A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 8EN

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Problem

Let *G* be a finite group, and *K* a *p*-Sylow subgroup of *G*. Let *X* be the set of all the conjugates of *K*. See Exercise M2. If $C_1, C_2 \in X$, let $C_1 \sim C_2$ iff $C_1 = aC_2a^{-l}$ for some $\alpha \in K$

Conclude: Let *G* be a finite group of order $p^k m$, where *p* is not a factor of *m*. Every *p*-Sylow subgroup *K of G* has order p^k.

Combining part 8 with Exercise L gives

Let G be a finite group and let p be a prime number. For each n such that p^n *divides* $|G|$*, G has a subgroup of order* p^n.

This is known as Sylow's theorem.

Step-by-step solution

Step 1 of 3

Assume that *G* is a finite group and *K* a *p*-Sylow subgroup of *G*. Consider the set *X* as the set of all the conjugates of *K*. Define an equivalence relation as:

If $C_1, C_2 \in X$, let $C_1 \approx C_2$ if and only if $C_1 = aC_2a^{-1}$ for some $a \in K$.

Then from the previous result one have:

(1) The number of elements in $[C]$ is either 1 or a power of *p*.

(2) There is only one single element class, that is, the only class with a single element is $[K]$.

(3) The number of elements in *X* is $kp+1$, for some integer *k*.

(4) The $(G:K)$ is not a multiple of *p*.

Objective is to conclude that the every *p*- Sylow subgroup *K* of *G* has order p^k, where $|G| = p^k m$ and *p* is not a factor of *m*.

Comment

Step 2 of 3

For any prime *p* there exists a *p*-Sylow subgroup in *G*, may be possible trivial. One knows that for any *p*-Sylow subgroup *K* in *G*, $(G:K)$ is not divisible by *p*. Let $|K| = p^j$. Then

$$
\begin{aligned}
(G:K) &= \frac{|G|}{|K|} \\
&= \frac{p^k m}{p^j} \\
&= p^{(k-j)} m
\end{aligned}
$$

is not divisible by *p*. So, $k - j = 0$ (because there is no common factor between *p* and *m*). Then, $k = j$.

Comment

Step 3 of 3

So, for any prime dividing the order of finite group *G*, there is a subgroup *K* along with the condition that order of *K* will be the maximal power *p* dividing *G*. Since *K* has a subgroup of order some power of *p*. Therefore, if *G* is a finite group then for all prime powers, say p^i for all $i < k$, dividing $|G|$, *G* has a subgroup of order p^i.

Comment