A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 6EN

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Problem

Let *G* be a finite group, and *K* a *p*-Sylow subgroup of *G*. Let *X* be the set of all the conjugates of *K*. See Exercise M2. If C_1, $C_2 \in X$, let $C_1 \sim C_2$ iff $C_1 = aC_2a^{-l}$ for some $\alpha \in K$

Prove that (*N: K*) is *not* a multiple of *p*. (Use Exercises K and M5.)

Step-by-step solution

Step 1 of 3

Assume that *G* is a finite group and *K* a *p*-Sylow subgroup of *G*. Consider the set *X* as the set of all the conjugates of *K*. Define an equivalence relation as:

If $C_1, C_2 \in X$, let $C_1 \approx C_2$ if and only if $C_1 = aC_2a^{-1}$ for some $a \in K$.

Note that, no non-identity element of N/K has order a power of *p*.

Objective is to prove that $(N:K)$is not a multiple of *p*, where $N = N(K)$.

Comment

Step 2 of 3

Suppose, for the sake of contradiction, that *p* divides $(N:K)$. Then, by the Cauchy theorem, N/K must have an element whose order is *p*. Let $mK \in N/K$ is a coset of order *p*. That is, $(mK)^p = e$, or $m^p K = e$.

So, $m^p \in K$ and also $(m^p)^{p^i} = e$, or equivalently, $m^{p^{i+1}} = e$.

This implies that $m \in K$ and order of mK equals 1, not *p*. This is because N/K does not have any element whose order is a power of *p*. But this contradicts the implication of Cauchy theorem here.

Comment

Step 3 of 3

Hence, $(N:K)$ is not a multiple of *p*.

Comment