A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 3EG

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Problem

If H is a subgroup of a group G, let X designate the set of all the left cosets of H in G. For each element $a \in G$, define p_a: $X \to X$ as follows:

$$p_a(xH) = (ax)H$$

Prove that the set $\{a \in H: xax^{-1} \in H$ for every $x \in G\}$, that is, the set of all the elements of H whose conjugates are all in H, is the kernel of h.

Step-by-step solution

Step 1 of 4

Assume that G be a group and H be its subgroup. Consider that X is the set of all the left cosets of H in G. Define a mapping, for some $a \in G$, $p_a : X \to X$ by

$$p_a(xH) = (ax)H .$$

Consider the following homomorphism mapping

$$h : G \to S_X$$

defined by

$$h(a) = p_a .$$

Objective is to prove that $\ker h = \{a \in H : xax^{-1} \in H \text{ for every } x \in G\}$. One can prove this result by containment property. That is, prove that $\ker h \subset H$ and $H \subset \ker h$.

Comment

Step 2 of 4

Let $x \in \ker h$. Then by the definition of kernel,

$$h(x) = p_e .$$

By the mapping *h*, $h(x) = p_x$. So, $p_x = p_e$. Then,

$$p_x(yH) = p_e(yH)$$
$$(xy)H = yH.$$

Now by the coset property, if $aH = bH$ then $b^{-1}a \in H$, it implies that

$$y^{-1}xy \in H.$$

Since *y* was arbitrary, therefore it implies an equivalent condition that all conjugates of $x \in \ker h$ must lie in *H*. Thus, $\ker h \subset H$.

Comment

Step 3 of 4

Conversely, let $x \in H$. Then

$$\begin{aligned} p_x(yH) &= (xy)H \\ &= xHyH \end{aligned}$$

Since $x \in H$, therefore by coset property $xH = H$. So,

$$\begin{aligned} p_x(yH) &= HyH \\ &= yH. \end{aligned}$$

It implies that, $p_x = p_e$ and then $x \in \ker h$. That is, $H \subset \ker h$. And therefore, $\ker h = H$.

Comment

Step 4 of 4

Hence, the kernel of homomorphism *h* will be the set of all elements of *H* whose conjugates are all in *H*.

Comment