CS536 Science of Programming - Assignment 5

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Problem 1

Full proof outline under partial correctness for q10 Assignment 4.

$\{n > 0\}$
$k := n - 1; \{n > 0 \land k = n - 1\} x := n; \{n > 0 \land k = n - 1 \land x = n\}$
$\{\textbf{inv}\ p \equiv 1 \le k \le n \land x = n!/k!\}$
$\textbf{while}\ k > 1\ \textbf{do}$
$\quad \{p \land k > 1\}\{p[x * k/x][k - 1/k]\} k := k - 1; \{p[x * k/x]\} x := x * k; \{p\}$
$\textbf{od}$
$\{p \land k \le 1\}\{x = n!\}$

Problem 2

Minimal proof outline under partial correctness for q10 Assignment 4.

$\{n > 0\}$
$k := n - 1; x := n;$
$\{\textbf{inv}\ p \equiv 1 \le k \le n \land x = n!/k!\}$
$\textbf{while}\ k > 1\ \textbf{do}$
$\quad k := k - 1; x := x * k;$
$\textbf{od}$
$\{x = n!\}$

Problem 3

Full proof outline with backward assignment.

$\{y \ge 1\}$
$\{1 \le 1 = 2^0 \le y\} x := 0; \{1 \le 1 = 2^x \le y\} r := 1;$
$\{\textbf{inv}\ p \equiv 1 \le r = 2^x \le y\}$
$\textbf{while}\ 2 * r \le y\ \textbf{do}$
$\quad \{p \land 2 * r \le y\}$
$\quad \{1 \le 2 * r = 2^{x+1} \le y\}$
$\quad r := 2 * r; \{1 \le r = 2^{x+1} \le y\}$
$\quad x := x + 1; \{1 \le r = 2^x \le y\}$
$\textbf{od}$
$\{p \land 2 * r > y\}$
$\{r = 2^x \le 2^{x+1}\}$

Problem 4

Full proof outline with forward assignment.

$\{y \geq 1\}$
$x := 0; \{y \geq 1 \wedge x = 0\} r := 1; \{y \geq 1 \wedge x = 0 \wedge r = 1\}$
$\{$**inv** $p \equiv 1 \leq r = 2^x \leq y\}$
while $2 * r \leq y$ **do**
$\{p \wedge 2 * r \leq y\}$
$r := 2 * r; \{1 \leq r_0 = 2^x \leq y \wedge 2 * r_0 \leq y \wedge r = 2 * r_0\}$
$x := x + 1; \{1 \leq r_0 = 2^{x_0} \leq y \wedge 2 * r_0 \leq y \wedge r = 2 * r_0 \wedge x = x_0 + 1\}$
od
$\{p \wedge 2 * r > y\}$
$\{r = 2^x \leq 2^{x+1}\}$

Problem 5

Bound expression and full proof outline.

$\{y \geq 1\}$
$\{1 \leq 1 = 2^0 \leq y\} x := 0; \{1 \leq 1 = 2^x \leq y\} r := 1;$
$\{$**inv** $p \equiv 1 \leq r = 2^x \leq y\}\{$**bd** $y - r\}$
while $2 * r \leq y$ **do**
$\{p \wedge 2 * r \leq y \wedge y - r = t_0\}$
$\{1 \leq 2 * r = 2^{x+1} \leq y \wedge y - (2 * r) < t_0\}$
$r := 2 * r; \{1 \leq r = 2^{x+1} \leq y \wedge y - r < t_0\}$
$x := x + 1; \{1 \leq r = 2^x \leq y \wedge y - r < t_0\}$
od
$\{p \wedge 2 * r > y\}$
$\{r = 2^x \leq 2^{x+1}\}$

Problem 6

- The postcondition $x = y \geq k$ is already safe.
- We can use conditional rule 2, and backward assignment rule to create p.

  $\{p\}$
  if $x > y$ **then**
  $\{b[x - y] = y \geq k \wedge 0 \leq (x - y) < size(b)\} x := b[x - y]\{x = y \geq k\}$
  else
  $\{x = k/b[y - x] \geq k \wedge 0 \leq (y - x) < size(b) \wedge b[y - x] \neq 0\} y := k/b[y - x]\{x = y \geq k\}$
  fi $\{x = y \geq k\}$

  Where $p \equiv (x > y \rightarrow b[x - y] = y \geq k \wedge 0 \leq (x - y) < size(b)) \wedge (x \leq y \rightarrow x = k/b[y - x] \geq k \wedge 0 \leq (y - x) < size(b) \wedge b[y - x] \neq 0)$

Problem 7

Decide true or false for each of the statements.
a. False. $t < 0$ doesn't necessarily imply that.
b. True. Because W is totally correct so pseudo state can't happen.

c. True.
d. False. t cannot be negative.
e. False. t can be more than 0 and still terminate.

Problem 8

Whether each of the following expressions can be the bound expression for W.

a. False. Although $n-k$ is positive, we are not given that x is positive, so the expression can be negative.
b. False. t cannot be a constant.
c. True. Because from the loop invariant, we know that $0 < C \leq k \leq n+C$ and also $B \equiv k \leq n$.
d. False. t will increase because k is increasing after each iteration.
e. True. It is similar to C.

Problem 9

Create 5 possible candidates for the loop invariant p and their corresponding loop condition B by replacing a constant by a variable in $q \equiv y \geq 0 \wedge z = 2*y \leq x < 2*(y+1)$.

