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Chapter 16, Problem 3EF

Problem

Let *G* be a group; let *H* and *K* be subgroups of *G*, with *H* a normal subgroup of *G*. Prove the following:

H is a normal subgroup of *HK*.

Step-by-step solution

Step 1 of 3

Suppose that *G* is any group and let *H, K* are the subgroups of *G*, with *H* a normal subgroup of *G*. Consider the following set:

$HK = \{xy : x \in H, y \in K\}$.

Objective is to prove that *H* is a normal subgroup of *K* (do correction in the question).

To show that *H* is a normal subgroup of *K*, there is a need to show that for some $k \in K$, and $h \in H$

$khk^{-1} \in H$.

Comment

Step 2 of 3

Before going to prove this, assume that $H \subseteq K \subseteq G$ (a subset).

Since *H* is normal in *G*, so *H* is a subgroup of *G* (obvious). But *K* is a subgroup of *G* and $H \subseteq K$. So, this implies that *H* is also a subgroup of *K*.

Let $k \in K$. Then $k \in G$. Since *H* is normal subgroup of *G*, by definition left and right cosets in *H* will be same. That is,

$Hk = kH$.

This condition holds for all $k \in K$.

Comment

Step 3 of 3

Since H is a subgroup of K and $Hk = kH$ for all $k \in K$, therefore it conclude that H is a normal subgroup of K.

Comment