A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 2EK

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Problem

If G is a group and p is any prime divisor of $|G|$, it will be shown here that G has at least one element of order p. This has already been shown for abelian groups in Chapter 15, Exercise H4. Thus, assume here that G is not abelian. The argument will proceed by induction; thus, let $|G| = k$, and assume our claim is true for any group of order less than k. Let C be the center of G, let Ca be the centralizer of a for each $a \in G$, and let $k = c + k_s + \cdots + k_t$ be the class equation of G, as in Chapter 15, Exercise G2.

Prove that for any $a \notin \mathbf{C}$ in G, if p is not a factor of $|C_a|$, then p is a factor of $(G: C_a)$.

Step-by-step solution

Step 1 of 3

Consider a non-abelian group G whose order is divisible by some prime p. Cauchy Theorem states that group G has at least one element whose order is p.

Assume that order of G is k. Let C be the center of G and C_a be the centralizer of $a \in G$. The class equation for group G is given by:

$$k = c + k_S + \quad + k_t.$$

Objective is to prove that if p is not a factor of $|C_a|$ for any $a \notin C$ in G, then p is a factor of $(G : C_a)$.

Comment

Step 2 of 3

Here $(G : C_a)$ denotes an index set of G by centralizer set C_a given by

$$(G : C_a) = \frac{G}{C_a}.$$

From the order of group G, one have

$$|G| = |G| \times \frac{|C_a|}{|C_a|}$$
$$= \frac{|G|}{|C_a|} \times |C_a|$$
$$= |(G:C_a)| \times |C_a|.$$

Since $p \mid |G|$ and $p \nmid |C_a|$. Therefore, $p \mid (|G|/|C_a|)$. Or equivalently, $p/(G:C_a)$ as required.

Comment

Step 3 of 3

Hence, *p* is a factor of $(G:C_a)$.

Comment