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Chapter 32, Problem 6EH   Bookmark   Show all steps: ON

Problem

Prove that the identity function and the function $a + bi \to a - bi$ are the only automorphisms $\mathbb{C}$ of which fix $\mathbb{Q}$.

Step-by-step solution

Step 1 of 2

The objective is to prove that the only automorphism of $\mathbb{C}$ are the identity function and the function $a+ib \to a-ib$.

Comment

Step 2 of 2

Let $f:\mathbb{C}\to\mathbb{C}$ be an automorphism.

Then $f(x)=x \quad \forall\, x\in\mathbb{Q}$.

Let $z=a+ib\in\mathbb{C}$ where $a,b\in\mathbb{R}$.

Then $f(z)=f(a+ib)$

$=f(a)+f(ib)$, since f is a homomorphism

$=f(a)+f(i)f(b)$

$=a+f(i)b$, since $f(x)=x \quad \forall\, x\in\mathbb{Q}$.

Consider $[f(i)]^2=f(i)f(i)$

$= f(i \cdot i)$

$= f(i^2)$

$= f(-1)$

$= -1.$

$[f(i)]^2 = -1.$

Therefore， $f(i) = \pm i.$

Thus， $f(z) = a + f(i)b$

$= a + ib$ or $a - ib.$

Hence， $f(z) = a + ib = z$ gives the identity mapping and $f(z) = a - ib = \bar{z}$ gives conjugate map.

Comment