A Book of Abstract Algebra | (2nd Edition)

Problem

Find the minimum polynomial of each of the following numbers over $\mathbb{Q}$. (Where appropriate, use the methods of Chapter 26, Exercises D, E, and F to ensure that your polynomial is irreducible.)

(*a*) 1 + 2*i*

(*b*) $1+\sqrt{2}$

(*c*) $1+\sqrt{2i}$

#(*d*) $\sqrt{2+\sqrt[3]{3}}$

(*e*) $\sqrt{3}+\sqrt{5}$

(*f*) $\sqrt{1+\sqrt{2}}$

Step-by-step solution

Step 1 of 7

(a)

Objective is to determine the minimal polynomial of the number $1+2i$ over *Q*.

Let $a=1+2i$. Then

$$a-1=2i$$
$$(a-1)^2=(2i)^2$$
$$a^2-2a+1=-4$$
$$a^2-2a+5=0.$$

Since the roots of this polynomial is complex, so it is irreducible over *Q*.

Thus, the required minimal polynomial is a^2-2a+5.

Comment

Step 2 of 7

(b)

Objective is to determine the minimal polynomial of the number $1+\sqrt{2}$ over *Q*.

Let $a=1+\sqrt{2}$. Then

$$a-1=\sqrt{2}$$
$$(a-1)^2=(\sqrt{2})^2$$
$$a^2-2a+1=2$$
$$a^2-2a-1=0.$$

Because of irrational roots this polynomial is irreducible over *Q*.

Thus, the required minimal polynomial is a^2-2a-1.

Comment

Step 3 of 7

(c)

Objective is to determine the minimal polynomial of the number $1+\sqrt{2}i$ over *Q*.

Let $a=1+\sqrt{2}i$. Then

$$a-1=\sqrt{2}i$$
$$(a-1)^2=(\sqrt{2}i)^2$$
$$a^2-2a+1=-2$$
$$a^2-2a+3=0.$$

Because of complex roots this polynomial is irreducible over *Q*.

Thus, the required minimal polynomial is a^2-2a+3.

Comment

Step 4 of 7

(d)

Determine the minimal polynomial of the number $\sqrt{2+\sqrt[3]{3}}$ over *Q*.

Let $a=\sqrt{2+\sqrt[3]{3}}$; then

$$a^2=2+\sqrt[3]{3}$$
$$(a^2-2)^3=(\sqrt[3]{3})^3$$
$$a^6-8-6a^4-12a^2=3$$
$$a^6-6a^4-12a^2-11=0$$

Thus, the required minimal polynomial is $a^6-6a^4-12a^2-11$.

Comments (1)

Step 5 of 7

(e)

Determine the minimal polynomial of the number $\sqrt{3}+\sqrt{5}$ over *Q*.

Let $a=\sqrt{3}+\sqrt{5}$; then

$$a^2=(\sqrt{3}+\sqrt{5})^2$$
$$a^2=3+5+2\sqrt{3}\sqrt{5}$$
$$a^2-8=2\sqrt{15}$$
$$(a^2-8)^2=(2\sqrt{15})^2$$

And finally $a^4+64-16a^2=60$. Thus, the required minimal polynomial is a^4-16a^2+4.

Comment

Step 6 of 7

(f)

Objective is to determine the minimal polynomial of the number $\sqrt{1+\sqrt{2}}$ over *Q*.

Let $a=\sqrt{1+\sqrt{2}}$. Then

$$a^2=1+\sqrt{2}$$
$$(a^2-1)^2=(\sqrt{2})^2$$
$$a^4-2a^2+1=2$$
$$a^4-2a^2-1=0.$$

Comment

Step 7 of 7

Thus, the required minimal polynomial is a^4-2a^2-1.

Comment