Contents

1 Structure of GL$(n, \mathbb{F})$

$$\mathrm{GL}(n, \mathbb{F}) = \mathcal{L}_n(\mathbb{F}) \cdot P(n) \cdot \mathcal{D}_n(\mathbb{F}) \cdot \mathcal{U}_n(\mathbb{F})$$

1.1 $P, Q \in \Pi_n : PN = MQ \Rightarrow P = Q$

1.1.1 $MQ = PN$ zero columns also in P, Q

Note that $M \in \mathcal{L}_n(\mathbb{F})$ and N is non-singular and upper triangular.

First we prove P and Q have the same zero columns. Let the jth column of Q be zero, so $q_{kj} = 0 \; \forall k \in [n]$.

Let $M = (\mathbf{m_1} \cdots \mathbf{m_n})$, then $(MQ)_{:j} = q_{1j}\mathbf{m_1} + \cdots + q_{nj}\mathbf{m_n} = \mathbf{0}$. But $MQ = PN \Rightarrow (MQ)_{:j} = (PN)_{:j} = \mathbf{0}$.

Next we prove that the jth column of P is also zero. Let $p_{ij} \neq 0$, but $p_{ir} = 0$ for all $r \neq j$ since it is a permutation matrix and there is only a single nonzero value per row.

$$\begin{aligned}(PN)_{ij} &= p_{i1}n_{1j} + \cdots + p_{ij}n_{jj} + \cdots + p_{in}n_{nj} \\ &= p_{ij}n_{jj}\end{aligned}$$

But N is nonsingular and it is upper trianglar so a basic property is that its diagonals are nonzero $\Rightarrow n_{jj} \neq 0$.

This means $(PN)_{:j}$ is nonzero which contradicts the first part of this proof.

Lastly by the same argument $QN^{-1} = M^{-1}P$ means every zero column of P is also a zero column of Q.

1.1.2 Final proof using the above result

Suppose the jth columns of P and Q are nonzero.

$$p_{rj} = q_{sj} = 1$$

$$\begin{aligned}(PN)_{rj} &= p_{rj}n_{jj} \neq 0 \\ &= (MQ)_{rj} \\ &= m_{r1}q_{1j} + \cdots + m_{rn}q_{nj}\end{aligned}$$

M does downward transvections so the row terminates after the center

$$(PN)_{rj} = m_{r1}q_{1j} + \cdots + m_{rr}q_{rj}$$

Since this cell is nonzero, one of $q_{1j}, \ldots, q_{rj}$ is also nonzero, and so it follows $s \leq r$.

Likewise applying the same logic to $QN^{-1} = M^{-1}P$, we see $r \leq s$.

Therefore $r = s \Rightarrow P = Q$.

2 Exercises

2.1 Ex 4.3.16

Matrix is symmetric so by eliminating mirror entries, we end up with $U = L^T$.

$LPDU = (LPDU)^T = U^T D^T P^T L^T = LD^T P^T U \Rightarrow D^T P^T = PD \Rightarrow P = P^T$ by $MP = NQ$. But $P(n) \subseteq O(n) \Rightarrow P^T = P^{-1}$ so $P = P^{-1} \Rightarrow D^T P = PD$ but $D^T = D$ so $DP = PD$.

Likewise $DP = D^T P = D^T P^T = PD = (PD)^T$ so is symmetric and $L = U^T$ preserves symmetry.