A Book of Abstract Algebra | (2nd Edition)

Chapter 30, Problem 2EF

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Problem

By de Moivre's theorem,

$$\omega = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$$

is a complex seventh root of unity. Since

$$x^7 - 1 = (x-1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$$

ω is a root of $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$.

Prove that

$$8\cos^3\frac{2\pi}{7} + 4\cos^2\frac{2\pi}{7} - 4\cos\frac{2\pi}{7} - 1 = 0$$

(Use part 1 and Exercise El.) Conclude that cos (2π/7) is a root of $8x^3 + 4x^2 - 4x - 1$.

Step-by-step solution

Step 1 of 5

Here, objective is to prove that $8\cos^3\frac{2\pi}{7}+4\cos^2\frac{2\pi}{7}-4\cos\frac{2\pi}{7}-1=0$ and $\cos\frac{2\pi}{7}$ is a root of

$8x^3+4x^2-4x-1=0$

Comment

Step 2 of 5

De Moivre's theorem:

$\omega=\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}$ is a complex seventh root of unity.

Since $x^7-1=(x-1)(x^6+x^5+x^4+x^3+x^2+x+1)$

ω is a root of $P(x)=(x^6+x^5+x^4+x^3+x^2+x+1)$

Comment

Step 3 of 5

Consider $\omega=\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}$

$$\frac{1}{\omega}=\cos\frac{2\pi}{7}-i\sin\frac{2\pi}{7}$$

$$\left(\omega+\frac{1}{\omega}\right)=\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}+\cos\frac{2\pi}{7}-i\sin\frac{2\pi}{7}$$

$$\left(\omega+\frac{1}{\omega}\right)=2\cos\frac{2\pi}{7}$$

Comment

Step 4 of 5

Consider ω is a root of $P(x) = (x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$

Then, $P(\omega) = 0$

$$(\omega^4 + \omega^3 + \omega^2 + \omega + 1) = 0$$
$$\omega^2(\omega^2 + \omega + 1 + \omega^{-1} + \omega^{-2}) = 0$$

$$\omega^2 = 0 \text{ or}$$
$$\omega^3 + \omega^2 + \omega + 1 + \omega^{-1} + \omega^{-2} + \omega^{-3} = 0$$

$$\left(\omega + \frac{1}{\omega}\right)^3 + \left(\omega + \frac{1}{\omega}\right)^2 - 2\left(\omega + \frac{1}{\omega}\right) - 1 = 0$$

$$\left(2\cos\frac{2\pi}{7}\right)^3 + \left(2\cos\frac{2\pi}{7}\right)^2 - 2\left(2\cos\frac{2\pi}{7}\right) - 1 = 0$$

$$8\cos^3\frac{2\pi}{7} + 4\cos^2\frac{2\pi}{7} - 4\cos\frac{2\pi}{7} - 1 = 0$$

Comment

Step 5 of 5

put $x = \cos\frac{2\pi}{7}$ in above equation, then

$$8x^3 + 4x^2 - 4x - 1 = 0$$

Hence, $8\cos^3\frac{2\pi}{7} + 4\cos^2\frac{2\pi}{7} - 4\cos\frac{2\pi}{7} - 1 = 0$

and $\cos\frac{2\pi}{7}$ is a root of $8x^3 + 4x^2 - 4x - 1 = 0$

Comment