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Chapter 32, Problem 8ED Bookmark Show all steps: ON

Problem

If $\alpha = \sqrt[4]{2}$ is a real fourth root of 2, then the four fourth roots of 2 are $\pm\alpha$ and $\pm i\alpha$. Explain parts 1–6, briefly but carefully:

Compute the table of the group $Gal(\mathbb{Q}(\alpha, i) : \mathbb{Q})$ and show that it is isomorphic to D_4, the group of symmetries of the square.

Step-by-step solution

Step 1 of 2

The objective is to compute the table of the group $Gal\left(\mathbb{Q}\left(\sqrt[4]{2}, i\right) : \mathbb{Q}\right)$ and show that it is isomorphic to D_4, the group of symmetries of the square.

Comment

Step 2 of 2

The Galois group of $\mathbb{Q}\left(\sqrt[4]{2}, i\right)$ over $\mathbb{Q}$ is

$Gal\left(\mathbb{Q}\left(\sqrt[4]{2}, i\right) : \mathbb{Q}\right) = \{id,\ r,\ r^2,\ r^3,\ s,\ rs,\ rs^2,\ rs^3\}$, where

$$id : \begin{cases} \sqrt[4]{2} \mapsto \sqrt[4]{2} \\ i \mapsto i \end{cases} \quad r : \begin{cases} \sqrt[4]{2} \mapsto i\sqrt[4]{2} \\ i \mapsto i \end{cases} \quad r^2 : \begin{cases} \sqrt[4]{2} \mapsto -\sqrt[4]{2} \\ i \mapsto i \end{cases} \quad r^3 : \begin{cases} \sqrt[4]{2} \mapsto -i\sqrt[4]{2} \\ i \mapsto i \end{cases}$$

$$s : \begin{cases} \sqrt[4]{2} \mapsto \sqrt[4]{2} \\ i \mapsto -i \end{cases} \quad rs : \begin{cases} \sqrt[4]{2} \mapsto i\sqrt[4]{2} \\ i \mapsto -i \end{cases} \quad r^2s : \begin{cases} \sqrt[4]{2} \mapsto -\sqrt[4]{2} \\ i \mapsto -i \end{cases} \quad r^3s : \begin{cases} \sqrt[4]{2} \mapsto -i\sqrt[4]{2} \\ i \mapsto -i \end{cases}.$$

The operation is composition, giving the table:

$\circ$	id	r	r^2	r^3	s	rs	r^2s	r^3s
id	id	r	r^2	r^3	s	rs	r^2s	r^3s
r	r	r^2	r^3	id	r^3s	r^2s	r	rs
r^2	r^2	r^3	id	r	rs	r	r^3s	r^2s
r^3	r^3	id	r	r^2	r^2s	r^3s	rs	s
s	s	r^2s	rs	r^3s	id	r^2	r	r^3
rs	rs	r^3s	s	r^2s	r^2	id	r^3	r
r^2s	r^2s	rs	r^3s	s	r^3	r	id	r^2
r^3s	r^3s	s	r^2s	rs	r	r^3	r^2	id

From the table， $r^4 = id$ ， $s^2 = id$.

Also， $rs = sr^{-1} = sr^3$.

So， $Gal\left(\mathbb{Q}\left(\sqrt[4]{2},i\right):\mathbb{Q}\right)$ is isomorphic to D_4 ，the group of symmetries of the square.

Comment