A Book of Abstract Algebra | (2nd Edition)

Chapter 32, Problem 2EA

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Problem

Find the degree of $\mathbb{Q}(i,\sqrt{2})$ over $\mathbb{Q}$.

Step-by-step solution

Step 1 of 2

The objective is to find the degree of $\mathbb{Q}\left(i,\sqrt{2}\right)$ over $\mathbb{Q}$.

Comment

Step 2 of 2

The minimal polynomial of $\sqrt{2}$ over $\mathbb{Q}$ is x^2-2 as it is monic and irreducible with $\sqrt{2}$ as a root.

Hence, $\left[\mathbb{Q}\left(\sqrt{2}\right):\mathbb{Q}\right]=2$; a basis is $\left\{1,\sqrt{2}\right\}$.

Show that $i\notin\mathbb{Q}\left(\sqrt{2}\right)$.

Assume that $i\in\mathbb{Q}\left(\sqrt{2}\right)$.

Then i must have the form $a+b\sqrt{2}$, for some $a,b\in\mathbb{Q}$.

It follows that $\left(a+b\sqrt{2}\right)^2=-1$ and thus $a^2+2\sqrt{2}ab+2b^2+1=0$.

Since $\left\{1,\sqrt{2}\right\}$ is a linear independent set as it is a basis for $\mathbb{Q}\left(\sqrt{2}\right)$ as a vector space over $\mathbb{Q}$, either $a=0$ or $b=0$.

If $a=0$ then $b=\pm\frac{1}{\sqrt{2}}i$ and if $b=0$ then $a=\pm i$.

This is a contradiction to $a,b\in\mathbb{Q}$.

Hence， x^2+1 is irreducible over $\mathbb{Q}(\sqrt{2})$; it is a minimal polynomial over $\mathbb{Q}(\sqrt{2})$.

So， $\left[\mathbb{Q}(i,\sqrt{2}):\mathbb{Q}(\sqrt{2})\right]=2$ and that $\{1,i\}$ is a basis for $\mathbb{Q}(i,\sqrt{2})$ over $\mathbb{Q}(\sqrt{2})$.

Therefore， $\left[\mathbb{Q}(i,\sqrt{2}):\mathbb{Q}\right]=\left[\mathbb{Q}(i,\sqrt{2}):\mathbb{Q}(\sqrt{2})\right]\left[\mathbb{Q}(\sqrt{2}):\mathbb{Q}\right]$

$=2\cdot 2$

$=4.$

Comment