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Chapter 16, Problem 1EJ

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Problem

Let *f* be a homomorphism from G onto *H* with kernel *K*:

$$f : G \xrightarrow[K]{} H$$

If *S* is any subgroup of *H*, let *S** = {*x* ∈ *G*: *f*(*x*) ∈ *S*}. Prove:

*S** is a subgroup of *G*.

Step-by-step solution

Step 1 of 4

Suppose that *G* is any group. Let the mapping

$$f : G_K \to H$$

is a homomorphism from *G* onto *H* with kernel *K*. Assume that *S* is any subgroup of *H* and consider the following set:

$$S^* = \{x \in G : f(x) \in S\}.$$

Objective is to prove that the set S^* forms a subgroup of *G*.

One step test: If *H* is a nonempty subset of group *G*, then *H* will be subgroup of *G* if and only if for all $a, b \in H$

$$ab^{-1} \in H.$$

Comment

Step 2 of 4

Since *S* is a subgroup, so identity will be there in *S*. Also identity is self-imaged element, so it will belong to S^* too. Because of the existence of identity, the set S^* is nonempty subset of *G*.

If one is able to show that $f(xy^{-1}) \in S$, then this will ensure that $xy^{-1} \in S^*$.

Let $x, y \in S^*$. Consider $f(xy^{-1})$ and expand it by the homomorphism rule as:

$$\begin{aligned} f(xy^{-1}) &= f(x)f(y^{-1}) \\ &= f(x)[f(y)]^{-1}. \end{aligned}$$

The second step is the well-known property of homomorphism.

Comment

Step 3 of 4

Since $x, y \in S^*$, therefore by the above definition $f(x), f(y) \in S$. Also *S* is a subgroup, so $[f(y)]^{-1} \in S$. Being operation closed, $f(x)[f(y)]^{-1} \in S$. Thus, $f(xy^{-1}) \in S$ and correspondingly $xy^{-1} \in S^*$.

Comment

Step 4 of 4

Hence, by one-step test it can be conclude that S^* forms a subgroup of *G*.

Comment