A Book of Abstract Algebra | (2nd Edition)

Chapter 24, Problem 2EE

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Problem

If B is an *ideal* of A, $B[x]$ is an ideal of $A[x]$.

Step-by-step solution

Step 1 of 4

Consider a ring A and ideal B of A.

Now show that $B[x]$ is an ideal of $A[x]$.

Comment

Step 2 of 4

Recall the definition of ideal.

Definition: A subring *A* of a ring *R* is said to be an ideal of *R* if for every $r \in R$ and for every $a \in A$ both $ra, ar \in A$.

According to definition of an ideal ,*B* is an ideal of *A* that implies two points. First point is *B* is a subring of *A* and second point is for every $a \in A$ and $b \in B$ both $ba, ab \in B$

To prove $B[x]$ is an ideal of $A[x]$, prove two points.

(1) $B[x]$ is a subring of $A[x]$.

(2) For every polynomial $a(x) \in A[x]$ and $b(x) \in B[x]$, $a(x)b(x), b(x)a(x) \in B[x]$

Comment

Step 3 of 4

Prove $B[x]$ is a subring of $A[x]$.

Recall definition of subring and the theorem know as subring test.

Definition: A subset *S* of a ring *R* is a subring of R if *S* itself a ring with the operation of R.

Theorem 1(Subring test): A nonempty subset *S* of a ring *R* is a subring if $a-b$ and ab are in *S* whenever *a* and *b* are in *S*.

First prove $B[x] \subseteq A[x]$.

B is a subset of *A* implies for every element of *B* is the element *A*.

That is if $b \in B$ implies $b \in A$.

Let any polynomial $p(x) \in B[x]$. Now prove $p(x) \in A[x]$.

If $p(x) \in B(x)$ implies every coefficient of $p(x)$ is in *B*. since *B* is a subset of *A* implies the coefficients of $p(x)$ are also elements of *A*.

Then $p(x)$ is an element of $A[x]$.

Since $p(x)$ is chosen arbitrary implies for every element in $B[x]$ is an element in $A[x]$.

Let two polynomials $p(x)$ and $q(x)$ in $B[x]$.

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_0$$
$$q(x) = b_n x^n + b_{n-1}x^{n-1} + \dots + b_0$$

To prove $B[x]$ is a subring of $A[x]$, it is sufficient to prove

$p(x)-q(x) \in B[x]$ and $p(x)q(x) \in B[x]$.

$$\begin{aligned} p(x)-q(x) &= \left(a_n x^n + a_{n-1}x^{n-1} + \dots + a_0\right) - \left(b_n x^n + b_{n-1}x^{n-1} + \dots + b_0\right) \\ &= (a_n - b_n)x^n + (a_{n-1} - b_{n-1})x^{n-1} + \dots + (a_0 - b_0) \end{aligned}$$

$a_n, a_{n-1}, \dots, a_0, b_n, b_{n-1}, \dots, b_0 \in B$ and *B* is a subring of *A*.

Then by using theorem 1 , $(a_n - b_n), (a_{n-1} - b_{n-1}), \dots, (a_0 - b_0) \in B$. Therefore all coefficients of $p(x)-q(x)$ belongs to *B* then $p(x)-q(x) \in B[x]$

$$p(x)q(x)=(a_nx^n+a_{n-1}x^{n-1}+\dots+a_0)(b_nx^n+b_{n-1}x^{n-1}+\dots+b_0)$$
$$=(a_nb_nx^{2n}+a_nb_{n-1}x^{2n-1}+\dots+b_0a_nx^n)+\dots+(a_0b_nx^n+\dots+a_0b_0)$$
$$=a_nb_nx^{2n}+\dots+(a_0b_2+a_1b_1+a_2b_0)x^2+(a_0b_1+b_0a_1)x+a_0b_0$$

$a_n,a_{n-1},\dots,a_0,b_n,b_{n-1},\dots,b_0\in B$ and B is a ring.

$a_nb_n,\dots,(a_0b_2+a_1b_1+a_2b_0),(a_0b_1+b_0a_1),a_0b_0\in B$

Therefore all coefficients of $p(x)q(x)$ belongs to *B*. Then $p(x)q(x)\in B[x]$

Thus, $B[x]\subseteq A[x]$, $p(x)-q(x)\in B[x]$ and $p(x)q(x)\in B[x]$ for every polynomial $p(x)$, $q(x)\in B[x]$.

Hence according to theorem 1 $B[x]$ is a subring of $A[x]$.

Comment

Step 4 of 4

Now prove subring $B[x]$ is an ideal.

Let $a(x)=a_nx^n+a_{n-1}x^{n-1}+\dots+a_0\in A[x]$ and $b(x)=b_nx^n+b_{n-1}x^{n-1}+\dots+b_0\in B[x]$

Then,

$$a(x)b(x)=(a_nx^n+a_{n-1}x^{n-1}+\dots+a_0)(b_nx^n+b_{n-1}x^{n-1}+\dots+b_0)$$
$$=(a_nb_nx^{2n}+a_nb_{n-1}x^{2n-1}+\dots+b_0a_nx^n)+\dots+(a_0b_nx^n+\dots+a_0b_0)$$
$$=a_nb_nx^{2n}+\dots+(a_0b_2+a_1b_1+a_2b_0)x^2+(a_0b_1+b_0a_1)x+a_0b_0$$

Here $a_i\in A$ and $b_i\in B\,\forall i=0,1,2,\dots,n$ and *B* is an ideal. It implies $a_ib_j,b_ja_i\in B$.

Then every coefficients of $a(x)b(x)$ belongs to *B* implies $a(x)b(x)$ is in $B[x]$.

Similarly $b(x)a(x)$ is in $B[x]$.

Therefore, $B[x]$ is an ideal of $A[x]$.

Comment