A Book of Abstract Algebra | (2nd Edition)

Chapter 24, Problem 3ED

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Problem

In each of the following, let A be an integral domain:

Prove: If A has characteristic 3, then $x + 2$ is a factor of $x^m + 2$ for all m. More generally, if A has characteristic p, then $x + (p - 1)$isa factor of $x^m + (p - 1)$ for all m.

Step-by-step solution

Step 1 of 2

Step 1 of 2

Here, we use induction method to prove

Basic case: for $m=1$

$x^m+2=x+2$, so obviously true

Hypothesis: assume $x+2$ is factor of x^m+2 for $m=k$

$\Rightarrow$ $x+2$ is a factor of x^k+2

To prove: $x+2$ is a factor of x^m+2 for $m=k+1$

As 3 is characteristic of A so

$$\begin{aligned}x^{k+1}+2 &\equiv x^{k+1}+3x^k+2\\ &\equiv x^{k+1}+2x^k+x^k+2\\ &\equiv x^k(x+2)+x^k+2\end{aligned}$$

$\Rightarrow$ $x+2$ is a factor of $x^{k+1}+2$ so true for $m=k+1$

Hence, $x+2$ is a factor of x^m+2 for all m

Comment

Step 2 of 2

Step 2 of 2

Just like in step 1, here also, we use induction method to prove.

Basic case: for $m=1$

$x^m+(p-1)=x+(p-1)$, so obviously true

Hypothesis: assume $x+(p-1)$ is a factor of $x^m+(p-1)$ for $m=k$

$\Rightarrow$ $x+(p-1)$ is a factor of $x^k+(p-1)$

To prove: $x+(p-1)$ is a factor of $x^m+(p-1)$ for $m=k+1$

As p is characteristic of A

$$\begin{aligned}x^{k+1}+(p-1) &\equiv x^{k+1}+px^k+(p-1)\\ &\equiv x^{k+1}+(p-1)x^k+x^k+(p-1)\\ &\equiv x^k(x+(p-1))+x^k+(p-1)\end{aligned}$$

$\Rightarrow$ $x+(p-1)$ is a factor of $x^{k+1}+(p-1)$ so true for $m=k+1$

Therefore, $x+(p-1)$ is a factor of $x^m+(p-1)$ for all m

Comment