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Chapter 17, Problem 1EB | Bookmark | Show all steps: ON

Problem

Verify that $\mathcal{F}(\mathbb{R})$ satisfies all the axioms for being a commutative ring with unity. Indicate the zero and unity, and describe the negative of any *f*.

Step-by-step solution

Step 1 of 5

Consider that the set $F(R)$ of all the function from real number *R* to *R*, with the following addition and multiplication:

$$(f+g)(x)=f(x)+g(x),$$
$$(fg)(x)=f(x)g(x),$$

for every real number *x*.

Objective is to show that $F(R)$ satisfies all the axioms to be a commutative ring with unity. Write explicitly the zero element, the unity, and the negative of any *f*.

First show that $(F(R),+)$ is an abelian group.

(1) Since sum of two real valued function is again a real function, therefore sum is closed in $F(R)$.

(2) Associative: Let $f,g,h\in F(R)$. Then

$$\begin{aligned}[(f+g)+h](x)&=[f+(g+h)](x)\\(f+g)(x)+h(x)&=f(x)+(g+h)(x)\\f(x)+g(x)+h(x)&=f(x)+g(x)+h(x)\end{aligned}$$

Since both the sides are equals, so addition is associative in $F(R)$.

(3) Since addition is commutative in real numbers, so

$$\begin{aligned}(f+g)(x)&=f(x)+g(x)\\&=g(x)+f(x)\\&=(g+f)(x).\end{aligned}$$

(4) Additive identity or zero element:

$$(f+g)(x)=f(x)$$

Consider the zero function $g(x)=0$ for all real number *x*. Then

$$\begin{aligned}(f+g)(x)&=f(x)+g(x)\\&=f(x)+0\\&=f(x).\end{aligned}$$

Thus, zero function will be the zero element of $F(R)$.

(5) Since

$$\begin{aligned}(f+(-f))(x)&=f(x)+(-f)(x)\\&=f(x)-f(x)\\&=0\end{aligned}$$

Therefore, negative of any $f\in F(R)$will be $-f$.

And from here it conclude that, $F(R)$ is an abelian group.

Comment

Step 2 of 5

Now, show that product of two function is associative. Let $f,g,h\in F(R)$. Then

$$\begin{aligned}((fg)h)(x)&=(fg)(x)h(x)\\&=f(x)g(x)h(x),\\(f(gh))(x)&=f(x)(gh)(x)\\&=f(x)g(x)h(x).\end{aligned}$$

Since both the sides are equals, so multiplication is associative in $F(R)$.

Comment

Step 3 of 5

Next is distributive law:

$$\begin{aligned}[f(g+h)](x)&=f(x)(g+h)(x)\\&=f(x)[g(x)+h(x)]\\&=f(x)g(x)+f(x)h(x).\end{aligned}$$

Next, show that product of functions is commutative. For this,

$$\begin{aligned}(fg)(x)&=f(x)g(x)\\&=g(x)f(x)\\&=(gf)(x)\end{aligned}$$

Since addition and multiplication both are commutative, therefore
$[(g+h)f](x)=g(x)f(x)+h(x)f(x)$automatically holds.

Comment

Step 4 of 5

The unity in $F(R)$ will be:

$$(fg)(x) = f(x)$$
$$f(x)g(x) = f(x)$$

Both the sides will be equal when $g(x)=1$ for all real number *x*. Thus, this *g* will work as a unity of any *f* in $F(R)$.

Comment

Step 5 of 5

Hence, $F(R)$ satisfies all the axioms to be a commutative ring with unity. The zero element is the zero function, the unity is constant function 1, and the negative of any *f* is $-f$ in $F(R)$.

Comment