A Book of Abstract Algebra | (2nd Edition)

Chapter 31, Problem 3EB

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Problem

Find the root field of $x^3 + x^2 + 1 \in \mathbb{Z}_2[x]$ over $\mathbb{Z}_2$. Write its addition and multiplication tables.

Step-by-step solution

Step 1 of 4

The objective is to find the root field of $x^3 + x^2 + 1$ over $\mathbb{Z}_2$, and write its addition and multiplication tables.

Comment

Step 2 of 4

Let $a(x)=x^3+x^2+1\in\mathbb{Z}_2$.

$$\begin{aligned}a(0)&=0^3+0^2+1\\&=1\\&\neq 0\end{aligned}$$

$$\begin{aligned}a(1)&=1^3+1^2+1\\&=1+1+1\\&=1\\&\neq 0\end{aligned}$$

Therefore, $a(x)$ has no roots in $\mathbb{Z}_2$.

This implies that x^3+x^2+1 is irreducible over $\mathbb{Z}_2$.

Therefore, the root field of x^3+x^2+1 over $\mathbb{Z}_2$ is $\mathbb{Z}_2[x]/(x^3+x^2+1)$.

Comment

Step 3 of 4

Let k be any root of $a(x)=x^3+x^2+1\in\mathbb{Z}_2$.

Then $k^3+k^2+1=0$.

$$\begin{aligned}k^3&=-k^2-1\\&=k^2+1\end{aligned}$$

The root field is of size $2^3=8$.

The addition table is given as follows:

+	0	1	k	$1+k$	k^2	$1+k^2$	$k+k^2$	
0	0	1	k	$1+k$	k^2	$1+k^2$	$k+k^2$	
1	1	0	$1+k$	k	$1+k^2$	k^2	$1+k+k^2$	
k	k	$1+k$	0	1	$k+k^2$	$1+k+k^2$	k^2	
$1+k$	$1+k$	k	1	0	$1+k+k^2$	$k+k^2$	$1+k^2$	
k^2	k^2	$1+k^2$	$k+k^2$	$1+k+k^2$	0	1	k	
$1+k^2$	$1+k^2$	k^2	$1+k+k^2$	$k+k^2$	1	0	$1+k$	
$k+k^2$	$k+k^2$	$1+k+k^2$	k^2	$1+k^2$	k	$1+k$	0	1

$1+k+k^2$	$1+k+k^2$	$k+k^2$	$1+k^2$	k^2	$1+k$	k	1	0

Comment

Step 4 of 4

The multiplication table is given as follows:

$\times$	0	1	k	$1+k$	k^2	$1+k^2$	$k+k^2$	$1+k+k^2$
0	0	0	0	0	0	0	0	0
1	0	1	k	$1+k$	k^2	$1+k^2$	$k+k^2$	$1+k+k^2$
k	0	k	k^2	$k+k^2$	$1+k^2$	$1+k+k^2$	1	$1+k$
$1+k$	0	$1+k$	$k+k^2$	$1+k^2$	1	k	$1+k+k^2$	k^2
k^2	0	k^2	$1+k^2$	1	$1+k+k^2$	$1+k$	k	$k+k^2$
$1+k^2$	0	$1+k^2$	$1+k+k^2$	k	$1+k$	$k+k^2$	k^2	1
$k+k^2$	0	$k+k^2$	1	$1+k+k^2$	k	k^2	$1+k$	$1+k^2$
$1+k+k^2$	0	$1+k+k^2$	$1+k$	k^2	$k+k^2$	1	$1+k^2$	k

Comment