A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 4EP

Problem

Let G be an abelian group of order p^km, where p^k and m are relatively prime (that is, p^k and m have no common factors except ±1). (REMARK: If two integers j and k are relatively prime, then there are integers s and t such that $sj + tk = 1$. This is proved on page 220.)

Let G_pk be the subgroup of G consisting of all elements whose order divides p^k. Let Gm be the subgroup of G consisting of all elements whose order divides ra. Prove:

$G \cong G_{p^k} \times G_{m'}$ (See Exercise H, Chapter 14.)

Step-by-step solution

Step 1 of 3

Assume that G is an abelian group of order $p^k m$, where p^k and m are relatively prime. Suppose that G_{p^k} be the subgroup of G consisting of all elements whose order divides p^k. Let G_m be the subgroup of G consisting of all elements whose order divides m.

Objective is to prove that $G \cong G_{p^k} \times G_m$.

Consider the following result:

If G is an internal direct product of H_1, $, H_k$, then $G \cong H_1 \times \quad \times H_k$.

Comment

Step 2 of 3

Definition of Internal direct product:

A group G is said to be the internal direct product of H_1, H_2 if the following conditions are satisfied:

(1) The H_1, H_2 are both normal subgroups of G.

(2) $H_1 \quad H_2 = \{e\}$..

(3) And the product subgroup $H_1 H_2 = G$.

Since one knows that, for any $x \in G$, there are $y \in G_{p^k}$ and $z \in G_m$ such that

$x = yz$.

Also the intersection of G_{p^k} and G_m is empty. And the fact that G_{p^k}, G_m are the subgroups of an abelian group *G*, therefore both are normal too.

Comment

Step 3 of 3

Thus, *G* is an internal direct product of G_{p^k}, G_m. Hence, from the above result

$G \cong G_{p^k} \times G_m$.

Comment