A Book of Abstract Algebra | (2nd Edition)

Chapter 28, Problem 5EB

Bookmark

Show all steps: ON

Problem

Prove that the set of all even functions [that is, functions *f* such that *f*(*x*) = *f*(–*x*)] is a subspace of $\mathcal{F}(\mathbb{R})$. Is the same true for the set of all the odd functions [that is, functions *f* such that *f*(–*x*) = –*f*(*x*)]?

Step-by-step solution

Step 1 of 3

It is already shown that $f(\mathbb{R})$ represents a vector space as it satisfies all conditions for vector space.

Given subset for $f(\mathbb{R})$ is set of all functions which are even functions.

Or given condition for subspace is

$$\{f(-x)=f(x)\}$$

Comment

Step 2 of 3

A 2 step check is needed to determine if given subset is a subspace. First determine if any linear combination of functions in subset is closed under given operation. Second, determine if 0 satisfies given conditions.

STEP 1: For any 2 functions *f* and *g*,

$$f(-x)=f(x) \qquad (1)$$
$$g(-x)=g(x) \qquad (2)$$

Combining above 2 equations, $s(1)+t(2)$ gives

$$sf(-x)+tg(-x)=sf(x)+tg(x)$$

As functions are vector space in themselves, any constant multiple of function is also a function. Also sum of 2 functions is also a function. Thus,

$$sf(-x)+tg(-x)=sf(x)+tg(x)$$
$$\Rightarrow f'(-x)+g'(-x)=f'(x)+g'(x)$$
$$\Rightarrow F(-x)=F(x)$$

Thus linear combination of 2 functions in subset lies in subset.

STEP 2: Check if 0 function (which is 0 everywhere) satisfies given condition,

$$f_0=0 \qquad \forall x$$
$$\Rightarrow f_0(-x)=f_0(x)=0$$

$$\boxed{\text{Hence given set of even functions represents a subspace}}$$

Comment

Step 3 of 3

Now check same steps for odd function which is represented by

$$\{f(-x)=-f(x)\}$$

STEP 1: For any 2 functions *f* and *g*,

$$f(-x)=-f(x) \qquad (1)$$
$$g(-x)=-g(x) \qquad (2)$$

Combining above 2 equations, $s(1)+t(2)$ gives

$$sf(-x)+tg(-x)=-sf(x)-tg(x)$$

As functions are vector space in themselves, any constant multiple of function is also a function. Also sum of 2 functions is also a function. Thus,

$$sf(-x)+tg(-x)=-(sf(x)+tg(x))$$
$$\Rightarrow f'(-x)+g'(-x)=-(f'(x)+g'(x))$$
$$\Rightarrow F(-x)=-F(x)$$

Thus linear combination of 2 functions in subset lies in subset.

STEP 2: Check if 0 function (which is 0 everywhere) satisfies given condition,

$$f_0=0 \qquad \forall x$$
$$\Rightarrow f_0(-x)=-f_0(x)=0$$

$$\boxed{\text{Hence given set of odd functions represents a subspace}}$$

Comment