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Chapter 32, Problem 3EB Bookmark Show all steps: ON

Problem

List the eight elements of $G = Gal(\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}) : \mathbb{Q})$ and write its table.

Step-by-step solution

Step 1 of 2

The objective is to list the eight elements of $G = Gal\left(\mathbb{Q}\left(\sqrt{2},\sqrt{3},\sqrt{5}\right):\mathbb{Q}\right)$ and write its table.

Comment

Step 2 of 2

The root field $\mathbb{Q}\left(\sqrt{2},\sqrt{3},\sqrt{5}\right)$ is of degree 8 over $\mathbb{Q}$.

Therefore，there are eight automorphism of $\mathbb{Q}\left(\sqrt{2},\sqrt{3},\sqrt{5}\right)$ which fix $\mathbb{Q}$，since the number of automorphism is equal to the degree of $\mathbb{Q}\left(\sqrt{2},\sqrt{3},\sqrt{5}\right)$ over $\mathbb{Q}$.

Since an automorphism is determined by its effect on $\sqrt{2}$，$\sqrt{3}$ and $\sqrt{5}$，there are eight possibilities，namely，

$$\sigma_2 : \begin{cases} \sqrt{2} \mapsto -\sqrt{2} \\ \sqrt{3} \mapsto \sqrt{3} \\ \sqrt{5} \mapsto \sqrt{5} \end{cases} \quad \sigma_3 : \begin{cases} \sqrt{2} \mapsto \sqrt{2} \\ \sqrt{3} \mapsto -\sqrt{3} \\ \sqrt{5} \mapsto \sqrt{5} \end{cases} \quad \sigma_5 : \begin{cases} \sqrt{2} \mapsto \sqrt{2} \\ \sqrt{3} \mapsto \sqrt{3} \\ \sqrt{5} \mapsto -\sqrt{5} \end{cases}$$

$$\sigma_2\sigma_3 : \begin{cases} \sqrt{2} \mapsto -\sqrt{2} \\ \sqrt{3} \mapsto -\sqrt{3} \\ \sqrt{5} \mapsto \sqrt{5} \end{cases} \quad \sigma_2\sigma_5 : \begin{cases} \sqrt{2} \mapsto -\sqrt{2} \\ \sqrt{3} \mapsto \sqrt{3} \\ \sqrt{5} \mapsto -\sqrt{5} \end{cases} \quad \sigma_3\sigma_5 : \begin{cases} \sqrt{2} \mapsto \sqrt{2} \\ \sqrt{3} \mapsto -\sqrt{3} \\ \sqrt{5} \mapsto -\sqrt{5} \end{cases}$$

$$\sigma_2\sigma_3\sigma_5:\begin{cases}\sqrt{2}\mapsto-\sqrt{2}\\ \sqrt{3}\mapsto-\sqrt{3}\\ \sqrt{5}\mapsto-\sqrt{5}\end{cases}\quad id:\begin{cases}\sqrt{2}\mapsto\sqrt{2}\\ \sqrt{3}\mapsto\sqrt{3}\\ \sqrt{5}\mapsto\sqrt{5}\end{cases}.$$

Thus, the Galois group of $\mathbb{Q}\left(\sqrt{2},\sqrt{3},\sqrt{5}\right)$ over $\mathbb{Q}$ is

$$Gal\left(\mathbb{Q}\left(\sqrt{2},\sqrt{3},\sqrt{5}\right):\mathbb{Q}\right)=\{id,\ \sigma_2,\ \sigma_3,\ \sigma_5,\ \sigma_2\sigma_3,\ \sigma_2\sigma_5,\ \sigma_3\sigma_5,\ \sigma_2\sigma_3\sigma_5\}.$$

The operation is composition, giving the table.

$\circ$	id	σ_2	σ_3	σ_5	$\sigma_2\sigma_3$	$\sigma_2\sigma_5$	$\sigma_3\sigma_5$	$\sigma_2\sigma_3\sigma_5$
id	id	σ_2	σ_3	σ_5	$\sigma_2\sigma_3$	$\sigma_2\sigma_5$	$\sigma_3\sigma_5$	$\sigma_2\sigma_3\sigma_5$
σ_2	σ_2	id	$\sigma_2\sigma_3$	$\sigma_2\sigma_5$	σ_3	σ_5	$\sigma_2\sigma_3\sigma_5$	$\sigma_3\sigma_5$
σ_3	σ_3	$\sigma_2\sigma_3$	id	$\sigma_3\sigma_5$	σ_2	$\sigma_2\sigma_3\sigma_5$	σ_5	$\sigma_2\sigma_5$
σ_5	σ_5	$\sigma_2\sigma_5$	$\sigma_3\sigma_5$	id	$\sigma_2\sigma_3\sigma_5$	σ_2	σ_3	$\sigma_2\sigma_3$
$\sigma_2\sigma_3$	$\sigma_2\sigma_3$	σ_2	σ_3	$\sigma_2\sigma_3\sigma_5$	id	$\sigma_2\sigma_5$	$\sigma_3\sigma_5$	σ_5
$\sigma_2\sigma_5$	$\sigma_2\sigma_5$	σ_2	σ_3	σ_5	$\sigma_2\sigma_3$	id	$\sigma_3\sigma_5$	$\sigma_2\sigma_3\sigma_5$
$\sigma_3\sigma_5$	$\sigma_3\sigma_5$	σ_2	σ_3	σ_5	$\sigma_2\sigma_3$	$\sigma_2\sigma_5$	id	$\sigma_2\sigma_3\sigma_5$
$\sigma_2\sigma_3\sigma_5$	$\sigma_2\sigma_3\sigma_5$	σ_2	σ_3	σ_5	$\sigma_2\sigma_3$	$\sigma_2\sigma_5$	$\sigma_3\sigma_5$	id

Comment