1. Replace 0 by k.

$\{$**inv** $y \geq k \wedge z = 2*y \leq x < 2*(y+1) \wedge k \leq 0\}\{$**bd** $y-k\}$
while $k \neq 0$ **do**
  ...make k larger or make y smaller
od
$\{y \geq k \wedge z = 2*y \leq x < 2*(y+1) \wedge k \leq 0 \wedge k = 0\}$
$\{y \geq 0 \wedge z = 2*y \leq x < 2*(y+1)\}$

2. Replace 2 by k.

$\{$**inv** $y \geq 0 \wedge z = k*y \leq x < 2*(y+1) \wedge k \leq 2\}\{$**bd** $x - k*y\}$
while $k \neq 2$ **do**
  ...make $k*y$ larger
od
$\{y \geq 0 \wedge z = k*y \leq x < 2*(y+1) \wedge k \leq 2 \wedge k = 2\}$
$\{y \geq 0 \wedge z = 2*y \leq x < 2*(y+1)\}$

3. Replace x by k.

$\{$**inv** $y \geq 0 \wedge z = 2*y \leq k < 2*(y+1) \wedge k \geq x\}\{$**bd** $k - 2*y\}$
while $k \neq x$ **do**
  ...make k smaller or make $2*y$ larger
od
$\{y \geq 0 \wedge z = 2*y \leq k < 2*(y+1) \wedge k \geq x \wedge k = x\}$
$\{y \geq 0 \wedge z = 2*y \leq x < 2*(y+1)\}$

4. Replace 2 by k.

$\{$**inv** $y \geq 0 \wedge z = 2*y \leq x < k*(y+1) \wedge k \geq 2\}\{$**bd** $k*(y+1) - x\}$
while $k \neq 2$ **do**
  ...make $k*(y+1)$ smaller
od
$\{y \geq 0 \wedge z = 2*y \leq x < k*(y+1) \wedge k \geq 2 \wedge k = 2\}$
$\{y \geq 0 \wedge z = 2*y \leq x < 2*(y+1)\}$

5. Replace 1 by k.

$\{\textbf{inv } y \geq 0 \wedge z = 2 * y \leq x < 2 * (y + k) \wedge k \geq 1\}\{\textbf{bd } 2 * (y + k) - x\}$
while $k \neq 1$ **do**
    ...make $2 * (y + k)$ *smaller*
od
$\{y \geq 0 \wedge z = 2 * y \leq x < 2 * (y + k) \wedge k \geq 1 \wedge k = 1\}$
$\{y \geq 0 \wedge z = 2 * y \leq x < 2 * (y + 1)\}$

Problem 10

Create 4 possible candidates for the loop invariant p and their corresponding loop condition B by dropping off one conjunct in $q \equiv (y \geq 0) \wedge (z = 2^y) \wedge (2^y \leq x) \wedge (x < 2^{y+1})$.

1. Drop $(y \geq 0)$ off.

$\{\textbf{inv } (z = 2^y) \wedge (2^y \leq x) \wedge (x < 2^{y+1})\}\{\textbf{bd } ...\}$
while $y < 0$ **do**
    $\{(z = 2^y) \wedge (2^y \leq x) \wedge (x < 2^{y+1}) \wedge (y < 0)\}$
    loop body
    $\{(z = 2^y) \wedge (2^y \leq x) \wedge (x < 2^{y+1})\}$
od
$\{(z = 2^y) \wedge (2^y \leq x) \wedge (x < 2^{y+1}) \wedge (y \geq 0)\}$
$\{(y \geq 0) \wedge (z = 2^y) \wedge (2^y \leq x) \wedge (x < 2^{y+1})\}$

2. Drop $(z = 2^y)$ off.

$\{\textbf{inv } (y \geq 0) \wedge (2^y \leq x) \wedge (x < 2^{y+1})\}\{\textbf{bd } ...\}$
while $z \neq 2^y$ **do**
    $\{(y \geq 0) \wedge (2^y \leq x) \wedge (x < 2^{y+1}) \wedge (z \neq 2^y)\}$
    loop body
    $\{(y \geq 0) \wedge (2^y \leq x) \wedge (x < 2^{y+1})\}$
od
$\{(y \geq 0) \wedge (2^y \leq x) \wedge (x < 2^{y+1}) \wedge (z = 2^y)\}$
$\{(y \geq 0) \wedge (z = 2^y) \wedge (2^y \leq x) \wedge (x < 2^{y+1})\}$

3. Drop $(2^y \leq x)$ off.

$\{\textbf{inv } (y \geq 0) \wedge (z = 2^y) \wedge (x < 2^{y+1})\}\{\textbf{bd } ...\}$
while $2^y > x$ **do**
    $\{(y \geq 0) \wedge (z = 2^y) \wedge (x < 2^{y+1}) \wedge (2^y > x)\}$
    loop body
    $\{(y \geq 0) \wedge (z = 2^y) \wedge (x < 2^{y+1})\}$
od
$\{(y \geq 0) \wedge (z = 2^y) \wedge (x < 2^{y+1}) \wedge (2^y \leq x)\}$
$\{(y \geq 0) \wedge (z = 2^y) \wedge (2^y \leq x) \wedge (x < 2^{y+1})\}$

4. Drop $(x < 2^{y+1})$ off.

$\{\textbf{inv } (y \geq 0) \wedge (z = 2^y) \wedge (2^y \leq x)\}\{\textbf{bd } ...\}$
while $x \geq 2^{y+1}$ **do**
    $\{(y \geq 0) \wedge (z = 2^y) \wedge (2^y \leq x) \wedge (x \geq 2^{y+1})\}$
    loop body
    $\{(y \geq 0) \wedge (z = 2^y) \wedge (2^y \leq x)\}$
od

$\{(y \geq 0) \wedge (z = 2^y) \wedge (2^y \leq x) \wedge (x < 2^{y+1})\}$
$\{(y \geq 0) \wedge (z = 2^y) \wedge (2^y \leq x) \wedge (x < 2^{y+1})\}$