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Chapter 16, Problem 5EI

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Problem

Conclude (using the FHT) that $(G/H)K/H) \cong G/K.$

Step-by-step solution

Step 1 of 4

Suppose that *G* is any group and let *H, K* are normal subgroups of *G* with $H \subseteq K$.

Consider a well-defined mapping $\phi : G/H \to G/K$ defined by

$\phi(Ha) = Ka$, for all $a \in G$.

Objective is to prove that $(G/H)/(K/H) \cong G/K$, by using fundamental homomorphism theorem.

According to the fundamental homomorphism theorem, if $f : G \to H$ is a homomorphism of *G* onto *H*, with kernel *K* then

$H \cong G/K$.

Comment

Step 2 of 4

First show that ϕ is an onto homomorphism mapping.

Assume that $Ha, Hb \in G/H$, for some $a, b \in G$. Then use the definition of mapping in the following manner:

$$\begin{aligned}\phi(Ha \cdot Hb) &= \phi(Hab) \\ &= Kab \\ &= (Ka)\cdot(Kb) \\ &= \phi(Ha)\cdot\phi(Hb).\end{aligned}$$

Assume that $Kx \in G/K$, for some $x \in G$. An element *x* is the member of *G* because of the coset of G/K. So from there it implies that

$$Hx \in G/H$$

such that

$$\phi(Hx) = Kx.$$

Thus, ϕ is an onto.

Comment

Step 3 of 4

Let $Ha \in G/H$, for some $a \in G$. If $Ha \in Ker\phi$ then by the define function,

$$\begin{aligned}\phi(Ha) &= K \\ Ka &= K.\end{aligned}$$

It implies that $a \in K$ and correspondingly $Ha \in K/H$. Thus,

$$\ker\phi \subseteq K/H.$$

Now let $Ha \in K/H$. Then $a \in K$ and

$$\begin{aligned}Ka &= K \\ \phi(Ha) &= K\end{aligned}$$

So,

$$K/H \subseteq \ker\phi.$$

On combining both the equations of containment,

$$\ker\phi = K/H.$$

Comment

Step 4 of 4

Since $\phi : G/H \to G/K$ is onto homomorphism with $\ker\phi = K/H$, therefore by fundamental homomorphism theorem it conclude that

$$(G/H)/(K/H) \cong G/K.$$

Comment