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## 1 Main Theorem

$$\det : K^{n \times n} \rightarrow K$$

1.  $\det(AB) = \det(A)\det(B)$
2. If  $A = (a_{ij})$  is upper or lower triangular then  $\det(A) = \prod_{i=1}^n a_{ii}$ .
3. If  $E$  is a row swap matrix then  $\det(E) = -1$ .
4.  $A$  is nonsingular iff  $\det(A) \neq 0$ .

Note that nonsingular means the rank of  $A \in K^{n \times n}$  is  $n$ . For the matrix to be invertible  $\text{rank}(A) = n$  and  $N(A) = \{0\}$ .

### 1.1 Simple Computation

Using row operations  $E_1, \dots, E_k$ , we can create an upper triangular matrix  $U = E_1 \cdots E_k A$  with  $\det U = u_{11} \cdots u_{nn} \Rightarrow \det A = (u_{11} \cdots u_{nn}) / (\det(E_1) \cdots \det(E_k))$ .

## 2 Signature of a Permutation

Define the signature  $\text{sgn}(\sigma)$  to be

$$\text{sgn}(\sigma) = \prod_{i < j} \frac{\sigma(i) - \sigma(j)}{i - j}$$

## 2.1 $\text{sgn}(\sigma) = \pm 1 \quad \forall \sigma \in S(n)$

By swapping the arbitrary symbols  $i, j$  we see

$$\begin{aligned}
 \prod_{i < j} \frac{\sigma(i) - \sigma(j)}{i - j} &= \prod_{j < i} \frac{\sigma(j) - \sigma(i)}{j - i} \\
 &= \prod_{j < i} \frac{\sigma(j) - \sigma(i)}{j - i} \\
 &= \prod_{j < i} \frac{\sigma(i) - \sigma(j)}{i - j} && \text{multiply prev line by } (-1/-1) \\
 \Rightarrow (\text{sgn}(\sigma))^2 &= \prod_{i < j} \frac{\sigma(i) - \sigma(j)}{i - j} \prod_{j < i} \frac{\sigma(i) - \sigma(j)}{i - j} \\
 &= \prod_{i \neq j} \frac{\sigma(i) - \sigma(j)}{i - j}
 \end{aligned}$$

Expanding this out gives us all possible combos  $i, j$ , so  $\text{sgn}(\sigma)^2 = 1$ .

## 2.2 $\text{sgn}(\tau\sigma) = \text{sgn}(\tau) \text{sgn}(\sigma)$

Let  $N(\sigma) = \{(i, j) \mid i < j, \sigma(i) > \sigma(j)\}$ , and  $n(\sigma) = |N(\sigma)|$ . Thus  $n(\sigma)$  counts the number of inversions in the set  $D = \{(i, j) \mid i < j\}$ . By the proposition above,

$$\text{sgn}(\sigma) = (-1)^{n(\sigma)}$$

Let  $\sigma D = \{(\sigma(i), \sigma(j)) \mid i < j\}$ , then for all  $k < l$ , either  $(k, l)$  or  $(l, k) \in \sigma D$ .

Now apply  $\tau\sigma D$  which contains either  $(\tau k, \tau l)$  or  $(\tau l, \tau k)$ . Thus  $\tau$  inverts  $n(\tau)$  pairs, and so  $D \rightarrow \sigma D \rightarrow \tau\sigma D$  has inverted  $n(\sigma) + n(\tau)$  pairs.

But  $D \rightarrow (\tau\sigma)D$  has inverted  $n(\tau\sigma)$  pairs.

We also see  $(i, j) \in N(\tau\sigma) \Leftrightarrow (i, j) \in N(\sigma)$  or  $(\sigma(i), \sigma(j)) \in N(\tau)$ . And there is no pair  $(i, j) \in N(\tau\sigma) : (i, j) \in N(\sigma)$  and  $(\sigma(i), \sigma(j)) \in N(\tau)$  so it follows

$$n(\tau\sigma) = n(\tau) + n(\sigma)$$

## 2.3 $\text{sgn}(\sigma) = \text{sgn}(\sigma^{-1})$

Observe that  $\text{sgn}(\sigma) \text{sgn}(\sigma^{-1}) = \text{sgn}(\sigma\sigma^{-1}) = \text{sgn}(e) = 1$ . Then since  $\text{sgn}(\sigma), \text{sgn}(\sigma^{-1}) \in \{-1, 1\} \Rightarrow \text{sgn}(\sigma) = \text{sgn}(\sigma^{-1})$ .

## 2.4 $\text{sgn}(\sigma)$ measures the number of transpositions

Note these facts:

- Every permutation is the product of distinct cycles.
- $\text{sgn}(\sigma) = \text{sgn}(\sigma^{-1})$  and  $\text{sgn}(\sigma\sigma^{-1}) = 1$
- So the parity  $\text{sgn}(\sigma)$  is always consistent for all representations.

