A Book of Abstract Algebra | (2nd Edition)

Chapter 24, Problem 1ED

Problem

In each of the following, let *A* be an integral domain:

Prove that if *A* has characteristic *p*, then *A*[*x*] has characteristic *p*.

Step-by-step solution

Step 1 of 2

Consider an integral domain *A* has characteristic *p*. objective of the problem is prove $A[x]$ has characteristics *p*.

Step 2 of 2

Now definition of characteristic is given below.

Definition: The characteristic of a ring *R* is the least positive integer *n* such that $na = 0$ for all *a* in *R*. if no such integer exists, then the characteristic is 0.

Here *A* is an integral domain having characteristic *p*. That implies $pa = 0$ for all *a* in *A*.

Consider a polynomial $q(x)$ in $A[x]$.

$q(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$

Here, $a_i \in A \text{ for } i = 0,1,2,\ldots,n$.

Then $pa_i = 0 \text{ for } i = 0,1,\ldots,n$ (since $a_i \in A \text{ for } i = 0,1,2,\ldots,n$)

To prove *p* is the characteristic of $A[x]$, prove $pq(x) = 0$.

$$\begin{aligned} pq(x) &= p\left(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0\right) \\ &= pa_n x^n + pa_{n-1}x^{n-1} + \ldots + pa_0 \end{aligned}$$

Since $pa_i = 0 \text{ for } i = 0,1,\ldots,n$ implies

$$\begin{aligned} pq(x) &= 0x^n + 0x^{n-1} + \ldots + 0a_0 \\ &= 0 \end{aligned}$$

Therefore, if an integral domain *A* has characteristic *p* then $A[x]$ has characteristics *p*.

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