A Book of Abstract Algebra | (2nd Edition)

Problem

Let N be the null space of h, and $\mathcal{R}$ the range space of h. Let $\{\mathbf{a}_1, \ldots, \mathbf{a}_r\}$ be a basis of N. Extend it to a basis $\{\mathbf{a}_1, \ldots, \mathbf{a}_r, \ldots, \mathbf{a}_n\}$ of U.

Prove part:

Let U and V have the same dimension n. Use part 7 to prove that h is injective iff h is surjective.

Step-by-step solution

Step 1 of 4

It is already known that U and V are vector spaces and so they satisfies all conditions for vector space. It is known that basis of U contains n elements. Thus, dimension of U is n.

Comment

Step 2 of 4

Linear transformation h is said to be surjective if all elements of V are map to some element of U.

Linear transformation h is said to be injective if,

$$h(\mathbf{a}) = h(\mathbf{b}) \Rightarrow \mathbf{a} = \mathbf{b}$$

Comment

Step 3 of 4

From question 7 of this section,

$$\text{Hence } \dim(\text{domain of } h) = n = r + n - r = \dim(\text{nullspace of } h) + \dim(\text{range space of } h)$$

It is also given that,

$$\dim V = \dim U = n$$

Since h is surjective, rangespace of h is same as V. Thus,

$$\dim(\text{range space of } h) = \dim V = n$$

Substituting this in result of question 7,

$$\dim(\text{domain of } h) = n = r + n - r = \dim(\text{nullspace of } h) + \dim(\text{range space of } h)$$
$$\Rightarrow \dim U = n = n - r = \dim(\text{nullspace of } h) + \dim V$$
$$\Rightarrow \dim(\text{nullspace of } h) + \dim V = n$$
$$\Rightarrow \dim(\text{nullspace of } h) + n = n$$
$$\Rightarrow \dim(\text{nullspace of } h) = 0$$
$$\Rightarrow \text{nullspace of } h = \{\mathbf{0}\}$$

Thus, $\boxed{h \text{ is injective if } h \text{ is surjective}}$

Comment

Step 4 of 4

Now if h is injective, then

nullspace of $h = \{\mathbf{0}\}$

$\Rightarrow \dim(\text{range space of } h) = n$

Or, $\boxed{h \text{ is surjective if } h \text{ is injective}}$

$\boxed{\text{Hence it can be said that } h \text{ is injective iff } h \text{ is surjective}}$

Comment