A Book of Abstract Algebra | (2nd Edition)

Chapter 24, Problem 3EE

Problem

Let *S* be the set of all the polynomials *a*(*x*) in *A*[*x*] for which every coefficient *ai* for *odd i* is equal to zero. Show that *S* is a subring of *A*[*x*]. Why is the same not true when "odd" is replaced by "even"?

Step-by-step solution

Step 1 of 2

Consider a ring $A[x]$. Let *S* be the set of all polynomials $a(x)$ in $A[x]$ for which every coefficient a_i for odd *i* is equal to zero.

Then $a(x)=a_n x^n+0x^{n-1}+...0x+a_0$

It implies every powers of *x* in the polynomials in *S* are even numbers because odd coefficients are zero.

Objective of the question is to prove *S* is a subring of $A[x]$.

Now prove *S* is a subring of $A[x]$.

Recall the theorem known as subring test.

Theorem 1: A non empty subset *S* of ring *R* is a subring if

$$a-b\in S\forall a,b\in S$$
$$ab\in S\forall a,b\in S$$

Let two polynomials $a(x)$ and $b(x)$ in *S*.

$$a(x)=a_n x^n+0x^{n-1}+...0x+a_0$$

$$b(x)=b_n x^n+0x^{n-1}+...+a_2x^2+0x+a_0$$

Then,

$$
\begin{aligned}
a(x)-b(x) &= \left(a_n x^n + 0x^{n-1} + \ldots 0x + a_0\right) - \left(b_n x^n + 0x^{n-1} + \ldots + b_2 x^2 + 0x + b_0\right) \\
&= \left(a_n - b_n\right)x^n + (0-0)x^{n-1} + \left(a_{n-2} - b_{n-2}\right)x^{n-2} + \ldots + (0-0)x + \left(a_0 - b_0\right) \\
&= \left(a_n - b_n\right)x^n + 0x^{n-1} + \left(a_{n-2} - b_{n-2}\right)x^{n-2} + \ldots + 0x + \left(a_0 - b_0\right)
\end{aligned}
$$

Then odd coefficients are zero. That implies $a(x)-b(x) \in S$

Now prove $a(x)b(x) \in S$.

$$a(x)b(x) = \left(a_n x^n + 0x^{n-1} + \ldots 0x + a_0\right)\left(b_n x^n + 0x^{n-1} + \ldots + b_2 x^2 + 0x + b_0\right)$$

To prove $a(x)b(x) \in S$ it suffices to prove the power of all *x* in the product is even number.

$$
\begin{aligned}
a(x)b(x) &= \left(a_n x^n + 0x^{n-1} + \ldots 0x + a_0\right)\left(b_n x^n + 0x^{n-1} + \ldots + b_2 x^2 + 0x + b_0\right) \\
&= \left(a_n b_n x^{2n} + 0x^{2n-1} + \ldots + a_n b_2 x^{n+2} + 0x^{n+1} + a_n b_0 x^n\right) + \ldots + \\
&\quad \left(a_0 b_n x^n + 0x^{n-1} + \ldots + a_0 b_2 x^2 + 0x + a_0 b_0\right)
\end{aligned}
$$

It can see that each term is in the form $a_i b_j x^{i+j}$.

Here each *i* and *j* are even numbers.

That is

$$
\begin{aligned}
i &= 2k \text{ for any } k \in \mathbb{Z}^+ \\
j &= 2l \text{ for any } l \in \mathbb{Z}^+
\end{aligned}
$$

Then,

$$
\begin{aligned}
i + j &= 2k + 2l \\
&= 2(k+l)
\end{aligned}
$$

It implies $i + j$ is an even number.

Here $i + j$ is the power of *x* in the polynomial $a(x)b(x)$.

Therefore power of all *x* in the product of polynomials is even number.

Hence $a(x)b(x) \in S$.

Then according to theorem 1 *S* is a subring.

Comment

Step 2 of 2

Objective of the question is to disprove the statement “If *S* be the set of all polynomials $a(x)$ in $A[x]$ for which every coefficient a_i for even *i* is equal to zero then *S* is a subring of $A[x]$”.

Let *S* be the subring of $A[x]$ such that every coefficient a_i for even *i* is equal to zero.

To disprove the statement it suffices to give examples which violating the definition of subring.

First recall the definition of subring.

Definition: A subset *S* is said to be a subring of a ring *R,* if *S* itself a ring.

Consider two polynomials $p(x)$ and $q(x)$ in *S.*

$$p(x) = x^3 + x$$
$$q(x) = x$$

Then,

$$\begin{aligned} p(x)q(x) &= x(x^3 + x) \\ &= x^4 + x^2 \end{aligned}$$

Here coefficients of even powers are not equal to zero.

Therefore $p(x)q(x) \notin S$.

It violates the definition of ring.

Hence *S* is not a ring.

It implies *S* is not a subring of $A[x]$.

Comment