A Book of Abstract Algebra | (2nd Edition)

Problem

By de Moivre's theorem,

$$\omega = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$$

is a complex seventh root of unity. Since

$$x^7 - 1 = (x-1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$$

ω is a root of $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$.

Prove that $\omega^3 + \omega^2 + \omega + 1 + \omega^{-1} + \omega^{-2} + \omega^{-3} = 0$.

Step-by-step solution

Step 1 of 3

Here, objective is to prove that $\omega^3 + \omega^2 + \omega + 1 + \omega^{-1} + \omega^{-2} + \omega^{-3} = 0$.

Comment

Step 2 of 3

De Moivre's theorem:

$\omega = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$ is a complex seventh root of unity.

Since $x^7 - 1 = (x-1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$

ω is a root of $P(x) = (x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$

Comment

Step 3 of 3

Consider ω is a root of $P(x) = (x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$

Then,

$$P(\omega) = 0$$
$$\omega^6 + \omega^5 + \omega^4 + \omega^3 + \omega^2 + \omega + 1 = 0$$
$$\omega^3 \cdot \omega^3 + \omega^3 \cdot \omega^2 + \omega^3 \cdot \omega + \omega^3 \cdot 1 + \omega^3 \cdot \omega^{-1} + \omega^3 \cdot \omega^{-2} + \omega^3 \cdot \omega^{-3} = 0$$
$$\omega^3(\omega^3 + \omega^2 + \omega + 1 + \omega^{-1} + \omega^{-2} + \omega^{-3}) = 0$$
$$\omega^3 = 0 \text{ or}$$
$$(\omega^3 + \omega^2 + \omega + 1 + \omega^{-1} + \omega^{-2} + \omega^{-3}) = 0$$

Hence, proved

Comment