# A Book of Abstract Algebra | (2nd Edition)

Chapter 24, Problem 2EF

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## Problem

Let *A* be an integral domain.

Explain why the kernel of *h* in part 1 consists of all the products *xa*(*x*), for all *a*(*x*) ∈ *A*[*x*]. Why is this the same as the principal ideal ⟨x⟩ in *A*[*x*]?

## Step-by-step solution

**Step 1** of 3

Consider an integral domain $A[x]$ and let $h: A[x] \to A$ map every polynomial to its constant coefficient.

That is $h\left(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0\right) = a_0$

Objective of the question to prove the kernel of *h* is $\langle x \rangle$.

Comment

**Step 2** of 3

Let any polynomial $a(x) \in A[x]$ and consider the product $xa(x)$.

$\{xa(x) \mid \text{for all } a(x) \in A[x]\}$ is the set of all elements generated by *x* in $A[x]$.

And it is denoted as $\langle x \rangle$.

First prove $\langle x \rangle \subseteq \text{kernel of } h$

Then,

$$h(xa(x)) = h(x)h(a(x)) \qquad (\text{since } h \text{ is a homomorphism})$$
$$h(x) = 0$$
$$h(a(x)) = k$$

Here, $k \in A$.

Then,

$$h(xa(x)) = 0 \times k$$
$$= 0$$

Hence $xa(x) \in \text{kernel of } h$

Since $xa(x) \in \text{kernel of } h$ implies

$\square x \square \subseteq \text{kernel of } h$ ......(1)

Comment

**Step 3** of 3

Now prove $\text{kernel of } h \subseteq \square x \square$

Let a $p(x) \in \text{kernel of } h$

Then $h(p(x)) = 0$

That is its constant term is zero.

Then the polynomial should be in the form of $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x$.

Take the common factor *x* outside.

$$p(x) = x\left(a_n x^{n-1} + a_{n-1}x^{n-2} + \ldots + a_2 x^1 + a_1\right)$$
$$= xq(x)$$

Therefore $p(x)$ is the form of elements in $\square x \square$.

Therefore $p(x) \in \square x \square$

It implies $\text{kernel of } h \subseteq \square x \square$ ......(2)

Combining both equation (1) and (2) then $\text{kernel of } h = \square x \square$

Comment