A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 4EO

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Problem

The purpose of this exercise is to prove a property of cosets which is needed in Exercise Q. Let *G* be a finite abelian group, and let *a* be an element of *G* such that ord(*a*) is a multiple of ord(*x*) for every $x \in G$. Let $H = \langle a \rangle$. We will prove:

For every $x \in G$, there is some $y \in G$ such that $Hx = Hy$ and ord(y) = *ord*(Hy).

This means that every coset of *H* contains an element *y* whose order is the same as the coset's order.

Let *x* be any element in *G*, and let ord $(a) = t$, ord$(x) = s$, and ord $(Hx) = r$.

Setting $y = xa^{-UZ}$, prove that $Hx = Hy$ and ord(y) = r, as required.

Step-by-step solution

Step 1 of 4

Consider that *G* is a finite abelian group. Let $a, x \in G$ and $H = \langle a \rangle$ is a subgroup of *G*. Suppose that order of the elements are:

$$\begin{aligned}\operatorname{ord}(a) &= t,\\ \operatorname{ord}(x) &= s,\\ \operatorname{ord}(Hx) &= r.\end{aligned}$$

Note that *r* is the least positive integer such that $x^r = a^m$. Also $a^{mu} = e$, and it follows that $mu = tz$ for some integer *z*. And, $m = rvz$.

Objective is to prove that $Hx = Hy$ and $\operatorname{ord}(y) = r$, if $y = xa^{-vz}$.

Comment

Step 2 of 4

Let $y = xa^{-vz}$. Since group *G* and *H* are abelian, therefore

$xa^{-vz} = a^{-vz}x \cdot$

Then

$$\begin{aligned} Hy &= Hxa^{-vz} \\ &= Ha^{-vz}x \\ &= Hx. \end{aligned}$$

Since $Ha = H$ if and only if $a \in H$. Therefore, $Ha^{-vz} = H$ because $a^{-vz} \in H$. Thus, $Hx = Hy$.

Comment

Step 3 of 4

Next for proving $\operatorname{ord}(y) = r$, consider

$$\begin{aligned} y^r &= \left(xa^{-vz}\right)^r \\ &= x^r a^{-rvz} \\ &= a^m \cdot a^{-rvz} \end{aligned}$$

From the hypothesis, $m = rvz$. Substitute this value in above and get,

$$\begin{aligned} y^r &= a^{rvz} \cdot a^{-rvz} \\ &= a^0 \\ &= e. \end{aligned}$$

Thus, $\operatorname{ord}(y) = r$.

Comment

Step 4 of 4

Hence, if $y = xa^{-vz}$ then $Hx = Hy$ and $\operatorname{ord}(y) = r$.

Comment