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Chapter 16, Problem 1EL

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Problem

Let pbea prime number. A *p-group* is any group whose order is a power of *p*. It will be shown here that if $|G| = p^k$ then *G* has a normal subgroup of order p^m for every *m* between 1 and *k*. The proof is by induction on |*G*|; we therefore assume our result is true for all /^-groups smaller than *G*. Prove parts 1 and 2:

There is an element *a* in the center of *G* such that ord *(a)* = *p*. (See Chapter 15, Exercises G and H.)

Step-by-step solution

Step 1 of 3

Consider a group *G* whose order is a power of *p*. That is, *G* is a *p*-group and

$$|G| = p^k,$$

for some integer *k*. With the help of mathematical induction on the order of group *G*, it can be prove that *G* has a normal subgroup of order p^m for every $1 < m < k$.

Consider the induction hypothesis that this statement is true for all *p*-groups whose order is less than *G*.

Objective is to prove the existence of an element *a* in the center of *G* such that order of *a* is *p*, that is, $\operatorname{ord}(a) = p$.

Comment

Step 2 of 3

The center *C* of any group *G* defined as:

$$C = \{a \in G : ax = xa \text{ for every } x \in G\}.$$

From the properties, the center *C* of any group *G* is always a normal subgroup of *G*. Then, by the

Lagrange's theorem, the order of subgroup *C* will divide the order of *G*. That is,

$$|C|\,|\,|G| = p^k.$$

It implies that order of *C* is also some power of *p*, that is, *C* is also a *p*-group. Note that, due to the containment of all the commutative elements, the center *C* is always an abelian group.

So, center *C* is an abelian *p*-group. Apply the Cauchy theorem for abelian group and get that there will definitely exists an element $a \in C$ such that

$a^p = e$ and $a^k \neq e$ for some $k < p$.

Comment

Step 3 of 3

Hence, there will exist an element $a \in C$ such that $\operatorname{ord}(a) = p$.

Comment