### 2.4.1 Example: $\sigma = (37)$

Rewriting  $\sigma$  as adjacent transpositions, we see

$$\sigma = \begin{pmatrix} \dots & 3 & 4 & 5 & 6 & 7 & \dots \\ \dots & 4 & 3 & 5 & 6 & 7 & \dots \\ \dots & 4 & 5 & 3 & 6 & 7 & \dots \\ \dots & 4 & 5 & 6 & 3 & 7 & \dots \\ \dots & 4 & 5 & 6 & 7 & 3 & \dots \\ \dots & 4 & 5 & 7 & 6 & 3 & \dots \\ \dots & 4 & 7 & 5 & 6 & 3 & \dots \\ \dots & 7 & 4 & 5 & 6 & 3 & \dots \end{pmatrix}$$

where we first do  $m - n = 7 - 3 = 4$  swaps corresponding to rows 2 – 5. Then we finally do  $m - n - 1$  swaps corresponding to the remaining rows.

We can thus rewrite the cycle as

$$\sigma = (47)(57)(67)(37)(36)(35)(34)$$

where we see

$$\sigma = \begin{pmatrix} \cdots & 3 & 4 & 5 & 6 & 7 & \cdots \\ \cdots & 7 & 4 & 5 & 6 & 3 & \cdots \end{pmatrix}$$

as desired.

The total is  $2(m - n) - 1$  adjacent transpositions.

### 2.4.2 Correspondence between formulas

Let  $\sigma = (mn)$ , then

$$\text{sgn}(\sigma) = \prod_{i < j} \frac{\sigma(i) - \sigma(j)}{i - j} = (-1)$$

We can see by above that the parity of  $\sigma$  can be evaluated by counting the number of swaps for all  $i < j$ . For a single transposition, this will be  $\text{sgn}(\sigma) = -1$ .

Since  $\text{sgn}(\sigma)$  is multiplicative therefore  $\text{sgn}(\tau) = \text{sgn}(\sigma_1) \cdots \text{sgn}(\sigma_k) = (-1)^k$  where  $\tau = \sigma_1 \cdots \sigma_k$ .

## 3 Leibniz Formula

$$\det(A) := \sum_{\pi \in S(n)} \text{sgn}(\pi) a_{\pi(1)1} \cdots a_{\pi(n)n}$$

When  $U$  is upper (or lower) triangular then every permutation along rows or columns will end up including a 0. Hence  $\det(U) = u_{11} \cdots u_{nn}$ .

When  $P = P_\mu$  is a permutation matrix then  $\det(P) = \text{sgn}(\mu)$ . Recall  $P_\mu = (\mathbf{e}_{(1)} \cdots \mathbf{e}_{(n)})$ . Then  $p_{\mu(i)i} = 1$  for all  $i$ , but is 0 otherwise. Therefore  $\det(P) = \text{sgn}(\mu) p_{\mu(1)1} \cdots p_{\mu(n)n} = \text{sgn}(\mu)$ .

### 3.1 $\det(A^T) = \det(A)$

Observe that  $\sigma(i) = j$  then  $\sigma^{-1}(j) = i$  which is bijective. So given the set of tuples  $\{(\sigma(1), 1), \dots, (\sigma(n), n)\}$ , then the set  $\{(1, \sigma^{-1}(1)), \dots, (n, \sigma^{-1}(n))\}$  is the same since  $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$  is bijective.

$$\text{sgn}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n} = \text{sgn}(\sigma) a_{1\sigma^{-1}(1)} \cdots a_{n\sigma^{-1}(n)}$$

Using the result that  $\text{sgn}(\sigma) = \text{sgn}(\sigma^{-1})$ , and relabelling  $\sigma^{-1}$  as  $\tau$ , we get

$$\begin{aligned} \det(A) &= \sum_{\sigma \in S(n)} \text{sgn}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n} \\ &= \sum_{\tau \in S(n)} \text{sgn}(\tau) a_{1\tau(1)} \cdots a_{n\tau(n)} \\ &= \det(A^T) \end{aligned}$$

## 4 Row Operations on the Determinant

### 4.1 Multiply Row by $r \Rightarrow \det(EA) = r \det(A)$

Let  $E$  be the matrix multiplying a single row by  $r$ , then  $\det(E) = r$ .

Likewise  $\det(EA) = r \det(A)$  just by looking at the formula.

## 4.2 Swap Rows $\Rightarrow \det(SA) = -\det(A)$

Let  $B = SA$ , and denote  $S = P_\tau$  where  $\tau$  is a transposition.

