A Book of Abstract Algebra (2nd Edition)

Chapter 32, Problem 3ED

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Problem

If $\alpha = \sqrt[4]{2}$ is a real fourth root of 2, then the four fourth roots of 2 are $\pm\alpha$ and $\pm i\alpha$. Explain parts 1–6, briefly but carefully:

$i \notin \mathbb{Q}(\alpha)$; hence $[\mathbb{Q}(\alpha, i) : \mathbb{Q}(\alpha)] = 2$.

Step-by-step solution

Step 1 of 2

The objective is to show that $i \notin \mathbb{Q}\left(\sqrt[4]{2}\right)$ and hence $\left[\mathbb{Q}\left(\sqrt[4]{2}, i\right) : \mathbb{Q}\left(\sqrt[4]{2}\right)\right] = 2$.

Comment

Step 2 of 2

Because $\mathbb{Q}\left(\sqrt[4]{2}\right)$ is a subfield of the reals and so , $i \notin \mathbb{Q}\left(\sqrt[4]{2}\right)$.

Hence , $x^2 + 1$ is irreducible over $\mathbb{Q}\left(\sqrt[4]{2}\right)$.

So , $\left[\mathbb{Q}\left(\sqrt[4]{2}, i\right) : \mathbb{Q}\left(\sqrt[4]{2}\right)\right]$ is at least 2.

But i is a root of $x^2 + 1 \in \mathbb{Q}\left(\sqrt[4]{2}\right)[X]$, so the degree of $\mathbb{Q}\left(\sqrt[4]{2}, i\right)$ over $\mathbb{Q}\left(\sqrt[4]{2}\right)$ is at most 2 ,and therefore ,is exactly 2.

Hence , $\left[\mathbb{Q}\left(\sqrt[4]{2}, i\right) : \mathbb{Q}\left(\sqrt[4]{2}\right)\right] = 2$.

Comment