Introduction to Chaos

Course: Complex Systems Modeling

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Outline

From Bifurcation to Chaos

- **Recap:** We explored bifurcations (Saddle-Node, Pitchfork) where parameter changes alter stability.
- **Question:** What happens if we increase the parameter beyond these points?
- **Result:** In nonlinear systems, this often leads to **Chaos**.
- **Definition of Chaos:**
  - Long-term behavior that never settles into a static or periodic trajectory.
  - Deterministic (no random inputs), yet appears random.
  - Exhibits sensitive dependence on initial conditions.

Discrete vs. Continuous

- **Discrete-Time ($x_t = F(x_{t-1})$):**
  - Chaos is possible even in **1D** systems.
  - Example: Logistic Map.
- **Continuous-Time ($\dot{x} = f(x)$):**
  - Chaos requires at least **3 dimensions** (Poincaré-Bendixson theorem).
  - Example: Lorenz Equations.

The Logistic Map

A canonical model of population dynamics:

$$x_t = rx_{t-1}(1 - x_{t-1})$$

- $x_t \in [0, 1]$: Population scale.
- r: Growth rate parameter ($0 < r < 4$).

Behavior as r increases:

1. Stable Equilibrium (Fixed Point).
2. Period-Doubling Bifurcation (Oscillation between 2 states).
3. Period-Doubling Cascade ($2 \to 4 \to 8 \dots$).
4. **Chaos** (Aperiodic behavior).

Bifurcation Diagram

[Place Bifurcation Diagram Here]

- **X-axis:** Parameter r.
- **Y-axis:** Asymptotic states of x.
- Shows the transition from order (single lines) to chaos (dense regions).

1. Sensitivity to Initial Conditions

The "Butterfly Effect"

- Two trajectories starting arbitrarily close together diverge exponentially fast.
- Divergence formula:

$$|\delta(t)| \approx |\delta_0| e^{\lambda t}$$

- Implication: Long-term prediction is impossible, despite the system being deterministic.

2. The Lyapunov Exponent

How do we measure chaos mathematically?

- The **Lyapunov Exponent (λ)** measures the rate of separation.
- Calculated as the time average of the log-derivative:

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \sum_{i=0}^{t-1} \ln \left| \frac{dF}{dx}(x_i) \right|$$

- **Criterion:** $\lambda > 0$ implies Chaos.

3. Stretching and Folding

The geometric mechanism of chaos:

- **Stretching:** Exponential divergence of nearby points ($\lambda > 0$).
- **Folding:** Confinement within a finite phase space requires trajectories to fold back.
- This process creates **fractal** structures (Strange Attractors).

The Lorenz Equations

Discovered by Edward Lorenz (1963) in atmospheric modeling.

$$\dot{x} = s(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

Standard Chaotic Parameters:

$$s = 10, \quad r = 30, \quad b = 3 \quad (\text{or } 8/3)$$

The Lorenz Attractor

[Place Lorenz Attractor Image Here]

- A **Strange Attractor**.
- **Properties:**
  - Trajectories loop infinitely but never intersect.
  - Has a **Fractal Dimension** (approx. 2.06).

Summary

1. **Conditions for Chaos:**
   - Nonlinearity is essential.
   - Discrete systems: Possible in 1D.
   - Continuous systems: Requires $\geq$ 3D.
2. **Signatures:**
   - Sensitivity to initial conditions ($\lambda > 0$).
   - Aperiodic behavior.
   - Stretching and folding mechanism.

Next Step: We will explore complex systems with many variables.