CS536 Science of Programming - Assignment 2

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Problem 1

Solution:

a) No. The right side of the logical implications doesn't always have to logically imply the left side. In other words, satisfying Pumping Lemma doesn't necessarily mean that A is a regular language.

b) To prove that a language is not regular, we can prove by contradiction. If a string doesn't satisfy one of the three conditions in $D(s, p)$, it is a witness.

$(\exists i \geq 0.xy^i z \notin A) \vee (|y| \leq 0) \vee (|xy| > p)$

Problem 2

Solution:

a) e is a not a legal expression if $a \equiv b$. Because in conditional statement **if** B **then** e_1 **else** e_2 **fi**, we require expression e_1 and e_2 to have the same type. But if we try to evaluate the expression e when $a \equiv b$, the first expression $b[0]$ is likely to be an array, and $a[1][3]$ is going to be evaluated to an integer and they have different types, thus e is illegal.

b) Yes it is proper for e. If we evaluate the predicate in state σ:

$$\begin{aligned}
\sigma(\textbf{if } x \geq 0 \textbf{ then } b[0] \textbf{ else } a[1][3] \textbf{ fi}) \quad &= \sigma(\textbf{if } -1 \geq 0 \textbf{ then } b[0] \textbf{ else } a[1][3] \textbf{ fi}) \\
&= \sigma(\textbf{if } F \textbf{ then } b[0] \textbf{ else } a[1][3] \textbf{ fi}) \\
&= \sigma(a[1][3]) \\
&= \sigma(a[1])[3] \\
&= \sigma(\beta[3]) \\
&= \perp_e
\end{aligned}$$

It evaluates to a pseudo state with out of bound exception. Therefore, it doesn't satisfy e.

Problem 3

Solution:

		$\sigma[u \mapsto \alpha][v \mapsto \beta] = \sigma[v \mapsto \beta][u \mapsto \alpha]$?	$\sigma[u \mapsto \alpha][v \mapsto \beta] \equiv \sigma[v \mapsto \beta][u \mapsto \alpha]$?
$u \equiv v$	$\alpha = \beta$	True, the resulting state will be equal, because the state of the same variables u and v will be the same.	True, because u and v are the same variables and updating the value of the variable with the same samantic value twice will result in practically the same state. The procedure is the same.
$u \equiv v$	$\alpha \neq \beta$	False, because the first expression will end up assigning β to the same variable u and v, and the second expression will assign α.	False, because the expressions are not even semantically equal.
$u \not\equiv v$	$\alpha = \beta$	True, because u and v are different variables and the order of the update doesn't matter.	False, because updating procedures are different.
$u \not\equiv v$	$\alpha \neq \beta$	True, because u and v are different variables and the order of the update doesn't matter.	False, because updating procedures are different.

Problem 4

Solution:

a) $\sigma = \{x = 2, y = 4\}$

$$\begin{aligned}\sigma[x \mapsto \sigma(y)][y \mapsto \sigma(x)] &= \sigma[x \mapsto 4][y \mapsto \sigma(x)] \\ &= \{x = 4\}[y \mapsto \sigma(x)] \\ &= \{x = 4\}[y \mapsto 4] \\ &= \{x = 4, y = 4\}\end{aligned}$$

b) $\sigma = \{x = 2, y = 4\}$, $\tau = \sigma[x \mapsto 3]$, and $\gamma = \tau[y \mapsto \tau(x) * 4]$, what is γ?

$$\begin{aligned}\tau = \sigma[x \mapsto 3] &= \{x = 3, y = 4\} \\ \gamma = \tau[y \mapsto \tau(x) * 4] &= \tau[y \mapsto 3 * 4] \\ &= \tau[y \mapsto 12] \\ &= \{x = 3, y = 12\}\end{aligned}$$

Problem 5

Solution:

a) Does $\{x = 1, b = (5, 3, 6)\}$ satisfy $\forall x. \forall 0 \leq k < 3. x < b[k]$?

