A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 6EH

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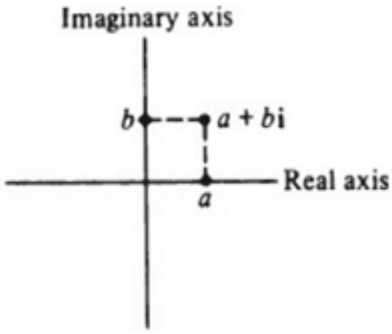
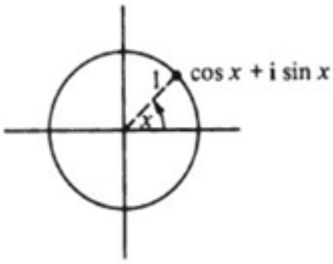
Problem

Every complex number $a + b\mathbf{i}$ may be represented as a point in the complex plane.





The *unit circle* in the complex plane consists of all the complex numbers whose distance from the origin is 1; thus, clearly, the unit circle consists of all the complex numbers which can be written in the form

$\cos x + \mathbf{i} \sin x$

for some real number x.

Prove that $g(x) = \text{cis } 2\pi x$ is a homomorphism from $\mathbb{R}$ onto T, with kernel $\mathbb{Z}$.

Step-by-step solution

Step 1 of 4

Consider the set T of all the complex numbers lying on the unit circle, with the operation multiplication as:

$$T = \{\text{cis } x : x \in R\},$$

where

$\text{cis}\, x = \cos x + i \sin x$.

Let $g: R \to T$ is a mapping defined by

$g(x) = \text{cis}\, 2\pi x$.

Objective is to prove that *g* is a homomorphism from *R* onto *T*, with kernel *Z* (a set of integers).

If *G* and *H* are two groups, a homomorphism from *G* to *H* is a function $f: G \to H$ such that for any two elements *a*, *b* in *G*,

$f(ab) = f(a)f(b)$.

Comment

Step 2 of 4

The mapping *f* is clearly onto because $\text{cis}\, x \in T$ corresponds to $x \in R$.

Let $x, y \in R$. Then, by the identity $\text{cis}\,(x+y) = (\text{cis}\, x)(\text{cis}\, y)$, one have

$$\begin{aligned} g(x)g(y) &= (\text{cis}\, 2\pi x)(\text{cis}\, 2\pi y) \\ &= \text{cis}\,(2\pi x + 2\pi y) \\ &= \text{cis}\,(2\pi(x+y)) \\ &= g(x+y). \end{aligned}$$

This is so because *R* is an additive group and *T* is a multiplicative group.

Comment

Step 3 of 4

According to the definition of kernel:

$\ker g = \{x \in R : g(x) = e\}$,

where $e = \text{cis}(0)$ is a multiplicative identity of *T*.

Since $g(x) = \text{cis}\, 2\pi x$, so equivalently

$\ker g = \{x \in G : \text{cis}\, 2\pi x = \text{cis}(0)\}$.

Also from the figure shown in definition of question, the condition $\text{cis}\, 2\pi x = \text{cis}(0)$ holds if and only if $x \in Z$. This implies that, the kernel of homomorphism *g* will be the set of integers.

Comment

Step 4 of 4

Thus, the mapping g is a homomorphism from R onto T, with kernel Z.

Comment