A Book of Abstract Algebra | (2nd Edition)

Chapter 30, Problem 3EF

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Problem

By de Moivre's theorem,

$$\omega = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$$

is a complex seventh root of unity. Since

$$x^7 - 1 = (x-1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$$

ω is a root of $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$.

Prove that $8x^3 + 4x^2 - 4x - 1$ has no rational roots. Conclude that it is irreducible over $\mathbb{Q}$.

Step-by-step solution

Step 1 of 4

Here, objective is to prove that $8x^3 + 4x^2 - 4x - 1$ has no rational roots.

Comment

Step 2 of 4

Rational root theorem:

Consider the polynomial $a_n x^n + a_{n-1}x^{n-1} + \ldots\ldots\ldots + a_0 = 0$, there is a rational solution and that could be determined by checking all the numbers $= \pm \dfrac{\text{dividers of } a_0}{\text{dividers of } a_n}$

Comment

Step 3 of 4

Consider the polynomial $8x^3 + 4x^2 - 4x - 1$

$a_n = 8, a_0 = -1$

Rational numbers $= \pm \dfrac{1}{1,2,4,8}$

Comment

Step 4 of 4

Validate all the rational roots:

For $x = 1$

$$8x^3 + 4x^2 - 4x - 1$$
$$= 8 + 4 - 4 - 1$$
$$= 7$$
$$\neq 0$$

For $x = -1$

$$8x^3 + 4x^2 - 4x - 1 = -1$$
$$\neq 0$$

For $x = -1/2$

$8x^3+4x^2-4x-1=1$

$\neq 0$

For $x=1/2$

$8x^3+4x^2-4x-1=-1$

$\neq 0$

For $x=-1/4$

$8x^3+4x^2-4x-1=1/8$

$\neq 0$

For $x=1/4$

$8x^3+4x^2-4x-1=-13/8$

$\neq 0$

For $x=1/8$

$8x^3+4x^2-4x-1=-91/64$

$\neq 0$

For $x=-1/8$

$8x^3+4x^2-4x-1=-29/64$

$\neq 0$

Therefore, $8x^3+4x^2-4x-1$ has no rational roots and it is irreducible over *Q.*

Hence, proved

Comment