No. Because, we can find $x = 4$ such that $\{x = 1, b = (5, 3, 6)\}[x \mapsto 4] = \{x = 4, b = (5, 3, 6)\}$ doesn't satisfy the predicate. When $k = 1$, $b[1] = 3$ which is not greater than 4.

b) Does $\{b = (2, 5, 4, 8)\}$ satisfy $\exists m. 0 \leq m < 4 \wedge b[m] < 2$?

No, we can't find an element in array b that is less than 2. In other words, all elements of b is greater or equal to 2.

Problem 6

Solution:

a) $m \coloneqq 0; x \coloneqq 0; y \coloneqq 1;$ **while** $m < n$ **do** $m \coloneqq m + 1; x \coloneqq x + 1; y \coloneqq x; y \coloneqq y * x;$ **od** $m \coloneqq m * m$

b) $m \coloneqq n; p \coloneqq 1; y \coloneqq 1; m \coloneqq m - 1;$ **while** $m < n$ **do** $y \coloneqq y + 1; p \coloneqq p * y; m \coloneqq m - 1;$ **od**

Problem 7

Solution:

$S \equiv$ **if** $x > 0$ **then** $x := x + 1$ **else** $y := -2 * x$ **fi** and $W \equiv$ **while** $x > y$ **do** S **od** .

a) Evaluate $\langle W, \sigma \rangle$ where $\sigma \vDash y < x \leq 0$.

$$\begin{aligned}
\langle W, \sigma \rangle \quad &= \langle \textbf{ while } x > y \textbf{ do } S \textbf{ od } , \sigma \rangle \\
&\to \langle S, W, \sigma \rangle \qquad //T \text{ since } y < x \leq 0 \\
&= \langle \textbf{ if } x > 0 \textbf{ then } x := x + 1 \textbf{ else } y := -2 * x \textbf{ fi } , W, \sigma \rangle \\
&\to \langle y := -2 * x, W, \sigma \rangle \\
&\to \langle W, \sigma[y \mapsto -2 * x] \rangle \\
&\to \langle E, \sigma[y \mapsto -2 * x] \rangle \qquad //y \text{ becomes positive number and the while loop terminates}
\end{aligned}$$

b) Evaluate $\langle W, \sigma \rangle$ where $\sigma \vDash x > 0 \wedge y \leq 0$.

$$\begin{aligned}
\langle W, \sigma \rangle \quad &= \langle \textbf{ while } x > y \textbf{ do } S \textbf{ od } , \sigma \rangle \\
&\to \langle S, W, \sigma \rangle \\
&= \langle \textbf{ if } x > 0 \textbf{ then } x := x + 1 \textbf{ else } y := -2 * x \textbf{ fi } , W, \sigma \rangle \\
&\to \langle x := x + 1, W, \sigma \rangle \\
&\to \langle W, \sigma[x \mapsto x + 1] \rangle \\
&\to^* \langle W, \sigma[x \mapsto x + 2] \rangle \\
&\to^* \langle W, \sigma[x \mapsto x + 3] \rangle \\
&... \\
&\to \langle W, \perp_d \rangle
\end{aligned}$$

It diverges, because the value of x will always increase and it will never be less than 0.

Problem 8

Solution:

Let $W \equiv$ **while** $x > 0$ **do** S **od** , where $S \equiv$ **if** $x < y$ **then** $x := y/x$ **else** $x := x - 1; y := b[y]$ **fi** .

a) $M(S, \sigma)$ where $\sigma(x) = -2$ and $\sigma(y) = -1$

$$\begin{aligned}
M(S, \sigma) \quad &= M(S, \{x = -2, y = -1\}) \\
&= M(\textbf{ if } x < y \textbf{ then } x := y/x \textbf{ else } x := x - 1; y := b[y] \textbf{ fi } , \{x = -2, y = -1\}) \\
&= M(x := y/x, \{x = -2, y = -1\}) \\
&= M(x := -1/-2, \{x = -2, y = -1\}) \\
&= M(x := 0.5, \{x = -2, y = -1\}) \\
&= \{\{x = 0.5, y = -1\}\}
\end{aligned}$$

