A Book of Abstract Algebra (2nd Edition)

Chapter 16, Problem 4EO

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Problem

The purpose of this exercise is to prove a property of cosets which is needed in Exercise Q. Let G be a finite abelian group, and let a be an element of G such that ord(a) is a multiple of ord(x)for every $x \in G$. Let $H = \langle a \rangle$. We will prove:

For every $x \in G$, there is some $y \in G$ such that Hx = Hy and ord(y) = ord(Hy).

This means that every coset of H contains an element y whose order is the same as the coset's order.

Let x be any element in G, and let ord (a) = t, ord (x) = s, and ord (Hx) = r.

Setting $y = xa^{-UZ}$, prove that Hx = Hy and ord(y) = r, as required.

Step-by-step solution

Step 1 of 4

Consider that G is a finite abelian group. Let $a, x \in G$ and $H = \langle a \rangle$ is a subgroup of G. Suppose that order of the elements are:

$$\operatorname{ord}(a) = t$$
,

$$\operatorname{ord}(x) = s$$
,

$$ord(Hx) = r$$
.

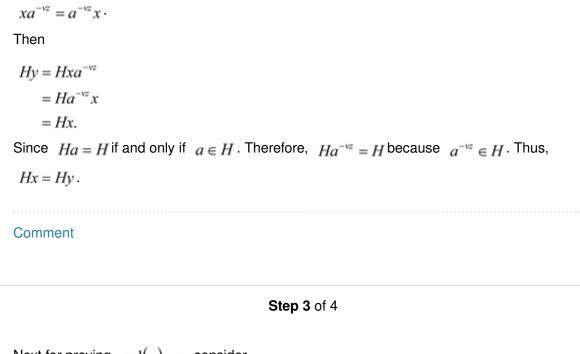
Note that r is the least positive integer such that $x^r = a^m$. Also $a^{mu} = e$, and it follows that mu = tz for some integer z. And, m = rvz.

Objective is to prove that Hx = Hy and ord(y) = r, if $y = xa^{-vz}$.

Comment

Step 2 of 4

Let $y = xa^{-vz}$. Since group G and H are abelian, therefore



Next for proving $\operatorname{ord}(y) = r$, consider

$$y'' = (xa^{-vz})^{r}$$
$$= x^{r}a^{-rvz}$$
$$= a^{m} \cdot a^{-rvz}$$

From the hypothesis, m = rvz. Substitute this value in above and get,

$$y^{r} = a^{rvz} \cdot a^{-rvz}$$
$$= a^{0}$$
$$= e.$$

Thus, $\operatorname{ord}(y) = r$.

Comment

Step 4 of 4

Hence, if $y = xa^{-yz}$ then Hx = Hy and ord(y) = r.

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