A Book of Abstract Algebra | (2nd Edition)

Chapter 28, Problem 4EA

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Problem

Prove that $\mathcal{M}_2(\mathbb{R})$, the set of all 2 × 2 matrices of real numbers, with matrix addition and the scalar multiplication

$$k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

is a vector space over $\mathbb{R}$.

Step-by-step solution

Step 1 of 2

There are 10 conditions which any vector space must satisfy. These are

1. For $u \in V, v \in V \Rightarrow u+v \in V$
2. For $u \in V, v \in V \Rightarrow u+v = v+u$
3. For $u \in V, v \in V, w \in V \Rightarrow (u+v)+w = u+(v+w)$
4. There exists $0 \in V$, such that $0+v=v$ for all $v \in V$
5. For all $u \in V$, there exists $x \in V$ such that $u+x=0$
6. For $c \in R, v \in V \Rightarrow cv \in V$
7. For $c \in R, u \in V, v \in V \Rightarrow c(u+v) = cu+cv$
8. For $c, d \in R, u \in V, v \in V \Rightarrow (c+d)u = cu+du$
9. For $c \in R, d \in R, v \in V \Rightarrow c(dv) = (cd)v$
10. There exists $1 \in R, v \in V \Rightarrow 1 \cdot v = v$

Comment

Step 2 of 2

$M_2(\mathbb{R})$ is matrix of order 2×2 .It can be represented by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Where *a, b, c, d* are real numbers. These can be thought of as 4 components of a vector. Addition of 2 matrices are done component wise. Multiplication of a matrix with a constant implies that all components are multiplied with that constant. It is also known that normal addition and multiplication is commutative.

Let

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$v = \begin{pmatrix} d & e \\ f & g \end{pmatrix}$$

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow -u = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$$

Then check aforementioned 8 properties or condition for this space.

$$u+v=\begin{pmatrix} a & b \\ c & d \end{pmatrix}+\begin{pmatrix} d & e \\ f & g \end{pmatrix}$$

1. $$\Rightarrow u+v=\begin{pmatrix} a+d & b+e \\ c+f & d+g \end{pmatrix}$$

$$\Rightarrow u+v=\begin{pmatrix} a+d & b+e \\ c+f & d+g \end{pmatrix}=w\in M_2(\mathbb{R})$$

$$u+v=\begin{pmatrix} a & b \\ c & d \end{pmatrix}+\begin{pmatrix} d & e \\ f & g \end{pmatrix}$$

$$\Rightarrow u+v=\begin{pmatrix} a+d & b+e \\ c+f & d+g \end{pmatrix}$$

2. $$v+u=\begin{pmatrix} d & e \\ f & g \end{pmatrix}+\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\Rightarrow v+u=\begin{pmatrix} d+a & e+b \\ f+c & g+d \end{pmatrix}$$

Or, $u+v=v+u$

$$(u+v)+w=(u_1+v_1,u_2+v_2,u_3+v_3,\ldots,u_n+v_n)+(w_1,w_2,w_3,\ldots,w_n)$$

$$\Rightarrow (u+v)+w=(u_1+v_1+w_1,u_2+v_2+w_2,u_3+v_3+w_3,\ldots,u_n+v_n+w_n)$$

3. $$u+(v+w)=(u_1,u_2,u_3,\ldots,u_n)+(v_1+w_1,v_2+w_2,v_3+w_3,\ldots,v_n+w_n)$$

$$\Rightarrow u+(v+w)=(u_1+v_1+w_1,u_2+v_2+w_2,u_3+v_3+w_3,\ldots,u_n+v_n+w_n)$$

4. $$u+0=\begin{pmatrix} a & b \\ c & d \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} a+0 & b+0 \\ c+0 & d+0 \end{pmatrix}=u$$

5. $$u+(-u)=\begin{pmatrix} a & b \\ c & d \end{pmatrix}+\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}=\begin{pmatrix} a-a & b-b \\ c-c & d-d \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}=0$$

6. $$ku=k\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}\in M_2(\mathbb{R})$$

$$k(u+v)=k\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}+\begin{pmatrix} d & e \\ f & g \end{pmatrix}\right)$$

$$\Rightarrow k(u+v)=k\begin{pmatrix} a+d & b+e \\ c+f & d+g \end{pmatrix}=\begin{pmatrix} ka+kd & kb+ke \\ kc+kf & kd+kg \end{pmatrix}$$

7. $$ku+kv=k\begin{pmatrix} a & b \\ c & d \end{pmatrix}+k\begin{pmatrix} d & e \\ f & g \end{pmatrix}$$

$$\Rightarrow cu+cv=\begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}+\begin{pmatrix} kd & ke \\ kf & kg \end{pmatrix}=\begin{pmatrix} ka+kd & kb+ke \\ kc+kf & kd+kg \end{pmatrix}$$

Or, $k(u+v)=ku+kv$

$$(k+l)u=(k+l)\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} (k+l)a & (k+l)b \\ (k+l)c & (k+l)d \end{pmatrix}$$

$$\Rightarrow (k+l)u=\begin{pmatrix} ka+la & kb+lb \\ kc+lc & kd+ld \end{pmatrix}$$

8. $$ku+lu=k\begin{pmatrix} a & b \\ c & d \end{pmatrix}+l\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\Rightarrow ku+lu=\begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}+\begin{pmatrix} la & lb \\ lc & ld \end{pmatrix}$$

$$\Rightarrow ku+lu=\begin{pmatrix} ka+la & kb+lb \\ kc+lc & kd+ld \end{pmatrix}$$

$$\text{Or,} (k+l)u=ku+lu$$

$$k(lv)=k\left(l\begin{pmatrix} d & e \\ f & g \end{pmatrix}\right)=k\begin{pmatrix} ld & le \\ lf & lg \end{pmatrix}=\begin{pmatrix} kld & kle \\ klf & klg \end{pmatrix}$$

9. $$(kl)v=kl\begin{pmatrix} d & e \\ f & g \end{pmatrix}=\begin{pmatrix} kld & kle \\ klf & klg \end{pmatrix}$$

$$\Rightarrow k(lv)=(kl)v$$

For, $k=1$

$$ku=1u=1\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

10. $$\Rightarrow ku=1u=\begin{pmatrix} 1\cdot a & 1\cdot b \\ 1\cdot c & 1\cdot d \end{pmatrix}$$

$$\Rightarrow ku=1u=\begin{pmatrix} a & b \\ c & d \end{pmatrix}=u$$

$$\boxed{\text{Hence } M_2(\mathbb{R}) \text{ satisfies all conditions for vector space and is a vector space}}$$

Comment