A Book of Abstract Algebra | (2nd Edition)

Problem

Let $F \subseteq K$ and $a, b \in K$. We have seen on page 295 that if a and b are algebraic over F, then $F(a, b)$ is a finite extension of F.

Use the above to prove part.

$\mathbb{Q}_1(c)$ is a finite extension of $\mathbb{Q}_1$ hence a finite extension of $\mathbb{Q}$. (Why?)

Step-by-step solution

Step 1 of 3

Consider a field Q and a field $\boldsymbol{A}$ of set of all algebraic numbers. Let

$a(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbf{A}[x]$,

and c be any root of $a(x)$. Let $Q(a_0, a_1, \ldots, a_n) = Q_1$. Since $a(x) \in Q_1[x]$, c is algebraic over Q_1. Objective is to prove that $Q_1(c)$ is a finite extension of Q_1 as well as finite extension of Q.

First show that Q_1 is a finite extension of Q. Since $a(x) \in \mathbf{A}[x]$, therefore all the coefficients $a_0, a_1, \ldots, a_n$ are algebraic over Q. Let Q_1 be the smallest field containing Q and $a_0, a_1, \ldots, a_n$. The formation of Q_1 can be done step by step by adjoining one a_i at a time.

Comment

Step 2 of 3

Note that, the degree of $F(c)$ over *F* is equal to the degree of the minimal polynomial of *c* over *F*.

If $a_0, a_1, \ldots, a_n$ are algebraic over *Q*, then by this result, each extension in

$$Q \subseteq Q(a_0) \subseteq Q(a_0, a_1) \subseteq Q(a_0, a_1, a_2) \subseteq \cdots \subseteq Q(a_0, a_1, \ldots, a_n)$$

is a finite extension. Again by the theorem of finite extension, $Q(a_0, a_1)$ is a finite extension of *Q*. Also, $Q(a_0, a_1, a_2)$ is a finite extension of *Q*, and so on.

Thus, Q_1 is a finite extension of *Q*.

Comment

Step 3 of 3

Since *c* is algebraic over Q_1, then an extension $Q_1(c)$ can be formed by adjoining a single element to Q_1. This is known as simple extension. Since simple extensions are always finite, therefore $Q_1(c)$ is a finite extension of Q_1.

Since $Q_1(c)$ is a finite extension of Q_1 and Q_1 is a finite extension of *Q*, therefore $Q_1(c)$ is also a finite extension of *Q*.

Comment