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$$t = q + 1 - \#E(\mathbb{F}_q)$$

So the characteristic polynomial of frobenius polynomial is $x^2 - tx + q$.

$$\Phi_q^2 - [t]\Phi_q + [q] = 0$$

Let α be that endomorphism so $\alpha \in \mathrm{End(E)}$.

If $\alpha \neq 0$ then $\# \ker \alpha \leq \deg \alpha$, so $\ker \alpha$ is finite.

We now show that if $\alpha \neq 0$ then $\# \ker \alpha = \infty$.

For any int n such that $p \nmid n$,

$$E[n] \cong \mathbb{Z}_n \times \mathbb{Z}_n$$

and we represent

$$\Phi_q|_{E[n]} : E[n] \to E[n]$$

since it's an endomorphism that is just restricted to $E[n]$ so we can represent this as a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

So by direct inspection

$$A_n^2 - \mathrm{tr}(A_n) \cdot A_n + \det(A_n)I = 0$$

We've shown that

$$\det(A_n) = \deg \Phi_q \mod n$$

Another calc shows

$$\mathrm{tr}(A_n) = 1 + \det(A_n) - \det(I - A_n)$$

so

$$\mathrm{tr}(A_n) = 1 + q - \deg(\mathrm{id} - \Phi_q) \mod n$$

since $\deg(\mathrm{id} - \Phi_q) = \#E(\mathbb{F}_q) = q + 1 - t$ so

$$A_n^2 - [1 + q - (q + 1 - t)]A_n + qI = 0$$

for 2x2 matrices. Remember that A_n is a matrix in $E[n]$ so the matrix is defined over mod n.

$$\underbrace{A_n^2 - [t]A_n + qI = 0}_{\text{representation of } \alpha|_{E[n]}}$$

This means that for any n such that $p \nmid n$ then for all $P \in E[n]$

$$\alpha(P) = 0$$

since the set

$$U_{p \nmid n} E[n]$$

is infinite (the U means union here),

$$\# \ker(\alpha) = \infty$$

contradiction.

Note: $t = \mathrm{tr}(A_n)\ \forall p \nmid n$ so is called the trace of Frobenius.

1 $\deg(\alpha \circ \alpha^{'}) = \deg(\alpha) \circ \deg(\alpha^{'})$

$$E \to E' \to E''$$

by the maps α', α.

For simplicity think $E = E' = E''$.

$$\alpha(x, y) = (R(x), yS(x))$$
$$\alpha'(x, y) = (R'(x), yS'(x))$$

Then $(\alpha \circ \alpha')(x, y)$ has repr:

$$(R''(x), yS''(x)) = (R(R'(x)), S(R'(x))S'(x)y)$$

So this already satisfies the property of canonical form. The other property is that both sides don't share a common root over the algebraic closure.

If $R(R'(x)) = \frac{u''(x)}{v''(x)}$ is a reduced rational function, then

$$\deg(\alpha \circ \alpha') = \max\{\deg u'', \deg v''\}$$

Reduced means no common roots over $\bar{K}$.

How do we prove $R(R'(x))$ is reduced? Lets write over $\bar{K}$

$$R(x) = \frac{\prod(x - \alpha_i)}{\prod(x - \beta_j)}$$

$$R'(x) = \frac{\prod(x - \alpha_i')}{\prod(x - \beta_j')}$$

$$R(R'(x)) = \frac{\prod(\frac{\prod(x-\alpha_i')}{\prod(x-\beta_j')} - \alpha_i)}{\prod(\frac{\prod(x-\alpha_i')}{\prod(x-\beta_j')} - \beta_j)}$$

if x_0 is such that

$$R'(x_0) = \alpha_i$$

for some i.

Then clearly since $a_i \neq \beta_j$ for all j.

$$R'(x_0) \neq \beta_j$$

Finally, a direct calculation shows that if

$$R''(x) = R(R'(x)) = \frac{u''(x)}{v''(x)}$$

then

$$\max\{\deg u'', \deg v''\} = \max\{\deg u, \deg v\} \max\{\deg u', \deg v'\}$$

$$R = u/v, R'' = u'/v'$$
$$R(R') = \frac{u(u'/v')}{v(u'/v')}$$

as rational functions,

$$\deg u(u'/v') = \deg u \max\{\deg u', \deg v'\}$$
$$\deg v(u'/v') = \deg v \max\{\deg u', \deg v'\}$$

(remember we are doing composition not multiplication)

$$R(R'(x)) = \frac{u''(x)}{v''(x)}$$

and

$$\begin{aligned}\max\{\deg u'', \deg v''\} &= \max\{u \max\{\deg u', \deg v'\}, v \max\{\deg u', \deg v'\}\} \\ &= \max\{u, v\} \max\{u', v'\} \\ &= \deg \alpha \deg \alpha'\end{aligned}$$

2 Isomorphic Isogeny

Isogeny $\alpha : E \to E'$ is called an isomorphism if $\exists$ an isogeny $\bar{\alpha}' : E' \to E$ such that $\alpha \circ \alpha^{-1} = \mathrm{id}_E$ and $\alpha^{-1} \circ \alpha = \mathrm{id}_E$.

2.1 $\deg \alpha = 1$ when α is an isomorphism

$$\begin{gathered}\deg \alpha \circ \deg \alpha^{-1} = \deg(\alpha \circ \alpha^{-1}) = \deg(\mathrm{id}_E) = 1 \\ \Rightarrow \deg \alpha = 1\end{gathered}$$

Remember E and E' might not be isomorphic over K but they might be isomorphic over an extension of K.

