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Chapter 16, Problem 1EK | 1 Bookmark | Show all steps: ON

Problem

If G is a group and p is any prime divisor of $|G|$, it will be shown here that G has at least one element of order p. This has already been shown for abelian groups in Chapter 15, Exercise H4. Thus, assume here that G is not abelian. The argument will proceed by induction; thus, let $|G| = k$, and assume our claim is true for any group of order less than k. Let C be the center of G, let Ca be the centralizer of a for each $a \in G$, and let $k = c + k_s + \cdots + k_t$ be the class equation of G, as in Chapter 15, Exercise G2.

Prove: If p is a factor of $|C_a|$ for any $a \in G$, where $a \notin$ **C**, we are done. (Explain why.)

Step-by-step solution

Step 1 of 3

Consider a non-abelian group G whose order is divisible by some prime p. Cauchy Theorem states that group G has at least one element whose order is p.

Assume that order of G is k. Let C be the center of G and C_a be the centralizer of $a \in G$. The class equation for group G is given by:

$$k = c + k_S + \quad + k_t.$$

Objective is to prove that if order of C_a is divisible by p, for any $a \in G$ and $a \notin C$, then the statement of Cauchy theorem will hold.

Comment

Step 2 of 3

Here in the class equation $k = c + k_S + \quad + k_t$, the $k_S, \quad , k_t$ are the sizes of all distinct conjugacy classes of elements $x \notin C$.

Consider the condition that order of C_a is divisible by p, for any $a \in G$ and $a \notin C$. As a does not belong to center C, it implies that $C_a \neq G$. That is, C_a is the proper subgroup of G. Then the order of C_a will be less than the order of group, that is,

$|C_a| < |G|$.

In the language of the question, it is mention that induction is getting used to proceed the argument. According to the hypothesis, Cauchy theorem holds for every group of order less than *k*, where $k = |G|$.

Comment

Step 3 of 3

Thus, if *p* is a factor of $|C_a|$ for any $a \in G$ and $a \notin C$, then the statement of Cauchy theorem will hold.

Comment