Contents

1 Signature of a Permutation

Define the signature $\operatorname{sgn}(\sigma)$ to be

$$\operatorname{sgn}(\sigma) = \prod_{i<j} \frac{\sigma(i) - \sigma(j)}{i - j}$$

1.1 $\operatorname{sgn}(\sigma) = \pm 1 \quad \forall \sigma \in S(n)$

By swapping the arbitrary symbols i, j we see

$$\begin{aligned}
\prod_{i<j} \frac{\sigma(i) - \sigma(j)}{i - j} &= \prod_{j<i} \frac{\sigma(j) - \sigma(i)}{j - i} \\
&= \prod_{j<i} \frac{\sigma(j) - \sigma(i)}{j - i} \\
&= \prod_{j<i} \frac{\sigma(i) - \sigma(j)}{i - j} && \text{multiply prev line by (-1/-1)} \\
\Rightarrow (\operatorname{sgn}(\sigma))^2 &= \prod_{i<j} \frac{\sigma(i) - \sigma(j)}{i - j} \prod_{j<i} \frac{\sigma(i) - \sigma(j)}{i - j} \\
&= \prod_{i \neq j} \frac{\sigma(i) - \sigma(j)}{i - j}
\end{aligned}$$

Expanding this out gives us all possible combos i, j, so $\operatorname{sgn}(\sigma)^2 = 1$.

1.2 $\operatorname{sgn}(\tau\sigma) = \operatorname{sgn}(\tau)\operatorname{sgn}(\sigma)$

Let $N(\sigma) = \{(i, j) \mid i < j, \sigma(i) > \sigma(j)\}$, and $n(\sigma) = |N(\sigma)|$. Thus $n(\sigma)$ counts the number of inversions in the set $D = \{(i, j) \mid i < j\}$. By the proposition above,

$$\operatorname{sgn}(\sigma) = (-1)^{n(\sigma)}$$

Let $\sigma D = \{(\sigma(i), \sigma(j)) \mid i < j\}$, then for all $k < l$, either (k, l) or $(l, k) \in \sigma D$.

Now apply $\tau\sigma D$ which contains either $(\tau k, \tau l)$ or $(\tau l, \tau k)$. Thus τ inverts $n(\tau)$ pairs, and so $D \to \sigma D \to \tau\sigma D$ has inverted $n(\sigma) + n(\tau)$ pairs.

But $D \to (\tau\sigma)D$ has inverted $n(\tau\sigma)$ pairs.

We also see $(i, j) \in N(\tau\sigma) \Leftrightarrow (i, j) \in N(\sigma)$ or $(\sigma(i), \sigma(j)) \in N(\tau)$. And there is no pair $(i, j) \in N(\tau\sigma) : (i, j) \in N(\sigma)$ and $(\sigma(i), \sigma(j)) \in N(\tau)$ so it follows

$$n(\tau\sigma) = n(\tau) + n(\sigma)$$