A Book of Abstract Algebra (2nd Edition)

Chapter 17, Problem 1EM   1 Bookmark   Show all steps: ON

Problem

An element *a* of a ring is *nilpotent* if $a^n = 0$ for some positive integer *n*.

In a ring with unity, prove that if *a* is nilpotent, then *a* +1 and *a* – 1 are both invertible. [HINT: Use the factorization

$$1 - a^n = (1 - a)(1 + a + a^2 + \cdots + a^{n-1})$$

for 1 – *a*, and a similar formula for 1 + *a*.]

Step-by-step solution

Step 1 of 3

Consider an arbitrary ring *R* with unity. Let an element $a \in R$ is nilpotent, that is,

$a^n = 0$,

for some positive integer *n*. Objective is to show that $a+1$ and $a-1$ both are invertible.

An element $a \in R$ is said to be an invertible element, if there exists $a^{-1} \in R$ such that

$aa^{-1} = 1, a^{-1}a = 1$,

where 1 stands for the unity of the ring.

The zero is always nilpotent element. If $a = 0$ then $a+1=1$, a unity.

And $a-1=-1$. Since $-1 \cdot -1 = 1$ therefore -1 will be self-inverse. Thus, when $a=0$ then $a+1$ and $a-1$ both are invertible.

Comment

Step 2 of 3

Suppose that $a \neq 0$. Since *R* is a ring with unity. Than unity 1 can be written as:

$1 = 1 - 0$.

By using the condition $a^n = 0$, one get

$$1 = 1 - a^n$$
$$1 = (1-a)(1 + a + a^2 + \cdots + a^{n-1})$$

Since *a* is nonzero element, so is $(1 + a + a^2 + \cdots + a^{n-1})$. Therefore, $1-a$ will be invertible.

Since $1 - a \in R$, therefore its negative $-(1-a) = a-1$ also belongs to *R* and invertible too (ring property).

Similarly,

$$\begin{aligned} 1 &= 1 + 0 \\ &= 1 + a^n \\ &= (1+a)(1 - a + a^2 - \cdots + (-1)^{n-1} a^{n-1}) \end{aligned}$$

By the same logic defined above, it conclude that $a+1$ is invertible.

Comment

Step 3 of 3

Hence, if *a* is nilpotent element then $a+1$ and $a-1$ both are invertible.

Comment