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Chapter 24, Problem 1EJ

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Problem

In the division algorithm, prove that $q(x)$ and $r(x)$ are uniquely determined. [HINT: Suppose $a(x) = b(x)q_1(x) + r_1(x) = b(x)q_2(x) + r_2(x)$, and subtract these two expressions, which are both equal to $a(x)$.]

Step-by-step solution

Step 1 of 1

Let $f(x)$ and $g(x)$ be polynomials in $A[x]$ with $g(x) \neq 0$, suppose there are $q_1(x), q_2(x)$ and $r_1(x), r_2(x)$ in $A[x]$ satisfying division algorithm

$f(x) = g(x)q_1(x) + r_1(x)$

Where $r_1(x) = 0$ or $\deg(r_1(x)) < \deg(g(x))$ and

$f(x) = g(x)q_2(x) + r_2(x)$

Where $r_2(x) = 0$ or $\deg(r_2(x)) < \deg(g(x))$

From equation (1.1) and (1.2) we can write,

$g(x)q_1(x) + r_1(x) = g(x)q_2(x) + r_2(x)$ implies

$g(x)(q_1(x) - q_2(x)) = (r_2(x) - r_1(x))$

$\Rightarrow$ $r_2(x) - r_1(x)$ is polynomial multiple of $g(x)$

From equation (1.1) and (1.2), if $r_2(x) \neq r_1(x)$ then

$\deg(r_2(x) - r_1(x)) = \max(\deg(r_2(x)), \deg(r_1(x))) < \deg(g(x))$

But as $r_2(x) - r_1(x)$ is polynomial multiple of $g(x)$ $\Rightarrow$ $\deg(r_2(x) - r_1(x)) \geq \deg(g(x))$ which is contradictory so $r_2(x) = r_1(x)$ is must $\Rightarrow$ $r_2(x) - r_1(x) = 0$, putting in equation (1.3)

We get $g(x)(q_1(x) - q_2(x)) = 0$ $\Rightarrow$ $q_2(x) = q_1(x)$

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