A Book of Abstract Algebra | (2nd Edition)

Problem

Find the minimum polynomial of the following numbers over the indicated fields:

$\sqrt{3}+i$ over $\mathbb{R}$; over $\mathbb{Q}$; over $\mathbb{Q}(i)$; over $\mathbb{Q}(\sqrt{3})$

$\sqrt{i+\sqrt{2}}$ over $\mathbb{R}$; over $\mathbb{Q}(i)$; over $\mathbb{Q}(\sqrt{2})$; over $\mathbb{Q}$

Step-by-step solution

Step 1 of 9

(a)

Objective is to determine the minimum polynomial of $\sqrt{3}+i$ over *R*.

Let $x=\sqrt{3}+i$. Then

$$\begin{aligned} x-\sqrt{3}&=i\\ \left(x-\sqrt{3}\right)^2&=i^2\\ x^2-2\sqrt{3}x+3&=-1. \end{aligned}$$

Thus, the minimum polynomial will be $x^2-2\sqrt{3}x+4$ over *R* as $\sqrt{3}\in R$.

Comment

Step 2 of 9

The minimum polynomial of $\sqrt{3}+i$ over *Q*:

$$\begin{aligned} x-i&=\sqrt{3}\\ (x-i)^2&=\left(\sqrt{3}\right)^2\\ x^2-2ix-4&=0\\ x^2-4&=2ix \end{aligned}$$

Also

$$\begin{aligned} \left(x^2-4\right)^2&=(2ix)^2\\ x^4-8x^2+16&=-4x^2\\ x^4-4x^2+16&=0. \end{aligned}$$

Thus, the minimum polynomial will be x^4-4x^2+16 over *Q*.

Comment

Step 3 of 9

The minimum polynomial of $\sqrt{3}+i$ over $Q(i)$:

$$\begin{aligned} x-i&=\sqrt{3}\\ (x-i)^2&=\left(\sqrt{3}\right)^2\\ x^2-2ix-4&=0\\ x^2-4&=2ix. \end{aligned}$$

Thus, the minimum polynomial will be $x^2-2ix-4$ over $Q(i)$ as $i\in Q(i)$.

Comment

Step 4 of 9

The minimum polynomial of $\sqrt{3}+i$ over $Q\left(\sqrt{3}\right)$:

$$\begin{aligned} x-\sqrt{3}&=i\\ \left(x-\sqrt{3}\right)^2&=i^2\\ x^2-2\sqrt{3}x+3&=-1. \end{aligned}$$

Thus, the minimum polynomial will be $x^2-2\sqrt{3}x+4$ over $Q\left(\sqrt{3}\right)$ as $\sqrt{3}\in Q\left(\sqrt{3}\right)$.

Comment

Step 5 of 9

(b)

Objective is to determine the minimum polynomial of $\sqrt{i+\sqrt{2}}$ over *R*.

Let $x=\sqrt{i+\sqrt{2}}$. Then

$$\begin{aligned} x^2&=i+\sqrt{2}\\ \left(x^2-\sqrt{2}\right)^2&=i^2\\ x^4-2\sqrt{2}x^2+3&=0. \end{aligned}$$

Thus, the minimum polynomial will be $x^4-2\sqrt{2}x^2+3$ over *R* as $\sqrt{2}\in R$.

Comment

Step 6 of 9

The minimum polynomial of $\sqrt{i+\sqrt{2}}$ over $Q(i)$:

$$\begin{aligned} x^2&=i+\sqrt{2}\\ \left(x^2-i\right)^2&=\left(\sqrt{2}\right)^2\\ x^4-2x^2i-3&=0. \end{aligned}$$

Thus, the minimum polynomial will be x^4-2x^2i-3 over $Q(i)$.

Comment

Step 7 of 9

The minimum polynomial of $\sqrt{i+\sqrt{2}}$ over $Q\left(\sqrt{2}\right)$:

$$\begin{aligned} x^2&=i+\sqrt{2}\\ \left(x^2-\sqrt{2}\right)^2&=i^2\\ x^4-2\sqrt{2}x^2+3&=0. \end{aligned}$$

Thus, the minimum polynomial will be $x^4-2\sqrt{2}x^2+3$ over $Q\left(\sqrt{2}\right)$ as $\sqrt{2}\in Q\left(\sqrt{2}\right)$.

Comment

Step 8 of 9

The minimum polynomial of $\sqrt{i+\sqrt{2}}$ over *Q*:

$$\begin{aligned} x^2&=i+\sqrt{2}\\ \left(x^2-i\right)^2&=\left(\sqrt{2}\right)^2\\ x^4-2x^2i-3&=0\\ \left(x^4-3\right)^2&=\left(2x^2i\right)^2\\ x^8-6x^4+9&=-4x^4\\ x^8-2x^4+9&=0. \end{aligned}$$

Comment

Step 9 of 9

Thus, the minimum polynomial will be x^8-2x^4+9 over *Q*.

Comment