A Book of Abstract Algebra | (2nd Edition)

Problem

In each of the following, use the fundamental homomorphism theorem to prove that the two given groups are isomorphic. Then display their tables.

P_2 and P_3/K, where $K = \{0, \{c\}\}$. [HINT: Consider the function $f(C) = C \cap \{a, b\}$. P_3 is the group of subsets of $\{a, b, c\}$, and P_2 of $\{a, b\}$.]

Step-by-step solution

Step 1 of 4

Consider that P_D is the power set of set *D*, that is, P_D is set of all subsets of *D*. Let $A, B \in P_D$, then the addition and multiplication in P_D will be defined as follows:

$$A + B = (A - B)\quad(B - A).$$

Note that, P_D is a commutative ring with unity. The zero element in P_D is an empty set ϕ.

Next consider the two groups P_2 and P_3/K, where $K = \{0, \{c\}\}$. Objective is to prove that these two groups are isomorphic by using the fundamental homomorphism theorem.

Step 2 of 4

According to the fundamental homomorphism theorem, if $f : G \to H$ is a homomorphism of *G* onto *H*, with kernel *K* then

$$H \cong G/K.$$

The elements of groups P_2 and P_3 are:

$$P_2 = \{\phi, a, b, ab\},$$
$$P_3 = \{\phi, a, b, c, ab, bc, ca, abc\}.$$

Consider the function $f : P_3 \to P_2$ defined by

$f(x)=x \quad \{a,b\}$

for all *x* in P_3. Then,

x	$f(x)$
ϕ	$\phi \quad \{a,b\}=\phi$
$\{a\}$	$\{a\} \quad \{a,b\}=a$
$\{b\}$	$\{b\} \quad \{a,b\}=b$
$\{c\}$	$\{c\} \quad \{a,b\}=\phi$
$\{ab\}$	$\{a,b\}$
$\{bc\}$	$\{b\}$
$\{ca\}$	$\{a\}$
$\{abc\}$	$\{a,b\}$

Comment

Step 3 of 4

Since empty set ϕ is the zero element in P_D, therefore the elements of kernel will be:

$K=\{0,\{c\}\}$.

From the table it implies that map *f* is onto, also the intersection operator preserves the define addition. Therefore, the map *f* is homomorphism from P_3 onto P_2 with kernel *K*.

Here, the addition in P_2 is the symmetric difference of two sets. Consider the two elements $\{a,b\}$, $\{a\}$ of P_2. Then their sum will be:

$$\begin{aligned}\{a,b\}+\{a\}&=(\{a,b\}-\{a\}) \quad (\{a\}-\{a,b\})\\&=\{b\} \quad \phi\\&=\{b\}\end{aligned}$$

Consider the following addition table of P_2 as:

$A+B$	ϕ	$\{a\}$	$\{b\}$	$\{a,b\}$
ϕ	ϕ	$\{a\}$	$\{b\}$	$\{a,b\}$

$\{a\}$	$\{a\}$	ϕ	$\{a, b\}$	$\{b\}$
$\{b\}$	$\{b\}$	$\{a, b\}$	ϕ	$\{a\}$
$\{a, b\}$	$\{a, b\}$	$\{b\}$	$\{a\}$	ϕ

Comment

Step 4 of 4

Hence, by the fundamental homomorphism theorem it concludes that

$P_2 \cong P_3 / K$.

Since both the groups are isomorphic therefore tables for the groups will have the same properties.

Comment