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Chapter 29, Problem 1EC | Bookmark | Show all steps: ON

Problem

By the proof of the basic theorem of field extensions, if $p(x)$ is an irreducible polynomial of degree n in $F[x]$, then $F[x]$, then $F[x]/\langle p(x)\rangle \cong F(c)$ where c is a root of $p(x)$. By Theorem 1 in this chapter, $F(c)$ is of degree η over F. Using the paragraph preceding Theorem 1:

Prove that every element of $F(c)$ can be written *uniquely* as $a_0 + a_1c + \cdots + a_{n-1}c^{n-1}$, for some $a_0, \ldots, a_{n-1} \in F$.

Step-by-step solution

Step 1 of 3

Consider that F is a field. Objective is to prove that every element of $F(c)$ can be written uniquely as $a_0 + a_1c + \cdots + a_{n-1}c^{n-1}$, for some $a_0, \ldots, a_{n-1} \in F$.

Note that, $F(c)$ denotes the smallest field which contains F and c, where c is the root of some polynomial of $F[x]$. Since c is algebraic over F, therefore $F(c)$ consists of all the elements of the form $a(c)$, for all $a(x) \in F[x]$.

Comment

Step 2 of 3

Since $F(c)$ is an extension of F, so $F(c)$ can be regarded as a vector space over F. According to the question details, $F(c)$ is of degree n over F. So, let $p(x)$ is the minimal polynomial of degree n of c over F. Then, the n elements

$$1, c, c^2, \ldots, c^{n-1}$$

are linearly independent and span $F(c)$. It shows that the set of n vectors $\{1, c, c^2, \ldots, c^{n-1}\}$ is a basis of $F(c)$.

By the basis property, each element of $F(c)$ can be written uniquely as a linear combination of basis elements, $1, c, c^2, \ldots, c^{n-1}$.

Comment

Step 3 of 3

Thus, every element of $F(c)$ can be written uniquely as $a_0 + a_1 c + \cdots + a_{n-1} c^{n-1}$, for some $a_0, \ldots, a_{n-1} \in F$.

Comment