Systems with a Small Number of Variables

Course: Complex Systems Modeling

Robert Flassig

Brandenburg University of Applied Sciences

November 25, 2025

Outline

What is a Bifurcation?

- A **bifurcation** occurs when a small smooth change in a parameter causes a sudden qualitative change in the long-term behavior of a system.
- For a 1D autonomous system:

$$\dot{x} = f(x, \mu)$$

  a bifurcation occurs when both:
  - An equilibrium $f(x^*, \mu) = 0$ is created/destroyed or changes stability.
  - The **linearization** $f_x(x^*, \mu)$ becomes zero.
- Bifurcations are the simplest building blocks of nonlinear phenomena (Strogatz, Ch. 3).

Why Bifurcations Cannot Occur in Linear Systems

Linear system:

$$\dot{x} = Ax$$

- Eigenvalues of A determine stability.
- Changing parameters changes eigenvalues *continuously*.
- But:

  origin is always the only equilibrium.

- No new fixed points can appear or disappear.

Conclusion: Bifurcations require **nonlinear** systems.

Example: Saddle–Node Bifurcation

Canonical model:

$$\dot{x} = \mu - x^2$$

Equilibria:

$$x^* = \pm\sqrt{\mu} \qquad (\mu > 0)$$

Stability:

$$f_x = -2x^* \;\Rightarrow\; \begin{cases} x^* = +\sqrt{\mu}: \; f_x < 0 \Rightarrow \text{stable} \\ x^* = -\sqrt{\mu}: \; f_x > 0 \Rightarrow \text{unstable} \end{cases}$$

At $\mu = 0$: Both equilibria collide and annihilate.

Saddle–Node Phase Portraits

saddlenode_diagram.png

Example: Transcritical Bifurcation

Canonical form:

$$\dot{x} = \mu x - x^2.$$

Equilibria:

$$x_1^* = 0, \qquad x_2^* = \mu.$$

Stability:

$$f_x = \begin{cases} \mu & \text{at } x^* = 0 \\ -\mu & \text{at } x^* = \mu \end{cases}$$

Interpretation: The two fixed points exchange their stability at $\mu = 0$.

Transcritical Bifurcation Diagram

transcritical.png

Supercritical Pitchfork Bifurcation

Canonical form:

$$\dot{x} = \mu x - x^3.$$

Equilibria:

$$x^* = 0, \qquad x^* = \pm\sqrt{\mu}\ (\mu > 0).$$

Stability:

$$f_x = \mu - 3x^2.$$

- At $x = 0$: $\mu < 0$ stable, $\mu > 0$ unstable.
- New stable equilibria appear when $\mu > 0$.

Pitchfork Bifurcation Diagram

pitchfork.png

Failure of Linear Stability at Bifurcations

At an equilibrium $f(x^*, \mu) = 0$, consider the linearization:

$$\dot{\eta} = f_x(x^*, \mu)\eta.$$

Bifurcation condition:

$$f_x(x^*, \mu_c) = 0.$$

Implication:

- Linearization predicts $\dot{\eta} = 0 \rightarrow$ **inconclusive**.
- Need higher-order terms $\rightarrow$ must use **nonlinear analysis**.
- Standard tool: **Taylor expand** near equilibrium:

$$f(x^* + \eta, \mu) = f_x\eta + \frac{1}{2}f_{xx}\eta^2 + \dots$$

Classification of Codimension-1 Bifurcations

Only three generic 1D continuous-time bifurcations:

1. **Saddle–Node**: Two equilibria collide and annihilate.
2. **Transcritical**: Two branches exchange stability.
3. **Pitchfork**:
   - Supercritical: one $\rightarrow$ three equilibria.
   - Subcritical: opposite direction.

All others are non-generic or higher-dimensional.

Exercise: Identify Bifurcation Type

Classify the bifurcation for each system:

1. $\dot{x} = \mu + x^2$
2. $\dot{x} = \mu x - x^2$
3. $\dot{x} = \mu x + x^3$

Tasks:

- Find equilibria as function of μ.
- Determine stability.
- Identify the bifurcation type.

Solutions

1. $\dot{x} = \mu + x^2$
   $\rightarrow$ Saddle–Node (at $\mu = 0$).
2. $\dot{x} = \mu x - x^2$
   $\rightarrow$ Transcritical (exchange of stability at $\mu = 0$).
3. $\dot{x} = \mu x + x^3$
   $\rightarrow$ Subcritical Pitchfork (unstable branches).

Reference: Strogatz (2018), *Nonlinear Dynamics and Chaos*, Chapter 3.