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Chapter 29, Problem 2EF

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Problem

Let *F* be a field, and *K* a finite extension of *F*. Prove each of the following:

If *b* is algebraic over *K*, then [*F*(*b* :*F*] | [*K*(*b*) :*F*].

Step-by-step solution

Step 1 of 3

Consider a field *F* and a finite extension *K* of *F*. Objective is to prove that if *b* is algebraic over *K*, then $[F(b):F]|[K(b):F]$.

Since *b* is algebraic of *K*, therefore $F(b)$ is a finite extension of field *F*, and $K(b)$ is a finite extension of field *K*. Then, by extension property

$$F \subseteq F(b),$$
$$K \subseteq K(b).$$

Also *K* is a finite extension of *F*, so $F \subseteq K$.

Comment

Step 2 of 3

On combining the above relationship, one gets

$F \subseteq F(b) \subseteq K(b)$.

Then by the formula for degree calculation of field,

$[K(b):F] = [K(b):F(b)] \cdot [F(b):F]$.

Here, $[K(b):F(b)] \neq 0$ and equals to some finite integer because *K* is a finite extension of *F*.
Thus, by the divisibility rule it implies that $[F(b):F]$ divides $[K(b):F]$.

Comment

Step 3 of 3

Hence, if *b* is algebraic over *K*, then $[F(b):F] \mid [K(b):F]$.

Comment