IE452/IE552: Algebraic and Geometric Methods in Data Analysis

Homework #1

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Question 1:

(a).

$$AA^T = \begin{bmatrix} 5 & 3 & -2 \\ 3 & 6 & 1 \\ -2 & 1 & 2 \end{bmatrix}$$

$$\begin{vmatrix} (5-\lambda) & 3 & -2 \\ 3 & (6-\lambda) & 1 \\ -2 & 1 & (2-\lambda) \end{vmatrix} = \lambda^3 - 13\lambda^2 + 38\lambda - 1 = 0$$

$$\lambda_{1,2,3} = 0.03, 4.38. 8.59$$

$$\text{Eigenvectors } v_{1,2,3} = \begin{bmatrix} -7.37 & 0.78 & 0.72 \\ -8.15 & 0.83 & -0.53 \\ 1 & 1 & 1 \end{bmatrix}, |v_{1,2,3}| = 11.04, 1.51, 1.34$$

$$U = \frac{v_{1,2,3}}{|v_{1,2,3}|} = [u_1, u_2, u_3] = \begin{bmatrix} -0.67 & -0.51 & 0.54 \\ -0.74 & 0.55 & -0.39 \\ 0.09 & 0.66 & 0.74 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 2 & 3 & 2 \\ 3 & 6 & 1 \\ 2 & 1 & 5 \end{bmatrix}$$

$$\begin{vmatrix} (2-\lambda) & 3 & 2 \\ 3 & (6-\lambda) & 1 \\ 2 & 1 & (5-\lambda) \end{vmatrix} = \lambda^3 - 13\lambda^2 + 38\lambda - 1 = 0$$

$$\lambda_{1,2,3} = 8.59, 4.38, 0.03$$

$$\text{Eigenvectors } v_{1,2,3} = \begin{bmatrix} 1.01 & 0.02 & -3.21 \\ 1.56 & -0.65 & 1.44 \\ 1 & 1 & 1 \end{bmatrix}, |v_{1,2,3}| = 2.11, 1.19, 3.66$$

$$V = \frac{v_{1,2,3}}{|v_{1,2,3}|} = [v_1, v_2, v_3] = \begin{bmatrix} 0.48 & 0.02 & -0.88 \\ 0.74 & -0.55 & 0.39 \\ 0.47 & 0.84 & 0.27 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & \sqrt{\lambda_3} \end{bmatrix} = \begin{bmatrix} 2.93 & 0 & 0 \\ 0 & 2.09 & 0 \\ 0 & 0 & 0.16 \end{bmatrix}$$

(b).

$$AA^T = \begin{bmatrix} 14 & 8 & 10 & 13 \\ 8 & 5 & 5 & 7 \\ 10 & 5 & 11 & 12 \\ 13 & 7 & 12 & 14 \end{bmatrix}$$

$$\begin{vmatrix} (14-\lambda) & 8 & 10 & 13 \\ 8 & (5-\lambda) & 5 & 7 \\ 10 & 5 & (11-\lambda) & 12 \\ 13 & 7 & 12 & (14-\lambda) \end{vmatrix} = \lambda^4 - 44\lambda^3 + 148\lambda^2 + 27\lambda = 0$$

$$\lambda_{1,2,3,4} = 40.35, 3.46, 0.19, 0$$

$$\text{Eigenvectors } v_{1,2,3,4} = \begin{bmatrix} 0.97 & -2.2 & -5.68 & -0.25 \\ 0.54 & -2.24 & 7.1 & -0.25 \\ 0.83 & 2.81 & 0.86 & 0.86 \\ 1 & 1 & 1 & 1 \end{bmatrix}, |v_{1,2,3,4}| = 1.71, 4.34, 9.19, 1.3$$

$$U = \frac{v_{1,2,3,4}}{|v_{1,2,3,4}|} = [u_1, u_2, u_3, u_4] = \begin{bmatrix} 0.57 & -0.51 & -0.62 & -0.19 \\ 0.31 & -0.52 & 0.77 & -0.19 \\ 0.49 & 0.65 & 0.09 & -0.59 \\ 0.59 & 0.23 & 0.11 & 0.77 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 3 & 8 & 6 \\ 8 & 23 & 17 \\ 6 & 17 & 18 \end{bmatrix}$$

$$\begin{vmatrix} (3-\lambda) & 8 & 6 \\ 8 & (23-\lambda) & 17 \\ 6 & 17 & (18-\lambda) \end{vmatrix} = \lambda^3 - 44\lambda^2 + 148\lambda - 27 = 0$$

$$\lambda_{1,2,3} = 8.59, 4.38, 0.03$$

$$\text{Eigenvectors } v_{1,2,3} = \begin{bmatrix} 0.41 & -0.26 & -2.86 \\ 1.17 & -0.76 & 1 \\ 1 & 1 & 1 \end{bmatrix}, |v_{1,2,3}| = 40.35, 3.46, 0.19$$

$$V = \frac{v_{1,2,3}}{|v_{1,2,3}|} = [v_1, v_2, v_3] = \begin{bmatrix} 0.26 & -0.02 & -0.95 \\ 0.73 & -0.6 & 0.33 \\ 0.63 & 0.78 & 0 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & \sqrt{\lambda_3} \end{bmatrix} = \begin{bmatrix} 6.35 & 0 & 0 \\ 0 & 1.86 & 0 \\ 0 & 0 & 0.44 \end{bmatrix}$$

