Systems with a Small Number of Variables

Course: Complex Systems Modeling

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Outline

Continuous-Time Models with Differential Equations 1/2

- Continuous-time models are described using differential equations.
- These models are prevalent in science and engineering due to their ability to represent natural phenomena occurring smoothly over time (e.g., motion of objects, flow of electric current).

General Formulation of a First-Order Continuous-Time Model

$$\frac{dx}{dt} = F(x, t)$$

- x: State of the system (scalar or vector variable).
- $\frac{dx}{dt}$: Time derivative of x, formally defined as:

$$\frac{dx}{dt} = \lim_{\delta t \to 0} \frac{x(t + \delta t) - x(t)}{\delta t}$$

Continuous-Time Models with Differential Equations 2/2

- Integrating the model over time t provides the system's trajectory.
- Integration can sometimes be performed algebraically, but computational simulation (numerical integration) is often used in practice.
- A fundamental assumption:
  - Trajectories of the system's state are smooth everywhere in the phase space.
  - Instantaneous abrupt changes, possible in discrete-time models, do not occur in continuous-time models.

$\rightarrow$ Recall discussion on the assumptions of the predator-prey model, e.g. 1.4 rabbit, 2.5 wolves. Also small vs. large number of entities - continuum assumption.

Classifications of Model Equations

- We may distinct between linear and nonlinear systems, as well as autonomous and non-autonomous systems.
  - First-order systems: Involve only first-order derivatives ($\frac{dx}{dt}$).
  - Higher-order systems: Involve higher-order derivatives ($\frac{d^2x}{dt^2}, \frac{d^3x}{dt^3}, \dots$).
- Non-autonomous, higher-order systems can always be transformed into:
  - Autonomous, first-order forms.
  - This is achieved by introducing additional state variables.

Classification of Systems: Linear vs. Nonlinear

Linear Systems

- $\dot{\mathrm{x}} = A\mathrm{x} + B\mathrm{u}$
- Scalar: $\dot{x} = ax + bu$
- Superposition principle holds

Nonlinear Systems

- $\dot{\mathrm{x}} = f(\mathrm{x}, \mathrm{u})$
- Scalar: $\dot{x} = ax^2 + bu$
- Complex dynamics: chaos, bifurcations, limit cycles

Classification of Systems: Autonomous vs. Non-Autonomous

Autonomous Systems

- Evolution depends only on the state, not explicitly on time.
- **Matrix form:**

$$\dot{\mathrm{x}} = A\mathrm{x}$$

- **Scalar example:**

$$\frac{dx}{dt} = ax$$

- System is *time-invariant* $\rightarrow$ suitable for phase-space or equilibrium analysis.

Non-Autonomous Systems

- Explicit time-dependence in the equations or inputs.
- **Matrix form:**

$$\dot{\mathrm{x}} = A(t)\mathrm{x} + B(t)\mathrm{u}(t)$$

- **Scalar example:**

$$\frac{dx}{dt} = a(t)x + b(t)$$

- Require special treatment for time-dependent forcing or varying parameters.

Example: Pendulum Dynamics

Second-order (nonlinear) equation of motion:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

Interpretation:

- The restoring torque is proportional to $-\sin\theta$.
- Equation is second-order and nonlinear.
- For large angles, motion deviates from simple harmonic behavior.

Variables

- θ: angular position
- g: gravitational acceleration
- L: pendulum length

Remarks

- Nonlinear due to $\sin\theta$
- Linearization possible for small angles

Converting to a First-Order System

Introduce angular velocity:

$$\omega = \frac{d\theta}{dt}$$

Substitute into the second-order equation:

$$\begin{aligned} \dot{\theta} &= \omega, \\ \dot{\omega} &= -\frac{g}{L}\sin\theta \end{aligned}$$

Compactly written as:

$$\dot{\mathrm{x}} = \begin{bmatrix} \omega \\ -\frac{g}{L}\sin\theta \end{bmatrix}, \quad \mathrm{x} = \begin{bmatrix} \theta \\ \omega \end{bmatrix}$$

Notes

- Two coupled first-order ODEs
- Nonlinear due to $\sin\theta$
- Enables phase-plane analysis and simulation

Linearized Pendulum (Small Angles)

For small angles, approximate $\sin\theta \approx \theta$:

$$\dot{\theta} = \omega,$$
$$\dot{\omega} = -\frac{g}{L}\theta$$

Matrix form:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{bmatrix}}_{A} x$$

Remarks

- Linear time-invariant (LTI) system
- Represents a *simple harmonic oscillator*
- Solution:

$$\theta(t) = \theta_0 \cos(\sqrt{g/L}\, t)$$

- Basis for control and small-signal stability analysis

Conversion of Higher-Order Equations

- The technique for converting higher-order equations to first-order forms works for:
  - 3rd/4th/.../-order equations,
  - Any higher-order equations (as long as the highest order is finite).
- This method ensures equations are represented as autonomous, first-order systems for analysis and simulation.

