A Book of Abstract Algebra | (2nd Edition)

Chapter 30, Problem 3EE

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Problem

We will show that $2\pi/5$ is a constructible angle, and it will follow that the regular pentagon is constructible.

Prove that

$$4\cos^2\frac{2\pi}{5} + 2\cos\frac{2\pi}{5} - 1 = 0$$

(HINT: Use parts 1 and 2.) Conclude that $\cos(2\pi/5)$ is a root of the quadratic $4x^2 - 2x - 1$.

Step-by-step solution

Step 1 of 5

Here, objective is to prove that $4\cos^2\frac{2\pi}{5} + 2\cos\frac{2\pi}{5} - 1 = 0$ and $\cos\frac{2\pi}{5}$ is a root of

$$4x^2 - 2x - 1 = 0$$

Comment

Step 2 of 5

De Moivre's theorem:

$\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$ is a complex fifth root of unity.

Since $x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)$

ω is a root of $P(x) = (x^4 + x^3 + x^2 + x + 1)$

Comment

Step 3 of 5

Consider $\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$$\left(\omega + \frac{1}{\omega}\right) = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} + \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}$$

$$\left(\omega + \frac{1}{\omega}\right) = 2\cos\frac{2\pi}{5}$$

Comment

Step 4 of 5

Consider ω is a root of $P(x) = (x^4 + x^3 + x^2 + x + 1)$

Then, $P(\omega) = 0$

$$(\omega^4 + \omega^3 + \omega^2 + \omega + 1) = 0$$

$$\omega^2(\omega^2 + \omega + 1 + \omega^{-1} + \omega^{-2}) = 0$$

$\omega^2 = 0$ or

$$\omega^2 + \omega + 1 + \omega^{-1} + \omega^{-2} = 0$$

$$\omega^2 + \omega^{-2} + 2 + \omega + \omega^{-1} - 1 = 0$$

$$\left(\omega + \frac{1}{\omega}\right)^2 + \left(\omega + \frac{1}{\omega}\right) - 1 = 0$$

$$\left(2\cos\frac{2\pi}{5}\right)^2+\left(2\cos\frac{2\pi}{5}\right)-1=0$$

$$4\cos^2\frac{2\pi}{5}+2\cos\frac{2\pi}{5}-1=0$$

Comment

Step 5 of 5

put $x=\cos\frac{2\pi}{5}$ in above equation, then

$$4x^2-2x-1=0$$

Hence, $4\cos^2\frac{2\pi}{5}+2\cos\frac{2\pi}{5}-1=0$ and $\cos\frac{2\pi}{5}$ is a root of $4x^2-2x-1=0$

Comment