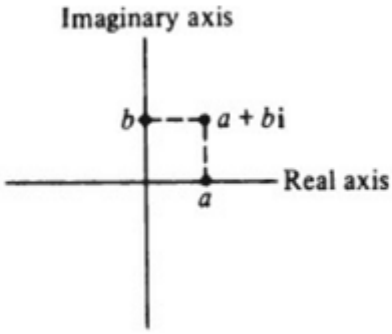
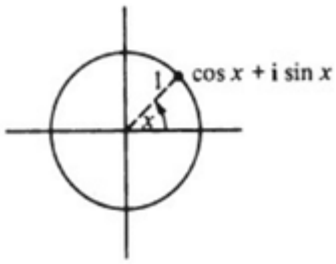
A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 1EH | Bookmark | Show all steps: ON

Problem

Every complex number $a + b\mathbf{i}$ may be represented as a point in the complex plane.





The *unit circle* in the complex plane consists of all the complex numbers whose distance from the origin is 1; thus, clearly, the unit circle consists of all the complex numbers which can be written in the form

cos x + $\mathbf{i}$ sin x

for some real number x.

For each each $x \in \mathbb{R}$, it is conventional to write cis x = cos x + i sin x. Prove that eis (x + y) = (cis x)(cis y).

Step-by-step solution

Step 1 of 3

Note that, the unit circle in the complex plane consists of all the complex numbers which can be written in the form

$\cos x + i \sin x$,

for some real number *x*. For the sake of convenience, write for some real number *x*,

$\text{cis } x = \cos x + i \sin x$.

Objective is to prove that $\text{cis }(x+y) = (\text{cis } x)(\text{cis } y)$.

Before starting proving this, consider the following trigonometric identities:

$$\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)],$$

$$\cos x \sin y = \frac{1}{2}[\sin(x+y) - \sin(x-y)],$$

$$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)],$$

$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$

Comment

Step 2 of 3

Now use the definition of $\text{cis } x$, and get

$$\begin{aligned}(\text{cis } x)(\text{cis } y) &= (\cos x + i \sin x)(\cos y + i \sin y) \\ &= \cos x \cos y + i(\cos x \sin y + \sin x \cos y) - \sin x \sin y\end{aligned}$$

Substitute all the identities defined above and solve in the following manner:

$$\begin{aligned}(\text{cis } x)(\text{cis } y) &= \cos x \cos y + i(\cos x \sin y + \sin x \cos y) - \sin x \sin y \\ &= \frac{1}{2}[\cos(x+y) + \cos(x-y)] - \frac{1}{2}[\cos(x-y) - \cos(x+y)] \\ &\quad + i\left(\frac{1}{2}[\sin(x+y) - \sin(x-y)] + \frac{1}{2}[\sin(x+y) + \sin(x-y)]\right)\end{aligned}$$

Then,

$$\begin{aligned}(\text{cis } x)(\text{cis } y) &= \frac{1}{2}[\cos(x+y) + \cos(x-y) - \cos(x-y) + \cos(x+y)] \\ &\quad + \frac{i}{2}(\sin(x+y) - \sin(x-y) + \sin(x+y) + \sin(x-y)) \\ &= \frac{1}{2}[2\cos(x+y)] + \frac{i}{2}(2\sin(x+y)) \\ &= \cos(x+y) + i\sin(x+y).\end{aligned}$$

It implies that, $(\text{cis } x)(\text{cis } y) = \text{cis }(x+y)$.

Comment

Step 3 of 3

Hence, $(\operatorname{cis} x)(\operatorname{cis} y) = \operatorname{cis}(x+y)$.

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