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Chapter 16, Problem 3EC

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Problem

Let *G* be an abelian group. Let $H = \{x^2 : x \in G\}$ and $K = \{x \in G : x^2 = e\}$.

Use the FHT to conclude that $H \cong G/K$

Step-by-step solution

Step 1 of 4

Suppose that *G* be an abelian group. Consider the following sets:

$$H = \{x^2 : x \in G\},$$
$$K = \{x \in G : x^2 = e\}$$

Let $f : G \to H$ is a mapping defined by $f(x) = x^2$. Objective is to prove that $H \cong G/K$ by using fundamental homomorphism theorem.

According to the fundamental homomorphism theorem, if $f : G \to H$ is a homomorphism of *G* onto *H*, with kernel *K* then

$$H \cong G/K.$$

Comment

Step 2 of 4

Let $x, y \in G$. Since *G* is an abelian group so for all $x, y \in G$, one have

$$xy = yx.$$

Then

$$\begin{aligned} f(xy) &= (xy)^2 \\ &= xy \cdot xy \\ &= x(xy)y \\ &= x^2 y^2 \end{aligned}$$

Thus, $f(xy) = f(x)f(y)$. Therefore, *f* is a homomorphism.

The function *f* is clearly onto because for all $y = x^2 \in H$ there exists $x \in G$ such that $f(x) = y$.

Comment

Step 3 of 4

According to the definition of kernel:

$\ker f = \{x \in G : f(x) = e\}$.

Since $f(x) = x^2$, so equivalently

$\ker f = \{x \in G : x^2 = e\}$.

That is, the defined set *K* is nothing but the kernel of mapping *f*.

Comment

Step 4 of 4

Thus, $f : G \to H$ is a homomorphism of *G* onto *H*, with kernel *K*. So, by the FHT

$H \cong G/K$.

Comment