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Chapter 24, Problem 3EB | Bookmark | Show all steps: ON

Problem

Write all the quadratic polynomials in $\mathbb{Z}_5[x]$. How many are there? How many cubic polynomials are there in $\mathbb{Z}_5[x]$? More generally, how many polynomials of degree *m* are there in $\mathbb{Z}_n[x]$?

Step-by-step solution

Step 1 of 3

The general form of second degree polynomial is ax^2+bx+c. Consider the polynomial ring $\mathbb{Z}_5[x]$. The elements of $\mathbb{Z}_5[x]$ are $\{0,1,2,3,4\}$.

Since ax^2+bx+c is a second degree polynomial then $a \neq 0$.

Then *a* can choose the numbers 1,2,3,or 4.

$$\text{Number ways } a \text{ can choose } = 4$$

b can choose the numbers 0,1,2,3, or 4.

$$\text{Number ways } b \text{ can choose } = 5$$

Similarly,

$$\text{Number ways } c \text{ can choose } = 5$$

Then ,

$$\begin{aligned}\text{Total number of second degree polynomial in } \mathbb{Z}_5[x] &= 4\times5\times5 \\ &= \boxed{100}\end{aligned}$$

The polynomials are the following.

$$x^2+0x+0,\quad x^2+x+0,\quad x^2+2x+0,\quad x^2+3x+0,\quad x^2+4x+0$$
$$x^2+0x+1,\quad x^2+x+1,\quad x^2+2x+1,\quad x^2+3x+1,\quad x^2+4x+1$$
$$x^2+0x+2,\quad x^2+x+2,\quad x^2+2x+2,\quad x^2+3x+2,\quad x^2+4x+2$$
$$x^2+0x+3,\quad x^2+x+3,\quad x^2+2x+3,\quad x^2+3x+3,\quad x^2+4x+3$$
$$x^2+0x+4,\quad x^2+x+4,\quad x^2+2x+4,\quad x^2+3x+4,\quad x^2+4x+4$$

And,

$$2x^2+0x+0,\quad 2x^2+x+0,\quad 2x^2+2x+0,\quad 2x^2+3x+0,\quad 2x^2+4x+0$$
$$2x^2+0x+1,\quad 2x^2+x+1,\quad 2x^2+2x+1,\quad 2x^2+3x+1,\quad 2x^2+4x+1$$
$$2x^2+0x+2,\quad 2x^2+x+2,\quad 2x^2+2x+2,\quad 2x^2+3x+2,\quad 2x^2+4x+2$$
$$2x^2+0x+3,\quad 2x^2+x+3,\quad 2x^2+2x+3,\quad 2x^2+3x+3,\quad 2x^2+4x+3$$
$$2x^2+0x+4,\quad 2x^2+x+4,\quad 2x^2+2x+4,\quad 2x^2+3x+4,\quad 2x^2+4x+4$$

And,

$$3x^2+0x+0,\quad 3x^2+x+0,\quad 3x^2+2x+0,\quad 3x^2+3x+0,\quad 3x^2+4x+0$$
$$3x^2+0x+1,\quad 3x^2+x+1,\quad 3x^2+2x+1,\quad 3x^2+3x+1,\quad 3x^2+4x+1$$
$$3x^2+0x+2,\quad 3x^2+x+2,\quad 3x^2+2x+2,\quad 3x^2+3x+2,\quad 3x^2+4x+2$$
$$3x^2+0x+3,\quad 3x^2+x+3,\quad 3x^2+2x+3,\quad 3x^2+3x+3,\quad 3x^2+4x+3$$
$$3x^2+0x+4,\quad 3x^2+x+4,\quad 3x^2+2x+4,\quad 3x^2+3x+4,\quad 3x^2+4x+4$$

And,

$$4x^2+0x+0,\quad 4x^2+x+0,\quad 4x^2+2x+0,\quad 4x^2+3x+0,\quad 4x^2+4x+0$$
$$4x^2+0x+1,\quad 4x^2+x+1,\quad 4x^2+2x+1,\quad 4x^2+3x+1,\quad 4x^2+4x+1$$
$$4x^2+0x+2,\quad 4x^2+x+2,\quad 4x^2+2x+2,\quad 4x^2+3x+2,\quad 4x^2+4x+2$$
$$4x^2+0x+3,\quad 4x^2+x+3,\quad 4x^2+2x+3,\quad 4x^2+3x+3,\quad 4x^2+4x+3$$
$$4x^2+0x+4,\quad 4x^2+x+4,\quad 4x^2+2x+4,\quad 4x^2+3x+4,\quad 4x^2+4x+4$$

Comment

Step 2 of 3

The general form of second degree polynomial is ax^3+bx^2+cx+d.

Since ax^3+bx^2+cx+d is a third degree polynomial then $a\neq 0$.

Then *a* can choose the numbers 1,2,3,or 4.

$$\text{Number ways } a \text{ can choose } =4$$

b can choose the numbers 0,1,2,3, or 4.

$$\text{Number ways } b \text{ can choose } =5$$

Similarly,

$$\text{Number ways } c \text{ can choose } =5$$

Similarly,

$$\text{Number ways } d \text{ can choose } =5$$

Then ,

$$\text{Total number of second degree polynomial in } \mathbb{Z}_5[x] = 4\times5\times5\times5$$
$$= \boxed{500}$$

Comment

Step 3 of 3

The general form of *m*th degree polynomial is $a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0$. Consider the polynomial ring $\mathbb{Z}_n[x]$. The elements of $\mathbb{Z}_n[x]$ are $\{0,1,2,3,4,\ldots,n-1\}$.

Since $a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0$ is a *m*th degree polynomial then $a_m \neq 0$.

Then a_m can choose the numbers $1,2,\ldots,\text{or } n-1$.

$$\text{Number ways } a_m \text{ can choose } = n-1$$

a_{m-1} can choose the numbers $0,1,2,\ldots,\text{or } n-1$.

$$\text{Number ways } a_{m-1} \text{ can choose } = n$$

Similarly,

$$\text{Number ways } a_{m-2} \text{ can choose } = n$$

Similarly for others can choose *n* ways.

Then ,

$$\text{Total number of } n\text{th degree polynomial in } \mathbb{Z}_n[x] = (n-1)\times n\times n\times \underset{m \text{ times}}{\ldots} \times n$$
$$= \boxed{n^m(n-1)}$$

Comment