A Book of Abstract Algebra | (2nd Edition)

Chapter 24, Problem 1EE

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Problem

Show that if *B* is a subring of *A*, then *B*[*x*] is a subring of *A*[*x*].

Step-by-step solution

Step 1 of 3

Consider a subring *B* of a ring *A*. the objective of the problem is to prove $B[x]$ is the subring of $A[x]$.

Comment

Step 2 of 3

Recall definition of subring and the theorem known as subring test.

Definition: A subset *S* of a ring *R* is a subring of R if *S* itself a ring with the operation of R.

Theorem 1(Subring test): A nonempty subset *S* of a ring *R* is a subring if $a-b$ and ab are in *S* whenever *a* and *b* are in *S*.

First prove $B[x]\subseteq A[x]$.

B is a subset of *A* implies for every element of *B* is the element *A*.

That is if $b\in B$ implies $b\in A$.

Let any polynomial $p(x)\in B[x]$. Now prove $p(x)\in A[x]$.

If $p(x)\in B(x)$ implies every coefficient of $p(x)$ is in *B*. since *B* is a subset of *A* implies the coefficients of $p(x)$ are also elements of *A*.

Then $p(x)$ is an element of $A[x]$.

Since $p(x)$ is chosen arbitrary implies for every element in $B[x]$ is an element in $A[x]$

Comment

Step 3 of 3

Let two polynomials $p(x)$ and $q(x)$ in $B[x]$.

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$$
$$q(x) = b_n x^n + b_{n-1}x^{n-1} + \ldots + b_0$$

To prove $B[x]$ is a subring of $A[x]$, it is sufficient to prove

$p(x) - q(x) \in B[x]$ and $p(x)q(x) \in B[x]$.

$$\begin{aligned} p(x)-q(x) &= \left(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0\right) - \left(b_n x^n + b_{n-1}x^{n-1} + \ldots + b_0\right) \\ &= \left(a_n - b_n\right)x^n + \left(a_{n-1} - b_{n-1}\right)x^{n-1} + \ldots + \left(a_0 - b_0\right) \end{aligned}$$

$a_n, a_{n-1}, \ldots, a_0, b_n, b_{n-1}, \ldots, b_0 \in B$ and *B* is a subring of *A*.

Then by using theorem 1 , $(a_n - b_n), (a_{n-1} - b_{n-1}), \ldots, (a_0 - b_0) \in B$. Therefore all coefficients of $p(x) - q(x)$ belongs to *B* then $p(x) - q(x) \in B[x]$

$$\begin{aligned} p(x)q(x) &= \left(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0\right)\left(b_n x^n + b_{n-1}x^{n-1} + \ldots + b_0\right) \\ &= \left(a_n b_n x^{2n} + a_n b_{n-1} x^{2n-1} + \ldots + b_0 a_n x^n\right) + \ldots + \left(a_0 b_n x^n + \ldots + a_0 b_0\right) \\ &= a_n b_n x^{2n} + \ldots + \left(a_0 b_2 + a_1 b_1 + a_2 b_0\right)x^2 + \left(a_0 b_1 + b_0 a_1\right)x + a_0 b_0 \end{aligned}$$

$a_n, a_{n-1}, \ldots, a_0, b_n, b_{n-1}, \ldots, b_0 \in B$ and *B* is a ring.

$$a_n b_n, \ldots, \left(a_0 b_2 + a_1 b_1 + a_2 b_0\right), \left(a_0 b_1 + b_0 a_1\right), a_0 b_0 \in B$$

Therefore all coefficients of $p(x)q(x)$ belongs to *B*. Then $p(x)q(x) \in B[x]$

Thus, $B[x] \subseteq A[x]$, $p(x) - q(x) \in B[x]$ and $p(x)q(x) \in B[x]$ for every polynomial $p(x)$, $q(x) \in B[x]$.

Hence according to theorem 1 $B[x]$ is a subring of $A[x]$.

Comment