A Book of Abstract Algebra | (2nd Edition)

Chapter AC, Problem 6E

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Problem

Use mathematical induction to prove the following:

$$1^2 + 2^2 + \cdots + (n-1)^2 < \frac{n^3}{3} < 1^2 + 2^2 + \cdots + n^2$$

Step-by-step solution

Step 1 of 2

Objective:-

The objective is to prove $1^2+2^2+\cdots+(n-1)^2 < \frac{n^3}{3} < 1^2+2^2+\cdots\cdots+n^2$ using mathematical induction.

Comment

Step 2 of 2

Proof:-

$$p(n): 1^2+2^2+\cdots+(n-1)^2 < \frac{n^3}{3} < 1^2+2^2+\cdots\cdots+n^2$$

Let consider statement for $n=1$.

$$p(1):\ (1-1)^2 < \frac{n^3}{3} < n^2$$

$$p(1):(1-1)^2 < \frac{1^3}{3} < 1^2$$

$$0 < \frac{1}{3} < 1$$

This rule is true for $n=1$.

Let this rule is true for $n = k$.

$$p(k): 1^2 + 2^2 + \cdots + (k-1)^2 < \frac{k^3}{3} < 1^2 + 2^2 + \cdots + k^2 \qquad ......(1)$$

Let us first take left hand side inequality (1).

$$1^2 + 2^2 + \cdots + (k-1)^2 < \frac{k^3}{3}$$

Let us add k^2 both sides.

$$1^2 + 2^2 + \cdots + (k-1)^2 + k^2 < \frac{k^3}{3} + k^2$$

$$1^2 + 2^2 + \cdots + (k-1)^2 + k^2 < \frac{k^3 + 3k^2}{3}$$

If one adds some quantity in the numerator on right side, then the inequality remains same.

So,

$$1^2 + 2^2 + \cdots + (k-1)^2 + k^2 < \frac{k^3 + 3k^2 + 3k + 1}{3}$$

$$1^2 + 2^2 + \cdots + (k-1)^2 + k^2 < \frac{k^3 + 3 \cdot k \cdot 1(k+1) + 1}{3}$$

$$1^2 + 2^2 + \cdots + (k-1)^2 + k^2 < \frac{(k+1)^3}{3} \qquad ......(2)$$

Let us take right hand side inequality (1).

$$\frac{k^3}{3} < 1^2 + 2^2 + \cdots + k^2$$

Let us add $(k+1)^2$ both sides.

$$\frac{k^3}{3} + (k+1)^2 < 1^2 + 2^2 + \cdots + k^2 + (k+1)^2$$

$$\frac{k^3}{3} + k^2 + 1^2 + 2k < 1^2 + 2^2 + \cdots + k^2 + (k+1)^2$$

$$\frac{k^3}{3} + k^2 + 1^2 + k + k < 1^2 + 2^2 + \cdots + k^2 + (k+1)^2$$

$$\frac{k^3 + 3k^2 + 3 + 3k + 3k}{3} < 1^2 + 2^2 + \cdots + k^2 + (k+1)^2$$

$$\frac{k^3 + 3k^2 + 3k + 1 + 2 + 3k}{3} < 1^2 + 2^2 + \cdots + k^2 + (k+1)^2$$

$$\frac{k^3 + 3 \cdot k \cdot 1(k+1) + 1^3 + 2 + 3k}{3} < 1^2 + 2^2 + \cdots + k^2 + (k+1)^2$$

$$\frac{(k+1)^3 + 2 + 3k}{3} < 1^2 + 2^2 + \cdots + k^2 + (k+1)^2$$

If one removes some quantity in the numerator on left side, then the inequality remains same.

$$\frac{(k+1)^3}{3} < 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 \qquad ......(3)$$

Let us combine the inequality (2) and (3).

$$p(k+1): 1^2 + 2^2 + \cdots + (k-1)^2 + k^2 < \frac{(k+1)^3}{3} < 1^2 + 2^2 + \cdots + k^2 + (k+1)^2$$

This result also true for $n = k+1$. Hence, by mathematical induction this rule is true for all positive integer *n*.

Proved

Comment