A Book of Abstract Algebra | (2nd Edition)

Problem

$A[x_1, x_2]$ denotes the ring of all the polynomials in *two letters* x_1 and x_2 with coefficients in A. For example, $x^2 - 2xy + y^2 + x - 5$ is a quadratic polynomial in $\mathbb{Q}[x, y]$. More generally, $A[x_1, \ldots, x_n]$ is the ring of the polynomials in *n letters* $x_1, \ldots, x_n$ with coefficients in A. Formally it is defined as follows:Let $A[x_1]$ be denoted by A_1; then $A_1[x_2]$ is $A[x_1, x_2]$. Continuing in this fashion, we may adjoin one new letter *xi* at a time, to get $A[x_1, \ldots, x_n]$.

Prove that if A is an integral domain, then $A[x_1, \ldots, x_n]$ is an integral domain.

Step-by-step solution

Step 1 of 2

Step 1 of 2

We use the concept that **if** A **is an integral domain then** $A[x]$ **is an integral domain**

Let A_1 be denoted by $A[x_1]$or $A_1 = A[x_1]$ and $A_2 = A_1[x_2] = A[x_1, x_2]$ similarly we write for general case $A_i = A_{i-1}[x_i] = A[x_1, x_2, .., x_i]$ where $i = 2, 3, 4, ...$

$\Rightarrow A_n = A_{n-1}[x_n] = A[x_1, x_2, .., x_n]$, here $x_1, x_2, ..., x_n$ are variables

$A_1 = A[x_1]$ is an integral domain as A is integral domain. Similarly, we can prove by induction that $A_n = A[x_1, x_2, .., x_n]$ is an integral domain

Comment

Step 2 of 2

Step 2 of 2

Basic case: for $n = 1$

$A_1 = A[x_1]$ is an integral domain (because A is integral domain)

Hypothesis: assume that for $n = k$ condition is true

$\Rightarrow A_k = A[x_1, x_2, .., x_k]$ is an integral domain

To prove: for $n = k+1$ the condition is true that mean $A_{k+1} = A[x_1, x_2, .., x_{k+1}]$ is an integral domain.

$A_{k+1} = A_k[x_{k+1}] = A[x_1, x_2, .., x_{k+1}]$

$\Rightarrow A_{k+1} = A[x_1, x_2, .., x_{k+1}]$ is an integral domain as $A_k[x_{k+1}]$ is an integral domain

Comment