A Book of Abstract Algebra | (2nd Edition)

Chapter 32, Problem 4EA

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Problem

Write the inclusion diagram for the subgroups of $Gal(\mathbb{Q}(i, \sqrt{2}) : \mathbb{Q})$, and the inclusion diagram for the fields intermediate between $\mathbb{Q}$ and $\mathbb{Q}(i, \sqrt{2})$. Indicate the Galois correspondence.

Step-by-step solution

Step 1 of 4

The objective is to write the inclusion diagram for the subgroups of $Gal\left(\mathbb{Q}\left(i,\sqrt{2}\right):\mathbb{Q}\right)$, the inclusion diagram for the fields intermediate between $\mathbb{Q}$ and $\mathbb{Q}\left(i,\sqrt{2}\right)$ and to indicate the Galois correspondence.

Comment

Step 2 of 4

The root field $\mathbb{Q}\left(i,\sqrt{2}\right)$ is of degree 4 over $\mathbb{Q}$.

Therefore, there are four automorphism of $\mathbb{Q}\left(i,\sqrt{2}\right)$ which fix $\mathbb{Q}$, since the number of automorphism is equal to the degree of $\mathbb{Q}\left(i,\sqrt{2}\right)$ over $\mathbb{Q}$.

Since an automorphism is determined by its effect on $\sqrt{2}$ and i, there are four possibilities, namely,

$$\varepsilon:\begin{cases}\sqrt{2}\mapsto\sqrt{2}\\ i\mapsto i\end{cases}\quad \alpha:\begin{cases}\sqrt{2}\mapsto-\sqrt{2}\\ i\mapsto i\end{cases}\quad \beta:\begin{cases}\sqrt{2}\mapsto\sqrt{2}\\ i\mapsto -i\end{cases}\quad \gamma:\begin{cases}\sqrt{2}\mapsto-\sqrt{2}\\ i\mapsto -i\end{cases}.$$

Thus, the Galois group of $\mathbb{Q}\left(i,\sqrt{2}\right)$ over $\mathbb{Q}$ is $Gal\left(\mathbb{Q}\left(i,\sqrt{2}\right):\mathbb{Q}\right)=\{\varepsilon,\alpha,\beta,\gamma\}$.

This group has exactly five subgroup-namely $\{\varepsilon\}$, $\{\varepsilon,\alpha\}$, $\{\varepsilon,\beta\}$, $\{\varepsilon,\gamma\}$, and the whole group G.

Inclusion Diagram for the Subgroups:

Comment

Step 3 of 4

Inclusion Diagram for the Intermediate fields:

On the other hand ,there are exactly five fields intermediate between $\mathbb{Q}$ and $\mathbb{Q}\left(i,\sqrt{2}\right)$,which may be represented in the inclusion diagram:

Comment

Step 4 of 4

The subgroup of G have the following fix fields:

$\{\varepsilon\}^o = \mathbb{Q}\left(i,\sqrt{2}\right)$ $\{\varepsilon,\alpha\}^o = \mathbb{Q}(i)$ $\{\varepsilon,\beta\}^o = \mathbb{Q}\left(\sqrt{2}\right)$

$\{\varepsilon,\gamma\}^o = \mathbb{Q}\left(\sqrt{2}i\right)$ $G^o = \mathbb{Q}$

The Galois correspondence may be represented as follows:

Comment