Systems with a Small Number of Variables

Course: Complex Systems Modeling

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Outline

Motivation for Scaling and Nondimensionalization

- Scaling transforms a model into a **canonical form** by removing units and redundant parameters.
- This highlights the **essential dynamics** independent of arbitrary units or parameter magnitudes.
- Especially useful in **asymptotic analysis** and **complex systems modeling**.

Goal: Reduce the number of parameters by suitable rescaling of state variables and time.

Simple ODE Example: Exponential Growth

Model

$$\frac{dx}{dt} = r\,x$$

- Exponential growth with rate r.
- Introduce scaled variables $x = \alpha x',\ t = \beta t'$.

Substitute:

$$\frac{d(\alpha x')}{d(\beta t')} = r\alpha x' \;\Rightarrow\; \frac{dx'}{dt'} = r\,\beta\,x'$$

Choose $\beta = 1/r$ to remove the parameter:

$$\boxed{\frac{dx'}{}\quad '}$$

Continuous-Time Logistic Growth Model

Original form

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right)$$

- Two parameters: growth rate r and carrying capacity K.
- In continuous time, we can also rescale **time**, not only x.

Apply scaling:

$$x = \alpha x', \quad t = \beta t'$$

Substitute and simplify:

$$\frac{d(\alpha x')}{d(\beta t')} = r\alpha x'\left(1 - \frac{\alpha x'}{K}\right) \Rightarrow \frac{dx'}{dt'} = r\beta x'\left(1 - \frac{\alpha x'}{K}\right)$$

Canonical Logistic Model via Rescaling

Choose:

$$\alpha = K, \quad \beta = \frac{1}{r}$$

Then:

$$\frac{dx'}{dt'} = x'(1 - x')$$

- All parameters (r, K) are removed.
- The resulting equation represents a **universal canonical form**.
- In continuous-time systems, rescaling **time** removes one additional parameter compared to discrete-time models.

Interpretation: Changing r or K only stretches trajectories along the time or x axes — the qualitative dynamics remain invariant.

Exercise: Rescale Quadratic ODE

Simplify the following differential equation by variable rescaling:

$$\frac{dx}{dt} = ax^2 + bx + c$$

Tasks

- (i) Shift x to remove the linear term.
- (ii) Rescale x, t to eliminate parameters.
- (iii) Write the canonical parameter-free form and mention the special case when the shift constant vanishes.

Hint: After shifting, you obtain an equation of the form

$$\dot{y} = ay^2 + d, \qquad d = c - \frac{b^2}{4a}.$$

Exercise: Rescale Reciprocal+Constant ODE

Simplify the following ODE by variable rescaling (assume $a > 0,\ b > 0$):

$$\frac{dx}{dt} = \frac{a}{x} + b$$

Tasks

- (i) Propose the substitutions $x = \alpha x',\ t = \beta t'$.
- (ii) Determine α, β so that the coefficients become unity.
- (iii) Derive the resulting canonical form.

Solution: Quadratic ODE

Given $\dot{x} = ax^2 + bx + c$.

1. Shift:

$$x = y - \frac{b}{2a} \;\Rightarrow\; \dot{y} = ay^2 + d, \quad d := c - \frac{b^2}{4a}.$$

2. Rescale:

$$y = \kappa u, \quad t = \frac{\tau}{\gamma}, \quad \text{where} \quad \kappa = \sqrt{\frac{|d|}{a}}, \;\; \gamma = a\kappa = \sqrt{a|d|}.$$

3. Substitute:

$$\frac{dy}{dt} = ay^2 + d \;\Rightarrow\; \frac{du}{d\tau} = u^2 + \mathrm{sgn}(d).$$

Special case: $d = 0 \Rightarrow \dot{y} = ay^2,\; t' = at \Rightarrow \frac{dy}{dt'} = y^2$.
Canonical forms:

$$\boxed{u' = u^2 + 1}, \quad \boxed{u' = u^2 - 1}, \quad \boxed{y' = y^2} \; (d = 0)$$

Solution: Reciprocal+Constant ODE

Given $\dot{x} = \frac{a}{x} + b$, with $a > 0,\ b > 0$.

