A Book of Abstract Algebra | (2nd Edition)

Chapter 32, Problem 5EC

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Problem

Use part 4 to prove that $Gal(\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3}) : \mathbb{Q}) \cong S_3$.

Step-by-step solution

Step 1 of 2

The objective is to show that $Gal\left(\mathbb{Q}\left(\sqrt[3]{2}, i\sqrt{3}\right) : \mathbb{Q}\right) \cong S_3$.

Comment

Step 2 of 2

The group $Gal\left(\mathbb{Q}\left(\sqrt[3]{2}, i\sqrt{3}\right) : \mathbb{Q}\right)$, being of order six, must be isomorphic to either $\mathbb{Z}/6\mathbb{Z}$ or S_3.

Claim: $Gal\left(\mathbb{Q}\left(\sqrt[3]{2}, i\sqrt{3}\right) : \mathbb{Q}\right) \cong S_3$.

Show that $Gal\left(\mathbb{Q}\left(\sqrt[3]{2}, i\sqrt{3}\right) : \mathbb{Q}\right)$ is not abelian.

Calculate the effect of both $\sigma_2\sigma_4$ and $\sigma_4\sigma_2$ on the element $\sqrt[3]{2}$:

The automorphism σ_2 and σ_4 is as follows:

$$\sigma_2 : \begin{cases} \sqrt[3]{2} \mapsto \omega\sqrt[3]{2} \\ i \mapsto i\sqrt{3} \end{cases} \quad \sigma_4 : \begin{cases} \sqrt[3]{2} \mapsto \sqrt[3]{2} \\ i \mapsto -i\sqrt{3} \end{cases}.$$

$$\begin{aligned} \sigma_2\sigma_4\left(\sqrt[3]{2}\right) &= \sigma_2\left(\sqrt[3]{2}\right) \\ &= \omega\sqrt[3]{2} \end{aligned}$$

whereas

$$\begin{aligned}\sigma_4\sigma_2\left(\sqrt[3]{2}\right) &= \sigma_4\left(\omega\sqrt[3]{2}\right)\\ &= \sigma_4(\omega)\sigma_4\left(\sqrt[3]{2}\right)\\ &= \omega^2\sqrt[3]{2}.\end{aligned}$$

Thus，$\sigma_2\sigma_4$ and $\sigma_4\sigma_2$ are not the same function，so this group of automorphism is not abelian.

Therefore，$Gal\left(\mathbb{Q}\left(\sqrt[3]{2}, i\sqrt{3}\right):\mathbb{Q}\right) \cong S_3$.

Comment