# A Book of Abstract Algebra | (2nd Edition)

Chapter 32, Problem 7ED Bookmark Show all steps: ON

## Problem

If $\alpha = \sqrt[4]{2}$ is a real fourth root of 2, then the four fourth roots of 2 are $\pm\alpha$ and $\pm i\alpha$. Explain parts 1–6, briefly but carefully:

Explain: $h(\alpha)$ must be a fourth root of 2 and $h(i)$ must be equal to $\pm i$. Combining the four possibilities for $h(\alpha)$ with the two possibilities for $h(i)$ gives eight possible automorphisms. List them in the format

$$\begin{Bmatrix} \alpha \to \alpha \\ i \to i \end{Bmatrix}, \quad \begin{Bmatrix} \alpha \to -\alpha \\ i \to i \end{Bmatrix}, \ldots$$

## Step-by-step solution

**Step 1** of 2

The objective is to list the automorphism of $\mathbb{Q}\left(\sqrt[4]{2}, i\right)$ over $\mathbb{Q}$.

Comment

**Step 2** of 2

Consider the Galois group of $x^4 - 2$ over $\mathbb{Q}$. The polynomial has $4$ roots :

$\sqrt[4]{2}, i\sqrt[4]{2}, -\sqrt[4]{2}, -i\sqrt[4]{2}$.

For any automorphism $h$ of $\mathbb{Q}\left(\sqrt[4]{2}, i\right)$ over $\mathbb{Q}$, $h\left(\sqrt[4]{2}\right)$ has to be a root of $x^4 - 2$ ( $4$ possible values) and $h(i)$ has to be a root of $x^2 + 1$ ( $2$ possible values).

Thus, there are at most $4 \cdot 2 = 8$ automorphism of $\mathbb{Q}\left(\sqrt[4]{2}, i\right)$ over $\mathbb{Q}$.

Because $\left[\mathbb{Q}\left(\sqrt[4]{2}, i\right) : \mathbb{Q}\right] = 8$, $Gal\left(\mathbb{Q}\left(\sqrt[4]{2}, i\right) : \mathbb{Q}\right)$ has size $8$ and therefore all assignments of $h\left(\sqrt[4]{2}\right)$ and $h(i)$ to roots of $x^4 - 2$ and $x^2 + 1$, respectively, must be realized by field automorphism.

Let $r$ and $s$ be the automorphism of $\mathbb{Q}\left(\sqrt[4]{2},i\right)$ over $\mathbb{Q}$ determined by

$r\left(\sqrt[4]{2}\right)=i\sqrt[4]{2},\ r(i)=i,\ s\left(\sqrt[4]{2}\right)=\sqrt[4]{2},\ s(i)=-i$.

Then the following $8$ different automorphism of $\mathbb{Q}\left(\sqrt[4]{2},i\right)$ over $\mathbb{Q}$ is obtained as follows:

$$id:\begin{cases}\sqrt[4]{2}\mapsto\sqrt[4]{2}\\ i\mapsto i\end{cases}\quad r:\begin{cases}\sqrt[4]{2}\mapsto i\sqrt[4]{2}\\ i\mapsto i\end{cases}\quad r^2:\begin{cases}\sqrt[4]{2}\mapsto -\sqrt[4]{2}\\ i\mapsto i\end{cases}\quad r^3:\begin{cases}\sqrt[4]{2}\mapsto -i\sqrt[4]{2}\\ i\mapsto i\end{cases}$$

$$s:\begin{cases}\sqrt[4]{2}\mapsto\sqrt[4]{2}\\ i\mapsto -i\end{cases}\quad rs:\begin{cases}\sqrt[4]{2}\mapsto i\sqrt[4]{2}\\ i\mapsto -i\end{cases}\quad r^2s:\begin{cases}\sqrt[4]{2}\mapsto -\sqrt[4]{2}\\ i\mapsto -i\end{cases}\quad r^3s:\begin{cases}\sqrt[4]{2}\mapsto -i\sqrt[4]{2}\\ i\mapsto -i\end{cases}.$$

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