(c).

$$AA^T = \begin{bmatrix} 104 & 8 & 90 \\ 8 & 38 & 9 \\ 90 & 9 & 90 \end{bmatrix}$$

$$\begin{vmatrix} (104-\lambda) & 8 & 90 \\ 8 & (38-\lambda) & 9 \\ 90 & 9 & (90-\lambda) \end{vmatrix} = \lambda^3 - 232\lambda^2 + 8487\lambda - 46656 = 0$$

$$\lambda_{1,2,3} = 188.23, 37.09. 6.68$$

$$\text{Eigenvectors } v_{1,2,3} = \begin{bmatrix} 1.08 & 3.28 & -0.92 \\ 0.12 & -38.71 & -0.05 \\ 1 & 1 & 1 \end{bmatrix}, |v_{1,2,3}| = 1.48, 38.86, 1.36$$

$$U = \frac{v_{1,2,3}}{|v_{1,2,3}|} = [u_1, u_2, u_3] = \begin{bmatrix} 0.73 & 0.08 & -0.68 \\ 0.08 & -1 & -0.04 \\ 0.68 & 0.03 & 0.74 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 14 & 8 & 10 & 13 \\ 8 & 5 & 5 & 7 \\ 10 & 5 & 11 & 12 \\ 13 & 7 & 12 & 14 \end{bmatrix}$$

$$\begin{vmatrix} (30-\lambda) & 6 & 51 & 33 \\ 6 & (36-\lambda) & 0 & 6 \\ 51 & 0 & (113-\lambda) & 76 \\ 33 & 6 & 76 & (53-\lambda) \end{vmatrix} = \lambda^4 - 232\lambda^3 + 8487\lambda^2 - 46656\lambda = 0$$

$$\lambda_{1,2,3,4} = 188.23, 37.09, 6.68, 0$$

$$\text{Eigenvectors } v_{1,2,3,4} = \begin{bmatrix} 0.69 & 1.81 & -1.97 & 0.38 \\ 0.07 & 15.47 & 0.2 & -0.23 \\ 1.48 & -2.22 & 0.23 & -0.85 \\ 1 & 1 & 1 & 1 \end{bmatrix}, |v_{1,2,3,4}| = 1.91, 15.77, 2.23, 1.38$$

$$V = \frac{v_{1,2,3}}{|v_{1,2,3}|} = [v_1, v_2, v_3] = \begin{bmatrix} 0.36 & 0.11 & -0.88 & 0.28 \\ 0.03 & 0.98 & 0.09 & -0.17 \\ 0.77 & -0.14 & 0.1 & -0.61 \\ 0.52 & 0.06 & 0.45 & 0.72 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 & 0 \\ 0 & 0 & \sqrt{\lambda_3} & 0 \end{bmatrix} = \begin{bmatrix} 13.72 & 0 & 0 & 0 \\ 0 & 6.09 & 0 & 0 \\ 0 & 0 & 2.59 & 0 \end{bmatrix}$$

Question 2:

(a). (Cramer-Chernoff Bound). For $x \geq E[X]$, we have $P[X \geq x] \leq e^{-\varphi_Z^*(x)}$

Let's say $t \geq E[X] = 0$

$=> P[X \geq t] \leq e^{-\varphi_X^*(t)} = e^{-\frac{\sigma t^2}{2}}$

where $\varphi_X^*(t) = \varphi_X(t) = \log m_x^{(t)}$

(b). $P[|X - E(X)| \leq \delta E(X)]$ where $0 < \delta < 1$ and $E(X) = \frac{1}{p} = \mu$

$= P[-\delta\mu \leq X - \mu \leq \delta\mu]$

$= P[\mu(1-\delta) \leq X \leq \mu(1+\delta)]$

$= 1 - P[X > \mu(1+\delta)]$

$= 1 - P[X < \mu(1-\delta)]$

By Chernoff bounds

- $P[X > \mu(1+\delta)] < (\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}})^{\mu}$ where $\delta > 0$
- $P[X < \mu(1-\delta)] < (\frac{e^{-\delta}}{(1+\delta)^{(1+\delta)}})^{\mu}$ where $0 < \delta \leq 1$

