A Book of Abstract Algebra | (2nd Edition)

Problem

Recall the definition of *F*(*a*). It is a field such that (i) $F \subseteq F(a)$; (ii) $a \in F(a)$; (iii) any field containing *F* and *a* contains *F*(*a*).

Use this definition to prove parts 1–5, where $F \subseteq K$, $c \in F$, and $a \in K$:

Let *p*(*x*) be irreducible, and let *a* be a root of *p*(*cx*). Then $F[x]/\langle p(cx)\rangle \cong F(a)$ and $F[x]/\langle p(x)\rangle \cong F(ca)$. Conclude that $F[x]/\langle p(cx)\rangle \cong F[x]/\langle p(x)\rangle$.

Step-by-step solution

Step 1 of 3

Consider that *F* is any arbitrary field and *K* is its extension field. Let $c \in F$, and $a \in K$. Assume that $p(x) \in F[x]$ be some irreducible polynomial, and *a* be the root of $p(cx)$.

Objective is to prove that

$F[x]/\langle p(cx)\rangle \cong F(a),$
$F[x]/\langle p(x)\rangle \cong F(ca).$

Consider the following result:

Let *F* is any arbitrary field. If $p(x) \in F[x]$ is an irreducible polynomial and *c* is some root of $p(x)$, then

$$\frac{F[x]}{\langle p(x)\rangle} \cong F(c).$$

Comment

Step 2 of 3

Note that, if $p(x) \in F[x]$ is some irreducible polynomial, then $p(cx)$ will also be irreducible. Because scalar multiplication by c does not change the behaviour of polynomial.

Since *a* is the root of $p(cx)$, so one can apply the above result and get that

$F[x]/\langle p(cx)\rangle \cong F(a).$

Again, since *a* is the root of $p(cx)$, therefore ca will be the root of $p(x)$. Use the above mentioned result, and obtained that

$F[x]/\langle p(x)\rangle \cong F(ca).$

Comment

Step 3 of 3

One knows that, for some nonzero *c*,

$F(a) = F(ca).$

Use this fact, and conclude that

$F[x]/\langle p(cx)\rangle \cong F[x]/\langle p(x)\rangle.$

Comment