A Book of Abstract Algebra | (2nd Edition)

Problem

Prove each of the following:

All the roots of the polynomial $x^4 - 3x^2 + 1$ are constructible numbers.

A line is called constructible if it passes through two constructible points. A circle is called constructible if its center and radius are constructible.

Step-by-step solution

Step 1 of 4

Here, objective is to prove that all the roots of the polynomial $x^4 - 3x^2 + 1 = 0$ are constructible numbers.

Comment

Step 2 of 4

A number is constructed if and only if, it can be obtained by arithmetic operations and square roots but not higher order roots.

The root of irreducible polynomial, whose degree is not a power of two is not a constructible number.

Comment

Step 3 of 4

Consider the polynomial $x^4 - 3x^2 + 1 = 0$

The roots of the polynomial are,

$$(x^2)^2 - 3(x^2) + 1 = 0$$

$$x^2 = \frac{3 \pm \sqrt{5}}{2}$$

$$x = \pm\sqrt{\frac{3 \pm \sqrt{5}}{2}}$$

Comment

Step 4 of 4

By observing the roots of the polynomial, they are Square roots of sums of square roots of rationales are constructible.

The degree of the polynomial is 4, which is a power of two. Therefore the roots of the polynomial are constructible.

Therefore, all the roots of the polynomial $x^4 - 3x^2 + 1 = 0$ are constructible numbers.

Hence, proved

Comment