A Book of Abstract Algebra | (2nd Edition)

Problem

Let F be any field.

Prove part:

Prove: $\mathbb{Q}(1+i) \cong \mathbb{Q}(1-i)$. However, $\mathbb{Q}(\sqrt{2}) \cong \mathbb{Q}(\sqrt{3})$.

Step-by-step solution

Step 1 of 4

Consider that F is any arbitrary field. Objective is to prove that

$Q(1+i) \cong Q(1-i)$.

Consider the following result:

Let F is any arbitrary field. If $p(x) \in F[x]$ is an irreducible polynomial and c is some root of $p(x)$, then $\frac{F[x]}{\langle p(x) \rangle} \cong F(c)$. Also if c and d are roots of the same irreducible polynomial $p(x)$ in $F[x]$, then

$F(c) \cong F(d)$.

Comment

Step 2 of 4

By the result, the task is to find the polynomial $p(x)$ whose roots are $1 \pm i$. Let $a = 1+i$. Then

$$\begin{aligned} a-1 &= i \\ (a-1)^2 &= (i)^2 \\ a^2-2a+1 &= -1 \\ a^2-2a+2 &= 0. \end{aligned}$$

Similarly, when $a = 1-i$

$$\begin{aligned} a-1 &= -i \\ (a-1)^2 &= (-i)^2 \\ a^2-2a+1 &= -1 \\ a^2-2a+2 &= 0. \end{aligned}$$

Thus, the polynomial x^2-2x+2 has roots $1 \pm i$.

Comment

Step 3 of 4

That is, $1+i$ and $1-i$ are roots of the same irreducible polynomial $p(x) = x^2-2x+2$ in $Q[x]$. Therefore, $Q(1+i) \cong Q(1-i)$.

Now, show that $Q(\sqrt{2}) \ncong Q(\sqrt{3})$. Let, if possible, there is an isomorphism $f : Q(\sqrt{2}) \to Q(\sqrt{3})$. Then $f(\sqrt{2}) = a + b\sqrt{3}$ satisfies $2 = (a+b\sqrt{3})^2$. That is, $a^2+3b^2+2ab\sqrt{3} = 2$. Note that 1 and $\sqrt{3}$ are linearly independent over Q. So, $2ab = 0$, either $a = 0$, or $b = 0$. Thus, either $a^2 = 2$, or $3b^2 = 2$. Neither of the equation have the solution in Q.

Comment

Step 4 of 4

Hence, $Q(\sqrt{2}) \ncong Q(\sqrt{3})$.

Comment