Since  $S$  swaps rows, we can observe that  $b_{ij} = a_{\tau(i)j}$ .

$$\det(B) = \sum_{\sigma \in S(n)} \text{sgn}(\sigma) b_{1\sigma(1)} \cdots b_{n\sigma(n)} = \sum_{\sigma \in S(n)} \text{sgn}(\sigma) a_{\tau(1)\sigma(1)} \cdots a_{\tau(n)\sigma(n)}$$

Now let  $\mu = \sigma\tau$  and since  $\tau$  is a transposition  $\Rightarrow \sigma = \mu\tau$

$$\begin{aligned} \det(B) &= \sum_{\sigma \in S(n)} \text{sgn}(\sigma) a_{\tau(1)\mu\tau(1)} \cdots a_{\tau(n)\mu\tau(n)} \\ &= \sum_{\sigma \in S(n)} \text{sgn}(\sigma) a_{1\mu(1)} \cdots a_{n\mu(n)} \\ &= \sum_{\mu \in S(n)} \text{sgn}(\mu\tau) a_{1\mu(1)} \cdots a_{n\mu(n)} \\ &= - \sum_{\mu \in S(n)} \text{sgn}(\mu) a_{1\mu(1)} \cdots a_{n\mu(n)} \\ &= -\det(A) \end{aligned}$$

We also see  $\det(S) = -1$  since  $\det(SI) = -\det(I) = -1$ .

## 4.3 Row Transvection $\Rightarrow \det(EA) = \det(A)$

The  $i$ th row of  $EA$  is  $\mathbf{a}_i + r\mathbf{a}_j$ .

$$(EA)_{ik} = a_{ik} + ra_{jk}$$

So we can see  $\det(EA) = \det(A) + r \det(C)$ , where  $C$  has the property that rows  $\mathbf{c}_i = \mathbf{c}_j$ .

### 4.3.1 $C$ has two rows the same $\Rightarrow \det(C) = 0$

Let  $S$  be the row swap matrix for rows  $i, j$ . Then  $\det(C) = \det(SC) = -\det(C) \Rightarrow 2\det(C) = 0 \Rightarrow \det(C) = 0$  (if the characteristic is not 2).

Using the transpose this also applies to columns.

### 4.3.2 $\det(EA) = \det(A)$

$$\begin{aligned} \det(EA) &= \det(A) + r \det(C) \\ &= \det(A) \end{aligned}$$

## 5 $\det(A) \neq 0 \Leftrightarrow A$ is Nonsingular

Write  $A$  in reduced form using elementary matrices

$$A_{\text{red}} = E_1 \cdots E_k A$$

Then by the results above, we know the product formula is valid for elementary matrices

$$\det(A_{\text{red}}) = \det(E_1) \cdots \det(E_k) \det(A)$$

So  $\det(A) \neq 0 \Leftrightarrow \det(A_{\text{red}}) \neq 0$ . But  $A_{\text{red}}$  is upper triangular so  $\det(A_{\text{red}}) \neq 0 \Leftrightarrow A_{\text{red}} = I$ .

### 5.1 $\det(AB) = \det(A) \det(B)$ for Nonsingular $A, B$

Now we prove the product formula. First for nonsingular  $A, B$ , then  $A_{\text{red}} = B_{\text{red}} = I$  and

$$\begin{aligned} AB &= E_1 \cdots E_k A_{\text{red}} F_1 \cdots F_j B_{\text{red}} \\ &= E_1 \cdots E_k F_1 \cdots F_j \end{aligned}$$

$$\det(AB) = \det(A) \det(B)$$

for nonsingular  $A, B$ .

## 5.2 $\det(AB) = \det(A)\det(B)$ for Singular $A, B$

Finally to prove  $\det(AB) = \det(A)\det(B)$  if  $A$  (or  $B$ ) is singular, we prove that  $AB$  is singular.

Assume  $AB$  is nonsingular. Then  $(AB)^{-1} = B^{-1}A^{-1}$  exists and is nonsingular. Then  $B(AB)^{-1}$  (or  $(AB)^{-1}A$ ) also exists and is a nonsingular inverse of  $A$  (or  $B$ ).

So  $AB$  is also singular, and hence  $\det(A)\det(B) = 0 \Rightarrow \det(AB) = 0$ .

## 6 If $A = LPDU$ is Nonsingular, then $\det(A) = \pm \det(D)$

Use product formula

## 7 Laplace Expansion

$$\det(A) = \sum_{i=1}^n (-1)^{i+j} a_{ij} \det(A_{ij})$$

This is the laplace expansion along the  $j$ th column. Because  $\det(A) = \det(A^T)$ , we can also do the same expansion along the  $i$ th row instead.

