# A Book of Abstract Algebra | (2nd Edition)

Chapter 32, Problem 5ED

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## Problem

If $\alpha = \sqrt[4]{2}$ is a real fourth root of 2, then the four fourth roots of 2 are ±*α* and ±*iα*. Explain parts 1–6, briefly but carefully:

{1, *α*, *α*$^2$, *α*$^3$, *i*, *iα*, *iα*$^2$, *iα*$^3$} is a basis for (*α*, *i*) over.

## Step-by-step solution

**Step 1** of 2

The objective is to explain $\{1,\alpha,\alpha^2,\alpha^3,i,i\alpha,i\alpha^2,i\alpha^3\}$ is a basis for $\mathbb{Q}(\alpha,i)$ over $\mathbb{Q}$.

Comment

**Step 2** of 2

Clearly , $\sqrt[4]{2}$ is the root of $x^4-2$.

Also , $x^4-2$ is irreducible polynomial of lowest degree $4$ over $\mathbb{Q}$ by Eisenstein $(p=2)$.

Therefore , $\left[\mathbb{Q}\left(\sqrt[4]{2}\right):\mathbb{Q}\right]=\deg\left(x^4-2\right)=4$.

Because $\mathbb{Q}\left(\sqrt[4]{2}\right)$ is a subfield of the reals and so , $i\notin\mathbb{Q}\left(\sqrt[4]{2}\right)$.

Hence , $x^2+1$ is irreducible over $\mathbb{Q}\left(\sqrt[4]{2}\right)$.

So , $\left[\mathbb{Q}\left(\sqrt[4]{2},i\right):\mathbb{Q}\left(\sqrt[4]{2}\right)\right]$ is at least $2$.

But $i$ is a root of $x^2+1\in\mathbb{Q}\left(\sqrt[4]{2}\right)[X]$ , so the degree of $\mathbb{Q}\left(\sqrt[4]{2},i\right)$ over $\mathbb{Q}\left(\sqrt[4]{2}\right)$ is at most $2$ ,and therefore ,is exactly $2$.

Hence , $\left[\mathbb{Q}\left(\sqrt[4]{2},i\right):\mathbb{Q}\left(\sqrt[4]{2}\right)\right]=2$.

Thus, $\left[\mathbb{Q}\left(\sqrt[4]{2},i\right):\mathbb{Q}\right]=\left[\mathbb{Q}\left(\sqrt[4]{2},i\right):\mathbb{Q}\left(\sqrt[4]{2}\right)\right]\left[\mathbb{Q}\left(\sqrt[4]{2}\right):\mathbb{Q}\right]$

$=2\cdot 4$

$=8.$

So, $\left[\mathbb{Q}\left(\sqrt[4]{2},i\right):\mathbb{Q}\left(\sqrt[4]{2}\right)\right]=2$ with basis $\{1,i\}$, and $\left[\mathbb{Q}\left(\sqrt[4]{2}\right):\mathbb{Q}\right]=4$ with basis $\left\{1,\alpha,\alpha^2,\alpha^3\right\}$.

Therefore, a basis for degree $6$ field over $\mathbb{Q}$ is obtained by multiplying the bases together: $\left\{1,\alpha,\alpha^2,\alpha^3,i,i\alpha,i\alpha^2,i\alpha^3\right\}$.

Comment