A Book of Abstract Algebra | (2nd Edition)

Chapter 33, Problem 1EC

Bookmark

Show all steps: ON

Problem

Let p be a prime number, and ω a primitive pth root of unity in the field F.

If d is any root of $x^p - a \in F[x]$, show that $F(\omega, d)$ is a root field of $x^p - a$. Suppose $x^p - a$ is *not* irreducible in $F[x]$.

Step-by-step solution

Step 1 of 4

Here, objective is to prove that $F(\omega,d)$ is a root field of $x^p - a$.

Consider p be a prime number.

Comment

Step 2 of 4

Root field:

The field contains a given field in which every polynomial can be written as a product of linear factors.

Comment

Step 3 of 4

Consider the polynomial $x^p - a$.

The root of above polynomial is a primitive p^{th} root of unity

$x^p - a = 0$

$x^p = a$

$x = \sqrt[p]{a}\ \omega$

ω is a primitive p^{th} root of unity

Consider d is a root of $x^p - a \in F(x)$.

Then, $d = \sqrt[p]{a}$

Comment

Step 4 of 4

Let F is the root filed of $x^p - a$.

Then, $\sqrt[p]{a}\ ,\sqrt[p]{a}\ \omega$ both are in F.

The quotient $\omega = \dfrac{\sqrt[p]{a}\ \omega}{\sqrt[p]{a}} \in F(x)$

Therefore $Q(\sqrt[p]{a}\ ,\omega) \subset F$

Since, the field $F(\omega, \sqrt[p]{a})$ is contains all the roots of the polynomial $x^p - a$'

$F(\omega, \sqrt[p]{a}) = F(\omega, d)$

Therefore,

The root field of $x^p - a$ is $F(\omega, d)$

Hence, proved

Comment