A Book of Abstract Algebra | (2nd Edition)

Chapter 28, Problem 2EE

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Problem

Let U and V be finite-dimensional vector spaces over a field F, and let $h : U \rightarrow V$ be a linear transformation. Prove part:

The range of h is a subspace of V. (It is called the *range space* of h.)

Step-by-step solution

Step 1 of 4

It is already known that U and V are vector spaces and so they satisfies all conditions for vector space.

Comment

Step 2 of 4

Given subset of V is set of all elements of V which are map of vectors of U.

Comment

Step 3 of 4

Or given subset is

$$\{\mathbf{r} \in V \mid h(\mathbf{u}) = \mathbf{r} \text{ for } \mathbf{u} \in U\}$$

Comment

Step 4 of 4

A 2 step check is needed to determine if given subset is a subspace. First determine if any linear combination of functions in subset is closed under given operation. Second, determine if 0 satisfies given conditions.

STEP 1: For any 2 elements **a** and **b** in *V*,

$$h(\mathbf{u}_1) = \mathbf{r}_1 \qquad (1)$$
$$h(\mathbf{u}_2) = \mathbf{r}_2 \qquad (2)$$

Combining above 2 equations, $s(1) + t(2)$ gives

$$s \cdot h(\mathbf{u}_1) + t \cdot h(\mathbf{u}_2) = s\mathbf{r}_1 + t\mathbf{r}_2$$

As *h* is a linear transformation,

$$\begin{aligned} & s \cdot h(\mathbf{u}_1) + t \cdot h(\mathbf{u}_2) = s\mathbf{r}_1 + t\mathbf{r}_2 \\ & \Rightarrow h(s\mathbf{u}_1) + h(t\mathbf{u}_2) = s\mathbf{r}_1 + t\mathbf{r}_2 \\ & \Rightarrow h(s\mathbf{u}_1 + t\mathbf{u}_2) = s\mathbf{r}_1 + t\mathbf{r}_2 \end{aligned}$$

Thus linear combination of 2 elements in subset lies in subset.

STEP 2: Check if 0 vector satisfies given condition,

$$h(\mathbf{0}_u) = \mathbf{0}_v \qquad \{\text{As } h \text{ is a linear transformation}\}$$

$$\boxed{\text{Hence given set or range represents a subspace}}$$

Comment