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Chapter 28, Problem 2EF

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Problem

Let U and V be vector spaces over the field F, with dim $U = n$ and dim $V = m$. Let $h : U \to V$ be a homomorphism.

Prove the following:

h is injective iff dim U = dim $h(U)$.

Step-by-step solution

Step 1 of 3

It is already known that U and V are vector spaces and so they satisfies all conditions for vector space. It is known that basis of U contains n elements. Thus, dimension of U is n.

Comment

Step 2 of 3

Linear transformation h is said to be injective if,

$$h(\mathbf{a}) = h(\mathbf{b}) \Rightarrow \mathbf{a} = \mathbf{b}$$

Comment

Step 3 of 3

From question 7 of section E,

$$\dim(\text{domain of } h) = n = r + n - r = \dim(\text{nullspace of } h) + \dim(\text{range space of } h)$$

If $\dim U = \dim h(U)$ or, dimension of domain of *h* is equal to dimension of range of *h*, substitute result in above mentioned result,

$$\dim U = n = \dim h(U) + \dim(\text{nullspace space of } h)$$

$$\Rightarrow \boxed{\dim(\text{nullspace space of } h) = 0}$$

Thus *h* is injective as proved in question 3 of section E

Now if *h* is injective or $\dim(\text{nullspace space of } h) = 0$,

It can be seen from same formula used above that,

$$\boxed{\dim U = \dim h(U)}$$

$$\boxed{\text{Hence it can be said that } h \text{ is injective iff } \dim U = \dim h(U)}$$

Comment