A Book of Abstract Algebra | (2nd Edition)

Problem

Let G be an abelian group. Let $H = \{x^2 : x \in G\}$ and $K = \{x \in G : x^2 = e\}$.

Prove that $f(x) = x^2$ is a homomorphism of G *onto* H.

Step-by-step solution

Step 1 of 3

Suppose that G be an abelian group. Consider the following sets:

$$H = \{x^2 : x \in G\},$$
$$K = \{x \in G : x^2 = e\}$$

Objective is to prove that $f(x) = x^2$ is a homomorphism of G onto H.

If G and H are two groups, a homomorphism from G to H is a function $f : G \to H$ such that for any two elements a, b in G,

$$f(ab) = f(a)f(b).$$

Comment

Step 2 of 3

Let $x, y \in G$. Then

$$\begin{aligned} f(xy) &= (xy)^2 \\ &= xy \cdot xy \end{aligned}$$

Since G is an abelian group therefore for all $x, y \in G$, one have

$$xy = yx.$$

Use this condition above and get,

$$
\begin{aligned}
f(xy) &= x(y \cdot x)y \\
&= x(xy)y \\
&= x^2 y^2 \\
&= f(x)f(y).
\end{aligned}
$$

Since $f(xy) = f(x)f(y)$, therefore *f* is a homomorphism.

The function *f* is clearly onto because for all $y = x^2 \in H$ there exists $x \in G$ such that $f(x) = y$.

Comment

Step 3 of 3

Thus, $f(x) = x^2$ is a homomorphism of *G* onto *H*.

Comment