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Chapter 16, Problem 1EE

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Problem

Let G and H be groups. Suppose J is a normal subgroup of G and K is a normal subgroup of H.

Show that the function $f(x, y) = (Jx, Ky)$ is a homomorphism from $G \times H$ *onto* $(G/J) \times (H/K)$.

Step-by-step solution

Step 1 of 4

Suppose that G and H are two arbitrary groups. Also let J is a normal subgroup of G and K is a normal subgroup of H.

Consider a mapping $f : G \times H \to (G/J) \times (H/K)$ defined by

$f(x, y) = (Jx, Ky)$.

Objective is to prove that f is an onto homomorphism from $G \times H$ to $(G/J) \times (H/K)$.

If G and H are two groups, a homomorphism from G to H is a function $f : G \to H$ such that for any two elements a, b in G,

$f(ab) = f(a)f(b)$.

Comment

Step 2 of 4

Since let J is a normal subgroup of G, therefore the group G/J is defined. Also Jx is the coset of G/J for some $x \in G$.

Consider two typical elements $f(x_1, y_1), f(x_2, y_2)$ of direct product group $G \times H$ such that $f(x_1, y_1) = (Jx_1, Ky_1)$ and $f(x_2, y_2) = (Jx_2, Ky_2)$. Then

$$\begin{aligned} f(x_1, y_1)\cdot f(x_2, y_2) &= (Jx_1, Ky_1)(Jx_2, Ky_2) \\ &= (Jx_1 \cdot Jx_2, Ky_1 \cdot Ky_2) \\ &= (Jx_1x_2, Ky_1y_2) \\ &= f(x_1x_2, y_1y_2) \end{aligned}$$

The third equality is obtained from the property of cosets. Therefore, *f* is a homomorphism.

Comment

Step 3 of 4

The elements of quotient group G/J will be the cosets of the form Jx, where $x \in G$. Also the elements of quotient group H/K will be the cosets of the form Ky, where $y \in G$.

So, for every $(Jx, Ky) \in (G/J)\times(H/K)$ there corresponds $(x, y) \in G \times H$ such that $f(x, y) = (Jx, Ky)$. This implies that function *f* is onto.

Comment

Step 4 of 4

Hence, the function *f* is a homomorphism from $G \times H$ onto $(G/J)\times(H/K)$.

Comment