A Book of Abstract Algebra | (2nd Edition)

Chapter 28, Problem 8EC

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Problem

Find a basis for the subspace of $\mathscr{P}\ell$ spanned by

$$\{x^3 + x^2 + x + 1, x^2 + 1, x^3 - x^2 + x - 1, x^2 - 1\}$$

Step-by-step solution

Step 1 of 3

Basis of a vector space is set of linearly independent vectors in that space. To determine if given set forms a basis of any vector space, it must be shown that these vectors are linearly independent. That is one set of vectors cannot be obtained from linear combination of other vectors.

Comment

Step 2 of 3

In present case all $x^3, x^2, x, 1$ represents different dimension where one cannot be obtained from other. Thus they can be considered 4 separate variables. And vectors with their combinations may be considered 4 linear equations. They will represent independent vectors if matrix formed by their coefficient is a full matrix or non-singular, which can be easily shown by reducing it to echelon form.

Comment

Step 3 of 3

Set given in question is $\left(x^3+x^2+x+1, x^2+1, x^3-x^2+x-1, x^2-1\right)$. Here observe that,

Matrix formed by coefficients of these vectors is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Row reducing this matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow[R_4 \to R_4 - R_2]{R_3 \to R_3 - R_1}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_2}$$

$$\begin{pmatrix} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{-2} & 0 & -2 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & \boxed{-2} \end{pmatrix}$$

Clearly, this matrix is full matrix with 4 pivots. Consequently given 4 vectors are independent.

$\boxed{\left(x^3+x^2+x+1, x^2+1, x^3-x^2+x-1, x^2-1\right) \text{ is basis of given subspace}}$

Comment