

# A Book of Abstract Algebra | (2nd Edition)

Chapter 29, Problem 1EB

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## Problem

Let  $F$  be a field of characteristic  $\neq 2$ . Let  $a \neq b$  be in  $F$ .

Prove that any field  $F$  containing  $\sqrt{a} + \sqrt{b}$  also contains  $\sqrt{a}$  and  $\sqrt{b}$ . [HINT: Compute  $(\sqrt{a} + \sqrt{b})^2$  and show that  $\sqrt{ab} \in F$ . Then compute  $\sqrt{ab}(\sqrt{a} + \sqrt{b})$ , which is also in  $F$ . Conclude that  $F(\sqrt{a} + \sqrt{b}) = F(\sqrt{a}, \sqrt{b})$ .

## Step-by-step solution

### Step 1 of 3

Consider a field  $F$  of characteristic  $\neq 2$ . Objective is to prove that any field  $F$  containing  $\sqrt{a} + \sqrt{b}$  also contains  $\sqrt{a}$  and  $\sqrt{b}$ , where  $a \neq b \in F$ . And then draw a conclusion that  $F(\sqrt{a} + \sqrt{b}) = F(\sqrt{a}, \sqrt{b})$ .

Suppose field  $F$  contains  $\sqrt{a} + \sqrt{b}$ . To show the required result, first show that  $\sqrt{ab} \in F$ . For this consider the following square:

$$(\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{ab} \in F.$$

Solve for  $\sqrt{ab}$  and get,

$$\sqrt{ab} = \frac{(\sqrt{a} + \sqrt{b})^2 - (a + b)}{2}.$$

That is,  $\sqrt{ab} \in F$  because  $a, b \in F$ .

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### Step 2 of 3

Since  $\sqrt{a} + \sqrt{b}, \sqrt{ab} \in F$ , therefore their product

$$\sqrt{ab}(\sqrt{a} + \sqrt{b}) = a\sqrt{b} + b\sqrt{a} \in F.$$

Since  $a \neq b$ , then

$$\begin{aligned} \frac{b(\sqrt{a} + \sqrt{b}) - \sqrt{ab}(\sqrt{a} + \sqrt{b})}{b - a} &= \frac{b\sqrt{a} + b\sqrt{b} - a\sqrt{b} - b\sqrt{a}}{b - a} \\ &= \frac{\sqrt{b}(b - a)}{b - a} \\ &= \sqrt{b} \in F. \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{a(\sqrt{a} + \sqrt{b}) - \sqrt{ab}(\sqrt{a} + \sqrt{b})}{a - b} &= \frac{a\sqrt{a} + a\sqrt{b} - a\sqrt{b} - b\sqrt{a}}{a - b} \\ &= \frac{\sqrt{a}(a - b)}{a - b} \\ &= \sqrt{a} \in F. \end{aligned}$$

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### Step 3 of 3

Note that,  $\sqrt{a}, \sqrt{b} \in F(\sqrt{a}, \sqrt{b})$ . Therefore,  $\sqrt{a} + \sqrt{b} \in F(\sqrt{a}, \sqrt{b})$ . It follows that

$$F(\sqrt{a} + \sqrt{b}) \subseteq F(\sqrt{a}, \sqrt{b}).$$

Also,  $\sqrt{a}, \sqrt{b} \in F(\sqrt{a} + \sqrt{b})$ . Therefore,

$$F(\sqrt{a}, \sqrt{b}) \subseteq F(\sqrt{a} + \sqrt{b}).$$

Thus,

$$F(\sqrt{a} + \sqrt{b}) = F(\sqrt{a}, \sqrt{b}).$$

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