A Book of Abstract Algebra | (2nd Edition)

Problem

Suppose $a(x) \in F[x]$, and K is an extension of F. An element $c \in K$ is called a multiple root of $a(x)$ if $(x - c)^m | a(x)$ for some $m > 1$. It is often important to know if all the roots of a polynomial are different, or not.

We now consider a method for determining whether an arbitrary polynomial $a(x) \in F[x]$ has multiple roots in any extension of F.

Let K be any field containing all the roots of $a(x)$. Suppose $a(x)$ has a multiple root c.

Conclude that $a(x)$ and $a'(x)$ have no common factor of degree >1 in $F[x]$.

This important result is stated as follows: *A polynomial $a(x)$ in $F[x]$ has a multiple root iff $a(x)$ and $a'(x)$ have a common factor of degree >1 in $F[x]$.*

Step-by-step solution

Step 1 of 3

Consider that K is any field that contains all the roots of polynomial $a(x) = a_0 + a_1x + \cdots + a_nx^n$. Assume that $a(x)$ has no multiple roots. Then polynomial $a(x)$ can be factored as

$a(x) = (x - c_1)\cdots(x - c_n)$

where $c_1, \ldots, c_n$ are all distinct. Objective is to prove that $a(x)$ and $a'(x)$ have no common factor of degree > 1 in $F[x]$.

Consider the following result:

If $a(x), b(x) \in F[x]$ have a common root c in some extension of F, they may have a common factor of positive degree in $F[x]$.

Comment

Step 2 of 3

The derivative of polynomial $a(x)$ will be the sum of terms of the following form:

$(x - c_1)\cdots(x - c_{i-1})(x - c_{i+1})\cdots(x - c_n)$.

Here, each time, differentiation of one term takes place.

Observe that, in $a'(x)$, the factor $(x - c_i)$ is not present at the $i - \text{th}$ term. So, if one substitute c_i in $a'(x)$, the derivative will not get vanish. This shows that no c_i is a root of $a'(x)$.

Thus, both $a(x)$ and $a'(x)$ have no roots in common.

Comment

Step 3 of 3

Hence, by this result it can be conclude that $a(x)$ and $a'(x)$ have no common factor of degree > 1 in $F[x]$.

Comment