A Book of Abstract Algebra | (2nd Edition)

Chapter 32, Problem 4ED

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Problem

If $\alpha = \sqrt[4]{2}$ is a real fourth root of 2, then the four fourth roots of 2 are $\pm\alpha$ and $\pm i\alpha$. Explain parts 1–6, briefly but carefully:

$[\mathbb{Q}(\alpha, i) : \mathbb{Q}] = 8.$

Step-by-step solution

Step 1 of 2

The objective is to show that $\left[\mathbb{Q}\left(\sqrt[4]{2}, i\right) : \mathbb{Q}\right] = 8$.

Comment

Step 2 of 2

Clearly, $\sqrt[4]{2}$ is the root of $x^4 - 2$.

Also, $x^4 - 2$ is irreducible polynomial of lowest degree 4 over $\mathbb{Q}$ by Eisenstein $(p = 2)$.

Therefore, $\left[\mathbb{Q}\left(\sqrt[4]{2}\right) : \mathbb{Q}\right] = \deg\left(x^4 - 2\right) = 4$.

Because $\mathbb{Q}\left(\sqrt[4]{2}\right)$ is a subfield of the reals and so, $i \notin \mathbb{Q}\left(\sqrt[4]{2}\right)$.

Hence, $x^2 + 1$ is irreducible over $\mathbb{Q}\left(\sqrt[4]{2}\right)$.

So, $\left[\mathbb{Q}\left(\sqrt[4]{2}, i\right) : \mathbb{Q}\left(\sqrt[4]{2}\right)\right]$ is at least 2.

But i is a root of $x^2 + 1 \in \mathbb{Q}\left(\sqrt[4]{2}\right)[X]$, so the degree of $\mathbb{Q}\left(\sqrt[4]{2}, i\right)$ over $\mathbb{Q}\left(\sqrt[4]{2}\right)$ is at most 2, and therefore, is exactly 2.

Hence, $\left[\mathbb{Q}\left(\sqrt[4]{2}, i\right) : \mathbb{Q}\left(\sqrt[4]{2}\right)\right] = 2$.

Thus， $\left[\mathbb{Q}\left(\sqrt[4]{2},i\right):\mathbb{Q}\right]=\left[\mathbb{Q}\left(\sqrt[4]{2},i\right):\mathbb{Q}\left(\sqrt[4]{2}\right)\right]\left[\mathbb{Q}\left(\sqrt[4]{2}\right):\mathbb{Q}\right]$

$=2\cdot 4$

$=8.$

Comment