A Book of Abstract Algebra (2nd Edition)

Chapter 16, Problem 4EG

Bookmark

Show all steps: ON

Problem

If H is a subgroup of a group G, let X designate the set of all the left cosets of H in G. For each element $a \in G$, define $p_a: X \to X$ as follows:

$$p_a(xH) = (ax)H$$

Prove that if H contains no normal subgroup of G except {e}, then G is isomorphic to a subgroup of S_X .

Step-by-step solution

Step 1 of 4

Assume that G be a group and H be its subgroup. Consider that X is the set of all the left cosets of *H* in *G*. Define a homomorphism mapping, for some $a \in G$, $p_a: X \to X$ by

$$p_a(xH) = (ax)H$$

Consider the following homomorphism mapping $h: G \to S_X$ defined by $h(a) = p_a$.

Objective is to prove that if H has no normal subgroup of G except the identity subgroup, then G is isomorphic to a subgroup of S_v .

To prove this, there is a need to show the existence of a bijective homomorphism between G and a subgroup of S_X .

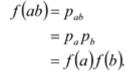
Comment

Step 2 of 4

Suppose that Y is some subgroup of S_X . Consider a mapping $f: G \to Y$ defined by:

$$f(g) = p_g$$

for some $g \in G$. Since p_a is homomorphism mapping, so



This shows that mapping f is homomorphism.

Comment

Step 3 of 4

Next, to show that f is injective suppose that f(a) = f(b), for some $a, b \in G$. Then apply the definition of f in the following manner:

$$f(a) = f(b)$$

$$p_a = p_b$$

$$p_a(xH) = p_b(xH)$$

$$(ax)H = (bx)H$$

And it implies that a = b, one-one mapping.

Let $x \in G$, then x^{-1} will also belong to G. Then

$$f(x^{-1}) = p_{x^{-1}}$$

$$= p_{x^{-1}}(xH)$$

$$= (x^{-1}x)H$$

$$= H$$

This shows that *f* is onto.

Comment

Step 4 of 4

Hence, group G is isomorphic to a subgroup of S_x .

Comment