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Chapter 16, Problem 5EM

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Problem

Let p be a prime number. A finite group G is called a *p-group* if the order of every element x in G is a power p. (The orders of different elements may be different powers of p.) If H is a subgroup of any finite group G, and H is a p-group, we call H a *p-subgroip* of G. Finally, if K is a p-subgroup of G, and K is maximal (in the sense that K is not contained in any larger p-subgroup of G), then K is called a *p-Sylow subgroup* of G.

Use parts 3 and 4 to prove: no element of N/K has order a power of p (except, trivially, the identity element).

Step-by-step solution

Step 1 of 3

Suppose that G is a p-group, so order of each element x in G will be the power of p. Let K is a p-Sylow subgroup of G and $N = N(K)$ be the normalizer of K.

Assume that $a \in N$, and the order of coset Ka in N/K is a power of p. Let $S = \langle Ka \rangle$ is the cyclic subgroup of N/K generated by Ka. Also N has a subgroup S^* such that S^*/K is a p-group.

Objective is to prove that no non-identity element of N/K has order a power of p.

Comment

Step 2 of 3

Suppose, if possible, that there is a non-identity element $Kn \in N/K$ whose order is some power of p, say p^j.

Note that, S^* is the set of all elements n of N such that $Kn = Ka^p$, so $K = S^*$ includes a. This similar argument can be make for arbitrary $a \in N$ such that

Order of $Ka = p^j$.

By this argument, it implies that n will belong to K. If $n \in K$, then

$Kn = K$.

Note that K is the identity element of quotient group N/K. So, $Kn \in N/K$ is nothing but an identity element.

Comment

Step 3 of 3

Hence, identity is the only element in N/K whose order is a power of p.

Comment