A Book of Abstract Algebra | (2nd Edition)

Chapter 33, Problem 4EE

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Problem

Let K be a finite extension of F, where K is a root field over F, with $G = Gal(K: F)$ a solvable group. As remarked in the text, we will assume that F contains the required roots of unity. By Exercise D, let $H_0, \ldots, H_n$ be a solvable series for G in which every quotient H_{i+1}/H_i is cyclic of prime order. For any $i = 1, \ldots, n$, let F_i. and F_{i+1} be the fixfields of H_i and H_{i+1}.

Prove that F_i is the root field of $x^p - c^p$ over F_{i+1}.

Step-by-step solution

Step 1 of 4

Here, objective is to prove that F_i is the root field of $x^p - c^p$ over F_{i+1}

Consider

ω is a primitive p^{th} root of unity and $c^p \in F_{i+1}$

Comment

Step 2 of 4

Root field:

The field contains a given field in which every polynomial can be written as a product of linear factors.

Comment

Step 3 of 4

$G = Gal(K:F)$ is a solvable group.

F is the fixed field of G.

Where, K is a the finite extension of F.

Consider F_i and F_{i+1} are the fixed fields of H_i and H_{i+1}

Comment

Step 4 of 4

Consider the polynomial $x^p - c^p$.

The root of above polynomial is a primitive p^{th} root of unity

$$x^p - c^p = 0$$
$$x^p = c^p$$
$$x = \sqrt[p]{c^p}\ \omega$$
$$x = \omega c$$
$$x = c$$
$$c^p \in F_{i+1}$$
$$c \in F_i$$

F_i is the root field of $x^p - c^p$ over F_{i+1}

Hence, proved

Comment