Contents

1 Desargues Theorem

1.1 $tA + (1-t)B = uA + vB$ (Parameterized Line)

$$\begin{aligned}\ell &= \{u(X_1 : Y_1 : Z_1 : W_1) + v(X_2 : Y_2 : Z_2 : W_2) : (u,v) \neq 0\} \\ &= \{(uX_1 + vX_1 : uY_1 + vY_2 : uZ_1 + vZ_2 : uW_1 + vW_2) : (u,v) \neq 0\}\end{aligned}$$

But since $u + v \neq 0$ then we see that for any $P \in \ell$

$$P(\frac{u}{u+v}X_1 + \frac{v}{u+v}X_1 : \frac{u}{u+v}Y_1 + \frac{v}{u+v}Y_2 : \frac{u}{u+v}Z_1 + \frac{v}{u+v}Z_2 : \frac{u}{u+v}W_1 + \frac{v}{u+v}W_2)$$

Let $t = \frac{u}{u+v}$, then we see $1 - t = \frac{v}{u+v}$ and so

$$P(tX_1 + (1-t)X_1 : tY_1 + (1-t)Y_2 : tZ_1 + (1-t)Z_2 : tW_1 + (1-t)W_2)$$

$$\Rightarrow \ell = \{tA + (1-t)B\}$$

so both representations are equivalent.

1.2 $BC, B'C'$ are in PQ

$$u_1(1 : 0 : 0 : \beta) + v_1(1 : \alpha : 0 : 0) = u_2(1 : 0 : \gamma : \beta) + v_2(1 : \alpha : \gamma : 0)$$

We have a system of equations to solve

$$u_1 + v_1 - u_2 - v_2 = 0$$

$$v_1\alpha - v_2\alpha = 0$$

$$-u_2\gamma - v_2\gamma = 0$$

$$u_1\beta - u_2\beta = 0$$

Generalizing to arbitrary $(X : Y : Z : W)$, we have

$$\begin{pmatrix} X_1 & X_2 & -X_3 & -X_4 \\ Y_1 & Y_2 & -Y_3 & -Y_4 \\ Z_1 & Z_2 & -Z_3 & -Z_4 \\ W_1 & W_2 & -W_3 & -W_4 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \mathbf{0}$$

