A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 2EP

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Problem

Let *G* be an abelian group of order $p^k m$, where p^k and *m* are relatively prime (that is, p^k and *m* have no common factors except ±1). (REMARK: If two integers *j* and *k* are relatively prime, then there are integers *s* and *t* such that $sj + tk = 1$. This is proved on page 220.)

Let G_pk be the subgroup of *G* consisting of all elements whose order divides p^k. Let *Gm* be the subgroup of *G* consisting of all elements whose order divides ra. Prove:

For every $x \in G$, there are $y \in G_pk$ and $z \in G_m$ such that $x = yz$.

Step-by-step solution

Step 1 of 3

Assume that *G* is an abelian group of order $p^k m$, where p^k and *m* are relatively prime. Suppose that G_{p^k} be the subgroup of *G* consisting of all elements whose order divides p^k. Let G_m be the subgroup of *G* consisting of all elements whose order divides *m*.

Objective is to prove that for any $x \in G$, there are $y \in G_{p^k}$ and $z \in G_m$ such that $x = yz$.

Comment

Step 2 of 3

Since p^k and *m* are relatively prime, so by the definition there exist integers *s* ant *t* such that

$$sp^k + tm = 1.$$

Let $y = x^{sp^k}$, $z = x^{tm}$. Substitute these values in right side of $x = yz$ and get,

$$\begin{aligned} yz &= x^{sp^k} \cdot x^{tm} \\ &= x^{sp^k + tm} \\ &= x^1 \\ &= x. \end{aligned}$$

The third step is obtained by using $sp^k + tm = 1$.

Comment

Step 3 of 3

Hence, $x = yz$ for some $x \in G$, where $y \in G_{p^k}$ and $z \in G_m$.

Comment