A Book of Abstract Algebra | (2nd Edition)

Chapter 33, Problem 5ED

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Problem

Let G be a group. The symbol $H \lhd G$ should be read, "H is a normal subgroup of G." A *maximal* normal subgroup of G is an $H \lhd G$ such that, if $H \lhd J \lhd G$, then necessarily $J = H$ or $J = G$. Prove the following:

If an abelian group G has no nontrivial subgroups, G must be a cyclic group of prime order. (Otherwise, choose some $a \in G$ such that $\langle a \rangle$ is a proper subgroup of G.)

Step-by-step solution

Step 1 of 4

Here, objective is to prove that G must be a cyclic group of prime order.

Comment

Step 2 of 4

Consider an abelian group G has no nontrivial subgroups.

$g \in G$; Where g is a proper subgroup of G.

So, the order of g is not infinite. Since G has no nontrivial subgroups.

G must be a cyclic group.

Comment

Step 3 of 4

Let $|G| = ord(g)$

$= n$

If $g \in G$ and $g \neq 1$.Since $|G|>1, \langle g \rangle = G$

Comment

Step 4 of 4

Consider

$k \in N; 1 < k < n$
such that
$n = mk; m \in N$

Then ,

$\text{ord}(g^k)>1$
$(g^k)^m = g^{mk}$
$= g^n$
$= 1$

And

$1 < k < n$
$1 < g^k < g = G$

It is clear that we have a nontrivial subgroup. So no such *k* exist.

Which implies *n* is a prime number.

Therefore, G must be a cyclic group of prime order.

Hence, proved

Comment