A Book of Abstract Algebra | (2nd Edition)

Chapter 28, Problem 4EF

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Problem

Let *U* and *V* be vector spaces over the field *F*, with dim *U* = *n* and dim *V* = *m*. Let $h : U \to V$ be a homomorphism.

Prove the following:

Any *n*-dimensional vector space *V* over *F* is isomorphic to the space *Fn* of all *n*-tupies of elements of *F*.

Step-by-step solution

Step 1 of 3

Isomorphism may be seen as type of equality where names and symbols of elements are not particularly important but 2 sets are otherwise same in abstract framework. These elements combine to produce element in similar way. For example

e	1
a	*i*

Is an isomorphism where $a^2 = e\ ;\ i^2 = 1$.

Comment

Step 2 of 3

To prove isomorphism one-to-one and onto bijection must be established between 2 sets.

Comment

Step 3 of 3

Given 2 sets are (i) n-dimensional vector space V over F and (ii) vector space F^n or n-tuples of F.

Standard basis for F^n is

$$\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix},\begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix},\begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix},\dots,\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}$$

Also dimension of V is n. Thus its basis will have n elements.

Let basis of V be

$$(\mathbf{b}_1,\mathbf{b}_2,\dots,\mathbf{b}_n)$$

Define a bijection T from F^n to V as

$$T:\begin{pmatrix}0\\\vdots\\1\\\vdots\\0\end{pmatrix}\to\mathbf{b}_r$$

This mapping is clearly bijective. $\boxed{\text{Hence there is homomorphism between } V \text{ and } F^n}$

Comment