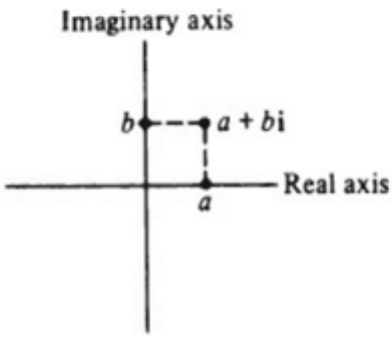
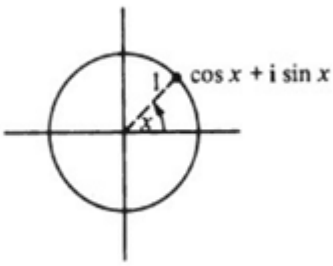
A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 4EH

Problem

Every complex number $a + b\mathbf{i}$ may be represented as a point in the complex plane.





The *unit circle* in the complex plane consists of all the complex numbers whose distance from the origin is 1; thus, clearly, the unit circle consists of all the complex numbers which can be written in the form

cos x + $\mathbf{i}$ sin x

for some real number x.

Prove that $f = \{2n\pi : n \in \mathbb{Z}\} = \langle 2\pi \rangle$.

Step-by-step solution

Step 1 of 3

Consider the set T of all the complex numbers lying on the unit circle, with the operation multiplication as:

$$T = \{\text{cis } x : x \in R\},$$

where

$\text{cis } x = \cos x + i \sin x$.

Let $f : R \to T$ is a homomorphism mapping from *R* onto *T*, defined by

$f(x) = \text{cis } x$.

Objective is to prove that $Kerf = \{2n\pi : n \in Z\}$.

Consider the following properties of trigonometric functions:

$$\begin{aligned}\sin(x+2\pi) &= \sin x \cos 2\pi + \cos x \sin 2\pi \\ &= \sin x, \\ \cos(x+2\pi) &= \cos x \cos 2\pi + \sin x \sin 2\pi \\ &= \cos x.\end{aligned}$$

Comment

Step 2 of 3

According to the definition of kernel:

$\ker f = \{x \in R : f(x) = e\}$,

where *e* is a multiplicative identity of *T*.

Since $f(x) = \text{cis } x$, so equivalently

$\ker f = \{x \in G : \text{cis } x = e\}$.

By the above identities, one have

$$\begin{aligned}\text{cis } (2n\pi) &= \cos(2n\pi) + i \sin(2n\pi) \\ &= 0.\end{aligned}$$

where $n \in Z$. Thus, $Kerf = \{2n\pi : n \in Z\}$

Comment

Step 3 of 3

Hence, $Kerf = \langle 2\pi \rangle$.

Comment