A Book of Abstract Algebra (2nd Edition)

Chapter 33, Problem 5EE | Bookmark | Show all steps: ON

Problem

Let K be a finite extension of F, where K is a root field over F, with $G = Gal(K: F)$ a solvable group. As remarked in the text, we will assume that F contains the required roots of unity. By Exercise D, let $H_0, \ldots, H_n$ be a solvable series for G in which every quotient H_{i+1}/H_i is cyclic of prime order. For any $i = 1, \ldots, n$, let F_i. and F_{i+1} be the fixfields of H_i and H_{i+1}.

Conclude that K is a radical extension of F.

Step-by-step solution

Step 1 of 4

Here, objective is to prove that K is a radical extension of F.

Consider

ω is a primitive p^{th} root of unity and $c^p \in F_{i+1}$

Comment

Step 2 of 4

Radical extension:

The radical extension of k is extension of k which is obtained by adjoining the sequence of p^{th} roots of elements..

Comment

Step 3 of 4

$G = Gal(K : F)$ is a solvable group.

F is the fixed field of G.

Where, K is a the finite extension of F.

Consider F_i and F_{i+1} are the fixed fields of H_i and H_{i+1}

Comment

Step 4 of 4

Consider the polynomial $x^p - c^p$.

The root of above polynomial is a primitive p^{th} root of unity

$x^p - c^p = 0$

$x^p = c^p$

$x = \sqrt[p]{c^p}\ \omega$

$x = \omega c$

$x = c$

F_i is the root field of $x^p - c^p$ over F_{i+1}

$c^p \in F_{i+1}$

$c \in F_i$

F_{i+1} is the simple radical extension of F_i

$F = F_0 \subset F_1 \subset ...F_n = K$

Therefore, K is a radical extension of F.

Comment