Contents

1 Matrix Multiplication

Let $A \in \mathbb{F}^{m\times n}$ and $B \in \mathbb{F}^{n\times p}$, then $AB \in \mathbb{F}^{m\times p}$

$$(AB)_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

1.1 Column Multiplication

$$A = (\mathbf{a}_1 \cdots \mathbf{a}_n)$$
$$(AB)_{:r} = b_{1r}\mathbf{a}_1 + b_{2r}\mathbf{a}_2 + \cdots + b_{n1}\mathbf{a}_n$$

1.2 Row Multiplication

$$B = \begin{pmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_n \end{pmatrix}$$
$$(AB)_{r:} = a_{r1}\mathbf{b}_1 + a_{r2}\mathbf{b}_2 + \cdots + a_{rn}\mathbf{b}_n$$

2 Uniqueness of Reduced Row Echelon Form

2.1 $A' = EA \Rightarrow \textbf{row}(A') = \textbf{row}(A)$

The row operations are:

1. interchange different rows
2. multiply rows by nonzero scalar
3. add a nonzero multiple of another row

We show A has equivalent row space under row operations.

Type 1 is immediate.

Type 2 replaces $\mathbf{a}_i$ by $r\mathbf{a}_i$, so we just rescale by $1/r$.

$$c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n = \frac{c_1}{r}\mathbf{a}_1' + \cdots + c_n\mathbf{a}_n$$

Type 3 replaces $\mathbf{a}_i$ by $\mathbf{a}_i + r\mathbf{a}_j$

$$\begin{aligned} c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n &= c_1(\mathbf{a}_1 + r\mathbf{a}_2) + (c_2 - rc_1)\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n \\ &= c_1\mathbf{a}_1' + (c_2 - rc_1)\mathbf{a}_2' + \cdots + c_n\mathbf{a}_n' \end{aligned}$$

So A and A' have the same row space.

2.2 $A = B : A, B \in \textbf{Red} \Leftrightarrow \textbf{row}(A) = \textbf{row}(B)$

$A = B \Rightarrow \text{row}(A) = \text{row}(B)$ is obvious so we prove the reverse direction.

Label the rows of A, B like so starting from the bottom.

$$A = \begin{pmatrix} \mathbf{a}_n \\ \vdots \\ \mathbf{a}_1 \end{pmatrix}, \qquad B = \begin{pmatrix} \mathbf{b}_n \\ \vdots \\ \mathbf{b}_1 \end{pmatrix}$$

We induct on the pivots starting with $\mathbf{a}_1, \mathbf{b}_1$.

1. the pivots for $\mathbf{a}_1, \mathbf{b}_2$ must be the same otherwise $\mathbf{a}_1 \notin \text{row}(B)$.
2. By symmetry, the pivots of $\mathbf{a}_1$ and $\mathbf{b}_1$ are in the same component.
3. $\mathbf{b}_1 = r_1\mathbf{a}_1 + \cdots + r_n\mathbf{a}_n$ but the other components don't share pivots $\Rightarrow \mathbf{b}_1 = r_1\mathbf{a}_1$.
4. $r_1 = 1$

Keep applying the same argument to see $A = B$.

2.3 Reduced Form is Unique

If two different sequences of elementary matrices corresponding to row operations yield two different reduced row echelon forms B and C for A, then by the previous propositions we get:

1. $\text{row}(A) = \text{row}(B) = \text{row}(C)$
2. $B = C$

3 Exercises

3.1 Ex 3.1.2

$$A = (a_{ij}), \qquad A^T = (a_{ij})^T = a_{ji}$$
$$(A + B)^T = ((a_{ij}) + (b_{ij}))^T = a_{ji} + b_{ji} = A^T + B^T$$

3.2 Ex 3.1.5

We use these simple rules:

$$(XY)^T = Y^T X^T$$
$$(X_{k,})^T = (X^T)_{,k}$$

and the column notation

$$(XY), k = Y_{1,k}X_{,1} + \cdots + Y_{n,k}X_{,n}$$

Putting this all together

$$\begin{aligned} (AB)^T_{,k} &= (B^T A^T)_{,k} = (A^T)_{1,k}(B^T)_{,1} + \cdots + (A^T)_{n,k}(B^T)_{,n} \\ &= A_{k,1}B_{1,} + \cdots + A_{k,n}B_{n,} \end{aligned}$$

but $(AB)^T_{,k} = (AB)_{k,}$

4 Forms of Matrix Multiplication

4.1 Column and Row Form

$$A = (\mathbf{a}_1 \cdots \mathbf{a}_n), \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$A\mathbf{x} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$$

$$A = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n \end{pmatrix}, \mathbf{x} = (x_1 \cdots x_n)$$

$$\mathbf{x}A = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$$

4.2 As a Dot Product

$$A = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$A\mathbf{x} = \begin{pmatrix} \langle \mathbf{a}_1, \mathbf{x} \rangle \\ \vdots \\ \langle \mathbf{a}_n, \mathbf{x} \rangle \end{pmatrix}$$

$$A = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n \end{pmatrix}, B = (\mathbf{b}_1 \cdots \mathbf{b}_m)$$

$$AB = \begin{pmatrix} \langle \mathbf{a}_1, \mathbf{b}_1 \rangle \ldots \langle \mathbf{a}_1, \mathbf{b}_m \rangle \\ \vdots \\ \langle \mathbf{a}_n, \mathbf{b}_1 \rangle \ldots \langle \mathbf{a}_n, \mathbf{b}_m \rangle \end{pmatrix}$$

A consequence of this is that $A^T A$, where $A = (\mathbf{a}_1 \cdots \mathbf{a}_n)$ is

$$A^T A = \begin{pmatrix} \langle \mathbf{a}_1, \mathbf{a}_1 \rangle \cdots \langle \mathbf{a}_n, \mathbf{a}_n \rangle \\ \vdots \\ \langle \mathbf{a}_n, \mathbf{a}_1 \rangle \cdots \langle \mathbf{a}_n, \mathbf{a}_n \rangle \end{pmatrix}$$

and likewise for AA^T when $A = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n \end{pmatrix}$.

4.3 Matrix as Map on Columns and Rows

$$A = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n \end{pmatrix}, B = (\mathbf{b}_1 \cdots \mathbf{b}_m)$$

$$\begin{aligned} AB &= (A\mathbf{b}_1 \cdots A\mathbf{b}_m) \\ &= \begin{pmatrix} \mathbf{a}_1 B \\ \vdots \\ \mathbf{a}_n B \end{pmatrix} \end{aligned}$$