A Book of Abstract Algebra | (2nd Edition)

Problem

Find a basis of $\mathbb{Q}\mathbb{Q}(\sqrt{5}, \sqrt{7})$ over $\mathbb{Q}$ and describe the elements of $\mathbb{Q}(\mathbb{Q}(\sqrt{5}, \sqrt{7}))$. (See the example at the end of this chapter.)

Step-by-step solution

Step 1 of 2

The objective is to find a basis of $\left(\sqrt{5}, \sqrt{7}\right)$ over and describe the elements of $\left(\sqrt{5}, \sqrt{7}\right)$.

Comment

Step 2 of 2

The minimal polynomial of $\sqrt{5}$ over is $x^2 - 5$ (It is monic and irreducible($5-$ Eisenstein) with $\sqrt{5}$ as a root.)

Hence, $[\ [\sqrt{5}]:\ \ ]=2$ and a basis is $\{1,\sqrt{5}\}$.

Claim: $\sqrt{7}\notin\ \ (\sqrt{5})$.

Suppose on the contrary that $\sqrt{7}\in\ \ (\sqrt{5})$.

Therefore, there exists $a,b\in\ \ $ with $\sqrt{7}=a+b\sqrt{5}$.

Squaring gives $49=a^2+2ab\sqrt{5}+5b^2$, from which it follows that $\sqrt{5}=\dfrac{49-a^2-5b^2}{2ab}\in\ \ $,

a contradiction.

Hence, x^2-7 is irreducible over $\ \ (\sqrt{5})$; it is the minimal polynomial over $\ \ (\sqrt{5})$.

So, $[\ \ (\sqrt{5},\sqrt{7}):\ \ (\sqrt{5})]=2$ and that $\{1,\sqrt{7}\}$ is a basis for $\ \ (\sqrt{5},\sqrt{7})$ over $\ \ (\sqrt{5})$.

$$\begin{aligned}[\ \ (\sqrt{5},\sqrt{7}):\ \ ]&=[\ \ (\sqrt{5},\sqrt{7}):\ \ (\sqrt{5})][\ \ (\sqrt{5}):\ \ ]\\&=2\cdot 2\\&=4\end{aligned}$$

The extension has basis $\{1,\sqrt{5},\sqrt{7},\sqrt{35}\}$ over $\ \ $.

Any element of $\ \ (\sqrt{5},\sqrt{7})$ is of the form :

$a+b\sqrt{5}+c\sqrt{7}+d\sqrt{21}:a,b,c,d\in\ \ $.

Comment