A Book of Abstract Algebra | (2nd Edition)

Chapter 23, Problem 4EF

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Problem

Prove part:

If p is a prime, $\phi(p^n) = p^n - p^{n-1} = p^{n-1}(p-1)$. (HINT: For any integer a, a and p^n have a common divisor $\neq \pm 1$ iff a is a multiple of p. There are exactly p^{n-1} multiples of p between 1 and p^n.)

Step-by-step solution

Step 1 of 3

Consider any arbitrary prime number p. Objective is to prove that

$$\phi(p^n) = p^n - p^{n-1}.$$

If p is any prime, then the only divisors of p will be 1 and p itself. So, the following numbers, that are less than p,

$$1, 2, 3, \ldots, p-1$$

will be relatively prime to p.

Thus, by the definition of Euler phi function, $\phi(p) = p-1$.

Comment

Step 2 of 3

If $p \nmid a$, then $\gcd(a, p) = 1$ and also $\gcd(a, p^n) = 1$. Note that, this is a necessary and sufficient condition. That is, $\gcd(a, p^n) = 1$ if and only if $p \nmid a$.

Observe that the following numbers

$$p, 2p, 3p, \ldots, (p^{n-1})p$$

are divisible by *p*, and there are total p^{n-1} integers between 1 and p^n. Thus, the set

$$\{1, 2, 3, \ldots, p^n\}$$

contains exactly $p^n - p^{n-1}$ integers that are relatively prime to p^n.

Comment

Step 3 of 3

Hence, by the definition of Euler phi function $\phi(p^n) = p^n - p^{n-1}$.

Comment