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Chapter 27, Problem 4EB   Bookmark   Show all steps: ON

Problem

For each of the following polynomials $p(x)$, find a number a such that $p(x)$ is the minimum polynomial of a over $\mathbb{Q}$:

(a) $x^2 + 2x - 1$

(b) $x^4 + 2x^2 - 1$

(c) $x^4 - 10x^2 + 1$

Step-by-step solution

Step 1 of 5

(a)

Objective is to determine a number a such that $p(x) = x^2 + 2x - 1$ is the minimum polynomial of a is over Q.

The required such number a will be the solution of equation $x^2 + 2x - 1 = 0$.

Let $ax^2 + bx + c = 0$ be some quadratic equation where a is nonzero, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Comment

Step 2 of 5

Use the method of solving the quadratic equation and get

$$\begin{aligned} x^2 + 2x - 1 &= 0 \\ x &= \frac{-2 \pm \sqrt{4+4}}{2} \\ x &= -1 \pm \sqrt{2}. \end{aligned}$$

Thus, the required a will be $-1 \pm \sqrt{2}$.

Comment

Step 3 of 5

(b)

Objective is to determine a number a such that $p(x) = x^4 + 2x^2 - 1$ is the minimum polynomial of a is over Q.

Let $x^2 = y$, then equation will become $y^2 + 2y - 1 = 0$. Use the method of solving the quadratic equation and get

$$\begin{aligned} y^2 + 2y - 1 &= 0 \\ y &= \frac{-2 \pm \sqrt{4+4}}{2} \\ y &= -1 \pm \sqrt{2} \\ x^2 &= -1 \pm \sqrt{2} \end{aligned}$$

Thus, the required a will be $\sqrt{-1 \pm \sqrt{2}}$.

Comment

Step 4 of 5

(c)

Objective is to determine a number a such that $p(x) = x^4 - 10x^2 + 1$ is the minimum polynomial of a is over Q.

Let $x^2 = y$, then equation will become $y^2 - 10y + 1 = 0$. Use the method of solving the quadratic equation and get

$$\begin{aligned} y^2 - 10y + 1 &= 0 \\ y &= \frac{10 \pm \sqrt{100-4}}{2} \\ y &= 5 \pm 2\sqrt{6} \\ x^2 &= 5 \pm 2\sqrt{6}. \end{aligned}$$

Comment

Step 5 of 5

Thus, the required a will be $\sqrt{5 \pm 2\sqrt{6}}$.

Comment