A Book of Abstract Algebra | (2nd Edition)

Problem

Let F be a field, and let c be transcendental over F. Prove the following:

If c is transcendental over F, so are $c + 1$, kc (where $k \in F$ and $k \neq 0$), c^2.

Step-by-step solution

Step 1 of 3

Consider that F is any arbitrary field and let $c \in F$ is transcendental over F. Assume that K is some extension field of F. Objective is to prove that $c+1, kc$ (where $k \in F, k \neq 0$), and c^2 will also be transcendental over F.

Suppose, by way of contradiction, that $a = c+1$ is not transcendental, that is, a is algebraic over K. Note that since 1 and $c \in F$, therefore $c+1 \in F$.

Comment

Step 2 of 3

The set of all elements of F which are algebraic over K form a field. Since $a, 1 \in F$, it implies that $a-1 = c$ is algebraic over F. but this contradicts the hypothesis that c is transcendental over F.

Similarly, let $a = kc$ is algebraic over F. Note that since $k, c \in F$, therefore $kc \in F$. Since $a, k \in F$, it implies that $a/k = c$ is algebraic over F. but this contradicts the hypothesis that c is transcendental over F.

Suppose that c^2 is algebraic over K. Then there is some $p(x) \in F[x]$ whose root will be c^2. This shows that c will be the root of square root of $p(x)$, that is,

$$\sqrt{p(c)} = 0,$$

a contradiction,

Comment

Step 3 of 3

Hence, if c is transcendental over F, so are $c+1, kc$ (where $k \in F, k \neq 0$), c^2.

Comment