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Chapter 16, Problem 7EN

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Problem

Let G be a finite group, and K a p-Sylow subgroup of G. Let X be the set of all the conjugates of K. See Exercise M2. If C_1, $C_2 \in X$, let $C_1 \sim C_2$ iff $C_1 = aC_2a^{-l}$ for some $\alpha \in K$

Use parts 5 and 6 to prove that $(G: K)$ is *not* a multiple of p.

Step-by-step solution

Step 1 of 3

Assume that G is a finite group and K a p-Sylow subgroup of G. Consider the set X as the set of all the conjugates of K. Define an equivalence relation as:

If $C_1, C_2 \in X$, let $C_1 \approx C_2$ if and only if $C_1 = aC_2a^{-1}$ for some $a \in K$.

Objective is to prove that $(G:K)$ is not a multiple of p, where $N = N(K)$.

Comment

Step 2 of 3

The order of quotient group G/K can be obtained as:

$$\begin{aligned}(G:K) &= \left|\frac{G}{K}\right| \\ &= \frac{|G|}{|K|} \\ &= \frac{|G|}{|N|}\cdot\frac{|N|}{|K|}\end{aligned}$$

The third step is so obtained because G/N and N/K both the groups are defined.

From the previous result, one knows that $(G:N)$ and $(N:K)$ both are not a multiple of p. That means, $(G:K)$ is a product of numbers not divisible by p. Thus, $(G:K)$ cannot be divisible by *p*.

Comment

Step 3 of 3

Hence, $(G:K)$ is not a multiple of *p*.

Comment