A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 2EF

Bookmark

Show all steps: ON

Problem

Let *G* be a group; let *H* and *K* be subgroups of *G*, with *H* a normal subgroup of *G*. Prove the following:

If $HK = \{xy: x \in H \text{ and } y \in K\}$, then *HK* is a subgroup of *G*.

Step-by-step solution

Step 1 of 4

Suppose that *G* is any group and let *H, K* are the subgroups of *G*, with *H* a normal subgroup of *G*.

Consider the following set:

$$HK = \{xy : x \in H, y \in K\}.$$

Objective is to prove that HK is a subgroup of *G*.

One step test: If *H* is a nonempty subset of group *G*, then *H* will be subgroup of *G* if and only if for all $a, b \in H$

$$ab^{-1} \in H.$$

Comment

Step 2 of 4

Since *H* and *K* are subgroups of *G*, therefore identity will belong to both the subgroups. Let $e_1 \in H$ and $e_2 \in K$. Then, $e_1 e_2 \in HK$. This ensures that HK is nonempty subset of *G*.

Now consider two typical elements $x_1 y_1, x_2 y_2 \in HK$ where $x_1, x_2 \in H$ and $y_1, y_2 \in K$. Since $xy \in HK$, its inverse

$$\begin{aligned}(xy)^{-1} &= y^{-1}x^{-1} \\ &= y^{-1}x^{-1}yy^{-1} \\ &= (y^{-1}x^{-1}y)y^{-1} \in HK\end{aligned}$$

Since *H* is a normal subgroup of *G*, so $y^{-1}x^{-1}y \in H$.

Comment

Step 3 of 4

Next, it is remaining to prove that product $(x_1y_1)(x_2y_2) \in HK$. For this, consider

$$\begin{aligned}(x_1y_1)(x_2y_2) &= (x_1y_1x_2)y_2 \\ &= (x_1y_1x_2)y_1^{-1}y_1y_2 \\ &= x_1(y_1x_2y_1^{-1})y_1y_2 \in HK\end{aligned}$$

as $x_1(y_1x_2y_1^{-1}) \in H,\ y_1y_2 \in K$. That is, product is closed in HK.

Comment

Step 4 of 4

Since product is closed in HK and inverse of each nonzero element exists, therefore HK forms a subgroup of *G*.

Comment