A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 6EQ

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Problem

As a provisional definition, let us call a finite abelian group “decomposable” if there are elements $a_1, \ldots, a_n \in G$ such that:

(DI) For every $x \in G$, there are integers $k_1, \ldots, k_n$ such that $x = a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}$. ($D_2$)

If there are integers $l_1, \ldots, l_n$ such that

$a_1^{l_1} a_2^{l_2} \cdots a_n^{l_n} = e$ then $a_1^{l_1} = a_2^{l_2} = \cdots = a_n^{l_n} = e$.

If (D_1) and (D_2) hold, we will write $G = [a_1, a_2, \ldots, a_n]$. Assume this in parts 1 and 2.

Use Exercise P5, together with parts 2 and 5 above, to prove: Every finite abelian group G is a direct product of cyclic groups of prime power order. (This is called the basis theorem of finite abelian groups.)

It can be proved that the above decomposition of a finite abelian group into cyclic p-groups is unique, except for the order of the factors. We leave it to the ambitious reader to supply the proof of uniqueness.

Step-by-step solution

Step 1 of 3

Objective is to prove that every finite abelian group G is a direct product of cyclic groups of prime power order. Also this decomposition is unique, except for the order of the factors.

If $a_1, \quad, a_n \in G$ and both the conditions $D1$, $D2$ holds, then $G = [a_1, a_2, \quad, a_n]$. And $G \cong \langle a_1 \rangle \times G'$, then $G \cong \langle a_1 \rangle \times \langle a_2 \rangle \times \quad \times \langle a_n \rangle$. That is, every finite abelian group is an inner direct product of p-Sylow subgroups. Also every p-group has a basis.

The intersection of any p-Sylow subgroups is trivial and the union of their basis elements is a basis for the complete (or whole) group.

Comment

Step 2 of 3

For uniqueness of basis: let P and Q be the products of p-cyclic groups. Then $P \cong Q$ only when they are the same powers of same primes.

If p^k is a factor of P and not Q then P has an element of order p^k (by Cauchy theorem) but Q does not. So, they are not isomorphic.

If P has more factors of p^k than Q. Then P has more elements of order p^k without $p\text{th}$ roots.

Comment

Step 3 of 3

Hence, every finite abelian group G can be written, in a unique way, as a direct product of cyclic groups of prime power order.

Comment