A Book of Abstract Algebra | (2nd Edition)

Problem

Prove that each of the following numbers is algebraic over $\mathbb{Q}$.

(a) i

(b) $\sqrt{2}$

(c) $2 + 3i$

(d) $\sqrt{1+\sqrt[4]{2}}$

(e) $\sqrt{i-\sqrt{2}}$

(f) $\sqrt{2}+\sqrt{3}$

(g) $\sqrt[3]{2}+\sqrt[3]{4}$

Step-by-step solution

Step 1 of 8

(a)

Objective is to prove that the number i is algebraic over Q.

Let $a = i$. Then $a^2 = i^2$, $a^2 = -1$. Thus, a satisfies $a^2 + 1 = 0$.

Comment

Step 2 of 8

(b)

Objective is to prove that the number $\sqrt{2}$ is algebraic over Q.

Let $a = \sqrt{2}$. Then $a^2 = \left(\sqrt{2}\right)^2$, $a^2 = 2$. Thus, a satisfies $a^2 - 2 = 0$.

Comment

Step 3 of 8

(c)

Objective is to prove that the number $2+3i$ is algebraic over Q.

Let $a = 2+3i$. Then

$$\begin{aligned} a-2 &= 3i \\ (a-2)^2 &= (3i)^2 \\ a^2-4a+4 &= -9 \\ a^2-4a+13 &= 0. \end{aligned}$$

Thus, a satisfies $a^2 - 4a + 13 = 0$.

Comment

Step 4 of 8

(d)

Objective is to prove that the number $\sqrt{1+\sqrt[4]{2}}$ is algebraic over Q.

Let $a = \sqrt{1+\sqrt[4]{2}}$; then

$$\begin{aligned} a^2 &= 1+\sqrt[4]{2} \\ a^2-1 &= \sqrt[4]{2} \\ \left(a^2-1\right)^4 &= \left(\sqrt[4]{2}\right)^4 \\ \left(a^2-1\right)^4 &= 2 \end{aligned}$$

And finally $a^6 - 1 - 3a^4 + 3a^2 = 2$. Thus, a satisfies $a^6 - 3a^4 + 3a^2 - 3 = 0$.

Comment

Step 5 of 8

(e)

Objective is to prove that the number $\sqrt{i-\sqrt{2}}$ is algebraic over Q.

Let $a = \sqrt{i-\sqrt{2}}$; then

$$\begin{aligned} a^2 &= i-\sqrt{2} \\ \left(a^2\right)^2 &= \left(i-\sqrt{2}\right)^2 \\ a^4 &= -1+2-2\sqrt{2}i \\ \left(a^4-1\right)^2 &= \left(-2\sqrt{2}i\right)^2 \end{aligned}$$

And finally $a^8 + 1 - 2a^4 = -8$. Thus, a satisfies $a^8 - 2a^4 + 9 = 0$.

Comment

Step 6 of 8

(f)

Objective is to prove that the number $\sqrt{2}+\sqrt{3}$ is algebraic over Q.

Let $a = \sqrt{2}+\sqrt{3}$; then

$$\begin{aligned} a^2 &= \left(\sqrt{2}+\sqrt{3}\right)^2 \\ a^2 &= 2+3+2\sqrt{2}\sqrt{3} \\ a^2-5 &= 2\sqrt{6} \\ \left(a^2-5\right)^2 &= \left(2\sqrt{6}\right)^2 \end{aligned}$$

And finally $a^4 + 25 - 10a^2 = 24$. Thus, a satisfies $a^4 - 10a^2 + 1 = 0$.

Comment

Step 7 of 8

(g)

Objective is to prove that the number $\sqrt[3]{2}+\sqrt[3]{4}$ is algebraic over Q.

Let $a = \sqrt[3]{2}+\sqrt[3]{4}$; then

$$\begin{aligned} a^3 &= \left(\sqrt[3]{2}+\sqrt[3]{4}\right)^3 \\ a^3 &= 2+4+3\left(2^{2/3}\right)\left(2^{2/3}\right)+3\left(2^{1/3}\right)\left(2^{4/3}\right) \\ a^3-6 &= 3\cdot 2^{4/3}\left(1+2^{1/3}\right) \\ a^3-6 &= 3\cdot 2\cdot 2^{1/3}\left(1+2^{1/3}\right) \end{aligned}$$

Rewrite a as:

$$\begin{aligned} a &= \sqrt[3]{2}+\sqrt[3]{4} \\ a &= 2^{1/3}+2^{2/3} \\ a &= 2^{1/3}\left(1+2^{1/3}\right) \end{aligned}$$

And finally substitute a and get,

$$a^3 - 6 = 3\cdot 2\cdot a.$$

Comment

Step 8 of 8

Thus, a satisfies $a^3 - 6a - 6 = 0$.

Comment