From System Reliability to Structural Reliability

Complex Systems Modeling – Structural Reliability Primer

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Course Material (Moodle)

Python Jupyter Notebook

A Python Jupyter notebook with all examples and code snippets for this lecture is available on Moodle.

- Includes: tension member example, Monte Carlo simulation, and cantilever beam reliability.
- You can run and modify all code interactively.

Outline

Component Reliability: $R(t)$

- We model the lifetime T of a component as a random variable.
- Reliability function:

$$R(t) = \Pr(T > t) = 1 - F_T(t).$$

- For many mechanical components we assume a Weibull model:

$$R(t) = \exp\left[-\left(\frac{t}{\eta}\right)^{\beta}\right],$$

  with scale parameter η and shape parameter β.
- Interpretation:
  - $\beta < 1$: early failures (infant mortality),
  - $\beta \approx 1$: random failures,
  - $\beta > 1$: wear-out.

System Reliability: Series and Parallel

- Consider n components with reliabilities $R_1(t), \dots, R_n(t)$.
- **Series system** ("weakest link"):

$$R_{\mathrm{sys}}^{\text{series}}(t) = \prod_{i=1}^{n} R_i(t).$$

- **Parallel system** (system works if at least one works):

$$R_{\mathrm{sys}}^{\text{parallel}}(t) = 1 - \prod_{i=1}^{n} \bigl(1 - R_i(t)\bigr).$$

- These formulas generalize to more complex block diagrams and fault trees.

Example: Airplane with 2 Wings and 4 Engines

- Simplified system model:
  - 2 wings (both must function) $\Rightarrow$ series.
  - 4 engines, at least 2 required $\Rightarrow$ k-out-of-n system.
- If engine reliabilities are $R_E(t)$ and independent:

$$R_{\text{engines}}(t) = \sum_{k=2}^{4} \binom{4}{k} R_E(t)^k \left(1 - R_E(t)\right)^{4-k}.$$

- Wing reliability $R_W(t)$ (per wing), series:

$$R_{\text{wings}}(t) = R_W(t)^2.$$

- Overall:

$$R_{\mathrm{sys}}(t) = R_{\text{wings}}(t) \cdot R_{\text{engines}}(t).$$

- **Key point:** So far we treated "components" abstractly; now we ask: *what determines the reliability of a structural component?*

From System Reliability to Structural Reliability

- System reliability answers: *How reliable is a system of components?*
- But each component (wing, beam, pressure vessel) is a *structure* loaded by uncertain loads and having uncertain strength.
- We need to connect:
  - material properties (f_y, E, S_y),
  - geometry (b, h, L),
  - loads (forces, stresses, environmental effects),

  to a probability of failure P_f.
- This is the realm of **structural reliability**.

Motivation: Limits of Deterministic Safety Factors

- Classical design: use safety factors, e.g.

$$\frac{R}{S} \geq \mathrm{SF},$$

  where R is resistance, S is load.
- Problems:
  - Safety factors can be *arbitrary* (1.5? 2.0? 1.2?).
  - No explicit probability of failure P_{f}.
  - Difficult to compare designs objectively or optimize cost vs. risk.
- Structural reliability replaces this by:
  - a probabilistic description of R and S,
  - an explicit failure event,
  - and a quantitative P_{f} or reliability index β.

Random Variables and Limit State Function

- Model key quantities as random variables:
  - Resistance R: strength, yield stress, capacity.
  - Load S: stresses from external loads, pressure, wind, etc.
- Define a **limit state function**

$$g(\boldsymbol{X}) = R - S.$$

- Failure occurs when

$$g(\boldsymbol{X}) < 0.$$

- Probability of failure:

$$P_\mathrm{f} = \Pr\left(g(\boldsymbol{X}) < 0\right).$$

- For simple problems, we can derive P_f analytically; for complex ones, we use Monte Carlo or FORM/SORM.

Reliability Index β

- Define the **reliability index** β via

$$\beta = \Phi^{-1}(1 - P_\mathrm{f}),$$

where Φ is the standard normal CDF.

- Interpretation:
  - β is the "distance" to the failure domain in standard normal space (number of standard deviations).
  - Example values:

$$\beta = 3.0 \Rightarrow P_\mathrm{f} \approx 1.35 \times 10^{-3}$$
$$\beta = 4.0 \Rightarrow P_\mathrm{f} \approx 3.17 \times 10^{-5}$$
$$\beta = 5.0 \Rightarrow P_\mathrm{f} \approx 2.87 \times 10^{-7}$$

- Higher β $\Rightarrow$ lower failure probability $\Rightarrow$ higher structural reliability.

Analytical Example: Normal R and S

- From the notebook: tension member with

$$R \sim \mathcal{N}(\mu_R, \sigma_R), \quad S \sim \mathcal{N}(\mu_S, \sigma_S),$$

  independent.
- Limit state: $g = R - S$.
- Then

$$M = R - S \sim \mathcal{N}\left(\mu_R - \mu_S, \sqrt{\sigma_R^2 + \sigma_S^2}\right).$$

- Failure event: $M < 0$.

$$P_\mathrm{f} = \Pr(M < 0) = \Phi\left(-\frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}\right).$$

$$\beta = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}.$$

Interpreting the Overlap of R and S

- Even if $\mu_R > \mu_S$, there is a non-zero overlap of the two distributions.
- The overlap region corresponds to states where $S > R$ and the component fails.
- Increasing reliability can be achieved by:
  - increasing μ_R (stronger material, larger cross-section),
  - reducing σ_R and σ_S (tighter controls),
  - reducing μ_S (lower design loads or better load management).
- In the notebook, this is illustrated with:
  - PDFs of R and S,
  - shaded failure region where $S > R$.

