A Book of Abstract Algebra | (2nd Edition)

Problem

Suppose $a(x) \in F[x]$, and K is an extension of F. An element $c \in K$ is called a multiple root of $a(x)$ if $(x - c)^m | a(x)$ for some $m > 1$. It is often important to know if all the roots of a polynomial are different, or not.

We now consider a method for determining whether an arbitrary polynomial $a(x) \in F[x]$ has multiple roots in any extension of F.

Let K be any field containing all the roots of $a(x)$. Suppose $a(x)$ has a multiple root c.

Prove that $a(x) = (x - c)^2 q(x) \in K[x]$.

Step-by-step solution

Step 1 of 3

Consider that K is any field that contains all the roots of polynomial $a(x) = a_0 + a_1 x + \cdots + a_n x^n$. Assume that $a(x)$ has a multiple root c.

Objective is to prove that

$$a(x) = (x-c)^2 q(x) \in K[x].$$

Since $a(x)$ has a multiple root c, so by definition of multiple root

$$(x-c)^m \mid a(x)$$

for some $m > 1$.

Comment

Step 2 of 3

Apply the definition of divisibility and get

$$a(x) = (x-c)^m b(x),$$

where $b(x) \in F[x]$ is any arbitrary polynomial. Rewrite this as:

$$a(x) = (x-c)^2 (x-c)^{m-2} b(x).$$

Then, $a(x) = (x-c)^2 q(x)$, where $q(x) = (x-c)^{m-2} b(x)$.

Comment

Step 3 of 3

Hence, if $a(x)$ has a multiple root c, then

$$a(x) = (x-c)^2 q(x) \in K[x].$$

Comment