A Book of Abstract Algebra | (2nd Edition)

Chapter 28, Problem 6EB

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Problem

Prove that the set of all polynomials of degree ≤*n* is a subspace of $\mathcal{P}\ell$

Step-by-step solution

Step 1 of 2

It is already shown that $P(\mathbb{R})$ represents a vector space as it satisfies all conditions for vector space.

Given subset for $P(\mathbb{R})$ is set of all polynomials which are of degree less than *n*.

Or given condition for subspace is

$$\{p \mid \deg(p) \le n\}$$

Comment

Step 2 of 2

This polynomial can be represented by

$$p(x)=a_0+a_1x+a_2x^2+a_3x^3+\ldots+a_nx^n$$

Where, any *a* can be zero.

A 2 step check is needed to determine if given subset is a subspace. First determine if any linear combination of functions in subset is closed under given operation. Second, determine if 0 satisfies given conditions.

STEP 1: For any 2 polynomials *f* and *g*,

$$\deg(f(x))\le n$$
$$\deg(g(x))\le n$$

Or,

$$f(x)=a_0+a_1x+a_2x^2+a_3x^3+\ldots+a_nx^n \qquad (1)$$
$$g(x)=b_0+b_1x+b_2x^2+b_3x^3+\ldots+b_nx^n \qquad (2)$$

Combining above 2 equations, $s(1)+t(2)$ gives

$$sf(x)+tg(x)=sa_0+sa_1x+sa_2x^2+\ldots+sa_nx^n+tb_0+tb_1x+tb_2x^2+\ldots+tb_nx^n$$

As polynomials are vector space in themselves, any constant multiple of polynomial is also a polynomial. Also sum of 2 polynomial is also a polynomial. Thus,

$$sf(x)+tg(x)=sa_0+sa_1x+sa_2x^2+\ldots+sa_nx^n+tb_0+tb_1x+tb_2x^2+\ldots+tb_nx^n$$
$$\Rightarrow sf(x)+tg(x)=(sa_0+tb_0)+(sa_1+tb_1)x+(sa_2+tb_2)x^2+\ldots+(sa_n+tb_n)x^n$$
$$\Rightarrow sf(x)+tg(x)=r(x)$$

Thus linear combination of 2 polynomials in subset lies in the subset.

STEP 2: Check if 0 function (which is 0 everywhere) satisfies given condition,

$$p_0(x)=0+0x+0x^2+\ldots+0x^n$$

Or 0 polynomial lies in subset.

$$\boxed{\text{Hence given set represents a subspace}}$$

Comment