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Chapter 17, Problem 1EA 3 Bookmarks Show all steps: ON

Problem

In each of the following, a set *A* with operations of addition and multiplication is given. *Prove that A satisfies all the axioms to be a commutative ring with unity. Indicate the zero element, the unity, and the negative of an arbitrary a.*

A is the set $\mathbb{Z}$ of the integers, with the following "addition" $\oplus$ and "multiplication" $\odot$:

$a \oplus b = a + b - 1$ $a \odot b = ab - (a + b) + 2$

Step-by-step solution

Step 1 of 5

Consider that the set *A* is the set of integers, with the following addition and multiplication:

$$a \oplus b = a + b - 1,$$
$$a \otimes b = ab - (a + b) + 2.$$

Objective is to show that *A* satisfies all the axioms to be a commutative ring with unity.

Write explicitly the zero element, the unity, and the negative of an arbitrary *a*.

First show that $(A, \oplus)$ is an abelian group.

(1) Since sum of integers is integers, therefore $a \oplus b$ is closed in *A*.

(2) Associative: Let $a, b, c \in A$. Then

$$\begin{aligned}(a \oplus b) \oplus c &= a \oplus (b \oplus c)\\ (a + b - 1) \oplus c &= a \oplus (b + c - 1)\\ (a + b - 1) + c - 1 &= a + (b + c - 1) - 1\\ a + b + c - 2 &= a + b + c - 2.\end{aligned}$$

Since both the sides are equals, so addition is associative in *A*.

(3) Since addition is commutative in integers, so

$$\begin{aligned}a \oplus b &= a + b - 1\\ &= b + a - 1\\ &= b \oplus a.\end{aligned}$$

(4) Additive identity or zero element:

$$\begin{aligned} a \oplus e &= a \\ a+e-1 &= a \\ e &= 1. \end{aligned}$$

Thus, zero element of *A* will be 1.

(5) Let for every *a* in *A*, the negative of *a* is *b* then

$$\begin{aligned} a \oplus b &= 1 \\ a+b-1 &= 1 \\ b &= 2-a. \end{aligned}$$

Thus, negative of *a* will be $2-a$.

And from here it conclude that, *A* is an abelian group.

Comment

Step 2 of 5

Now, show that $\otimes$ is associative. Let $a, b, c \in A$. Then

$$\begin{aligned} (a \otimes b) \otimes c &= (ab-(a+b)+2) \otimes c \\ &= (ab-(a+b)+2)c-(ab-(a+b)+2+c)+2 \\ &= abc-ac-bc+2c-ab+a+b+c \\ &= abc-ac-bc-ab+a+b+c, \end{aligned}$$

and

$$\begin{aligned} a \otimes (b \otimes c) &= a \otimes (bc-(b+c)+2) \\ &= a(bc-(b+c)+2)-(a+bc-(b+c)+2)+2 \\ &= abc-ab-ac+2a-a-bc+b+c \\ &= abc-ac-bc-ab+a+b+c. \end{aligned}$$

Since both the sides are equals, so multiplication is associative in *A*.

Comment

Step 3 of 5

Next is distributive law:

$$\begin{aligned} a \otimes (b \oplus c) &= a \otimes (b+c-1) \\ &= a(b+c-1)-(a+b+c-1)+2 \\ &= ab+ac-a-a-b-c+3 \\ &= ab+ac-2a-b-c+3 \end{aligned}$$

And

$$\begin{aligned} (a \otimes b) \oplus (a \otimes c) &= (ab-(a+b)+2) \oplus (ac-(a+c)+2) \\ &= ab-(a+b)+2+ac-(a+c)+2-1 \\ &= ab+ac-2a-b-c+3. \end{aligned}$$

Next, show that $\otimes$ is commutative. Let $a, b \in A$. Then

$$\begin{aligned} a \otimes b &= ab - (a+b) + 2 \\ &= ba - (b+a) + 2 \\ &= b \otimes a. \end{aligned}$$

Since addition $\oplus$, multiplication $\otimes$ both are commutative, therefore $(b \oplus c) \otimes a = (b \otimes a) \oplus (c \otimes a)$ automatically holds.

Comment

Step 4 of 5

Let the unity of non-identity element *a* in *A* is *b* then,

$$\begin{aligned} a \otimes b &= a \\ ab - (a+b) + 2 &= a \\ ab - b &= 2a - 2 \\ b(a-1) &= 2(a-1) \end{aligned}$$

Since *a* is non-identity element, so $b = 2$. Thus, $a \otimes 2 = a$.

Comment

Step 5 of 5

Hence, $(A, \oplus, \otimes)$ form a commutative ring with the zero element 1, the unity is 2, and the negative of an arbitrary *a* is $2 - a$.

Comment