A Book of Abstract Algebra | (2nd Edition)

Problem

Let *f* be a homomorphism from G onto *H* with kernel *K*:

$$f : G \underset{K}{\twoheadrightarrow} H$$

If *S* is any subgroup of *H*, let $S^* = \{x \in G\text{: } f(x) \in S\}$. Prove:

$K \subseteq S^*$.

Step-by-step solution

Step 1 of 4

Suppose that *G* is any group. Let the mapping

$$f : G_K \to H$$

is a homomorphism from *G* onto *H* with kernel *K*. Assume that *S* is any subgroup of *H* and consider the following set:

$$S^* = \{x \in G : f(x) \in S\}.$$

Objective is to prove that $K \subseteq S^*$, that is, kernel forms a subgroup of S^*.

One step test: If *H* is a nonempty subset of group *G*, then *H* will be subgroup of *G* if and only if for all $a, b \in H$

$$ab^{-1} \in H \cdot$$

Comment

Step 2 of 4

The kernel of *f* will be defined as:

$$\ker f = \{x \in G : f(x) = e\}.$$

where *e* is the identity of *H*. Note that identity maps to identity and *G* has the identity. Therefore,

$e \in \ker f$. Because of the existence of identity, the kernel of *f* will be nonempty.

Let $x, y \in \ker f$ such that $f(x), f(y) =$ identity of *H*. Since *S* is a subgroup of *H*, so identity of *H* will belong to *S*. It implies that, $f(x) \in S$.

Comment

Step 3 of 4

Now it is remaining to prove that $xy^{-1} \in \ker f$. For this, consider $f(xy^{-1})$ and expand it by the homomorphism rule as:

$$\begin{aligned} f(xy^{-1}) &= f(x)f(y^{-1}) \\ &= f(x)[f(y)]^{-1} \\ &= e \cdot e \\ &= e \end{aligned}$$

Thus according to the definition of kernel, $xy^{-1} \in \ker f$.

Comment

Step 4 of 4

Hence, by one-step test it can be conclude that $K \subseteq S^*$.

Comment