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Chapter 33, Problem 3EC

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Problem

Let p be a prime number, and ω a primitive pth root of unity in the field F.

If deg $p(x) = m$, explain why the constant term of $p(x)$ (let us call it b) is equal to the product of m pth roots of a. Conclude that $b = \omega^k d^m$ for some k.

Step-by-step solution

Step 1 of 5

Here, objective is to explain why the constant term of $p(x)$ is equal to product of m p^{th} roots of a.

The polynomial $x^p - a \in F(x)$

Where, p is a prime and $x^p - a$ is reducible in $F(x)$

Consider $\text{degree } p(x) = m$

Comment

Step 2 of 5

Consider the polynomial $x^p - a$.

The root of above polynomial is a primitive p^{th} root of unity

$$x^p - a = 0$$
$$x^p = a$$
$$x = \sqrt[p]{a}\ \omega$$

Consider d is a root of $x^p - a \in F(x)$.

Then, $d = \sqrt[p]{a}$, ω is the p^{th} root of unity

Comment

Step 3 of 5

Let us assume $d_1, d_2, \ldots\ldots\ldots\ldots d_p$ are the roots of $x^p - a$

F has an extension *K* contains all the roots

$d_1, d_2, \ldots\ldots\ldots\ldots d_p$ of $x^p - a$.

Comment

Step 4 of 5

Consider

$$x^p - a = p(x) f(x)$$

Then, write into linear factors

$$x^p - a = (x - d_1)(x - d_2)\ldots\ldots.(x - d_p)$$

$p(x)$ is equal to the product of *m* number of these factors.

$$p(x) = (x - d_1)(x - d_2)\ldots\ldots.(x - d_m)$$

Since, $\text{degree } p(x) = m$

$f(x)$ is equal to the product of remaining these factors

Comment

Step 5 of 5

Let the Constant term of $p(x)$ is *b,* which is the product of $d_1, d_2, \ldots\ldots d_m$

$$b = (d_1 d_2 \ldots\ldots d_m)$$

$$b = \sqrt[p]{a}\ \ldots\ldots \sqrt[p]{a}$$

$$b = \omega^k (\sqrt[p]{a})^m$$

$$b=\omega^k d^m$$

Hence, the constant term of $p(x)$ is equal to product of m p^{th} roots of *a* and $b=\omega^k d^m$.

Comment