Key Advantage

Autonomous, first-order forms are universal and cover all dynamics of higher-order and non-autonomous systems.

A Non-Autonomous Driven Pendulum

Second-Order Non-Autonomous Equation

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta + k\sin(2\pi f t + \phi) \tag{6.7}$$

- Describes a driven pendulum with:
  - $\frac{g}{L}$: Gravitational term.
  - $k\sin(2\pi f t + \phi)$: Periodic external force, e.g., from an electromagnet.
- This equation is nonlinear and non-autonomous due to explicit time-dependence (t).

- Goal: Convert it into a first-order, autonomous system.

Step 1: First-Order Conversion

Introduce $\omega = \frac{d\theta}{dt}$

$$\frac{d\theta}{dt} = \omega \tag{6.8}$$

$$\frac{d\omega}{dt} = -\frac{g}{L}\sin\theta + k\sin(2\pi f t + \phi) \tag{6.9}$$

- ω: Angular velocity of the pendulum.
- The system is now in first-order form but still non-autonomous due to explicit t in the sine term.

Step 2: Eliminating Time Dependency

Introduce a clock variable τ:

$$\frac{d\tau}{dt} = 1, \qquad \tau(0) = 0$$

- Ensures $\tau(t) = t$.
- Replaces explicit time dependence with state variable τ.
- Converts a non-autonomous system into an autonomous one.

Original (time-dependent) term:

$$k \sin(2\pi f t + \phi)$$

becomes

$$k \sin(2\pi f \tau + \phi)$$

Resulting autonomous system:

$$\begin{aligned}
\dot{\theta} &= \omega, \\
\dot{\omega} &= -\frac{g}{L}\sin\theta + k\sin(2\pi f\tau + \phi), \\
\dot{\tau} &= 1
\end{aligned}$$

Notes

- System now autonomous in (θ, ω, τ).
- Useful for phase-space analysis and numerical integration.

Key Insights from the Conversion Process

- Any higher-order, non-autonomous system can be converted into a first-order, autonomous system by:
  - Introducing new state variables for higher-order derivatives.
  - Replacing explicit time dependencies with a clock variable (τ).
- The resulting equations:
  - Are always first-order.
  - Have no explicit time dependency (autonomous).

General Utility

This method ensures compatibility with standard tools for analyzing first-order, autonomous systems.

Exercise 1: Convert to First-Order Form

Given Equation:

$$\frac{d^2x}{dt^2} - x\frac{dx}{dt} + x^2 = 0$$

Introduce a new variable:

$$v = \frac{dx}{dt}$$

Substitute and rewrite:

$$\dot{x} = v,$$
$$\dot{v} = xv - x^2$$

First-Order System:

$$\dot{x} = v,$$
$$\dot{v} = xv - x^2$$

Remarks:

- Nonlinear due to product term xv.
- Expressed as a system of two coupled ODEs.
- Easier to simulate or visualize in the phase plane.

Exercise 2: Convert to Autonomous, First-Order Form

Given Equation:

$$\frac{d^2x}{dt^2} - a\cos(bt) = 0$$

Step 1: Introduce $v = \frac{dx}{dt}$

$$\dot{x} = v, \qquad \dot{v} = a\cos(bt)$$

Step 2: Eliminate explicit time dependency:

$$\frac{d\tau}{dt} = 1, \quad \tau(0) = 0$$

Replace $t \to \tau$ in the forcing term:

$$\cos(bt) \to \cos(b\tau)$$

Autonomous, First-Order Form:

$$\begin{aligned}\dot{x} &= v,\\ \dot{v} &= a\cos(b\tau),\\ \dot{\tau} &= 1\end{aligned}$$

Remarks:

- System is now autonomous in (x, v, τ).
- Clock variable replaces explicit time.
- Technique generalizes to many driven systems.

Summary: Differential Equations and System Behavior

1. Typical Behaviors of Linear Systems

- Exponential growth or decay
- Periodic oscillations
- Stationary (equilibrium) states
- Combined forms, e.g. exponentially damped oscillations

2. Extended Behaviors (Non-Diagonalizable Systems)

- Solutions may include
  - Polynomials in time
  - Products of polynomials and exponentials
- Occurs when the coefficient matrix cannot be fully diagonalized

3. Solvability and Methods

- Linear ODEs: typically analytically solvable
- Nonlinear ODEs: often require numerical integration
- Many nonlinear systems exhibit complex phenomena:
  - Chaos, bifurcations, limit cycles

Key Insight

Understanding these distinctions guides model selection and the choice of analytical vs. numerical solution methods.