A Book of Abstract Algebra | (2nd Edition)

Chapter 28, Problem 5EE

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Problem

Let N be the null space of h, and $\mathcal{R}$ the range space of h. Let $\{\mathbf{a}_1, \ldots, \mathbf{a}_r\}$ be a basis of N. Extend it to a basis $\{\mathbf{a}_1, \ldots, \mathbf{a}_r, \ldots, \mathbf{a}_n\}$ of U.

Prove part:

$\{h(\mathbf{a}_{r+1}), \ldots, h(\mathbf{a}_n)\}$ is linearly independent.

Step-by-step solution

Step 1 of 5

It is already known that U and V are vector spaces and so they satisfies all conditions for vector space.

Comment

Step 2 of 5

Range space of h is subspace of V is set of all elements of V which are map of vectors of U.

Comment

Step 3 of 5

Or given subspace is

$$\{\mathbf{r} \in V \mid h(\mathbf{u}) = \mathbf{r} \text{ for } \mathbf{u} \in U\}$$

Comment

Step 4 of 5

Thus any element in range is a map of some vector in *U*

Comment

Step 5 of 5

For any element r in range of *h,* we can find a element *u* in *U* such that

$$h(\mathbf{u}) = \mathbf{r}$$

Since *U* is a vector space, every element in *U* can be expressed as linear combination of basis of *U*. So,

$$\mathbf{u} = t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \dots t_r\mathbf{a}_r + \dots + t_n\mathbf{a}_n$$

Taking linear transformation

$$h(\mathbf{u}) = h(t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \dots t_r\mathbf{a}_r + \dots + t_n\mathbf{a}_n)$$
$$\Rightarrow h(\mathbf{u}) = t_1h(\mathbf{a}_1) + t_2h(\mathbf{a}_2) + \dots + t_rh(\mathbf{a}_r) + \dots + t_nh(\mathbf{a}_n) \qquad \dots(1)$$

Since $(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_r)$ is null basis of *h*,

$$h(\mathbf{a}_r) = \mathbf{0}\,\forall r \in (0, 1, \dots, r)$$

Therefore (1) can be rewritten as,

$$h(\mathbf{u}) = t_{r+1}h(\mathbf{a}_{r+1}) + \ldots + t_n h(\mathbf{a}_n)$$

$h(\mathbf{u})$ represents a subspace, all element of which can be expressed as linear combinations of $h(\mathbf{a}_{r+1}),\ldots,h(\mathbf{a}_n)$. In other words $h(\mathbf{a}_{r+1}),\ldots,h(\mathbf{a}_n)$ forms basis of range of *h*. As vectors in basis are always independent, $\boxed{h(\mathbf{a}_{r+1}),\ldots,h(\mathbf{a}_n) \text{ are independent vectors}}$.

Comment