A Book of Abstract Algebra | (2nd Edition)

Problem

Let K be a finite extension of F, where K is a root field over F, with $G = Gal(K: F)$ a solvable group. As remarked in the text, we will assume that F contains the required roots of unity. By Exercise D, let $H_0, \ldots, H_n$ be a solvable series for G in which every quotient H_{i+1}/H_i is cyclic of prime order. For any $i = 1, \ldots, n$, let F_i and F_{i+1} be the fixfields of H_i and H_{i+1}.

Prove: F_i is a normal extension of F_{i+1}, and $[F_i: F_{i+1}]$ is a prime p.

Step-by-step solution

Step 1 of 4

Here, objective is to prove that F_i is a normal extension of F_{i+1} and $[F_i : F_{i+1}]$ is a prime *p.*

Comment

Step 2 of 4

A G is a group of automorphism of K. The set of elements fixed by every element of G called the fixed field.

Comment

Step 3 of 4

$G = Gal(K : F)$ is a solvable group.

F is the fixed field of G.

Where, K is a the finite extension of F.

$H_0, H_1, \ldots\ldots H_n$ is the solvable series for G.

Every quotient $H_{i+1}/H_i; i = 1,2,...n$ is a cyclic of prime order.

That is $|H_{i+1}/H_i| = p$

Comment

Step 4 of 4

Consider F_i and F_{i+1} are the fixed fields of H_i and H_{i+1}

$F_0, F_1, ...F_n$ is the solvable series for *H.*

$$F_i = L^{H_i}$$
$$\begin{aligned}[F_i : F_{i+1}] &= |H_{i+1}/H_i| \\ &= p\end{aligned}$$

So, $[F_i : F_{i+1}]$ is a prime *p.*

$F_0, F_1, ...F_n$ is the solvable series for *G*

Then,

$$G = Gal[F_i : F_{i+1}]$$

F_i is a simple normal extension of F_{i+1}.

Therefore, F_i is a normal extension of F_{i+1} and $[F_i : F_{i+1}]$ is a prime *p.*

Hence, proved

Comment