A Book of Abstract Algebra | (2nd Edition)

Chapter 17, Problem 1EF

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Problem

Let G be an abelian group in additive notation. An *endomorphism* of G is a homomorphism from G to G. Let End(G) denote the set of all the endomorphisms of G, and define addition and multiplication of endomorphisms as follows:

$$[f+g](x)=f(x)+g(x) \qquad \text{for every } x \text{ in } G$$
$$fg=f\circ g \qquad \text{the composite of } f \text{ and } g$$

Prove that End(G) with these operations is a ring with unity.

Step-by-step solution

Step 1 of 5

Suppose that G is an abelian additive group. Let $\mathrm{End}(G)$ is the set of all the endomorphisms of G, that is, all the homomorphism from G to G. Consider the following addition and multiplication of endomorphisms:

$$\begin{aligned}[f+g](x)&=f(x)+g(x),\\ (fg)(x)&=f\circ g(x)\\ &=f(g(x))\end{aligned}$$

for every x in G.

Objective is to show that $\mathrm{End}(G)$ satisfies all the axioms to be a ring with unity.

Comment

Step 2 of 5

First show that $(\mathrm{End}(G),+)$ is an abelian group.

(1) Since sum of two real valued function is again a real function, therefore sum is closed in $\mathrm{End}(G)$.

(2) Associative: Let $f, g, h \in \mathrm{End}(G)$. Then

$$\begin{aligned}[(f+g)+h](x) &= [f+(g+h)](x)\\ (f+g)(x)+h(x) &= f(x)+(g+h)(x)\\ f(x)+g(x)+h(x) &= f(x)+g(x)+h(x)\end{aligned}$$

Since both the sides are equals, so addition is associative in $\mathrm{End}(G)$.

(3) Since addition is commutative in real numbers, so

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ &= g(x)+f(x)\\ &= (g+f)(x).\end{aligned}$$

(4) Additive identity or zero element:

$$(f+g)(x) = f(x)$$

Consider the zero function $g(x)=0$ for all real number *x*. Then

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ &= f(x)+0\\ &= f(x).\end{aligned}$$

Thus, zero function will be the zero element of $\mathrm{End}(G)$.

(5) Since

$$\begin{aligned}(f+(-f))(x) &= f(x)+(-f)(x)\\ &= f(x)-f(x)\\ &= 0\end{aligned}$$

Therefore, negative of any $f \in \mathrm{End}(G)$ will be $-f$.

And from here it conclude that, $\mathrm{End}(G)$ is an abelian group.

Comment

Step 3 of 5

Now, show that product of two function is associative. Let $f, g, h \in \mathrm{End}(G)$. Then

$$\begin{aligned}[(fg)h] &= (fg)\circ h\\ &= f\circ g\circ h\\ [f(gh)] &= f\circ(gh)\\ &= f\circ g\circ h.\end{aligned}$$

Since both the sides are equals, so multiplication is associative in $\mathrm{End}(G)$.

Next is distributive law:

$$\begin{aligned}[f(g+h)](x) &= [f\circ(g+h)](x)\\ &= f(x)\circ(g+h)(x)\\ &= f(x)\circ(g(x)+h(x))\\ &= f\circ g(x)+f\circ h(x).\end{aligned}$$

Thus, $[f(g+h)](x) = fg(x)+fh(x)$. Similarly, $[(g+h)f](x) = gf(x)+hf(x)$.

Comment

Step 4 of 5

The unity in $\mathrm{End}(G)$ will be:

$$(fg)(x) = f(x)$$
$$f \circ g(x) = f(x)$$
$$f(g(x)) = f(x)$$

The last equality will hold when $g(x) = x$ for all real number *x*. Thus, this *g* will work as a unity of any *f* in $\mathrm{End}(G)$.

Comment

Step 5 of 5

Hence, $\mathrm{End}(G)$ is a ring with unity.

Comment