Contents

1 Main Theorem

$$\det : K^{n\times n} \to K$$

1. $\det(AB) = \det(A)\det(B)$
2. If $A = (a_{ij})$ is upper or lower triangular then $\det(A) = \prod_{i=1}^{n} a_{ii}$.
3. If E is a row swap matrix then $\det(E) = -1$.
4. A is nonsingular iff $\det(A) \neq 0$.

Note that nonsingular means the rank of $A \in K^{n\times n}$ is n. For the matrix to be invertible $\mathrm{rank}(A) = n$ and $N(A) = \{\mathbf{0}\}$.

1.1 Simple Computation

Using row operations $E_1, ..., E_k$, we can create an upper triangular matrix $U = E_1 \cdots E_k A$ with $\det U = u_{11} \cdots u_{nn} \Rightarrow \det A = (u_{11} \cdots u_{nn})/(\det(E_1) \cdots \det(E_k))$.

2 Signature of a Permutation

Define the signature $\mathrm{sgn}(\sigma)$ to be

$$\mathrm{sgn}(\sigma) = \prod_{i<j} \frac{\sigma(i) - \sigma(j)}{i - j}$$

2.1 $\operatorname{sgn}(\sigma) = \pm 1 \quad \forall \sigma \in S(n)$

By swapping the arbitrary symbols i, j we see

$$\begin{aligned}
\prod_{i<j} \frac{\sigma(i)-\sigma(j)}{i-j} &= \prod_{j<i} \frac{\sigma(j)-\sigma(i)}{j-i} \\
&= \prod_{j<i} \frac{\sigma(j)-\sigma(i)}{j-i} \\
&= \prod_{j<i} \frac{\sigma(i)-\sigma(j)}{i-j} && \text{multiply prev line by (-1/-1)} \\
\Rightarrow (\operatorname{sgn}(\sigma))^2 &= \prod_{i<j} \frac{\sigma(i)-\sigma(j)}{i-j} \prod_{j<i} \frac{\sigma(i)-\sigma(j)}{i-j} \\
&= \prod_{i \neq j} \frac{\sigma(i)-\sigma(j)}{i-j}
\end{aligned}$$

Expanding this out gives us all possible combos i, j, so $\operatorname{sgn}(\sigma)^2 = 1$.

2.2 $\operatorname{sgn}(\tau\sigma) = \operatorname{sgn}(\tau)\operatorname{sgn}(\sigma)$

Let $N(\sigma) = \{(i,j) \mid i < j, \sigma(i) > \sigma(j)\}$, and $n(\sigma) = |N(\sigma)|$. Thus $n(\sigma)$ counts the number of inversions in the set $D = \{(i,j) \mid i < j\}$. By the proposition above,

$$\operatorname{sgn}(\sigma) = (-1)^{n(\sigma)}$$

Let $\sigma D = \{(\sigma(i), \sigma(j)) \mid i < j\}$, then for all $k < l$, either (k,l) or $(l,k) \in \sigma D$.

Now apply $\tau\sigma D$ which contains either $(\tau k, \tau l)$ or $(\tau l, \tau k)$. Thus τ inverts $n(\tau)$ pairs, and so $D \to \sigma D \to \tau\sigma D$ has inverted $n(\sigma) + n(\tau)$ pairs.

But $D \to (\tau\sigma)D$ has inverted $n(\tau\sigma)$ pairs.

We also see $(i,j) \in N(\tau\sigma) \Leftrightarrow (i,j) \in N(\sigma)$ or $(\sigma(i), \sigma(j)) \in N(\tau)$. And there is no pair $(i,j) \in N(\tau\sigma) : (i,j) \in N(\sigma)$ and $(\sigma(i), \sigma(j)) \in N(\tau)$ so it follows

$$n(\tau\sigma) = n(\tau) + n(\sigma)$$

2.3 $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma^{-1})$

Observe that $\operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma^{-1}) = \operatorname{sgn}(\sigma\sigma^{-1}) = \operatorname{sgn}(e) = 1$. Then since $\operatorname{sgn}(\sigma), \operatorname{sgn}(\sigma^{-1}) \in \{-1, 1\} \Rightarrow \operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma^{-1})$.

