A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 1EA

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Problem

In each of the following, use the fundamental homomorphism theorem to prove that the two given groups are isomorphic. Then display their tables.

$\mathbb{Z}_5$; and $\mathbb{Z}_{20}/\langle 5\rangle$.

Step-by-step solution

Step 1 of 4

Consider the two groups Z_3 and $Z_6/\langle 3\rangle$, where $\langle 3\rangle$ denotes the subgroup generated by 3. Objective is to prove that these two groups are isomorphic by using the fundamental homomorphism theorem.

According to the fundamental homomorphism theorem, if $f:G\to H$ is a homomorphism of *G* onto *H*, with kernel *K* then

$$H\cong G/K.$$

Comment

Step 2 of 4

Consider the function $f:Z_6\to Z_3$ given by

$$f=\begin{pmatrix}0&1&2&3&4&5\\0&1&2&0&1&2\end{pmatrix}.$$

To show that this mapping *f* is homomorphism, one must show that

$$f(a+b)=f(a)+f(b)$$

for all choices of *a* and *b* in Z_6.

Since both Z_6 and Z_3 are commutative, even that brute force approach needs lesser additions. Observe that for all $a\in Z_6$,

$a \cong f(a)(\bmod 3)$.

According to congruence law, if $x \cong f(x)(\bmod 3)$ and $y \cong f(y)(\bmod 3)$, then

$x + y \cong f(x) + f(y)(\bmod 3)$.

But, by the defined *f*,

$x + y \cong f(x + y)(\bmod 3)$.

Comment

Step 3 of 4

Since congruence relation is always transitive, it gives

$f(x + y) \cong f(x) + f(y)(\bmod 3)$.

Therefore, *f* preserves sums and is a homomorphism. Since each element of Z_3 has the pre-image, so *f* is onto.

By the definition of *f*, only element 0, 3 of Z_6 maps to identity. Therefore, $\ker f = \{0, 3\}$, that is, kernel of *f* is generated by 3. So, $\ker f = \langle 3 \rangle$.

Hence, the map *f* is homomorphism from Z_6 onto Z_3 with kernel $\ker f = \langle 3 \rangle$.

The addition table of Z_3 will be:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Comment

Step 4 of 4

Hence, by the fundamental homomorphism theorem it concludes that

$Z_3 \cong Z_6 / \langle 3 \rangle$.

Comment