A Book of Abstract Algebra | (2nd Edition)

Problem

Let $F \subseteq K$ and $a, b \in K$. We have seen on page 295 that if a and b are algebraic over F, then $F(a, b)$ is a finite extension of F.

Use the above to prove part.

Prove: $\mathbb{Q}(a_0, a_1, \ldots, a_n)$ is a finite extension of $\mathbb{Q}$.

Let $\mathbb{Q}(a_0, \ldots, a_n) = \mathbb{Q}_1$ Since $a(x) \in \mathbb{Q}_1[x]$, c is algebraic over 1 Prove parts 4 and 5:

Step-by-step solution

Step 1 of 3

Consider a field Q and a field $\boldsymbol{A}$ of set of all algebraic numbers. Let

$$a(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbf{A}[x],$$

and c be any root of $a(x)$. Objective is to prove that $Q(a_0, a_1, \ldots, a_n)$ is a finite extension of Q.

Since $a(x) \in \mathbf{A}[x]$, therefore all the coefficients $a_0, a_1, \ldots, a_n$ are algebraic over Q. Let $Q(a_0, a_1, \ldots, a_n)$ be the smallest field containing Q and $a_0, a_1, \ldots, a_n$. The formation of $Q(a_0, a_1, \ldots, a_n)$ can be done step by step by adjoining one a_i at a time.

Comment

Step 2 of 3

Note that, the degree of $F(c)$ over *F* is equal to the degree of the minimal polynomial of *c* over *F*.

If $a_0, a_1, \ldots, a_n$ are algebraic over *Q*, then by this result, each extension in

$$Q \subseteq Q(a_0) \subseteq Q(a_0, a_1) \subseteq Q(a_0, a_1, a_2) \subseteq \cdots \subseteq Q(a_0, a_1, \ldots, a_n)$$

is a finite extension. Again by the theorem of finite extension, $Q(a_0, a_1)$ is a finite extension of *Q*. Also, $Q(a_0, a_1, a_2)$ is a finite extension of *Q*, and so on.

Comment

Step 3 of 3

Hence, if $a_0, a_1, \ldots, a_n$ are algebraic over *Q*, then $Q(a_0, a_1, \ldots, a_n)$ is a finite extension of *Q*.

Comment