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Chapter 16, Problem 1EO

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Problem

The purpose of this exercise is to prove a property of cosets which is needed in Exercise Q. Let G be a finite abelian group, and let a be an element of G such that ord(a) is a multiple of ord(x) for every $x \in G$. Let $H = \langle a \rangle$. We will prove:

For every $x \in G$, there is some $y \in G$ such that $Hx = Hy$ and ord(y) = *ord*(Hy).

This means that every coset of H contains an element y whose order is the same as the coset's order.

Let x be any element in G, and let ord $(a) = t$, ord$(x) = s$, and ord $(Hx) = r$.

Explain why r is the least positive integer such that xr equals some power of a, say $x^r = a^m$.

Step-by-step solution

Step 1 of 3

Consider that G is a finite abelian group. Let $a, x \in G$ and $H = \langle a \rangle$ is a subgroup of G. Suppose that order of the elements are:

$$\begin{aligned} &\operatorname{ord}(a) = t, \\ &\operatorname{ord}(x) = s, \\ &\operatorname{ord}(Hx) = r. \end{aligned}$$

Objective is to explain the reason that r is the least positive integer such that x^r equals some power of a, say $x^r = a^m$.

Comment

Step 2 of 3

Observe that the Hx denotes the coset of the quotient group G/H. By the definition of quotient group, one have that the identity of group G/H is equal to H.

Since $\operatorname{ord}(Hx) = r$, so

$$(Hx)^r = H$$
$$Hx^r = H.$$

Since $Ha = H$ if and only if $a \in H$, therefore the equation $Hx^r = H$ implies that

$x^r \in H$.

Since $H = \langle a \rangle$, therefore every element of *H* will be some power of *a*. So, x^r will also be equal to some power of *a*, say a^m.

Since *r* is the order of coset Hx, therefore *r* will be the least positive integer such that $Hx^r = H$.

Comment

Step 3 of 3

Hence, the *r* is the least positive integer such that $x^r = a^m$.

Comment