# A Book of Abstract Algebra | (2nd Edition)

Chapter 28, Problem 5ED

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## Problem

Let $V$ be a finite-dimensional vector space. Let dim $V$ designate the dimension of $V$. Prove each of the following:

Any subset of an independent set is independent. Any set of vectors containing a dependent set is dependent.

## Step-by-step solution

**Step 1** of 4

A set of vectors which is said to be linearly independent if there exists no combination of these vectors which can give **0** vector apart from a combination in which all coefficients are 0.

Comment

**Step 2** of 4

If $(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n)$ are $n$ vectors of a vector space and these are linearly independent. Then for.

$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \ldots + a_n\mathbf{u}_n = \mathbf{0}$

All $a_i$ have to be zero.

Comment

**Step 3** of 4

Now consider any set - $(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n)$ which is independent. Then for

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \dots + a_n\mathbf{u}_n = \mathbf{0} \qquad \dots(1)$$

All $a_i = 0$

Now any subset of given set can be obtained from equation (1). Vectors which are not required are removed from equation as their coefficients are 0. This makes given subset satisfy condition for linear independency. $\boxed{\text{Hence any subset of linear independent vectors is also independent}}$.

Comment

**Step 4** of 4

Now consider any set of vectors which contain dependent set. Let $(\mathbf{u}_t, \mathbf{u}_{t+1}, \dots, \mathbf{u}_{t+k})$ be *k* dependent vectors. Also assume that this set is part of set of *n* vectors $(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n)$.

Since *k* vectors are dependent, for

$$a_t\mathbf{u}_t + a_{t+1}\mathbf{u}_{t+1} + \dots + a_{t+k}\mathbf{u}_{t+k} = \mathbf{0} \qquad \dots(2)$$

Not all $a_i$ are 0.

Considering bigger set of *n* vectors. Here

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \dots + a_t\mathbf{u}_t + a_{t+1}\mathbf{u}_{t+1} + \dots + a_{t+k}\mathbf{u}_{t+k} \dots + a_n\mathbf{u}_n = \mathbf{0} \qquad \dots(3)$$

Here not all $a_i$ are 0. As it is already proved from (2) that one of $(a_t, a_{t+1}, \dots, a_{t+k})$ is not 0.

Thus this combination fails to satisfy condition for being linearly independent.

$\boxed{\text{Hence any set containing dependent set is not linearly independent}}$.

Comment