3 Leibniz Formula

$$\det(A) := \sum_{\pi \in S(n)} \operatorname{sgn}(\pi) a_{\pi(1)1} \cdots a_{\pi(n)n}$$

When U is upper (or lower) triangular then every permutation along rows or columns will end up including a 0. Hence $\det(U) = u_{11} \cdots u_{nn}$.

When $P = P_\mu$ is a permutation matrix then $\det(P) = \operatorname{sgn}(\mu)$. Recall $P_\mu = (\mathbf{e}_{(\mathbf{1})} \cdots \mathbf{e}_{(\mathbf{n})})$. Then $p_{\mu(i)i} = 1$ for all i, but is 0 otherwise. Therefore $\det(P) = \operatorname{sgn}(\mu) p_{\mu(1)1} \cdots p_{\mu(n)n} = \operatorname{sgn}(\mu)$.

3.1 $\det(A^T) = \det(A)$

Observe that $\sigma(i) = j$ then $\sigma^{-1}(j) = i$ which is bijective. So given the set of tuples $\{(\sigma(1), 1), ..., (\sigma(n), n)\}$, then the set $\{(1, \sigma^{-1}(1)), ..., (n, \sigma^{-1}(n))\}$ is the same since $\sigma : \{1, ..., n\} \to \{1, ..., n\}$ is bijective.

$$\operatorname{sgn}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n} = \operatorname{sgn}(\sigma) a_{1\sigma^{-1}(1)} \cdots a_{n\sigma^{-1}(n)}$$

Using the result that $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma^{-1})$, and relabelling σ^{-1} as τ, we get

$$\begin{aligned}\det(A) &= \sum_{\sigma\in S(n)} \operatorname{sgn}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n} \\ &= \sum_{\tau\in S(n)} \operatorname{sgn}(\tau) a_{1\tau(1)} \cdots a_{n\tau(n)} \\ &= \det(A^T)\end{aligned}$$

4 Row Operations on the Determinant

4.1 Multiply Row by $r \Rightarrow \det(EA) = r\det(A)$

Let E by the matrix multiplying a single row by r, then $\det(E) = r$.

Likewise $\det(EA) = r\det(A)$ just by looking at the formula.

4.2 Swap Rows $\Rightarrow \det(SA) = -\det(A)$

Let $B = SA$, and denote $S = P_\tau$ where τ is a transposition.

Since S swaps rows, we can observe that $b_{ij} = a_{\tau(i)j}$.

$$\det(B) = \sum_{\sigma\in S(n)} \operatorname{sgn}(\sigma) b_{1\sigma(1)} \cdots b_{n\sigma(n)} = \sum_{\sigma\in S(n)} \operatorname{sgn}(\sigma) a_{\tau(1)\sigma(1)} \cdots a_{\tau(n)\sigma(n)}$$

Now let $\mu = \sigma\tau$ and since τ is a transposition $\Rightarrow \sigma = \mu\tau$

$$\begin{aligned}\det(B) &= \sum_{\sigma\in S(n)} \operatorname{sgn}(\sigma) a_{\tau(1)\mu\tau(1)} \cdots a_{\tau(n)\mu\tau(n)} \\ &= \sum_{\sigma\in S(n)} \operatorname{sgn}(\sigma) a_{1\mu(1)} \cdots a_{n\mu(n)} \\ &= \sum_{\mu\in S(n)} \operatorname{sgn}(\mu\tau) a_{1\mu(1)} \cdots a_{n\mu(n)} \\ &= -\sum_{\mu\in S(n)} \operatorname{sgn}(\mu) a_{1\mu(1)} \cdots a_{n\mu(n)} \\ &= -\det(A)\end{aligned}$$

We also see $\det(S) = -1$ since $\det(SI) = -\det(I) = -1$.

