A Book of Abstract Algebra | (2nd Edition)

Chapter 24, Problem 3EA

2 Bookmarks

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Problem

REMARK ON NOTATION: In some of the problems which follow, we consider polynomials with coefficients in $\mathbb{Z}_n$ for various n. To simplify notation, we denote the elements of $\mathbb{Z}_n$ by 1, 2, …, $n - 1$ rather than the more correct $\bar{1}, \bar{2}, \ldots, \overline{n-1}$.

Find the quotient and remainder when $x^3 + 2$ is divided by $2x^2 + 3x + 4$ in

$\mathbb{Z}[x]$, in $\mathbb{Z}_3[x]$, and in $\mathbb{Z}_5[x]$.

We call $b(x)$ a *factor* of $a(x)$ if $a(x) = b(x)q(x)$ for some $q(x)$, that is, if the remainder when $a(x)$ is divided by $b(x)$ is equal to zero.

Step-by-step solution

Step 1 of 3

Consider the polynomial $p(x) = x^3 + 2$ and $q(x) = 2x^2 + 3x + 4$.

Objective of this question is to find quotient and remainder when $p(x) = x^3 + 2$ divided by $q(x) = 2x^2 + 3x + 4$ in $\mathbb{Z}[x]$, $\mathbb{Z}_5[x]$ and $\mathbb{Z}_5[x]$.

First consider the ring $\mathbb{Z}[x]$.

Given polynomials $p(x)$ and $q(x)$ are the elements of $\mathbb{Z}[x]$.

Now do long division.

$$2x^2 + 3x + 4 \overline{\big)\, x^3 + 0x^2 + 0x + 2}$$

Since,

$$2k \neq 1 \forall k \in \mathbb{Z}$$

$p(x)$ is not divisible by $q(x)$.

Comment

Step 2 of 3

Consider the ring $\mathbb{Z}_3[x]$.

Change polynomials $p(x)$ and $q(x)$ as the element of $\mathbb{Z}_3[x]$.

$$\begin{aligned} p(x) &= 1(\operatorname{mod} 3)x^3 + 0(\operatorname{mod} 3)x^2 + 0(\operatorname{mod} 3)x + 2(\operatorname{mod} 3) \\ &= x^3 + 2 \end{aligned}$$

$$\begin{aligned} q(x) &= 2(\operatorname{mod} 3)x^2 + 3(\operatorname{mod} 3)x + 4(\operatorname{mod} 3) \\ &= 2x^2 + 1 \end{aligned}$$

Now do long division. Here the operations multiplication and addition are multiplication modulo 3 and addition modulo 3.

$$\begin{array}{r|l} & 2x \\ \cline{2-2} & x^3 + 0x^2 + 0x + 2 \\ 2x^2 + 0x + 1 & \underline{x^3 + 0x^2 + 2x} \\ & \qquad\quad -2x + 2 \end{array}$$

Then, quotient and remainder when $p(x) = x^3 + 2$ divided by $q(x) = 2x^2 + 3x + 4$ in $\mathbb{Z}_3[x]$ are $\boxed{2x}$ and $\boxed{-2x+2}$ respectively.

Comment

Step 3 of 3

Consider the ring $\mathbb{Z}_5[x]$.

Change polynomials $p(x)$ and $q(x)$ as the element of $\mathbb{Z}_5[x]$.

$$\begin{aligned} p(x) &= 1(\operatorname{mod} 5)x^3 + 0(\operatorname{mod} 5)x^2 + 0(\operatorname{mod} 5)x + 2(\operatorname{mod} 5) \\ &= x^3 + 2 \end{aligned}$$

$$\begin{aligned} q(x) &= 2(\operatorname{mod} 5)x^2 + 3(\operatorname{mod} 5)x + 4(\operatorname{mod} 5) \\ &= 2x^2 + 3x + 4 \end{aligned}$$

Now do long division. Here the operations multiplication and addition are multiplication modulo 5 and addition modulo 5.

$$\begin{array}{r|l} & 3x - 2 \\ \cline{2-2} & x^3 + 0x^2 + 0x + 2 \\ & \underline{x^3 + 4x^2 + 2x} \\ 2x^2 + 3x + 4 & \quad -4x^2 - 2x + 2 \\ & \quad \underline{-4x^2 - x - 3} \\ & \qquad\quad -x + 0 \end{array}$$

Then, quotient and remainder when $p(x)=x^3+2$ divided by $q(x)=2x^2+3x+4$ in $\mathbb{Z}_5[x]$ are $\boxed{3x-2}$ and $\boxed{-x}$ respectively.

Comment