A Book of Abstract Algebra | (2nd Edition)

Problem

Let *F* be a field, and let *c* be transcendental over *F*. Prove the following:

If *c* is transcendental over *F*, every element in *F*(*c*) but not in *F* is transcendental over *F*.

Step-by-step solution

Step 1 of 3

Consider that *F* is any arbitrary field and let $c \in F$ is transcendental over *F*. Assume that *K* is some extension field of *F*. Objective is to prove that every element in $F(c)$ but not in *F* is transcendental over *F*.

It is equivalent to show the contrapositive: if $d \in F(c)$ is algebraic then *d* is in *F*. Since by the result $F(c)$ is isomorphic to $F(x)$, one may assume that $F(c) = F(x)$.

Comment

Step 2 of 3

If $d = 0$ then result is true, so assume that $d \neq 0$. If $d = f(x)/g(x) \in F(x)$ is algebraic over *F* then let $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in F[x]$ be its minimal polynomial. Since it is irreducible, therefore a_0, a_n both are nonzero.

Since $p(d) = 0$ it follows that

$$\begin{aligned} a_0 g(x)^n + a_1 f(x) g(x)^{n-1} + \cdots + a_n f(x)^n &= g(x)^n p(d) \\ &= 0 \end{aligned}$$

It implies that $f(x) \backslash a_0 g(x)^n$, $g(x) \backslash a_n f(x)^n$ and hence $f(x) \backslash a_0$, $g(x) \backslash a_n$ since $\gcd(f(x), g(x)) = 1$. As a_0, a_n both are nonzero, it follows that $f(x), g(x) \in F$ and thus $d = f(x)/g(x) \in F$.

Comment

Step 3 of 3

Hence, if *c* is transcendental over *F*, every element in $F(c)$ but not in *F* is transcendental over *F*.

Comment