A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 5EH

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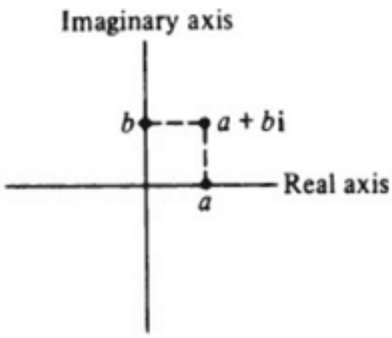
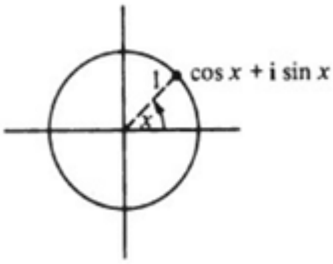
Problem

Every complex number $a + b\mathbf{i}$ may be represented as a point in the complex plane.





The *unit circle* in the complex plane consists of all the complex numbers whose distance from the origin is 1; thus, clearly, the unit circle consists of all the complex numbers which can be written in the form

$\cos x + \mathbf{i} \sin x$

for some real number x.

Use the FHT to conclude that $T \cong \mathbb{R}/\square 2\square$.

Step-by-step solution

Step 1 of 4

Consider the set T of all the complex numbers lying on the unit circle, with the operation multiplication as:

$T = \{\operatorname{cis} x : x \in R\}$,

where

$\text{cis } x = \cos x + i \sin x$.

Let $f : R \to T$ is a mapping defined by

$f(x) = \text{cis } x$.

Objective is to prove that $T \cong R/\langle 2 \rangle$ by using fundamental homomorphism theorem.

According to the fundamental homomorphism theorem, if $f : G \to H$ is a homomorphism of *G* onto *H*, with kernel *K* then

$H \cong G/K$.

Comment

Step 2 of 4

First show that the mapping *f* is a homomorphism from *R* onto *T*.

Let $x, y \in R$. Then, by the identity $\text{cis }(x+y) = (\text{cis } x)(\text{cis } y)$, one have

$$\begin{aligned} f(x)f(y) &= \text{cis } x \text{ cis } y \\ &= \text{cis }(x+y) \\ &= f(x+y). \end{aligned}$$

This is so because *R* is an additive group and *T* is a multiplicative group. The mapping *f* is clearly onto because $\text{cis } x \in T$ corresponds to $x \in R$.

Comment

Step 3 of 4

According to the definition of kernel:

$\ker f = \{x \in R : f(x) = e\}$,

where *e* is a multiplicative identity of *T*. Since $f(x) = \text{cis } x$, so equivalently

$\ker f = \{x \in G : \text{cis } x = e\}$.

By the trigonometric identities, one have

$$\begin{aligned} \text{cis }(2n\pi) &= \cos(2n\pi) + i \sin(2n\pi) \\ &= 0. \end{aligned}$$

where $n \in Z$. Thus, $Ker f = \{2n\pi : n \in Z\}$ or $Ker f = \langle 2\pi \rangle$.

Comment

Step 4 of 4

Thus, $f : R \to T$ is a homomorphism of *R* onto *T*, with kernel $Ker f = \langle 2\pi \rangle$. So, by the FHT

$T \cong R/\langle 2 \rangle$.

Comment