A Book of Abstract Algebra | (2nd Edition)

Problem

Let pbea prime number. A *p-group* is any group whose order is a power of *p*. It will be shown here that if $|G| = p^k$ then *G* has a normal subgroup of order p^m for every *m* between 1 and *k*. The proof is by induction on $|G|$; we therefore assume our result is true for all /^-groups smaller than *G*. Prove parts 1 and 2:

Use Exercise J4 to prove that *G* has a normal subgroup of order p^m.

Step-by-step solution

Step 1 of 5

Consider a group *G* whose order is a power of *p*. That is, *G* is a *p*-group and

$$|G| = p^k,$$

for some integer *k*. With the help of mathematical induction on the order of group *G*, it can be prove that *G* has a normal subgroup of order p^m for every $1 < m < k$.

Consider the induction hypothesis that this statement is true for all *p*-groups whose order is less than *G*.

Objective is to prove that *G* has a normal subgroup of order p^m.

A nonempty subset *H* of group *G* is said to be a normal subgroup of *G* if $g \in G$ and $h \in H$

$$ghg^{-1} \in H.$$

Comment

Step 2 of 5

Consider the following statement of referring exercise:

Suppose that *G* is any group. Let the mapping

$$f : G_K \to H$$

is a homomorphism from *G* onto *H* with kernel *K*. Assume that *S* is any subgroup of *H* and consider the following set:

$$S^{*} = \{x \in G : f(x) \in S\}.$$

Then the set S^{*} forms a subgroup of *G*.

Comment

Step 3 of 5

Consider a natural homomorphism $f : G \to G/\langle a \rangle$ with kernel $\langle a \rangle$. Let *S* be the normal subgroup of order p^{m-1} of $G/\langle a \rangle$. Now, referring to the above exercise, task is to show that S^{*} is a normal subgroup of *G* whose order is p^{m}.

If one is able to show that $f(gsg^{-1}) \in S$, then this will ensure that $gsg^{-1} \in S^{*}$ for some $g \in G$ and $s \in S^{*}$.

Consider $f(gsg^{-1})$ and expand it by the homomorphism rule as:

$$\begin{aligned} f(gsg^{-1}) &= f(g)f(s)f(g^{-1}) \\ &= f(g)f(s)[f(g)]^{-1}. \end{aligned}$$

The second step is the well-known property of homomorphism.

Comment

Step 4 of 5

Since $s \in S^{*}$, therefore $f(x) \in S$. Also $f(g) \in G/\langle a \rangle$, and *S* is the subgroup of codomain. So, $f(g) \in S$. By the subgroup property, its inverse will also be the member of *S*, that is,

$$[f(g)]^{-1} \in S.$$

Thus, by the closeness of product, it implies that $f(g)f(s)[f(g)]^{-1} \in S$, or $f(gsg^{-1}) \in S$. And thus, $gsg^{-1} \in S^{*}$ for some $g \in G$ and $s \in S^{*}$.

Comment

Step 5 of 5

Hence, *G* has a normal subgroup S^{*} whose order is p^{m}.

Comment