A Book of Abstract Algebra | (2nd Edition)

Problem

Show that the minimum polynomial of $\sqrt{2}+i$ is

(a) $x^2-2\sqrt{2}x+3$ over $\mathbb{R}$

(b) x^4-2x^2+9 over $\mathbb{Q}$

(c) $x^2-2ix-3$ over $\mathbb{Q}(i)$

Step-by-step solution

Step 1 of 4

(a)

Objective is to prove that the minimum polynomial of $\sqrt{2}+i$ is $x^2-2\sqrt{2}x+3$ over *R*.

Let $x=\sqrt{2}+i$. Then

$$\begin{aligned}x-\sqrt{2}&=i\\(x-\sqrt{2})^2&=i^2\\x^2-2\sqrt{2}x+2&=-1.\end{aligned}$$

Thus, the minimum polynomial will be $x^2-2\sqrt{2}x+3$ over *R* as $\sqrt{2}\in R$.

Comment

Step 2 of 4

(b)

Objective is to prove that the minimum polynomial of $\sqrt{2}+i$ is x^4-2x^2+9 over *Q*.

Let $x=\sqrt{2}+i$. Then

$$\begin{aligned}x-i&=\sqrt{2}\\(x-i)^2&=(\sqrt{2})^2\\x^2-2ix+i^2&=2\\x^2-3-2ix&=0.\end{aligned}$$

Also

$$\begin{aligned}x^2-3&=2ix\\(x^2-3)^2&=(2ix)^2\\x^4-6x^2+9&=-4x^2\\x^4-2x^2+9&=0.\end{aligned}$$

Thus, the minimum polynomial will be x^4-2x^2+9 over *Q*.

Comment

Step 3 of 4

(c)

Objective is to prove that the minimum polynomial of $\sqrt{2}+i$ is $x^2-2ix-3$ over $Q(i)$.

Let $x=\sqrt{2}+i$. Then

$$\begin{aligned}x-i&=\sqrt{2}\\(x-i)^2&=(\sqrt{2})^2\\x^2-2ix+i^2&=2\\x^2-3-2ix&=0.\end{aligned}$$

Comment

Step 4 of 4

Thus, the minimum polynomial will be $x^2-2ix-3$ over $Q(i)$ as $i\in Q(i)$.

Comment