A Book of Abstract Algebra | (2nd Edition)

Problem

Let $F \subseteq K$ and $a, b \in K$. We have seen on page 295 that if a and b are algebraic over F, then $F(a, b)$ is a finite extension of F.

Use the above to prove part.

$c \in \mathbb{A}$.

Conclusion: The roots of any polynomial whose coefficients are algebraic numbers are themselves algebraic numbers.

A field F is called *algebraically closed* if the roots of every polynomial in $F[x]$ are in F. We have thus proved that is algebraically closed.

Step-by-step solution

Step 1 of 3

Consider a field Q and a field ***A*** of set of all algebraic numbers. Let

$a(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbf{A}[x]$,

and c be any root of $a(x)$. Let

$Q(a_0, a_1, \ldots, a_n) = Q_1$.

Since $a(x) \in Q_1[x]$, *c* is algebraic over Q_1. Objective is to prove that $c \in \mathbf{A}$.

Comment

Step 2 of 3

Consider the following results:

(1) Q_1 is a finite extension of *Q*.

(2) $Q_1(c)$ is a finite extension of Q_1 as well as finite extension of *Q*.

Since *c* is algebraic over Q_1, and Q_1 is a finite extension of *Q*. Therefore, *c* is algebraic over *Q* as well. Also, the set of all the algebraic numbers forms a field *A*. This implies that *c* will also be the element of field ***A***.

Comment

Step 3 of 3

Hence, $c \in \mathbf{A}$.

Comment