# A Book of Abstract Algebra | (2nd Edition)

Chapter 27, Problem 4EI

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**Problem**

Let $a(x) = a_0 + a_1x + \cdots + a_nx^n \in F[x]$. The *derivative* of $a(x)$ is the following polynomial $a'(x) \in F[x]$:

$$a'(x) = a_1 + 2a_2x + \ldots + na_nx^{n-1}$$

(This is the same as the derivative of a polynomial in calculus.) We now prove the analogs of the formal rules of differentiation, familiar from calculus.

Let $a(x), b(x) \in F[x]$, and let $k \in F$.

Prove part:

If $F$ has characteristic 0 and $a'(x) = 0$, then $a(x)$ is a constant polynomial. Why is this conclusion not necessarily true if $F$ has characteristic $p \neq 0$?

**Step-by-step solution**

**Step 1** of 3

Consider the arbitrary field $F$ and let $a(x) = a_0 + a_1x + \cdots + a_nx^n \in F(x)$. The derivative of $a(x)$ will be given by

$$a'(x) = a_1 + 2a_2x + \cdots + na_nx^{n-1} \in F(x).$$

Suppose that $F$ has characteristic 0 and $a'(x) = 0$. Objective is to prove that $a(x)$ is a constant polynomial.

Since characteristic of $F$ is 0, therefore there does not exist any positive integer $n$ such that $n \cdot a = 0$ for any $a \in F$.

Comment

**Step 2** of 3

Also $a_1 + 2a_2x + \cdots + na_nx^{n-1} = 0$. Since $n \cdot a \neq 0$ for any $a \in F$, therefore the condition $a'(x) = 0$ implies that all the coefficients must be zero. That is, $a_i = 0$ for all $i = 1, \ldots, n$. Thus, in $a(x)$, only the constant term is arbitrary rest all will be zero.

Hence, $a(x)$ will be a constant polynomial.

Consider the field $Z_5$, whose characteristic is 5. Let

$$a(x) = x^{15} + 3x^{10} + 4x^5 + 1 \in Z_5[x].$$

Note that $a'(x) = 15x^{14} + 30x^9 + 20x^4$. Since $15, 20, 30 \equiv 0$ in $Z_5$, therefore $a'(x) = 0$.

Comment

**Step 3** of 3

Hence, conclusion is not necessarily true if characteristic of $F$ is nonzero.

Comment