A Book of Abstract Algebra | (2nd Edition)

Problem

Show that $a(x) = x^5 - 10x^4 + 40x^3 - 80x^2 + 79x - 30$ is solvable by radicals over, and give its root field. [HINT: Compute $(x - 2)^5 - (x - 2)$.]

Step-by-step solution

Step 1 of 4

Here, objective is to prove that the given polynomial is solvable by radicals over *Q*.

Comment

Step 2 of 4

A polynomial equation is solvable by radicals, if its roots are determined by applying finite number of additions, subtractions, multiplications, divisions, n^{th} roots to the integers.

Galois group:

If the polynomial whose Galois group is S_5 they are not solvable by radicals.

Generally the polynomials with degree five cannot be solved by radicals, except the polynomials of the form $x^5 - px + q$

Comment

Step 3 of 4

Consider the polynomial $a(x) = x^5 - 10x^4 + +40x^3 - 80x^2 + 79x - 30$

$$a(x) = x^5 - 10x^4 + 40x^3 - 80x^2 + 79x - 30$$
$$a(x) = (x-2)^5 - (x-2) = 0$$

Let $y = x - 2$

Then, the equation becomes,

$y^5 - y = 0$

The above equation is of the form $x^5 - px + q$. So it can be solved by radicals.

Comment

Step 4 of 4

To find the root field:

$$(x-2)^5 - (x-2) = 0$$
$$y^5 - y = 0$$
$$y(y^4 - 1) = 0$$
$$y = 0$$
$$(x-2) = 0$$
$$x = 2$$
$$y^4 - 1 = 0$$
$$y^4 = 1$$
$$y = \pm 1$$
$$x = 3, x = 1$$
$$y^2 = \pm 1$$
$$(x-2)^2 = \pm 1$$
$$x - 2 = i, -i$$
$$x = 2 + i, 2 - i$$

Roots are $\{1, 2, 3, 2+i, 2-i\}$

Root field of $a(x) = x^5 - 10x^4 + +40x^3 - 80x^2 + 79x - 30$ is $Q(i)$

Hence, the polynomial is solvable by radicals and its root field is determined.

Comment