A Book of Abstract Algebra | (2nd Edition)

Problem

Let F be a field of characteristic $\neq 2$. Let $a \neq b$ be in F.

Prove that if $b \neq x^2 a$ for any $x \in F$, then $\sqrt{b} \notin F(\sqrt{a})$. Conclude that $F(\sqrt{a}, \sqrt{b})$ is of degree 4 over F.

Step-by-step solution

Step 1 of 3

Consider a field F of characteristic $\neq 2$. Suppose that $a \neq b \in F$. Objective is to prove that if $b \neq x^2 a$ for any $x \in F$, then $\sqrt{b} \notin F(\sqrt{a})$. Also show that degree of $F(\sqrt{a}, \sqrt{b})$ over F is 4.

Suppose, for the sake of contradiction, that $\sqrt{b} \in F(\sqrt{a})$. By the definition of extension field, the elements of $F(\sqrt{a})$ will be of the following form:

$$F(\sqrt{a}) = \{x + y\sqrt{a} : x, y \in F\}.$$

Comment

Step 2 of 3

Since $\sqrt{b} \in F(\sqrt{a})$, therefore $\sqrt{b} = x + y\sqrt{a}$. On squaring both the sides, one get

$$\begin{aligned} b &= \left(x + y\sqrt{a}\right)^2 \\ &= x + ay^2 + 2xy\sqrt{a} \end{aligned}$$

$$\sqrt{a} = \frac{b - \left(x + ay^2\right)}{2xy}.$$

This shows that $\sqrt{a}$ can be rational, which is not possible for any field *F*.

Thus, $\sqrt{b} \notin F(\sqrt{a})$.

Comment

Step 3 of 3

Since $\sqrt{b} \notin F(\sqrt{a})$, therefore $\sqrt{b}$ cannot be a root of a polynomial of degree 1 over $F(\sqrt{a})$ (such polynomial would have to be $x - \sqrt{b}$). But $\sqrt{b}$ is a root of $x^2 - b$, which is therefore the minimal polynomial of $\sqrt{b}$ over $F(\sqrt{a})$. Thus, $F(\sqrt{a}, \sqrt{b})$ is of degree 2 over $F(\sqrt{a})$. And therefore by theorem, $F(\sqrt{a}, \sqrt{b})$ is of degree 4 over *F*.

Comment