A Book of Abstract Algebra | (2nd Edition)

Problem

Suppose $a(x) \in F[x]$, and *K* is an extension of *F*. An element $c \in K$ is called a multiple root of *a*(*x*) if $(x - c)^m | a(x)$ for some $m > 1$. It is often important to know if all the roots of a polynomial are different, or not.

We now consider a method for determining whether an arbitrary polynomial $a(x) \in F[x]$ has multiple roots in any extension of *F*.

Let *K* be any field containing all the roots of *a*(*x*). Suppose *a*(*x*) has a multiple root *c*.

Show that $x - c$ is a common factor of *a*(*x*) and *a*′(*x*). Use Exercise hi to conclude that *a*(*x*) and *a*′(*x*) have a common factor of degree >1 in *F*[*x*].

Thus, if *a*(*x*) has a multiple root, then *a*(*x*) and *a*′(*x*) have a common factor in *F*[*x*]. To prove the converse, suppose *a*(*x*) has *no* multiple roots. Then *a*(*x*) can be factored as $a(x) = (x - c_1) \cdots (x - c_n)$ where $c_1, \ldots, c_n$ are all different.

Step-by-step solution

Step 1 of 4

Consider that *K* is any field that contains all the roots of polynomial $a(x) = a_0 + a_1 x + \cdots + a_n x^n$. Assume that $a(x)$ has a multiple root *c*. Then polynomial $a(x)$ will be

$a(x) = (x-c)^2 q(x) \in K[x]$.

Objective is to prove that $x - c$ is a common factor of $a(x)$ and $a'(x)$. Also conclude that $a(x)$ and $a'(x)$ have a common factor of degree > 1 in $F[x]$.

Consider the following result:

If $a(x), b(x) \in F[x]$ have a common root *c* in some extension of *F*, they may have a common factor of positive degree in $F[x]$.

Comment

Step 2 of 4

Use the following formula: for some $a(x), b(x) \in F[x]$,

$[a(x)b(x)]' = a'(x)b(x) + a(x)b'(x)$.

The derivative $a'(x)$ will be:

$$\begin{aligned} a'(x) &= \left[(x-c)^2\right]' q(x) + (x-c)^2 q'(x) \\ &= 2(x-c)q(x) + (x-c)^2 q'(x) \\ &= (x-c)[2q(x) + (x-c)q'(x)] \end{aligned}$$

Thus, $x - c$ is a common factor of $a(x)$ and $a'(x)$.

Comment

Step 3 of 4

Suppose that $a(x)$ and $a'(x)$ have no common factor in $F[x]$, that is, both are relatively prime. Then there exist some $f(x), g(x) \in F[x]$ such that

$f(x)a(x) + g(x)a'(x) = 1$.

Since $x - c$ is a common factor of $a(x)$ and $a'(x)$, therefore $x - c$ is a common factor of $1 \in K[x]$. This cannot be possible.

Comment

Step 4 of 4

Hence, $a(x)$ and $a'(x)$ have a common factor of degree > 1 in $F[x]$.

Comment