A TUTORIAL INTRODUCTION TO SCALING, BIFURCATION ANALYSIS AND CHAOS: LASER RATE EQUATIONS AND BEYOND

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1. Introduction

Many nonlinear dynamical systems in physics contain parameters with physical units. Before analyzing equilibria, bifurcations, or stability, it is best practice to *nondimensionalize* the equations: convert them to a form involving only dimensionless variables and dimensionless parameters. This simplifies the mathematics, exposes universal behavior, and reduces the number of relevant parameters.

In this tutorial we will:

(1) review best practices for nondimensionalization with a simple example,
(2) apply these ideas to the semiclassical laser rate equations,
(3) perform a fixed point and stability analysis,
(4) sketch the bifurcation structure of the laser threshold,

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(5) briefly discuss how one might extend the system toward chaotic behavior.

2. A Brief Reminder: How to Nondimensionalize a System

Consider a system written in dimensional variables,

$$\dot{x} = f(x, t; \text{parameters}).$$

To nondimensionalize, we choose characteristic scales:

$$x = X_0\,\tilde{x}, \qquad t = T_0\,\tilde{t},$$

where X_0, T_0 have the same units as x and t and are constructed from the physical parameters. After substituting and dividing out physical units, we obtain an equation of the form

$$\tilde{x}' = F(\tilde{x}; \text{dimensionless parameters}).$$

2.1. **Simple Example.**

Take the equation

$$\dot{x} = ax - bx^2, \qquad a, b > 0.$$

The natural scale for x is the steady state $x^* = a/b$, so write

$$x = \frac{a}{b}\,\tilde{x}.$$

The natural time scale is $1/a$, so write $t = \tilde{t}/a$. Substituting yields

$$\tilde{x}' = \tilde{x} - \tilde{x}^2.$$

All parameters have disappeared: the model is now universal. This is considered *best practice*: identify natural physical scales coming from steady states, decay rates, or characteristic times, and scale variables accordingly.

3. Laser Rate Equations

We now consider the semiclassical laser equations

$$\dot{n} = G\,n\,N - kn, \tag{1}$$

$$\dot{N} = -G\,n\,N - fN + p, \tag{2}$$

where:

$$n(t) = \text{photon number}, \qquad N(t) = \text{excited atoms},$$

$$G > 0 \text{ gain}, \quad k > 0 \text{ cavity loss}, \quad f > 0 \text{ spontaneous decay}, \quad p \in \mathbb{R} \text{ pump}.$$

4. Nondimensionalization

A natural time scale is the photon decay time $1/k$, so we define

$$\tau = kt.$$

For n and N, we choose scales suggested by the nonlinear coupling GnN:

$$x = \frac{G}{k}n, \qquad y = \frac{G}{f}N.$$

Finally define dimensionless parameters

$$\alpha = \frac{f}{k}, \qquad \beta = \frac{G}{kf}p.$$

Using derivatives with respect to τ (denoted by $'$), the system becomes

$$x' = x(y-1), \tag{3}$$

$$y' = -\alpha y - \alpha xy + \alpha\beta. \tag{4}$$

This is a two-dimensional nonlinear system with only two parameters $\alpha > 0$ and $\beta \in \mathbb{R}$.

5. Fixed Points and Their Classification

To find equilibria, solve $x' = 0$ and $y' = 0$.

5.1. (a) Trivial equilibrium.

If $x = 0$, then $y' = -\alpha y + \alpha\beta = 0$ gives

$$E_0 = (0, \beta).$$

This corresponds to zero photons in the cavity.

5.2. (b) Lasing equilibrium.

If $y = 1$ (from $x' = 0$), substitute into $y' = 0$:

$$0 = -\alpha(1+x) + \alpha\beta \quad \Rightarrow \quad x = \beta - 1.$$

Thus, a second equilibrium exists:

$$E_L = (\beta - 1,\, 1), \qquad \text{only if } \beta > 1.$$

This is the operating point of a laser above threshold.

5.3. Jacobian and Stability.

The Jacobian is

$$J(x,y) = \begin{pmatrix} y-1 & x \\ -\alpha y & -\alpha(1+x) \end{pmatrix}.$$

Stability of $E_0 = (0, \beta)$.

$$J(E_0) = \begin{pmatrix} \beta - 1 & 0 \\ -\alpha\beta & -\alpha \end{pmatrix}.$$

Eigenvalues:

$$\lambda_1 = \beta - 1, \qquad \lambda_2 = -\alpha.$$

Thus

$$\begin{cases} \beta < 1 & : E_0 \text{ is a stable node,} \\ \beta = 1 & : E_0 \text{ is nonhyperbolic (bifurcation point),} \\ \beta > 1 & : E_0 \text{ becomes a saddle.} \end{cases}$$

Stability of $E_L = (\beta - 1, 1)$.

$$J(E_L) = \begin{pmatrix} 0 & \beta - 1 \\ -\alpha & -\alpha\beta \end{pmatrix}.$$

Eigenvalues satisfy

$$\lambda^2 + \alpha\beta\,\lambda + \alpha(\beta - 1) = 0.$$

The real part is always negative for $\beta > 1$, so E_L is *always stable* (node or spiral).

