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Chapter 32, Problem 3EC | Bookmark | Show all steps: ON

Problem

Explain why $x2 + 3$ is irreducible over $\mathbb{Q}(\sqrt[3]{2})$, then show that $[\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3}) : \mathbb{Q}(\sqrt[3]{2})] = 2$. Conclude that $[\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3}) : \mathbb{Q}] = 6$.

Step-by-step solution

Step 1 of 2

The objective is to explain why x^2+3 is irreducible over $\mathbb{Q}\left(\sqrt[3]{2}\right)$, show that $\left[\mathbb{Q}\left(\sqrt[3]{2}, i\sqrt{3}\right) : \mathbb{Q}\left(\sqrt[3]{2}\right)\right] = 2$ and then to conclude that $\left[\mathbb{Q}\left(\sqrt[3]{2}, i\sqrt{3}\right) : \mathbb{Q}\right] = 6$.

Comment

Step 2 of 2

Because $\mathbb{Q}\left(\sqrt[3]{2}\right)$ is a subfield of the reals and so, $i\sqrt{3} \notin \mathbb{Q}\left(\sqrt[3]{2}\right)$.

Hence, x^2+3 is irreducible over $\mathbb{Q}\left(\sqrt[3]{2}\right)$.

So, $\left[\mathbb{Q}\left(\sqrt[3]{2}, i\sqrt{3}\right) : \mathbb{Q}\left(\sqrt[3]{2}\right)\right]$ is at least 2.

But $i\sqrt{3}$ is a root of $x^2+3 \in \mathbb{Q}\left(\sqrt[3]{2}\right)[X]$, so the degree of $\mathbb{Q}\left(\sqrt[3]{2}, i\sqrt{3}\right)$ over $\mathbb{Q}\left(\sqrt[3]{2}\right)$ is at most 2, and therefore, is exactly 2.

Hence, $\left[\mathbb{Q}\left(\sqrt[3]{2}, i\sqrt{3}\right) : \mathbb{Q}\left(\sqrt[3]{2}\right)\right] = 2$.

Thus, $\left[\mathbb{Q}\left(\sqrt[3]{2}, i\sqrt{3}\right) : \mathbb{Q}\right] = \left[\mathbb{Q}\left(\sqrt[3]{2}, i\sqrt{3}\right) : \mathbb{Q}\left(\sqrt[3]{2}\right)\right]\left[\mathbb{Q}\left(\sqrt[3]{2}\right) : \mathbb{Q}\right]$

$$= 2 \cdot 3$$
$$= 6.$$

Comment