4.3 Row Transvection $\Rightarrow \det(EA) = \det(A)$

The ith row of EA is $\mathbf{a}_i + r\mathbf{a}_j$.

$$(EA)_{ik} = a_{ik} + ra_{jk}$$

So we can see $\det(EA) = \det(A) + r\det(C)$, where C has the property that rows $\mathbf{c}_i = \mathbf{c}_j$.

4.3.1 C has two rows the same $\Rightarrow \det(C) = 0$

Let S be the row swap matrix for rows i, j. Then $\det(C) = \det(SC) = -\det(C) \Rightarrow 2\det(C) = 0 \Rightarrow \det(C) = 0$ (if the characteristic is not 2).

Using the transpose this also applies to columns.

4.3.2 $\det(EA) = \det(A)$

$$\begin{aligned}\det(EA) &= \det(A) + r\det(C) \\ &= \det(A)\end{aligned}$$

5 $\det(A) \neq 0 \Leftrightarrow A$ is Nonsingular

Write A in reduced form using elementary matrices

$$A_{\text{red}} = E_1 \cdots E_k A$$

Then by the results above, we know the product formula is valid for elementary matrices

$$\det(A_{\text{red}}) = \det(E_1) \cdots \det(E_k) \det(A)$$

So $\det(A) \neq 0 \Leftrightarrow \det(A_{\text{red}}) \neq 0$. But A_{red} is upper triangular so $\det(A_{\text{red}}) \neq 0 \Leftrightarrow A_{\text{red}} = I$.

5.1 $\det(AB) = \det(A)\det(B)$ for Nonsingular A, B

Now we prove the product formula. First for nonsingular A, B, then $A_{\text{red}} = B_{\text{red}} = I$ and

$$\begin{aligned} AB &= E_1 \cdots E_k A_{\text{red}} F_1 \cdots F_j B_{\text{red}} \\ &= E_1 \cdots E_k F_1 \cdots F_j \end{aligned}$$

$$\det(AB) = \det(A)\det(B)$$

for nonsingular A, B.

5.2 $\det(AB) = \det(A)\det(B)$ for Singular A, B

Finally to prove $\det(AB) = \det(A)\det(B)$ if A (or B) is singular, we prove that AB is singular.

Assume AB is nonsingular. Then $(AB)^{-1} = B^{-1}A^{-1}$ exists and is nonsingular. Then $B(AB)^{-1}$ (or $(AB)^{-1}A$) also exists and is a nonsingular inverse of A (or B).

So AB is also singular, and hence $\det(A)\det(B) = 0 \Rightarrow \det(AB) = 0$.

6 If $A = LPDU$ is Nonsingular, then $\det(A) = \pm\det(D)$

Use product formula

7 Laplace Expansion

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

This is the laplace expansion along the jth column. Because $\det(A) = \det(A^T)$, we can also do the same expansion along the ith row instead.

Assume $j = 1$ then

$$\begin{aligned} \det(A) &= \sum_{\sigma \in S(n)} \operatorname{sgn}(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n} \\ &= a_{11} \sum_{\sigma(1)=1} \operatorname{sgn}(\sigma) a_{\sigma(2)2} \cdots a_{\sigma(n)n} + \cdots + a_{n1} \sum_{\sigma(1)=n} \operatorname{sgn}(\sigma) a_{\sigma(2)2} \cdots a_{\sigma(n)n} \end{aligned}$$

Now we have $\sigma \in S(n)$ where $\sigma(1) = r$. Take $P_\sigma \in \mathbb{F}^{n \times n}$ and delete column 1 and row r. Note that since every row and column contains a single 1, the new $P'_\sigma \in \mathbb{F}^{(n-1)\times(n-1)}$ is also a valid permutation. So $P'_\sigma = P_{\sigma'}$ for some $\sigma \in S(n-1)$.

