A Book of Abstract Algebra | (2nd Edition)

Problem

Let *F* be a field, and *K* a finite extension of *F*. Prove each of the following:

If *b* is algebraic over *K*, then $[K(b):F(b)] \le [K:F]$. [HINT: Note that $F \subseteq K \subseteq K(b)$ and $F \subseteq F(b) \subseteq K(b)$. Relate the degrees of the four extensions involved here, using part 3.]

Step-by-step solution

Step 1 of 4

Consider a field *F* and a finite extension *K* of *F*. Objective is to prove that if *b* is algebraic over *K*, then $[K(b):F(b)] \le [K:F]$.

Consider the result that if *b* is algebraic over *K*, then $[K(b):K] \le [F(b):F]$.

Since *b* is algebraic of *K*, therefore $F(b)$ is a finite extension of field *F*, and $K(b)$ is a finite extension of field *K*. Then, by extension property

$$F \subseteq F(b),$$
$$K \subseteq K(b).$$

Comment

Step 2 of 4

On combining the above relationship, one gets

$F \subseteq F(b) \subseteq K(b)$.

Also K is a finite extension of F, so $F \subseteq K$. And then,

$F \subseteq K \subseteq K(b)$.

Now by the formula for degree calculation of field,

$[K(b):F]=[K(b):F(b)]\cdot[F(b):F]$,

and

$[K(b):F]=[K(b):K]\cdot[K:F]$.

Comment

Step 3 of 4

Equate both the relations solve as follows:

$$\begin{aligned}[K(b):F(b)]\cdot[F(b):F]&=[K(b):K]\cdot[K:F]\\ \frac{[K(b):F(b)]}{[K:F]}&=\frac{[K(b):K]}{[F(b):F]},\\ \frac{[K(b):F(b)]}{[K:F]}&\leq 1,\\ [K(b):F(b)]&\leq[K:F]\end{aligned}$$

The third step is obtained from the condition that $[K(b):K]\leq[F(b):F]$, because it implies that $\frac{[K(b):K]}{[F(b):F]}\leq 1$.

Comment

Step 4 of 4

Hence, if *b* is algebraic over *K*, then $[K(b):F(b)] \leq [K:F]$.

Comment