A Book of Abstract Algebra | (2nd Edition)

Problem

Suppose $a(x) \in F[x]$, and K is an extension of F. An element $c \in K$ is called a multiple root of $a(x)$ if $(x - c)^m | a(x)$ for some $m > 1$. It is often important to know if all the roots of a polynomial are different, or not.

We now consider a method for determining whether an arbitrary polynomial $a(x) \in F[x]$ has multiple roots in any extension of F.

Let K be any field containing all the roots of $a(x)$. Suppose $a(x)$ has a multiple root c.

Explain why $a'(x)$ is a sum of terms of the form

$$(x - c_1)\cdots(x - c_{i-1})(x - c_{i+1})\cdots(x - c_n)$$

Step-by-step solution

Step 1 of 3

Consider that K is any field that contains all the roots of polynomial $a(x) = a_0 + a_1 x + \cdots + a_n x^n$. Assume that $a(x)$ has no multiple roots. Then polynomial $a(x)$ can be factored as

$$a(x) = (x - c_1)\cdots(x - c_n)$$

where $c_1, \ldots, c_n$ are all distinct.

Objective is to explain the reason that $a'(x)$ is a sum of terms of the following form:

$$(x - c_1)\cdots(x - c_{i-1})(x - c_{i+1})\cdots(x - c_n).$$

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Step 2 of 3

To prove this result one can use induction on n. Let $n = 2$. That is, $a(x)$ has only two roots. Then

$$a(x) = (x - c_1)(x - c_2),$$
$$a'(x) = (x - c_1) + (x - c_2).$$

Thus, result is true for $n = 2$. Here, each time, differentiation of one term takes place.

Now, assume that result holds for $n = k$ roots. That is, $a'(x)$ is a sum of terms of the following form:

$$(x - c_1)\cdots(x - c_{i-1})(x - c_{i+1})\cdots(x - c_k).$$

Next, show the result for $n = k + 1$ roots. The

$$a(x) = (x - c_1)\cdots(x - c_k)(x - c_{k+1}).$$

Let $(x - c_1)\cdots(x - c_k) = A$. Since result hold for two roots, also by induction hypothesis it implies that $a'(x)$ is a sum of the following terms:

$$(x - c_1)\cdots(x - c_{i-1})(x - c_{i+1})\cdots(x - c_n).$$

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Step 3 of 3

Hence, by mathematical induction result holds for all n.

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