A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 3EA

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Problem

In each of the following, use the fundamental homomorphism theorem to prove that the two given groups are isomorphic. Then display their tables.

$\mathbb{Z}_2$ and $S_3/\{\varepsilon, \beta, \delta\}$..

Step-by-step solution

Step 1 of 4

Consider the two groups Z_2 and $S_3/\{\varepsilon, \beta, \delta\}$. Objective is to prove that these two groups are isomorphic by using the fundamental homomorphism theorem.

According to the fundamental homomorphism theorem, if $f : G \to H$ is a homomorphism of *G* onto *H*, with kernel *K* then

$H \cong G/K$.

Comment

Step 2 of 4

Consider the function $f : S_3 \to Z_2$ given by

$$f = \begin{pmatrix} \varepsilon & \alpha & \beta & \gamma & \delta & \kappa \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

To show that this mapping *f* is homomorphism, one must show that

$f(a+b) = f(a) + f(b)$

for all choices of *a* and *b* in S_3.

Since $f(\varepsilon) = 0$, it can be seen that

$$\begin{aligned} f(\varepsilon \ \ x) &= f(x) \\ &= f(0)+f(x) \\ &= f(x)+f(0) \\ &= f(x \ \ \varepsilon). \end{aligned}$$

Use the same steps for the compositions of remaining elements and consider the following table:

$/+$	$\alpha/1$	$\beta/0$	$\gamma/1$	$\delta/0$	$\kappa/1$
$\alpha/1$	$\varepsilon/0$	$\gamma/1$	$\beta/0$	$\kappa/1$	$\delta/0$
$\beta/0$	$\kappa/1$	$\delta/0$	$\alpha/1$	$\varepsilon/0$	$\gamma/1$
$\gamma/1$	$\delta/0$	$\kappa/1$	$\varepsilon/0$	$\alpha/1$	$\beta/0$
$\delta/0$	$\gamma/1$	$\varepsilon/0$	$\kappa/1$	$\beta/0$	$\alpha/1$b
$\kappa/1$	$\beta/0$	$\alpha/1$	$\delta/0$	$\gamma/1$	$\varepsilon/0$

Here $\alpha/1$ represents that the image of $\alpha \in S_3$ is 1 in Z_2, and so on.

Comment

Step 3 of 4

In each case,

$$f(a \ \ b) \cong f(a)+f(b).$$

Therefore, *f* preserves composition and is a homomorphism. Since each element of Z_2 has the pre-image, so *f* is onto.

By the definition of *f*, only elements $\varepsilon, \beta, \delta$ of S_3 maps to identity. Therefore, the map *f* is homomorphism from S_3 onto Z_2 with $\ker f = \{\varepsilon, \beta, \delta\}$.

The addition table of Z_2 will be:

$+$	0	1
0	0	1
1	1	0

Comment

Step 4 of 4

Hence, by the fundamental homomorphism theorem it concludes that

$Z_2 \cong S_3 / \{\varepsilon, \beta, \delta\}$.

Comment