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Chapter 27, Problem 2EG | Bookmark | Show all steps: ON

Problem

Let F be a field, and let c be transcendental over F. Prove the following:

$F(c)$ is the field of quotients of $\{a(c): a(x) \in F[x]\}$, and is isomorphic to $F(x)$, the field of quotients of $F[x]$.

Step-by-step solution

Step 1 of 3

Consider that F is any arbitrary field and let $c \in F$ is transcendental over F. Objective is to prove that $F(c)$ is the field of quotients of $\{a(c): a(x) \in F[x]\}$. Also it is isomorphic to $F(x)$, the field of quotients of $F[x]$.

Assume that A is the field of quotients of $\{a(c): a(x) \in F[x]\}$. Since $F(c)$ is a field which contains F and c, therefore $F(c)$ will contain each

$$a(c) = a_0 + a_1 c + \cdots + a_n c^n$$

where $a_0, \ldots, a_n \in F$.

Comment

Step 2 of 3

Therefore, $A \subseteq F(c)$.

Next by definition, $F(c)$ is contained in any field containing F and c both. Thus, $F(c) \subseteq A$. On combining both the relation, one get $F(c) = A$.

Hence, $F(c)$ is the field of quotients of $\{a(c): a(x) \in F[x]\}$.

The other part directly holds by the following Theorem:

Let K be an extension field of F and $c \in K$. Then c is transcendental over F if and only if $F(c)$ is isomorphic to $F(x)$.

Comment

Step 3 of 3

Hence, $F(c)$ is the field of quotients of $\{a(c): a(x) \in F[x]\}$, and is isomorphic to $F(x)$, the field of quotients of $F[x]$.

Comment