Let $P_{\sigma'}$ take t row swaps to become the identity I_{n-1}. Then $\operatorname{sgn}(\sigma') = \det(P_{\sigma'}) = (-1)^t$.

Adding back row r, and noting $\sigma(r) = 1$, we see that we require $r - 1$ row swaps to bring it to the first row. That means we need $t + r - 1$ row swaps to bring P_σ to the identity I_n. So $\operatorname{sgn}(\sigma) = (-1)^{r-1} \operatorname{sgn}(\sigma')$

$$
\begin{aligned}
\sum_{\sigma(1)=r} \operatorname{sgn}(\sigma) a_{\sigma(2)2} \cdots a_{\sigma(n)n} &= \sum_{\sigma' \in S(n-1)} (-1)^{r-1} \operatorname{sgn}(\sigma') a_{\sigma(2)2} \cdots a_{\sigma(n)n} \\
&= (-1)^{r-1} \det(A_{r1}) \\
&= (-1)^{r+1} \det(A_{r1})
\end{aligned}
$$

where the last line we note $(-1)^{-j} = (-1)^{+j}$.

8 Cramer's Rule (3x3 Case)

Let $M(A) \in \mathbb{F}^{n \times n}$ be the matrix whose ij-entry is $(-1)^{i+j} \det(A_{ij})$. The **adjoint** matrix of A is $\operatorname{adj}(A) = M(A)^T$.

$$
\operatorname{adj}(A) = \begin{pmatrix} \det(A_{11}) & -\det(A_{21}) & \det(A_{31}) \\ -\det(A_{12}) & \det(A_{22}) & -\det(A_{32}) \\ \det(A_{13}) & -\det(A_{23}) & \det(A_{33}) \end{pmatrix}
$$

Since we want to prove $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$, we can also show $I = A^{-1}A = \frac{1}{\det(A)} \operatorname{adj}(A)A$ or rather

$$
\det(A)I = \operatorname{adj}(A)A
$$

$$
\operatorname{adj}(A)A = \begin{pmatrix} \det(A_{11}) & -\det(A_{21}) & \det(A_{31}) \\ -\det(A_{12}) & \det(A_{22}) & -\det(A_{32}) \\ \det(A_{13}) & -\det(A_{23}) & \det(A_{33}) \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}
$$

Expanding the diagonal entries, we see

$$
\begin{aligned}
(\operatorname{adj}(A)A)_{11} &= a_{11} \det(A_{11}) - a_{21} \det(A_{21}) + a_{31} \det(A_{31}) = \det(A) \\
(\operatorname{adj}(A)A)_{22} &= a_{12} \det(A_{12}) - a_{22} \det(A_{22}) + a_{32} \det(A_{32}) = \det(A) \\
(\operatorname{adj}(A)A)_{33} &= a_{13} \det(A_{13}) - a_{23} \det(A_{23}) + a_{33} \det(A_{33}) = \det(A)
\end{aligned}
$$

The remaining non-diagonal entries $(\operatorname{adj}(A)A)_{ij}$ are of the form

$$
\begin{aligned}
(\operatorname{adj}(A)A)_{ij} &= \sum_{k=1}^{n} (\operatorname{adj}(A))_{ik} a_{kj} \\
&= \sum_{k=1}^{n} (-1)^{k+i} a_{kj} \det(A_{ki})
\end{aligned}
$$

Let $B = (A \leftsquigarrow^i \mathbf{a}_j)$ be the matrix, where we replace column i in A with column j. We can then see that $(\operatorname{adj}(A)A)_{ij} = \det(B)$ for $i \neq j$. But B has 2 columns that are the same so $(\operatorname{adj}(A)A)_{ij} = \det(B) = 0$.

So finally we have proved the relation and hence the inverse of A by

$$
\det(A)I = \operatorname{adj}(A)A
$$