Assume  $j = 1$  then

$$\begin{aligned} \det(A) &= \sum_{\sigma \in S(n)} \text{sgn}(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n} \\ &= a_{11} \sum_{\sigma(1)=1} \text{sgn}(\sigma) a_{\sigma(2)2} \cdots a_{\sigma(n)n} + \cdots + a_{n1} \sum_{\sigma(1)=n} \text{sgn}(\sigma) a_{\sigma(2)2} \cdots a_{\sigma(n)n} \end{aligned}$$

Now we have  $\sigma \in S(n)$  where  $\sigma(1) = r$ . Take  $P_\sigma \in \mathbb{F}^{n \times n}$  and delete column 1 and row  $r$ . Note that since every row and column contains a single 1, the new  $P'_\sigma \in \mathbb{F}^{(n-1) \times (n-1)}$  is also a valid permutation. So  $P'_\sigma = P_{\sigma'}$  for some  $\sigma' \in S(n-1)$ .

Let  $P_{\sigma'}$  take  $t$  row swaps to become the identity  $I_{n-1}$ . Then  $\text{sgn}(\sigma') = \det(P_{\sigma'}) = (-1)^t$ .

Adding back row  $r$ , and noting  $\sigma(r) = 1$ , we see that we require  $r-1$  row swaps to bring it to the first row. That means we need  $t+r-1$  row swaps to bring  $P_\sigma$  to the identity  $I_n$ . So  $\text{sgn}(\sigma) = (-1)^{r-1} \text{sgn}(\sigma')$

$$\begin{aligned} \sum_{\sigma(1)=r} \text{sgn}(\sigma) a_{\sigma(2)2} \cdots a_{\sigma(n)n} &= \sum_{\sigma' \in S(n-1)} (-1)^{r-1} \text{sgn}(\sigma') a_{\sigma(2)2} \cdots a_{\sigma(n)n} \\ &= (-1)^{r-1} \det(A_{r1}) \\ &= (-1)^{r+1} \det(A_{r1}) \end{aligned}$$

where the last line we note  $(-1)^{-j} = (-1)^{+j}$ .

## 8 Cramer's Rule (3x3 Case)

Let  $M(A) \in \mathbb{F}^{n \times n}$  be the matrix whose  $ij$ -entry is  $(-1)^{i+j} \det(A_{ij})$ . The **adjoint** matrix of  $A$  is  $\text{adj}(A) = M(A)^T$ .

$$\text{adj}(A) = \begin{pmatrix} \det(A_{11}) & -\det(A_{21}) & \det(A_{31}) \\ -\det(A_{12}) & \det(A_{22}) & -\det(A_{32}) \\ \det(A_{13}) & -\det(A_{23}) & \det(A_{33}) \end{pmatrix}$$

Since we want to prove  $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$ , we can also show  $I = A^{-1}A = \frac{1}{\det(A)} \text{adj}(A)A$  or rather

$$\det(A)I = \text{adj}(A)A$$

$$\text{adj}(A)A = \begin{pmatrix} \det(A_{11}) & -\det(A_{21}) & \det(A_{31}) \\ -\det(A_{12}) & \det(A_{22}) & -\det(A_{32}) \\ \det(A_{13}) & -\det(A_{23}) & \det(A_{33}) \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Expanding the diagonal entries, we see

$$\begin{aligned}(\operatorname{adj}(A)A)_{11} &= a_{11} \det(A_{11}) - a_{21} \det(A_{21}) + a_{31} \det(A_{31}) = \det(A) \\(\operatorname{adj}(A)A)_{22} &= a_{12} \det(A_{12}) - a_{22} \det(A_{22}) + a_{32} \det(A_{32}) = \det(A) \\(\operatorname{adj}(A)A)_{33} &= a_{13} \det(A_{13}) - a_{23} \det(A_{23}) + a_{33} \det(A_{33}) = \det(A)\end{aligned}$$

The remaining non-diagonal entries  $(\operatorname{adj}(A)A)_{ij}$  are of the form

$$\begin{aligned}(\operatorname{adj}(A)A)_{ij} &= \sum_{k=1}^n (\operatorname{adj}(A))_{ik} a_{kj} \\&= \sum_{k=1}^n (-1)^{k+i} a_{kj} \det(A_{ki})\end{aligned}$$

Let  $B = (A \leftarrow^i \mathbf{a}_j)$  be the matrix, where we replace column  $i$  in  $A$  with column  $j$ . We can then see that  $(\operatorname{adj}(A)A)_{ij} = \det(B)$  for  $i \neq j$ . But  $B$  has 2 columns that are the same so  $(\operatorname{adj}(A)A)_{ij} = \det(B) = 0$ .

So finally we have proved the relation and hence the inverse of  $A$  by

$$\det(A)I = \operatorname{adj}(A)A$$