A Book of Abstract Algebra | (2nd Edition)

Chapter 30, Problem 6EA

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Problem

If *O* and *I* are any two points in the plane, consider a coordinate system such that the interval *OI* coincides with the unit interval on the *x* axis. Let $\mathbb{D}$ be the set of real numbers such that $a \in \mathbb{D}$ iff the point (*a*, 0) is constructible from {*O, I*}. Prove the following:

If *a* is a real root of any quadratic polynomial with coefficients in $\mathbb{D}$, then $a \in \mathbb{D}$. (HINT: Complete the square and use part 4.)

Step-by-step solution

Step 1 of 4

Here, objective is to prove that $a \in D$, if *a* is a real root of any quadratic polynomial with coefficients in *D.*

Comment

Step 2 of 4

Constructible point:

This point is either the end point of given unit segment or it is the intersection of two lines determined by previous constructible points.

Comment

Step 3 of 4

Consider $a \in D$ if and only if the point $(a,0)$ is constructible from $\{O,I\}$

Consider the below figure:

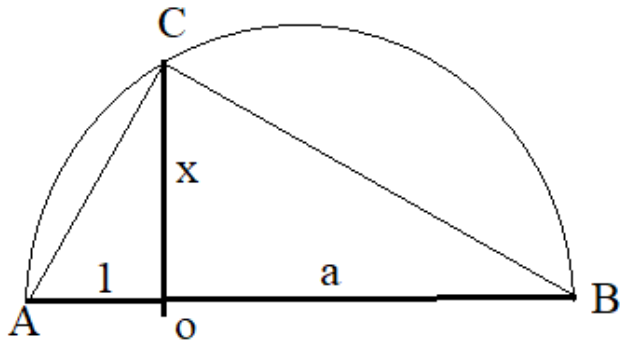

figure:construction of sqare root of a

By observing there exist two equal triangles

ΔAOB and ΔBOC

Using the property of similar triangles, we have

$$\frac{1}{x} = \frac{x}{a}$$

$$x^2 = a$$

$$x = \sqrt{a}$$

Then, the length $\sqrt{a}$ is constructed from $\{O,I\}$ which implies $\sqrt{a} \in D$.

So, D is the field of constructible numbers.

Comment

Step 4 of 4

Consider *a* is a root of the quadratic polynomial

$$x^2 + bx + c = 0$$

$$a = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}$$

$$b^2 - 4ac \geq 0$$

D is a field of closed under square roots of positive elements.

Then, $a \in D$

Hence, proved

Comment