A Book of Abstract Algebra | (2nd Edition)

Problem

We will show that $2\pi/5$ is a constructible angle, and it will follow that the regular pentagon is constructible.

Prove that $\omega^2 + \omega + 1 + \omega^{-1} + \omega^{-2} = 0$.

Step-by-step solution

Step 1 of 3

Here, objective is to prove that $\omega^2 + \omega + 1 + \omega^{-1} + \omega^{-2} = 0$.

Comment

Step 2 of 3

De Moivre's theorem:

$\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$ is a complex fifth root of unity.

Since $x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)$

ω is a root of $P(x)=(x^4+x^3+x^2+x+1)$

Comment

Step 3 of 3

Consider ω is a root of $P(x)=(x^4+x^3+x^2+x+1)$

Then,

$$P(\omega)=0$$
$$(\omega^4+\omega^3+\omega^2+\omega+1)=0$$
$$\omega^2\cdot\omega^2+\omega^2\cdot\omega+\omega^2\cdot 1+\omega^2\cdot\omega^{-1}+\omega^2\cdot\omega^{-2}=0$$
$$\omega^2(\omega^2+\omega+1+\omega^{-1}+\omega^{-2})=0$$

$$\omega^2=0 \text{ or}$$
$$\omega^2+\omega+1+\omega^{-1}+\omega^{-2}=0$$

Hence, proved

Comment