A Book of Abstract Algebra | (2nd Edition)

Problem

Recall the definition of $F(a)$. It is a field such that (i) $F \subseteq F(a)$; (ii) $a \in F(a)$; (iii) any field containing F and a contains $F(a)$.

Use this definition to prove parts 1–5, where $F \subseteq K$, $c \in F$, and $a \in K$:

$F(a^2) \subseteq F(a)$ and $F(a+b) \subseteq F(a, b)$. [$F(a, b)$ is the field containing F, a, and b, and contained in any other field containing F, a and b.] Why are the reverse inclusions not necessarily true?

Step-by-step solution

Step 1 of 5

Using definition of $F(a)$ and $F \subseteq K$, $c \in F$, $a \in K$. Prove that $F(a^2) \subseteq F(a)$ and $F(a+b) \subseteq F(a,b)$. [$F(a,b)$ Is the field containing F, a and b, and contained in any other field containing F, a and b]. Why are the reverse inclusions not necessarily true?

Comment

Step 2 of 5

$F(a)$ is a filed such that $F \subseteq F(a)$, $a \in F(a)$ and any field containing F and a contains $F(a)$.

Comment

Step 3 of 5

By above definition of, $F \subseteq F(a)$ and $a \in F(a)$.

Therefore, by the properties of field, $a \cdot a = a^2 \in F(a)$.

Hence, $F(a)$ is field containing F and a^2. Therefore by above definition $F(a^2) \subseteq F(a)$.

Inverse inclusion is not necessarily true.

$\mathbb{Q}(i^2) \subseteq \mathbb{Q}(i)$ But, $\mathbb{Q}(i) \not\subset \mathbb{Q}(i^2) = \mathbb{Q}$.

Comment

Step 4 of 5

Comment

Step 5 of 5

By above definition of , $F \subseteq F(a,b)$ and $a, b \in F(a,b)$.

Therefore, by the properties of field, $a+b \in F(a,b)$.

Hence, $F(a,b)$ is field containing F and $a+b$. Therefore by above definition $F(a+b) \subseteq F(a,b)$.

Inverse inclusion is not necessarily true.

$\mathbb{Q}(\sqrt{2}+\sqrt{3}) \subseteq \mathbb{Q}(\sqrt{2},\sqrt{3})$ But, $\mathbb{Q}(\sqrt{2},\sqrt{3}) \not\subset \mathbb{Q}(\sqrt{2}+\sqrt{3})$.

Comment