A Book of Abstract Algebra | (2nd Edition)

Chapter 32, Problem 6EG

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Problem

In the next three parts, let ω be a primitive pth root of unity, where p is a prime.

Use part 5 to prove that $Gal(\mathbb{Q}(\omega):\mathbb{Q})$ is an abelian group.

Step-by-step solution

Step 1 of 2

Consider a primitive pth root of unity ω, where p is a prime. The objective is to prove that $Gal(\mathbb{Q}(w):\mathbb{Q})$ is an abelian group.

Comment

Step 2 of 2

Suppose that $\alpha,\beta \in Gal(\mathbb{Q}(\omega):\mathbb{Q})$.

Then $\alpha(\omega)=\omega^i$ and $\beta(\omega)=\omega^j$ for $1 \le i, j \le p-1$.

$$(\alpha \circ \beta)(\omega)=\alpha(\beta(w))$$
$$=\alpha(w^j)$$

$=(\alpha(\omega))^j$, since α is a homomorphism

$$=(\omega^i)^j$$
$$=w^{ij}$$

and similarly

$$(\beta \circ \alpha)(\omega)=\beta(\alpha(w))$$
$$=\beta(w^i)$$

$= (\beta(\omega))^i$, since β is a homomorphism

$= (\omega^j)^i$

$= w^{ij}$

This shows that $Gal(\mathbb{Q}(w):\mathbb{Q})$ is an abelian group.

Comment