A Book of Abstract Algebra | (2nd Edition)

Chapter 32, Problem 5EB

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Problem

For each subgroup of **G**, find its fixfield.

Step-by-step solution

Step 1 of 2

The objective is to find the fix field for each subgroup of $G = Gal\left(\mathbb{Q}\left(\sqrt{2},\sqrt{3},\sqrt{5}\right)\right)$.

Comment

Step 2 of 2

The extension $K = \mathbb{Q}\left(\sqrt{2},\sqrt{3},\sqrt{5}\right)$ is the root field of the polynomial

$f(x) = \left(x^2-2\right)\left(x^2-3\right)\left(x^2-5\right)$ over $\mathbb{Q}$.

Moreover, $\left\{1,\sqrt{2},\sqrt{3},\sqrt{5},\sqrt{6},\sqrt{10},\sqrt{15},\sqrt{30}\right\}$ is a $\mathbb{Q}-$basis for K.

Thus, $[K:\mathbb{Q}]=8$. So, if $G = Gal(K:\mathbb{Q})$ then $|G|=8$.

Since G is abelian, all its subgroups are normal.

Now, by the fundamental theorem of Galois theory, every normal subgroup H corresponds to a subfield K^H, which is a root field over $\mathbb{Q}$.

By Lagrange's theorem, $|H|$ divides 8. So, there are four cases.

Case I: $|H|=1$, then clearly $K^H = \mathbb{Q}\left(\sqrt{2},\sqrt{3},\sqrt{5}\right)$.

Case II: $|H|=2$.

Then H contain the identity and an element of order 2, so it can be any of the following 7 groups:

$\{id,\sigma_2\}$, $\{id,\sigma_3\}$, $\{id,\sigma_5\}$, $\{id,\sigma_2\sigma_3\}$, $\{id,\sigma_2\sigma_5\}$, $\{id,\sigma_3\sigma_5\}$, $\{id,\sigma_2\sigma_3\sigma_5\}$.

By looking at the action on the basis elements, the fixed subfields of the above groups are:

$\mathbb{Q}(\sqrt{3},\sqrt{5})$, $\mathbb{Q}(\sqrt{2},\sqrt{5})$, $\mathbb{Q}(\sqrt{2},\sqrt{3})$, $\mathbb{Q}(\sqrt{5},\sqrt{6})$, $\mathbb{Q}(\sqrt{2},\sqrt{15})$, $\mathbb{Q}(\sqrt{3},\sqrt{10})$, $\mathbb{Q}(\sqrt{6},\sqrt{10})$.

Case III: $|H|=4$.

Then H contain the identity, two distinct elements of order 2, and their product, so it can be any of the following 7 groups:

$\{id,\sigma_2,\sigma_3,\sigma_2\sigma_3\}$, $\{id,\sigma_2,\sigma_5,\sigma_2\sigma_5\}$, $\{id,\sigma_3,\sigma_5,\sigma_3\sigma_5\}$, $\{id,\sigma_2,\sigma_3\sigma_5,\sigma_2\sigma_3\sigma_5\}$, $\{id,\sigma_3,\sigma_2\sigma_5,\sigma_2\sigma_3\sigma_5\}$, , $\{id,\sigma_5,\sigma_2\sigma_3,\sigma_2\sigma_3\sigma_5\}$, $\{id,\sigma_2\sigma_3,\sigma_3\sigma_5,\sigma_2\sigma_5\}$.

Their corresponding fixed subfields are $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$, $\mathbb{Q}(\sqrt{15})$, $\mathbb{Q}(\sqrt{10})$, $\mathbb{Q}(\sqrt{6})$, $\mathbb{Q}(\sqrt{30})$.

Case IV: $|H|=8$ **.**

Then $K^H=\mathbb{Q}$.

Comment