A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 5ED

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Problem

Let G be a group. By an *automorphism* of G we mean an isomorphism $f: G \to G$.

By the *center* of G we mean the set of all those elements of G which commute with every element of G, that is, the set C defined by

$$C = \{a \in G: ax = xa \text{ for every } x \in G\}$$

Prove that $a \in C$ if and only if $axa^{-1} = x$ for every $x \in G$.

Step-by-step solution

Step 1 of 3

Consider the center of any group G defined as:

$$C = \{a \in G : ax = xa \text{ for every } x \in G\}.$$

Objective is to prove that $a \in C$ if and only if $axa^{-1} = x$ for every $x \in G$.

First suppose that $a \in C$. Then for all $x \in G$, one have

$$ax = xa.$$

Since center C is a subgroup of G, so all the elements in C will be the elements of G. That is, $a \in G$ also $a^{-1} \in G$.

Post-multiply by a^{-1} in the condition $ax = xa$ yields:

$$axa^{-1} = xaa^{-1}$$

$$axa^{-1} = x,$$

for all $x \in G$.

Thus, if $a \in C$ then $axa^{-1} = x$ for every $x \in G$.

Comment

Step 2 of 3

Conversely, let $axa^{-1} = x$ for every $x \in G$. Post-multiplication by a in this condition yields:

$$\begin{aligned} axa^{-1} &= x \\ axa^{-1}a &= xa \\ axe &= xa \\ ax &= xa \end{aligned}$$

for all $x \in G$. This implies that $a \in C$.

Comment

Step 3 of 3

Hence, $a \in C$ if and only if $axa^{-1} = x$ for every $x \in G$.

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