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Chapter 16, Problem 2EN

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Problem

Let G be a finite group, and K a p-Sylow subgroup of G. Let X be the set of all the conjugates of K. See Exercise M2. If C_1, $C_2 \in X$, let $C_1 \sim C_2$ iff $C_1 = aC_2a^{-l}$ for some $\alpha \in K$

For each $C \in X$, prove that the number of elements in $[C]$ is a divisor of $/K/$. (HINT: Use Exercise I10 of Chapter 14.) Conclude that for each $C \in X$, the number of elements in $[C]$ is either 1 or a power of p.

Step-by-step solution

Step 1 of 4

Assume that G is a finite group and K a p-Sylow subgroup of G. Consider the set X as the set of all the conjugates of K. Define an equivalence relation as:

If $C_1, C_2 \in X$, let $C_1 \approx C_2$ if and only if $C_1 = aC_2a^{-1}$ for some $a \in K$.

Objective is to prove that for each $C \in X$, the number of elements in $[C]$ is a divisor of $|K|$.

Comment

Step 2 of 4

Before proving the above statement, consider the following result:

Let K be any subgroup of G. Let $K^* = \{Na : a \in K\}$ and $X_K = \{aHa^{-1} : a \in K\}$. The defined set X_K is in one-one correspondence with K^*. And the number of element in X_K is a divisor of $|K|$.

By the previous result, one knows that every conjugate of K is also a p-Sylow subgroup of G. So, the set X contains the p-Sylow subgroups of G. Now, by the above result, it directly implies that the number of elements in $[C]$ is a divisor of $|K|$, where

$[C] = \{aCa^{-1} : a \in K\}$.

Comment

Step 3 of 4

Since *K* is a *p*-Sylow subgroup of *G*, or can say maximal *p*-subgroup of *G*. Therefore, the order of *K* will be some power of *p*. Since the number of elements in $[C]$ is a divisor of $|K|$, so the number of elements will be the divisors of some power of *p*.

Note that identity element has order as $|e| = p^0$, that is, 1. So, in trivial case, $[C]$ may have only identity element.

Comment

Step 4 of 4

Hence, the number of elements in $[C]$ is either 1 or a power of *p*.

Comment