A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 1EM

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Problem

Let *p* be a prime number. A finite group *G* is called a *p-group* if the order of every element *x* in *G* is a power *p*. (The orders of different elements may be different powers of *p*.) If *H* is a subgroup of any finite group *G*, and *H* is a *p*-group, we call *H* a *p-subgroip* of *G*. Finally, if *K* is a *p*-subgroup of *G*, and *K* is maximal (in the sense that *K* is not contained in any larger *p*-subgroup of *G*), then *K* is called a *p-Sylow subgroup* of *G*.

Prove that the order of any *p*-group is a power of *p*.(HINT:Use Exercise K.)

Step-by-step solution

Step 1 of 3

Consider that *G* is a *p*-group, so order of each element *x* in *G* will be the power of *p*. Objective is to prove that the order of any *p*-group is a power of *p*. That is, if *G* is a *p*-group then

$$|G| = p^k,$$

for some integer *k*.

Suppose that *q* is some arbitrary prime such that it divides the order of *G*, that is,

$$q \mid |G|.$$

Comment

Step 2 of 3

Since Cauchy theorem holds for any finite group, so one can apply it here. By Cauchy theorem, there exists $x \in G$ such that order of *x* will be *q*, then

$$x^q = e.$$

Since *G* is a *p*-group, and *x* is the element of *G*. Therefore, the order of *x* must be some power of *p*. So,

$$\mathrm{ord}(x) = p^r,$$

where r is some nonnegative integer. Then by the above condition that $\mathrm{ord}(x) = q$, it implies that

$$q = p^r.$$

Since q and p both are prime, therefore it conclude that

$$q = p.$$

Thus, p is the only divisor of order of G.

Comment

Step 3 of 3

Hence, the order of any p-group is always a power of p.

Comment