A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 2EE

Bookmark

Show all steps: ON

Problem

Let *G* and *H* be groups. Suppose *J* is a normal subgroup of *G* and *K* is a normal subgroup of *H*.

Find the kernel of *f*.

Step-by-step solution

Step 1 of 4

Suppose that *G* and *H* are two arbitrary groups. Also let *J* is a normal subgroup of *G* and *K* is a normal subgroup of *H*.

Consider a mapping $f: G\times H \to (G/J)\times(H/K)$ defined by

$f(x, y) = (Jx, Ky)$.

Since let *J* is a normal subgroup of *G*, therefore the group G/J is defined. Also Jx is the coset of G/J for some $x \in G$. Note that *f* is an onto homomorphism from $G\times H$ to $(G/J)\times(H/K)$.

Objective is to determine the kernel of homomorphism *f*.

Comment

Step 2 of 4

According to the definition of kernel,

$\ker f = \{(x, y)\in G\times H : f(x, y) = e\}$,

where *e* is the identity of $(G/J)\times(H/K)$.

Note that, the identity of quotient group G/J is *J* and the identity of H/K is *K*. And then the identity of direct product $(G/J)\times(H/K)$ will be an ordered pair (J, K).

Substitute $f(x, y) = (Jx, Ky)$ and $e = (J, K)$ in kernel set and get,

$\ker f = \{(x, y)\in G\times H : (Jx, Ky) = (J, K)\}$.

Comment

Step 3 of 4

On comparing the equation $(Jx, Ky) = (J, K)$, one get,

$Jx = J, Ky = K$.

According to the coset property: if H is a subgroup of G and let $a, b \in G$. Then $aH = H$ if and only if $a \in H$.

So, by this property, the condition $Jx = J$ implies that $x \in J$. Similarly, from $Ky = K$ it implies that $y \in K$.

Thus,

$$\begin{aligned}\ker f &= \{(x, y) \in G \times H : (Jx, Ky) = (J, K)\} \\ &= \{(x, y) \in G \times H : x \in J, y \in K\}\end{aligned}$$

Since $x \in J, y \in K$, therefore $(x, y) \in J \times K$. So, $\ker f = J \times K$.

Comment

Step 4 of 4

Hence, the required kernel of homomorphism f will be:

$\ker f = J \times K$.

Comment