

# A Book of Abstract Algebra | (2nd Edition)

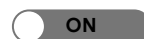


Chapter 33, Problem 1EC



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## Problem

Let  $p$  be a prime number, and  $\omega$  a primitive  $p$ th root of unity in the field  $F$ .

If  $d$  is any root of  $x^p - a \in F[x]$ , show that  $F(\omega, d)$  is a root field of  $x^p - a$ . Suppose  $x^p - a$  is *not* irreducible in  $F[x]$ .

## Step-by-step solution

### Step 1 of 4

Here, objective is to prove that  $F(\omega, d)$  is a root field of  $x^p - a$ .

Consider  $p$  be a prime number.

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### Step 2 of 4

Root field:

The field contains a given field in which every polynomial can be written as a product of linear factors.

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### Step 3 of 4

Consider the polynomial  $x^p - a$ .

The root of above polynomial is a primitive  $p^{\text{th}}$  root of unity

$$x^p - a = 0$$

$$x^p = a$$

$$x = \sqrt[p]{a} \omega$$

$\omega$  is a primitive  $p^{\text{th}}$  root of unity

Consider  $d$  is a root of  $x^p - a \in F(x)$ .

$$\text{Then, } d = \sqrt[p]{a}$$

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#### Step 4 of 4

Let  $F$  is the root field of  $x^p - a$ .

Then,  $\sqrt[p]{a}, \sqrt[p]{a} \omega$  both are in  $F$ .

The quotient  $\omega = \frac{\sqrt[p]{a} \omega}{\sqrt[p]{a}} \in F(x)$

Therefore  $Q(\sqrt[p]{a}, \omega) \subset F$

Since, the field  $F(\omega, \sqrt[p]{a})$  is contains all the roots of the polynomial  $x^p - a$ ,

$$F(\omega, \sqrt[p]{a}) = F(\omega, d)$$

Therefore,

The root field of  $x^p - a$  is  $F(\omega, d)$

Hence, proved

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