CS536 Science of Programming - Assignment 4

Batkhishig Dulamsurankhor - A20543498

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Problem 1

$p \equiv x > y \to \forall z.y \leq z < x \to \exists w.w > 0 \wedge (w * y < 0 \to f(w,z))$. Finish the syntactic substitutions.

a) $p[y+z/x] \equiv (x > y \to \forall z.y \leq z < x \to \exists w.w > 0 \wedge (w * y < 0 \to f(w,z)))[y+z/x]$
$\equiv (y+z) > y \to (\forall z.y \leq z < x \to \exists w.w > 0 \wedge (w * y < 0 \to f(w,z)))[y+z/x]$
$\equiv (y+z) > y \to \forall v.(y \leq z < x \to \exists w.w > 0 \wedge (w * y < 0 \to f(w,z)))[v/z][y+z/x]$
$\equiv (y+z) > y \to \forall v.(y \leq v < x \to \exists w.w > 0 \wedge (w * y < 0 \to f(w,v)))[y+z/x]$
$\equiv (y+z) > y \to \forall v.y \leq v < y+z \to \exists w.(w > 0 \wedge (w * y < 0 \to f(w,v)))[y+z/x]$
$\equiv (y+z) > y \to \forall v.y \leq v < y+z \to \exists w.w > 0 \wedge (w * y < 0 \to f(w,v))$

b) $p[x+w/y] \equiv (x > y \to \forall z.y \leq z < x \to \exists w.w > 0 \wedge (w * y < 0 \to f(w,z)))[x+w/y]$
$\equiv x > (x+w) \to \forall z.(y \leq z < x \to \exists w.w > 0 \wedge (w * y < 0 \to f(w,z)))[x+w/y]$
$\equiv x > (x+w) \to \forall z.x+w \leq z < x \to (\exists w.w > 0 \wedge (w * y < 0 \to f(w,z)))[x+w/y]$
$\equiv x > (x+w) \to \forall z.x+w \leq z < x \to (\exists v.(w > 0 \wedge (w * y < 0 \to f(w,z)))[v/w])[x+w/y]$
$\equiv x > (x+w) \to \forall z.x+w \leq z < x \to \exists v.(v > 0 \wedge (v * y < 0 \to f(v,z)))[x+w/y]$
$\equiv x > (x+w) \to \forall z.x+w \leq z < x \to \exists v.v > 0 \wedge (v * (x+w) < 0 \to f(v,z))$

c) $p[w+z/y] \equiv (x > y \to \forall z.y \leq z < x \to \exists w.w > 0 \wedge (w * y < 0 \to f(w,z)))[w+z/y]$
$\equiv x > (w+z) \to (\forall z.y \leq z < x \to \exists w.w > 0 \wedge (w * y < 0 \to f(w,z)))[w+z/y]$
$\equiv x > (w+z) \to (\forall v.(y \leq z < x \to \exists w.w > 0 \wedge (w * y < 0 \to f(w,z)))[v/z])[w+z/y]$
$\equiv x > (w+z) \to \forall v.(y \leq v < x \to \exists w.w > 0 \wedge (w * y < 0 \to f(w,v)))[w+z/y]$
$\equiv x > (w+z) \to \forall v.w+z \leq v < x \to (\exists w.w > 0 \wedge (w * y < 0 \to f(w,v)))[w+z/y]$
$\equiv x > (w+z) \to \forall v.w+z \leq v < x \to (\exists u.(w > 0 \wedge (w * y < 0 \to f(w,v)))[u/w])[w+z/y]$
$\equiv x > (w+z) \to \forall v.w+z \leq v < x \to \exists u.(u > 0 \wedge (u * y < 0 \to f(u,v)))[w+z/y]$
$\equiv x > (w+z) \to \forall v.w+z \leq v < x \to \exists u.u > 0 \wedge (u * (w+z) < 0 \to f(u,v))$

d) $p[x+y/z] \equiv (x > y \to \forall z.y \leq z < x \to \exists w.w > 0 \wedge (w * y < 0 \to f(w,z)))[x+y/z]$
$\equiv x > y \to (\forall z.y \leq z < x \to \exists w.w > 0 \wedge (w * y < 0 \to f(w,z)))[x+y/z]$

Problem 2

Solution:
a) True.
b) False.

