A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 2EH

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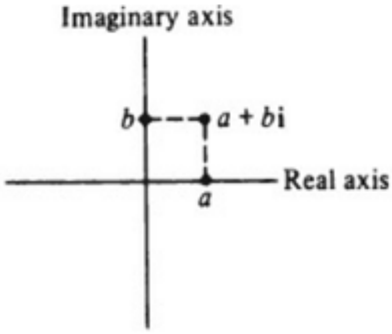
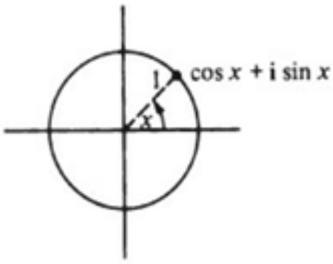
Problem

Every complex number a + $b\mathbf{i}$ may be represented as a point in the complex plane.





The *unit circle* in the complex plane consists of all the complex numbers whose distance from the origin is 1; thus, clearly, the unit circle consists of all the complex numbers which can be written in the form

cos x + $\mathbf{i}$ sin x

for some real number x.

Let T designate the set $\{\text{cis } x : x \in \mathbb{R}\}$, that is, the set of all the complex numbers lying on the unit circle, with the operation of multiplication. Use part 1 to prove that T is a group. (T is called the *circle group*.)

Step-by-step solution

Step 1 of 4

Consider the set T of all the complex numbers lying on the unit circle, with the operation multiplication as:

$T = \{\text{cis } x : x \in R\}$,

where

$\text{cis } x = \cos x + i \sin x$.

Objective is to prove that *T* forms a group.

Before starting proving this, consider the following identity:

$\text{cis } (x+y) = (\text{cis } x)(\text{cis } y)$.

It ensures that multiplication is closed under *T*.

Comment

Step 2 of 4

Now to check that multiplication is associative in *T*, use the above identity as:

$$\begin{aligned}(\text{cis } x \text{ cis } y)(\text{cis } z) &= \text{cis } (x+y)(\text{cis } z) \\ &= \text{cis } (x+y+z), \\ (\text{cis } x)(\text{cis } y \text{ cis } z) &= (\text{cis } x)\text{cis } (y+z) \\ &= \text{cis } (x+y+z).\end{aligned}$$

Since both the sides are same, therefore operation is associative.

The set *T* has the multiplicative identity $\mathrm{e} = \text{cis } (0)$, because

$$\begin{aligned}(\text{cis } x)(\text{cis } 0) &= \text{cis } (x+0) \\ &= \text{cis } (x)\end{aligned}$$

for all real number *x*.

Comment

Step 3 of 4

Note that,

$$\begin{aligned}\text{cis}(x)\text{cis}(-x) &= \text{cis}(x-x) \\ &= \text{cis}(0) \\ &= e.\end{aligned}$$

Thus, inverse of each nonzero element in *T* will be:

$(\text{cis}(x))^{-1} = \text{cis}(-x)$.

Comment

Step 4 of 4

Hence, T will form a group with identity $\text{cis}(0)$, inverse of $\text{cis}(x)$ is $\text{cis}(-x)$.

Comment