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Chapter 33, Problem 3EB | Bookmark | Show all steps: ON

Problem

Let G be a group. The symbol $H \lhd G$ is commonly used as an abbreviation of "H is a *normal* subgroup of G." A *normal series* of G is a finite sequence $H_0, H_1, \ldots, H_n$ of subgroups of G such that

$$\{e\} = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G$$

Such a series is called a *solvable series* if each quotient group H_{i+1}/H_i is abelian. G is called a *solvable group* if it has a solvable series.

Use the remark immediately preceding Theorem 2 to prove that $J_0, \ldots, J_n$ is a solvable series of K.

Step-by-step solution

Step 1 of 4

Here, objective is to prove that $J_0, J_1, \ldots\ldots\ldots\ldots, J_n$ is a solvable series of K.

Comment

Step 2 of 4

A group G is solvable, if there exist a finite chain of successive subgroups $1 = G_0 \le G_1 \le G_2 \le \ldots\ldots\ldots\ldots \le G_n$ having the following properties.

G_i is the normal subgroup of $G_{i+1}; \forall \quad 0 \le i \le n-1$

$\frac{G_{i+1}}{G_i}$ is an Abelian group $\forall \quad 0 \le i \le n-1$

Theorem:

Any homomorphic image of solvable group is a solvable group.

Comment

Step 3 of 4

Consider $J_0, J_1, \ldots\ldots\ldots\ldots, J_n$ is a normal series of *K*.

Where, $J_0 = K \cap H_0, \ldots\ldots\ldots\ldots, J_n = K \cap H_n$

Comment

Step 4 of 4

Let $f : K \to X$ is a homomorphism from *K* on to group *X*.

Then $f(J_0), f(J_1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots, f(J_n)$ are subgroups of *X* and it is clear that $f(J_0) \subseteq f(J_1) \subseteq \ldots\ldots\ldots \subseteq f(J_n) = X$

For each *i*, we have, if $f(a) \in f(J_i)$, then $a \in f(J_i), x \in f(J_{i+1})$

Hence, $xax^{-1} \in J_i$. Therefore $f(x)f(a)f(x^{-1}) \in f(J_i)$

So, $f(J_i)$ is a normal subgroup of $f(J_{i+1})$ trivially.

Hence, $\frac{J_{i+1}}{J_i}$ is a abelian group $\forall \quad 0 \leq i \leq n-1$

Therefore, $J_0, J_1, \ldots\ldots\ldots\ldots, J_n$ is a solvable series of *K*.

Hence, proved

Comment