A Book of Abstract Algebra | (2nd Edition)

Chapter 32, Problem 4EE

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Problem

List all the subgroups of $Gal(K:\mathbb{Q})$, with their fixfields. Exhibit the Galois correspondence.

Step-by-step solution

Step 1 of 3

The objective is to list all the subgroups of $Gal\left(\mathbb{Q}(\omega):\mathbb{Q}\right)$ where ω is the primitive seventh root of unity , with their fix fields and exhibit the Galois correspondence.

Comment

Step 2 of 3

The Galois group $Gal\left(\mathbb{Q}(\omega):\mathbb{Q}\right)=\{\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6\}$,where $\sigma_j(\omega)=\omega^j$ for $j=1,2,3,4,5,6$ .Clearly $\alpha=\omega+\omega^2+\omega^4$ is left fixed by $\{\sigma_1,\sigma_2,\sigma_4\}$.

Now , $\alpha^2=\omega^2+\omega^4+\omega+2\omega^3+2\omega^6+2\omega^5$ and $\alpha=\omega+\omega^2+\omega^4$.

Thus , $\alpha^2+\alpha=2\left(\omega^6+\omega^5+\omega^4+\omega^3+\omega^2+\omega+1\right)$.

Because ω is a zero of $\Phi_7(x)=x^6+x^5+x^4+x^3+x^2+x+1$, $\alpha^2+\alpha+2=0$.

The zeros of x^2+x+2 are $\left(-1\pm i\sqrt{7}\right)/2$ and so , $\mathbb{Q}(\alpha)=\mathbb{Q}\left(i\sqrt{7}\right)$.

Working in an analogous way for the subgroup $\{\sigma_1,\sigma_6\}$, $\beta=\omega+\omega^6$ is left fixed by this subgroup.

Now , $\beta^3=\left(\omega+\omega^6\right)^3=\omega^3+3\omega+3\omega^6+\omega^4$, $\beta=\left(\omega+\omega^6\right)^2=\omega^2+2+\omega^5$ and

$\beta=\omega+\omega^6$.

Because ω is a zero of $\Phi_7(x)=x^6+x^5+x^4+x^3+x^2+x+1$, $\beta^3+\beta^2-2\beta-1=0$.

Thus，β is a zero of x^3+x^2-2x-1 which is irreducible because it has no zero in $\mathbb{Z}$.

Comment

Step 3 of 3

The Galois Correspondence may be represented as follows:

Comment