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Chapter 29, Problem 3EF

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Problem

Let *F* be a field, and *K* a finite extension of *F*. Prove each of the following:

If fe is algebraic over *K*, then [*K*(*b*) :*K*] ≤ [*F*(*b*) : *F*]. (HINT: The minimum polynomial of *b* over *F* may factor in *K*[*x*], and 6 will then be a root of one of its irreducible factors.)

Step-by-step solution

Step 1 of 3

Consider a field *F* and a finite extension *K* of *F*. Objective is to prove that if *b* is algebraic over *K*, then $[K(b):K] \le [F(b):F]$.

Since *b* is algebraic of *K*, therefore $F(b)$ is a finite extension of field *F*, and $K(b)$ is a finite extension of field *K*. Then, by extension property

$$F \subseteq F(b),$$
$$K \subseteq K(b).$$

Also *K* is a finite extension of *F*, so $F \subseteq K$.

Comment

Step 2 of 3

Note that, it may possible that the minimal polynomial of *b* over *F* has factor in $K[x]$, because *b* is algebraic of *K*. Also, if this happens then *b* will be the root of one of its irreducible factors.

But, there does not exist any root of minimal polynomial in $F[x]$. Therefore, the degree of extension field $K(b)$ over *K* will be less or equal to the degree of extension field $F(b)$ over *F*.

Comment

Step 3 of 3

Hence, if *b* is algebraic over *K*, then $[K(b):K] \leq [F(b):F]$.

Comment