A Book of Abstract Algebra | (2nd Edition)

Problem

Let p be a prime number. A finite group G is called a *p-group* if the order of every element x in G is a power p. (The orders of different elements may be different powers of p.) If H is a subgroup of any finite group G, and H is a p-group, we call H a *p-subgroip* of G. Finally, if K is a p-subgroup of G, and K is maximal (in the sense that K is not contained in any larger p-subgroup of G), then K is called a *p-Sylow subgroup* of G.

Prove that every conjugate of a p-Sylow subgroup of G is a p-Sylow subgroup of G.

Let K be a p-Sylow subgroup of G, and $N = N(K)$ the normalizer of K.

Step-by-step solution

Step 1 of 3

Consider that G is a p-group, so order of each element x in G will be the power of p. Let K is a p-Sylow subgroup of G. Objective is to prove that every conjugate of a p-Sylow subgroup of G is a p-Sylow subgroup of G.

Definition of conjugate subgroups:

Let H be a subgroup of G. For any $a \in G$, let $aHa^{-1} = \{axa^{-1} : x \in H\}$; aHa^{-1} is called a conjugate of H.

Comment

Step 2 of 3

Consider the following result that conjugation by a fixed element is an automorphism.

Suppose that K and K' are conjugate subgroups of G and consider an inner automorphism of group G that takes K to K'. Then this inner automorphism will also take any subgroup between K and G to a similar (or isomorphic) subgroup between K' and G.

It implies that if K' is a maximal p-subgroup then K will also be a maximal p-subgroup, and vice versa. Since maximal *p*-subgroup is known as *p*-Sylow subgroup, therefore if K' is a *p*-Sylow subgroup then *K* will also be a *p*-Sylow subgroup, and vice versa.

Comment

Step 3 of 3

Hence, every conjugate of a *p*-Sylow subgroup of *G* is a *p*-Sylow subgroup of *G*.

Comment