A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 6EF

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Problem

Let *G* be a group; let *H* and *K* be subgroups of *G*, with *H* a normal subgroup of *G*. Prove the following:

By the FHT, *K*/(*H K*) ≅ *HK*/*H*. (This is referred to as the *first isomorphism theorem*.)

Step-by-step solution

Step 1 of 4

Suppose that *G* is any group and let *H*, *K* are the subgroups of *G*, with *H* a normal subgroup of *G*. Consider a mapping $f : K \to HK / H$ defined by

$f(k) = Hk$, for all $x \in K$.

Objective is to prove that $K / (H \quad K) \cong HK / H \quad H \quad K$, by using fundamental homomorphism theorem.

According to the fundamental homomorphism theorem, if $f : G \to H$ is a homomorphism of *G* onto *H*, with kernel *K* then

$H \cong G / K$.

Consider the following result: the set HK will form the subgroup of *G* if and only if $HK = KH$.

Comment

Step 2 of 4

First prove that *f* is a homomorphism from *K* onto HK / H.

Let $x, y \in K$. Then

$$\begin{aligned} f(xy) &= Hxy \\ &= Hx \cdot Hy \\ &= f(x) \cdot f(y) \end{aligned}$$

This holds for all $x, y \in K$. Therefore, *f* is a homomorphism.

Next, let $Hx \in HK / H$. It implies that $x \in HK$. Or for some $h \in H$ and $k \in K$, $x = hk$.

Since $HK = KH$, there exists $h' \in H$ and $k' \in K$ such that

$hk = k'h'$.

Now,

$$\begin{aligned} f(k') &= Hk' \\ &= (Hh')k' \\ &= H(h'k') \\ &= H(k'h'). \end{aligned}$$

The second and last equality is obtained by the coset property. At last, it implies that

$f(k') = Hx$.

That is, *f* is onto.

Comment

Step 3 of 4

Now, it is remaining to show that $\ker f = H \quad K$.

Let $x \in K$. If $x \in Kerf$ then

$f(x) = H$

because *H* is the identity of quotient group HK / H. And then,

$Hx = H$.

It implies that $x \in H$. Therefore, $x \in H \quad K$. Thus, $\ker f \subseteq H \quad K$.

Now let $x \in H \quad K$. Then $x \in H$, $x \in K$. Also then $Hx = H$ and $f(x) = H$. It implies that $x \in Kerf$. On combining the equations, one gets

$\ker f = H \quad K$.

Comment

Step 4 of 4

Hence, by fundamental homomorphism theorem it conclude that

$K / (H \quad K) \cong HK / H$.

Comment