A Book of Abstract Algebra | (2nd Edition)

Chapter 30, Problem 5EC

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Problem

An angle α is called *constructible* iff there exist constructible points *A, B*, and *C* such that $\angle LABC = \alpha$.

Prove the following:

If α and β are constructible angles, so are $\alpha + \beta,\ \alpha - \beta,\ \frac{1}{2}\alpha,$ and $n\alpha$ for any positive integer *n*.

Step-by-step solution

Step 1 of 5

Here, objective is to prove that $\alpha+\beta, \alpha-\beta, \frac{1}{2}\alpha$ ans $n\alpha$ for any integer *n* are constructible.

Comment

Step 2 of 5

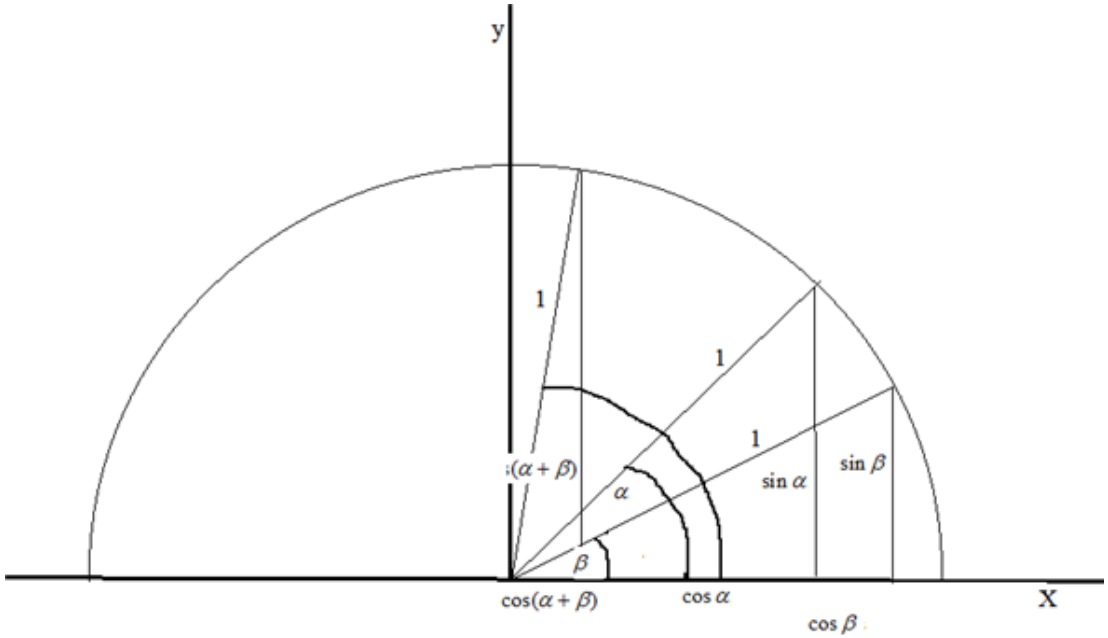
Constructible point is the end point of given unit segment or it is intersection of two lines

determined by constructional points.

Comment

Step 3 of 5

Consider the below figure:



Consider, the angles α, β are constructible from $\{O, I\}$

Then the lengths $\cos\alpha, \cos\beta$ are constructible.

If $(\cos\alpha, 0), (\cos\beta, 0)$ are constructible, then $\sin\alpha, \sin\beta$ are constructible, by using the identities

$$\sin\alpha = 1 - \cos^2\alpha$$

$$\sin\beta = 1 - \cos^2\beta$$

The lengths

$\sin\alpha, \sin\beta$ is constructible.

Comment

Step 4 of 5

Consider

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

So,

$\cos(\alpha+\beta),\cos(\alpha-\beta)$ are constructible from $\cos\alpha,\cos\beta$ and $\sin\alpha,\sin\beta$.

Then draw the perpendicular lines from $(\cos(\alpha+\beta),0),(\cos(\alpha-\beta),0)$ to circumference of unit circle and draw the line joining from origin. The angles making with x-axis are $\alpha+\beta,\alpha-\beta$.

Therefore, $\alpha+\beta,\alpha-\beta$ are also constructible.

Comment

Step 5 of 5

If α is constructible, then bisecting the angle we will construct $\frac{\alpha}{2}$

Similarly, $n\alpha$ is also constructible by multiplication of angle with any integer.

Therefore, $\alpha+\beta,\alpha-\beta,\frac{1}{2}\alpha$ ans $n\alpha$ for any integer *n* are constructible, if α,β are constructible.

Hence, proved

Comment