# A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 3EL

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## Problem

Let pbea prime number. A *p-group* is any group whose order is a power of *p*. It will be shown here that if $|G| = p^k$ then *G* has a normal subgroup of order $p^m$ for every *m* between 1 and *k*. The proof is by induction on $|G|$; we therefore assume our result is true for all /^-groups smaller than *G*. Prove parts 1 and 2:

Explain why it may be assumed that $G/\langle a\rangle$ has a normal subgroup of order $p^{m-1}$.

## Step-by-step solution

**Step 1** of 3

Consider a group *G* whose order is a power of *p*. That is, *G* is a *p*-group and

$$|G| = p^k,$$

for some integer *k*. With the help of mathematical induction on the order of group *G*, it can be prove that *G* has a normal subgroup of order $p^m$ for every $1 < m < k$.

Consider the induction hypothesis that this statement is true for all *p*-groups whose order is less than *G*.

Objective is to describe the reason behind the assumption that $G/\langle a\rangle$ has a normal subgroup of order $p^{m-1}$.

Comment

**Step 2** of 3

Before going to prove this statement consider the following results:

(1) There exists an element $a \in C$ (center of *G*) such that $\operatorname{ord}(a) = p$.

(2) $\langle a\rangle$ is a normal subgroup of *G*.

Note that, the elements of subgroup generated by *a* will be some powers of *a*. That is,

$\langle a \rangle = \{a^k : k \in Z\}$.

Since order of *a* is *p*, therefore the elements generated by *a* also have the order as *p*. So, $|\langle a \rangle| = p$. Now, use the basic property to calculate the order of quotient group $G/\langle a \rangle$ as:

$$\begin{aligned}\left|\frac{G}{\langle a \rangle}\right| &= \frac{|G|}{|\langle a \rangle|} \\ &= \frac{p^k}{p} \\ &= p^{k-1}.\end{aligned}$$

Since $p^{k-1} < p^k$, therefore by induction hypothesis it can be conclude that there exists a normal subgroup whose order can be $p^{m-1}$.

Comment

**Step 3** of 3

Hence, it can be assume that $G/\langle a \rangle$ has a normal subgroup of order $p^{m-1}$.

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