A Book of Abstract Algebra | (2nd Edition)

Problem

We will show that $2\pi/5$ is a constructible angle, and it will follow that the regular pentagon is constructible.

Use part 3 and A6 to prove that cos $(2\pi/5)$ is a constructible number.

Step-by-step solution

Step 1 of 5

Here, objective is to prove that $\cos\frac{2\pi}{5}$ is a constructible number.

Comment

Step 2 of 5

if the point $(a,0)$is constructible from $\{O,I\}$, then $a \in D$

$a \in D$, if *a* is a real root of any quadratic polynomial with coefficients in *D*.

De Moivre's theorem: $\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$ is a complex fifth root of unity.

Since $x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)$. ω is a root of $P(x) = (x^4 + x^3 + x^2 + x + 1)$

Comment

Step 3 of 5

Consider $\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$$\frac{1}{\omega} = \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}$$

$$\left(\omega + \frac{1}{\omega}\right) = 2\cos\frac{2\pi}{5}$$

Comment

Step 4 of 5

Consider ω is a root of $P(x)=(x^4+x^3+x^2+x+1)$

Then,

$$(\omega^4+\omega^3+\omega^2+\omega+1)=0$$

$$\omega^2+\omega+1+\omega^{-1}+\omega^{-2}=0$$

$$\left(\omega+\frac{1}{\omega}\right)^2+\left(\omega+\frac{1}{\omega}\right)-1=0$$

$$\left(2\cos\frac{2\pi}{5}\right)^2+\left(2\cos\frac{2\pi}{5}\right)-1=0$$

$$4\cos^2\frac{2\pi}{5}+2\cos\frac{2\pi}{5}-1=0$$

Comment

Step 5 of 5

Consider the polynomial $4x^2-2x-1$

put $x=2\cos\frac{2\pi}{5}$ in above equation , then

$$4x^2-2x-1=$$

$$=4\cos^2\frac{2\pi}{5}+2\cos\frac{2\pi}{5}-1$$

$$=0$$

Hence, $\cos\frac{2\pi}{5}$ is a root of quadratic polynomial $4x^2-2x-1=0$, then $\cos\frac{2\pi}{5}\in D$

Therefore, $\cos\frac{2\pi}{5}$ is a constructible from $\{O,I\}$.

Hence, proved.

Comment