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Chapter 27, Problem 6EE   Bookmark   Show all steps: ON

Problem

Use parts 4 and 5 to prove the following:

(*a*) $\mathbb{Z}_{11}[x]/\langle x^2+1\rangle \cong \mathbb{Z}_{11}[x]/\langle x^2+x+4\rangle$.

(*b*) If *a* is a root of x^2-2 and *b* is a root of x^2-4x+2, then $\mathbb{Q}(a) \cong \mathbb{Q}(b)$.

(*c*) If a is a root of x^2-2 and *b* is a root of $x^2-\frac{1}{2}$, then $\mathbb{Q}(a) \cong \mathbb{Q}(b)$.

Step-by-step solution

Step 1 of 4

Consider that *F* is any arbitrary field and *K* is its extension field. Let $c \in F$, and $a \in K$. Assume that $p(x) \in F[x]$ be some irreducible polynomial, and *a* be the root of $p(x+c)$. Then

$$F[x]/\langle p(x+c)\rangle \cong F(a),$$
$$F[x]/\langle p(x)\rangle \cong F(a+c).$$

If *a* be the root of $p(cx)$ then

$$F[x]/\langle p(cx)\rangle \cong F(a),$$
$$F[x]/\langle p(x)\rangle \cong F(ca).$$
$$F[x]/\langle p(cx)\rangle \cong F[x]/\langle p(x)\rangle.$$

Comment

Step 2 of 4

(a)

Objective is to prove that $Z_{11}[x]/\langle x^2+1\rangle \cong Z_{11}[x]/\langle x^2+x+4\rangle$.

Note that polynomial x^2+1 and x^2+x+4 both are irreducibles over $Z_{11}[x]$ because no element of Z_{11} satisfies these polynomials. Thus, there extension fields will be isomorphic. Hence, by the above result $Z_{11}[x]/\langle x^2+1\rangle \cong Z_{11}[x]/\langle x^2+x+4\rangle$.

Comment

Step 3 of 4

(b)

Objective is to prove that if *a* is a root of x^2-2 and *b* is a root of x^2-4x+2, then $Q(a) \cong Q(b)$.

By the above result, one have

$$\frac{Q[x]}{\langle x^2-2\rangle} \cong Q(a),$$
$$\frac{Q[x]}{\langle x^2-4x+2\rangle} \cong Q(b).$$

Let $p(x)=x^2-2$. Then

$$\begin{aligned} p(x-2) &= (x-2)^2-2 \\ &= x^2-4x+2 \end{aligned}$$

Thus, $\frac{Q[x]}{\langle p(x+c)\rangle} \cong Q(b)$, for some $c=-2$. Hence, from the first result it follows that $Q(a) \cong Q(b)$.

Comment

Step 4 of 4

(c)

Objective is to prove that if *a* is a root of x^2-2 and *b* is a root of $x^2-\frac{1}{2}$, then $Q(a) \cong Q(b)$

Let $p(x)=x^2-2$. Then *b* will be the root of $p(2x)$ because

$$\begin{aligned} p(2x) &= 4x^2-2 \\ &= 4\left(x^2-\frac{1}{2}\right) \end{aligned}$$

Hence, by the second result it follows that $Q(a) \cong Q(b)$.

Comment