A Book of Abstract Algebra | (2nd Edition)

Problem

List the subgroups of **G**. (By Lagrange's theorem, any proper subgroup of **G** has either two or four elements.)

Step-by-step solution

Step 1 of 2

The objective is to find the subgroups of $G = Gal\left(\mathbb{Q}\left(\sqrt{2},\sqrt{3},\sqrt{5}\right):\mathbb{Q}\right)$.

Comment

Step 2 of 2

The extension $K = \mathbb{Q}\left(\sqrt{2},\sqrt{3},\sqrt{5}\right)$ is the root field of the polynomial

$f(x) = \left(x^2-2\right)\left(x^2-3\right)\left(x^2-5\right)$ over $\mathbb{Q}$.

Moreover, $\left\{1,\sqrt{2},\sqrt{3},\sqrt{5},\sqrt{6},\sqrt{10},\sqrt{15},\sqrt{30}\right\}$ is a $\mathbb{Q}-$basis for K.

Thus, $\left[K:\mathbb{Q}\right]=8$. So, if $G = Gal\left(K:\mathbb{Q}\right)$ then $|G|=8$.

Let H be a subgroup of G.

By Lagrange's theorem, $|H|$ divides 8. So, there are four cases.

Case I: $|H|=1$, then clearly $H=\{id\}$.

Case II: $|H|=2$.

Then H contain the identity and an element of order 2, so it can be any of the following 7 groups:

$\{id,\sigma_2\}$, $\{id,\sigma_3\}$, $\{id,\sigma_5\}$, $\{id,\sigma_2\sigma_3\}$, $\{id,\sigma_2\sigma_5\}$, $\{id,\sigma_3\sigma_5\}$, $\{id,\sigma_2\sigma_3\sigma_5\}$.

Case III: $|H|=4$.

Then H contain the identity ，two distinct elements of order 2 ，and their product ，so it can be any of the following 7 groups:

$\{id,\sigma_2,\sigma_3,\sigma_2\sigma_3\}$, $\{id,\sigma_2,\sigma_5,\sigma_2\sigma_5\}$, $\{id,\sigma_3,\sigma_5,\sigma_3\sigma_5\}$, $\{id,\sigma_2,\sigma_3\sigma_5,\sigma_2\sigma_3\sigma_5\}$ ，

$\{id,\sigma_3,\sigma_2\sigma_5,\sigma_2\sigma_3\sigma_5\}$, , $\{id,\sigma_5,\sigma_2\sigma_3,\sigma_2\sigma_3\sigma_5\}$, $\{id,\sigma_2\sigma_3,\sigma_3\sigma_5,\sigma_2\sigma_5\}$.

Case IV: $|H|=8$ **.**

Then $H=G$.

Comment