# A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 1EN

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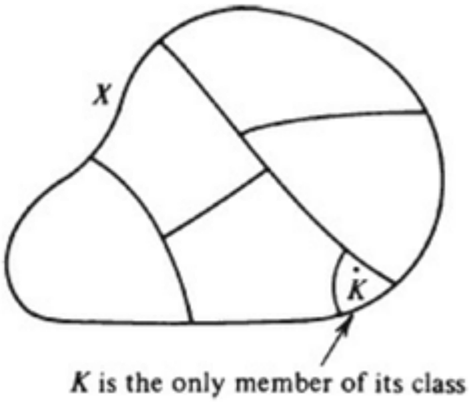
## Problem

Let $G$ be a finite group, and $K$ a $p$-Sylow subgroup of $G$. Let $X$ be the set of all the conjugates of $K$. See Exercise M2. If $C_1$, $C_2 \in X$, let $C_1 \sim C_2$ iff $C_1 = aC_2a^{-l}$ for some $\alpha \in K$

Prove that ~ is an equivalence relation on $X$.

Thus, ~ partitions $X$ into equivalence classes. If $C \in$, $X$ let the equivalence class of $C$ be denoted by




## Step-by-step solution

**Step 1** of 4

Assume that $G$ is a finite group and $K$ a $p$-Sylow subgroup of $G$. Consider the set $X$ as the set of all the conjugates of $K$. Define a relation as:

If $C_1, C_2 \in X$, let $C_1 \approx C_2$ if and only if $C_1 = aC_2a^{-1}$ for some $a \in K$.

Objective is to prove that $\approx$ is an equivalence relation on $X$. That is, show that $\approx$ is a reflexive, symmetric and transitive relation.

Comment

**Step 2** of 4

Reflexive: since *K* is a subgroup, so identity will belong to *K* and identity is the conjugate of *K*. Also,

$$eC_1e^{-1} = C_1.$$

This shows that $C_1 \approx C_1$. And thus relation is reflexive.

Symmetric: let $C_1 \approx C_2$. Then

$$C_1 = aC_2a^{-1}$$
$$C_1a = aC_2a^{-1}a$$
$$a^{-1}C_1a = a^{-1}aC_2e$$
$$a^{-1}C_1a = C_2.$$

Also the last equation can be rewrite as:

$$a^{-1}C_1\left(a^{-1}\right)^{-1} = C_2.$$

This implies that relation is symmetric.

Comment

**Step 3** of 4

Transitive: Let $C_1 \approx C_2$ and $C_2 \approx C_3$. Then

$$C_1 = aC_2a^{-1}, C_2 = bC_3b^{-1}.$$

Substitute the value of $C_2$ here and get,

$$C_1 = a\left(bC_3b^{-1}\right)a^{-1}$$
$$C_1 = abC_3b^{-1}a^{-1}$$
$$C_1 = (ab)C_3(ab)^{-1}.$$

That is, $C_1 \approx C_3$, a transitive relation.

Comment

**Step 4** of 4

Hence, $\approx$ is an equivalence relation on *X*.

Comment