A Book of Abstract Algebra | (2nd Edition)

Chapter 31, Problem 4EA

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Problem

Explain: $\mathbb{Q}(i, \sqrt{2})$ is the root field of $x^4 - 2x^2 + 9$ over $\mathbb{Q}$, and is the root field of $x^2 - 2\sqrt{2}x + 3$ over $\mathbb{Q}(\sqrt{2})$.

Step-by-step solution

Step 1 of 3

The objective is to find the root field of $x^4 - 2x^2 + 9$ over $\mathbb{Q}$ and the root field of $x^2 - 2\sqrt{2}x + 3$ over $\mathbb{Q}(\sqrt{2})$.

Comment

Step 2 of 3

Take $y = x^2$.

$x^4 - 2x^2 + 9 = y^2 - 2y + 9$

Use $y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to find roots of $y^2 - 2y + 9$.

Here, $a = 1$, $b = -2$, $c = 9$.

$$y = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 9}}{2 \cdot 1}$$
$$= \frac{2 \pm \sqrt{-32}}{2}$$
$$= 1 \pm 2\sqrt{2}i$$

$x^2 = 1 \pm 2\sqrt{2}i$

$x = \sqrt{2} + i, x = -\sqrt{2} - i, x = -\sqrt{2} + i, x = \sqrt{2} - i$

The roots of $x^4 - 2x^2 + 9$ are $\sqrt{2}+i$, $-\sqrt{2}-i$, $-\sqrt{2}+i$, $\sqrt{2}-i$.

Therefore, the root field of $x^4 - 2x^2 + 9$ over $\mathbb{Q}$ is $\mathbb{Q}\left(\sqrt{2}+i, -\sqrt{2}-i, -\sqrt{2}+i, \sqrt{2}-i\right)$. This can be written simply as $\mathbb{Q}\left(\sqrt{2}, i\right)$.

Comment

Step 3 of 3

Use $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to find roots of $x^2 - 2\sqrt{2}x + 3$.

Here, $a = 1$, $b = -2\sqrt{2}$, $c = 3$.

$$\begin{aligned} x &= \frac{-\left(-2\sqrt{2}\right) \pm \sqrt{\left(-2\sqrt{2}\right)^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1} \\ &= \frac{2\sqrt{2} \pm \sqrt{-4}}{2} \\ &= \sqrt{2} \pm i \end{aligned}$$

In $\mathbb{Q}\left(\sqrt{2}\right)$, $\sqrt{2} \pm i$ are roots of $x^2 - 2\sqrt{2}x + 3$.

Therefore, the root field of $x^2 - 2\sqrt{2}x + 3$ over $\mathbb{Q}\left(\sqrt{2}\right)$ is $\mathbb{Q}\left(\sqrt{2} \pm i\right)$. This can be written simply as $\mathbb{Q}\left(\sqrt{2}, i\right)$.

Comment