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Chapter 16, Problem 2EG

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Problem

If H is a subgroup of a group G, let X designate the set of all the left cosets of H in G. For each element $a \in G$, define $p_a: X \to X$ as follows:

$$p_a(xH) = (ax)H$$

Prove that $h: G \to S_X$ defined by $h(a) = p_a$ is a homomorphism.

Step-by-step solution

Step 1 of 3

Assume that G be a group and H be its subgroup. Consider that X is the set of all the left cosets of H in G. Define a mapping, for some $a \in G$, $p_a : X \to X$ by

$p_a(xH) = (ax)H$.

Objective is to prove that the mapping $h : G \to S_X$ defined by $h(a) = p_a$ is a homomorphism.

If G and H are two groups, a homomorphism from G to H is a function $f : G \to H$ such that for any two elements a, b in G,

$f(ab) = f(a)f(b)$.

Comment

Step 2 of 3

Let $x, y \in G$. Then $h(xy) = p_{xy}$.

Let $p_x, p_y \in S_X$. Then

$$\begin{aligned}(p_x p_y)(aH) &= p_x(p_y(aH)) \\ &= p_x((ya)H) \\ &= (xya)H \\ &= p_{xy}(aH).\end{aligned}$$

That is, $p_x p_y = p_{xy}$. Use this condition in $h(xy) = p_{xy}$ and get,

$$\begin{aligned}h(xy) &= p_{xy} \\ &= p_x p_y \\ &= h(x)h(y).\end{aligned}$$

Comment

Step 3 of 3

Hence, the mapping $h : G \to S_X$ defined by $h(a) = p_a$ is a homomorphism.

Comment