A Book of Abstract Algebra | (2nd Edition)

Chapter 33, Problem 2EE

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Problem

Let K be a finite extension of F, where K is a root field over F, with $G = Gal(K: F)$ a solvable group. As remarked in the text, we will assume that F contains the required roots of unity. By Exercise D, let $H_0, \dots, H_n$ be a solvable series for G in which every quotient H_{i+1}/H_i is cyclic of prime order. For any $i = 1, \dots, n$, let F_i. and F_{i+1} be the fixfields of H_i and H_{i+1}.

Let π be a generator of $Gal(Fi: Fi + 1)$, ω a pth root of unity in F_{i+1}, and $b \in F_i$. Set

$$c = b + \omega\pi^{-1}(b) + \omega^2\pi^{-2}(b) + \cdots + \omega^{p-1}\pi^{-(p-1)}(b)$$

Show that $\pi(c) = \omega c$.

Step-by-step solution

Step 1 of 4

Here, objective is to prove that $\pi(c) = \omega c$.

Comment

Step 2 of 4

A G is a group of automorphism of K. The set of elements fixed by every element of G called the fixed field.

Comment

Step 3 of 4

$G = Gal(K:F)$ is a solvable group.

F is the fixed field of *G.*

Where, *K* is a the finite extension of *F.*

Consider F_i and F_{i+1} are the fixed fields of H_i and H_{i+1}

Comment

Step 4 of 4

Consider π is the generator of $Gal[F_i : F_{i+1}]$

$F_i = F_{i+1}(\pi)$

$b \in F_i$

Then,

$b = F_{i+1}(\pi)$

$\pi^{-1}(b) = F_{i+1}$

ω is a p^{th} root of unity in F_{i+1} and $b \in F_i$

Consider

$$c = b + \omega\pi^{-1}(b) + \omega^2\pi^{-2}(b) + ... + \omega^{p-1}\pi^{-(p-1)}(b)$$

$$\begin{aligned}\pi(c) &= \pi(b) + b + \omega\pi^{-1}(b) + \omega^2\pi^{-2}(b) + ... + \omega^{p-1}\pi^{-(p-2)}(b) \\ &= \pi(b) + c \\ &= c \qquad (\because b \in F_i) \\ &= \omega c \qquad (\because \omega = \sqrt[p]{1})\end{aligned}$$

Hence, proved

Comment