A Book of Abstract Algebra | (2nd Edition)

Chapter 29, Problem 3EB

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Problem

Let F be a field of characteristic $\neq 2$. Let $a \neq b$ be in F.

Show that $x = \sqrt{a} + \sqrt{b}$ satisfies $x^4 - 2(a + b)x^2 + (a - b)^2 = 0$. Show that $\sqrt{a + b + 2\sqrt{ab}}$ also satisfies this equation. Conclude that

$$F(\sqrt{a + b + 2\sqrt{ab}}) = F(\sqrt{a}, \sqrt{b})$$

Step-by-step solution

Step 1 of 4

Consider a field F of characteristic $\neq 2$. Suppose that $a \neq b \in F$. Objective is to prove that $x = \sqrt{a} + \sqrt{b}$ and $x = \sqrt{a+b+2\sqrt{ab}}$ satisfy $x^4 - 2(a+b)x^2 + (a-b)^2 = 0$.

And then draw a conclusion that $F\left(\sqrt{a+b+2\sqrt{ab}}\right) = F\left(\sqrt{a}, \sqrt{b}\right)$.

If c and d are the roots of some irreducible polynomial then $F(c) = F(d)$. Also, consider $F\left(\sqrt{a}+\sqrt{b}\right) = F\left(\sqrt{a}, \sqrt{b}\right)$.

First check for $x = \sqrt{a} + \sqrt{b}$. Note that

$$
\begin{aligned}
x^2 &= \left(\sqrt{a}+\sqrt{b}\right)^2 \\
&= a+b+2\sqrt{ab}, \\
x^4 &= \left((a+b)+2\sqrt{ab}\right)^2 \\
&= (a+b)^2+4ab+4\sqrt{ab}(a+b).
\end{aligned}
$$

Comment

Step 2 of 4

Substitute these values in the quartic equation $x^4-2(a+b)x^2+(a-b)^2=0$ and get

$$
\begin{aligned}
x^4-2(a+b)x^2+(a-b)^2 &= (a+b)^2+4ab+4\sqrt{ab}(a+b)-2(a+b)\left(a+b+2\sqrt{ab}\right)+(a-b)^2 \\
&= (a+b)^2+4ab+4\sqrt{ab}(a+b)-2(a+b)^2-4\sqrt{ab}(a+b)+(a-b)^2 \\
&= -(a+b)^2+4ab+(a-b)^2 \\
&= -a^2-b^2-2ab+4ab+a^2+b^2-2ab
\end{aligned}
$$

That is, $x^4-2(a+b)x^2+(a-b)^2=0$.

Thus, $x=\sqrt{a}+\sqrt{b}$ satisfy $x^4-2(a+b)x^2+(a-b)^2=0$.

Comment

Step 3 of 4

Now assume that $x^2=y$, then equation $x^4-2(a+b)x^2+(a-b)^2=0$ will become quadratic:

$y^2-2(a+b)y+(a-b)^2=0$.

The roots of this equation will be:

$$
\begin{aligned}
y &= \frac{2(a+b)\pm\sqrt{4(a+b)^2-4(a-b)^2}}{2} \\
&= (a+b)\pm\sqrt{a^2+b^2+2ab-a^2-b^2+2ab} \\
&= (a+b)\pm 2\sqrt{ab}.
\end{aligned}
$$

And thus, $x=\sqrt{(a+b)\pm 2\sqrt{ab}}$. That is, being a root, $x=\sqrt{a+b+2\sqrt{ab}}$ satisfies the equation $x^4-2(a+b)x^2+(a-b)^2=0$.

Comment

Step 4 of 4

Since $x=\sqrt{a}+\sqrt{b}$ and $x=\sqrt{a+b+2\sqrt{ab}}$ are the roots of irreducible polynomial $x^4-2(a+b)x^2+(a-b)^2=0$, therefore by the result $F\left(\sqrt{a+b+2\sqrt{ab}}\right)=F\left(\sqrt{a}+\sqrt{b}\right)$. Since $F\left(\sqrt{a}+\sqrt{b}\right)=F\left(\sqrt{a},\sqrt{b}\right)$, therefore,

$$F\left(\sqrt{a+b+2\sqrt{ab}}\right)=F\left(\sqrt{a},\sqrt{b}\right).$$

Comment