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Chapter 28, Problem 4EC

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Problem

If $\mathcal{P}\ell_n$ is the subspace of $\mathcal{P}\ell$ consisting of all polynomials of degree $\le n$, prove that $\{1, x, x^2, \ldots, x^n\}$ is a basis of $\mathcal{P}\ell_n$. Then find another basis of $\mathcal{P}\ell_n$.

Step-by-step solution

Step 1 of 2

For polynomial with basis $\left(1, x, x^2, x^3, \ldots, x^n\right)$, it can be thought of as n dimensional vector space with x^i representing i dimension.

In vector form this basis is $\begin{pmatrix}1\\ \vdots\\ 0\end{pmatrix}, \cdots, \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}, \cdots \begin{pmatrix}0\\ \vdots\\ 1\end{pmatrix}$. Here a 1 is placed at every pivot position.

This matrix is full rank matrix with rank equal to number of rows and columns.

Comment

Step 2 of 2

Now to construct another basis, we have to imagine a matrix which is full rank matrix but different from the one given above,

One of simple way of doing is by making some 1 in above matrix 2's. Thus one possible matrix which is full rank is

$$\boxed{\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix},\begin{pmatrix}0\\2\\\vdots\\0\end{pmatrix}\cdots,\begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix},\cdots\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}}$$

Here all 1's except at first and last are replaced by 2's. This matrix have rank equal to number of rows and number of columns, Thus rows and columns form basis for polynomial with degree less than or equal to *n*.

$$\boxed{\text{Hence another set of basis is } \left(1, 2x, 2x^2, 2x^3, \ldots, x^n\right)}.$$

Since there is no unique basis for any vector space, this answer is not unique.

Comment