A Book of Abstract Algebra | (2nd Edition)

Chapter 28, Problem 1EA

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Problem

Prove that $\mathbb{R}^n$, as defined on page 283, satisfies all the conditions for being a vector space over $\mathbb{R}$.

Step-by-step solution

Step 1 of 3

There are 10 conditions which any vector space must satisfy. These are

1. For $u \in V, v \in V \Rightarrow u+v \in V$
2. For $u \in V, v \in V \Rightarrow u+v = v+u$
3. For $u \in V, v \in V, w \in V \Rightarrow (u+v)+w = u+(v+w)$
4. There exists $0 \in V$, such that $0+v = v$ for all $v \in V$
5. For all $u \in V$, there exists $x \in V$ such that $u+x=0$
6. For $c \in R, v \in V \Rightarrow cv \in V$
7. For $c \in R, u \in V, v \in V \Rightarrow c(u+v) = cu+cv$
8. For $c, d \in R, u \in V, v \in V \Rightarrow (c+d)u = cu+du$
9. For $c \in R, d \in R, v \in V \Rightarrow c(dv) = (cd)v$
10. There exists $1 \in R, v \in V \Rightarrow 1 \cdot v = v$

Comment

Step 2 of 3

$\mathbb{R}^n$ is defined as an array with *n* components. This is represented by $(a_1, a_2, a_3, ..., a_n)$ where all a_i are real numbers. Addition of 2 arrays are done component wise. Multiplication of an array with a constant implies that all components are multiplies with that constant.

Let

$v = (v_1, v_2, v_3, ..., v_n)$

$u = (u_1, u_2, u_3, ..., u_n)$

$\Rightarrow -u = (-u_1, -u_2, -u_3, ..., -u_n)$

Then check aforementioned 8 properties or condition for this space.

1. $u + v = (u_1, u_2, u_3, ..., u_n) + (v_1, v_2, v_3, ..., v_n) = (u_1 + v_1, u_2 + v_2, u_3 + v_3, ..., u_n + v_n) \in V$
2. $u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3, ..., u_n + v_n) = (v_1 + u_1, v_2 + u_2, v_3 + u_3, ..., v_n + u_n) = v + u$
3. $$\begin{aligned} &(u+v)+w = (u_1 + v_1, u_2 + v_2, u_3 + v_3, ..., u_n + v_n) + (w_1, w_2, w_3, ..., w_n) \\ &\Rightarrow (u+v)+w = (u_1 + v_1 + w_1, u_2 + v_2 + w_2, u_3 + v_3 + w_3, ..., u_n + v_n + w_n) \\ &u+(v+w) = (u_1, u_2, u_3, ..., u_n) + (v_1 + w_1, v_2 + w_2, v_3 + w_3, ..., v_n + w_n) \\ &\Rightarrow u+(v+w) = (u_1 + v_1 + w_1, u_2 + v_2 + w_2, u_3 + v_3 + w_3, ..., u_n + v_n + w_n) \end{aligned}$$
4. $u + 0 = (u_1 + 0, u_2 + 0, u_3 + 0, ..., u_n +) = (u_1, u_2, u_3, ..., u_n) = u$
5. $u + (-u) = (u_1 + (-u_1), u_2 + (-u_2), u_3 + (-u_3), ..., u_n + (-u_n)) = (0, 0, 0, ..., 0) = 0$
6. $cv = c(v_1, v_2, v_3, ..., v_n) = (cv_1, cv_2, cv_3, ..., cv_n) \in V$
7. $$\begin{aligned} &c(u+v) = c(u_1 + v_1, u_2 + v_2, u_3 + v_3, ..., u_n + v_n) = \\ &\Rightarrow c(u+v) = (cv_1 + cu_1, cv_2 + cu_2, cv_3 + cu_3, ..., cv_n + cu_n) \\ &cu + cv = c(u_1, u_2, u_3, ..., u_n) + c(v_1, v_2, v_3, ..., v_n) \\ &\Rightarrow cu + cv = (cv_1 + cu_1, cv_2 + cu_2, cv_3 + cu_3, ..., cv_n + cu_n) \\ &\text{Or, } c(u+v) = cu + cv \end{aligned}$$
8. $$\begin{aligned} &(c+d)u = (c+d)(u_1, u_2, u_3, ..., u_n) = \\ &\Rightarrow (c+d)(u_1, u_2, u_3, ..., u_n) = ((c+d)u_1, (c+d)u_2, (c+d)u_3, ..., (c+d)u_n) \\ &cu + du = c(u_1, u_2, u_3, ..., u_n) + d(u_1, u_2, u_3, ..., u_n) \\ &\Rightarrow cu + du = (cu_1, cu_2, cu_3, ..., cu_n) + (du_1, du_2, du_3, ..., du_n) \\ &cu + du = ((c+d)u_1, (c+d)u_2, (c+d)u_3, ..., (c+d)u_n) \\ &\text{Or, } (c+d)u = cu + du \end{aligned}$$
9. $$\begin{aligned} &c(dv) = c(d(v_1, v_2, v_3, ..., v_n)) = c(dv_1, dv_2, dv_3, ..., dv_n) = (cdv_1, cdv_2, cdv_3, ..., cdv_n) \\ &(cd)v = cd(v_1, v_2, v_3, ..., v_n) = (cdv_1, cdv_2, cdv_3, ..., cdv_n) \\ &\Rightarrow c(dv) = (cd)v \end{aligned}$$
10. $1v = (v_1, v_2, v_3, ..., v_n) = (1 \cdot v_1, 1 \cdot v_2, 1 \cdot v_3, ..., 1 \cdot v_n) = v$

$\boxed{\text{Hence } \mathbb{R}^n \text{ satisfies all conditions for vector space}}$

Comment

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Comment