A Book of Abstract Algebra | (2nd Edition)

Chapter 33, Problem 6EB | Bookmark | Show all steps: ON

Problem

Let G be a group. The symbol $H \lhd G$ is commonly used as an abbreviation of "H is a *normal* subgroup of G." A *normal series* of G is a finite sequence $H_0, H_1, \ldots, H_n$ of subgroups of G such that

$$\{e\} = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G$$

Such a series is called a *solvable series* if each quotient group H_{i+1}/Hi is abelian. G is called a *solvable group* if it has a solvable series.

In S_4, let A_4 be the group of all the even permutations, and let

$$B = \{\varepsilon, (12)(34), (13)(24), (14)(23)\}$$

Show that $\{\varepsilon\} \subseteq B \subseteq A_4 \subseteq S_4$ is a solvable series for S_4. Conclude that S_4 and all its subgroups are solvable.

The next three sets of exercises are devoted to proving the converse of Theorem 3: If the Galois group of $a(x)$ is solvable, then $a(x)$ is solvable by radicals.

Step-by-step solution

Step 1 of 4

Here, objective is to prove that $\{\varepsilon\} \subseteq B \subseteq A_4 \subseteq S_4$ is a solvable series for S_4.

Comment

Step 2 of 4

A group G is solvable, if there exist a finite chain of successive subgroups.

Abelian groups are solvable.

Comment

Step 3 of 4

Alternating group A_4 is the group of even permutations of four elements.

Symmetric group S_4 is the group of all permutations of four elements.

Consider in S_4 , A_4 be the group of all even permutations.

Then,

$A_4 \subseteq S_4$

Comment

Step 4 of 4

Consider $B = \{\varepsilon, (12)(34), (13)(24), (14)(23)\}$, then B has order four.

A_4 has the normal subgroup of $B = \{\varepsilon, (12)(34), (13)(24), (14)(23)\}$ of order four.

So,

$B \subseteq A_4$

Then,

$\{\varepsilon\} \subseteq B \subseteq A_4 \subseteq S_4$

Is a subnormal sequence with Abelian quotient.

So,

$\{\varepsilon\} \subseteq B \subseteq A_4 \subseteq S_4$ is a solvable series for S_4.

Therefore, all of its groups are solvable.

Hence, proved

Comment