A Book of Abstract Algebra | (2nd Edition)

Chapter 28, Problem 7ED

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Problem

Let V be a finite-dimensional vector space. Let dim V designate the dimension of V. Prove each of the following:

If $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is a basis of V, so is $\{k_1\mathbf{a}_1, \ldots, k_n a_n\}$ for any nonzero scalars

Step-by-step solution

Step 1 of 3

For any vector space with basis $(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$, it can be thought of as n dimensional vector space with $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ representing different directions. These vectors are linearly independent.

In vector form basis of this subspace with respect to $(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ is $\begin{pmatrix}1\\ \vdots\\ 0\end{pmatrix}, \cdots, \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}, \cdots \begin{pmatrix}0\\ \vdots\\ 1\end{pmatrix}$. Here a 1 is placed at every pivot position with $\mathbf{a}_1$ taking 1st position, $\mathbf{a}_2$ taking second position and $\mathbf{a}_n$ **c** taking n^{th} position.

Comment

Step 2 of 3

This matrix is full rank matrix with rank equal to number of rows and columns.

Comment

Step 3 of 3

Now, another set of vectors given is $(k_1\mathbf{a}_1, k_2\mathbf{a}_2, ..., k_n\mathbf{a}_n)$. This set can be represented in vector from with respect to basis $(\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n)$ by placing coefficients of $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n$ at 1st, 2nd and *n*th position.

$$\begin{pmatrix} k_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ k_2 \\ \vdots \\ 0 \end{pmatrix} \cdots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots \begin{pmatrix} 0 \\ 0 \\ \vdots \\ k_n \end{pmatrix}$$

To check linear independency of these vectors a matrix with these vectors as rows is reduced to echelon form. If that matrix is full row/ column matrix or is a non-singular matrix then these vectors are independent.

This matrix is already in row reduced echelon form with *n* pivots.

$\boxed{\text{Hence vectors } (k_1\mathbf{a}_1, k_2\mathbf{a}_2, ..., k_n\mathbf{a}_n) \text{ are linearly independent and forms a basis}}$.

Comment