A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 3EP

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Problem

Let G be an abelian group of order $p^k m$, where p^k and m are relatively prime (that is, p^k and m have no common factors except ±1). (REMARK: If two integers j and k are relatively prime, then there are integers s and t such that $sj + tk = 1$. This is proved on page 220.)

Let G_{p^k} be the subgroup of G consisting of all elements whose order divides p^k. Let Gm be the subgroup of G consisting of all elements whose order divides ra. Prove:

$$G_{p^k} \cap G_m = \{e\}.$$

Step-by-step solution

Step 1 of 3

Assume that G is an abelian group of order $p^k m$, where p^k and m are relatively prime. Suppose that G_{p^k} be the subgroup of G consisting of all elements whose order divides p^k. Let G_m be the subgroup of G consisting of all elements whose order divides m.

Objective is to prove that

$$G_{p^k} \quad G_m = \{e\}.$$

Prove this result by method of contradiction.

Comment

Step 2 of 3

Suppose, for the sake of contradiction, that $G_{p^k} \quad G_m \neq \{e\}$. Let $b \in G_{p^k} \quad G_m$, where b is some non-identity element. Then $b \in G_{p^k}$ and $b \in G_m$.

Since order of each element in G_{p^k} divides p^k, so

$$|b| \mid p^k.$$

Similarly, order of *b* will divide *m* because $b \in G_m$ and order of each element in G_m divides *m*. That is,

$$|b| \mid m.$$

On combining both the equations, one can conclude that order of *b* is the common factor of p^k and *m* (order of *b* is finite). This generates a huge contradiction because p^k and *m* are relatively prime. Thus, assumed condition was wrong.

Comment

Step 3 of 3

Hence, $G_{p^k} \quad G_m = \{e\}$.

Comment