A Book of Abstract Algebra | (2nd Edition)

Chapter 29, Problem 2EG

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Problem

Let $F \subseteq K$ and $a, b \in K$. We have seen on page 295 that if a and b are algebraic over F, then $F(a, b)$ is a finite extension of F.

Use the above to prove part.

The set $\{x \in K : x \text{ is algebraic over } F\}$ is a subfield of K, containing F.

Any complex number which is algebraic over $\mathbb{Q}$ is called an *algebraic number*. By part 2, the set of all the algebraic numbers is a field, which we shall designate by $\mathbb{A}$.

Let $a(x) = a_0 + a_1x + \cdots + a_nx^n$ be in $\mathbb{A}[x]$, and let c be any root of $a(x)$. We will prove that $c \in \mathbb{A}$.

To begin with, all the coefficients of $a(x)$ are in $\mathbb{Q}(a_0, a_1, \ldots, a_n)$.

Step-by-step solution

Step 1 of 2

Consider a field F and an extension K of F. The objective is to prove that the set $E = \{x \in K : x \text{ is algebraic over } F\}$ is a subfield of K, containing F.

Comment

Step 2 of 2

Take $a, b \in E$.

Consider the subfield $F(a,b)$ of K.

$F(a,b)$ is a finite extension ,since a and b are algebraic over F and hence an algebraic extension.

Thus ,all the elements of $F(a,b)$ are algebraic over F and so $F(a,b) \subseteq E$.

The elements $a+b$, $a-b$, ab and $1/a(a \neq 0)$ lie in $F(a,b)$,and thus also in E.

So , E is a subfield of K.

Clearly , $F \subseteq E$,since any $\alpha \in F$ is a zero of polynomial $X-\alpha \in F[x]$ and therefore algebraic over F.

Comment