Monte Carlo Approximation of P_f

- Analytical solution works here, but we use this example to introduce **Monte Carlo**:
  1. Sample $R_i \sim \mathcal{N}(\mu_R, \sigma_R)$, $S_i \sim \mathcal{N}(\mu_S, \sigma_S)$.
  2. Compute $g_i = R_i - S_i$.
  3. Count failures N_fail where $g_i < 0$.
  4. Estimate

$$\widehat{P}_\mathrm{f} = \frac{N_\mathrm{fail}}{N_\mathrm{samples}}.$$

- In the notebook:
  - $N \approx 10^5$ samples,
  - comparison of analytical P_f and Monte Carlo $\widehat{P}_\mathrm{f}$,
  - scatter plot in R-S space and histogram of g.

Effect of Design Parameters μ_R on β and P_f

- In the notebook: sensitivity of β and P_f to the mean resistance μ_R.
- For each μ_R in a range, compute:

$$\beta(\mu_R) = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}, \quad P_\mathrm{f}(\mu_R) = \Phi\big(-\beta(\mu_R)\big).$$

- Plot:
  - β vs. μ_R (with target line $\beta = 3$),
  - P_f vs. μ_R in log scale (with target P_f levels).
- **Interpretation:**
  - Small increases in μ_R can significantly decrease P_f.
  - This gives a rational basis for choosing safety margins.

Target Reliability Levels

- Structural reliability connects P_f/β to consequences of failure.

Application	Target β	Target P_f	Consequence
Reusable spacecraft	5.0–6.0	10^{-7}–10^{-9}	catastrophic, loss of life
Aircraft structures	4.0–5.0	10^{-5}–10^{-7}	loss of life, high cost
Pressure vessels	3.5–4.5	10^{-4}–10^{-6}	potential loss of life
Building structures	3.0–3.5	10^{-3}–10^{-4}	economic loss, repairable
Machine components	2.5–3.0	10^{-2}–10^{-3}	local failure, replaceable

- Choice of target β depends on:
  - safety and societal risk acceptance,
  - economic and environmental consequences,
  - cost of increasing reliability.

Cantilever Beam Reliability Problem

- From the notebook: Cantilever beam of length $L = 2\,\mathrm{m}$, load P at free end.
- Max bending stress:

$$\sigma = \frac{M_{\text{max}}}{W} = \frac{P \cdot L}{W}.$$

- For rectangular cross-section:

$$W = \frac{bh^2}{6}.$$

- Random variables:
  - Yield strength $S_y \sim \mathcal{N}(250\,\mathrm{MPa}, 25\,\mathrm{MPa})$,
  - Load $P \sim \mathcal{N}(10\,\mathrm{kN}, 2\,\mathrm{kN})$,
  - Height $h \sim \mathcal{N}(100\,\mathrm{mm}, 2\,\mathrm{mm})$,
  - Width $b = 50\,\mathrm{mm}$ (deterministic),
  - Length $L = 2\,\mathrm{m}$ (deterministic).

Cantilever Beam: Limit State and Tasks

- Limit state function:

$$g = S_y - \sigma_{\text{applied}}.$$

- With

$$\sigma_{\text{applied}} = \frac{PL}{W} = \frac{PL}{bh^2/6} = \frac{6PL}{bh^2}.$$

- Student tasks (from notebook):
  1. Implement Monte Carlo simulation for g.
  2. Estimate P_{f} and β (target $\beta \geq 2.5$ for machine component).
  3. Sensitivity analysis: Which variable dominates?
  4. Propose design changes to achieve $\beta \approx 3.0$.

Cantilever Beam: Hints and Expected Behaviour

- Hints from the notebook:
  - Keep units consistent (Pa, N, m).
  - W is random since h is random.
- Expected findings:
  - Baseline $\beta \approx$ 2.0–2.5.
  - Height h typically has strong influence (enters as h^2).
  - Increasing μ_h by 10–20% can raise β to ≈ 3.0.
- This example links:
  - structural mechanics (bending stress),
  - random variables (material, load, geometry),
  - and system-level reliability targets.

Discussion Questions

- Why is the coefficient of variation (CoV) critical for reliability?
- What are the limitations of pure safety-factor design?
- In the cantilever example: which random variable drives the failure most?
- How many Monte Carlo samples do we need for:
  - $P_\mathrm{f} \approx 10^{-2}$?
  - $P_\mathrm{f} \approx 10^{-6}$?
- How would you add time-dependent effects (fatigue, corrosion) to the limit state?

Summary

- Started from component and system reliability ($R(t)$, Weibull, series/parallel, airplane example).
- Introduced structural reliability:
  - random resistance and load,
  - limit state function g,
  - failure probability P_f and index β.
- Worked through notebook examples:
  - normal R–S tension member,
  - Monte Carlo approximation,
  - sensitivity of β to design parameters,
  - cantilever beam exercise.
- Next step: connect structural reliability to *system reliability of complex structures* (e.g. full aircraft, turbines, networks).

Further Reading

- Melchers, R. E., Beck, A. T.: *Structural Reliability Analysis and Prediction*, 3rd ed.
- Haldar, A., Mahadevan, S.: *Probability, Reliability and Statistical Methods in Engineering Design*.
- Nowak, A. S., Collins, K. R.: *Reliability of Structures*, 2nd ed.