A Book of Abstract Algebra | (2nd Edition)

Chapter AB, Problem 15E

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Problem

Prove that the following are true for any integers a, b, and c:

$a \cdot \text{lcm}(b, c) = \text{lcm}(ab, ac)$.

Step-by-step solution

Step 1 of 3

Objective:-

The objective is to prove $a \cdot lcm(b,c) = lcm(ab,ac)$.

Comment

Step 2 of 3

Proof:-

Let us suppose $\gcd(b,c) = t.$

Let us consider the theorem.

Theorem:-Any two nonzero integers *r* and *s* have a unique positive greatest common divisor *t*, Moreover, *t* is equal to a "Linear combination" of *r* and *s*. That is,

$t = kr + ls \quad for \ \ some \ \ \text{integer} \ \ k \ and \ \ l$

According to this definition:-

$t = mb + nc \quad for \ \ some \ \ \text{integer} \ \ m \ and \ \ n \qquad .....(1)$

Let us multiply by a both sides.

$ta = mba + nca \quad for \ \ some \ \ \text{integer} \ \ m \ and \ \ n$

$ta = mab + nac \quad for \ \ some \ \ \text{integer} \ \ m \ and \ \ n$

Thus, according to the definition ta is greatest common divisor of $ab \ \ and \ \ ac.$

$$\gcd(ab,ac)=at \qquad \ldots\ldots(2)$$

Comment

Step 3 of 3

Let us consider the theorem.

Theorem:-If *p* and *q* are two integers with greatest common divisor $\gcd(p,q)$ and least common multiple $lcm(p,q)$, then

$$p\times q=\gcd(p,q)\times lcm(p,q)$$

According to this theorem:-

$$b\times c=\gcd(b,c)\times lcm(b,c)$$

$$b\times c=t\times lcm(b,c) \qquad \ldots\ldots(3)$$

And,

$$ab\times ac=at\times lcm(ab,ac) \qquad \ldots\ldots(4)$$

Let us divide the equation (3) by (4)

$$\frac{b\times c}{ab\times ac}=\frac{t\times lcm(b,c)}{at\times lcm(ab,ac)}$$

$$\frac{\cancel{(b\times c)}}{a^2\cancel{(b\times c)}}=\frac{\cancel{t}\times lcm(b,c)}{a\cancel{t}\times lcm(ab,ac)}$$

$$\frac{1}{a\times\cancel{a}}=\frac{lcm(b,c)}{\cancel{a}\times lcm(ab,ac)}$$

$$\frac{1}{a}=\frac{lcm(b,c)}{lcm(ab,ac)}$$

$$lcm(ab,ac)=a\cdot lcm(b,c)$$

Proved

Comment