# A Book of Abstract Algebra | (2nd Edition)



## Problem

If the minimum polynomial of *a* over *F* has degree 2, we call *F*(*a*) a quadratic extension of *F*.

Prove that, if *F* is a field whose characteristic is ≠2, any quadratic extension of *F* is of the form $F(\sqrt{a})$, for some $a \in F$ (HINT: Complete the square, and use Exercise E4.)

## Step-by-step solution

### Step 1 of 3

Consider the field *F* whose characteristic is $\neq 2$, that is, $2 \neq 0$ in *F*. Objective is to prove that any quadratic extension of *F* is of form $F(\sqrt{a})$, for some $a \in F$.

Suppose that *K* is the quadratic extension of *F*. Let $d \in K, d \notin F$. Then

$$[K:F] = [K:F(d)][F(d):F].$$

By some choice of *d*, $[F(d):F] \geq 2$. Also *K* is quadratic extension, so $[K:F] = 2$. Thus, one must have

$$[K:F(d)] = 1.$$

Comment

### Step 2 of 3

It implies that $K = F(d)$. So, the minimal polynomial of *d* over *F* must be quadratic, say $x^2 + cx + b$. Now, complete the square in $d^2 + cd + b = 0$ as:

$$d^2 + cd + b + \left(\frac{c}{2}\right)^2 - \left(\frac{c}{2}\right)^2 = 0$$

$$\left(d^2 + cd + \frac{c^2}{4}\right) - \left(\frac{c^2}{4} - b\right) = 0$$

$$\left(d + \frac{c}{2}\right)^2 - \left(\frac{c^2}{4} - b\right) = 0$$

$$\left(d + \frac{c}{2}\right) = \sqrt{\frac{c^2}{4} - b}.$$

Then

$$\begin{aligned} K &= F(d) \\ &= F\left(d + \frac{c}{2}\right) \\ &= F\left(\sqrt{\frac{c^2}{4} - b}\right). \end{aligned}$$

Comment

### Step 3 of 3

Hence, $K = F(\sqrt{a})$, where $a = \frac{c^2}{4} - b \in F$.

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