A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 3ED

Bookmark

Show all steps: ON

Problem

Let *G* be a group. By an *automorphism* of *G* we mean an isomorphism *f*: *G* → *G*.

Prove that, for arbitrary *a*, *b* ∈ *G*.

$$\phi_a \circ \phi_b = \phi_{ab} \quad \text{and} \quad (\phi_a)^{-1} = \phi_{a-1}$$

Step-by-step solution

Step 1 of 4

Suppose that *G* is a group. Consider an inner automorphism of *G* as the function $\phi_a : G \to G$ of the following form:

for every $x \in G$, $\phi_a(x) = axa^{-1}$.

Objective is to prove that, for arbitrary $a, b \in G$,

$$\phi_a \quad \phi_b = \phi_{ab},$$
$$(\phi_a)^{-1} = \phi_{a^{-1}}.$$

Comment

Step 2 of 4

Let $a, b \in G$. Then by the above definition of inner automorphism,

$$\phi_a(x) = axa^{-1}, \phi_b(x) = bxb^{-1}.$$

Now, to show that $\phi_a \quad \phi_b = \phi_{ab}$ consider the left hand side and solve in the following manner:

$$\begin{aligned} \phi_a \quad \phi_b(x) &= \phi_a(\phi_b(x)) \\ &= \phi_a(bxb^{-1}) \\ &= a(bxb^{-1})a^{-1} \\ &= (ab)x(b^{-1}a^{-1}) \end{aligned}$$

Since $(ab)^{-1}=b^{-1}a^{-1}$, so

$$\begin{aligned}\phi_a \;\; \phi_b(x) &= (ab)x(ab)^{-1}\\ &= \phi_{ab}.\end{aligned}$$

Comment

Step 3 of 4

Next, to show that $(\phi_a)^{-1}=\phi_{a^{-1}}$ there is a need to show that $\phi_a \;\; \phi_{a^{-1}}=\phi_e$, where ϕ_e is identity element given by

$$\begin{aligned}\phi_e(x) &= exe^{-1}\\ &= x.\end{aligned}$$

So solve the left side of $\phi_a \;\; \phi_{a^{-1}}=\phi_e$ as:

$$\begin{aligned}\phi_a \;\; \phi_{a^{-1}}(x) &= \phi_a\left(\phi_{a^{-1}}(x)\right)\\ &= \phi_a\left(a^{-1}x\left(a^{-1}\right)^{-1}\right)\\ &= \phi_a\left(a^{-1}xa\right)\\ &= a\left(a^{-1}xa\right)a^{-1}\end{aligned}$$

Use the fact that $aa^{-1}=e$ and get,

$$\begin{aligned}\phi_a \;\; \phi_{a^{-1}}(x) &= aa^{-1}xaa^{-1}\\ &= exe\\ &= x\\ &= \phi_e.\end{aligned}$$

Comment

Step 4 of 4

Thus, $\phi_a \;\; \phi_b=\phi_{ab}$, $(\phi_a)^{-1}=\phi_{a^{-1}}$.

Comment