The discriminant

$$\Delta = (\alpha\beta)^2 - 4\alpha(\beta - 1)$$

determines node vs. spiral.

6. Phase Portraits and Qualitative Dynamics

The system exhibits qualitatively different behavior depending on β:

(1) **Subthreshold ($\beta < 1$):** only E_0 exists and is stable. Photon number decays to zero.
(2) **Threshold ($\beta = 1$):** a *transcritical bifurcation*. The laser turns on at this point.
(3) **Above threshold ($\beta > 1$):** E_0 becomes a saddle and E_L becomes stable. Approach to E_L may be monotone (node) or oscillatory (spiral).

7. Stability Diagram and Bifurcation

The curve separating nodes from spirals is given by $\Delta = 0$:

$$\alpha = \frac{4(\beta - 1)}{\beta^2}.$$

Below this curve: stable spiral (damped laser oscillations). Above this curve: stable node (non-oscillatory relaxation).

The line $\beta = 1$ is the *laser threshold bifurcation*, a classical **transcritical bifurcation**.

8. Can the Laser System Become Chaotic?

The nondimensional laser rate equations

$$x' = x(y-1), \qquad y' = -\alpha y - \alpha xy + \alpha\beta,$$

form a two–dimensional autonomous dynamical system. By the Poincaré–Bendixson theorem, such systems cannot exhibit deterministic chaos: their long–term behavior consists only of fixed points or limit cycles. Thus, *the simplest semiclassical laser model cannot become chaotic.*

Nevertheless, real lasers *do* display chaotic intensity fluctuations, and several extensions of the basic model lead naturally to chaos. The most famous example is the *Haken–Lorenz laser model*, which has exactly the same mathematical structure as the classical Lorenz equations.

8.1. **A Concrete Example: The Haken–Lorenz Equations.** To move from the 2–variable model to a chaotic one, we include a third dynamical variable representing the *polarization* of the atomic medium. Typical laser physics leads to the three coupled equations

$$\dot{E} = \kappa(P - E), \tag{5}$$

$$\dot{P} = \gamma(EN - P), \tag{6}$$

$$\dot{N} = \rho(\beta - N - EP), \tag{7}$$

where:

$$E = \text{electric field amplitude}, \quad P = \text{medium polarization}, \quad N = \text{population inversion}.$$

The parameters κ, γ, ρ measure the decay rates of the field, polarization, and inversion, respectively. After suitable nondimensionalization (dropping tildes), the system can be cast into the form

$$\dot{X} = \sigma(Y - X), \tag{8}$$

$$\dot{Y} = rX - Y - XZ, \tag{9}$$

$$\dot{Z} = XY - bZ, \tag{10}$$

which is exactly the classical Lorenz system, with the identifications:

$$X \leftrightarrow E, \quad Y \leftrightarrow P, \quad Z \leftrightarrow N.$$

For appropriate parameter ranges (e.g. $\sigma = 10$, $r = 28$, $b = 8/3$), the system exhibits the well-known butterfly attractor.

Thus a laser with field–polarization coupling can display the full range of Lorenz-type behavior: fixed points, limit cycles, period doubling, intermittency, and fully developed chaos.

8.2. **Other Mechanisms for Chaos in Lasers.** Beyond the Lorenz structure, there are several other standard routes to chaos in laser physics:
1. Delayed optical feedback (Ikeda-type laser). The output of the laser is partially reflected back after a time delay τ. This produces a delay-differential equation,

$$\dot{x}(t) = F\big(x(t), x(t-\tau)\big),$$

which is infinite-dimensional and easily becomes chaotic. This is a common model for chaotic diode lasers and mode-locked lasers.
2. External periodic modulation. Modulating pump strength or cavity loss,

$$\dot{x} = x(y-1), \qquad \dot{y} = -\alpha y - \alpha xy + \alpha(\beta + A\cos\omega t),$$

creates a periodically forced nonlinear oscillator. Poincaré sections reduce this to a 2D map, where period doubling cascades and chaos readily occur.
3. Injection locking (two-laser coupling). With injected field E_{inj}, the laser evolution includes an additional phase equation. The resulting 3D system can undergo:

- quasiperiodicity,
- torus breakdown,
- Shilnikov chaos.

4. Multimode lasers. Allowing multiple longitudinal or transverse modes introduces additional field amplitudes:

$$E_1, E_2, \ldots$$

so even two-mode lasers lead to 4 or more state variables and chaotic dynamics.
5. Reduction to discrete maps. The laser system may be sampled stroboscopically (Poincaré map) or reduced to a 1D or 2D discrete map under suitable slow–fast assumptions. Typical examples include logistic-type maps,

$$x_{n+1} = rx_n(1-x_n),$$

arising in periodically pumped lasers.

8.3. **Summary Chaotic Laser.** While the basic two-dimensional rate equations cannot display chaos, even minimal physical extensions—such as including polarization, phase, time-delay, periodic forcing, or multimode structure—produce systems that are mathematically equivalent to classical chaotic models, most famously the Lorenz system.