A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 7ED

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Problem

Let *G* be a group. By an *automorphism* of *G* we mean an isomorphism *f*: *G* → *G*.

Use the FHT to conclude that *I*(*G*) is isomorphic with *G*/*C*.

Step-by-step solution

Step 1 of 4

Suppose that $I(G)=\{\phi_a : a\in G\}$ is the set of all the inner automorphisms of *G*. Consider a mapping $h:G\to I(G)$ defined by

$$h(a)=\phi_a.$$

Objective is to prove that $I(G)\cong G/C$, where *C* is the center of *G*, by using fundamental homomorphism theorem.

According to the fundamental homomorphism theorem, if $f:G\to H$ is a homomorphism of *G* onto *H*, with kernel *K* then

$$H\cong G/K.$$

Comment

Step 2 of 4

First show that *h* is a homomorphism from *G* onto $I(G)$ and kernel of this homomorphism is the center *C* of *G*.

Let $x, y\in G$. Then

$$\begin{aligned} h(xy) &= \phi_{xy} \\ &= \phi_x \quad \phi_y \\ &= h(x) \quad h(y). \end{aligned}$$

The second step is obtained from the property that $\phi_a \quad \phi_b = \phi_{ab}$. Therefore, *h* is a homomorphism.

Let $\phi_a \in I(G)$. Then correspondingly the element *a* will belong to *G*. That is, for all $\phi_a \in I(G)$ there exists $a \in G$ such that $h(a) = \phi_a$, a onto mapping.

Comment

Step 3 of 4

According to the definition of kernel,

$$\ker h = \{a \in G : h(a) = e\},$$

where *e* is the identity of $I(G)$. Since $h(a) = \phi_a$, so

$$\begin{aligned}\ker h &= \{a \in G : \phi_a = e\} \\ &= \{a \in G : \phi_a(x) = e(x)\} \\ &= \{a \in G : axa^{-1} = x\}\end{aligned}$$

The last equality is obtained by definition of inner automorphism and identity function. Solve the condition $axa^{-1} = x$ by multiplying both the sides by *a* as:

$$\begin{aligned}axa^{-1}a &= xa \\ ax &= xa\end{aligned}$$

for all $x \in G$. That is, $a \in \ker h$ if it satisfies the condition that for all *x* in *G*, $ax = xa$. Therefore, $a \in C$ and thus kernel of *h* contains all the center elements.

Comment

Step 4 of 4

Since the function *h* is a homomorphism from *G* onto $I(G)$ with $\ker h = C$, therefore by FHT it can be conclude that $I(G) \cong G/C$.

Comment