A Book of Abstract Algebra | (2nd Edition)

Problem

Let F be any field.

Prove part:

Suppose $F \subseteq K$ and $a \in K$. If $p(x)$ is a monic irreducible polynomial in $F[x]$, and $p(a) = 0$, then $p(x)$ is the minimum polynomial of a over F.

Step-by-step solution

Step 1 of 2

The objective is to prove that if $p(x)$ is a monic irreducible polynomial in $F[x]$, $p(a)=0$, then $p(x)$ is the minimum polynomial of a over F.

Comment

Step 2 of 2

For $p(x)$ to be the minimum polynomial of a over F, it needs to be the unique monic irreducible polynomial in $F[x]$ such that $p(a)=0$.

Let, if possible, there exists a monic irreducible polynomial $p'(x) \neq p(x)$ such that $p'(a)=0$.

Then, $p(a)=0=p'(a)$.

Since $p'(x) \neq p(x)$, then a is also a root of the polynomial $h(x)=p'(x)-p(x)$.

The leading terms of $p(x)$ and $p'(x)$ are same. Therefore, the degree of $h(x)$ is less than that of $p(x)$ and $p'(x)$. This is a contradiction because $p(x)$ and $p'(x)$ are of the least degree. Therefore, the assumption is wrong and $p(x)$ is the unique monic irreducible polynomial in $F[x]$ such that $p(a)=0$.

Hence, it is proved that if $p(x)$ is a monic irreducible polynomial in $F[x]$, $p(a)=0$, then $p(x)$ is the minimum polynomial of a over F.

Comment