A Book of Abstract Algebra | (2nd Edition)

Chapter 31, Problem 5EA

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Problem

Find irreducible polynomials *a*(*x*) over $\mathbb{Q}$, and *b*(*x*) over $\mathbb{Q}(i)$, such that $\mathbb{Q}(i, \sqrt{3})$ is the root field of *a*(*x*) over $\mathbb{Q}$, and is the root field of *b*(*x*) over $\mathbb{Q}(i)$. Then do the same for $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.

Step-by-step solution

Step 1 of 3

The objective is to find the irreducible polynomials $a(x)$ over $\mathbb{Q}$, and $b(x)$ over $\mathbb{Q}(i)$, such that $\mathbb{Q}(i, \sqrt{3})$ is the root field of $a(x)$ over $\mathbb{Q}$, and is the root field of $b(x)$ over $\mathbb{Q}(i)$. Then do the same for $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.

Comment

Step 2 of 3

Take $x = \sqrt{3} + i$.

Then

$$x^2 = 2 + 2\sqrt{3}i$$
$$x^2 - 2 = 2\sqrt{3}i$$
$$(x^2 - 2)^2 = (2\sqrt{3}i)^2$$
$$x^4 - 4x^2 + 4 = -12$$
$$x^4 - 4x^2 + 16 = 0$$

The polynomial $a(x) = x^4 - 4x^2 + 16$ is irreducible over $\mathbb{Q}$ and $\mathbb{Q}(i, \sqrt{3})$ is the root field of $a(x)$ over $\mathbb{Q}$.

The polynomial $b(x)=x^2-3$ is irreducible over $\mathbb{Q}(i)$ and $\mathbb{Q}(i,\sqrt{3})$ is the root field of $b(x)$ over $\mathbb{Q}(i)$.

Comment

Step 3 of 3

Take $x=\sqrt{2}+\sqrt{3}$.

Then

$$\begin{aligned} x^2 &= 5+2\sqrt{6} \\ x^2-5 &= 2\sqrt{6} \\ \left(x^2-5\right)^2 &= \left(2\sqrt{6}\right)^2 \\ x^4-10x^2+25 &= 24 \\ x^4-10x^2+1 &= 0 \end{aligned}$$

The polynomial $a(x)=x^4-10x^2+1$ is irreducible over $\mathbb{Q}$ and $\mathbb{Q}\left(\sqrt{2},\sqrt{3}\right)$ is the root field of $a(x)$ over $\mathbb{Q}$.

The polynomial $b(x)=x^2-3$ is irreducible over $\mathbb{Q}(i)$ and $\mathbb{Q}\left(\sqrt{2},\sqrt{3}\right)$ is the root field of $b(x)$ over $\mathbb{Q}\left(\sqrt{2}\right)$.

Comment