A Book of Abstract Algebra | (2nd Edition)

Chapter 29, Problem 6EA

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Problem

Find a basis of $\mathbb{Q}(\sqrt{2},\sqrt{3},\sqrt{5})$ over $\mathbb{Q}$, and describe the elements of $\mathbb{Q}(\sqrt{2},\sqrt{3},\sqrt{5})$.

Step-by-step solution

Step 1 of 2

The objective is to find a basis of $\left(\sqrt{2},\sqrt{3},\sqrt{5}\right)$ over , and describe the elements of $\left(\sqrt{2},\sqrt{3},\sqrt{5}\right)$.

Comment

Step 2 of 2

The minimal polynomial of $\sqrt{2}$ over is x^2-2 as it is monic and irreducible with $\sqrt{2}$ as a root.

Hence, $\left[\ \left(\sqrt{2}\right):\ \ \right]=2$; a basis is $\left\{1,\sqrt{2}\right\}$.

Show that $\sqrt{3}\notin\ \left(\sqrt{2}\right)$.

Assume that $\sqrt{3}\in\ \left(\sqrt{2}\right)$.

Then $\sqrt{3}$ must have the form $a+b\sqrt{2}$, for some $a,b\in\ $.

It follows that $\left(a+b\sqrt{2}\right)^2=3$ and thus $a^2+2\sqrt{2}ab+2b^2-3=0$.

Since $\left\{1,\sqrt{2}\right\}$ is a linear independent set as it is a basis for $\ \left(\sqrt{2}\right)$ as a vector space over , either $a=0$ or $b=0$.

If $a=0$ then $b=\pm\frac{\sqrt{3}}{\sqrt{2}}$ and if $b=0$ then $a=\pm\sqrt{3}$.

This is a contradiction to $a,b\in\ $.

Hence, x^2-3 is irreducible over $\ \left(\sqrt{2}\right)$; it is a minimal polynomial over $\ \left(\sqrt{2}\right)$.

So, $\left[\ \left(\sqrt{3},\sqrt{2}\right):\ \left(\sqrt{2}\right)\right]=2$ and that $\left\{1,\sqrt{3}\right\}$ is a basis for $\ \left(\sqrt{3},\sqrt{2}\right)$ over $\ \left(\sqrt{2}\right)$.

Show that $\sqrt{5}\notin\ \left(\sqrt{2},\sqrt{3}\right)$.

Suppose that it were, then

$\sqrt{5}=c+d\sqrt{2}+f\sqrt{3}+g\sqrt{6}$.

Squaring on both sides and rearrange the terms such that the constant term is equal to 5, and the other three terms in front of the radicals are equal to 0.

$$\begin{aligned}
&c^2+2d^2+3f^2+6g^2=5\\
&cd+3fg=0\\
&cf+2dg=0\\
&cg+df=0
\end{aligned}$$

Any (c,d,f,g) that satisfies these relationships implies that $(c,d,-f,-g)$,

$(c,-d,f,-g)$ and $(c,-d,-f,g)$.

Therefore, $c+d\sqrt{2}+f\sqrt{3}+g\sqrt{6}=\sqrt{5}$

$c+d\sqrt{2}-f\sqrt{3}-g\sqrt{6}=\pm\sqrt{5}$

$$c-d\sqrt{2}+f\sqrt{3}-g\sqrt{6}=\pm\sqrt{5}$$
$$c-d\sqrt{2}-f\sqrt{3}+g\sqrt{6}=\pm\sqrt{5}$$

Add the first two equations ，the result implies $c=d=0$ as $c,d\in$.

Add the first and third ，the result implies $f\sqrt{3}=0$ or $\sqrt{5}$ and so $f\ =0$.

Finally $g=0$ since $g\sqrt{6}$ cannot be equal to $\sqrt{5}$.

So，$\left[\ \left(\sqrt{2},\sqrt{3},\sqrt{5}\right):\ \left(\sqrt{2},\sqrt{3}\right)\right]=2$ and its basis is $\left\{1,\sqrt{5}\right\}$.

Therefore，

$$\begin{aligned}\left[\ \left(\sqrt{2},\sqrt{3},\sqrt{5}\right):\ \right]&=\left[\ \left(\sqrt{2},\sqrt{3},\sqrt{5}\right):\ \left(\sqrt{2},\sqrt{3}\right)\right]\left[\ \left(\sqrt{2},\sqrt{3}\right):\ \left(\sqrt{2}\right)\right]\left[\ \left(\sqrt{2}\right):\ \right]\\&=2\cdot 2\cdot 2\\&=8\end{aligned}$$

Therefore，$\left\{1,\sqrt{2},\sqrt{3},\sqrt{5},\sqrt{6},\sqrt{10},\sqrt{15},\sqrt{30}\right\}$ is a basis for $\left(\sqrt{2},\sqrt{3},\sqrt{5}\right)$ over .

Any element of $\left(\sqrt{2},\sqrt{3},\sqrt{5}\right)$ is of the form:

$$a+b\sqrt{2}+c\sqrt{3}+d\sqrt{5}+e\sqrt{6}+f\sqrt{10}+g\sqrt{15}+h\sqrt{30}:a,b,c,d,e,f,g,h\in\ .$$

Comment