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Chapter 29, Problem 1EB

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Problem

Let *F* be a field of characteristic ≠ 2. Let $a \neq b$ be in *F*.

Prove that any field *F* containing $\sqrt{a}+\sqrt{b}$ also contains $\sqrt{a}$ and $\sqrt{b}$. [HINT: Compute $(\sqrt{a}+\sqrt{b})^2$ and show that $\sqrt{ab} \in F$. Then compute $\sqrt{ab}(\sqrt{a}+\sqrt{b})$, which is also in *F*] Conclude that $F(\sqrt{a}+\sqrt{b}) = F(\sqrt{a}, \sqrt{b})$.

Step-by-step solution

Step 1 of 3

Consider a field *F* of characteristic $\neq 2$. Objective is to prove that any field *F* containing $\sqrt{a}+\sqrt{b}$ also contains $\sqrt{a}$ and $\sqrt{b}$, where $a \neq b \in F$. And then draw a conclusion that $F(\sqrt{a}+\sqrt{b}) = F(\sqrt{a}, \sqrt{b})$.

Suppose field *F* contains $\sqrt{a}+\sqrt{b}$. To show the required result, first show that $\sqrt{ab} \in F$. For this consider the following square:

$$(\sqrt{a}+\sqrt{b})^2 = a+b+2\sqrt{ab} \in F.$$

Solve for $\sqrt{ab}$ and get,

$$\sqrt{ab} = \frac{(\sqrt{a}+\sqrt{b})^2-(a+b)}{2}.$$

That is, $\sqrt{ab} \in F$ because $a, b \in F$.

Comment

Step 2 of 3

Since $\sqrt{a}+\sqrt{b}$, $\sqrt{ab}\in F$, therefore their product

$$\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)=a\sqrt{b}+b\sqrt{a}\in F.$$

Since $a\neq b$, then

$$\begin{aligned}\frac{b\left(\sqrt{a}+\sqrt{b}\right)-\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)}{b-a}&=\frac{b\sqrt{a}+b\sqrt{b}-a\sqrt{b}-b\sqrt{a}}{b-a}\\&=\frac{\sqrt{b}(b-a)}{b-a}\\&=\sqrt{b}\in F.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{a\left(\sqrt{a}+\sqrt{b}\right)-\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)}{a-b}&=\frac{a\sqrt{a}+a\sqrt{b}-a\sqrt{b}-b\sqrt{a}}{a-b}\\&=\frac{\sqrt{a}(a-b)}{a-b}\\&=\sqrt{a}\in F.\end{aligned}$$

Comment

Step 3 of 3

Note that, $\sqrt{a},\sqrt{b}\in F\left(\sqrt{a},\sqrt{b}\right)$. Therefore, $\sqrt{a}+\sqrt{b}\in F\left(\sqrt{a},\sqrt{b}\right)$. It follows that

$$F\left(\sqrt{a}+\sqrt{b}\right)\subseteq F\left(\sqrt{a},\sqrt{b}\right).$$

Also, $\sqrt{a},\sqrt{b}\in F\left(\sqrt{a}+\sqrt{b}\right)$. Therefore,

$$F\left(\sqrt{a},\sqrt{b}\right)\subseteq F\left(\sqrt{a}+\sqrt{b}\right).$$

Thus,

$$F\left(\sqrt{a}+\sqrt{b}\right)=F\left(\sqrt{a},\sqrt{b}\right).$$

Comment