A Book of Abstract Algebra | (2nd Edition)

Chapter 29, Problem 5EE

Bookmark

Show all steps: ON

Problem

Show that part 4 remains true for $\sqrt[q]{m/n}$ where $q > 1$.

Step-by-step solution

Step 1 of 3

Let *F* be a field. Consider the following result:

If a real number *c* is a root of an irreducible polynomial of degree >1 in $Q[x]$, then *c* is irrational.

Objective is to prove that $\sqrt[q]{m/n}$, where $m, n \in Z$ and $q > 1$, is irrational if there is a prime which divides *m* but not *n*, and whose square does not divide *m*.

Comment

Step 2 of 3

Let $x = \sqrt[q]{m/n}$. That is, $x = \left(\frac{m}{n}\right)^{1/q}$. Then

$x^q = \frac{m}{n}$, and $nx^q - m = 0$.

Assume that $p(x) = nx^q - m$. By Eisenstein's irreducible criterion, the polynomial $p(x)$ will be irreducible if there is a prime number *p* such that $p \mid m$, $p \nmid n$ and $p^2 \nmid m$. Let these conditions holds and $p(x)$ is irreducible.

Since $x = \sqrt[q]{m/n}$ is a root of an irreducible polynomial $p(x)$ of degree >1 in $Q[x]$, therefore by the above result $\sqrt[q]{m/n}$ will be irrational.

Comment

Step 3 of 3

Hence, $\sqrt[q]{m/n}$, where $m, n \in Z$ and $q > 1$, is irrational if there is a prime which divides *m* but not *n*, and whose square does not divide *m*.

Comment