A Book of Abstract Algebra | (2nd Edition)

Chapter 32, Problem 1EH

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Problem

Prove: The only automorphism of $\mathbb{Q}$ is the identity function. [HINT: If *h* is an automorphism, *h*(1) = 1; hence *h*(2) = 2, and so on.]

Step-by-step solution

Step 1 of 2

The objective is to prove that the only automorphism of $\mathbb{Q}$ is the identity function.

Comment

Step 2 of 2

Let $f:\mathbb{Q}\rightarrow\mathbb{Q}$ be an automorphism .

$f(1)=f(1\times 1)$

$f(1)\times 1=f(1)\times f(1)$,since f is a homomorphism

$f(1)=1$.

Let $n\in\mathbb{Z}^{+}$.

$$f(n)=f\left(\underbrace{1+1+...+1}_{n\text{ times}}\right)$$
$$=\underbrace{f(1)+f(1)+...+f(1)}_{n\text{ times}}$$
$$=nf(1)$$
$$=n\times 1$$
$$=n.$$

Let $n=-m$,$m\in\mathbb{Z}^{+}$.

$$\begin{aligned} f(n) &= f(-m) \\ &= -f(m) \\ &= -m \\ &= n. \end{aligned}$$

Therefore，$f(a) = a \quad \forall\, a \in \mathbb{Z}$.

Let $\dfrac{a}{b} \in \mathbb{Q}$ ，$a \in \mathbb{Z}$ and $b \in \mathbb{Z}^{+}$.

$$\begin{aligned} a &= f(a) \\ &= f\left(\frac{a}{b} \times b\right) \\ &= f\left(\underbrace{\frac{a}{b} + \frac{a}{b} + \ldots + \frac{a}{b}}_{n \text{ times}}\right) \\ &= \underbrace{f\left(\frac{a}{b}\right) + f\left(\frac{a}{b}\right) + \ldots + f\left(\frac{a}{b}\right)}_{n \text{ times}} \\ &= bf\left(\frac{a}{b}\right) \end{aligned}$$

Therefore，$\dfrac{a}{b} = f\left(\dfrac{a}{b}\right)$.

Thus，$f(x) = x \quad \forall\, x \in \mathbb{Q}$.

This shows that the only automorphism of $\mathbb{Q}$ is the identity function.

Comment