A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 2EQ

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Problem

As a provisional definition, let us call a finite abelian group “decomposable” if there are elements $a_1, \ldots, a_n \in G$ such that:

(DI) For every $x \in G$, there are integers $k_1, \ldots, k_n$ such that $x = a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}$. ($D_2$)

If there are integers $l_1, \ldots, l_n$ such that

$a_1^{l_1} a_2^{l_2} \cdots a_n^{l_n} = e$ then $a_1^{l_1} = a_2^{l_2} = \cdots = a_n^{l_n} = e$.

If (D_1) and (D_2) hold, we will write $G = [a_1, a_2, \ldots, a_n]$. Assume this in parts 1 and 2.

Prove: $G \cong \langle a_1 \rangle \times G'$. Conclude that $G \cong \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times a_n$

In the remaining exercises of this set, let p be a prime number, and assume G is a finite abelian group such that the order of every element in G is some power of p. Let $a \in G$ be an element whose order is the highest possible in G. We will argue by induction to prove that G is “decomposable.” Let $H = \langle a \rangle$.

Step-by-step solution

Step 1 of 4

Assume that a finite abelian group G, of order $p^k m$, is decomposable. That is, if $a_1, \quad , a_n \in G$ and both the conditions $D1$, $D2$ holds, then $G = [a_1, a_2, \quad , a_n]$.

Objective is to prove that $G \cong \langle a_1 \rangle \times G'$ and then $G \cong \langle a_1 \rangle \times \langle a_2 \rangle \times \quad \times \langle a_n \rangle$.

Consider the following result:

If G is an internal direct product of $H_1, \quad , H_k$, then $G \cong H_1 \times \quad \times H_k$.

Comment

Step 2 of 4

To show the required result, prove that G is an internal direct product of $\langle a_1 \rangle, G'$.

Since G is an abelian group, so all subgroups of G will be normal.

Let $x \in G$. Then

$x = a_1^{k_1} a_2^{k_2} \quad a_n^{k_n}$,

for some integers $k_1, k_2, \quad, k_n$, and $a_1^{k_1} \in \langle a_1 \rangle, a_2^{k_2} \quad a_n^{k_n} \in G'$.

Now, the remaining work is to prove that groups $\langle a_1 \rangle, G'$ are distinct. If $x \in \langle a_1 \rangle \quad G'$, then by the $D2$ from the decomposable definition will lead to a contradiction.

Thus, $G \cong \langle a_1 \rangle \times G'$.

Comment

Step 3 of 4

Now use induction to prove that $G \cong \langle a_1 \rangle \times \langle a_2 \rangle \times \quad \times \langle a_n \rangle$.

Define $G^{(m)} = \{a_{m+1}^{l_{m+1}} a_{m+2}^{l_{m+2}} \quad a_n^{l_n} \mid l_{m+1}, ..., l_n \in Z\}$ for $m \in \{0, 1, ..., n-1\}$ such that $G = G^{(0)}, G' = G^{(1)}$. Then one have $G \cong \langle a_1 \rangle \times G^{(1)}$. Next, assume, as an induction step, that $G \cong \langle a_1 \rangle \times \quad \times \langle a_{m-1} \rangle \times G^{(m-1)}$.

If $G^{(m-1)} \cong \langle a_m \rangle \times G^{(m)}$, then it implies that $G \cong \langle a_1 \rangle \times \quad \times \langle a_m \rangle \times G^{(m)}$. Note that this isomorphism holds for all $m \in \{0, 1, ..., n-1\}$, thus, one can conclude that

$G \cong \langle a_1 \rangle \times \quad \times \langle a_{n-1} \rangle \times G^{(n-1)}$,

where $G^{(n-1)} = \langle a_n \rangle$.

Comment

Step 4 of 4

Hence, $G \cong \langle a_1 \rangle \times \langle a_2 \rangle \times \quad \times \langle a_n \rangle$.

Comment