A Book of Abstract Algebra | (2nd Edition)

Chapter 28, Problem 2EC

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Problem

If **a** = (1, 2, 3,4) and **b** = (4, 3,2,1), explain why {**a**, **b**} may be extended to a basis of $\mathbb{R}^4$. Then find a basis of $\mathbb{R}^4$ which includes **a** and **b**.

Step-by-step solution

Step 1 of 2

Any set of basis is a set of vectors which are linearly independent and their number equals dimension of vector space. And any set is linearly independent if there exists no combination of these vectors which can give 0 vectors.

If $u_1, u_2, \ldots, u_n$ are *n* vectors of a vector space and these are linearly independent. Then for.

$$a_1u_1 + a_2u_2 + \ldots + a_nu_n = 0$$

All a_i have to be zero.

Any given basis is linearly independent if matrix formed with vectors as row of matrix same rank as number of rows of matrix.

Comment

Step 2 of 2

Matrix formed by given set of vectors is,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

Row reducing this matrix,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2 \to R_2 - 4R_1}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -5 & -10 & -15 \end{pmatrix}$$

$$\xrightarrow{R_2 \to R_2 / -5}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

$$\xrightarrow{R_1 \to R_1 - 2R_2}$$

$$\begin{pmatrix} \boxed{1} & 0 & -1 & -2 \\ 0 & \boxed{1} & 2 & 3 \end{pmatrix}$$

It can be easily seen that this matrix has 2 pivots and thus have rank of 2, equal to number of vectors. But numbers of vectors is not equal to dimension of $\mathbb{R}^4$ which is 4.

$\boxed{\text{It can be extended as given 2 vectors are linearly independent}}$. To make given set of vectors a basis of $\mathbb{R}^4$, vectors which will give pivot position in next 2 rows should be added.

$$\boxed{\text{Hence given set of extended vectors which are basis of } \mathbb{R}^4 \text{ is } \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$$

Comment