A Book of Abstract Algebra | (2nd Edition)

Problem

Throughout this set of questions, let *K* be a root field over *F*, let **G** = *Gal*(*K* : *F*), and let *I* be any intermediate field. Prove the following:

Let *I* be a normal extension of *F* (that is, a root field of some polynomial over *F*). If **G** is abelian, then *Gal*(*K* : *I*) and *Gal*(*I* : *F*) are abelian. (HINT: Use Theorem 4.)

Step-by-step solution

Step 1 of 2

Consider a root field K over F, let $G = Gal(K:F)$, and let I be any intermediate field which is a normal extension of F. The objective is to prove that if G is abelian, then $Gal(K:I)$ and $Gal(I:F)$ are abelian.

Comment

Step 2 of 2

Since $G(K:I)$ is a normal subgroup of $G(K:F)$ and $G(K:F)$ is abelian, $G(K:I)$ is abelian as a subgroup of an abelian group is abelian.

Also, $G(I:F) \cong \dfrac{G(K:F)}{G(K:I)}$ and $G(K:F)$ is abelian, $G(K:I)$ is abelian as a factor group of an abelian group, where multiplication is done by choosing representative, must be abelian.

This proves that if G is abelian, then $Gal(K:I)$ and $Gal(I:F)$ are abelian.

Comment