A Book of Abstract Algebra | (2nd Edition)

Chapter 28, Problem 1EF

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Problem

Let U and V be vector spaces over the field F, with dim $U = n$ and dim $V = m$. Let $h : U \to V$ be a homomorphism.

Prove the following:

Let h be injective. If $\{\mathbf{a}_1 .. ., \mathbf{a}_r\}$ is a linearly independent subset of U, then $\{h(\mathbf{a}_1), \ldots, h(\mathbf{a}_r)\}$ is a linearly independent subset of V.

Step-by-step solution

Step 1 of 3

It is already known that U and V are vector spaces and so they satisfies all conditions for vector space. It is known that basis of U contains n elements. Thus, dimension of U is n.

Comment

Step 2 of 3

Linear transformation h is said to be injective if,

$$h(\mathbf{a}) = h(\mathbf{b}) \Rightarrow \mathbf{a} = \mathbf{b}$$

Comment

Step 3 of 3

It is known that subset $(\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_r)$ is linearly independent. Then for,

$$t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + ...t_r\mathbf{a}_r = \mathbf{0} \qquad ...(1)$$

All t_i are 0.

Taking linear transformation of equation (1).

$$h(t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + ...t_r\mathbf{a}_r) = h(\mathbf{0})$$
$$\Rightarrow h(t_1\mathbf{a})_1 + h(t_2\mathbf{a}_2) + ...h(t_r\mathbf{a}_r) = h(\mathbf{0})$$
$$\Rightarrow t_1 h(\mathbf{a})_1 + t_2 h(\mathbf{a}_2) + ...t_r h(\mathbf{a}_r) = h(\mathbf{0})$$
$$\Rightarrow t_1 h(\mathbf{a})_1 + t_2 h(\mathbf{a}_2) + ...t_r h(\mathbf{a}_r) = \mathbf{0}_v$$

Since *h* is injective, each independent vector is mapped into a different vector. Thus only possible solution for t_i is

$$t_i = 0 \ \forall i$$

$$\boxed{\text{Hence } \{h(\mathbf{a})_1, h(\mathbf{a}_2), ...h(\mathbf{a}_r)\} \text{ is independent subset of } V}$$

Comment