A Book of Abstract Algebra | (2nd Edition)

Chapter 32, Problem 4EC

Bookmark

Show all steps: ON

Problem

Use part 3 to explain why $Gal(\mathbb{Q}(\sqrt[3]{2}, \sqrt{3}) : \mathbb{Q})$ has six elements. Then use the discussion following Rule (ii) on page 323 to explain why every element of $Gal(\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3}) : \mathbb{Q})$ may be identifed with a permutation of the three cube roots of 2.

Step-by-step solution

Step 1 of 2

The objective is to explain why $Gal\left(\mathbb{Q}\left(\sqrt[3]{2}, i\sqrt{3}\right) : \mathbb{Q}\right)$ has six elements.

Comment

Step 2 of 2

The root field $\mathbb{Q}\left(\sqrt[3]{2}, i\sqrt{3}\right)$ is of degree 6 over $\mathbb{Q}$.

An automorphism is determined by its effect on $\sqrt[3]{2}$ and $i\sqrt{3}$.

The first must map to one of $\left\{\sqrt[3]{2}, \omega\sqrt[3]{2}, \omega^2\sqrt[3]{2}\right\}$ and $i\sqrt{3}$ must map to one of $\left\{\pm i\sqrt{3}\right\}$.

Since $\omega = \dfrac{-1+i\sqrt{3}}{2}$, negating $i\sqrt{3}$ has the effect of sending ω to $\omega^2 = \varpi$.

Moreover, this extension is normal, since the conjugates of both generators are in the field.

So, there will be six automorphism as the degree of $\mathbb{Q}\left(\sqrt[3]{2}, i\sqrt{3}\right)$ over $\mathbb{Q}$ is 6 which is as follows:

$$\sigma_1 : \begin{cases} \sqrt[3]{2} \mapsto \sqrt[3]{2} \\ i\sqrt{3} \mapsto i\sqrt{3} \end{cases} \quad \sigma_2 : \begin{cases} \sqrt[3]{2} \mapsto \omega\sqrt[3]{2} \\ i\sqrt{3} \mapsto i\sqrt{3} \end{cases} \quad \sigma_3 : \begin{cases} \sqrt[3]{2} \mapsto \omega^2\sqrt[3]{2} \\ i\sqrt{3} \mapsto i\sqrt{3} \end{cases}$$

$$\sigma_4:\begin{cases}\sqrt[3]{2}\mapsto\sqrt[3]{2}\\ i\sqrt{3}\mapsto -i\sqrt{3}\end{cases}\quad \sigma_5:\begin{cases}\sqrt[3]{2}\mapsto w\sqrt[3]{2}\\ i\sqrt{3}\mapsto -i\sqrt{3}\end{cases}\quad \sigma_6:\begin{cases}\sqrt[3]{2}\mapsto \omega^2\sqrt[3]{2}\\ i\sqrt{3}\mapsto -i\sqrt{3}\end{cases}$$

Thus，the Galois group of $\mathbb{Q}\left(\sqrt[3]{2},i\sqrt{3}\right)$ over $\mathbb{Q}$ is

$$Gal\left(\mathbb{Q}\left(\sqrt[3]{2},i\sqrt{3}\right):\mathbb{Q}\right)=\{\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6\}.$$

Comment