b) $M(W, \sigma)$ where $\sigma = \{x = 1, y = 2, b = (4, 2, 0)\}$

$$\begin{aligned}
M(W, \sigma) \quad &= M(W, \{x = 1, y = 2, b = (4, 2, 0)\}) \\
&= M(W, M(S, \{x = 1, y = 2, b = (4, 2, 0)\})) \\
&= M(W, M(x := y/x, \{x = 1, y = 2, b = (4, 2, 0)\})) \\
&= M(W, M(x := 2/1, \{x = 1, y = 2, b = (4, 2, 0)\})) \\
&= M(W, M(x := 2, \{x = 1, y = 2, b = (4, 2, 0)\})) \\
&= M(W, \{x = 2, y = 2, b = (4, 2, 0)\}) \\
&= M(W, M(S, \{x = 2, y = 2, b = (4, 2, 0)\})) \\
&= M(W, M(x := x - 1; y := b[y], \{x = 2, y = 2, b = (4, 2, 0)\})) \\
&= M(W, M(x := 1; y := b[y], \{x = 2, y = 2, b = (4, 2, 0)\})) \\
&= M(W, M(y := b[2], \{x = 1, y = 2, b = (4, 2, 0)\})) \\
&= M(W, M(y := 0, \{x = 1, y = 2, b = (4, 2, 0)\})) \\
&= M(W, \{x = 1, y = 0, b = (4, 2, 0)\}) \\
&= M(W, M(S, \{x = 1, y = 0, b = (4, 2, 0)\})) \\
&= M(W, M(x := x - 1; y := b[y], \{x = 1, y = 0, b = (4, 2, 0)\})) \\
&= M(W, M(x := 0; y := b[y], \{x = 1, y = 0, b = (4, 2, 0)\})) \\
&= M(W, M(y := b[0], \{x = 0, y = 0, b = (4, 2, 0)\})) \\
&= M(W, M(y := 4, \{x = 0, y = 0, b = (4, 2, 0)\})) \\
&= M(W, \{x = 0, y = 4, b = (4, 2, 0)\}) \\
&= \{\{x = 0, y = 4, b = (4, 2, 0)\}\}
\end{aligned}$$

c) $M(W, \sigma)$ where $\sigma = \{x = 2, y = 2, b = (0,1,2)\}$

$$
\begin{aligned}
M(W, \sigma) \quad & = M(W, \{x = 2, y = 2, b = (0,1,2)\}) \\
& = M(W, M(S, \{x = 2, y = 2, b = (0,1,2)\})) \\
& = M(W, M(x \coloneqq x - 1; y \coloneqq b[y], \{x = 2, y = 2, b = (0,1,2)\})) \\
& = M(W, M(x \coloneqq 1; y \coloneqq b[y], \{x = 2, y = 2, b = (0,1,2)\})) \\
& = M(W, M(y \coloneqq b[y], \{x = 1, y = 2, b = (0,1,2)\})) \\
& = M(W, M(y \coloneqq b[2], \{x = 1, y = 2, b = (0,1,2)\})) \\
& = M(W, M(y \coloneqq 2, \{x = 1, y = 2, b = (0,1,2)\})) \\
& = M(W, \{x = 1, y = 2, b = (0,1,2)\}) \\
& = M(W, M(S, \{x = 1, y = 2, b = (0,1,2)\})) \\
& = M(W, M(x \coloneqq y/x, \{x = 1, y = 2, b = (0,1,2)\})) \\
& = M(W, M(x \coloneqq 2, \{x = 1, y = 2, b = (0,1,2)\})) \\
& = M(W, \{x = 2, y = 2, b = (0,1,2)\}) \\
& = ... \\
& = \{\perp_d\}
\end{aligned}
$$