3 j-invariant

EC should be non-singular means $\Delta = 4A^3 + 27B^2 \neq 0$.

$$j = 1728 \frac{4A^3}{\Delta}$$

determines E up to isomorphism over $\bar{K}$.

A twist is you have two curves where $K \subseteq K'$

$$\begin{gathered}E(K), \quad E'(K') \\ E(K') \cong E'(K')\end{gathered}$$

It also turns out $[K' : K]$ is only 2, 4 or 6 (quadratic, quartic, sextic twists).

For $E(K)$, you can calculate $\#\mathrm{Aut}_{\bar{K}}(E) \leq 24$.

Remark: if $A = 0$ then $j = 0$. If $B = 0$, then $j = 1728$.

3.1 Proof of j invariant

If $j = 0$ or 1728, then take $E : y^2 = x^3 + 1$ or $E : y^2 = x^3 + x$, otherwise

$$A = 3j_0(1728 - j_0),\ B = 2j_0(1728 - j_0)^2$$

Then we see the j-invariants are consistent.

3.2 We cannot use rational maps, only polynomials for isogenies

All well defined rational maps which map $R(x)$ or $S(x)$ to ∞ must map to (∞, ∞). To observe this just look at $y^2 = x^3 + Ax + B$.

Let $R(x) = \frac{p(x)}{q(x)}$, then there's a root of $q(x)$ which is x_0. Then $R(x_0) = \infty$, but $\alpha(\infty) = \infty$ so we have a contradiction.

3.3 Showing $A' = \mu^4 A, B' = \mu^6 B$

Since deg $\alpha = 1$, $R(x) = ax + b$ by the definition of degree for a rational map.

$$S^2(x)(x^3 + Ax + B) = (ax + b)^3 + A'(ax + b) + B'$$

so comparing coefficients, we see $c^2 = a^3$ so $\mu = c/a \in K^\times$ so $a = \mu^2$.

$$\begin{gathered}\mu^6(x^3 + Ax + B) = \mu^6 x^3 + A'\mu^2 x + B' \\ \Rightarrow A' = \mu^4 A, B' = \mu^6 B\end{gathered}$$

3.4 Converse

Let $A' = \mu^4 A, B' = \mu^6 B$, $\alpha(x, y) = (\mu^2 x, \mu^3 y)$. Then α is a rational map that preserves ∞, so α is an isogeny.

Also α has an inverse $\alpha^{-1}(x, y) = (x/\mu^2, y/\mu^3)$.

And then composing them clearly gives the identity.

4 Tower of Field Extensions

$$A'/A = \mu^4$$

consider $g(x) = A'/A - x^4$, a root of $g(x)$ is our desired μ.

Curves sharing same j-invariant are isomorphic over some finite extension of K. This field extension is of degree 2, 4, or 6 when char $\neq 2, 3$.

Recall $E \cong E'$ then exists $\mu \in K^\times$ with $A' = \mu^4 A, B' = \mu^6 B$

$$\begin{aligned} j(E') &= 1728 \frac{4(\mu^4 A)^3}{4(\mu^4 A)^3 + 27(\mu^6 B)^2} \\ &= 1728 \frac{4A^3}{4A^3 + 27B^2} = j(E) \end{aligned}$$

Conversely, suppose $j(E) = j(E') = j_0$.

If $j_0 = 0$ then $A = A' = 0$ and $B, B' \neq 0$, we want $\mu \in \bar{K}$ such that $B' = \mu^6 B$. Such μ is a root of the polynomial $x^6 - B'/B$.

Likewise $j_0 = 1728$, then $B = B' = 0$ and $A, A' \neq 0$ so $A' = \mu^4 B$ which is the root of $x^4 - A'/A$.

For the remaining case $A, A', B, B' \neq 0$, then let

$$A'' = 3j_0(1728 - j_0)$$

$$B'' = 2j_0(1728 - j_0)^2$$

so that $j(A'', B'') = j_0$.

Now take

$$j(E) = 1728 \frac{4A^3}{4A^3 + 27B^2} = j_0$$

$$\begin{aligned} A'' &= 3 \cdot 1728 \frac{4A^3}{4A^3 + 27B^2} (1728 - 1728 \frac{4A^3}{4A^3 + 27B^2}) \\ &= \left(\frac{2^7 3^5 AB}{4A^3 + 27B^2} \right)^2 \cdot A \end{aligned}$$

$$B'' = ... = \left(\frac{2^7 3^5 AB}{4A^3 + 27B^2} \right)^3 \cdot B$$

(these terms are the u below)

Analogously we can plug in

$$j_0 = 1728 \frac{4{A'}^3}{4{A'}^3 + 27{B'}^2}$$

into A'' and B'' and get expressions for A'' and B'' in terms of A' and B'.

Now if we let

$$u = \left(\frac{2^7 3^5 AB}{4A^3 + 27B^2} \right)^2 \cdot \left(\frac{4{A'}^3 + 27B'}{2^7 3^5 A'B'} \right)$$

then $A' = u^2 A$, $B' = u^3 B$ so we choose $\mu \in K^\times$ such that $\mu^2 = u$ so

$$A' = \mu^4 A, B' = \mu^6 B$$

and μ exists in an extension of degree at most 2.