A Book of Abstract Algebra | (2nd Edition)

Chapter 32, Problem 4EH

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Problem

Use parts 1 and 3 to prove that the only automorphism of $\mathbb{R}$ is the identity function.

Step-by-step solution

Step 1 of 2

The objective is to prove that the only automorphism of $\mathbb{R}$ is the identity function.

Comment

Step 2 of 2

Let $f:\mathbb{R}\to\mathbb{R}$ be an automorphism.

Since the only automorphism of $\mathbb{Q}$ is the identity function, $f(a)=a \quad \forall\, a\in\mathbb{Q}$.

Let $x>0$.

$f(x)=f\left(\sqrt{x}\sqrt{x}\right)$

$=f\left(\sqrt{x}\right)f\left(\sqrt{x}\right)$, since f is a homomorphism

$=\left[f\left(\sqrt{x}\right)\right]^2>0$

Let $\alpha>\beta$.

Therefore, $\alpha-\beta>0$.

$$f(\alpha-\beta)>0$$
$$f(\alpha)-f(\beta)>0$$
$$f(\alpha)>f(\beta).$$

Let $x\in\mathbb{R}$.

Let $\alpha, \beta \in \mathbb{Q}$ be such that $\alpha < x < \beta$ ......$(*)$.

Now， $f(\alpha) < f(x) < f(\beta)$ implies $\alpha < f(x) < \beta$ ，since $f(a) = a \quad \forall\, a \in \mathbb{Q}$.

Thus，comparing with $*$ ， $f(x) = x \quad \forall\, x \in \mathbb{R}$.

This shows that the only automorphism of $\mathbb{R}$ is the identity function.

Comment