A Book of Abstract Algebra | (2nd Edition)

Chapter 24, Problem 1EA

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Problem

REMARK ON NOTATION: In some of the problems which follow, we consider polynomials with coefficients in $\mathbb{Z}_n$ for various n. To simplify notation, we denote the elements of $\mathbb{Z}_n$ by 1, 2, …, n – 1 rather than the more correct $\bar{1}, \bar{2}, \ldots, \overline{n-1}$.

Let $a(x) = 2x^2 + 3x + 1$ and $b(x) = x^3 + 5x^2 + x$. Compute $a(x) + b(x)$, $a(x) - b(x)$ and $a(x)b(x)$ in $\mathbb{Z}[x]$, $\mathbb{Z}_5[x]$, $\mathbb{Z}_6[x]$, and $\mathbb{Z}_7[x]$.

Step-by-step solution

Step 1 of 4

Consider two polynomials $a(x)=2x^2+3x+1$ and $b(x)=x^3+5x^2+x$.

Consider the ring $\mathbb{Z}[x]$.

Now evaluate $a(x)+b(x)$ by adding corresponding coefficients.

$$\begin{aligned} a(x)+b(x) &= 2x^2+3x+1+x^3+5x^2+x \\ &= 0x^3+2x^2+3x+1+x^3+5x^2+x+0 \\ &= x^3+7x^2+4x+1 \end{aligned}$$

Thus,

$a(x)+b(x)$ in the ring $\mathbb{Z}[x]$ is $\boxed{x^3+7x^2+4x+1}$.

Now evaluate $a(x)-b(x)$.

$$\begin{aligned} a(x)-b(x) &= 2x^2+3x+1-(x^3+5x^2+x) \\ &= 0x^3+2x^2+3x+1-(x^3+5x^2+x+0) \\ &= -x^3-3x^2+2x+1 \end{aligned}$$

Thus,

$a(x)-b(x)$ in the ring $\mathbb{Z}[x]$ is $\boxed{-x^3-3x^2+2x+1}$.

Now evaluate $a(x)b(x)$.

$$\begin{aligned} a(x)b(x) &= (2x^2+3x+1)(x^3+5x^2+x) \\ &= 2x^5+10x^4+2x^3+3x^4+15x^3+3x^2+x^3+5x^2+x \\ &= 2x^5+13x^4+18x^3+8x^2+x \end{aligned}$$

Thus,

$a(x)b(x)$ in the ring $\mathbb{Z}[x]$ is $\boxed{2x^5+13x^4+18x^3+8x^2+x}$.

Comment

Step 2 of 4

Consider the ring $\mathbb{Z}_5[x]$.

Write the polynomials $a(x)=2x^2+3x+1$ and $b(x)=x^3+5x^2+x$ as elements in $\mathbb{Z}_5[x]$.

For this compute the remainder get by dividing each coefficients by 5.

Thus,

$$\begin{aligned} a(x) &= 2(\text{mod}\,5)x^2+3(\text{mod}\,5)x+1(\text{mod}\,5) \\ &= 2x^2+3x+1 \end{aligned}$$

And,

$$\begin{aligned} b(x) &= 1(\text{mod}\,5)x^3+5(\text{mod}\,5)x^2+1(\text{mod}\,5)x \\ &= x^3+x \end{aligned}$$

Now evaluate $a(x)+b(x)$ by adding corresponding coefficients. Here operation addition is addition modulo 5.

$$\begin{aligned} a(x)+b(x) &= 2x^2+3x+1+x^3+x \\ &= x^3+2x^2+4x+1 \end{aligned}$$

Thus,

$a(x)+b(x)$ in the ring $\mathbb{Z}_5[x]$ is $\boxed{x^3+2x^2+4x+1}$.

Now evaluate $a(x)-b(x)$.

$$\begin{aligned} a(x)-b(x) &= 2x^2+3x+1-(x^3+x) \\ &= 0x^3+2x^2+3x+1-(x^3+0x^2+x+0) \\ &= -x^3+2x^2+2x+1 \end{aligned}$$

Thus,

$a(x)-b(x)$ in the ring $\mathbb{Z}_5[x]$ is $\boxed{-x^3+2x^2+2x+1}$.

Now evaluate $a(x)b(x)$.

$$\begin{aligned} a(x)b(x) &= (2x^2+3x+1)(x^3+x) \\ &= 2x^5+2x^3+3x^4+3x^2+x^3+x \\ &= 2x^5+3x^3+3x^4+3x^2+x \end{aligned}$$

Thus,

$a(x)b(x)$ in the ring $\mathbb{Z}_5[x]$ is $\boxed{2x^5+3x^3+3x^4+3x^2+x}$.

