A Book of Abstract Algebra | (2nd Edition)

Chapter 28, Problem 3EC

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Problem

Let *A* be the set of eight vectors (*x*, *y*, *z*) where *x*, *y*, *z* = 1, 2. Prove that *A* spans $\mathbb{R}^3$, , and find a subset of *A* which is a basis of $\mathbb{R}^3$.

Step-by-step solution

Step 1 of 2

Any set of basis is a set of vectors which are linearly independent and their number equals dimension of vector space. And any set is linearly independent if there exists no combination of these vectors which can give 0 vectors.

If $u_1, u_2, \ldots, u_n$ are *n* vectors of a vector space and these are linearly independent. Then for.

$$a_1u_1 + a_2u_2 + \ldots + a_nu_n = 0$$

All a_i have to be zero.

Comment

Step 2 of 2

Any given basis is linearly independent if matrix formed with vectors as row of matrix same rank as number of rows of matrix.

Matrix formed by given set of vectors as rows is,

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$$

Row reducing this matrix,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 2 \end{pmatrix} \xrightarrow{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \\ R_5 \to R_5 - R_1 \\ R_6 \to R_6 - R_1 \\ R_8 \to R_8 - R_1 \\ R_7 \to R_7 - R_1}} \begin{pmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{1} & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & \boxed{1} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

It can be easily seen that this matrix has many combination of pivots. One such combination have been highlighted. As there are 3 pivots, given matrix have rank 3 equal to dimension of $\mathbb{R}^3$.

$$\boxed{\text{Hence given set of vectors spans } \mathbb{R}^3}$$

Original vectors in pivot row positions forms one set of basis of $\mathbb{R}^3$

$$\boxed{\text{Hence one of many subsets that forms basis of } \mathbb{R}^3 \text{ is } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}}$$

Comment