A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 5EP

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Problem

Let G be an abelian group of order $p^k m$, where p^k and m are relatively prime (that is, p^k and m have no common factors except ±1). (REMARK: If two integers j and k are relatively prime, then there are integers s and t such that $sj + tk = 1$. This is proved on page 220.)

Let G_pk be the subgroup of G consisting of all elements whose order divides p^k. Let Gm be the subgroup of G consisting of all elements whose order divides ra. Prove:

Suppose $|G|$ has the following factorization into primes: $|G| = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}$. Then $G \cong G_1 \times G_2 \times \cdots \times G_n$ where for each $i = 1, \ldots, n$, G_i is a p_i-group.

Step-by-step solution

Step 1 of 4

Assume that G is an abelian group of order $p^k m$, where p^k and m are relatively prime. Suppose that the order of G has the following prime factorization:

$$|G| = p_1^{k_1} p_2^{k_2} \quad p_n^{k_n}.$$

Objective is to prove that $G \cong G_1 \times G_2 \times \quad \times G_n$ where for each $i = 1, 2, \quad , n$, G_i is a p_i group.

Consider the following result:

if G is an internal direct product of $H_1, \quad , H_k$, then $G \cong H_1 \times \quad \times H_k$.

Comment

Step 2 of 4

To show the required result, prove that G is an internal direct product of its p_i group. Since G is an abelian group, so all subgroups of G will be normal. Next show that

$$G = G_1 G_2 \quad G_n.$$

By the internal direct product property, $G_1G_2 \quad G_n \subseteq G$ and the order of such product is defined as:

$$\begin{aligned} |G_1G_2 \quad G_n| &= |G_1||G_2| \quad |G_n| \\ &= p_1^{k_1} p_2^{k_2} \quad p_n^{k_n} \\ &= |G|. \end{aligned}$$

Hence, $G = G_1G_2 \quad G_n$.

Comment

Step 3 of 4

Now, the remaining work is to prove that all p_i group are distinct, that is,

$$G_i \quad (G_1, \quad G_{i-1}, G_{i+1}, \quad ,G_n) = \{e\},$$

for all $i \in \{1, 2, ..., k\}$. For some fix *i*, the order of *G* will be:

$$|G| = p_i^{k_i} a,$$

where p_i and *a* are relatively prime. Also by internal direct product property,

$$|G_1, \quad G_{i-1}, G_{i+1}, \quad ,G_n| = a.$$

And thus $|G_i| = p^{k_i}$. Then

$$(|G_1, \quad G_{i-1}, G_{i+1}, \quad ,G_n|, |G_i|) = 1$$

and thus

$$G_i \quad (G_1, \quad G_{i-1}, G_{i+1}, \quad ,G_n) = \{e\}.$$

It shows that *G* is an internal direct product of $G_1, G_2, \quad ,G_n$.

Comment

Step 4 of 4

Hence, $G \cong G_1 \times G_2 \times \quad \times G_n$.

Comment