Trial Exam: Complex Systems Modeling

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Time limit: 90 Minutes

Instructions:

- Answer all questions.
- You may use a non-programmable calculator.
- Show all intermediate steps for mathematical derivations.
- Illustrations should be clearly labeled.

Problem	**Topic**	**Points**
1	Fundamentals of Modeling	15
2	Scaling and Nondimensionalization	15
3	Dynamical Systems & Stability	20
4	Bifurcation Analysis	20
5	Phase Space Reconstruction	20
6	Chaos Theory	10
Total		**100**
Bonus	*Poincaré-Bendixson Theorem*	*(+5)*

Problem 1: Fundamentals of Modeling (15 Points)

a) **(5 pts)** The statistician George Box stated, "All models are wrong, but some are useful." Explain this statement in the context of engineering simulation. What differentiates a "useful" model from a "correct" one?

b) **(5 pts)** Define the principle of *Occam's Razor* and explain how it applies to selecting between two competing models that describe the same dataset with equal accuracy.

c) **(5 pts)** In the context of model quality, define the terms **Validity** and **Robustness**. How do they differ?

Problem 2: Scaling and Nondimensionalization (15 Points)

Consider the differential equation for a chemical reaction with a threshold effect:

$$\frac{dx}{dt} = -rx\left(1 - \frac{x}{K}\right) \tag{1}$$

where x is the state variable, t is time, and $r, K > 0$ are physical parameters.

a) **(5 pts)** Propose a substitution $x = \alpha u$ and $t = \beta\tau$ to nondimensionalize the equation.

b) **(10 pts)** Determine the values of α and β such that the parameters r and K are eliminated from the equation. Write down the final canonical dimensionless form of the ODE.

Problem 3: Linear Dynamical Systems (20 Points)

Consider the following two-dimensional linear dynamical system:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{with} \quad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ -4 & -2 \end{pmatrix} \tag{2}$$

a) **(5 pts)** Calculate the trace $\mathrm{Tr}(\mathbf{A})$ and the determinant $\det(\mathbf{A})$ of the coefficient matrix.

b) **(10 pts)** Determine the eigenvalues $\lambda_{1,2}$ of the matrix $\mathbf{A}$. Based on the eigenvalues, classify the stability of the origin $(0,0)$ (e.g., stable spiral, saddle, unstable node).

c) **(5 pts)** Physically, does this matrix represent an underdamped, overdamped, or critically damped mass-spring-damper system? Justify your answer.

Problem 4: Bifurcation Analysis (20 Points)

Consider the nonlinear system dependent on parameter μ:

$$\dot{x} = \mu x - x^3 \tag{3}$$

a) **(5 pts)** Find all fixed points x^* as a function of μ for both $\mu < 0$ and $\mu > 0$.

b) **(5 pts)** Analyze the linear stability of the fixed point at the origin ($x^* = 0$) for $\mu < 0$ and $\mu > 0$.

c) **(5 pts)** Identify the type of bifurcation at $\mu = 0$ (Saddle-Node, Transcritical, or Pitchfork) and whether it is supercritical or subcritical.

d) **(5 pts)** Sketch the bifurcation diagram (x^* vs. μ) below. Use solid lines for stable branches and dashed lines for unstable branches.

Problem 5: Phase Space Reconstruction (20 Points)

Consider the nonlinear oscillator described by the second-order equation:

$$\frac{d^2x}{dt^2} - x\frac{dx}{dt} + x^2 = 0 \tag{4}$$

By defining $y = \frac{dx}{dt}$, this system can be converted into the first-order system:

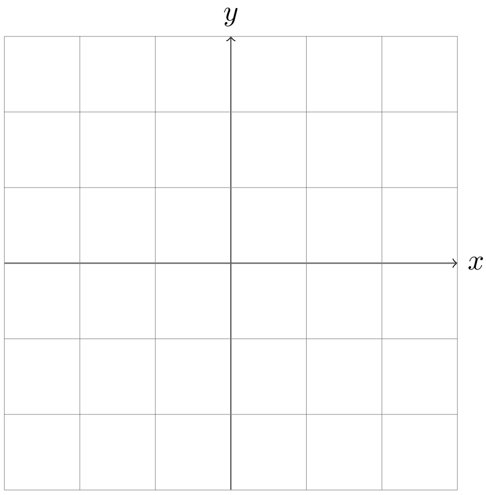
$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= xy - x^2 \end{aligned}$$

a) **(5 pts)** Determine the equations for the $\dot{x}$-nullclines and the $\dot{y}$-nullclines.

b) **(5 pts)** Find all fixed points (equilibrium points) of the system by finding the intersection(s) of the nullclines.

c) **(10 pts)** Sketch the phase space in the (x, y) plane.

- Draw and label the nullclines.
- Mark the fixed point(s).
- In the regions created by the nullclines, draw arrows indicating the qualitative direction of the flow (e.g., North-East, South-West, etc.).



Problem 6: Chaos Theory (10 Points)

a) **(5 pts)** Define **Chaos** in deterministic dynamical systems. Explain the concept of "sensitivity to initial conditions".

b) **(5 pts)** What is the **Lyapunov Exponent** (λ)? If a system has a positive Lyapunov exponent ($\lambda > 0$), what does this imply about its long-term predictability?.

Bonus Question: Continuous-Time Chaos (+5 Points)

(Optional) The Lorenz system is a continuous-time system with 3 state variables (x, y, z). Can chaos exist in a continuous-time system with only 2 state variables? Explain why or why not, referencing the **Poincaré-Bendixson theorem**.