A Book of Abstract Algebra | (2nd Edition)

Problem

Let p be a prime number. A finite group G is called a *p-group* if the order of every element x in G is a power p. (The orders of different elements may be different powers of p.) If H is a subgroup of any finite group G, and H is a p-group, we call H a *p-subgroip* of G. Finally, if K is a p-subgroup of G, and K is maximal (in the sense that K is not contained in any larger p-subgroup of G), then K is called a *p-Sylow subgroup* of G.

Use part 6 to prove: if $aKa^{-1} = K$ and the order of a is a power of p, then $a \in K$.

Step-by-step solution

Step 1 of 3

Consider that G is a p-group, so order of each element x in G will be the power of p. Let K is a p-Sylow subgroup of G and $N = N(K)$is the normalizer of K. Consider the following result:

If $a \in N$ and the order of a is a power of p (prime), then the order of coset Ka in N/K is also a power of p. And also $Ka = K$.

Objective is to prove that if $aKa^{-1} = K$ and the order of a is a power of p, then $a \in K$.

Comment

Step 2 of 3

Consider the given condition that $aKa^{-1} = K$. Apply a both the sides from the right and get,

$$\begin{aligned} aKa^{-1}a &= Ka \\ aKe &= Ka \\ aK &= Ka. \end{aligned}$$

From the last obtained equation $aK = Ka$, it implies that left and right cosets in K are equal. Since $a \in N$ and $aK = Ka$. This shows that K is normal in N and then

$a \in N(K)$.

Since the order of *a* is a power of *p*, by the mentioned result one gets that the order of coset Ka in N/K is also a power of *p*. And thus, again by the same result

$Ka = K$.

At last, by the coset property: the $Ha = H$ if and only if $a \in H$, it implies that

$a \in K$.

Comment

Step 3 of 3

Hence, if $aKa^{-1} = K$ and the order of *a* is a power of *p*, then $a \in K$.

Comment