A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 4ED

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Problem

Let G be a group. By an *automorphism* of G we mean an isomorphism $f: G \to G$.

Let $I(G)$ designate the set of all the inner automorphisms of G. That is, $I(G) = \{\phi_a: a \in G\}$. Use part 3 to prove that $I(G)$ is a subgroup of Aut(G). Explain why $I(G)$ is a group.

Step-by-step solution

Step 1 of 3

Suppose that $I(G)=\{\phi_a : a\in G\}$ is the set of all the inner automorphisms of G.

Objective is to prove that $I(G)$ is a subgroup of $Aut(G)$.

Consider following properties of inner automorphism of group G:

For arbitrary $a, b \in G$,

$$\phi_a \quad \phi_b = \phi_{ab},$$
$$(\phi_a)^{-1} = \phi_{a^{-1}}.$$

One step test: If H is a nonempty subset of group G, then H will be subgroup of G if and only if for all $a, b \in H$

$$ab^{-1} \in H.$$

Comment

Step 2 of 3

Since every inner automorphism of G is an automorphism of G. Also, identity $\phi_e \in I(G)$.

Therefore, $I(G)$ is a nonempty subset of $Aut(G)$.

Let $\phi_a, \phi_b \in I(G)$ such that $a, b \in G$. Consider the composition $\phi_a \quad (\phi_b)^{-1}$ and use the properties defined above as :

$$\phi_a \quad (\phi_b)^{-1} = \phi_a \quad \phi_{b^{-1}}$$
$$= \phi_{ab^{-1}}.$$

Since $a, b \in G$, so $ab^{-1} \in G$ as *G* is a group. This implies that

$\phi_{ab^{-1}} \in I(G)$.

Comment

Step 3 of 3

Hence, by one step test of subgroup it implies that $I(G)$ is a subgroup of $Aut(G)$.

Note that, identity ϕ_e and inverse of each nonzero element $(\phi_a)^{-1} = \phi_{a^{-1}}$ exists in $I(G)$. Also composition of functions is closed as well as associative. Therefore, $I(G)$ forms a group.

Comment