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Chapter 27, Problem 5EJ　Bookmark　Show all steps: ON

Problem

Suppose $a(x) \in F[x]$, and *K* is an extension of *F*. An element $c \in K$ is called a multiple root of $a(x)$ if $(x - c)^m | a(x)$ for some $m > 1$. It is often important to know if all the roots of a polynomial are different, or not.

We now consider a method for determining whether an arbitrary polynomial $a(x) \in F[x]$ has multiple roots in any extension of *F*.

Let *K* be any field containing all the roots of $a(x)$. Suppose $a(x)$ has a multiple root *c*.

Using part 4, explain why none of the roots $c_1, \ldots, c_n$ of $a(x)$ are roots of $a'(x)$.

Step-by-step solution

Step 1 of 3

Consider that *K* is any field that contains all the roots of polynomial $a(x) = a_0 + a_1x + \cdots + a_nx^n$. Assume that $a(x)$ has no multiple roots. Then polynomial $a(x)$ can be factored as

$$a(x) = (x - c_1)\cdots(x - c_n)$$

where $c_1, \ldots, c_n$ are all distinct.

Objective is to prove that the roots $c_1, \ldots, c_n$ of $a(x)$ will not be the roots of $a'(x)$.

Comment

Step 2 of 3

The derivative of polynomial $a(x)$ will be the sum of terms of the following form:

$$(x - c_1)\cdots(x - c_{i-1})(x - c_{i+1})\cdots(x - c_n).$$

Here, each time, differentiation of one term takes place.

Observe that, in $a'(x)$, the factor $(x - c_i)$ is not present at the $i - \mathrm{th}$ term. So, if one substitute c_i in $a'(x)$, the derivative will not get vanish. This shows that no c_i is a root of $a'(x)$.

Comment

Step 3 of 3

Hence, none of the roots $c_1, \ldots, c_n$ of $a(x)$ are the roots of $a'(x)$.

Comment