d) $M(W, \sigma)$ where $\sigma = \{x = 8, y = 2, b = (4,2,0)\}$

$$
\begin{aligned}
M(W, \sigma) \quad & = M(W, \{x = 8, y = 2, b = (0,1,2)\}) \\
& = M(W, M(S, \{x = 8, y = 2, b = (4,2,0)\})) \\
& = M(W, M(x \coloneqq x - 1; y \coloneqq b[y], \{x = 8, y = 2, b = (4,2,0)\})) \\
& = M(W, M(x \coloneqq 7; y \coloneqq b[y], \{x = 8, y = 2, b = (4,2,0)\})) \\
& = M(W, M(y \coloneqq b[2], \{x = 7, y = 2, b = (4,2,0)\})) \\
& = M(W, M(y \coloneqq 0, \{x = 7, y = 2, b = (4,2,0)\})) \\
& = M(W, \{x = 7, y = 0, b = (4,2,0)\}) \\
& = M(W, M(S, \{x = 7, y = 0, b = (4,2,0)\})) \\
& = M(W, M(x \coloneqq x - 1; y \coloneqq b[y], \{x = 7, y = 0, b = (4,2,0)\})) \\
& = M(W, M(x \coloneqq 6; y \coloneqq b[y], \{x = 7, y = 0, b = (4,2,0)\})) \\
& = M(W, M(y \coloneqq b[0], \{x = 6, y = 0, b = (4,2,0)\})) \\
& = M(W, M(y \coloneqq 4, \{x = 6, y = 0, b = (4,2,0)\})) \\
& = M(W, \{x = 6, y = 4, b = (4,2,0)\}) \\
& = M(W, M(S, \{x = 6, y = 4, b = (4,2,0)\})) \\
& = M(W, M(x \coloneqq x - 1; y \coloneqq b[y], \{x = 6, y = 4, b = (4,2,0)\})) \\
& = M(W, M(x \coloneqq 5; y \coloneqq b[y], \{x = 6, y = 4, b = (4,2,0)\})) \\
& = M(W, M(y \coloneqq b[4], \{x = 5, y = 0, b = (4,2,0)\})) \\
& = \{\perp_e\} \qquad\qquad \text{// out of bounds}
\end{aligned}
$$

e) No. Division by zero $= \{\perp_e\}$ state can't occur, because the condition of while loop gaurantees x must always be greater than 0.

Problem 9

Solution:

$S \equiv x \coloneqq sqrt(x)/b[y]$, $\sigma = \{b = (3, 0, -2, 4), x = \alpha, y = \beta\}$

1. Array index out of bounds: Since the size of the array is 4, when $\sigma(\beta) < 0$ or $\sigma(\beta) \geq 4$ it results in the pseudo state.

2. Division by zero: in $\sigma(b[y]) = 0$ state, there will be a runtime error. Since 0 is the second element of the array b, $\sigma(\beta) = 1$ will produce $\{\perp_e\}$.

3. Square root of negative number: $\sigma(\alpha) < 0$ will result in $\{\perp_e\}$.

Problem 10

Solution:

a) $\sigma \models \exists x \in S.p$ means for **this** state σ and for **some** $\alpha \in S$, it is the case that $\sigma[x \mapsto \alpha] \models p$.
b) $\sigma \models \forall x \in S.p$ means for **this** state σ and for **every** $\alpha \in S$, it is the case that $\sigma[x \mapsto \alpha] \models p$.
c) $\sigma \not\models \exists x \in S.p$ means for **this** state σ and for **every** $\alpha \in S$, it is the case that $\sigma[x \mapsto \alpha] \not\models p$.
d) $\sigma \not\models \forall x \in S.p$ means for **this** state σ and for **some** $\alpha \in S$, it is the case that $\sigma[x \mapsto \alpha] \not\models p$.
e) $\models \exists x \in S.p$ means for **every** state σ, we have $\sigma \models \exists x \in S.p$.
f) $\models \forall x \in S.p$ means for **every** state σ, we have $\sigma \models \forall x \in S.p$.
g) $\not\models \exists x \in S.p$ means for **some** state σ, we have $\sigma \not\models \exists x \in S.p$.
h) $\not\models \forall x \in S.p$ means for **some** state σ, we have $\sigma \not\models \forall x \in S.p$.