A Book of Abstract Algebra (2nd Edition)

Chapter 32, Problem 1ED

Bookmark

Show all steps: ON

Problem

If $\alpha = \sqrt[4]{2}$ is a real fourth root of 2, then the four fourth roots of 2 are $\pm\alpha$ and $\pm i\alpha$. Explain parts 1–6, briefly but carefully:

$\mathbb{Q}(\alpha, i)$ is the root field of $x^4 - 2$ over $\mathbb{Q}$.

Step-by-step solution

Step 1 of 2

The objective is to show that $\mathbb{Q}\left(\sqrt[4]{2}, i\right)$ is the root field of $x^4 - 2$ over $\mathbb{Q}$.

Comment

Step 2 of 2

First, note that:

$$x^4 - 2 = \left(x^2 + \sqrt{2}\right)\left(x^2 - \sqrt{2}\right) = \left(x + i\sqrt[4]{2}\right)\left(x - i\sqrt[4]{2}\right)\left(x + \sqrt[4]{2}\right)\left(x - \sqrt[4]{2}\right).$$

Next, note that:

$$\mathbb{Q}\left(\pm\sqrt[4]{2},\ \pm i\sqrt[4]{2}\right) = \mathbb{Q}\left(\sqrt[4]{2},\ i\sqrt[4]{2}\right) = \mathbb{Q}\left(\sqrt[4]{2}, i\right).$$

$\mathbb{Q}\left(\sqrt[4]{2}, i\right)$ is clearly the splitting field of $x^4 - 2$ over $\mathbb{Q}$, because it is generated by the four roots of $x^4 - 2$.

The equalities obviously hold because $i \in \mathbb{Q}\left(\sqrt[4]{2},\ i\sqrt[4]{2}\right)$ and $i\sqrt[4]{2} \in \mathbb{Q}\left(\sqrt[4]{2}, i\right)$.

Comment