A Book of Abstract Algebra | (2nd Edition)

Problem

Prove that $\mathcal{P}\ell_n$, as defined on page 284, is a vector space over $\mathbb{R}$.

Step-by-step solution

Step 1 of 2

There are 10 conditions which any vector space must satisfy. These are

1. For $u \in V, v \in V \Rightarrow u+v \in V$
2. For $u \in V, v \in V \Rightarrow u+v = v+u$
3. For $u \in V, v \in V, w \in V \Rightarrow (u+v)+w = u+(v+w)$
4. There exists $0 \in V$, such that $0+v=v$ for all $v \in V$
5. For all $u \in V$, there exists $x \in V$ such that $u+x=0$
6. For $c \in R, v \in V \Rightarrow cv \in V$
7. For $c \in R, u \in V, v \in V \Rightarrow c(u+v) = cu+cv$
8. For $c,d \in R, u \in V, v \in V \Rightarrow (c+d)u = cu+du$
9. For $c \in R, d \in R, v \in V \Rightarrow c(dv) = (cd)v$
10. There exists $1 \in R, v \in V \Rightarrow 1 \cdot v = v$

Comment

Step 2 of 2

$P(\mathbb{R})$ is polynomial of degree *n* .This can be represented by $a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n$

where all a_i are real numbers. These a_i can be thought of as *n* components of a vector. Addition of 2 polynomials are done component wise. Multiplication of a polynomial with a constant implies that all constants or a_i are multiplied with that constant. It is also known that normal addition and multiplication is commutative.

Let

$$v=(v_1,v_2,v_3,\ldots,v_n)$$

$$u=(u_1,u_2,u_3,\ldots,u_n)$$

$$\Rightarrow -u=(-u_1,-u_2,-u_3,\ldots,-u_n)$$

Then check aforementioned 8 properties or condition for this space.

1. $$\begin{aligned}&u+v=(u_1,u_2,u_3,\ldots,u_n)+(v_1,v_2,v_3,\ldots,v_n)\\&\Rightarrow u+v=u_1+u_2x+u_3x^2+\ldots+u_nx_n^{\,n}+v_1+v_2x+v_3x^2+\ldots+v_nx_n^{\,n}\\&\Rightarrow u+v=(u_1+v_1)+(u_2+v_2)x+(u_3+v_3)x^2+\ldots+(u_n+v_n)x_n^{\,n}\\&\Rightarrow u+v=(u_1+v_1,u_2+v_2,u_3+v_3,\ldots,u_n+v_n)\in V\end{aligned}$$

2. $$\begin{aligned}&u+v=(u_1+v_1,u_2+v_2,u_3+v_3,\ldots,u_n+v_n)\\&\Rightarrow u+v=u_1+u_2x+u_3x^2+\ldots+u_nx_n^{\,n}+v_1+v_2x+v_3x^2+\ldots+v_nx_n^{\,n}\\&\Rightarrow u+v=(u_1+v_1)+(u_2+v_2)x+(u_3+v_3)x^2+\ldots+(u_n+v_n)x_n^{\,n}\\&v+u=v_1+v_2x+v_3x^2+\ldots+v_nx_n^{\,n}+u_1+u_2x+u_3x^2+\ldots+u_nx_n^{\,n}\\&\Rightarrow v+u=(v_1+u_1)+(v_2+u_2)x+(v_3+u_3)x^2+\ldots+(v_n+u_n)x_n^{\,n}\\&\text{Or, } u+v=v+u\end{aligned}$$

3. $$\begin{aligned}&(u+v)+w=(u_1+v_1,u_2+v_2,u_3+v_3,\ldots,u_n+v_n)+(w_1,w_2,w_3,\ldots,w_n)\\&\Rightarrow(u+v)+w=(u_1+v_1+w_1,u_2+v_2+w_2,u_3+v_3+w_3,\ldots,u_n+v_n+w_n)\\&u+(v+w)=(u_1,u_2,u_3,\ldots,u_n)+(v_1+w_1,v_2+w_2,v_3+w_3,\ldots,v_n+w_n)\\&\Rightarrow u+(v+w)=(u_1+v_1+w_1,u_2+v_2+w_2,u_3+v_3+w_3,\ldots,u_n+v_n+w_n)\end{aligned}$$

4. $u+0=(u_1+0,u_2+0,u_3+0,\ldots,u_n+)=(u_1,u_2,u_3,\ldots,u_n)=u$

5. $u+(-u)=(u_1+(-u_1),u_2+(-u_2),u_3+(-u_3),\ldots,u_n+(-u_n))=(0,0,0,\ldots,0)=0$

6. $cv=c(v_1,v_2,v_3,\ldots,v_n)=(cv_1,cv_2,cv_3,\ldots,cv_n)\in P(\mathbb{R})$

7. $$\begin{aligned}&c(u+v)=c(u_1+v_1,u_2+v_2,u_3+v_3,\ldots,u_n+v_n)=\\&\Rightarrow c(u+v)=(cv_1+cu_1,cv_2+cu_2,cv_3+cu_3,\ldots,cv_n+cu_n)\\&cu+cv=c(u_1,u_2,u_3,\ldots,u_n)+c(v_1,v_2,v_3,\ldots,v_n)\\&\Rightarrow cu+cv=(cv_1+cu_1,cv_2+cu_2,cv_3+cu_3,\ldots,cv_n+cu_n)\\&\text{Or, } c(u+v)=cu+cv\end{aligned}$$

8. $$\begin{aligned}&(c+d)u=(c+d)(u_1,u_2,u_3,\ldots,u_n)=\\&\Rightarrow(c+d)u=(c+d)u_1+(c+d)u_2+(c+d)u_3+\ldots+(c+d)u_n\\&cu+du=c(u_1,u_2,u_3,\ldots,u_n)+d(u_1,u_2,u_3,\ldots,u_n)\\&\Rightarrow cu+du=cu_1+cu_2+cu_3+\ldots+cu_n+du_1+du_2+du_3+\ldots+du_n\\&\Rightarrow cu+du=(c+d)u_1+(c+d)u_2+(c+d)u_3+\ldots+(c+d)u_n\\&\text{Or,}(c+d)u=cu+du\end{aligned}$$

$c(dv) = c(d(v_1, v_2, v_3, ..., v_n)) = c(dv_1, dv_2, dv_3, ..., dv_n) = (cdv_1, cdv_2, cdv_3, ..., cdv_n)$

9. $(cd)v = cd(v_1, v_2, v_3, ..., v_n) = (cdv_1, cdv_2, cdv_3, ..., cdv_n)$

   $\Rightarrow c(dv) = (cd)v$

   For, $c = 1$

   $cu = 1 \cdot (u_1, u_2, u_3, ..., u_n)$

10. $\Rightarrow cu = 1 \cdot (u_1 + u_2 x + u_3 x^2 + ... + u_n x_n^{\,n})$

    $\Rightarrow cu = u_1 + u_2 x + u_3 x^2 + ... + u_n x_n^{\,n}$

    $\Rightarrow cu = u$

$\boxed{\text{Hence } P(\mathbb{R}) \text{ satisfies all conditions for vector space and is a vector space}}$

Comment