A Book of Abstract Algebra | (2nd Edition)

Chapter 28, Problem 3EF

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Problem

Let U and V be vector spaces over the field F, with dim $U = n$ and dim $V = m$. Let $h : U \to V$ be a homomorphism.

Prove the following:

Suppose dim U = dim V; h is an isomorphism (that is, a bijective homomorphism) iff h is injective iff h is surjective.

Step-by-step solution

Step 1 of 5

It is already known that U and V are vector spaces and so they satisfies all conditions for vector space. It is known that basis of U contains n elements. Thus, dimension of U is n.

Comment

Step 2 of 5

Linear transformation h is said to be surjective if all elements of V are map to some element of U.

Linear transformation h is said to be injective if,

$$h(\mathbf{a}) = h(\mathbf{b}) \Rightarrow \mathbf{a} = \mathbf{b}$$

Comment

Step 3 of 5

If h is homomorphism, then it is isomorphism if there are equal elements in U and V with **0** mapped to **0.**

Comment

Step 4 of 5

Comment

Step 5 of 5

From question 7 of this section,

$$\text{Hence } \dim(\text{domain of } h) = n = r + n - r = \dim(\text{nullspace of } h) + \dim(\text{range space of } h)$$

It is also given that,

$$\dim V = \dim U = n$$

PART 1: h is surjective and injective, prove that h is isomorphism.

$$\dim U = \dim V = n$$

Also, $\dim(\text{nullspace of } h) = 0$

Thus, h is isomorphism.

PART 2: h is isomorphism then prove that h is injective and surjective.

Since h is isomorphism,

h^{-1} exists. This means there is some element in V which is mapped to some element in U. In other words h is surjective.

$$\text{If } h(\mathbf{a}-\mathbf{b}) = \mathbf{0}_v$$
$$\Rightarrow h(\mathbf{a}-\mathbf{b}) = h(\mathbf{0}) = h(\mathbf{a}) - h(\mathbf{b})$$
$$\Rightarrow h(\mathbf{a}) = h(\mathbf{b}) \text{ iff } \mathbf{a}-\mathbf{b} = 0 \text{ or, } \mathbf{a} = \mathbf{b}$$
$$\Rightarrow h \text{ is injective}$$

$$\boxed{\text{Hence it can be said that } h \text{ is isomorphism iff } h \text{ is surjective and iff } h \text{ is injective}}$$

Comment