A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 1EI

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Problem

Let *H* and *K* be normal subgroups of a group *G*, with $H \subseteq k$ Define ϕ: $G/H \to G/K$ by $\phi(Ha) = Ka$. Prove part:

ϕ is a well-defined function. [That is, if $Ha = Hb$, then $\phi(Ha) = \phi(Hb)$.]

Step-by-step solution

Step 1 of 3

Suppose that *G* is any group and let *H, K* are normal subgroups of *G* with $H \subseteq K$.

Consider a mapping $\phi : G/H \to G/K$ defined by

$\phi(Ha) = Ka$, for all $a \in G$.

Objective is to prove that function ϕ is well defined. That is, there is a need to show that if two cosets Ha, Hb are equal, then $\phi(Ha) = \phi(Hb)$ for some $a, b \in G$.

Comment

Step 2 of 3

Assume that $Ha, Hb \in G/H$, for some $a, b \in G$, such that

$Ha = Hb$.

By the coset property, the $Hx = Hy$ if and only if $xy^{-1} \in H$, it implies that

$ab^{-1} \in H$.

Since $H \subseteq K$, therefore

$ab^{-1} \in K$.

Again by applying the same property as in the previous step, the condition $ab^{-1} \in K$ implies that

$Ka = Kb$.

At last by the defined function ϕ, $Ka = Kb$ implies that

$\phi(Ha) = \phi(Hb)$.

Comment

Step 3 of 3

Since $Ha = Hb$ implies $\phi(Ha) = \phi(Hb)$, therefore ϕ is well defined mapping.

Comment