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Chapter 33, Problem 2EA

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Problem

Show that the following polynomials in $\mathbb{Q}[x]$ are not solvable by radicals:

(*a*) $2x^5 - 5x^4 + 5$

(*b*) $x^5 - 4x^2 + 2$

(*c*) $x^5 - 4x^4 + 2x + 2$

Step-by-step solution

Step 1 of 6

Here, objective is to prove that the polynomials in $Q(x)$ are not solvable by radicals.

Comment

Step 2 of 6

A polynomial equation is solvable by radicals, if its roots are determined by applying finite number of additions, subtractions, multiplications, divisions, n^{th} roots to the integers.

Galois group:

If a polynomial $f(x) \in Q(x)$ has degree *p* and has only two non real roots , then its Galois group is S_p.

If the polynomial whose Galois group is S_5 they are not solvable by radicals.

Comment

Step 3 of 6

Generally the polynomials with degree five cannot be solved by radicals, except the polynomials of the form $x^5 - px + q$

Comment

Step 4 of 6

(a)

Consider the polynomial $2x^5 - 5x^4 + 5$

The above polynomial is irreducible over Q by Eisenstein's criterion.

Then the Galois group is symmetric

And has two complex roots and three real roots.

The polynomial has degree $p = 5$ and has only two non real roots, then its Galois group is S_5.

$G(K / F) = S_5$

So, they are not solvable by radicals.

Hence,

The polynomials in $Q(x)$ are not solvable by radicals.

Comment

Step 5 of 6

(b)

Consider the complex number $x^5 - 4x^2 + 2$

The above polynomial is a degree five and it is irreducible over Q by Eisenstein's criterion.

Its Galois group is S_5.

The polynomial is not converted in to the form $x^5 - px + q$

So, the polynomial is not solvable by radicals.

Hence,

The polynomials in $Q(x)$ are not solvable by radicals.

Comment

Step 6 of 6

(c)

Consider the complex number $x^5 - 4x^4 + 2x + 2$

The above polynomial is having degree five and it is irreducible over Q by Eisenstein's criterion.

Its Galois group is S_5.

$$G(K/F) = S_5$$

The polynomial is not converted in to the form $x^5 - px + q$

So, the polynomial is not solvable by radicals.

Hence,

The polynomials in $Q(x)$ are not solvable by radicals.

Comment