Setting the coefficient matrix M for the system of equations

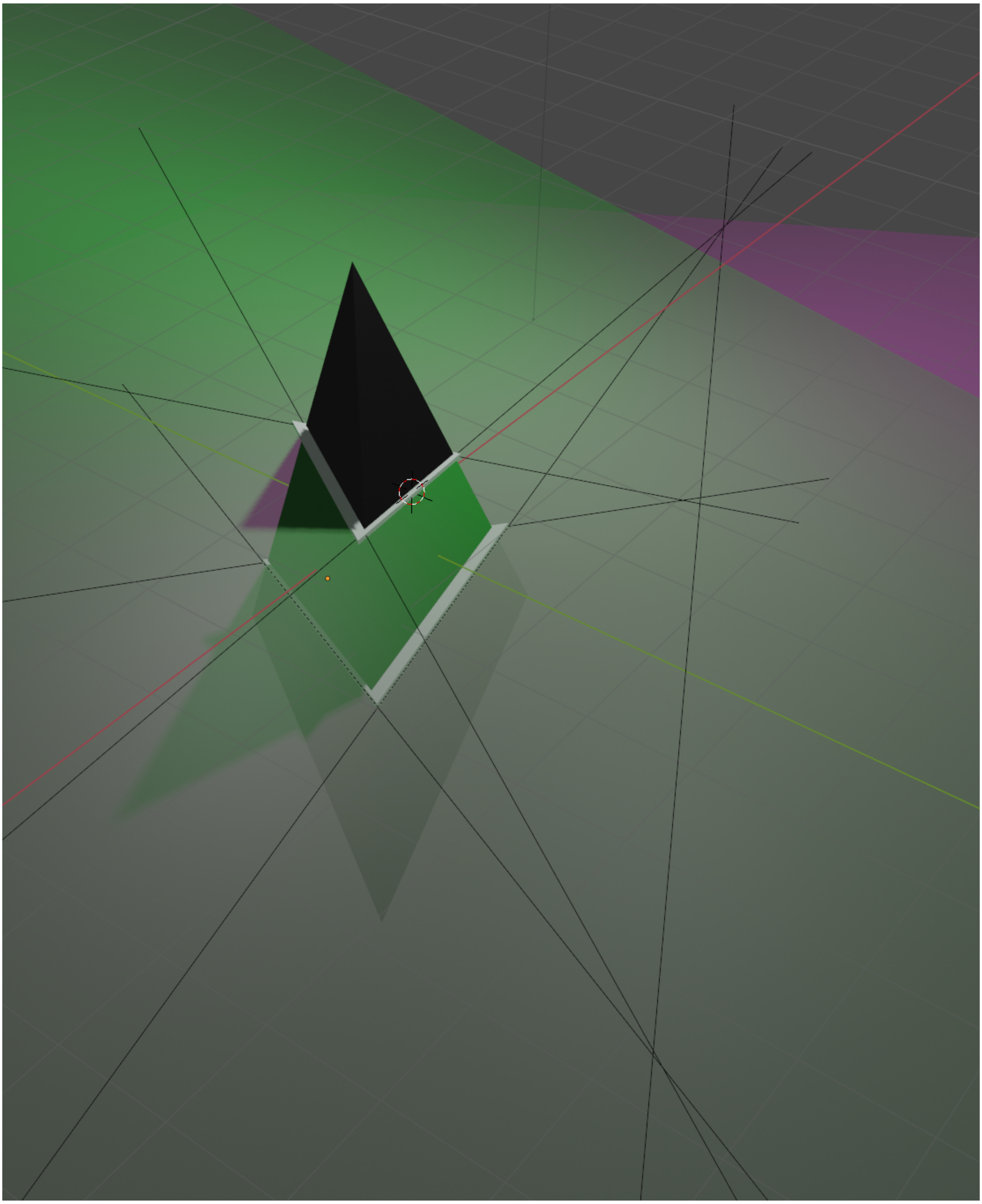
$$M = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & \alpha & 0 & -\alpha \\ 0 & 0 & -\gamma & -\gamma \\ \beta & 0 & -\beta & 0 \end{pmatrix}$$

```
sage: var("a b c")
(a, b, c)
sage: M = matrix([
....: [1, 1, -1, -1],
....: [0, a, 0, -a],
....: [0, 0, -c, -c],
....: [b, 0, -b, 0]
```

```
....: ])
sage: M * vector([1, -1, 1, -1])
(0, 0, 0, 0)
sage: M.right_kernel()
Vector space of degree 4 and dimension 1 over Symbolic Ring
Basis matrix:
[ 1 -1  1 -1]
```

1.3 Visualization

See `descargues.blend`.



2 Duality

$$\mathbb{P}^2, \qquad \mathbb{R}[X, Y, Z]$$

- P - statement about lines and points
- P' - `s/line/point` and `s/point/line`

2.0.1 Example

- P : through points Q_1 and Q_2 there is a unique line ℓ.
- P' : through lines q_1 and q_2 there is a unique point L.

Let A be a point such that

$$A = [A_0 : A_1 : A_2]$$

then the statement is equivalent to

$$\alpha_0 A_0 + \alpha_1 A_1 + \alpha_2 A_2 = 0$$

$$f = \alpha_0 X + \alpha_1 Y + \alpha_2 Z$$

$$f(A) = 0$$

$$f = [\alpha_0 : \alpha_1 : \alpha_2]$$

Now observe a startling result.

$$g = A_0 X + A_1 Y + A_2 Z$$

$$B = [\alpha_0 : \alpha_1 : \alpha_2]$$

$$\begin{aligned} g(B) &= A_0\alpha_0 + A_1\alpha_1 + A_2\alpha_2 \\ &= 0 \end{aligned}$$

In other words, the dot product is symmetric due to the ring's commutativity.