A Book of Abstract Algebra | (2nd Edition)

Chapter 30, Problem 4EA

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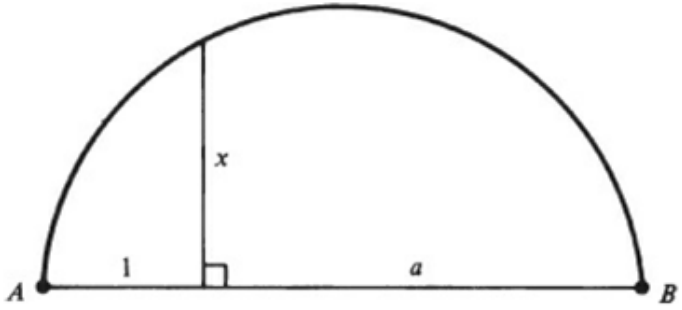
Problem

If O and I are any two points in the plane, consider a coordinate system such that the interval OI coincides with the unit interval on the x axis. Let $\mathbb{D}$ be the set of real numbers such that $a \in \mathbb{D}$ iff the point $(a, 0)$ is constructible from $\{O, I\}$. Prove the following:

If $a > 0$ and $a \in \mathbb{D}$, then $\sqrt{a} \in \mathbb{D}$. (HINT: In the accompanying figure, AB is the diameter of a circle. Use an elementary property of chords of a circle to show that $x = \sqrt{a}$.)



It follows from parts 1 to 4 that $\mathbb{D}$ is a field, closed with respect to taking square roots of positive numbers. $\mathbb{D}$ is called the *field of constructible numbers*.

Step-by-step solution

Step 1 of 3

Here, objective is to prove that, if $a \in D, a > 0$, then $\sqrt{a} \in D$.

Comment

Step 2 of 3

Constructible point:

This point is either the end point of given unit segment or it is the intersection of two lines determined by previous constructible points.

Comment

Step 3 of 3

Consider $a \in D$ if and only if the point $(a,0)$ is constructible from $\{O,I\}$

Now, we have to prove that length $\sqrt{a}$ is constructed from the length a

Consider the below figure:

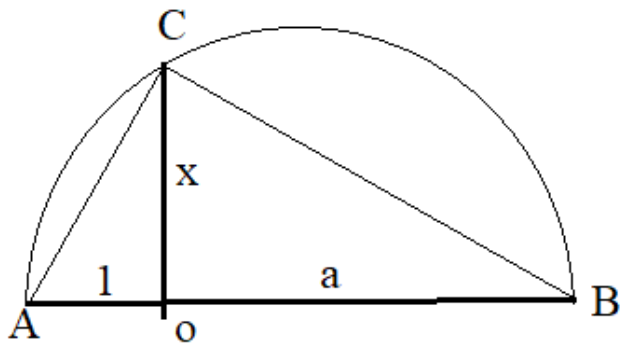


figure:construction of sqare root of a

AB is the diameter of a circle.

By observing there exist two similar triangles

ΔAOB and ΔBOC

Using the property of similar triangles, we have

$$\frac{1}{x} = \frac{x}{a}$$

$$x^2 = a$$

$$x = \sqrt{a}$$

Then, the length $\sqrt{a}$ is constructed from $\{O, I\}$ which implies $\sqrt{a} \in D$

Therefore, if $a \in D, a > 0$, then $\sqrt{a} \in D$.

Hence, proved

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