A Book of Abstract Algebra | (2nd Edition)

Problem

Let $a(x) = a_0 + a_1x + \cdots + a_nx^n \in F[x]$. The *derivative* of $a(x)$ is the following polynomial $a'(x) \in F[x]$:

$$a'(x) = a_1 + 2a_2x + \ldots + na_nx^{n-1}$$

(This is the same as the derivative of a polynomial in calculus.) We now prove the analogs of the formal rules of differentiation, familiar from calculus.

Let $a(x), b(x) \in F[x]$, and let $k \in F$.

Prove part:

$[a(x) + b(x)]' = a'(x) + b'(x)$

Step-by-step solution

Step 1 of 3

Consider the arbitrary field F and let $a(x) = a_0 + a_1x + \cdots + a_nx^n \in F(x)$. The derivative of $a(x)$ will be given by

$$a'(x) = a_1 + 2a_2x + \cdots + na_nx^{n-1} \in F(x).$$

Objective is to prove that

$$[a(x) + b(x)]' = a'(x) + b'(x),$$

where $a(x), b(x) \in F[x]$.

Comment

Step 2 of 3

Suppose that $a(x) = a_0 + a_1x + \cdots + a_nx^n$ and $b(x) = b_0 + b_1x + \cdots + b_nx^n$. Then their derivatives will be:

$$a'(x) = a_1 + 2a_2x \cdots + na_nx^{n-1},$$
$$b'(x) = b_1 + 2b_2x \cdots + nb_nx^{n-1}.$$

Now, consider the left hand side of $[a(x) + b(x)]' = a'(x) + b'(x)$ as:

$$\begin{aligned}[a(x)+b(x)]' &= [(a_0+b_0)+(a_1+b_1)x+\cdots+(a_n+b_n)x^n]' \\ &= (a_1+b_1)+\cdots+n(a_n+b_n)x^{n-1} \\ &= (a_1+\cdots+na_nx^{n-1})+(b_1+\cdots+nb_nx^{n-1}) \\ &= a'(x)+b'(x).\end{aligned}$$

Comment

Step 3 of 3

Hence, $[a(x) + b(x)]' = a'(x) + b'(x)$.

Comment