A Book of Abstract Algebra | (2nd Edition)

Problem

Find a monic irreducible polynomial $p(x)$ such that $\mathbb{Q}[x]/\langle p(x)\rangle$ is isomorphic to

(*a*) $\mathbb{Q}(\sqrt{2})$

(*b*) $\mathbb{Q}(1+\sqrt{2})$

(*c*) $\mathbb{Q}(\sqrt{1+\sqrt{2}})$

Step-by-step solution

Step 1 of 4

Consider the following result:

Let *F* is any arbitrary field. If $p(x)\in F[x]$is an irreducible polynomial and *c* is some root of $p(x)$, then $\frac{F[x]}{\langle p(x)\rangle}\cong F(c)$.

(a)

Objective is to determine a monic irreducible polynomial $p(x)$ such that

$\frac{Q[x]}{\langle p(x)\rangle}\cong Q(\sqrt{2})$.

By the above result, the polynomial $p(x)$ will be the minimal polynomial of number $\sqrt{2}$ over *Q*.

For this, let $x=\sqrt{2}$. Then $x^2=(\sqrt{2})^2$, $x^2=2$. Thus, *x* satisfies $x^2-2=0$.

Thus, the required monic irreducible polynomial is $p(x)=x^2-2$.

Comment

Step 2 of 4

(b)

Objective is to determine a monic irreducible polynomial $p(x)$ such that

$\frac{Q[x]}{\langle p(x)\rangle}\cong Q(1+\sqrt{2})$.

Using the same argument as above, let $a=1+\sqrt{2}$. Then

$$\begin{aligned} a-1&=\sqrt{2}\\ (a-1)^2&=(\sqrt{2})^2\\ a^2-2a+1&=2\\ a^2-2a-1&=0. \end{aligned}$$

Thus, the required monic irreducible polynomial is $p(x)=x^2-2x-1$.

Comment

Step 3 of 4

(c)

Objective is to determine a monic irreducible polynomial $p(x)$ such that

$\frac{Q[x]}{\langle p(x)\rangle}\cong Q\left(\sqrt{1+\sqrt{2}}\right)$.

Let $a=\sqrt{1+\sqrt{2}}$. Then

$$\begin{aligned} a^2&=1+\sqrt{2}\\ (a^2-1)^2&=(\sqrt{2})^2\\ a^4-2a^2+1&=2\\ a^4-2a^2-1&=0. \end{aligned}$$

Comment

Step 4 of 4

Thus, the required monic irreducible polynomial is $p(x)=x^4-2x^2-1$.

Comment