A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 1EB

Bookmark

Show all steps: ON

Problem

Let $\mathscr{F}(\mathbb{R}) \to \mathbb{R}$ be defined by $\alpha(f) = f(1)$ and let $\beta: \mathscr{F}(\mathbb{R}) \to \mathbb{R}$ be defined by $\beta(f) = f(2)$.

Prove that α and β are homomorphisms from $\mathscr{F}(\mathbb{R})$ *onto* $\mathbb{R}$.

Step-by-step solution

Step 1 of 3

Consider the two functions

$$\alpha : F(R) \to R,$$
$$\beta : F(R) \to R,$$

defined by

$$\alpha(f) = f(1),$$
$$\beta(f) = f(2).$$

Here, $F(R)$represents the group of all functions from *R* to *R* with the following addition:

$$(f+g)(x) = f(x) + g(x)$$

for all real numbers *x*.

Objective is to prove that α, β both are homomorphism from $F(R)$onto *R*.

Comment

Step 2 of 3

If *G* and *H* are two groups, a homomorphism from *G* to *H* is a function $f : G \to H$ such that for any two elements *a*, *b* in *G*,

$$f(ab) = f(a)f(b).$$

Since $F(R)$is an additive group therefore α, β will be homomorphism if

$$\alpha(f+g)=\alpha(f)+\alpha(g)$$
$$\beta(f+g)=\beta(f)+\beta(g).$$

To check this, consider the left side and use the definition of above defined function as:

$$\begin{aligned}\alpha(f+g)&=(f+g)(1)\\&=f(1)+g(1)\\&=\alpha(f)+\alpha(g).\end{aligned}$$

The function α is onto because for all $x\in R$, one can define a function $f:R\to R$ such that $f(1)=x$.

Similarly,

$$\begin{aligned}\beta(f+g)&=(f+g)(2)\\&=f(2)+g(2)\\&=\beta(f)+\beta(g).\end{aligned}$$

The function β is onto because for all $x\in R$, one can define a function $f:R\to R$ such that $f(2)=x$.

Comment

Step 3 of 3

Hence, α,β both are homomorphism from $F(R)$ onto *R*.

Comment