A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 2ED

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Problem

Let G be a group. By an *automorphism* of G we mean an isomorphism $f: G \to G$.

By an *inner automorphism* of G we mean any function ϕ_a of the following form:

$$\text{for every } x \in G \quad \phi_a(x) = axa^{-1}$$

Prove that every inner automorphism of G is an automorphism of G.

Step-by-step solution

Step 1 of 4

Suppose that G is a group. Consider an inner automorphism of G as the function $\phi_a : G \to G$ of the following form:

for every $x \in G$, $\phi_a(x) = axa^{-1}$.

Objective is to prove that every inner automorphism of G is an automorphism of G. That is, the function ϕ_a is one-one, onto and homomorphism.

Comment

Step 2 of 4

Let $x, y \in G$. Then to show that ϕ_a is one-one, suppose that $\phi_a(x) = \phi_a(y)$. Then by the use of definition of inner automorphism one have,

$$\phi_a(x) = \phi_a(y)$$
$$axa^{-1} = aya^{-1}$$

Now pre-multiply both the sides by a^{-1} and then do the post-multiply by a in the following manner:

$$
\begin{aligned}
a^{-1}\cdot axa^{-1} &= a^{-1}\cdot aya^{-1} \\
exa^{-1} &= eya^{-1} \\
xa^{-1}a &= ya^{-1}a \\
x &= y.
\end{aligned}
$$

Since the condition $\phi_a(x)=\phi_a(y)$ implies that $x=y$, therefore ϕ_a is one-one.

Comment

Step 3 of 4

Since for each $y\in G$ there exists $x=a^{-1}ya$ such that

$\phi_a(x)=y$.

Thus, the mapping ϕ_a is onto.

For homomorphism, consider

$$
\begin{aligned}
\phi_a(xy) &= a(xy)a^{-1} \\
&= ax(a^{-1}a)ya^{-1} \\
&= (axa^{-1})(aya^{-1}) \\
&= \phi_a(x)\phi_a(y).
\end{aligned}
$$

This implies that ϕ_a is homomorphism.

Comment

Step 4 of 4

Since ϕ_a is one-one, onto and homomorphism, therefore ϕ_a of *G* is an automorphism of *G*.

Comment