(b) \subseteq F(a) \subseteq K$ and $F \subseteq F(a)\cap F(b) \subseteq F(b) \subseteq K$

$$=\left[F(a):L\right]\left[F(b):L\right]\left[L:F\right]^2$$
$$=\left[F(a,b):L\right]\left[L:F\right]^2$$
$$=\left[F(a,b):F\right]\left[L:F\right]$$

Thus, $\left[L:F\right]=1$ and hence $L=F$, that is, $F(a)\cap F(b)=F$.

Comment