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Chapter 16, Problem 1EG

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Problem

If H is a subgroup of a group G, let X designate the set of all the left cosets of H in G. For each element $a \in G$, define p_a: $X \to X$ as follows:

$$p_a(xH) = (ax)H$$

Prove that each p_a is a permutation of X.

Step-by-step solution

Step 1 of 4

Assume that G be a group and H be its subgroup. Consider that X is the set of all the left cosets of H in G. Define a mapping, for some $a \in G$, $p_a : X \to X$ by

$$p_a(xH) = (ax)H .$$

Objective is to prove that each p_a is a permutation of X.

The defined mapping p_a will said to be a permutation on X if it is bijective. That is, it is sufficient to prove that p_a is one-one and onto map.

Comment

Step 2 of 4

To show that mapping p_a is one-one, consider two typical elements $x, y \in G$ such that

$$\begin{aligned} p_a(xH) &= p_a(yH) \\ (ax)H &= (ay)H \\ axH &= ayH \\ xH &= yH. \end{aligned}$$

The last step is obtained by the left cancellation law of group G. Since the condition

$p_a(xH) = p_a(yH)$ implies $xH = yH$, therefore p_a is one-one.

Comment

Step 3 of 4

Let xH is the left coset of *H* in *G*, then $a^{-1}xH \in X$. This is so because *G* is a group and $a \in G$, so $a^{-1} \in G$. Then,

$$\begin{aligned} p_a(a^{-1}xH) &= (a(a^{-1}x))H \\ &= (aa^{-1}x)H \\ &= exH \\ &= xH. \end{aligned}$$

Note that $xH \in X$. This tells that every element of *X* has a pre-image, and so p_a is an onto map.

Comment

Step 4 of 4

Thus, p_a is a permutation of *X*.

Comment