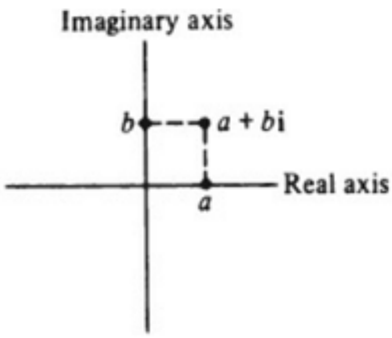
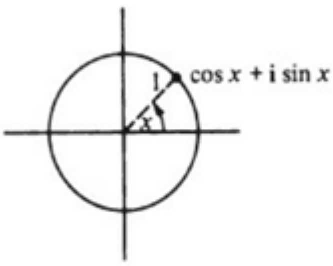
A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 7EH

Problem

Every complex number $a + b\mathbf{i}$ may be represented as a point in the complex plane.





The *unit circle* in the complex plane consists of all the complex numbers whose distance from the origin is 1; thus, clearly, the unit circle consists of all the complex numbers which can be written in the form

$\cos x + \mathbf{i} \sin x$

for some real number x.

Conclude that $T \cong \mathbb{R}/\mathbb{Z}$.

Step-by-step solution

Step 1 of 4

Consider the set T of all the complex numbers lying on the unit circle, with the operation multiplication as:

$$T = \{\text{cis}\, x : x \in R\},$$

where

$\text{cis } x = \cos x + i \sin x$.

Let $g : R \to T$ is a mapping defined by

$g(x) = \text{cis } 2\pi x$.

Objective is to prove that $T \cong R / Z$ by using fundamental homomorphism theorem.

According to the fundamental homomorphism theorem, if $f : G \to H$ is a homomorphism of *G* onto *H*, with kernel *K* then

$H \cong G / K$.

Comment

Step 2 of 4

First show that the mapping *g* is a homomorphism from *R* onto *T*.

Let $x, y \in R$. Then, by the identity $\text{cis } (x+y) = (\text{cis } x)(\text{cis } y)$, one have

$$\begin{aligned} g(x)g(y) &= (\text{cis } 2\pi x)(\text{cis } 2\pi y) \\ &= \text{cis } (2\pi x + 2\pi y) \\ &= \text{cis } (2\pi(x+y)) \\ &= g(x+y). \end{aligned}$$

This is so because *R* is an additive group and *T* is a multiplicative group. The mapping *f* is clearly onto because $\text{cis } x \in T$ corresponds to $x \in R$.

Comment

Step 3 of 4

According to the definition of kernel:

$\ker g = \{x \in R : g(x) = e\}$,

where $e = \text{cis}(0)$ is a multiplicative identity of *T*.

Since $g(x) = \text{cis } 2\pi x$, so equivalently

$\ker g = \{x \in G : \text{cis } 2\pi x = \text{cis}(0)\}$.

Also from the figure shown in definition of question, the condition $\text{cis } 2\pi x = \text{cis}(0)$ holds if and only if $x \in Z$. This implies that, the kernel of homomorphism *g* will be the set of integers

Comment

Step 4 of 4

Thus, $g : R \to T$ is a homomorphism of *R* onto *T*, with kernel *Z*. So, by the FHT

$T \cong R / Z$.

Comment