A Book of Abstract Algebra | (2nd Edition)

Problem

By de Moivre's theorem,

$$\omega = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$$

is a complex seventh root of unity. Since

$$x^7 - 1 = (x-1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$$

ω is a root of $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$.

Conclude from part 3 that cos (2π/7) is not a constructible number.

Step-by-step solution

Step 1 of 4

Here, objective is to prove that $\cos\frac{2\pi}{7}$ is not a constructible number.

Comment

Step 2 of 4

De Moivre's theorem:

$\omega = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$ is a complex seventh root of unity.

Since $x^7 - 1 = (x-1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$

ω is a root of $P(x) = (x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$

Then,

$$\left(\omega + \frac{1}{\omega}\right) = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7} + \cos\frac{2\pi}{7} - i\sin\frac{2\pi}{7}$$

$$\left(\omega + \frac{1}{\omega}\right) = 2\cos\frac{2\pi}{7}$$

Comment

Step 3 of 4

Consider ω is a root of $P(x) = (x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$

Then.

$$(\omega^4 + \omega^3 + \omega^2 + \omega + 1) = 0$$

$$\omega^3 + \omega^2 + \omega + 1 + \omega^{-1} + \omega^{-2} + \omega^{-3} = 0$$

$$\left(\omega + \frac{1}{\omega}\right)^3 + \left(\omega + \frac{1}{\omega}\right)^2 - 2\left(\omega + \frac{1}{\omega}\right) - 1 = 0$$

$$\left(2\cos\frac{2\pi}{7}\right)^3 + \left(2\cos\frac{2\pi}{7}\right)^2 - 2\left(2\cos\frac{2\pi}{7}\right) - 1 = 0$$

$$8\cos^3\frac{2\pi}{7} + 4\cos^2\frac{2\pi}{7} - 4\cos\frac{2\pi}{7} - 1 = 0$$

Comment

Step 4 of 4

For $x = 2\cos\frac{2\pi}{7}$

The equation is $8x^3 + 4x^2 - 4x - 1 = 0$

The above polynomial is irreducible. Since this is a cubic equation.

The roots cannot be constructed. $\cos\frac{2\pi}{7}$ is not constructible.

Hence, $\cos\frac{2\pi}{7}$ is not a constructible number.

Comment