A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 2EB

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Problem

Let $\mathscr{F}(\mathbb{R}) \to \mathbb{R}$ be defined by $\alpha(f) = f(1)$ and let $\beta : \mathscr{F}(\mathbb{R}) \to \mathbb{R}$ be defined by $\beta(f) = f(2)$.

Let J be the set of all the functions from $\mathbb{R}$ to $\mathbb{R}$ whose graph passes through the point (1, 0) and let K be the set of all the functions whose graph passes through (2, 0). Use the FHT to prove that $\mathbb{R} \cong \mathscr{F}(\mathbb{R})/J$ and $\mathbb{R} \cong \mathscr{F}(\mathbb{R})/K$

Step-by-step solution

Step 1 of 4

Consider the two functions

$$\alpha : F(R) \to R,$$
$$\beta : F(R) \to R,$$

defined by

$$\alpha(f) = f(1),$$
$$\beta(f) = f(2).$$

Here, $F(R)$ represents the group of all functions from R to R.

Suppose that J is the set of all real functions whose graph passes through the point $(1, 0)$ and let K is the set of all functions whose graph passes through the point $(2, 0)$. Objective is to use the fundamental homomorphism theorem and prove that

$R \cong f(R)/J$ and $R \cong f(R)/K$.

Comment

Step 2 of 4

According to the fundamental homomorphism theorem, if $f : G \to H$ is a homomorphism of G onto H, with kernel K then

$H \cong G/K$.

The function α is onto because for all $x \in R$, one can define a function $f : R \to R$ such that $f(1) = x$. If one finds the kernel of mapping α, it will be

$\ker \alpha = \{h \in F(R) : \alpha(h) = 0\}$

The condition $\alpha(h) = 0$ implies that $h(1) = 0$. That is, the function *h* will be the member of $\ker \alpha$ if it satisfies the condition $h(1) = 0$. Equivalently, the kernel is the set of all real functions whose graph passes through the point $(1, 0)$. Thus, *J* will form the kernel of α.

Comment

Step 3 of 4

Since $\alpha : F(R) \to R$ is an onto homomorphism with kernel *J*, therefore by the fundamental homomorphism theorem $R \cong f(R)/J$.

Similarly, The function β is onto because for all $x \in R$, one can define a function $f : R \to R$ such that $f(2) = x$. If one finds the kernel of mapping β, it will be

$\ker \beta = \{h \in F(R) : \beta(h) = 0\}$

The condition $\beta(h) = 0$ implies that $h(2) = 0$. That is, the function *h* will be the member of $\ker \beta$ if it satisfies the condition $h(2) = 0$. Equivalently, the kernel is the set of all real functions whose graph passes through the point $(2, 0)$. Thus, *K* will form the kernel of β.

Comment

Step 4 of 4

Thus, $R \cong f(R)/K$.

Comment