A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 1EP

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Problem

Let G be an abelian group of order $p^k m$, where p^k and m are relatively prime (that is, p^k and m have no common factors except ±1). (REMARK: If two integers j and k are relatively prime, then there are integers s and t such that $sj + tk = 1$. This is proved on page 220.)

Let G_pk be the subgroup of G consisting of all elements whose order divides p^k. Let Gm be the subgroup of G consisting of all elements whose order divides ra. Prove:

For any $x \in G$ and integers s and t, $x^{sp^k} \in G_m$ and $x^{tm} \in G_{p^k}$.

Step-by-step solution

Step 1 of 4

Assume that G is an abelian group of order $p^k m$, where p^k and m are relatively prime. Suppose that G_{p^k} be the subgroup of G consisting of all elements whose order divides p^k. Let G_m be the subgroup of G consisting of all elements whose order divides m.

Objective is to prove that for any $x \in G$ and integers s and t,

$x^{sp^k} \in G_m$ and $x^{tm} \in G_{p^k}$.

Comment

Step 2 of 4

Since p^k and m are relatively prime, so by the definition there exist integers s ant t such that

$sp^k + tm = 1$.

Let $y = x^{sp^k}$. Then find the following power of y as:

$$\begin{aligned} y^m &= \left(x^{sp^k}\right)^m \\ &= \left(x^s\right)^{p^k m} \\ &= \left(x^s\right)^{|G|}. \end{aligned}$$

because order of group *G* is $p^k m$. Use the identity $x^{|G|} = e$ for some $x \in G$ and get,

$y^m = e$.

It implies that order of *y* divides *m*. And so, $y = x^{sp^k} \in G_m$.

Comment

Step 3 of 4

Similarly, let $z = x^{tm}$. Then

$$\begin{aligned} z^{p^k} &= \left(x^{tm}\right)^{p^k} \\ &= \left(x^t\right)^{p^k m} \\ &= \left(x^t\right)^{|G|}. \end{aligned}$$

because order of group *G* is $p^k m$. Use the identity $x^{|G|} = e$ for some $x \in G$ and get,

$z^{p^k} = e$.

It implies that order of *z* divides p^k. And so, $z = x^{tm} \in G_{p^k}$

Comment

Step 4 of 4

Hence, for any $x \in G$ and integers *s* and *t*, $x^{sp^k} \in G_m$ and $x^{tm} \in G_{p^k}$.

Comment