Since our δ cannot satisfy ≤ 1 the equality part use the first bound.

$$P[X > \mu(1+\delta)] < \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}$$

$$1 - P[X > \mu(1+\delta)] > 1 - (\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}})^{\mu}$$

Question 3:

(Random Projection Theorem)

$$\boldsymbol{P\left[\left|f(v) - \sqrt{k}|v|\right| \geq \varepsilon\sqrt{k}|v|\right] \leq 3e^{-ck\varepsilon^2}}$$ **there exists constant**

$$\boldsymbol{c > 0\ such\ that\ for\ \varepsilon\ \epsilon(0,1)}$$

Where $\boldsymbol{f(v) = (u_1.v, u_2.v, \ldots, u_k.v)\ , The\ projection\ f:\ R^d \rightarrow R^k\ , k \ll d}$

$$\boldsymbol{u_1, u_2, u_3 \ldots, u_k\ are\ Gaussian\ vectors, v\ be\ a\ fixed\ vector\ in\ R^d}$$

$P[|\|\mathrm{v}\|_2^2 - \|\mathrm{u}\|_2^2| \geq \varepsilon\|\mathrm{u}\|_2^2]$

$= P\left[\sqrt{k}|\|\mathrm{v}\|_2^2 - \|\mathrm{u}\|_2^2| \geq \sqrt{k}\varepsilon\|\mathrm{u}\|_2^2\right]$

$= P\left[\left|\sqrt{k}\|\mathrm{v}\|_2^2 - \sqrt{k}\|\mathrm{u}\|_2^2\right| \geq \sqrt{k}\varepsilon\|\mathrm{u}\|_2^2\right]$

$$= P\left[\left|\frac{\sqrt{k}\|\mathrm{Ru}\|_2^2}{\sqrt{k}} - \sqrt{k}\|\mathrm{u}\|_2^2\right| \geq \sqrt{k}\varepsilon\|\mathrm{u}\|_2^2\right]$$

$= P\left[\left|\|\mathrm{Ru}\|_2^2 - \sqrt{k}\|\mathrm{u}\|_2^2\right| \geq \sqrt{k}\varepsilon\|\mathrm{u}\|_2^2\right]$

Since $\|\mathrm{Ru}\|_2^2$ works as f(v) in the theorem and $\|\mathrm{v}\|_2^2 = |v|$ when v is a vector or in other words one column matrix.

$$P[|\|\mathrm{v}\|_2^2 - \|\mathrm{u}\|_2^2| \geq \varepsilon\|\mathrm{u}\|_2^2] = P\left[\left|\|\mathrm{Ru}\|_2^2 - \sqrt{k}\|\mathrm{u}\|_2^2\right| \geq \sqrt{k}\varepsilon\|\mathrm{u}\|_2^2\right] \leq 3\mathrm{e}^{-\mathrm{ck}\varepsilon^2} = f(.)$$

$$for\ a\ c > 0$$

By the random projection theorem.

Question 4:

Lets define i^{th} bernoulli trials of meeting friends action as X_i, and studying the research as Y_i where;

$$X_i = \begin{cases} 2, & with\ p = 3/4 \\ -1, & with\ p = 1/4 \end{cases}'$$

$$E(X_i) = 2 * \frac{3}{4} - 1 * \frac{1}{4} = \frac{5}{4} \quad AND$$

$$Var(X_i) = \left(2 - \frac{5}{4}\right)^2 * \frac{3}{4} + \left(-1 - \frac{5}{4}\right)^2 * \frac{1}{4} = 1.6875$$

$$Y_i = \begin{cases} 1002, & with\ p = 2/3 \\ -2001, & with\ p = 1/3 \end{cases}$$

$$E(Y_i) = 1002 * \frac{2}{3} - 2001 * \frac{1}{3} = 1 \quad AND$$

$$Var(X_i) = (1002 - 1)^2 * \frac{2}{3} + (-2001 - 1)^2 * \frac{1}{3} = 2004002$$

$$X = \sum_{i=1}^{1000} X_i \sim Binomial\ (\mu = 1250, \sigma^2 = 1687.5)$$

$$Y = \sum_{i=1}^{1000} Y_i \sim Binomial\ (\mu = 1000, \sigma^2 = 2004002000)$$

(Chebyysev Inequality). $\boldsymbol{P[|X - \mu| \geq k\sigma] \leq \frac{1}{k^2}}$

$P[|X - 1250| \geq 100] \leq 0.16875$

by Chebysev Inequality where $k = 2.434322477800736$

$P[|X - 1000| \geq 100] \leq 200400.2\ OR\ 1$

by Chebysev Inequality where $k = 0.0022338341433564$

Loss probabilty estimations are in hw1.rmd code.