A Book of Abstract Algebra | (2nd Edition)

Chapter 32, Problem 3EE | Bookmark | Show all steps: ON

Problem

Use part 2 to list explicitly the six elements of $Gal(K:\mathbb{Q})$. Then write the table of $Gal(K:\mathbb{Q})$ and show that it is cyclic.

Step-by-step solution

Step 1 of 2

The objective is to list explicitly the six elements of $Gal(\mathbb{Q}(\omega):\mathbb{Q})$ where ω is the primitive seventh root of unity , write the table of $Gal(\mathbb{Q}(\omega):\mathbb{Q})$ and show that it is cyclic.

Comment

Step 2 of 2

Let $\sigma_j \in Gal(K:\mathbb{Q})$, where $K=\mathbb{Q}(\omega)$ be an automorphism such that $\sigma_j(\omega)=\omega^j$ for $j=1,2,3,4,5,6$.

Then $\sigma_j\sigma_k(\omega)=\sigma_j(\omega^k)$

$=(\omega^j)^k$

$=\omega^{jk}$

$=\sigma_m(\omega)$ where m is the product jk in $\mathbb{Z}_7$.

Thus $Gal(K:\mathbb{Q})$ is isomorphic to the group $\{1,2,3,4,5,6\}$ of nonzero elements of $\mathbb{Z}_7$ under multiplication.

It is cyclic of order 6 generated by σ_3 , since $\sigma_3(\omega)=\omega^3$, $\sigma_3^2(\omega)=\omega^2$, $\sigma_3^3(\omega)=\omega^6$, $\sigma_3^4(\omega)=\omega^4$, $\sigma_3^5(\omega)=\omega^5$ and $\sigma_3^6(\omega)=\omega$.

So，$Gal(K:\mathbb{Q})=\{\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6\}$.

The Galois group $Gal(K:\mathbb{Q})=\{\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6\}$ is described by the following table:

$\times$	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
σ_1	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
σ_2	σ_2	σ_4	σ_6	σ_1	σ_3	σ_5
σ_3	σ_3	σ_6	σ_2	σ_5	σ_1	σ_4
σ_4	σ_4	σ_1	σ_5	σ_2	σ_6	σ_3
σ_5	σ_5	σ_3	σ_1	σ_6	σ_4	σ_2
σ_6	σ_6	σ_5	σ_4	σ_3	σ_2	σ_1

Comment