1. Substitute:

$$x = \alpha x',\ t = \beta t' \quad \Rightarrow \quad \frac{\alpha\, dx'}{\beta\, dt'} = \frac{a}{\alpha x'} + b$$

2. Simplify:

$$\frac{dx'}{dt'} = \frac{\beta a}{\alpha^2}\frac{1}{x'} + \frac{\beta b}{\alpha}$$

3. Choose scales:

$$\frac{\beta a}{\alpha^2} = 1, \quad \frac{\beta b}{\alpha} = 1 \ \Rightarrow\ \alpha = \frac{a}{b}, \quad \beta = \frac{a}{b^2}.$$

4. Canonical form:

$$x = \frac{a}{b}x', \quad t = \frac{a}{b^2}t' \quad \Longrightarrow \quad \boxed{\frac{dx'}{dt'} = \frac{1}{x'} + 1.}$$

- Scaling transforms physical models into **dimensionless forms**.
- Continuous-time models allow **time rescaling**:

$$t \to \beta t'$$

$\to$ one additional parameter can be eliminated.

- The resulting canonical equations are suitable for:
  - Asymptotic analysis,
  - Bifurcation and stability studies,
  - Comparing systems of different scales.
- Example: Logistic growth reduces from (r, K) to a single invariant form:

$$\dot{x}' = x'(1 - x')$$

Asymptotic Behavior of Continuous-Time Linear Dynamical Systems

A general formula for continuous-time linear dynamical systems is given by

$$\frac{dx}{dt} = Ax, \tag{1}$$

where x is the state vector of the system and A is the coefficient matrix.

Asymptotic Behavior of Continuous-Time Linear Dynamical Systems

Adding a constant vector a on the right-hand side can always be converted into a constant-free form by increasing the dimensions of the system: Define

$$y = \begin{pmatrix} x \\ 1 \end{pmatrix}. \tag{2}$$

Then we have

$$\frac{dy}{dt} = \begin{pmatrix} A & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = By. \tag{3}$$

Asymptotic Behavior of Continuous-Time Linear Dynamical Systems

What is the asymptotic behavior of $\frac{dx}{dt} = Ax$? Although not immediately intuitive, there is a closed-form solution:

$$x(t) = e^{At}x(0), \tag{4}$$

where e^X is the matrix exponential for a square matrix X and $X^0 = \mathbb{I}$ defined as

$$e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}. \tag{5}$$

Matrix Exponential: Generalization of Exponential Function

The matrix exponential e^X is a generalization of the scalar exponential function based on the Taylor series:

$$e^X = I + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \dots$$

with higher-order terms involving derivatives like:

$$f(x) \approx f(x_0) + \nabla f(x_0)^\top X + \frac{1}{2} X^\top H_f(x_0) X.$$

- In the scalar case, X is a 1×1 square matrix.
- For a general square matrix X, this infinite series always converges to a well-defined square matrix.
- The resulting matrix e^X has the same size as X.

This generalization is fundamental in solving linear differential equations and analyzing

Properties of the Matrix Exponential

The matrix exponential e^X has some interesting properties:

- **Eigenvalues**: The eigenvalues of e^X are the exponentials of the eigenvalues of X.
- **Eigenvectors**: The eigenvectors of e^X are the same as the eigenvectors of X.

Mathematically, if X has an eigenvector v corresponding to the eigenvalue λ, i.e.,

$$Xv = \lambda v,$$

then for the matrix exponential e^X, we have:

$$e^X v = e^\lambda v.$$

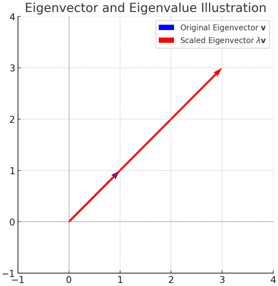
Eigenvector and Eigenvalue Illustration in 2D

The eigenvector v and its scaled version under a linear transformation (eigenvalue λ) are illustrated as follows:

- The original eigenvector v is shown in **blue**.
- The scaled eigenvector λv, where λ is the eigenvalue, is shown in **red**.

The transformation preserves the direction of the eigenvector but scales its magnitude:

$$A\mathrm{v} = \lambda\mathrm{v}.$$



Asymptotic Behavior of Continuous-Time Systems

Given a continuous-time linear system:

$$x(t) = e^{At}x(0),$$

where A is diagonalizable, the initial state can be expressed as:

$$x(0) = b_1 v_1 + b_2 v_2 + \cdots + b_n v_n,$$

where v_i are the eigenvectors of A and e^A, and b_i are coefficients. Applying this to the solution, we get:

$$x(t) = b_1 e^{\lambda_1 t} v_1 + b_2 e^{\lambda_2 t} v_2 + \cdots + b_n e^{\lambda_n t} v_n.$$

Summation of Exponential Terms

So: the solution $x(t)$ is a summation of multiple exponential terms:

$$x(t) = b_1 e^{\lambda_1 t} v_1 + b_2 e^{\lambda_2 t} v_2 + \cdots + b_n e^{\lambda_n t} v_n.$$

- λ_i: Eigenvalues of A.
- $e^{\lambda_i t}$: Exponential growth or decay term corresponding to λ_i.
- The absolute magnitude of each term is determined by $|e^{\lambda_i t}| = e^{\mathrm{Re}(\lambda_i) t}$.

