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Chapter 29, Problem 4EB

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Problem

Let F be a field of characteristic $\neq 2$. Let $a \neq b$ be in F.

Using parts 1 to 3, find an uncomplicated basis for $\mathbb{Q}(d)$ over $\mathbb{Q}$, where d is a root of $x^4 - 14x^2 + 9$. Then find a basis for $\mathbb{Q}(\sqrt{7+2\sqrt{10}})$ over $\mathbb{Q}$.

Step-by-step solution

Step 1 of 3

Consider a field F of characteristic $\neq 2$. Suppose that $a \neq b \in F$. Let d is the root of quartic equation $x^4 - 14x^2 + 9 = 0$. Objective is to determine the uncomplicated basis for $Q(d)$ over Q.

Compare $x^4 - 14x^2 + 9 = 0$ with $x^4 - 2(a+b)x^2 + (a-b)^2 = 0$ and get,

$(a+b) = 7, (a-b)^2 = 9.$

Or, $a+b = 7, a-b = 3$. On adding and subtracting both the equations, one get $a = 5$, $b = 2$ respectively.

Comment

Step 2 of 3

Since $x=\sqrt{a}+\sqrt{b}$ and $x=\sqrt{a+b+2\sqrt{ab}}$ satisfy $x^4-2(a+b)x^2+(a-b)^2=0$. Therefore, $x=\sqrt{5}+\sqrt{2}$ and $x=\sqrt{7+2\sqrt{10}}$ are the roots of $x^4-14x^2+9=0$. Let $d=\sqrt{5}+\sqrt{2}$.

Consider the result that any field *F* containing $\sqrt{a}+\sqrt{b}$ also contains $\sqrt{a}$ and $\sqrt{b}$. So, field *F* containing $\sqrt{5}+\sqrt{2}$ also contains $\sqrt{5},\sqrt{2}$.

Also, x^2-2 and x^2-5 are minimal polynomials of $\sqrt{2},\sqrt{5}$ because both are irreducible (by Eisenstein's criterion).

Now by the result: if $b\neq x^2a$ for any $x\in F$, then $\sqrt{b}\notin F(\sqrt{a})$; one have $\sqrt{5}\notin Q(\sqrt{2})$.

Comment

Step 3 of 3

Thus the basis for $Q(d)$ over *Q* will be:

$\{1,\sqrt{2},\sqrt{5},\sqrt{10}\}$.

Since $x=\sqrt{7+2\sqrt{10}}$ satisfies $x^4-14x^2+9=0$, therefore

$Q\left(\sqrt{7+2\sqrt{10}}\right)=Q(\sqrt{2},\sqrt{5})$.

Hence, the basis for $Q\left(\sqrt{7+2\sqrt{10}}\right)$ over *Q* will be $\{1,\sqrt{2},\sqrt{5},\sqrt{10}\}$.

Comment