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Chapter 16, Problem 3EM

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Problem

Let *p* be a prime number. A finite group *G* is called a *p-group* if the order of every element *x* in *G* is a power *p*. (The orders of different elements may be different powers of *p*.) If *H* is a subgroup of any finite group *G*, and *H* is a *p*-group, we call *H* a *p-subgroip* of *G*. Finally, if *K* is a *p*-subgroup of *G*, and *K* is maximal (in the sense that *K* is not contained in any larger *p*-subgroup of *G*), then *K* is called a *p-Sylow subgroup* of *G*.

Let $a \in N$, and suppose the order of Ka in N/K is a power of *p*. Let $S = \langle Ka \rangle$ be the cyclic subgroup of N/K generated by Ka. Prove that *N* has a subgroup S^* such that S^*/K is a *p*-group. (HINT: See Exercise J4.)

Step-by-step solution

Step 1 of 3

Suppose that *G* is a *p*-group, so order of each element *x* in *G* will be the power of *p*. Let *K* is a *p*-Sylow subgroup of *G* and $N = N(K)$be the normalizer of *K*.

Assume that $a \in N$, and the order of coset Ka in N/K is a power of *p*. Let $S = \langle Ka \rangle$ is the cyclic subgroup of N/K generated by Ka.

Objective is to prove that *N* has a subgroup S^* such that S^*/K is a *p*-group.

Consider the following result:

Suppose that *G* is any group. Let the mapping

$f : G_K \to H$

is a homomorphism from *G* onto *H* with kernel *K*. Assume that *S* is any subgroup of *H* and consider the following set:

$S^* = \{x \in G : f(x) \in S\}$.

Note that, the set S^* forms a subgroup of *G*. Consider the following restriction map $g : S^* \to S$ defined as

$g(x) = f(x)$for every $x \in S^*$.

Then $S \cong S^*/K$.

Comment

Step 2 of 3

According to the question, $S=\langle Ka\rangle$ is the cyclic subgroup of quotient group N/K. Choose S^* to be the set of pre-images of this subgroup. Then, by the above mentioned result, it can be conclude that S^*/K is isomorphic to *S*. That is, S^*/K will also form the cyclic subgroup of N/K.

Observe that N/K is well defined group, so *K* will be normal in *N*. Since *K* is a *p*-Sylow subgroup of *G*, so the order of *N* will also be some power of *p*. Thus, the order of S^*/K will be some power of *p*, that is, a *p*-group. Since S^*/K is a *p*-group of N/K, therefore S^* will be the subgroup of *N*.

Comment

Step 3 of 3

Hence, *N* has a subgroup S^* such that S^*/K is a *p*-group.

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