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Chapter 16, Problem 4EM

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Problem

Let p be a prime number. A finite group G is called a *p-group* if the order of every element x in G is a power p. (The orders of different elements may be different powers of p.) If H is a subgroup of any finite group G, and H is a p-group, we call H a *p-subgroip* of G. Finally, if K is a p-subgroup of G, and K is maximal (in the sense that K is not contained in any larger p-subgroup of G), then K is called a *p-Sylow subgroup* of G.

Prove that S^* is a p-subgroup of G (use Exercise D3, Chapter 15). Then explain why $S^* = K$, and why it follows that $Ka = K$.

Step-by-step solution

Step 1 of 3

Suppose that *G* is a *p*-group, so order of each element *x* in *G* will be the power of *p*. Let *K* is a *p*-Sylow subgroup of *G* and $N = N(K)$ be the normalizer of *K*.

Assume that $a \in N$, and the order of coset Ka in N/K is a power of *p*. Let $S = \langle Ka \rangle$ is the cyclic subgroup of N/K generated by Ka. Then *N* has a subgroup S^* such that S^*/K is a *p*-group.

Objective is to prove that S^* is a *p*-subgroup of *G*. Then $S^* = K$, and also $Ka = K$.

Since S^*/K is a *p*-group, therefore it will be well defined. So, *K* will form a subgroup of S^*. Since, *K* and S^*/K both are *p*-groups, therefore $|K|$ and $|S^*/K|$ will be equal to some power of *p*. Then,

$$\begin{aligned} |S^*| &= \frac{|S^*|}{|K|} \times |K| \\ &= \left|\frac{S^*}{K}\right| \times |K|. \end{aligned}$$

This implies that, $|S^*|$ is also a power of *p*. And hence, S^* is a *p*-subgroup of *G*.

Comment

Step 2 of 3

Since *K* is a *p*-Sylow subgroup of *G*, so it is not contained in any larger *p*-group, because *K* is the maximal *p*-group. So, only possibility is of equality, that is,

$S^* = K$.

Note that, S^* is the set of all elements *n* of *N* such that $Kn = Ka^p$, so $K = S^*$ includes *a*. This similar argument can be make for arbitrary $a \in N$ such that

Order of $Ka = p^j$.

Since $a \in K$, therefore by the coset property $Ka = K$ as desired.

Comment

Step 3 of 3

Hence, S^* is a *p*-subgroup of *G*, with $S^* = K$, and $Ka = K$.

Comment