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Chapter 28, Problem 3EE

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Problem

Let U and V be finite-dimensional vector spaces over a field F, and let $h : U \to V$ be a linear transformation. Prove part:

h is injective iff the null space of h is equal to $\{\mathbf{0}\}$.

Step-by-step solution

Step 1 of 5

It is already known that U and V are vector spaces and so they satisfies all conditions for vector space.

Comment

Step 2 of 5

Any transformation is injective if,

$$h(\mathbf{a}) = h(\mathbf{b}) \Rightarrow \mathbf{a} = \mathbf{b}$$

Comment

Step 3 of 5

It is given that null space which is subset of U is $\{\mathbf{0}\}$.

Comment

Step 4 of 5

Or,

$$\{\text{if } h(\mathbf{u}) = \mathbf{0}_v, \text{ then, } \mathbf{u} = \mathbf{0}_u\}$$

Comment

Step 5 of 5

We prove required result by contradiction. Assume that kernel of *U* is not $\{\mathbf{0}\}$ and contain some other element **p**. And let *h* be injective.

Then for any element **u** in *U,*

$$\mathbf{u} = \mathbf{u} + \mathbf{0}$$

Taking linear transformation

$$h(\mathbf{u}) = h(\mathbf{u} + \mathbf{0})$$
$$\Rightarrow h(\mathbf{u}) = h(\mathbf{u}) + h(\mathbf{0})$$

Then,

$$h(\mathbf{u}) = h(\mathbf{u}) + \mathbf{p}$$

Or,

$$h(\mathbf{u}) = h(\mathbf{u}) + \mathbf{0}_v = h(\mathbf{u})$$

Thus there is a contradiction that *h* is injective. This is due to wrong assumption that nullspace of *U* is not $\{\mathbf{0}\}$ but contains other element.

$$\boxed{\text{Hence } h \text{ is injective if nullspace of } U \text{ is } \{\mathbf{0}\}}$$

Comment