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Chapter 31, Problem 1EB Bookmark Show all steps: ON

Problem

Show that, in any extension of $\mathbb{Z}_3$ which contains a root u of

$$a(x) = x^3 + 2x + 1 \in \mathbb{Z}_3[x]$$

it happens that $u + 1$ and $u + 2$ are the remaining two roots of $a(x)$. Use this fact to find the root field of $x^3 + 2x + 1$ over $\mathbb{Z}_3$. List the elements of the root field.

Step-by-step solution

Step 1 of 3

The objective is to show that in any extension of $\mathbb{Z}_3$ which contains a root u of $a(x) = x^3 + 2x + 1 \in \mathbb{Z}_3[x]$, it happens that $u+1$ and $u+2$ are the remaining roots of $a(x)$. Also find the root field of $x^3 + 2x + 1$ over $\mathbb{Z}_3$ and list its elements.

Comment

Step 2 of 3

Let u be a root of $a(x)=x^3+2x+1\in\mathbb{Z}_3[x]$.

Then $a(u)=u^3+2u+1=0$. … (1)

$$\begin{aligned} a(u+1) &= (u+1)^3+2(u+1)+1 \\ &= u^3+3u^2+3u+1+2u+2+1 \\ &= u^3+3u^2+5u+4 \\ &= u^3+2u+1 \end{aligned}$$

By (1), $a(u+1)=0$.

$$\begin{aligned} a(u+2) &= (u+2)^3+2(u+2)+1 \\ &= u^3+6u^2+12u+8+2u+4+1 \\ &= u^3+6u^2+14u+13 \\ &= u^3+2u+1 \end{aligned}$$

By (1), $a(u+2)=0$.

Therefore, it is shown that $u+1$ and $u+2$ are the remaining roots of $a(x)$.

Comment

Step 3 of 3

In $\mathbb{Z}_3$, u, $u+1$, and $u+2$ are the roots of x^3+2x+1.

Also, $\mathbb{Z}_3$ contains u, $u+1$, and $u+2$.

Therefore, $\mathbb{Z}_3(u,u+1,u+2)$ is the root field of x^3+2x+1 over $\mathbb{Z}_3$ and u, $u+1$, and $u+2$ are its elements.

Comment