A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 2EA

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Problem

In each of the following, use the fundamental homomorphism theorem to prove that the two given groups are isomorphic. Then display their tables.

$\mathbb{Z}_3$ and $\mathbb{Z}_6/\langle 3\rangle$.

Step-by-step solution

Step 1 of 4

Consider the two groups Z_5 and $Z_{20}/\langle 5\rangle$, where $\langle 5\rangle$ denotes the subgroup generated by 5. Objective is to prove that these two groups are isomorphic by using the fundamental homomorphism theorem.

According to the fundamental homomorphism theorem, if $f: G \to H$ is a homomorphism of *G* onto *H*, with kernel *K* then

$$H \cong G/K.$$

Comment

Step 2 of 4

Consider the function $f: Z_{20} \to Z_5$ given by

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & & 15 & 16 & 17 & 18 & 19 \\ 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & & 0 & 1 & 2 & 3 & 4 \end{pmatrix}.$$

To show that this mapping *f* is homomorphism, one must show that

$$f(a+b) = f(a) + f(b)$$

for all choices of *a* and *b* in Z_{20}.

Since both Z_{20} and Z_5 are commutative, even that brute force approach needs lesser additions. Observe that for all $a \in Z_{20}$,

$a \cong f(a)(\text{mod}\, 5)$.

According to congruence law, if $x \cong f(x)(\text{mod}\, 5)$ and $y \cong f(y)(\text{mod}\, 5)$, then

$x + y \cong f(x) + f(y)(\text{mod}\, 5)$.

But, by the defined *f*,

$x + y \cong f(x + y)(\text{mod}\, 5)$.

Comment

Step 3 of 4

Since congruence relation is always transitive, it gives

$f(x + y) \cong f(x) + f(y)(\text{mod}\, 5)$.

Therefore, *f* preserves sums and is a homomorphism. Since each element of Z_5 has the pre-image, so *f* is onto.

By the definition of *f*, only element 0, 5, 10, 15 of Z_{20} maps to identity. Therefore, $\ker f = \{0, 5, 10, 15\}$, that is, kernel of *f* is generated by 5. So, $\ker f = \langle 5 \rangle$.

Hence, the map *f* is homomorphism from Z_{20} onto Z_5 with kernel $\ker f = \langle 5 \rangle$.

The addition table of Z_5 will be:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	14
3	3	4	0	1	2
4	4	0	1	2	3

Comments (1)

Step 4 of 4

Hence, by the fundamental homomorphism theorem it concludes that

$Z_5 \cong Z_{20} / \langle 5 \rangle$.

Comment