A Book of Abstract Algebra | (2nd Edition)

Problem

Let *G* and *H* be groups. Suppose *J* is a normal subgroup of *G* and *K* is a normal subgroup of *H*.

Use the FHT to conclude that $(G \times H)/(J \times K) \cong (G/J) \times (H/K)$.

Step-by-step solution

Step 1 of 4

Suppose that *G* and *H* are two arbitrary groups. Also let *J* is a normal subgroup of *G* and *K* is a normal subgroup of *H*.

Consider a mapping $f: G \times H \to (G/J) \times (H/K)$ defined by

$f(x, y) = (Jx, Ky)$.

Objective is to prove that $(G \times H)/(J \times K) \cong (G/J) \times (H/K)$, by using fundamental homomorphism theorem.

According to the fundamental homomorphism theorem, if $f: G \to H$ is a homomorphism of *G* onto *H*, with kernel *K* then

$H \cong G/K$.

Comment

Step 2 of 4

First prove that *f* is a homomorphism from $G \times H$ onto $(G/J) \times (H/K)$.

Consider two typical elements $f(x_1, y_1), f(x_2, y_2)$ of direct product group $G \times H$ such that $f(x_1, y_1) = (Jx_1, Ky_1)$ and $f(x_2, y_2) = (Jx_2, Ky_2)$. Then

$$\begin{aligned} f(x_1, y_1)\cdot f(x_2, y_2) &= (Jx_1, Ky_1)(Jx_2, Ky_2) \\ &= (Jx_1 \cdot Jx_2, Ky_1 \cdot Ky_2) \\ &= (Jx_1x_2, Ky_1y_2) \\ &= f(x_1x_2, y_1y_2) \end{aligned}$$

The third equality is obtained from the property of cosets. Therefore, *f* is a homomorphism.

Also, for every $(Jx, Ky) \in (G/J)\times(H/K)$ there corresponds $(x, y) \in G\times H$ such that $f(x, y) = (Jx, Ky)$. That is, function *f* is onto.

Comment

Step 3 of 4

Now, the kernel of *f* will be:

$$\ker f = \{(x, y) \in G\times H : f(x, y) = e\},$$

where $e = (J, K)$ is the identity of $(G/J)\times(H/K)$. Substitute $f(x, y) = (Jx, Ky)$ and $e = (J, K)$ in kernel set and get,

$$\ker f = \{(x, y) \in G\times H : (Jx, Ky) = (J, K)\}.$$

On comparing the equation $(Jx, Ky) = (J, K)$, one get,

$$Jx = J, Ky = K.$$

By the coset property, the condition $Jx = J$ implies that $x \in J$. Similarly, from $Ky = K$ it implies that $y \in K$.

Thus,

$$\begin{aligned} \ker f &= \{(x, y) \in G\times H : x \in J, y \in K\} \\ &= J\times K. \end{aligned}$$

Since $x \in J, y \in K$, therefore $(x, y) \in J\times K$.

Comment

Step 4 of 4

Hence, by fundamental homomorphism theorem it conclude that

$$(G\times H)/(J\times K) \cong (G/J)\times(H/K).$$

Comment