A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 5EQ

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Problem

As a provisional definition, let us call a finite abelian group "decomposable" if there are elements $a_1, \ldots, a_n \in G$ such that:

(DI) For every $x \in G$, there are integers $k_1, \ldots, k_n$ such that $x = a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}$. (D2)

If there are integers $l_1, \ldots, l_n$ such that

$a_1^{l_1} a_2^{l_2} \cdots a_n^{l_n} = e$ then $a_1^{l_1} = a_2^{l_2} = \cdots = a_n^{l_n} = e$.

If (D_1) and (D_2) hold, we will write $G = [a_1, a_2, \ldots, a_n]$. Assume this in parts 1 and 2.

Prove that if $a^{l_0} b_1^{l_1} \cdots b_n^{l_n} = e$. then $a^{l_0} = b_1^{l_1} = \cdots = b_n^{l_n} = e$.

Conclude that $G = [a, b_{,1}, \ldots, b_n]$.

Step-by-step solution

Step 1 of 3

Assume that G is a finite abelian group, and order of each element in G is some power of prime p. Let a is the highest possible order element in G and $H = \langle a \rangle$.

Objective is to prove that if $a^{l_0} b_1^{l_1} b_2^{l_2} \quad b_n^{l_n} = e$, then $a^{l_0}, b_1^{l_1}, \quad , b_n^{l_n} = e$. Also conclude that $G = [a, b_1, \ldots, b_n]$.

According to the statement of decomposable group:

If $a_1, \quad , a_n \in G$ and both the conditions $D1$, $D2$ holds, then $G = [a_1, a_2, \quad , a_n]$.

Comment

Step 2 of 3

One have seen that the following assumption is valid

$G / H = [Hb_1, ..., Hb_n]$,

for some $b_1, ..., b_n \in G$. Also, $G = [a, b_1, ..., b_n]$.

That is, $[a, b_1, ..., b_n]$ forms a basis of *G*, also it is known that the conditions *D*1, *D*2 holds. So, any element *x* in *G* can be written as a product of some powers of $a, b_1, ..., b_n$. Thus,

for every $x \in G$, there are integers $k_0, k_1, ..., k_n$ such that

$x = a^{k_0} b_1^{k_1} \quad b_n^{k_n}$.

Comment

Step 3 of 3

On combining the above statement with the *D*1, *D*2 conditions, it implies that if $a^{l_0} b_1^{l_1} b_2^{l_2} \quad b_n^{l_n} = e$, then $a^{l_0}, b_1^{l_1}, \quad , b_n^{l_n} = e$. And $G = [a, b_1, ..., b_n]$.

Comment