A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 3EK

Problem

If G is a group and p is any prime divisor of $|G|$, it will be shown here that G has at least one element of order p. This has already been shown for abelian groups in Chapter 15, Exercise H4. Thus, assume here that G is not abelian. The argument will proceed by induction; thus, let $|G| = k$, and assume our claim is true for any group of order less than k. Let C be the center of G, let Ca be the centralizer of a for each $a \in G$, and let $k = c + k_S + \cdots + k_t$ be the class equation of G, as in Chapter 15, Exercise G2.

Solving the equation $k = c + k_S + \cdots + k_t$ for c, explain why p is a factor of c. We are now done. (Explain why.)

Step-by-step solution

Step 1 of 4

Consider a non-abelian group G whose order is divisible by some prime p. Cauchy Theorem states that group G has at least one element whose order is p.

Assume that order of G is k. Let C be the center of G and C_a be the centralizer of $a \in G$. The class equation for group G is given by:

$$k = c + k_S + \quad + k_t.$$

Objective is to solve the class equation of G and show that p is a factor of c.

Comment

Step 2 of 4

Rewrite the class equation $k = c + k_S + \quad + k_t$ in the following manner:

$$\begin{aligned} c &= k - (k_S + \quad + k_t) \\ &= k - k_S - \quad - k_t. \end{aligned}$$

It is known that p divides the order of G, that is, $p \mid k$.

If $p \mid |C_a|$, that is, if *p* is a factor of $|C_a|$ for any $a \in G$ and $a \notin C$, then the statement of Cauchy theorem will hold (previous exercise result).

Let $p \nmid |C_a|$ for all $a \notin C$. Then one have the result that

$$p \mid (G : C_a)$$

for all $a \notin C$. Where $(G : C_a)$ denotes an index set of *G* by centralizer set C_a. Thus, it implies that $p \mid k_s, \quad , p \mid k_t$. And then $p \mid c$.

Comment

Step 3 of 4

Since group *G* is non abelian, so the center *C* of *G* will be proper subgroup of *G*. That is,

$$|C| < |G|.$$

In the language of the question, it is mention that induction is getting used to proceed the argument. According to the hypothesis, Cauchy theorem holds for every group of order less than *k*, where $k = |G|$. So, by induction, there exists $x \in C$ such that

$$x^p = e.$$

Comment

Step 4 of 4

Hence, if *p* is a factor of *c*, then statement of Cauchy theorem holds.

Comment