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Chapter 28, Problem 2EA | Bookmark | Show all steps: ON

Problem

Prove that $\mathcal{F}(\mathbb{R})$, as defined on page 284, is a vector space over $\mathbb{R}$.

Step-by-step solution

Step 1 of 2

There are 10 conditions which any vector space must satisfy. These are

1. For $u \in V, v \in V \Rightarrow u + v \in V$
2. For $u \in V, v \in V \Rightarrow u + v = v + u$
3. For $u \in V, v \in V, w \in V \Rightarrow (u+v)+w = u+(v+w)$
4. There exists $0 \in V$, such that $0 + v = v$ for all $v \in V$
5. For all $u \in V$, there exists $x \in V$ such that $u + x = 0$
6. For $c \in R, v \in V \Rightarrow cv \in V$
7. For $c \in R, u \in V, v \in V \Rightarrow c(u+v) = cu + cv$
8. For $c, d \in R, u \in V, v \in V \Rightarrow (c+d)u = cu + du$
9. For $c \in R, d \in R, v \in V \Rightarrow c(dv) = (cd)v$
10. There exists $1 \in R, v \in V \Rightarrow 1 \cdot v = v$

Comment

Step 2 of 2

$f(\mathbb{R})$ is any function which gives a real value for any real value as input. By definition of function $f(\mathbb{R})$ is onto as well as into. In other words there exists an output for each input in $\mathbb{R}$. Also no two different outputs can be given by same input.

Let,

v be a random function defined across $\mathbb{R}$.

u be another function defined across $\mathbb{R}$.

Then,

$-u$ represents negative function of *u* in $\mathbb{R}$.

0 function represents such a function for which output is 0 for all inputs.

Then check aforementioned 8 properties or condition for this space.

1. $u+v$ represents another function as sum of 2 function is also a function. Thus, $u+v \in f(\mathbb{R})$
.

2. Since addition is commutative, outcome of addition of 2 functions does not depend on order of addition. Thus, $u+v=v+u$.

3. Again adding 3 functions is like adding 3 numbers for which order of addition is immaterial. Thus, $(u+v)+w=u+(v+w)$.

4. By definition of 0 function, $u+0=u$.

5. By definition of negative of a function, $u+(-u)=0$

6. $cv = c\cdot(\text{function value}) = \text{new function}$

7. As value of functions is just a real number and real numbers follow distributive law. It can be said that $c(u+v)=cu+cv$

8. Again *u* represents a function which is just a real number. Thus, $(c+d)u=cu+du$

9. Ordinary multiplication is associative as well as commutative. Thus, $c(dv)=(cd)v$

10. There exists a constant $c=1$ such that $1\cdot u=u$

$\boxed{\text{Hence } f(\mathbb{R}) \text{ satisfies all conditions for vector space and is a vector space}}$

Comment