A Book of Abstract Algebra | (2nd Edition)

Chapter 29, Problem 4EA

Bookmark

Show all steps: ON

Problem

Find a basis of $\mathbb{Q}(\mathbb{Q}(\sqrt{2}+\sqrt[3]{4}))$ over $\mathbb{Q}$, and describe the elements of $\mathbb{Q}(\sqrt{2}+\sqrt[3]{4})$.

Step-by-step solution

Step 1 of 3

Objective is to determine the basis of $Q\left(\sqrt{2}+\sqrt[3]{4}\right)$ over *Q*, and describe the elements too.

Assume that $a=\sqrt{2}+\sqrt[3]{4}$.

Let $x=\sqrt{2}$. Then $x^2-2=0$. One knows that, x^2-2 is irreducible over *Q* and thus a minimal polynomial of $\sqrt{2}$. Since polynomial is of degree 2, therefore

$$\left[Q\left(\sqrt{2}\right):Q\right]=2$$

with the basis $\left\{1,\, 2^{1/2}\right\}$.

Comment

Step 2 of 3

Let $y=\sqrt[3]{4}$. Then $y^3-4=0$. Substitute $y=y+1$ and get,

$$\begin{aligned}(y+1)^3-4&=0\\ y^3+1+3y^2+3y-4&=0\\ y^3+3y^2+3y-3&=0.\end{aligned}$$

The last polynomial is irreducible by Eisenstein's criterion. Thus, y^3-4 is a minimal polynomial of $\sqrt[3]{4}$. Since polynomial is of degree 3, therefore

$$[Q(\sqrt[3]{4}):Q]=3.$$

The basis for this will be:

$\{1,4^{1/3},4^{2/3}\}$, or $\{1,2^{2/3},2^{4/3}\}$.

Comments (1)

Step 3 of 3

Also $a=\sqrt{2}+\sqrt[3]{4}$, it implies that $\sqrt{2},\sqrt[3]{4}\in Q(a)$. Next, in $Q(\sqrt{2},\sqrt[3]{4})$, *a* satisfies $a=\sqrt{2}+\sqrt[3]{4}$. So, $Q(\sqrt{2},\sqrt[3]{4})=Q(a)$.

Then, the required basis, with the help of theorem, will be:

$$\{1,2^{1/2},2^{2/3},2^{4/3},2^{2/3}\cdot 2^{1/2},2^{4/3}\cdot 2^{1/2}\}.$$

And the elements of $Q(a)$ will be of the form:

$$Q(a)=\{p+q\cdot 2^{1/2}+r\cdot 2^{2/3}+s\cdot 2^{4/3}+t\cdot 2^{2/3}\cdot 2^{1/2}+u\cdot 2^{4/3}\cdot 2^{1/2}:p,q,r,s,t,u\in Q\}.$$

Comment