# A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 5EN

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## Problem

Let $G$ be a finite group, and $K$ a $p$-Sylow subgroup of $G$. Let $X$ be the set of all the conjugates of $K$. See Exercise M2. If $C_1$, $C_2 \in X$, let $C_1 \sim C_2$ iff $C_1 = aC_2a^{-l}$ for some $\alpha \in K$

Use part 4 to prove that $(G:N)$ is *not* a multiple of $p$.

## Step-by-step solution

**Step 1** of 4

Assume that $G$ is a finite group and $K$ a $p$-Sylow subgroup of $G$. Consider the set $X$ as the set of all the conjugates of $K$. Define an equivalence relation as:

If $C_1, C_2 \in X$, let $C_1 \approx C_2$ if and only if $C_1 = aC_2a^{-1}$ for some $a \in K$.

Note that the number of elements in $X$ is $kp+1$, for some integer $k$. Objective is to prove that $(G:N)$ is not a multiple of $p$, where $N = N(K)$.

Consider the following result:

If $aKa^{-1} = K$ and the order of $a$ is some power of $p$. Then $a \in K$.

Comment

**Step 2** of 4

First show that $|X| = (G:N)$. For this define the mapping $f : X \to (G:N)$ as:

$f(gKg^{-1}) = gN$.

The mapping $f$ is well defined because if

$$\begin{aligned} gKg^{-1} &= hKh^{-1} \\ g^{-1}(gKg^{-1})g &= g^{-1}(hKh^{-1})g \\ K &= (g^{-1}h)K(g^{-1}h)^{-1} \end{aligned}$$

It implies that $g^{-1}h$ is in *N*. And then

$$N = g^{-1}hN$$
$$gN = hN$$
$$f\left(gKg^{-1}\right) = f\left(hKh^{-1}\right)$$

Thus, there is a bijection between the conjugates of *K* and the cosets of *N* in *G*. Hence, the number of conjugates is nothing but equal to $(G : N)$.

Comment

**Step 3** of 4

Now, by using the equivalence relation, one have partitioned the order of *X* into some equivalence classes of size $p^i$, for some integer *i*, with exactly one class of size 1. This size one class is of identity class as $p^0 = 1$.

Therefore, the number of elements in $(G : N)$ is one more than a multiple of *p* and thus it cannot be divisible by *p*.

Comment

**Step 4** of 4

Hence, $(G : N)$is not a multiple of *p*.

Comment