A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 5EF

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Problem

Let *G* be a group; let *H* and *K* be subgroups of *G*, with *H* a normal subgroup of *G*. Prove the following:

The function *f*(*k*) = *Hk* is a homomorphism from *K* onto *HK*/*H*, and its kernel is *H K*

Step-by-step solution

Step 1 of 4

Suppose that *G* is any group and let *H, K* are the subgroups of *G*, with *H* a normal subgroup of *G*.

Consider a mapping $f : K \to HK/H$ defined by

$f(k) = Hk$, for all $x \in K$.

Objective is to prove that function *f* is a homomorphism from *K* onto HK/H with kernel $H \quad K$.

Consider the following result: the set HK will form the subgroup of *G* if and only if $HK = KH$.

Comment

Step 2 of 4

First prove that *f* is a homomorphism from *K* onto HK/H.

Let $x, y \in K$. Then

$$\begin{aligned} f(xy) &= Hxy \\ &= Hx \cdot Hy \\ &= f(x) \cdot f(y) \end{aligned}$$

This holds for all $x, y \in K$. Therefore, *f* is a homomorphism.

Next, let $Hx \in HK/H$. It implies that $x \in HK$. Or for some $h \in H$ and $k \in K$, $x = hk$.

Since $HK = KH$, there exists $h' \in H$ and $k' \in K$ such that

$hk = k'h'$.

Now,

$$\begin{aligned} f(k') &= Hk' \\ &= (Hh')k' \\ &= H(h'k') \\ &= H(k'h'). \end{aligned}$$

The second and last equality is obtained by the coset property. At last, it implies that

$$f(k') = Hx.$$

That is, *f* is onto.

Comment

Step 3 of 4

Now, it is remaining to show that $\ker f = H \quad K$.

Let $x \in K$. If $x \in Kerf$ then

$$f(x) = H$$

because *H* is the identity of quotient group HK/H. And then,

$$Hx = H.$$

It implies that $x \in H$. Therefore, $x \in H \quad K$. Thus, $\ker f \subseteq H \quad K$.

Now let $x \in H \quad K$. Then $x \in H$, $x \in K$. Also then $Hx = H$ and $f(x) = H$. It implies that $x \in Kerf$. On combining the equations, one gets

$$\ker f = H \quad K.$$

Comment

Step 4 of 4

Hence, the defined function *f* is a homomorphism from *K* onto HK/H with kernel $H \quad K$.

Comment