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Chapter 16, Problem 3EN

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Problem

Let G be a finite group, and K a p-Sylow subgroup of G. Let X be the set of all the conjugates of K. See Exercise M2. If C_1, $C_2 \in X$, let $C_1 \sim C_2$ iff $C_1 = aC_2a^{-l}$ for some $\alpha \in K$

Use Exercise M7 to prove that the only class with a single element is [K],

Step-by-step solution

Step 1 of 3

Assume that G is a finite group and K a p-Sylow subgroup of G. Consider the set X as the set of all the conjugates of K. Define an equivalence relation as:

If $C_1, C_2 \in X$, let $C_1 \approx C_2$ if and only if $C_1 = aC_2a^{-1}$ for some $a \in K$.

Consider the following result:

If $aKa^{-1} = K$ and the order of a is some power of p. Then $a \in K$.

Objective is to use this result and prove that the only class with a single element is $[K]$.

Comment

Step 2 of 3

Let K be any subgroup of G. Let $K^* = \{Na : a \in K\}$ and $X_K = \{aHa^{-1} : a \in K\}$. The defined set X_K is in one-one correspondence with K^*. And the number of element in X_K is a divisor of $|K|$.

If $aKa^{-1} = K$ for all a, then C will form a subset of K. But by the above statement K and C have the same number of elements. Since both are isomorphic (as there exists an one-one correspondence), therefore $K = C$.

Comment

Step 3 of 3

Thus, the only class of size one is K itself, that is, $[K]$.

Comment