A Book of Abstract Algebra | (2nd Edition)

Chapter 33, Problem 7EC

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Problem

Let p be a prime number, and ω a primitive pth root of unity in the field F.

Conclude: If $x^p - a$ is *not* irreducible in $F[x]$, it has a root (namely, $b^s a^t$) in F.

We have proved: *$x^p - a$ either has a root in. F or is irreducible over F.*

Step-by-step solution

Step 1 of 4

Here, objective is to prove $x^p - a$ has a root $b^s a^t$ in F.

Consider $x^p - a$ is not irreducible in $F(x)$

Comment

Step 2 of 4

Consider the polynomial $x^p - a$.

The root of above polynomial is a primitive p^{th} root of unity

$$x^p - a = 0$$
$$x = \sqrt[p]{a}\ \omega$$

Then, the root $d = \sqrt[p]{a}$, ω is the p^{th} root of unity.

Comment

Step 3 of 4

Consider the polynomial $x^p - a \in F(x)$

P is a prime and $x^p - a$ is reducible in $F(x)$

Let us assume $d_1, d_2, ...d_p$ are the roots of $x^p - a$

Then,

$$x^p - a = (x - d_1)(x - d_2).......(x - d_p)$$

$p(x)$ is equal to the product of *m* number of these factors.

$p(x) = (x - d_1)(x - d_2).......(x - d_m)$. Since, degree $p(x) = m$

Comment

Step 4 of 4

Let the Constant term of above equation is *b,*

Which is the product of $d_1, d_2, ...d_m$

$$b = (d_1 d_2 .....d_m)$$

$$b = \sqrt[p]{a} \;......\sqrt[p]{a}$$

$$b = \omega^k (\sqrt[p]{a}\,)^m$$

$$b = \omega^k d^m$$

$$b = \left(\sqrt[p]{a}\,\right)^m \qquad (\because \omega^k = 1)$$

$$b^p = a^m$$

$$b^{sp} = a^{sm}$$

$$\begin{aligned} \text{Consider } (b^s a^t)^p &= (b^{sp} a^{tp}) \\ &= (a^{sm} a^{tp}) \\ &= a^{sm+tp} \\ &= a \qquad (\because sm + tp = 1) \end{aligned}$$

Then, $b^s a^t = \sqrt[p]{a}$

Hence, $x^p - a$ has a root $b^s a^t$ in *F.*

Comment