A Book of Abstract Algebra | (2nd Edition)

Chapter 24, Problem 3EF

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Problem

Let *A* be an integral domain.

Using parts 1 and 2, explain why $A[x]/\langle x\rangle \cong A$.

Step-by-step solution

Step 1 of 8

consider an integral domain $A[x]$ and let $h: A[x] \to A$ map every polynomial to its constant coefficient.

That is $h(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0) = a_0$

Objective of the question is to prove $A[x]/\langle x\rangle \cong A$

Comment

Step 2 of 8

To prove $A[x]/\langle x\rangle \cong A$ following statements.

1) Prove *h* is a on to homomorphism.

2) Prove $\text{kernel of } h = \langle x\rangle$

3) Prove $A[x]/\langle x\rangle \cong A$

First prove *h* is a homomorphism.

Definition: A ring homomorphism *f* from a ring *R* to a ring *S* is a mapping from *R* to *S* that preserves the two operations. That is for all $a,b \in R$

$$f(a+b)=f(a)+f(b)$$
$$f(ab)=f(a)f(b)$$

Let

$$p(x)=a_n x^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a_0$$
$$q(x)=b_n x^n+b_{n-1}x^{n-1}+\ldots+b_2x^2+b_1x+b_0$$

Prove *h* is a ring homomorphism.

$$\begin{aligned} h(p(x))&=h(a_n x^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a_0)\\ &=a_0 \end{aligned}$$

$$\begin{aligned} h(q(x))&=h(b_n x^n+b_{n-1}x^{n-1}+\ldots+b_2x^2+b_1x+b_0)\\ &=b_0 \end{aligned}$$

$$\begin{aligned} h(p(x)+q(x))&=h((a_n+b_n)x^n+(a_{n-1}+b_{n-1})x^{n-1}+\ldots+(a_0+b_0))\\ &=a_0+b_0\\ &=h(p(x))+h(q(x)) \end{aligned}$$

$$\begin{aligned} h(p(x)q(x))&=h((a_n x^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a_0)(b_n x^n+b_{n-1}x^{n-1}+\ldots+b_2x^2+b_1x+b_0))\\ &=h(a_nb_nx^n+\ldots+(a_0b_2+a_1b_1+a_2b_0)x^2+(a_0b_1+a_1b_0)x+a_0b_0)\\ &=a_0b_0\\ &=h(p(x))h(q(x)) \end{aligned}$$

According to definition of ring homomorphism *h* is a ring homomorphism.

Comment

Step 3 of 8

Now show that *h* is a onto function.

Recall the definition of onto function.

Definition: *f* is a function from *A* to *B* is said to be onto function if for all $b\in B$ there exists at least one $a\in A$ such that $f(a)=b$.

Let any element $a\in A$.

Now construct a polynomial as follows

$$p(x)=a_n x^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a$$

Then $h(a_n x^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a)=a$

Therefore for all $a\in A$ there exists a polynomial in $A[x]$ such that $h(p(x))=a$.

Comment

Step 4 of 8

Now find kernel of the function *h*.

Recall the definition of kernel of a homomorphism.

Definition: Let a homomorphism *f* from *A* to *B*. A subset *K* of *A* is said to be kernel of homomorphism *f* if for every $t \in K$, $f(t) = 0$.

Now find set of elements such that $h(p(x)) = 0$

Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$.

$$\begin{aligned} h(p(x)) &= h\left(a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0\right) \\ &= a_0 \end{aligned}$$

Then $h(p(x)) = 0$ if $a_0 = 0$.

There for the kernel of this homomorphism is the set of all polynomials whose constant term is zero.

That is $K = \left\{ a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 \mid a_i \in A, a_0 = 0, i = 0, 1, \ldots, n \right\}$

Comment

Step 5 of 8

Now prove the kernel of *h* is $\square x \square$.

Let any polynomial $a(x) \in A[x]$ and consider the product $xa(x)$.

$\{xa(x) \mid \text{for all } a(x) \in A[x]\}$ is the set of all elements generated by *x* in $A[x]$.

And it is denoted as $\square x \square$.

Comment

Step 6 of 8

First prove $\square x \square \subseteq \ker h$

Then,

$$\begin{aligned} h(xa(x)) &= h(x) h(a(x)) \qquad (\text{since } h \text{ is a homomorphism}) \\ h(x) &= 0 \\ h(a(x)) &= k \end{aligned}$$

Here, $k \in A$.

Then,

$$\begin{aligned} h(xa(x)) &= 0 \times k \\ &= 0 \end{aligned}$$

Hence $xa(x) \in \ker h$

Since $xa(x) \in \ker h$ implies

$□x□ \subseteq \ker h$ ......(1)

Comment

Step 7 of 8

Now prove $\ker h \subseteq □x□$

Let a $p(x) \in \ker h$

Then $h(p(x)) = 0$

That is its constant term is zero.

Then the polynomial should be in the form of $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x$.

Take the common factor *x* outside.

$$\begin{aligned} p(x) &= x\left(a_n x^{n-1} + a_{n-1}x^{n-2} + ... + a_2 x^1 + a_1\right) \\ &= xq(x) \end{aligned}$$

Therefore $p(x)$ is the form of elements in $□x□$.

Therefore $p(x) \in □x□$

It implies $\text{kernel of } h \subseteq □x□$ ......(2)

Combining both equation (1) and (2) then $\ker h = □x□$

Comment

Step 8 of 8

Now prove $A[x]/□x□ \cong A$.

Recall the first isomorphism theorem for ring.

Theorem: *h* is a homomorphism from *R* to *S* then $R/\ker h \cong h(R)$.

h is an on to homomorphism then $h(A[x]) = A$.

And $\ker h = □x□$.

Then according to theorem $A[x]/□x□ \cong A$

Comment