A Book of Abstract Algebra | (2nd Edition)

Chapter 17, Problem 2EA | 2 Bookmarks | Show all steps: ON

Problem

In each of the following, a set *A* with operations of addition and multiplication is given. *Prove that A satisfies all the axioms to be a commutative ring with unity. Indicate the zero element, the unity, and the negative of an arbitrary a.*

A is the set $\mathbb{Q}$ of the rational numbers, and the operations are $\oplus$ and $\odot$ defined as follows:

$a \oplus b = a + b + 1 \quad a \odot b = ab + a + b$

Step-by-step solution

Step 1 of 5

Consider the set *A* is the set of rational numbers, with the following addition and multiplication:

$$a \oplus b = a + b + 1,$$
$$a \otimes b = ab + a + b.$$

Objective is to show that *A* satisfies all the axioms to be a commutative ring with unity.

Write explicitly the zero element, the unity, and the negative of an arbitrary *a*.

First show that $(A, \oplus)$ is an abelian group.

(1) Since sum of rationals is rational number, therefore $a \oplus b$ is closed in *A*.

(2) Associative: Let $a, b, c \in A$. Then

$$\begin{aligned}(a \oplus b) \oplus c &= a \oplus (b \oplus c)\\ (a+b+1) \oplus c &= a \oplus (b+c+1)\\ (a+b+1)+c+1 &= a+(b+c+1)+1\\ a+b+c+2 &= a+b+c+2.\end{aligned}$$

Since both the sides are equals, so addition is associative in *A*.

(3) Since addition is commutative in integers, so

$$\begin{aligned}a \oplus b &= a+b+1\\ &= b+a+1\\ &= b \oplus a.\end{aligned}$$

(4) Additive identity or zero element: take $b=-1$,

$$\begin{aligned} a \oplus e &= a \\ a+e+1 &= a \\ e &= -1 \end{aligned}$$

Thus, zero element of *A* will be -1.

(5) Let for every *a* in *A*, the negative of *a* is *b* then

$$\begin{aligned} a \oplus b &= e \\ a+b+1 &= -1 \\ b &= -2-a. \end{aligned}$$

Thus, negative of *a* will be $-2-a$.

And from here it conclude that, *A* is an abelian group.

Comment

Step 2 of 5

Now, show that $\otimes$ is associative. Let $a,b,c \in A$. Then

$$\begin{aligned} (a \otimes b) \otimes c &= (ab+a+b) \otimes c \\ &= (ab+a+b)c+ab+a+b+c \\ &= abc+ac+bc+ab+a+b+c, \end{aligned}$$

and

$$\begin{aligned} a \otimes (b \otimes c) &= a \otimes (bc+b+c) \\ &= a(bc+b+c)+a+bc+b+c \\ &= abc+ac+bc+ab+a+b+c \end{aligned}$$

Since both the sides are equals, so multiplication is associative in *A*.

Comment

Step 3 of 5

Next is distributive law:

$$\begin{aligned} a \otimes (b \oplus c) &= a \otimes (b+c+1) \\ &= a(b+c+1)+a+b+c+1 \\ &= ab+ac+2a+b+c+1 \end{aligned}$$

And

$$\begin{aligned} (a \otimes b) \oplus (a \otimes c) &= (ab+a+b) \oplus (ac+a+c) \\ &= ab+a+b+ac+a+c+1 \\ &= ab+ac+2a+b+c+1 \end{aligned}$$

Next, show that $\otimes$ is commutative. Let $a,b \in A$. Then

$$\begin{aligned} a \otimes b &= ab + a + b \\ &= ba + b + a \\ &= b \otimes a. \end{aligned}$$

Since addition $\oplus$, multiplication $\otimes$ both are commutative, therefore $(b \oplus c) \otimes a = (b \otimes a) \oplus (c \otimes a)$ automatically holds.

Comment

Step 4 of 5

Let the unity of non-identity element *a* in *A* is *b* then,

$$\begin{aligned} a \otimes b &= a \\ ab + a + b &= a \\ b(a+1) &= 0 \end{aligned}$$

Since *a* is non-identity arbitrary element, so $b = 0$. Thus, $a \otimes 0 = a$.

Comment

Step 5 of 5

Hence, $(A, \oplus, \otimes)$ form a commutative ring with the zero element -1, the unity is 0, and the negative of an arbitrary *a* is $-2-a$.

Comment