A Book of Abstract Algebra | (2nd Edition)

Problem

Let $a(x) = a_0 + a_1x + \cdots + a_nx^n \in F[x]$. The *derivative* of $a(x)$ is the following polynomial $a'(x) \in F[x]$:

$$a'(x) = a_1 + 2a_2x + \ldots + na_nx^{n-1}$$

(This is the same as the derivative of a polynomial in calculus.) We now prove the analogs of the formal rules of differentiation, familiar from calculus.

Let $a(x)$, $b(x) \in F[x]$, and let $k \in F$.

Prove part:

$[a(x)b(x)]' = a'(x)b(x) + a(x)b'(x)$

Step-by-step solution

Step 1 of 3

Consider the arbitrary field F and let $a(x) = a_0 + a_1x + \cdots + a_nx^n \in F(x)$. The derivative of $a(x)$ will be given by

$$a'(x) = a_1 + 2a_2x + \cdots + na_nx^{n-1} \in F(x).$$

Objective is to prove that

$$[a(x)b(x)]' = a'(x)b(x) + a(x)b'(x).$$

where $a(x), b(x) \in F[x]$.

Comment

Step 2 of 3

Let $y = a(x)b(x)$. Take logarithm both the sides and then calculate its derivatives as:

$$\begin{aligned} \log y &= \log[a(x)b(x)] \\ \log y &= \log a(x) + \log b(x) \\ \frac{y'}{y} &= \frac{a'(x)}{a(x)} + \frac{b'(x)}{b(x)}. \end{aligned}$$

The derivative is obtained by the chain rule of differentiation.

Now, solve this relation for y' and use the assumption that $y = a(x)b(x)$ as:

$$\begin{aligned} y' &= y\left[\frac{a'(x)b(x) + a(x)b'(x)}{a(x)b(x)}\right] \\ &= a(x)b(x)\left[\frac{a'(x)b(x) + a(x)b'(x)}{a(x)b(x)}\right] \\ &= a'(x)b(x) + a(x)b'(x). \end{aligned}$$

Comment

Step 3 of 3

Hence, $[a(x)b(x)]' = a'(x)b(x) + a(x)b'(x)$.

Comment