A Book of Abstract Algebra | (2nd Edition)

Problem

Prove that each of the following numbers is algebraic over the given field:

(a) $\sqrt{\pi}$ over $\mathbb{Q}(\pi)$

(b) $\sqrt{\pi}$ over $\mathbb{Q}(\pi^2)$

(c) $\pi^2 - 1$ over $\mathbb{Q}(\pi^3)$

NOTE: Recognizing a transcendental element is much more difficult, since it requires proving that the element cannot be a root of *any* polynomial over the given field. In recent times it has been proved, using sophisticated mathematical machinery, that π and e are transcendental over $\mathbb{Q}$.

Step-by-step solution

Step 1 of 4

(a)

Objective is to prove that the number $\sqrt{\pi}$ is algebraic over $Q(\pi)$.

Let $a = \sqrt{\pi}$. Then $a^2 = \left(\sqrt{\pi}\right)^2$, $a^2 = \pi$. Thus, *a* satisfies $a^2 - \pi = 0$ as $\pi \in Q(\pi)$.

Comment

Step 2 of 4

(b)

Objective is to prove that the number $\sqrt{\pi}$ is algebraic over $Q(\pi^2)$.

Let $a = \sqrt{\pi}$. Then $a^2 = \left(\sqrt{\pi}\right)^2$, $a^2 = \pi$. Also, $a^4 = \pi^2$.

Thus, *a* satisfies $a^4 - \pi^2 = 0$ as $\pi^2 \in Q(\pi^2)$.

Comment

Step 3 of 4

(c)

Objective is to prove that the number $\pi^2 - 1$ is algebraic over $Q(\pi^3)$.

Let $a = \pi^2 - 1$. Then

$$\begin{aligned} a+1 &= \pi^2 \\ (a+1)^3 &= (\pi^2)^3 \\ a^3 + 1 + 3a^2 + 3a &= \pi^6 \end{aligned}$$

Comment

Step 4 of 4

Thus, *a* satisfies $a^3 + 3a^2 + 3a + 1 = \pi^6$ as $\pi^6 \in Q(\pi^3)$.

Comment