A Book of Abstract Algebra | (2nd Edition)

Chapter 28, Problem 3EG

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Problem

Let T and U be subspaces of V. The *sum* of T and U, denoted by $T + U$, is the set of all vectors **a** + **b**, where $\mathbf{a} \in T$ and $\mathbf{b} \in U$.

Let T be a k-dimensional subspace of an n-dimensional space V. Prove that an $(n - k)$-dimensional subspace U exists such that $V = T \oplus U$.

Step-by-step solution

Step 1 of 2

V is finite dimensional vector space. Any subspace of V will also be finite dimensional. Consider a subspace T of dimension k.

So there is a basis for $T = (\mathbf{t}_1, \mathbf{t}_2, \ldots \mathbf{t}_k)$

This basis can be extended to V such that,

Basis of $V = (\mathbf{t}_1, \mathbf{t}_2, \ldots \mathbf{t}_k, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n-k})$

Comment

Step 2 of 2

It can be easily seen that $(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n-k})$ form independent set of vectors and there is a subspace which is spanned by this set

Also it can be easily seen that any vector **v** in V, can be expressed as

$$\mathbf{v} = a_1\mathbf{t}_1 + a_2\mathbf{t}_2 + \ldots + a_k\mathbf{t}_k + b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + \ldots + b_{n-k}\mathbf{u}_{n-k}$$

This can be rearranged as

$$\mathbf{v} = (a_1\mathbf{t}_1 + a_2\mathbf{t}_2 + \ldots + a_k\mathbf{t}_k) + (b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + \ldots + b_{n-k}\mathbf{u}_{n-k})$$
$$\Rightarrow \mathbf{v} = \mathbf{t}' + \mathbf{u}'$$

Dimension of subspace *U* is $n-k$ and all **v** can be expressed as sum of *T* and *U*.

$$\boxed{\text{Hence there exists a subspace } U \text{ of dimension } n-k \text{ such that } V = T \oplus K}$$

Comment