Problem 3

Solution:
a) True.
b) False.
c) False.

d) False.
e) False.

Problem 4

Calculate $sp(i \leq j \land j < n+1, i \coloneqq f(i+j); j \coloneqq g(i-j))$.

$$\begin{aligned}
&sp(i \leq j \land j < n+1, i \coloneqq f(i+j); j \coloneqq g(i-j)) \\
&\qquad \equiv sp(sp(i \leq j \land j < n+1, i \coloneqq f(i+j)), j \coloneqq g(i-j)) \\
&\qquad \equiv sp(i_0 \leq j \land j < n+1 \land i = (f(i+j))[i_0, i], j \coloneqq g(i-j)) \\
&\qquad \equiv sp(i_0 \leq j \land j < n+1 \land i = f(i_0+j), j \coloneqq g(i-j)) \\
&\qquad \equiv i_0 \leq j_0 \land j_0 < n+1 \land i = f(i_0+j) \land j = g(i-j_0)
\end{aligned}$$

Problem 5

Calculate $sp(T, y \coloneqq x;\ \textbf{if } x < 0 \textbf{ then } y \coloneqq -y \textbf{ fi})$.

Let $p \equiv T$, $S \equiv y \coloneqq x;\ \textbf{if } x < 0 \textbf{ then } y \coloneqq -y \textbf{ fi}$

- $lhs(S) = \{y\}$
- $rhs(S) = \{x, y\}$
- $free(p) = \varnothing$
- $aged(p, S) = \{y\}$

$$\begin{aligned}
&sp(T, y \coloneqq x;\ \textbf{if } x < 0 \textbf{ then } y \coloneqq -y \textbf{ fi}) \\
&\qquad \equiv sp(sp(T, y \coloneqq x),\ \textbf{if } x < 0 \textbf{ then } y \coloneqq -y \textbf{ fi}) \\
&\qquad \equiv sp(T \land y = x,\ \textbf{if } x < 0 \textbf{ then } y \coloneqq -y \textbf{ fi}) \\
&\qquad \equiv sp(T \land y = x \land y = y_0 \land x < 0, y \coloneqq -y) \lor sp(T \land y = x \land y = y_0 \land x \geq 0, skip) \\
&\qquad \equiv (T \land y_0 = x \land y_0 = y_0 \land x < 0 \land y = -y_0) \lor (T \land y = x \land y = y_0 \land x \geq 0) \\
&\qquad \equiv (y_0 = x \land x < 0 \land y = -y_0) \lor (y = x \land y = y_0 \land x \geq 0)
\end{aligned}$$

Problem 6

Calculate $sp(x = y,\ \textbf{if } x \geq 0 \to x \coloneqq y+1; z \coloneqq x \ \square\ x \leq 0 \to y \coloneqq x-1; z \coloneqq y \textbf{ fi})$.

Let $p \equiv x = y$, $S \equiv\ \textbf{if } x \geq 0 \to x \coloneqq y+1; z \coloneqq x \ \square\ x \leq 0 \to y \coloneqq x-1; z \coloneqq y \textbf{ fi}$

- $lhs(S) = \{x, y, z\}$
- $rhs(S) = \{x, y\}$
- $rhs(p) \lor free(p, S) = \{x, y\}$
- $aged(p, S) = \{x, y\}$

$sp(x = y,\ \textbf{if } x \geq 0 \to x \coloneqq y+1; z \coloneqq x \ \square\ x \leq 0 \to y \coloneqq x-1; z \coloneqq y \textbf{ fi})$

- $sp(x = y \land x = x_0 \land y = y_0 \land x \geq 0, x \coloneqq y+1; z \coloneqq x)$
  $$\begin{aligned}
  &\equiv sp(sp(x = y \land x = x_0 \land y = y_0 \land x \geq 0, x \coloneqq y+1), z \coloneqq x) \\
  &\equiv sp(x_0 = y \land x_0 = x_0 \land y = y_0 \land x_0 \geq 0 \land x = y+1, z \coloneqq x) \\
  &\equiv x_0 = y \land x_0 = x_0 \land y = y_0 \land x_0 \geq 0 \land x = y+1 \land z = x
  \end{aligned}$$

