A Book of Abstract Algebra | (2nd Edition)

Problem

Let *f* be a homomorphism from G onto *H* with kernel *K*:

$$f : G \underset{K}{\twoheadrightarrow} H$$

If *S* is any subgroup of *H*, let $S^* = \{x \in G: f(x) \in S\}$. Prove:

$$S \cong S^*/K.$$

Step-by-step solution

Step 1 of 3

Suppose that *G* is any group. Let the mapping

$$f : G_K \to H$$

is a homomorphism from *G* onto *H* with kernel *K*. Assume that *S* is any subgroup of *H* and consider the following set:

$$S^* = \{x \in G : f(x) \in S\}.$$

Note that, the set S^* forms a subgroup of *G*. Consider the following restriction map $g : S^* \to S$ defined as

$g(x) = f(x)$ for every $x \in S^*$.

Objective is to prove that $S \cong S^* / K$.

According to the fundamental homomorphism theorem, if $f : G \to H$ is a homomorphism of *G* onto *H*, with kernel *K* then

$$H \cong G / K.$$

Comment

Step 2 of 3

First prove that the restriction map *g* is a homomorphism from S^* onto *S* with $K = \ker g$.

Assume that $x, y \in S^*$. Then use the homomorphism of mapping *f* in the following manner:

$$\begin{aligned} g(ab) &= f(ab) \\ &= f(a)\cdot f(b) \\ &= g(a)\cdot g(b). \end{aligned}$$

This shows that *g* is homomorphism also onto because *f* is onto.

Since codomain of *g* is same as mapping *f* and *K* is the kernel of *f*, therefore $K = \ker g$.

Comment

Step 3 of 3

Hence, by fundamental homomorphism theorem it conclude that $S \cong S^* / K$.

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