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Chapter 32, Problem 1EC

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Problem

Show that $\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3})$ is the root field of $x^3 - 2$ over $\mathbb{Q}$, where $\sqrt[3]{2}$ designates the *real* cube root of 2. (HINT: Compute the complex cube roots of unity.)

Step-by-step solution

Step 1 of 2

The objective is to show that $\mathbb{Q}\left(\sqrt[3]{2}, i\sqrt{3}\right)$ is the root field of $x^3 - 2$ over $\mathbb{Q}$, where $\sqrt[3]{2}$ designates the real cube root of 2.

Comment

Step 2 of 2

Consider $x^3 - 2 = 0$.

$$\Rightarrow x = 2^{\frac{1}{3}}$$

$$\Rightarrow x = 2^{\frac{1}{3}}(1)^{\frac{1}{3}}$$

$$\Rightarrow x = 2^{\frac{1}{3}}(\cos 0 + i\sin 0)^{\frac{1}{3}}.$$

By De Moivre's theorem, $x = 2^{\frac{1}{3}}\left(\cos\left(\frac{2k\pi}{3}\right) + i\sin\left(\frac{2k\pi}{3}\right)\right)$ where $k = 0,1,2$.

Therefore, all roots of $x^3 - 2$ are:

$$2^{\frac{1}{3}},\ 2^{\frac{1}{3}}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right),\ 2^{\frac{1}{3}}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right);\ i = \sqrt{-1}.$$

The smallest field containing $\mathbb{Q}$ and the above roots is $\mathbb{Q}\left(\sqrt[3]{2}, i\sqrt{3}\right)$.

Hence, $\mathbb{Q}\left(\sqrt[3]{2}, i\sqrt{3}\right)$ is the root field of $x^3 - 2$ over $\mathbb{Q}$.

Comment