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1 $\dim \text{col}(A) = \dim \text{row}(A)$

Let $A = (\mathbf{c}_1 \cdots \mathbf{c}_n)$ with basis for column space $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$, then

$$\begin{aligned}\mathbf{c}_i &= \gamma_{1i}\mathbf{v}_1 + \cdots + \gamma_{ki}\mathbf{v}_k \\ B &:= (\mathbf{v}_1 \cdots \mathbf{v}_k) \in \mathbb{F}^{m \times k} \\ C &:= (\gamma_{ij}) = \begin{pmatrix} \gamma_{1i} \\ \vdots \\ \gamma_{ki} \end{pmatrix} \in \mathbb{F}^{k \times n} \\ &\Rightarrow A = BC\end{aligned}$$

so A is a linear combo of rows of $C \Rightarrow \dim \text{row}(A) \leq \dim \text{row}(C) = k = \dim \text{col}(A)$.

Now applying the same argument to A^T we see that $\dim \text{col}(A) \leq \dim \text{row}(A) \Rightarrow \dim \text{col}(A) = \dim \text{row}(A)$. ■

2 \mathbf{a} and \mathbf{b} are orthogonal $\Leftrightarrow |a + b|^2 = |a|^2 + |b|^2$

$$\begin{aligned}|\mathbf{a} + \mathbf{b}|^2 &= \langle \mathbf{a} + \mathbf{b}, \mathbf{a} + \mathbf{b} \rangle = \langle \mathbf{a}, \mathbf{a} \rangle + 2\langle \mathbf{a}, \mathbf{b} \rangle + \langle \mathbf{b}, \mathbf{b} \rangle \\ &= |\mathbf{a}|^2 + 2\langle \mathbf{a}, \mathbf{b} \rangle + |\mathbf{b}|^2\end{aligned}$$

but note that $\langle \mathbf{a}, \mathbf{b} \rangle = 0$ since \mathbf{a}, \mathbf{b} are orthogonal.

3 Orthogonal Decomposition

There exists a unique λ such that $\mathbf{a} = \lambda \mathbf{b} + \mathbf{c}$ and $\langle \mathbf{b}, \mathbf{c} \rangle = 0$.

Write $\mathbf{c} = \mathbf{a} - \lambda \mathbf{b}$, then

$$\begin{aligned}\langle \mathbf{b}, \mathbf{c} \rangle &= 0 \Leftrightarrow \langle \mathbf{b}, \mathbf{a} \rangle - \lambda \langle \mathbf{b}, \mathbf{b} \rangle = 0 \\ &\Leftrightarrow \lambda = \frac{\langle \mathbf{b}, \mathbf{a} \rangle}{\langle \mathbf{b}, \mathbf{b} \rangle}\end{aligned}$$

4 Cachy-Schwarz Inequality

Since $\langle \lambda \mathbf{b}, \mathbf{c} \rangle = 0$, then $|\lambda \mathbf{b} + \mathbf{c}|^2 = \lambda^2 |\mathbf{b}|^2 + |\mathbf{c}|^2$. So $|\mathbf{c}|^2 = |\mathbf{a}|^2 - \lambda^2 |\mathbf{b}|^2$.

But $|\mathbf{c}|^2 \geq 0 \Rightarrow$

$$\begin{aligned} |\mathbf{c}|^2 &= |\mathbf{a}|^2 - \lambda^2 |\mathbf{b}|^2 \\ &= \langle \mathbf{a}, \mathbf{a} \rangle - \frac{\langle \mathbf{a}, \mathbf{b} \rangle^2}{\langle \mathbf{b}, \mathbf{b} \rangle} \langle \mathbf{b}, \mathbf{b} \rangle \\ &= |\mathbf{a}|^2 - \frac{\langle \mathbf{a}, \mathbf{b} \rangle^2}{|\mathbf{b}|^2} \geq 0 \end{aligned}$$

Multiplying both sides, rearranging and taking roots,

$$|\langle \mathbf{a}, \mathbf{b} \rangle| \leq |\mathbf{a}| |\mathbf{b}|$$

From this we get that $-1 \leq \frac{\langle \mathbf{a}, \mathbf{b} \rangle}{|\mathbf{a}| |\mathbf{b}|} \leq 1$, hence there exists a unique $\theta \in [0, \pi]$ such that

$$\langle \mathbf{a}, \mathbf{b} \rangle = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

5 Fourier Coefficients

Let $\mathbf{u}_1, \dots, \mathbf{u}_n$ be an orthonormal basis of V . Then if $w \in V$

$$\mathbf{w} = \sum_{i=1}^n \langle \mathbf{w}, \mathbf{u}_i \rangle \mathbf{u}_i$$

Proof. Let $\mathbf{w} = \sum_{i=1}^n x_i \mathbf{u}_i$. Then $\langle \mathbf{w}, \mathbf{u}_j \rangle = \sum_{i=1}^n x_i \langle \mathbf{u}_i, \mathbf{u}_j \rangle = x_j$ since the \mathbf{u}_i are orthonormal.

6 Orthogonal Complement of a Subspace

$$U^\perp = \{ \mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{u} \rangle = 0 \ \forall \mathbf{u} \in U \}$$

Furthermore if $W = \text{span } U$, then $W \cap U^\perp = \{ \mathbf{0} \}$.

We can visualize W^\perp in matrix terms. Let $A \in K^{m \times n} : \text{col}(A) = \text{span } U = W$ where $A = (\mathbf{u}_1 \dots \mathbf{u}_n)$.

