A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 3EQ

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Problem

As a provisional definition, let us call a finite abelian group "decomposable" if there are elements $a_1, \ldots, a_n \in G$ such that:

(DI) For every $x \in G$, there are integers $k_1, \ldots, k_n$ such that $x = a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}$. (D2)

If there are integers $l_1, \ldots, l_n$ such that

$a_1^{l_1} a_2^{l_2} \cdots a_n^{l_n} = e$ then $a_1^{l_1} = a_2^{l_2} = \cdots = a_n^{l_n} = e$.

If (D_1) and (D_2) hold, we will write $G = [a_1, a_2, \ldots, a_n]$. Assume this in parts 1 and 2.

Explain why we may assume that $G/H = [Hb_1, \ldots, Hb_n]$ for some $b_1, \ldots, b_n \in G$.

By Exercise O, we may assume that for each $i = 1, \ldots, n$, ord (b_i) = (Hb_i). We will show that $G = [a, b_1, \ldots, b_n]$.

Step-by-step solution

Step 1 of 3

Assume that G is a finite abelian group, and order of each element in G is some power of prime p. Let a is the highest possible order element in G and $H = \langle a \rangle$.

Objective is to explain the reason that for some $b_1, \ldots, b_n \in G$, the

$G/H = [Hb_1, \ldots, Hb_n]$.

From the result of lifting elements from cosets, one may assume that for each $i = 1, \ldots, n$, $\operatorname{ord}(b_i) = \operatorname{ord}(Hb_i)$. So, it is sufficient to show that $G = [a, b_1, \ldots, b_n]$.

Comment

Step 2 of 3

Task is to show prove that G has a basis and by induction hypothesis every p-group smaller than

G has a basis.

By the inductive hypothesis choose a basis for quotient group G/H as

$G/H = [Hb_1, ..., Hb_m]$.

If *G* is abelian and $a \in G$ such that for any $g \in G$, $|\langle g \rangle| \mid |\langle a \rangle|$. Then for any $x \in G$ there exists *y* such that $Hx = Hy$ and $\text{ord}(y) = \text{ord}(Hy)$ where $H = \langle a \rangle$.

So from here one may assume that every b_i is chosen so that $\text{ord}(b_i) = \text{ord}(Hb_i)$.

Comment

Step 3 of 3

Since $[Hb_i]$ is a basis of G/H, one have $Hx = H(b_{ij}^{k_{ij}})$ for some product of b_i's. Therefore, $x = a^{k_0}(b_{ij}^{k_{ij}})$. Since the product of b_i's is unique the power of *a* and is fully determined. So, the entire product is unique. Thus, $[a, b_1, ..., b_n]$ is a basis of *G*.

Comment