A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 4EG

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Problem

If H is a subgroup of a group G, let X designate the set of all the left cosets of H in G. For each element $a \in G$, define p_a: $X \to X$ as follows:

$$p_a(xH) = (ax)H$$

Prove that if H contains no normal subgroup of G except $\{e\}$, then G is isomorphic to a subgroup of S_X.

Step-by-step solution

Step 1 of 4

Assume that G be a group and H be its subgroup. Consider that X is the set of all the left cosets of H in G. Define a homomorphism mapping, for some $a \in G$, $p_a : X \to X$ by

$p_a(xH) = (ax)H$.

Consider the following homomorphism mapping $h : G \to S_X$ defined by $h(a) = p_a$.

Objective is to prove that if H has no normal subgroup of G except the identity subgroup, then G is isomorphic to a subgroup of S_X.

To prove this, there is a need to show the existence of a bijective homomorphism between G and a subgroup of S_X.

Comment

Step 2 of 4

Suppose that Y is some subgroup of S_X. Consider a mapping $f : G \to Y$ defined by:

$f(g) = p_g$,

for some $g \in G$. Since p_a is homomorphism mapping, so

$$f(ab) = p_{ab}$$
$$= p_a p_b$$
$$= f(a)f(b).$$

This shows that mapping *f* is homomorphism.

Comment

Step 3 of 4

Next, to show that *f* is injective suppose that $f(a) = f(b)$, for some $a, b \in G$. Then apply the definition of *f* in the following manner:

$$f(a) = f(b)$$
$$p_a = p_b$$
$$p_a(xH) = p_b(xH)$$
$$(ax)H = (bx)H$$

And it implies that $a = b$, one-one mapping.

Let $x \in G$, then x^{-1} will also belong to *G*. Then

$$f(x^{-1}) = p_{x^{-1}}$$
$$= p_{x^{-1}}(xH)$$
$$= (x^{-1}x)H$$
$$= H$$

This shows that *f* is onto.

Comment

Step 4 of 4

Hence, group *G* is isomorphic to a subgroup of S_X.

Comment