A Book of Abstract Algebra | (2nd Edition)

Chapter 33, Problem 4EB

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Problem

Let *G* be a group. The symbol $H \lhd G$ is commonly used as an abbreviation of "*H* is a *normal* subgroup of *G*." A *normal series* of *G* is a finite sequence $H_0, H_1, \ldots, H_n$ of subgroups of *G* such that

$$\{e\} = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G$$

Such a series is called a *solvable series* if each quotient group H_{i+1}/H_i is abelian. *G* is called a *solvable group* if it has a solvable series.

Use parts 2 and 3 to prove: Every subgroup of a solvable group is solvable.

Step-by-step solution

Step 1 of 4

Here, objective is to prove that every subgroup of a solvable group is a solvable group.

Comment

Step 2 of 4

A group *G* is solvable, if there exist a finite chain of successive subgroups $1 = G_0 \le G_1 \le G_2 \le \ldots\ldots\ldots\ldots \le G_n$ having the following properties.

G_i is the normal subgroup of $G_{i+1}; \forall \quad 0 \le i \le n-1$

$\frac{G_{i+1}}{G_i}$ is an Abelian group $\forall \quad 0 \le i \le n-1$

Comment

Step 3 of 4

Let the group G is solvable and H is a subgroup of G.

Comment

Step 4 of 4

Let $\{e\} = G_0 \lhd G_1 \lhd \ldots\ldots\ldots \lhd G_n = G$ Which is a normal series of G with abelian quotients.

$$(H \cap G_i) \cap G_{i-1} = (H \cap G_{i-1}) \qquad (\because \text{second isomorphism theorem for groups})$$
$$\text{where } i = 1, 2, \ldots. n$$
$$\frac{(H \cap G_i) \cap G_{i-1}}{G_{i-1}} \cong \frac{(H \cap G_i)}{(H \cap G_{i-1})}$$

That is $(H \cap G_{i-1})$ is a normal subgroup of $(H \cap G_i)$

$$(H \cap G_i) \cap G_{i-1} \subseteq G_i$$

$$\frac{(H \cap G_i) \cap G_{i-1}}{G_{i-1}} \leq \frac{G_i}{G_{i-1}}$$

We have, $\dfrac{G_i}{G_{i-1}}$ is a abelian group.

We know that subgroup of Abelian group is abelian.

$\dfrac{(H \cap G_i) \cap G_{i-1}}{G_{i-1}}$ is abelian group, So $\dfrac{(H \cap G_i)}{(H \cap G_{i-1})}$ is abelian group.

Then, the series $H \bigcap G_0 \lhd H \bigcap G_1 \lhd \ldots\ldots\ldots \lhd H \bigcap G_n = G$ is a normal series of H.

Therefore, H is solvable.

Comment