A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 2EL

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Problem

Let pbea prime number. A *p-group* is any group whose order is a power of *p*. It will be shown here that if $|G| = p^k$ then *G* has a normal subgroup of order p^m for every *m* between 1 and *k*. The proof is by induction on $|G|$; we therefore assume our result is true for all /^-groups smaller than *G*. Prove parts 1 and 2:

$\langle a \rangle$ is a normal subgroup of *G*.

Step-by-step solution

Step 1 of 4

Consider a group *G* whose order is a power of *p*. That is, *G* is a *p*-group and

$|G| = p^k$,

for some integer *k*. With the help of mathematical induction on the order of group *G*, it can be prove that *G* has a normal subgroup of order p^m for every $1 < m < k$. Also there exists an element $a \in C$ (center) such that $\text{ord}(a) = p$.

Consider the induction hypothesis that this statement is true for all *p*-groups whose order is less than *G*.

Objective is to prove that $\langle a \rangle$ is a normal subgroup of *G*.

Comment

Step 2 of 4

The center *C* of any group *G* defined as:

$C = \{a \in G : ax = xa \text{ for every } x \in G\}$.

To show that *H* is a normal subgroup of *K*, there is a need to show that for some $k \in K$, and $h \in H$

$khk^{-1} \in H$.

Note that, the elements of subgroup generated by *a* will be some powers of *a*. That is,

$\langle a \rangle = \{a^k : k \in Z\}$.

Comment

Step 3 of 4

Since $a \in C$, so $\langle a \rangle \subset C$ because any integer power of *a* is also a commutative element. Now, for proving the normal subgroup let $g \in G$ and $a^k \in \langle a \rangle$, then

$$\begin{aligned} ga^k g^{-1} &= a^k g g^{-1} \\ &= a^k e \\ &= a^k \in \langle a \rangle \end{aligned}$$

The second step is obtained from the condition that $a \in C$ (a commutative element). Thus, $ga^k g^{-1} \in \langle a \rangle$.

Comment

Step 4 of 4

Hence, $\langle a \rangle$ is a normal subgroup of *G*.

Comment