A Book of Abstract Algebra | (2nd Edition)

Chapter 24, Problem 1EF

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Problem

Let *A* be an integral domain.

Let *h* : *A*[*x*]→ *A* map every polynomial to its constant coefficient; that is,

$$h(a_0 + a_1x + \cdots + a_nx^n) = a_0$$

Prove that *h* is a homomorphism from *A*[*x*] *onto A*, and describe its kernel.

Step-by-step solution

Step 1 of 3

Consider an integral domain $A[x]$ and let $h: A[x] \to A$ map every polynomial to its constant coefficient.

That is $h\left(a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0\right) = a_0$

Objective of the question is to prove *h* is a homomorphism from $A[x]$ on to *A*.

Recall the definition of homomorphism.

Definition: A ring homomorphism *f* from a ring *R* to a ring *S* is a mapping from *R* to *S* that preserves the two operations. That is for all $a, b \in R$

$$f(a+b) = f(a) + f(b)$$
$$f(ab) = f(a)f(b)$$

Let

$$p(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$$
$$q(x) = b_nx^n + b_{n-1}x^{n-1} + \ldots + b_2x^2 + b_1x + b_0$$

Prove *h* is a ring homomorphism.

$$
\begin{aligned}
h(p(x)) &= h\left(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0\right) \\
&= a_0
\end{aligned}
$$

$$
\begin{aligned}
h(q(x)) &= h\left(b_n x^n + b_{n-1}x^{n-1} + \ldots + b_2x^2 + b_1x + b_0\right) \\
&= b_0
\end{aligned}
$$

$$
\begin{aligned}
h(p(x)+q(x)) &= h\left((a_n+b_n)x^n + (a_{n-1}+b_{n-1})x^{n-1} + \ldots + (a_0+b_0)\right) \\
&= a_0 + b_0 \\
&= h(p(x)) + h(q(x))
\end{aligned}
$$

$$
\begin{aligned}
h(p(x)q(x)) &= h\left(\left(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0\right)\left(b_n x^n + b_{n-1}x^{n-1} + \ldots + b_2x^2 + b_1x + b_0\right)\right) \\
&= h\left(a_n b_n x^n + \ldots + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + (a_0b_1 + a_1b_0)x + a_0b_0\right) \\
&= a_0b_0 \\
&= h(p(x))h(q(x))
\end{aligned}
$$

According to definition of ring homomorphism *h* is a ring homomorphism.

Comment

Step 2 of 3

Now show that *h* is a onto function.

Recall the definition of onto function.

Definition: *f* is a function from *A* to *B* is said to be onto function if for all $b \in B$ there exists at least one $a \in A$ such that $f(a) = b$.

Let any element $a \in A$.

Now construct a polynomial as follows

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a$$

Then $h\left(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a\right) = a$

Therefore for all $a \in A$ there exists a polynomial in $A[x]$ such that $h(p(x)) = a$.

Comment

Step 3 of 3

Now find kernel of the function *h*.

Recall the definition of kernel of a function.

Definition: Let a homomorphism *f* from *A* to *B*. A subset *K* of *A* is said to be kernel of homomorphism *f* if for every $t \in K$, $f(t) = 0$.

Now find set of elements such that $h(p(x)) = 0$

Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$.

$$\begin{aligned} h(p(x)) &= h\left(a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0\right) \\ &= a_0 \end{aligned}$$

Then $h(p(x)) = 0$ if $a_0 = 0$.

There for the kernel of this homomorphism is the set of all polynomials whose constant term is zero.

That is $K = \left\{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 \mid a_i \in A, a_0 = 0, i = 0, 1, \ldots, n\right\}$

Comment