A Book of Abstract Algebra | (2nd Edition)

Problem

Let $p(x)$ be an irreducible polynomial of degree n over F. Let c denote a root of $p(x)$ in some extension of F (as in the basic theorem on field extensions).

If $s(c) = t(c)$ in $F(c)$, where $s(x)$ and $t(x)$ have degree $< n$, prove that $s(x) = t(x)$.

Step-by-step solution

Step 1 of 3

Consider any arbitrary field F. Suppose $s(x), t(x) \in F[x]$ are the polynomials having degree $< n$. Objective is to prove that if $s(c) = t(c)$ in some extension $F(c)$ of F, then $s(x) = t(x)$.

Consider the following result:

If any polynomial $a(x)$ has degree n, it has at most n roots.

Comment

Step 2 of 3

It is known that, $s(c) = t(c)$ in $F(c)$. That is,

$s(c) - t(c) = 0$ in $F(c)$.

Let $p(x) = s(x) - t(x)$. Then, the degree of this polynomial will be $< n$ for sure, because $\deg s(x), \deg t(x) < n$. According to the result, the roots of $p(x)$ will be less than n in F. But

$p(c) = 0$

in $F(c)$(n-values). It implies that, $p(x) = 0$ for all values of x in extension field $F(c)$. Thus, $s(c) = t(c)$ for all values of c.

Comments (2)

Step 3 of 3

Hence, $s(x) = t(x)$.

Comment