A Book of Abstract Algebra | (2nd Edition)

Chapter 31, Problem 5EB

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Problem

Find the root field of $x^3 + x^2 + x + 2$ over $\mathbb{Z}_3$. Find a basis for this root field over $\mathbb{Z}_3$.

Step-by-step solution

Step 1 of 3

The objective is to find the root field of x^3+x^2+x+2 over $\mathbb{Z}_3$, and a basis for this root field over $\mathbb{Z}_3$.

Comment

Step 2 of 3

Let $a(x)=x^3+x^2+x+2\in\mathbb{Z}_3$.

$$\begin{aligned} a(0) &= 0^3+0^2+0+2 \\ &= 2 \\ &\neq 0 \end{aligned}$$

$$\begin{aligned} a(1) &= 1^3+1^2+1+2 \\ &= 1+1+1+2 \\ &= 2 \\ &\neq 0 \end{aligned}$$

$$\begin{aligned} a(2) &= 2^3+2^2+2+2 \\ &= 8+4+2+2 \\ &= 1 \\ &\neq 0 \end{aligned}$$

Therefore, $a(x)$ has no roots in $\mathbb{Z}_3$.

This implies that x^3+x^2+x+2 is irreducible over $\mathbb{Z}_3$.

Therefore, the root field of x^3+x^2+x+2 over $\mathbb{Z}_3$ is $\mathbb{Z}_3[x]/(x^3+x^2+x+2)$.

Comment

Step 3 of 3

The root field has $3^3=27$ times.

If k is any root of $a(x)=x^3+x^2+x+2$, then the basis for this root field is

$$\left\{\begin{array}{l}0,1,2,k,k+1,k+2,k^2,k^2+1,k^2+2,k^2+k,k^2+k+1,k^2+k+2,k^3,\\ k^3+1,k^3+2,k^3+k,k^3+k+1,k^3+k+2,k^3+k^2,k^3+k^2+1,k^3+k^2+2,\\ k^3+k^2+k,k^3+k^2+k+1,k^3+k^2+k+2\end{array}\right\}.$$

Comment