A Book of Abstract Algebra | (2nd Edition)

Problem

Which of the following extensions are root fields over $\mathbb{Q}$? Justify your answer:

$\mathbb{Q}(i); \mathbb{Q}(\sqrt{2}); \mathbb{Q}(\sqrt[3]{2})$, where $\sqrt[3]{2}$ is the *real* cube root of 2;

$\mathbb{Q}(2+\sqrt{5}); \mathbb{Q}(i+\sqrt{3}); \mathbb{Q}(i, \sqrt{2}, \sqrt{3})$.

Step-by-step solution

Step 1 of 2

The objective is to find whether the following extensions are root fields over $\mathbb{Q}$ or not, with justification.

$\mathbb{Q}(i)$; $\mathbb{Q}(\sqrt{2})$; $\mathbb{Q}(\sqrt[3]{2})$ where $\sqrt[3]{2}$ is the real cube root of 2; $\mathbb{Q}(2+\sqrt{5})$; $\mathbb{Q}(i+\sqrt{3})$; $\mathbb{Q}(i, \sqrt{2}, \sqrt{3})$.

Comment

Step 2 of 2

$\mathbb{Q}(i)$ is the root field of x^2+1.

$\mathbb{Q}(\sqrt{2})$ is the root field of x^2-2.

$\mathbb{Q}(2+\sqrt{5})$ is same as $\mathbb{Q}(\sqrt{5})$. $\mathbb{Q}(\sqrt{5})$ is the root field of x^2-5.

$\mathbb{Q}(i, \sqrt{2}, \sqrt{3})$ is the root field of $x^6-4x^4+x^2+6$.

$\mathbb{Q}(\sqrt[3]{2})$ is not a root field over $\mathbb{Q}$ because it does not contains complex cube roots of 2.

$\mathbb{Q}(i+\sqrt{3})$ is not a root field over $\mathbb{Q}$ because it does not contains the complex root $i-\sqrt{3}$.

Therefore, $\mathbb{Q}(i)$; $\mathbb{Q}(\sqrt{2})$; $\mathbb{Q}(2+\sqrt{5})$; and $\mathbb{Q}(i, \sqrt{2}, \sqrt{3})$ are root field over $\mathbb{Q}$, and

$\mathbb{Q}\left(\sqrt[3]{2}\right)$ where $\sqrt[3]{2}$ is the real cube root of 2 and $\mathbb{Q}\left(i+\sqrt{3}\right)$ are not root field over $\mathbb{Q}$.

Comment