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Chapter 28, Problem 7EE

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Problem

Let N be the null space of h, and $\mathcal{R}$ the range space of h. Let $\{\mathbf{a}_1, \ldots, \mathbf{a}_r\}$ be a basis of N. Extend it to a basis $\{\mathbf{a}_1, \ldots, \mathbf{a}_r, \ldots, \mathbf{a}_n\}$ of U.

Prove part:

Conclude as follows: for any linear transformation h, dim (domain h) = dim (null space of h) + dim (range space of h).

Step-by-step solution

Step 1 of 5

It is already known that U and V are vector spaces and so they satisfies all conditions for vector space. It is known that basis of U contains n elements. Thus, dimension of U is n.

Comment

Step 2 of 5

Range space of h is subspace of V is set of all elements of V which are map of vectors of U.

Comment

Step 3 of 5

Or given subspace is

$$\{\mathbf{r} \in V \mid h(\mathbf{u}) = \mathbf{r} \text{ for } \mathbf{u} \in U\}$$

Comment

Step 4 of 5

Thus any element in range is a map of some vector in U

Comment

Step 5 of 5

For any element r in range of h, we can find a element u in U such that

$$h(\mathbf{u}) = \mathbf{r}$$

Since U is a vector space, every element in U can be expressed as linear combination of basis of U. So,

$$\mathbf{u} = t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \dots t_r\mathbf{a}_r + \dots + t_n\mathbf{a}_n$$

Taking linear transformation

$$h(\mathbf{u}) = h(t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \dots t_r\mathbf{a}_r + \dots + t_n\mathbf{a}_n)$$

$$\Rightarrow h(\mathbf{u}) = t_1h(\mathbf{a}_1) + t_2h(\mathbf{a}_2) + \dots + t_rh(\mathbf{a}_r) + \dots + t_nh(\mathbf{a}_n) \qquad \dots(1)$$

Since $(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_r)$ is null basis of h,

$h(\mathbf{a}_r) = \mathbf{0}\ \forall r \in (0,1,\dots,r)$

Also number of elements in basis of *h* is *r*. Thus, dimension of null-space is *r*.

Therefore (1) can be rewritten as,

$$h(\mathbf{u}) = t_{r+1}h(\mathbf{a}_{r+1}) + \dots + t_n h(\mathbf{a}_n)$$

$h(\mathbf{u})$ represents a subspace, all element of which can be expressed as linear combinations of $h(\mathbf{a}_{r+1}),\dots,h(\mathbf{a}_n)$. In other words $h(\mathbf{a}_{r+1}),\dots,h(\mathbf{a}_n)$ forms basis of range of *h*. Dimension of subspace is number of elements is basis of a subspace. Here range subspace have $n-r$ elements in its basis. In other words dimension of range space is $(n-r)$

$$\boxed{\text{Hence } \dim(\text{domain of } h) = n = r + n - r = \dim(\text{nullspace of } h) + \dim(\text{range space of } h)}$$

Comment