A Book of Abstract Algebra | (2nd Edition)

Chapter 17, Problem 1EC

1 Bookmark

Show all steps: ON

Problem

Let $\mathcal{M}_2(\mathbb{R})$ designate the set of all 2 × 2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

whose entries are real numbers *a*, *b*, *c*, and *d*, with the following addition and multiplication:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} a+r & b+s \\ c+t & d+u \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} ar+bt & as+bu \\ cr+dt & cs+du \end{pmatrix}$$

Verify that $\mathcal{M}_2(\mathbb{R})$ satisfies the ring axioms.

Step-by-step solution

Step 1 of 5

Consider that $M_2(R)$ is the set of all 2×2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, b, c, d \in R$ (real number), with the following addition and multiplication:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}+\begin{pmatrix} r & s \\ t & u \end{pmatrix}=\begin{pmatrix} a+r & b+s \\ c+t & d+u \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} r & s \\ t & u \end{pmatrix}=\begin{pmatrix} ar+bt & as+bu \\ cr+dt & cs+du \end{pmatrix}.$$

Objective is to show that $M_2(R)$ satisfies all the ring axioms.

Comment

Step 2 of 5

First show that $(M_2(R),+)$ is an abelian group.

(1) The sum is again a 2×2 real matrix, so addition is closed.

(2) Associative:

$$\begin{aligned}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}+\begin{pmatrix} r & s \\ t & u \end{pmatrix}\right)+\begin{pmatrix} x & y \\ z & w \end{pmatrix}&=\begin{pmatrix} a+r & b+s \\ c+t & d+u \end{pmatrix}+\begin{pmatrix} x & y \\ z & w \end{pmatrix}\\&=\begin{pmatrix} a+r+x & b+s+y \\ c+t+z & d+u+w \end{pmatrix},\end{aligned}$$

$$\begin{aligned}\begin{pmatrix} a & b \\ c & d \end{pmatrix}+\left(\begin{pmatrix} r & s \\ t & u \end{pmatrix}+\begin{pmatrix} x & y \\ z & w \end{pmatrix}\right)&=\begin{pmatrix} a & b \\ c & d \end{pmatrix}+\begin{pmatrix} r+x & s+y \\ t+z & u+w \end{pmatrix}\\&=\begin{pmatrix} a+r+x & b+s+y \\ c+t+z & d+u+w \end{pmatrix}.\end{aligned}$$

Since both the sides are equals, so addition is associative in $M_2(R)$.

(3) Addition is commutative

$$\begin{aligned}\begin{pmatrix} a & b \\ c & d \end{pmatrix}+\begin{pmatrix} r & s \\ t & u \end{pmatrix}&=\begin{pmatrix} a+r & b+s \\ c+t & d+u \end{pmatrix}\\&=\begin{pmatrix} r+a & s+b \\ t+c & u+d \end{pmatrix}\\&=\begin{pmatrix} r & s \\ t & u \end{pmatrix}+\begin{pmatrix} a & b \\ c & d \end{pmatrix}.\end{aligned}$$

(4) Additive identity or zero element is the 2×2 zero matrix because

$$\begin{aligned}\begin{pmatrix} a & b \\ c & d \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}&=\begin{pmatrix} a+0 & b+0 \\ c+0 & d+0 \end{pmatrix}\\&=\begin{pmatrix} a & b \\ c & d \end{pmatrix}.\end{aligned}$$

(5) The negative of every 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ will be $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ because

$$\begin{aligned}\begin{pmatrix} a & b \\ c & d \end{pmatrix}+\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}&=\begin{pmatrix} a-a & b-b \\ c-c & d-d \end{pmatrix}\\ &=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

Thus, $M_2(R)$ is an abelian group.

Comment

Step 3 of 5

Now, show that product is associative. So,

$$\begin{aligned}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} r & s \\ t & u \end{pmatrix}\right)\cdot\begin{pmatrix} x & y \\ z & w \end{pmatrix}&=\begin{pmatrix} ar+bt & as+bu \\ cr+dt & cs+du \end{pmatrix}\cdot\begin{pmatrix} x & y \\ z & w \end{pmatrix}\\ &=\begin{pmatrix} arx+btx+asz+buz & ary+bty+asw+buw \\ crx+dtx+csz+duz & cry+dty+csw+duw \end{pmatrix}\end{aligned}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\left(\begin{pmatrix} r & s \\ t & u \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix}\right)=\begin{pmatrix} arx+btx+asz+buz & ary+bty+asw+buw \\ crx+dtx+csz+duz & cry+dty+csw+duw \end{pmatrix}.$$

Since both the sides are equals, so multiplication is associative in $M_2(R)$.

Comment

Step 4 of 5

Next is distributive law:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \left(\begin{pmatrix} r & s \\ t & u \end{pmatrix} + \begin{pmatrix} x & y \\ z & w \end{pmatrix} \right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} r+x & s+y \\ t+z & u+w \end{pmatrix}$$
$$= \begin{pmatrix} ar+bt+ax+bz & as+bu+ay+bw \\ cr+dt+cx+dz & cs+du+cy+dw \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} r & s \\ t & u \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} ar+bt & as+bu \\ cr+dt & cs+du \end{pmatrix} + \begin{pmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{pmatrix}$$
$$= \begin{pmatrix} ar+bt+ax+bz & as+bu+ay+bw \\ cr+dt+cx+dz & cs+du+cy+dw \end{pmatrix}$$

Similarly,

$$\left(\begin{pmatrix} r & s \\ t & u \end{pmatrix} + \begin{pmatrix} x & y \\ z & w \end{pmatrix} \right) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Comment

Step 5 of 5

Hence, $M_2(R)$ is the ring.

Comment