A Book of Abstract Algebra | (2nd Edition)

Problem

Let *F* be any field.

Prove part:

If *p*(*x*) is irreducible and has degree 2, prove that $F[x]/\langle p(x)\rangle$ contains *both* roots of *p*(*x*).

Step-by-step solution

Step 1 of 3

Suppose that *F* is any arbitrary field. Let $p(x)$ is some irreducible polynomial of degree 2.

Objective is to prove that $\frac{F[x]}{\langle p(x)\rangle}$ contains both the roots of $p(x)$.

Consider the following result:

Let *F* is any arbitrary field. If $p(x)\in F[x]$ is an irreducible polynomial and *c* is some root of $p(x)$, then

$$\frac{F[x]}{\langle p(x)\rangle}\cong F(c).$$

Comment

Step 2 of 3

Suppose that α is the root of $p(x)$ in some extension field of *F*. Then by the above result:

$$\frac{F[x]}{\langle p(x)\rangle}\cong F(\alpha).$$

Let $p(x)=ax^2+bx+c$ (quadratic polynomial) whose one root is α and other root is β. Then the sum of roots will be:

$$\alpha+\beta=-\frac{b}{a}.$$

And $\beta=-\frac{b}{a}-\alpha$. Since $p(x)\in F[x]$ and $F(\alpha)\supset F$ contains a root α of $p(x)$. Therefore, $F(\alpha)$ must contain β (field property).

Comment

Step 3 of 3

Hence, $F[x]/\langle p(x)\rangle$ contains both the roots of $p(x)$.

Comment