A Book of Abstract Algebra (2nd Edition)

Chapter 16, Problem 2EF

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Problem

Let G be a group; let H and K be subgroups of G, with H a normal subgroup of G. Prove the following:

If $HK = \{xy: x \in H \text{ and } y \in K\}$, then HK is a subgroup of G.

Step-by-step solution

Step 1 of 4

Suppose that G is any group and let H, K are the subgroups of G, with H a normal subgroup of G.

Consider the following set:

$$HK = \{xy : x \in H, y \in K\}$$

Objective is to prove that HK is a subgroup of G.

One step test: If H is a nonempty subset of group G, then H will be subgroup of G if and only if for all $a, b \in H$

$$ab^{-1} \in H$$

Comment

Step 2 of 4

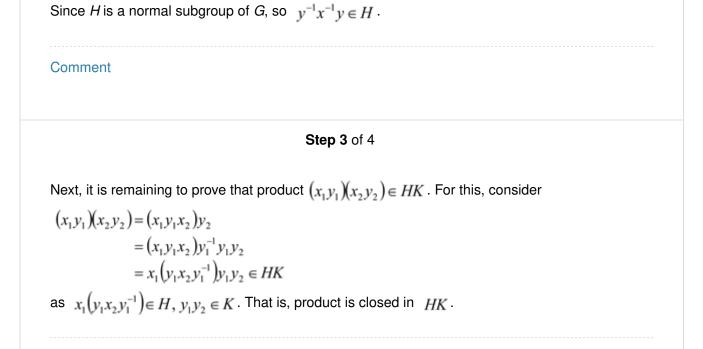
Since *H* and *K* are subgroups of *G*, therefore identity will belong to both the subgroups. Let $e_1 \in H$ and $e_2 \in K$. Then, $e_1e_2 \in HK$. This ensures that HK is nonempty subset of G.

Now consider two typical elements $x_1y_1, x_2y_2 \in HK$ where $x_1, x_2 \in H$ and $y_1, y_2 \in K$. Since $xy \in HK$, its inverse

$$(xy)^{-1} = y^{-1}x^{-1}$$

$$= y^{-1}x^{-1}yy^{-1}$$

$$= (y^{-1}x^{-1}y)y^{-1} \in HK$$



Comment

Step 4 of 4

Since product is closed in HK and inverse of each nonzero element exists, therefore HK forms a subgroup of G.

Comment