A Book of Abstract Algebra | (2nd Edition)

Chapter 17, Problem 1EE

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Problem

A *quaternion*(in matrix form) is a 2 × 2 matrix of complex numbers of the form

$$\alpha = \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$$

Prove that the set of all the quaternions, with the matrix addition and multiplication explained on pages 7 and 8, is a ring with unity. This ring is denoted by the symbol $\mathcal{Q}$. Find an example to show that $\mathcal{Q}$ is not commutative. (You may assume matrix addition and multiplication are associative and obey the distributive law.)

Step-by-step solution

Step 1 of 5

Consider a quaternion in matrix form, that is, a 2×2 matrix of complex numbers of the form:

$$\alpha = \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}.$$

The addition of any two elements is defined as follows:

$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} + \begin{pmatrix} x+yi & z+wi \\ -z+wi & x-yi \end{pmatrix} = \begin{pmatrix} a+x+bi+yi & c+z+di+wi \\ -c-z+di+wi & a+x-bi-yi \end{pmatrix}$$

and the multiplication is same as ordinary multiplication of matrices.

Objective is to show that the set of all the quaternions is a ring with unity.

Comment

Step 2 of 5

First show that the set of all the quaternions with addition is an abelian group.

(1) The sum is again a 2×2 complex matrix, so addition is closed.

(2) Associative: since matrix addition is associative, so here also. Thus, the addition of complex matrices is associative.

(3) Addition is commutative

$$\begin{aligned}\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}+\begin{pmatrix} x+yi & z+wi \\ -z+wi & x-yi \end{pmatrix} &= \begin{pmatrix} a+x+bi+yi & c+z+di+wi \\ -c-z+di+wi & a+x-bi-yi \end{pmatrix} \\ &= \begin{pmatrix} x+a+yi+bi & z+c+wi+di \\ -z-c+wi+di & x+a-yi-bi \end{pmatrix} \\ &= \begin{pmatrix} x+yi & z+wi \\ -z+wi & x-yi \end{pmatrix}+\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}.\end{aligned}$$

(4) Additive identity or zero element is the 2×2 zero matrix because

$$\begin{aligned}\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} a+bi+0 & c+di+0 \\ -c+di+0 & a-bi+0 \end{pmatrix} \\ &= \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}.\end{aligned}$$

Note that $0=0+0i$ is the complex number with the imaginary part zero.

(5) The negative of every 2×2 matrix $\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$ will be $\begin{pmatrix} -a-bi & -c-di \\ c-di & -a+bi \end{pmatrix}$ because

$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}+\begin{pmatrix} -a-bi & -c-di \\ c-di & -a+bi \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, the set of all the quaternions is an abelian additive group.

Comment

Step 3 of 5

Now, since product of matrices is associative. So, multiplication is associative here also.

Thus, set of all the quaternions will satisfy the distributive law because addition and multiplication both are associative.

Next, the unity of set of all the quaternions will be $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ because

$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}\cdot\begin{pmatrix} 1+0i & 0 \\ 0 & 0+0i \end{pmatrix}=\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}.$$

Thus, set of all the quaternions is a ring with unity. But this set is not commutative.

Comments (1)

Step 4 of 5

To check this consider the following example:

$$\begin{pmatrix} i & 1+i \\ -1+i & 0-i \end{pmatrix} \cdot \begin{pmatrix} 1+i & i \\ i & 1-i \end{pmatrix} = \begin{pmatrix} i-1+i-1 & -1+1+1 \\ -1-1+1 & -i-1-i-1 \end{pmatrix}$$

$$= \begin{pmatrix} 2i-2 & 1 \\ -1 & -2i-2 \end{pmatrix}$$

$$\begin{pmatrix} 1+i & i \\ i & 1-i \end{pmatrix} \cdot \begin{pmatrix} i & 1+i \\ -1+i & 0-i \end{pmatrix} = \begin{pmatrix} -2 & 1+2i \\ 2i-1 & -2 \end{pmatrix}$$

Since both the product are not same, therefore it is not commutative.

Comment

Step 5 of 5

Hence, the set of all the quaternions is a non-commutative ring with unity.

Comment