Comment

Step 3 of 4

Consider the ring $\mathbb{Z}_6[x]$.

Write the polynomials $a(x)=2x^2+3x+1$ and $b(x)=x^3+5x^2+x$ as elements in $\mathbb{Z}_6[x]$.

For this compute the remainder get by dividing each coefficients by 6.

Thus,

$$\begin{aligned} a(x) &= 2(\text{mod}\,6)x^2+3(\text{mod}\,6)x+1(\text{mod}\,6) \\ &= 2x^2+3x+1 \end{aligned}$$

And,

$$\begin{aligned} b(x) &= 1(\text{mod}\,6)x^3+5(\text{mod}\,6)x^2+1(\text{mod}\,6)x \\ &= x^3+x^2+x \end{aligned}$$

Now evaluate $a(x)+b(x)$ by adding corresponding coefficients. Here operation addition is addition modulo 6.

$$\begin{aligned} a(x)+b(x) &= 2x^2+3x+1+x^3+5x^2+x \\ &= x^3+x^2+4x+1 \end{aligned}$$

Thus,

$a(x)+b(x)$ in the ring $\mathbb{Z}_6[x]$ is $\boxed{x^3+x^2+4x+1}$.

Now evaluate $a(x)-b(x)$.

$$\begin{aligned} a(x)-b(x) &= 2x^2+3x+1-(x^3+5x^2+x) \\ &= 0x^3+2x^2+3x+1-(x^3+5x^2+x+0) \\ &= -x^3-3x^2+2x+1 \end{aligned}$$

Thus,

$a(x)-b(x)$ in the ring $\mathbb{Z}_6[x]$ is $\boxed{-x^3-3x^2+2x+1}$.

Now evaluate $a(x)b(x)$. Here multiplication is multiplication modulo 6.

$$\begin{aligned} a(x)b(x) &= (2x^2+3x+1)(x^3+5x^2+x) \\ &= 2x^5+4x^4+2x^3+3x^4+3x^3+3x^2+x^3+5x^2+x \\ &= 2x^5+x^4+0x^3+2x^2+x \\ &= 2x^5+x^4+2x^2+x \end{aligned}$$

Thus,

$a(x)b(x)$ in the ring $\mathbb{Z}_6[x]$ is $\boxed{2x^5+x^4+2x^2+x}$.

Comment

Step 4 of 4

Consider the ring $\mathbb{Z}_7[x]$.

Write the polynomials $a(x)=2x^2+3x+1$ and $b(x)=x^3+5x^2+x$ as elements in $\mathbb{Z}_6[x]$.

For this compute the remainder get by dividing each coefficients by 7.

Thus,

$$\begin{aligned} a(x) &= 2(\bmod 7)x^2 + 3(\bmod 7)x + 1(\bmod 7) \\ &= 2x^2 + 3x + 1 \end{aligned}$$

And,

$$\begin{aligned} b(x) &= 1(\bmod 7)x^3 + 5(\bmod 7)x^2 + 1(\bmod 7)x \\ &= x^3 + x^2 + x \end{aligned}$$

Now evaluate $a(x)+b(x)$ by adding corresponding coefficients. Here operation addition is addition modulo 7.

$$\begin{aligned} a(x)+b(x) &= 2x^2 + 3x + 1 + x^3 + 5x^2 + x \\ &= x^3 + 0x^2 + 4x + 1 \\ &= x^3 + 4x + 1 \end{aligned}$$

Thus,

$a(x)+b(x)$ in the ring $\mathbb{Z}_7[x]$ is $\boxed{x^3 + 4x + 1}$.

Now evaluate $a(x)-b(x)$.

$$\begin{aligned} a(x)-b(x) &= 2x^2 + 3x + 1 - (x^3 + 5x^2 + x) \\ &= 0x^3 + 2x^2 + 3x + 1 - (x^3 + 5x^2 + x + 0) \\ &= -x^3 - 3x^2 + 2x + 1 \end{aligned}$$

Thus,

$a(x)-b(x)$ in the ring $\mathbb{Z}_7[x]$ is $\boxed{-x^3 - 3x^2 + 2x + 1}$.

Now evaluate $a(x)b(x)$. Here multiplication is multiplication modulo 7.

$$\begin{aligned} a(x)b(x) &= (2x^2 + 3x + 1)(x^3 + 5x^2 + x) \\ &= 2x^5 + 3x^4 + 2x^3 + 3x^4 + x^3 + 3x^2 + x^3 + 5x^2 + x \\ &= 2x^5 + 6x^4 + 4x^3 + x^2 + x \end{aligned}$$

Thus,

$a(x)b(x)$ in the ring $\mathbb{Z}_7[x]$ is $\boxed{2x^5 + 6x^4 + 4x^3 + x^2 + x}$.

Comment