A Book of Abstract Algebra | (2nd Edition)

Problem

Let F be any field.

Prove part:

If $c \neq 0$ and c is algebraic over F, so is $1/c$.

Step-by-step solution

Step 1 of 3

Consider the arbitrary field F. Objective is to show that if $c \neq 0$ and c is algebraic over F, so is $1/c$.

Consider the following result:

If $a(x) = a_0 + a_1x + \cdots + a_nx^n$, $\hat{a}(x) = a_n + a_{n-1}x + \cdots + a_0x^n \in F(x)$, then $a(c) = 0$ if and only if $\hat{a}(1/c) = 0$, where $c \in F$.

Comment

Step 2 of 3

The number c is algebraic over F, if it is the root of some polynomial in $F[x]$. Suppose that

$p(x) = a_0 + a_1x + \cdots + a_nx^n \in F[x]$

such that $p(c) = 0$. That is,

$a_0 + a_1c + \cdots + a_nc^n = 0$.

Now, by the above if and only if result, if $a_0 + a_1c + \cdots + a_nc^n = 0$ then there is a polynomial, namely $\hat{a}(x) = a_n + a_{n-1}x + \cdots + a_0x^n \in F(x)$ such that $\hat{a}(1/c) = 0$.

Since $\hat{a}(x) \in F(x)$ and $\hat{a}(1/c) = 0$, therefore by the definition it implies that $1/c$ is also algebraic over F.

Comment

Step 3 of 3

Hence, if $c \neq 0$ and c is algebraic over F, so is $1/c$.

Comment