A Book of Abstract Algebra | (2nd Edition)

Problem

Let F be a field, and let $a(x)$, $b(x) \in F[x]$. Prove the following:

If $a(x)$ and $b(x)$ have a common root c in some extension of F, they have a common factor of positive degree in $F[x]$. [Use the fact that $a(x)$, $b(x) \in \ker \sigma_c$.]

Step-by-step solution

Step 1 of 3

Consider that F is any field and $a(x), b(x) \in F[x]$. Objective is to prove that if $a(x)$ and $b(x)$ have a common root c in some extension of F, they have a common factor of positive degree in $F[x]$.

Let $a(x) = a_0 + a_1x + \cdots + a_nx^n$ and $b(x) = b_0 + b_1x + \cdots + b_nx^n$. Since $a(x)$ has a root c, therefore

$a(c) = 0$.

Similarly, $b(c) = 0$.

Comment

Step 2 of 3

Consider the following result:

In F every polynomial $a(x)$ of degree n has exactly n roots say $c_1, \ldots c_n$. Then polynomial can be factored as

$a(x) = k(x - c_1)\cdots(x - c_n)$.

Using this result, one can factor the $a(x)$ as:

$a(x) = (x - c)q(x)$,

where $q(x) \in F[x]$. Similarly,

$b(x) = (x - c)r(x)$,

for some polynomial $r(x) \in F[x]$.

Comment

Step 3 of 3

Thus, the factor $x - c$ is common for $a(x)$ and $b(x)$ both.

Comment