A Book of Abstract Algebra | (2nd Edition)

Chapter 24, Problem 1EG

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Problem

Let *A* and *B* be rings and let $h : A \to B$ be a homomorphism with kernel *K*. Define $\bar{h} : A[x] \to B[x]$ by

$$\bar{h}(a_0 + a_1x + \cdots + a_nx^n) = h(a_0) + h(a_1)x + \cdots + h(a_n)x^n$$

(We say that $\bar{h}$ is *induced by h*.)

Prove that $\bar{h}$ is a homomorphism from *A*[*x*] to *B*[*x*].

Step-by-step solution

Step 1 of 1

Let $a(x), b(x) \in A[x]$ and

$$a(x) = a_0 + a_1x + \ldots\ldots + a_nx^n$$
$$b(x) = b_0 + b_1x + \ldots\ldots + b_mx^m$$

$$\begin{aligned}\bar{h}(a(x)+b(x)) &= h(a_0)+h(a_1)x+\ldots+h(a_n)x^n+h(b_0)+h(b_1)x+\ldots+h(b_m)x^m \\ &= \bar{h}(a_0+a_1x+\ldots+a_nx^n)+\bar{h}(b_0+b_1x+\ldots+b_mx^m) \\ &= \bar{h}(a(x)+b(x))\end{aligned}$$

$$\begin{aligned}\bar{h}(a(x)b(x)) &= \bar{h}\left(\sum_{j=0}^{m}\sum_{i=0}^{n} a_ib_jx^{i+j}\right) \\ &= \sum_{j=0}^{m}\sum_{i=0}^{n} h(a_ib_j)x^{i+j} = \sum_{j=0}^{m}\sum_{i=0}^{n} h(a_i)h(b_j)x^{i+j} \\ &= \sum_{j=0}^{m}\sum_{i=0}^{n} h(a_ix^i)h(b_jx^j)\end{aligned}$$

$$\Rightarrow \quad \bar{h}(a(x)b(x)) = \sum_{j=0}^{m} h(a_i)x^i \sum_{i=0}^{n} h(b_j)x^j$$

$$= \bar{h}(a(x))\bar{h}(b(x))$$

$\Rightarrow$ $\bar{h}$ is a homomorphism from $A[x]$ to $B[x]$

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