A Book of Abstract Algebra | (2nd Edition)

Problem

Let V be a finite-dimensional vector space. Let dim V designate the dimension of V. Prove each of the following:

The space spanned by $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is the same as the space spanned by $\{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ iff each $\mathbf{a}_i$ is a linear combination of $\mathbf{b}_1, \ldots, \mathbf{b}_m$, and each $\mathbf{b}j$ is a linear combination of $\mathbf{a}_1 \ldots, \mathbf{a}_n$.

Step-by-step solution

Step 1 of 4

Here it is to be proved that a vector space is spanned by set of vectors $(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ and $(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m)$ if each $\mathbf{a}_i$ is combination of $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m$ and each $\mathbf{b}_j$ is combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$.

Comment

Step 2 of 4

To prove this we assume that given vector space is spanned by both $(\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n)$ and $(\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_m)$. Then prove that each $\mathbf{a}_i$ is combination of $\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_m$ and each $\mathbf{b}_j$ is combination of $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n$.

Comment

Step 3 of 4

Let **v** be any vector in vector space *V*, then

$$\mathbf{v} = t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + ... + t_n\mathbf{a}_n$$
$$\mathbf{v} = k_1\mathbf{b}_1 + k_2\mathbf{b}_2 + ... + k_m\mathbf{b}_m$$

Since, all $(\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n)$ lies in vector space *V*, it can be said that for all $\mathbf{a}_i$,

$$\mathbf{a}_i = k_1\mathbf{b}_1 + k_2\mathbf{b}_2 + ... + k_m\mathbf{b}_m \ ; i \in (1, 2, ..., n)$$

$$\boxed{\text{Hence each } \mathbf{a}_i \text{ is a combination of } \mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_m}$$

Comment

Step 4 of 4

Also, all $(\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_m)$ lies in vector space *V*, it can be said that for all $\mathbf{b}_j$,

$$\mathbf{b}_j = t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + ... + t_n\mathbf{a}_n \ ; j \in (1, 2, ..., m)$$

$$\boxed{\text{Hence each } \mathbf{b}_j \text{ is a combination of } \mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n}$$

Comment