A Book of Abstract Algebra | (2nd Edition)

Problem

Let F be a finite field, and F^* the multiplicative group of nonzero elements of F. Obviously $H = \{x^2 : x \in F^*\}$ is a subgroup of F^*; since every square x^2 in F^* is the square of only two different elements, namely $\pm x$, exactly half the elements of F^* are in H. Thus, H has exactly two cosets: H itself, containing all the squares, and aH (where $a \notin H$), containing all the nonsquares. If a and b are nonsquares, then by Chapter 15, Theorem 5(i),

$$ab^{-1} = \frac{a}{b} \in H$$

Thus: if a and b are nonsquares, a/b is a square. Use these remarks in the following:

If the minimum polynomial of a over F has degree 2, we call $F(a)$ a quadratic extension of F.

If a and b are nonsquares in $\mathbb{R}$, a/b is a square (why?). Use the same argument as in part 4 to prove that any two simple extensions of $\mathbb{R}$ are isomorphic (hence isomorphic to $\mathbb{C}$).

Step-by-step solution

Step 1 of 2

Objective is to prove that if a and b non-squares real numbers then a/b is a square.

Since a and b are some real numbers, then corresponding $\sqrt{a}, \sqrt{b}$ will also be the members of R. Then

$$\begin{aligned}\frac{a}{b} &= \frac{\left(\sqrt{a}\right)^2}{\left(\sqrt{b}\right)^2} \\ &= \left(\sqrt{\frac{a}{b}}\right)^2.\end{aligned}$$

Let $c = \sqrt{\frac{a}{b}}$. Then $\frac{a}{b} = c^2$. This shows that a/b is a square in the field of real numbers.

Comment

Step 2 of 2

Note that, any simple extension of R will be quadratic. Also, polynomials of degree ≤ 2 are the only irreducibles in R.

Also from the result: if polynomials $p(x)$ and $q(x)$ are arbitrary irreducibles in $F[x]$ of degree 2, then any two quadratic extensions of a field are isomorphic.

Thus, any two simple extensions of R are isomorphic. Since quadratic extension of R is the set of all complex numbers, therefore any two simple extensions of R are isomorphic to $\mathbf{C}$.

Comment