A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 3EH

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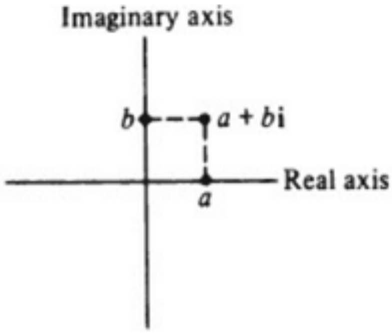
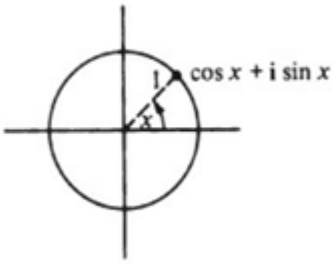
Problem

Every complex number $a + b\mathbf{i}$ may be represented as a point in the complex plane.





The *unit circle* in the complex plane consists of all the complex numbers whose distance from the origin is 1; thus, clearly, the unit circle consists of all the complex numbers which can be written in the form

$\cos x + \mathbf{i} \sin x$

for some real number x.

Prove that $f(x) = \text{cis } x$ is a homomorphism from $\mathbb{R}$ onto T.

Step-by-step solution

Step 1 of 3

Consider the set T of all the complex numbers lying on the unit circle, with the operation multiplication as:

$$T = \{\text{cis } x : x \in R\},$$

where

$\text{cis } x = \cos x + i \sin x$.

Let $f : R \to T$ is a mapping defined by

$f(x) = \text{cis } x$.

Objective is to prove that *f* is a homomorphism from *R* onto *T*.

If *G* and *H* are two groups, a homomorphism from *G* to *H* is a function $f : G \to H$ such that for any two elements *a*, *b* in *G*,

$f(ab) = f(a)f(b)$.

Comment

Step 2 of 3

The mapping *f* is clearly onto because $\text{cis } x \in T$ corresponds to $x \in R$.

Let $x, y \in R$. Then, by the identity $\text{cis }(x+y) = (\text{cis } x)(\text{cis } y)$, one have

$$\begin{aligned} f(x)f(y) &= \text{cis } x \text{ cis } y \\ &= \text{cis }(x+y) \\ &= f(x+y). \end{aligned}$$

This is so because *R* is an additive group and *T* is a multiplicative group.

Comment

Step 3 of 3

Thus, the mapping *f* is a homomorphism from *R* onto *T*.

Comment