A Book of Abstract Algebra | (2nd Edition)

Problem

Let $a(x) = a_0 + a_1x + \cdots + a_nx^n \in F[x]$. The *derivative* of $a(x)$ is the following polynomial $a'(x) \in F[x]$:

$$a'(x) = a_1 + 2a_2x + \ldots + na_nx^{n-1}$$

(This is the same as the derivative of a polynomial in calculus.) We now prove the analogs of the formal rules of differentiation, familiar from calculus.

Let $a(x), b(x) \in F[x]$, and let $k \in F$.

Prove part:

If F has characteristic $p \neq 0$, and $a'(x) = 0$, prove that the only nonzero terms of $a(x)$ are of the form $a_{mp}x^{mp}$ for some m. [That is, $a(x)$ is a polynomial in powers of x^p.]

Step-by-step solution

Step 1 of 3

Consider the arbitrary field F and let $a(x) = a_0 + a_1x + \cdots + a_nx^n \in F(x)$. The derivative of $a(x)$ will be given by

$a'(x) = a_1 + 2a_2x + \cdots + na_nx^{n-1} \in F(x)$.

Suppose that F has characteristic $p \neq 0$ and $a'(x) = 0$. Objective is to prove that the only nonzero terms of $a(x)$ will be of the form $a_{mp}x^{mp}$ for some m.

Since characteristic of F is $p \neq 0$, so $p \cdot a = 0$ for any $a \in F$.

Comment

Step 2 of 3

Also $a_1 + 2a_2x + \cdots + na_nx^{n-1} = 0$. This implies that, either coefficients are zero, or coefficients are multiples of p.

If all the coefficients are zero then in $a(x)$ only the constant term will be arbitrary rest all will be zero. That is, $a(x)$ will be constant polynomial. And thus, nonzero term of $a(x)$ will be of the form $a_{0p}x^{0p} = a_0$.

And if coefficients are multiples of p in $a'(x)$, then corresponding to that coefficient the power of x will be multiple of p in polynomial $a(x)$. That is, $a(x)$ is a polynomial in powers of x^p.

Comment

Step 3 of 3

Hence, the only nonzero terms of $a(x)$ will be of the form $a_{mp}x^{mp}$ for some m.

Comment