A Book of Abstract Algebra | (2nd Edition)

Chapter 16, Problem 1ED   Bookmark   Show all steps: ON

Problem

Let *G* be a group. By an *automorphism* of *G* we mean an isomorphism *f*: *G* → *G*.

The symbol Aut(*G*) is used to designate the set of all the automorphisms of *G*. Prove that the set Aut (*G*), with the operation ∘ of composition, is a group *by proving that Aut*(*G*) *is a subgroup of* S_G.

Step-by-step solution

Step 1 of 4

Suppose that *G* is a group and let $\mathrm{Aut}(G)$ is the set of all the automorphisms of *G*.

Objective is to prove that $\mathrm{Aut}(G)$with composition as a binary operation forms a group by proving that $\mathrm{Aut}(G)$is a subgroup of S_G. Here $S_G$denotes the group of all the permutations of *G*, that is, symmetric group on *G*.

One step test: If *H* is a nonempty subset of group *G*, then *H* will be subgroup of *G* if and only if for all $a, b \in H$

$ab^{-1} \in H$.

Comment

Step 2 of 4

The identity is an automorphism, so $\mathrm{Aut}(G)$is not empty. There is a need to show that composition of two automorphisms is an automorphism and that the inverse of an automorphism is an automorphism.

Suppose that $x, y \in \mathrm{Aut}(G)$. Let $z = x \quad y$. If $a, b \in G$ then

$$
\begin{aligned}
z(ab) &= x \quad y(ab) \\
&= x(y(ab)) \\
&= x(y(a)y(b)) \\
&= x(y(a))x(y(b))
\end{aligned}
$$

then

$$
\begin{aligned}
z(ab) &= x \quad y(a) \cdot x \quad y(b) \\
&= z(a)z(b).
\end{aligned}
$$

Thus permutation *z* is a homomorphism. Since *z* is a bijection (member of $\mathrm{Aut}(G)$), therefore *z* is an isomorphism and so $\mathrm{Aut}(G)$ is closed under composition.

Comment

Step 3 of 4

Now, for next let $c = ab$. Because of the permutation *x*, one can find a', b' and c' such that $x(a') = a, x(b') = b$ and $x(c') = c$.

Since *x* is a homomorphism, so

$$
\begin{aligned}
x(a'b') &= x(a') \cdot x(b') \\
&= ab.
\end{aligned}
$$

Apply x^{-1} both the sides and get

$$
\begin{aligned}
x^{-1}x(a'b') &= x^{-1}(ab) \\
a'b' &= x^{-1}(c) \\
a'b' &= c',
\end{aligned}
$$

that is, $x^{-1}(ab) = x^{-1}(a)x^{-1}(b)$. It shows that x^{-1} is an automorphism.

Comment

Step 4 of 4

Hence, by the one step test it conclude that $\mathrm{Aut}(G)$ is a subgroup of S_G and thus $\mathrm{Aut}(G)$ forms a group under the composition of mapping.

Comment