A Book of Abstract Algebra | (2nd Edition)

Chapter 28, Problem 4EG

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Problem

Let *T* and *U* be subspaces of *V*. The *sum* of *T* and *U*, denoted by *T* + *U*, is the set of all vectors **a** + **b**, where **a** $\in$ *T* and **b** $\in$ *U*.

If *T* and *U* are arbitrary subspaces of *V*, prove that

dim $(T + U) = \dim T + \dim U - \dim (T \cap U)$

Step-by-step solution

Step 1 of 2

V is finite dimensional vector space. Any subspace of *V* will also be finite dimensional. Let there be 2 subspaces *T* and *U*.

Consider intersection of these 2 subspaces. Assume there are some elements in this intersection. Since *T* and *U* are subspaces, it can be easily shown and have been shown in previous problems that their intersection is also a subspace.Let *k* be dimension of this intersection subspace.

Then basis of intersection subspace or $T \cap U$ is

$$\text{Basis for } T \cap U = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\}$$

Observe that this intersection is either equal to or less than both of subspaces. Also $T \cap U$ is wholly contained in both *U* and *V*. Thus this basis can be extended to that of *T* and *U*.

Extended basis for $T = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k, \mathbf{t}_1, \mathbf{t}_2, \ldots \mathbf{t}_p\}$

Similarly, extended basis for $U = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k, \mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_q\}$

Thus,

$$\dim T = k + p$$

$$\dim U = k + q$$

Comment

Step 2 of 2

Now, $T+U$ also forms a subspace where each vector of $T+U$ can be expressed as sum of vectors from *T* and *U*.

Basis of $T+U$ is just basis of *U* and *V* with repeated elements removed.

Or, basis of $T+U = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k, \mathbf{t}_1, \mathbf{t}_2, \ldots \mathbf{t}_p, \mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_q\}$

Thus,

$$\dim T+U = k+p+q$$

Now it can be easily seen that, $\dim(T+U)+\dim(T\cap U)=\dim T+\dim U$. Rewriting this to desired result, $\boxed{\dim(T+U)=\dim T+\dim U-\dim(T\cap U)}$.

Comment