A Book of Abstract Algebra | (2nd Edition)

Chapter 28, Problem 2EG

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Problem

Let T and U be subspaces of V. The *sum* of T and U, denoted by $T + U$, is the set of all vectors **a** + **b**, where **a** $\in T$ and **b** $\in U$.

Prove: $V = T \oplus U$ iff every vector **c** $\in V$ can be written, in a unique manner, as a sum **c** = **a** + **b** where **a** $\in = U$ and **b** $\in U$.

Step-by-step solution

Step 1 of 2

Sum of 2 subspaces is equal to vector space if number of elements are same in both of them. In this only distinct elements should be counted.

Direct sum of 2 subspaces is set in which every element can be expressed as sum of those 2 subspaces. Direct sum of subspaces may be extended to any number of subspaces. One condition on these subspaces is that they must be exclusive. This is

$$S_1 \cap S_2 \cap S_3 \cap S_4 \cap \ldots \cap S_n = \{0\}$$

Comment

Step 2 of 2

If every vector **c** in V can be expressed as follows,

$$\mathbf{c} = \mathbf{a} + \mathbf{b} \mid \mathbf{a} \in T, \mathbf{b} \in U$$

Thus every element in V can be expressed as $S + T$.

Conversely, if every element in V can be expressed as $T + U$ and $T \cap U = \{0\}$, then every element **c** in V can be expressed as

$\mathbf{c} = \mathbf{a} + \mathbf{b} \mid \mathbf{a} \in T, \mathbf{b} \in U$

$\boxed{\text{Hence } V = T \oplus U \text{ iff every } \mathbf{c} \in V \text{ can be written as } \mathbf{a} + \mathbf{b}}$

Comment