A Book of Abstract Algebra | (2nd Edition)

Chapter 23, Problem 6ED

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Problem

Prove the following for an integers *a*, *b*, *c* and all positive integers *m* and *n*:

If $ab \equiv 1 \pmod{c}$, $ac \equiv 1 \pmod{b}$ and $bc \equiv 1 \pmod{a}$, then $ab + bc + ac \equiv 1 \pmod{abc}$. (Assume $a, b, c > 0$.)

Step-by-step solution

Step 1 of 5

Here, objective is to prove that $ab + bc + ac = 1(\text{mod}\, abc)$

Comment

Step 2 of 5

Consider a, b are integers, m, n are positive integer.

If *a* is congruent to *b* modulo *m* which is represented by $a = b(\text{mod m})$

Then, we can say that $a - b$ divided by *m*,

Comment

Step 3 of 5

Consider

$ab = 1(\bmod c)$

$ac = 1(\bmod b)$

$bc = 1(\bmod a))$

From above three equations, we can say that

$(ab-1)$ divided by a

$(ac-1)$ divided by b

$(bc-1)$ divided by c

Then,

$(ab-1)(ac-1)(bc-1)$ divided by abc

Comment

Step 4 of 5

Consider the product

$$(ab-1)(ac-1)(bc-1) = (a^2bc - ab - ac + 1)(bc - 1)$$
$$= a^2b^2c^2 - a^2bc - ab^2c + ab - ac^2b + ac + bc - 1$$
$$= abc(abc - a - b - c) + (ab + bc + ca - 1)$$
$$(ab-1)(ac-1)(bc-1) = abc(abc - a - b - c) + (ab + bc + ca - 1)$$

Comment

Step 5 of 5

if $(ab-1)(ac-1)(bc-1)$ divided by abc, then

$abc(abc-a-b-c)$ divided by abc,

$(ab+bc+ca-1)$ must be divided by abc

Therefore,

$ab + bc + ca = 1(\bmod abc)$

Therefore,

If $ab = 1(\bmod c),\ ac = 1(\bmod b), bc = 1(\bmod a))$, then $ab + bc + ac = 1(\bmod abc)$

Hence, proved

Comment