A Book of Abstract Algebra | (2nd Edition)

Chapter 28, Problem 6ED

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Problem

Let *V* be a finite-dimensional vector space. Let dim *V* designate the dimension of *V*. Prove each of the following:

If {**a**, **b**, **c**} is linearly independent, so is {**a** + **b**, **b** + **c**, **a** + **c**}.

Step-by-step solution

Step 1 of 3

For any subspace with basis $(\mathbf{a},\mathbf{b},\mathbf{c})$, it can be thought of as 3 dimensional vector space with **a, b, c** representing different directions.

In vector form basis of this subspace with respect to $(\mathbf{a},\mathbf{b},\mathbf{c})$ is $\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}$. Here a 1 is placed at every pivot position with **a** taking 1^{st} position, **b** taking second position and **c** taking 3^{rd} position.

Comment

Step 2 of 3

This matrix is full rank matrix with rank equal to number of rows and columns.

Comment

Step 3 of 3

Now, another set of vectors given is $(\mathbf{a}+\mathbf{b},\mathbf{b}+\mathbf{c},\mathbf{a}+\mathbf{c})$. This set can be represented in vector from with respect to basis $(\mathbf{a},\mathbf{b},\mathbf{c})$ by placing coefficients of **a, b, c** at 1st, 2nd or 3rd position.

$$\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}$$

To check linear independency of these vectors a matrix with these vectors as rows is reduced to echelon form. If that matrix is full row/ column matrix or is a non-singular matrix then these vectors are independent.

Row reducing matrix with these vectors as column vectors

$$\begin{pmatrix}1&1&0\\0&1&1\\1&0&1\end{pmatrix}$$
$$\xrightarrow{R_3\to R_3-R_1}$$
$$\begin{pmatrix}1&1&0\\0&1&1\\0&-1&1\end{pmatrix}$$
$$\xrightarrow{R_3\to R_3+R_2}$$
$$\begin{pmatrix}\boxed{1}&1&0\\0&\boxed{1}&1\\0&0&\boxed{2}\end{pmatrix}$$

As there are 3 pivots, this matrix is non-singular.

$\boxed{\text{Hence vectors } (\mathbf{a}+\mathbf{b},\mathbf{b}+\mathbf{c},\mathbf{a}+\mathbf{c}) \text{ are linearly independent}}$.

Comment