A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 6EM

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Problem

Let *p* be a prime number. A finite group *G* is called a *p-group* if the order of every element *x* in *G* is a power *p*. (The orders of different elements may be different powers of *p*.) If *H* is a subgroup of any finite group *G*, and *H* is a *p*-group, we call *H* a *p-subgroip* of *G*. Finally, if *K* is a *p*-subgroup of *G*, and *K* is maximal (in the sense that *K* is not contained in any larger *p*-subgroup of *G*), then *K* is called a *p-Sylow subgroup* of *G*.

If *a* ∈ *N* and the order of *p* is a power of *p*, then the order of *Ka* (in *N*/*K*) is also a power of *p*. (Why?) Thus, *Ka* = *K*.(Why?)

Step-by-step solution

Step 1 of 3

Suppose that *G* is a *p*-group, so order of each element *x* in *G* will be the power of *p*. Let *K* is a *p*-Sylow subgroup of *G* and $N = N(K)$be the normalizer of *K*.

Assume that $a \in N$, and the order of coset Ka in N/K is a power of *p*. Let $S = \langle Ka \rangle$ is the cyclic subgroup of N/K generated by Ka.

Objective is to prove that if $a \in N$ and the order of *a* is a power of *p*, then the order of Ka in N/K is also a power of *p*. And then $Ka = K$.

Comment

Step 2 of 3

Let $|a| = p^j$, for some integer *j*. Consider arbitrary element (coset) $Ka \in N/K$. Now calculate the following power of this coset:

$$\begin{aligned}(Ka)^{p^j} &= Ka^{p^j} \\ &= Ke \\ &= K.\end{aligned}$$

The second step is obtained from the condition that $|a| = p^j$, so $a^{p^j} = e$.

Note that K is the identity element of quotient group N/K. So, the equation $(Ka)^{p^j} = K$ implies that the order of coset Ka must be a divisor of p^j. That is, order of Ka will be some power of p.

Since no non-identity element of N/K has order a power of p and order of Ka is some power of p, so $Ka = K$.

Comment

Step 3 of 3

Hence, if $a \in N$ and $|a| = p^j$ then the order of Ka in N/K is also a power of p.

Comment