- $sp(x = y \wedge x = x_0 \wedge y = y_0 \wedge x \leq 0, y := x - 1; z := y)$
  $\equiv sp(sp(x = y \wedge x = x_0 \wedge y = y_0 \wedge x \leq 0, y := x - 1), z := y)$
  $\equiv sp(x = y_0 \wedge x = x_0 \wedge y_0 = y_0 \wedge x \leq 0 \wedge y = x - 1, z := y)$
  $\equiv x = y_0 \wedge x = x_0 \wedge y_0 = y_0 \wedge x \leq 0 \wedge y = x - 1 \wedge z = y$

- $sp(x = y,\ \textbf{if } x \geq 0 \to x := y + 1; z := x \ \Box\ x \leq 0 \to y := x - 1; z := y \textbf{ fi})$
  $\equiv (x_0 = y \wedge x_0 = x_0 \wedge y = y_0 \wedge x_0 \geq 0 \wedge x = y + 1 \wedge z = x) \vee (x = y_0 \wedge x = x_0 \wedge y_0 = y_0 \wedge x \leq 0 \wedge y = x - 1 \wedge z = y)$

Problem 7

Let's calculate $p \Leftrightarrow wlp(S, L \leq R)$.

$wlp(S, L \leq R) \equiv (b[M] < x \to wlp(L := M + 1, L \leq R)) \wedge (b[M] \geq x \to wlp(R := M, L \leq R))$
$\equiv (b[M] < x \to M + 1 < R) \wedge (b[M] > x \to L \leq M)$

1. $\{M + 1 \leq R\} L := M + 1 \{L \leq R\}$ backward assignment
2. $(b[M] < x \to M + 1 \leq R) \wedge b[M] < x \Rightarrow M + 1 \leq R$ modus ponens
3. $p \wedge b[M] < x \Rightarrow (b[M] < x \to M + 1 \leq R) \wedge b[M] > x$ predicate logic
4. $p \wedge b[M] < x \Rightarrow M + 1 < R$ predicate logic
5. $\{p \wedge b[M] < x\} L := M + 1 \{L \leq R\}$ strengthen precondition 4,1
6. $\{L \leq M\} R := M \{L \leq R\}$ backward assignment
7. $p \wedge b[M] \geq x \Rightarrow L \leq M$ predicate logic
8. $\{p \wedge b[M] \geq x\} R := m \{L \leq R\}$ strengthen precondition 7,6
9. $\{p\} S \{L \leq R\}$ 5,8

Problem 8

Solution:

1. $\{p_1\} S_1 \{q_1\}$
2. $\{p_2\} S_2 \{q_2\}$
3. $\{(B \to p_1) \wedge (\neg B \to p_2)\}$ **if** B **then** S_1 **else** S_2 **fi** $\{q_1 \vee q_2\}$

Problem 9

Solution:

1. $\{p_1\} x := x * 2 \{p_2\}$
2. $\{p_2\} k := k + 1 \{p_3\}$
3. $\{p_1\} x := x * 2; k := k + 1 \{p_3\}$ sequence 1,2
4. $p_3 \to p$ predicate logic
5. $\{p_1\} x := x * 2; k := k + 1 \{p\}$ weaken postcondition 4, 3
6. $\{$**inv** $p\}$ **while** $k < n$ **do** $x := x * 2; k := k + 1$ **od** $\{p_4\}$ loop 5

Problem 10

Solution:

1. forward assignment
2. forward assignment
3. predicate logic
4. weaken postcondition 2,3
5. sequence 1,4
6. backward assignment
7. backward assignment

8. predicate logic
9. strengthen precondition 8,7
10. sequence 9,6
11. loop 10
12. sequence 5,11
13. predicate logic
14. weaken postcondition 12, 13