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Chapter 29, Problem 5ED

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Problem

Let F be a field, and *K* a field extension of *F*. Prove the following:

If the degree of *F*(*a*) over *F* is a prime, then *F*(*a*) = *F*(a^n) for any *n* (on the condition that $a^n \notin F$).

Step-by-step solution

Step 1 of 3

Consider a field *F* and a field extension *K* of *F*. Objective is to prove that if the degree of $F(a)$ over *F* is a prime, then $F(a)=F(a^n)$ for any *n* (where $a^n \notin F$).

Let degree of $F(a)$ over *F* is a *p*, where *p* is prime number. That is,

$[F(a):F]=p$.

Then, it shows that *a* will be the root of polynomial whose degree is *p*. Suppose, without loss of generality, that such minimal polynomial of *a* is x^p-a.

Comment

Step 2 of 3

The corresponding equation of irreducible polynomial $x^p - a$ will be

$x^p - a = 0$.

Then $x = a^{1/p}$. The basis for $F(a)$ over *F* will be:

$\{1, a^{1/p}, a^{2/p}, \ldots, a^{(p-1)/p}\}$.

For some integer *n*, $\gcd(n, p) = 1$ because *p* is prime. Observe that, remainder of *n* when divided by *p* will be one of the $\{1, 2, 3, \ldots, p-1\}$. That is, the basis for $F(a^n)$ will be same as $\{1, a^{1/p}, a^{2/p}, \ldots, a^{(p-1)/p}\}$ when $a^n \notin F$.

Comment

Step 3 of 3

Hence, if degree of $F(a)$ over *F* is a prime, then $F(a) = F(a^n)$ for any *n* (where $a^n \notin F$).

Comment