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Chapter 29, Problem 5EF

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Problem

Let F be a field, and K a finite extension of F. Prove each of the following:

Let $p(x)$ be irreducible in $F[x]$. If $[K : F]$ and deg $p(x)$ are relatively prime, then $p(x)$ is irreducible in $K[x]$.

Step-by-step solution

Step 1 of 2

Consider an irreducible polynomial $p(x)$ in $F[x]$. Suppose $[K:F]$ and $\deg p(x)$ are relatively prime. The objective is to prove that $p(x)$ is irreducible in $K[x]$.

Comment

Step 2 of 2

Without loss of generality, suppose that $p(x)$ is monic.

Let u be a root of the polynomial $p(x)$.

Then $p(x) = p_F(x)$ is the minimal polynomial of u over F.

Compute $[K(u):F]$ in wo ways:

$$[K(u):K]\cdot[K:F] = [K(u):F] = [K(u):F(u)]\cdot[F(u):F].$$

It follows that $[F(u):F] \mid [K(u):K]\cdot[K:F]$.

By hypothesis, $\deg(p_F(x)) = [F(u):F]$ is relatively prime to $[K:F]$.

Therefore, $[F(u):F] \mid [K(u):K]$; note that $[F(u):F] \le [K(u):K]$.

But $F \subseteq K$ implies that $p_K(x) \mid p_F(x)$ in $K[x]$, so

$$\deg p_F = [F(u):F] \ge [K(u):K] = \deg p_K.$$

Hence, $\deg p_F = \deg p_K$.

Since $p_K(x) \mid p_F(x)$ and both polynomials are monic, $p_K(x) = p_F(x)$.

That is, as the minimal polynomial of u over K, $p(x) = p_F$ is irreducible over K.

Comment