Dominant Eigenvalue

The term with the largest real part of λ_i dominates as $t \to \infty$:

$$|e^{\lambda_i t}| = e^{\mathrm{Re}(\lambda_i)t}.$$

If λ_1 has the largest real part:

$$\mathrm{Re}(\lambda_1) > \mathrm{Re}(\lambda_2), \mathrm{Re}(\lambda_3), \ldots, \mathrm{Re}(\lambda_n),$$

then as $t \to \infty$:

$$x(t) \approx e^{\lambda_1 t} b_1 v_1.$$

Key Insight: The eigenvalue with the largest real part determines the system's asymptotic behavior.

Role of Eigenvalues in System Dynamics

An eigenvalue λ determines whether a system's state component (given by its corresponding eigenvector) grows, shrinks, or remains constant over time:

- $\mathrm{Re}(\lambda) > 0$: The component grows exponentially.
- $\mathrm{Re}(\lambda) < 0$: The component shrinks exponentially.
- $\mathrm{Re}(\lambda) = 0$: The component is conserved (no growth or decay).

For continuous-time models, the behavior of individual state components is tied to the real part of their eigenvalues.

System Stability and the Dominant Eigenvalue

The real part of the dominant eigenvalue λ_d determines the overall stability of a continuous-time system:

- $\mathrm{Re}(\lambda_d) > 0$: The system is **unstable**, diverging to infinity.
- $\mathrm{Re}(\lambda_d) < 0$: The system is **stable**, converging to the origin.
- $\mathrm{Re}(\lambda_d) = 0$: The system is **marginally stable**:
  - The dominant eigenvector component is conserved.
  - The system may converge to a non-zero equilibrium point.

Key Insight: Stability is governed by the real part of the dominant eigenvalue λ_d.

Two-Dimensional Linear Dynamical System

The "love affairs" model, proposed by Strogatz, is given as:

$$\frac{dx}{dt} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} x = Ax, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Key properties of the coefficient matrix A determine the system's stability:

- Eigenvalues λ are critical to understanding the dynamics.
- Stability depends on the real part of the dominant eigenvalue λ_d.

Eigenvalue Calculation

The eigenvalues λ of A are obtained by solving the characteristic equation:

$$\det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0.$$

Simplifying:

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0,$$

where:

- $\mathrm{Tr}(A) = a + d$ (trace of A).
- $\det(A) = ad - bc$ (determinant of A).

The eigenvalues are:

$$\lambda_\pm = \frac{\mathrm{Tr}(A) \pm \sqrt{\mathrm{Tr}(A)^2 - 4\det(A)}}{2}.$$

Dominant Eigenvalue

The dominant eigenvalue is the one with the largest real part.
Conditions:

- If $\mathrm{Tr}(A)^2 < 4\det(A)$: The real part of λ is:

$$\mathrm{Re}(\lambda_d) = \frac{\mathrm{Tr}(A)}{2}.$$

- If $\mathrm{Tr}(A)^2 \geq 4\det(A)$: The real part of λ is:

$$\mathrm{Re}(\lambda_d) = \frac{\mathrm{Tr}(A) + \sqrt{\mathrm{Tr}(A)^2 - 4\det(A)}}{2}.$$

The term with the *plus* sign always dominates because it has the largest real part.

Stability Diagram for Two-Dimensional Systems

The stability of a two-dimensional linear dynamical system depends on the trace $\mathrm{Tr}(A)$ and determinant $\det(A)$ of the coefficient matrix A.

- $\mathrm{Tr}(A)$: Sum of diagonal elements.
- $\det(A)$: Determinant of the matrix.

Stability Regions:

- $\mathrm{Tr}(A) < 0$ and $\det(A) > 0$: Stable system (converges to origin).
- $\mathrm{Tr}(A) > 0$: Unstable system (diverges).
- $\det(A) < 0$: Saddle point (unstable).
- $\mathrm{Tr}(A)^2 < 4\det(A)$: Complex eigenvalues (oscillatory behavior).

