A Book of Abstract Algebra | (2nd Edition)

Chapter AB, Problem 14E

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Problem

Prove that the following are true for any integers a, b, and c:

lcm(*a*, *ab*) = *ab*.

Step-by-step solution

Step 1 of 4

Objective:-

The objective is to prove $lcm(a,ab)=ab.$

Comment

Step 2 of 4

Proof:-

Let us first prove a theorem which helps in prove this result.

Let us suppose $\gcd(b,c)=t.$

Let us consider the theorem.

Theorem:-Any two nonzero integers *r* and *s* have a unique positive greatest common divisor *t*, Moreover, *t* is equal to a “Linear combination” of *r* and *s*. That is,

$$t=kr+ls \quad for \ some \ \text{integer} \ k \ and \ l$$

According to this definition:-

$$t=mb+nc \quad for \ some \ \text{integer} \ m \ and \ n \qquad ......(1)$$

Let us multiply by a both sides.

$$ta=mba+nca \quad for \ some \ \text{integer} \ m \ and \ n$$
$$ta=mab+nac \quad for \ some \ \text{integer} \ m \ and \ n$$

Thus, according to the definition ta is greatest common divisor of ab and ac.

$$\gcd(ab,ac)=at \qquad \ldots\ldots(2)$$

Comment

Step 3 of 4

Let us consider the theorem.

Theorem:- If p and q are two integers with greatest common divisor $\gcd(p,q)$ and least common multiple $lcm(p,q)$, then

$$p\times q=\gcd(p,q)\times lcm(p,q)$$

According to this theorem:-

$$b\times c=\gcd(b,c)\times lcm(b,c)$$
$$b\times c=t\times lcm(b,c) \qquad \ldots\ldots(3)$$

And,

$$ab\times ac=at\times lcm(ab,ac) \qquad \ldots\ldots(4)$$

Let us divide the equation (3) by (4)

$$\frac{b\times c}{ab\times ac}=\frac{t\times lcm(b,c)}{at\times lcm(ab,ac)}$$
$$\frac{\cancel{(b\times c)}}{a^2\cancel{(b\times c)}}=\frac{\cancel{t}\times lcm(b,c)}{a\cancel{t}\times lcm(ab,ac)}$$
$$\frac{1}{a\times\cancel{a}}=\frac{lcm(b,c)}{\cancel{a}\times lcm(ab,ac)}$$
$$\frac{1}{a}=\frac{lcm(b,c)}{lcm(ab,ac)}$$
$$lcm(ab,ac)=a\cdot lcm(b,c)$$

Proved

Comment

Step 4 of 4

According to this theorem:-

$$lcm(a,ab)=lcm(a\cdot 1,a\cdot b)$$
$$lcm(a,ab)=a\cdot lcm(1,b)$$

The least common multiply of 1 and any integer is always equal to that integer.

So,

$$lcm(a,ab) = a \cdot b$$

$$lcm(a,ab) = ab$$

Proved

Comment