A Book of Abstract Algebra | (2nd Edition)

Chapter 29, Problem 3EA

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Problem

If $a=\sqrt{1+\sqrt[3]{2}}$, show that {1, $2^{1/3}$, $2^{2/3}$, a,$2^{1/3}a$, $2^{2/3}a$} is a basis of $\mathbb{Q}(a)$ over $\mathbb{Q}$. Describe the elements of $\mathbb{Q}(a)$.

Step-by-step solution

Step 1 of 3

Consider that $a=\sqrt{1+\sqrt[3]{2}}$. Objective is to show that $\{1, 2^{1/3}, 2^{2/3}, a, 2^{1/3}a, 2^{2/3}a\}$is a basis of $Q(a)$over Q.

Let $x=\sqrt[3]{2}$. Then $x^3=2$, and $x^3-2=0$. Note that, x^3-2 is a minimal polynomial of $\sqrt[3]{2}$, because it is irreducible by Eisenstein's irreducible criterion.

Since polynomial is of degree 3, therefore $\left[Q\left(\sqrt[3]{2}\right):Q\right]=3$. The basis for this will be:

$\{1, 2^{1/3}, 2^{2/3}\}$

Comment

Also from $a=\sqrt{1+\sqrt[3]{2}}$, it implies that $a^2-1=\sqrt[3]{2}$. Then $\sqrt[3]{2}\in Q(a)$, and therefore

$Q(a)=Q(\sqrt[3]{2},a)$.

Next, in $Q(\sqrt[3]{2})$, *a* satisfies $a^2-1-\sqrt[3]{2}=0$. So, *a* is a root of the polynomial $x^2-1-\sqrt[3]{2}=0.$ Since the root of this quadratic equation is some irrational number, therefore it is irreducible over $Q(\sqrt[3]{2})[x]$. Hence, quadratic polynomial $x^2-1-\sqrt[3]{2}=0$ is the minimal polynomial of *a* over $Q(\sqrt[3]{2})[x]$. Thus,

$[Q(\sqrt[3]{2},a):Q(\sqrt[3]{2})]=2$.

And the basis will be $\{1,a\}$.

Comment

Then,

$$\begin{aligned}[Q(\sqrt[3]{2},a):Q]&=[Q(\sqrt[3]{2},a):Q(\sqrt[3]{2})]\cdot[Q(\sqrt[3]{2}):Q]\\&=2\cdot 3\\&=6\end{aligned}$$

The required basis, with the help of theorem, will be:

$\{1,2^{1/3},2^{2/3},a,2^{1/3}a,2^{2/3}a\}$.

And the elements of $Q(a)$ will be of the form:

$Q(a)=\{p+q\cdot 2^{1/3}+r\cdot 2^{2/3}+s\cdot a+t\cdot 2^{1/3}a+u\cdot 2^{2/3}a : p,q,r,s,t,u\in Q\}$.

Comment