A Book of Abstract Algebra | (2nd Edition)

Chapter 28, Problem 1EE

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Problem

Let U and V be finite-dimensional vector spaces over a field F, and let $h : U \to V$ be a linear transformation. Prove part:

The kernel of h is a subspace of U. (It is called the *null space* of h.)

Step-by-step solution

Step 1 of 4

It is already known that U is a vector space and so it satisfies all conditions for vector space.

Comment

Step 2 of 4

Given subset of U is set of all elements of U which maps to zero-vector of V.

Comment

Step 3 of 4

Or given subset is

$$\{\mathbf{k} \in U \mid h(\mathbf{k}) = \mathbf{0}_v\}$$

Comment

Step 4 of 4

A 2 step check is needed to determine if given subset is a subspace. First determine if any linear combination of functions in subset is closed under given operation. Second, determine if 0 satisfies given conditions.

STEP 1: For any 2 elements **a** and **b** in *U*,

$$h(\mathbf{a}) = \mathbf{0}_v \qquad (1)$$
$$h(\mathbf{b}) = \mathbf{0}_v \qquad (2)$$

Combining above 2 equations, $s(1)+t(2)$ gives

$$s \cdot h(\mathbf{a}) + t \cdot h(\mathbf{b}) = \mathbf{0}_v$$

As functions or linear transformations are vector space in themselves, any constant multiple of function is also a function. Also sum of 2 functions is also a function. Thus,

$$\begin{aligned} &s \cdot h(\mathbf{a}) + t \cdot h(\mathbf{b}) = \mathbf{0}_v \\ &\Rightarrow h(s\mathbf{a}) + h(t\mathbf{b}) = \mathbf{0}_v \\ &\Rightarrow h(s\mathbf{a} + t\mathbf{b}) = \mathbf{0}_v \end{aligned}$$

Thus linear combination of 2 elements in subset lies in subset.

STEP 2: Check if **0** vector satisfies given condition,

$$h(\mathbf{0}_u) = \mathbf{0}_v \qquad \{\text{As } h \text{ is a linear transformation}\}$$

$$\boxed{\text{Hence given set or kernel represents a subspace}}$$

Comment