Now we are interested in $\mathbf{v} \in U^\perp : \langle \mathbf{v}, \mathbf{u}_i \rangle = 0$ for all i . This is equivalent to $A^T \mathbf{v} = (\langle \mathbf{u}_1, \mathbf{v} \rangle \dots \langle \mathbf{u}_n, \mathbf{v} \rangle)^T = \mathbf{0}$.

$$\Rightarrow W^\perp = \mathcal{N}(A^T)$$

We also have $\dim \mathcal{N}(A^T) = \dim \mathcal{N}(A)$ and $\dim \text{row}(A^T) = \dim \text{col}(A)$ so

$$\dim W + \dim W^\perp = m$$

6.1 $V = W \oplus W^\perp$

Let $\mathbf{v} \in V$ and $\mathbf{u}_1, \dots, \mathbf{u}_k$ be an orthonormal basis of W . Then we can put $\mathbf{y} = \mathbf{v} - \sum_{i=1}^k \langle \mathbf{v}, \mathbf{u}_i \rangle \mathbf{u}_i$. Since the \mathbf{u}_i are orthonormal, we have $\langle \mathbf{y}, \mathbf{u}_i \rangle = 0$ and so $\mathbf{y} \in W^\perp$.

Thus $V = W + W^\perp$ and since $W \cap W^\perp = \{ \mathbf{0} \}$, we get $V = W \oplus W^\perp$. ■

7 Gram-Schmidt Method

This is an algorithm for producing orthonormal basis from a general basis. Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be a basis.

1. Normalize \mathbf{v}_1 by putting $\mathbf{u}_1 = \frac{\mathbf{v}_1}{|\mathbf{v}_1|}$.
2. Remove the projection of \mathbf{u}_1 from \mathbf{v}_2 as follows:
 1. Set $\mathbf{v}'_2 = \mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{u}_1 \rangle \mathbf{u}_1$.
 2. Normalize by letting $\mathbf{u}_2 = \frac{\mathbf{v}'_2}{|\mathbf{v}'_2|}$.
3. Now repeat the process:
 1. $\mathbf{v}'_3 = \mathbf{v}_3 - \langle \mathbf{v}_3, \mathbf{u}_1 \rangle \mathbf{u}_1 - \langle \mathbf{v}_3, \mathbf{u}_2 \rangle \mathbf{u}_2$.
 2. $\mathbf{u}_3 = \frac{\mathbf{v}'_3}{|\mathbf{v}'_3|}$.
4. And so on.

8 $d(\mathbf{a}, \mathbf{c}) \leq d(\mathbf{a}, \mathbf{b}) + d(\mathbf{b}, \mathbf{c})$

Denote the sides of a triangle by $\mathbf{a} = \mathbf{b} + \mathbf{c}$.

$$\begin{aligned}\langle \mathbf{a}, \mathbf{a} \rangle &= \langle \mathbf{b}, \mathbf{b} \rangle + 2\langle \mathbf{b}, \mathbf{c} \rangle + \langle \mathbf{c}, \mathbf{c} \rangle \\ &\leq \langle \mathbf{b}, \mathbf{b} \rangle + 2|\langle \mathbf{b}, \mathbf{c} \rangle| + \langle \mathbf{c}, \mathbf{c} \rangle \\ &\leq \langle \mathbf{b}, \mathbf{b} \rangle + 2|\mathbf{b}||\mathbf{c}| + \langle \mathbf{c}, \mathbf{c} \rangle \\ &= |\mathbf{b}| + 2|\mathbf{b}||\mathbf{c}| + |\mathbf{c}| \\ &= (|\mathbf{b}| + |\mathbf{c}|)^2\end{aligned}$$

using the Cauchy-Schwarz inequality.

9 Least Squares Principle

Let $\mathbf{v} = \mathbf{w} + \mathbf{y}$, with $\mathbf{w} \in W$ and $\mathbf{y} \in W^\perp$. Then $d(\mathbf{v}, W) = |\mathbf{y}|$.

Let $\mathbf{w}' = \mathbf{w} + \mathbf{m}$, then $\mathbf{v} - \mathbf{w}' = \mathbf{y} + \mathbf{m}$

$$d(\mathbf{v}, \mathbf{w}')^2 = |\mathbf{y} + \mathbf{m}|^2 = |\mathbf{y}|^2 + |\mathbf{m}|^2 \geq 0$$

since \mathbf{y} and \mathbf{m} are orthogonal.

But

$$d(\mathbf{v}, \mathbf{w})^2 = |\mathbf{y}|^2 \geq 0$$

so $d(\mathbf{v}, \mathbf{w}) \leq d(\mathbf{v}, \mathbf{w}')$ for all $\mathbf{w}' \in W$.

10 Vector Space Quotients

$$\dim V/W = \dim V - \dim W$$

Suppose $A \in \mathbb{F}^{m \times n}$, then $\mathbb{F}^n/\mathcal{N}(A)$ are the solution sets for $A\mathbf{x} = \mathbf{b}$ where \mathbf{b} varies through \mathbb{F}^m .

$\mathcal{N}(A)$ corresponds to $A\mathbf{x} = \mathbf{0}$, and $\mathbf{p} + \mathcal{N}(A)$ corresponds to $A\mathbf{p} = \mathbf{b}$.

10.1 $\dim(V + W)/W = \dim V/(V \cap W)$

$$\begin{aligned}\dim(V + W)/W &= (\dim V + \dim W - \dim(V \cap W)) - \dim W \\ &= \dim V - \dim(V \cap W) \\ &= \dim V/(V \cap W)\end{aligned}$$