Stability Diagram for Two-Dimensional Systems

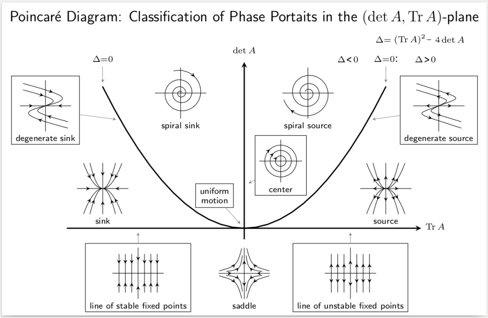


Figure: Stability Diagram. Credit: Freesodas, Wikimedia.

Note: This diagram applies only to two-dimensional systems and is not generalizable to higher dimensions.

Exercise: Stability Analysis

Determine the stability of the following continuous-time linear systems:

1. $\frac{dx}{dt} = Ax, \quad A = \begin{pmatrix} -1 & 2 \\ 2 & -2 \end{pmatrix}.$
2. $\frac{dx}{dt} = Ax, \quad A = \begin{pmatrix} 0.5 & -1.5 \\ 1 & -1 \end{pmatrix}.$

Tasks:

- Find the eigenvalues of the system matrix A.
- Analyze the real parts of the eigenvalues to determine stability.

Solution to System 1

For the first system:

$$A = \begin{pmatrix} -1 & 2 \\ 2 & -2 \end{pmatrix}.$$

Step 1: Characteristic Equation

$$\det(A - \lambda I) = 0 \implies \det \begin{pmatrix} -1-\lambda & 2 \\ 2 & -2-\lambda \end{pmatrix} = 0.$$

Simplify:

$$(-1-\lambda)(-2-\lambda) - (2)(2) = 0.$$
$$\lambda^2 + 3\lambda = 0 \implies \lambda(\lambda+3) = 0.$$

Eigenvalues:

$$\lambda_1 = 0, \quad \lambda_2 = -3.$$

Solution to System 1

Step 2: Stability Analysis

- $\lambda_1 = 0$: Marginally stable (eigenvalue on the imaginary axis).
- $\lambda_2 = -3$: Negative real part (stable).

Conclusion: The system is **marginally stable** because one eigenvalue is zero.

Solution to System 2

For the second system:

$$A = \begin{pmatrix} 0.5 & -1.5 \\ 1 & -1 \end{pmatrix}.$$

Step 1: Characteristic Equation

$$\det(A - \lambda I) = 0 \implies \det \begin{pmatrix} 0.5 - \lambda & -1.5 \\ 1 & -1 - \lambda \end{pmatrix} = 0.$$

Simplify:

$$(0.5 - \lambda)(-1 - \lambda) - (-1.5)(1) = 0.$$

$$\lambda^2 + 0.5\lambda + 1 = 0.$$

Solution to System 2

Step 2: Solve for Eigenvalues

$$\lambda = \frac{-0.5 \pm \sqrt{(0.5)^2 - 4(1)(1)}}{2}.$$

$$\lambda = \frac{-0.5 \pm \sqrt{-3.75}}{2}.$$

$$\lambda = -0.25 \pm i\frac{\sqrt{3.75}}{2}.$$

Eigenvalues:

$$\lambda_1, \lambda_2 = -0.25 \pm i0.968.$$

Step 3: Stability Analysis

- Real part of both eigenvalues is -0.25 (negative).
- The system is stable (converges to the origin).

Summary of Results

The stability of the two systems is as follows:

1. $A = \begin{pmatrix} -1 & 2 \\ 2 & -2 \end{pmatrix}$:
   - Eigenvalues: $\lambda_1 = 0$, $\lambda_2 = -3$.
   - Stability: **Marginally stable** (zero eigenvalue).
2. $A = \begin{pmatrix} 0.5 & -1.5 \\ 1 & -1 \end{pmatrix}$:
   - Eigenvalues: $\lambda = -0.25 \pm i0.968$.
   - Stability: **Stable** (negative real part).

Key Insight: Stability depends on the real parts of the eigenvalues.

Phase Space of Both Systems

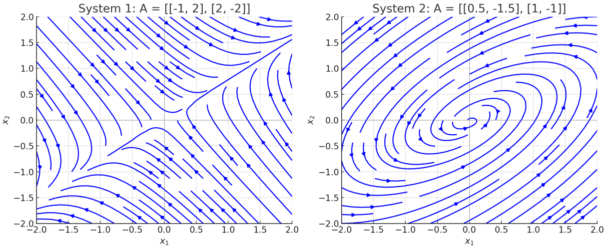


Figure: Phase space of System 1 (left) and System 2 (right).

Observations:

- **System 1:** Shows marginal stability with trajectories moving toward a line due to the zero eigenvalue.
- **System 2:** Exhibits stable spiraling trajectories converging to the origin, indicating stability with complex eigenvalues.