A Book of Abstract Algebra | (2nd Edition)

Chapter 17, Problem 1ED

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Problem

If D is a set, then the power set of D is the set P_D of all the subsets of D. Addition and multiplication are defined as follows: If A and B are elements of P_D (that is, subsets of D), then

$$A + B = (A - B) \cup (B - A) \text{ and } AB = A \cap B$$

It was shown in Chapter 3, Exercise C, that P_D with addition alone is an abelian group.

Prove: P_D is a commutative ring with unity. (You may assume ∩ is associative; for the distributive law, use the same diagram and approach as was used to prove that addition is associative in Chapter 3, Exercise C.)

Step-by-step solution

Step 1 of 4

Consider that P_D is the power set of set D, that is, P_D is set of all subsets of D. Let $A, B \in P_D$, then the addition and multiplication in P_D will be defined as follows:

$$A + B = (A - B) \cup (B - A),$$
$$AB = A \cap B.$$

Objective is to show that P_D is a commutative ring with unity.

It is given that $(P_D, +)$ forms an abelian group.

Comment

Step 2 of 4

Now, show that product of elements in P_D is associative. Let $A, B, C \in P_D$. Then

$$\begin{aligned}(AB)C &= (AB) \cap C \\ &= (A \cap B) \cap C \\ &= A \cap (B \cap C) \\ &= A(BC)\end{aligned}$$

The third step is obtained from the fact that intersection is associative.

Next is distributive law:

$$\begin{aligned} A(B+C) &= A\cap((B-C)\cup(C-B)) \\ &= (A\cap B - A\cap C)\cup(A\cap C - A\cap B) \\ &= (A\cap B)+(A\cap C) \\ &= AB + AC \end{aligned}$$

The third and fourth step is obtained from the given definition of addition and multiplication. Similarly, $(B+C)A = BA + CA$.

Comment

Step 3 of 4

Since intersection operation is commutative,

$$\begin{aligned} AB &= A\cap B \\ &= B\cap A \\ &= BA. \end{aligned}$$

Thus, P_D is commutative. The unity in P_D will be:

$$\begin{aligned} AB &= A \\ A\cap B &= A \end{aligned}$$

The condition $A\cap B = A$ will true for all $A \in P_D$ if $B = D$ because

$$\begin{aligned} AD &= A\cap D \\ &= A, \\ DA &= D\cap A \\ &= A. \end{aligned}$$

Thus, *D* will work as a unity in P_D.

Comment

Step 4 of 4

Hence, P_D is a commutative ring with unity.

Comment