A Book of Abstract Algebra | (2nd Edition)

Chapter 33, Problem 3EE

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Problem

Let K be a finite extension of F, where K is a root field over F, with $G = Gal(K: F)$ a solvable group. As remarked in the text, we will assume that F contains the required roots of unity. By Exercise D, let $H_0, \ldots, H_n$ be a solvable series for G in which every quotient H_{i+1}/H_i is cyclic of prime order. For any $i = 1, \ldots, n$, let F_i. and F_{i+1} be the fixfields of H_i and H_{i+1}.

Use part 2 to prove that $\pi^k(c^p) = c^p$ for every k, and deduce from this that $c^p \in F_{i+1}$

Step-by-step solution

Step 1 of 5

Here, objective is to prove that $\pi^k(c^p) = c^p$.

Comment

Step 2 of 5

A G is a group of automorphism of K. The set of elements fixed by every element of G called the fixed field.

Comment

Step 3 of 5

$G = Gal(K:F)$ is a solvable group.

F is the fixed field of *G.*

Where, *K* is a the finite extension of *F.*

Consider F_i and F_{i+1} are the fixed fields of H_i and H_{i+1}

Comment

Step 4 of 5

Consider π is the generator of $Gal[F_i : F_{i+1}]$

$F_i = F_{i+1}(\pi)$

$b \in F_i$

Then,

$b = F_{i+1}(\pi)$

$\pi^{-1}(b) = F_{i+1}$

ω is a p^{th} root of unity in F_{i+1} and $b \in F_i$

Comment

Step 5 of 5

$$\text{Consider } \pi(c) = \omega c$$
$$\pi(c) = c$$

$\pi(c^p) = c^p$

$\pi^2(c^p) = c^p$

.

.

$\pi^k(c^p) = c^p; \forall k$

$(c^p) = \pi^{-k}(c^p)$

We have $F_{i+1} = \pi^{-1}(b)$

Then, by comparing $c^p \in F_{i+1}$

Therefore,

$\pi^k(c^p) = c^p$ and $c^p \in F_{i+1}$.

Hence, proved

Comment