A Book of Abstract Algebra | (2nd Edition)

Chapter 31, Problem 2EA

Bookmark

Show all steps: ON

Problem

Prove that $x^2 - 3$ and $x^2 - 2x - 2$ are both irreducible over $\mathbb{Q}$. Then find their root fields over $\mathbb{Q}$ and show they are the same.

Step-by-step solution

Step 1 of 3

The objective is to prove that x^2-3 and x^2-2x-2 are irreducible over $\mathbb{Q}$. Also, find their roots fields over $\mathbb{Q}$ and show that they are same.

Comment

Step 2 of 3

Eisenstein's irreducibility criteria- "Let $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_0$ be a polynomial with integer coefficients. If there exists a prime number p such that $p \nmid a_n$, $p \mid a_{n-1}, a_{n-2}, \ldots, a_1, a_0$, and $p^2 \nmid a_0$, then $f(x)$ is irreducible over $\mathbb{Q}$".

For $f(x)=x^2-3$, $a_2=1, a_1=0, a_0=-3$.

$3 \nmid 1$, $3 \mid 0, -3$, and $9 \nmid -3$. Therefore, by Eisenstein's irreducibility criteria, x^2-3 is irreducible over $\mathbb{Q}$.

For $f(x)=x^2-2x-2$, $a_2=1, a_1=-2, a_0=-2$.

$2 \nmid 1$, $2 \mid -2, -2$, and $4 \nmid -2$. Therefore, by Eisenstein's irreducibility criteria, x^2-2x-2 is irreducible over $\mathbb{Q}$.

Comment

Step 3 of 3

The roots of x^2-3 are $\pm\sqrt{3}$. Therefore, the root field of x^2-3 over $\mathbb{Q}$ is $\mathbb{Q}\left(\pm\sqrt{3}\right)$. This can be written simply as $\mathbb{Q}\left(\sqrt{3}\right)$.

Use the formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ to find roots of x^2-2x-2.

Here, $a=1$, $b=-2$, and $c=-2$.

$$\begin{aligned} x &= \frac{-(-2)\pm\sqrt{(-2)^2-4\cdot1\cdot(-2)}}{2\cdot1} \\ &= \frac{2\pm\sqrt{12}}{2} \\ &= 1\pm\sqrt{3} \end{aligned}$$

Therefore, the root field of x^2-2x-2 over $\mathbb{Q}$ is $\mathbb{Q}\left(1\pm\sqrt{3}\right)$. This can be written simply as $\mathbb{Q}\left(\sqrt{3}\right)$.

Therefore, it is shown that x^2-3 and x^2-2x-2 have same root fields over $\mathbb{Q}$.

Comment