Contents

1 No Integer Solutions for $x^3 = y^2 + k$

Suppose $k = 1, 2 \pmod 4$, that k is squarefree, and k is not of the form $3t^2 \pm 1$ for some $t \in \mathbb{Z}$.

Also assume $3 \nmid \mathrm{cl}(\mathbb{Q}(\sqrt{-k}))$.

Then $x^3 = y^2 + k$ has no integer solution.

1.1 x is Odd

We start by brute-forcing all possible values mod 4 for x, y.

```
sage: for x in range(4):
....:     for y in range(4):
....:         if (x^3 - (y^2 + 1)) % 4 == 0:
....:             print(x, y)
....:
1 0
1 2
sage: for x in range(4):
....:     for y in range(4):
....:         if (x^3 - (y^2 + 2)) % 4 == 0:
....:             print(x, y)
....:
3 1
3 3
```

So in both cases x is odd.

1.2 (x, y) are Coprime

Let $p|(x, y)$ then $p|x^3 - y^2$ so $p|k$.

We also see $p^3|x^3 \Rightarrow p^2|x^3$ but $p^2 \nmid k$ since k is squarefree, so $p^2 \nmid y^2 + k$.

Hence (x, y) are coprime.

1.3 $y + \sqrt{-k}$ and $y - \sqrt{-k}$ are in the Same Ideal

$$x^3 = (y + \sqrt{-k})(y - \sqrt{-k})$$

Suppose there is a prime ideal $\mathfrak{p}$ such that $(y \pm \sqrt{-k}) \in \mathfrak{p}$ which means they are both coprime. This means $x^3 \in \mathfrak{p} \Rightarrow x \in \mathfrak{p}$, also by summing the ideals we see also $2y \in \mathfrak{p}$. Since x is odd, 2 is not in $\mathfrak{p}$ otherwise it would be the whole ring. But $\mathfrak{p}$ is prime $\Rightarrow y \in \mathfrak{p}$. But both x, y are coprime so this cannot be true.

1.4 Both Ideals are Principal

Next we see both ideals are principal.

$$\langle y + \sqrt{-k} \rangle = \mathfrak{a}^3, \qquad \langle y - \sqrt{-k} \rangle = \mathfrak{b}^3$$

We see $[\mathfrak{a}^3] = [1]$ in the class group since it is principal. Therefore $[\mathfrak{a}]^3 = [1]$ means that $3|\mathrm{ord}([\mathfrak{a}])$, but by lagrange's theorem $\mathrm{ord}([\mathfrak{a}])|\mathrm{cl}(\mathbb{Q}(\sqrt{-k}))$ which means also $3|\mathrm{cl}(\mathbb{Q}(\sqrt{-k}))$. But we stated this is not true in the beginning so we conclude $\mathfrak{a}$ and likewise $\mathfrak{b}$ are both principal.

1.5 Result

Lastly we see our result.

$y + \sqrt{-k} = u\alpha^3$ for some unit u. Note $k \equiv 1, 2 \pmod 4$ means $-k \equiv 3, 2 \pmod 4$. For all $-k$, the units are $\{\pm 1\}$ except $-k = -1$ which includes $\{\pm i\}$. But $k = 1$ is of the form $3t^2 + 1$ so we ignore that value.

In all cases, these units have integer cube roots so $y + \sqrt{-k} = \alpha^3$ for some $\alpha = a + b\sqrt{-k}$. Then

$$y + \sqrt{-k} = (a + b\sqrt{-5})^3$$

```
sage: var("a b k")
(a, b, k)
sage: ( (a + b*sqrt(-k))^3 ).expand()
b^3*(-k)^(3/2) - 3*a*b^2*k + 3*a^2*b*sqrt(-k) + a^3
```

By comparing coefficients, we see

$$\begin{aligned} \sqrt{-k} &= b^3\sqrt{-k}^3 + 3a^2b\sqrt{-k} \\ &= (b^3\sqrt{-k}^2 + 3a^2b)\sqrt{-k} \\ &= (-kb^3 + 3a^2b)\sqrt{-k} \\ \Rightarrow 1 &= b(3a^2 - kb^2) \end{aligned}$$

So $b = \pm 1$ and so $3a^2 - kb^2 = 3a^2 - k = \pm 1$, which means

$$k = 3a^2 \mp 1$$

which has no solutions as stated at the beginning.