A Book of Abstract Algebra | (2nd Edition)

Chapter 33, Problem 3ED

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Problem

Let *G* be a group. The symbol $H \lhd G$ should be read, "*H* is a normal subgroup of *G*." A *maximal* normal subgroup of *G* is an $H \lhd G$ such that, if $H \lhd J \lhd G$, then necessarily *J* = *H* or *J* = *G*. Prove the following:

Let $K \lhd G$. If $\mathcal{J}$ is a subgroup of *G*/*K*, let $\hat{\mathcal{J}}$ denote the union of all the cosets which are members of $\mathcal{J}$.. If $\mathcal{J} \lhd G/K$, then $\hat{\mathcal{J}} \lhd G$. (Use part 2.)

Step-by-step solution

Step 1 of 4

Here, objective is to prove that $\hat{g} \lhd G$

Comment

Step 2 of 4

If *G* is a finite group, then the group *H* is normal subgroup of *G* is denoted by $H \lhd G$

Consider $f: G \to H$ is a homomorphism, then $f(xy) = f(x).f(y); \forall x, y \in G.$

Comment

Step 3 of 4

Consider $K \lhd G$, g is a subgroup of $\frac{G}{K}$ and

$\hat{g}$ is a union of all cosets which are members of *g.*

Consider $g \lhd \frac{G}{K}$

Then, $f: G \to \frac{G}{K}$ is a natural homomorphism

Let $x, y \in f^{-1}(g)$

Then, $f(x), f(y) \in g$

$f(xy) \in g$

$xy \in f^{-1}(g)$

$f^{-1}(g)$ is closed under multiplication.

Comment

Step 4 of 4

Let $z \in f^{-1}(g)$

$f(z) \in g$

$f(c)^{-1} \in g$

$f(c^{-1}) \in g$

$c^{-1} \in f^{-1}(g)$

$f^{-1}(g)$ is closed under inversion.

So, $f^{-1}(g)$ is closed under multiplication and inversion.

Therefore, $f^{-1}(g) < G$ is a subgroup of *G*, implies

$$f^{-1}(g) < G$$
$$\hat{g} < G$$
.

Hence, proved

Comment