A Book of Abstract Algebra (2nd Edition)

Chapter 28, Problem 1EG

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Problem

Let T and U be subspaces of V. The *sum* of T and U, denoted by $T + U$, is the set of all vectors $\mathbf{a} + \mathbf{b}$, where $\mathbf{a} \in T$ and $\mathbf{b} \in U$.

Prove that $T + U$ and $T \cap U$ are subspaces of V.

V is said to be the *direct sum* of T and U if $V = T + U$ and $T \cap U = \{\mathbf{0}\}$. In that case, we write $V = T \oplus U$.

Step-by-step solution

Step 1 of 5

It is already shown that V represents a vector space and T and U represents sub space.

Comment

Step 2 of 5

Given subsets are set of all vectors which can be expressed as

Comment

Step 3 of 5

$$\{T + U = \mathbf{c} = \mathbf{a} + \mathbf{b} \mid \mathbf{a} \in T, \mathbf{b} \in U\}$$
$$\{T \cap U = \mathbf{d} \mid \mathbf{d} \in T \text{ and } \mathbf{d} \in U\}$$

Comment

Step 4 of 5

A 2 step check is needed to determine if given subset is a subspace. First determine if any linear combination of functions in subset is closed under given operation. Second, determine if 0 satisfies given conditions.

For *T+U,*

STEP 1: For any 2 vectors $\mathbf{c}_1$ and $\mathbf{c}_2$,

$$\mathbf{c}_1 = \mathbf{a}_1 + \mathbf{b}_1 \qquad (1)$$
$$\mathbf{c}_2 = \mathbf{a}_2 + \mathbf{b}_2 \qquad (2)$$

Combining above 2 equations, $s(1)+t(2)$ gives

$$\begin{aligned}
&s\mathbf{c}_1 + t\mathbf{c}_2 = s(\mathbf{a}_1 + \mathbf{b}_1) + t(\mathbf{a}_2 + \mathbf{b}_2) \\
&\Rightarrow s\mathbf{c}_1 + t\mathbf{c}_2 = s\mathbf{a}_1 + s\mathbf{b}_1 + t\mathbf{a}_2 + t\mathbf{b}_2 \\
&\Rightarrow s\mathbf{c}_1 + t\mathbf{c}_2 = s\mathbf{a}_1 + t\mathbf{a}_2 + s\mathbf{b}_1 + t\mathbf{b}_2 \\
&\Rightarrow s\mathbf{c}_1 + t\mathbf{c}_2 = \mathbf{a}' + \mathbf{b}'
\end{aligned}$$

Thus linear combination of 2 element in subset lies in subset.

STEP 2: Check if **0** element satisfies given condition,

$$\mathbf{0} = \mathbf{0} + \mathbf{0}$$

$$\boxed{\text{Hence given set } (S+T) \text{ represents a subspace}}$$

Comment

Step 5 of 5

For, $T \cap U$

STEP 1: For any 2 vectors $\mathbf{c}_1$ and $\mathbf{c}_2$,

$$\mathbf{d}_1 \,|\, \mathbf{d}_1 \in S,\ \mathbf{d}_1 \in T \qquad (1)$$
$$\mathbf{d}_2 \,|\, \mathbf{d}_2 \in S,\ \mathbf{d}_1 \in T \qquad (2)$$

Combining above 2 equations, $s(1)+t(2)$ gives

$$\begin{aligned}
&s\mathbf{d}_1 + t\mathbf{d}_2 \in S,\ s\mathbf{d}_1 + t\mathbf{d}_2 \in T \\
&\Rightarrow s\mathbf{d}_1 + t\mathbf{d}_2 \in S \cap T
\end{aligned}$$

As *S* and *T* are vector spaces themselves, linear combinations of vectors in *S* and *T* also lies in *S* and *T*.

Thus linear combination of 2 element in subset lies in subset.

STEP 2: Check if **0** element satisfies given condition,

$$\mathbf{0} \in S, \mathbf{0} \in T \Rightarrow \mathbf{0} + \mathbf{0} = \mathbf{0} \in S \cap T$$

$$\boxed{\text{Hence given set } (S \cap T) \text{ represents a subspace}}$$

Comment