A Book of Abstract Algebra | (2nd Edition)

Chapter 23, Problem 8EH

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Problem

An integer a is called a *quadratic residue* modulo m if there is an integer x such that $x^2 \equiv a \pmod{m}$. This is the same as saying that $\bar{a}$ is a square in $\mathbb{Z}_m$. If a is not a quadratic residue modulo m, then a is called a *quadratic nonresidue* modulo m. Quadratic residues are important for solving quadratic congruences, for studying sums of squares, etc. Here, we will examine quadratic residues modulo an arbitrary prime $p > 2$.

Let $h : \mathbb{Z}_p^* \to \mathbb{Z}_p^*$ be defined by $h(\bar{a}) = \bar{a}^2$.

Use parts 5 to 7 and the law of quadratic reciprocity to find:

$$\left(\frac{30}{101}\right) \quad \left(\frac{10}{151}\right) \quad \left(\frac{15}{41}\right) \quad \left(\frac{14}{59}\right) \quad \left(\frac{379}{401}\right)$$

Is 14 a quadratic residue, modulo 59?

Step-by-step solution

Step 1 of 7

Here, objective is to find the given Legendre symbols by using law of reciprocity.

Comment

Step 2 of 7

Law of reciprocity :

$$\left(\frac{p}{q}\right)=\begin{cases}-\left(\frac{q}{p}\right) & \text{if } p,q \text{ are } 3(\text{mod } 4)\\ \left(\frac{q}{p}\right) & \text{otherwise}\end{cases}$$

Comment

Step 3 of 7

Consider the Legendre symbol $\frac{30}{101}$

$$\begin{aligned}\frac{30}{101}&=-\frac{15}{101}\\&=-\frac{11}{15}\\&=\frac{4}{11}\\&=-\frac{2}{11}\\&=\frac{1}{11}\\&=1\end{aligned}$$

Hence, $\frac{30}{101}=1$

Comment

Step 4 of 7

Consider the Legendre symbol $\frac{10}{151}$

$$\begin{aligned}\frac{10}{151}&=\frac{5}{151}\\&=\frac{1}{5}\\&=1\end{aligned}$$

Hence, $\frac{10}{151}=1$

Comment

Step 5 of 7

Consider the Legendre symbol $\frac{15}{41}$

$$\frac{15}{41}=\frac{11}{15}$$
$$=-\frac{4}{11}$$
$$=\frac{2}{11}$$
$$=-\frac{1}{11}$$
$$=-1$$

Hence, $\frac{15}{41}=-1$

Comment

Step 6 of 7

Consider the Legendre symbol $\frac{14}{59}$

$$\frac{14}{59}=-\frac{7}{59}$$
$$=\frac{3}{7}$$
$$=-\frac{1}{3}$$
$$=-1$$

Hence, $\frac{14}{59}=-1$

Therefore, 14 is non-quadratic residue, modulo 59

Comment

Step 7 of 7

Consider the Legendre symbol $\frac{379}{401}$

$$\frac{379}{401}=\frac{22}{379}$$
$$=-\frac{11}{379}$$
$$=\frac{1}{5}$$
$$=1$$

Hence, $\frac{379}{401} = 1$

Comment