A Book of Abstract Algebra | (2nd Edition)

Problem

If *O* and *I* are any two points in the plane, consider a coordinate system such that

the interval *OI* coincides with the unit interval on the *x* axis. Let $\mathbb{D}$ be the set of real numbers such that $a \in \mathbb{D}$ iff the point (*a*, 0) is constructible from {*O, I*}. Prove the following:

If $a, b \in \mathbb{D}$, then $a + b \in \mathbb{D}$ and $a - b \in \mathbb{D}$.

Step-by-step solution

Step 1 of 4

Here, objective is to prove that, $a+b \in D$ and $a-b \in D$.

Consider $a \in D$ if and only if the point $(a,0)$ is constructible from $\{O,I\}$

Comment

Step 2 of 4

Constructible point:

This point is either the end point of given unit segment or it is the intersection of two lines determined by previous constructible points.

Comment

Step 3 of 4

D is a set of real numbers. Therefore, we can add, subtract, multiply any two points of them.

Let $a,b \in D$, then the points $(a,0)$ and $(b,0)$ are constructible from $\{O,I\}$

Consider the point $(a,0)$is constructible from $\{O,I\}$

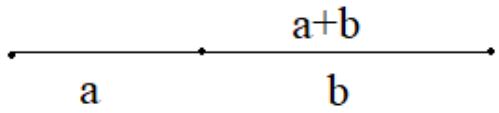

fig:addition

Draw a line through the two points $\{O,I\}$ with a distance $|a|$, then a circle of length $|b|$around *I*

This circle intersects the line at a distance of $|a+b|$ from O

Then, the point $(a+b,0)$is constructible from $\{O,I\}$

Hence, $a+b \in D$

Comment

Step 4 of 4

Consider the points $(a,0),(b,0)$are constructible from $\{O,I\}$

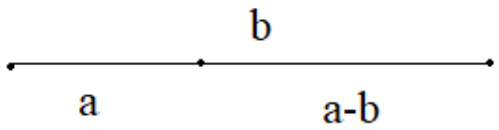

fig 2:subtraction

Draw a line through the two points $\{O, I\}$ with a distance $|a|$, then a circle of length $|a-b|$ around *I*. This circle intersects the line at a distance of $|b|$ from O

Then, the point $(a-b, 0)$ is constructible from $\{O, I\}$

Hence, $a-b \in D$

Therefore, if $a, b \in D$, then $a+b \in D$ and

Hence, proved

Comment