Systems with a Small Number of Variables

Course: Complex Systems Modeling

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November 4, 2025

Outline

Equilibrium Points

Definition

Equilibrium points of a continuous-time model $\frac{dx}{dt} = F(x)$ are found by solving:

$$0 = F(x_{\text{eq}})$$

- At equilibrium, the system's state does not change over time.
- Example: Logistic growth model:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right)$$

Equilibrium points:

$$x_{\text{eq}} = 0, \quad x_{\text{eq}} = K$$

Phase Space Analysis

Phase Space

The phase space represents the state of the system as a trajectory in a coordinate space.

- Continuous-time models have smooth trajectories in phase space.
- Key features:
  - Equilibrium points: Points where $\frac{dx}{dt} = 0$.
  - Trajectory directions: Show how the state evolves over time.
- Example: A predator-prey model:

$$\frac{dx}{dt} = x - xy, \quad \frac{dy}{dt} = -y + xy$$

The trajectories rotate around equilibrium points.

Example: Phase Space Analysis

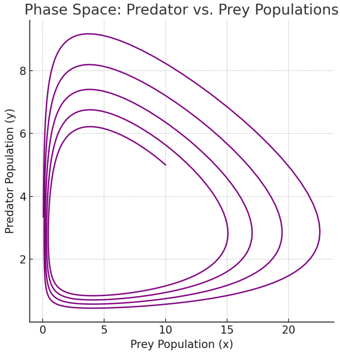

Figure: Phase space representation of the Lotka–Volterra system.

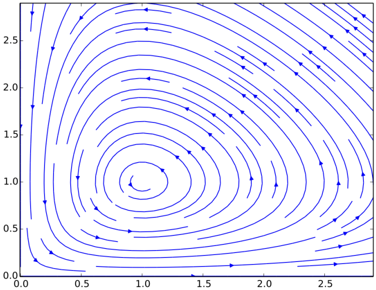

Figure: Alternative phase space trajectory visualization. Credit: H. Sayama 2015

Nullclines

Definition

A nullcline is a set of points where one of the time derivatives becomes zero.

- For $\frac{dx}{dt} = 0$: Horizontal movement stops, trajectories flow vertically.
- For $\frac{dy}{dt} = 0$: Vertical movement stops, trajectories flow horizontally.
- Intersection points of nullclines are equilibrium points.

Nullclines

Example: Lotka-Volterra Model

$$\frac{dx}{dt} = ax - bxy, \quad \frac{dy}{dt} = -cy + dxy$$

- Nullclines:

$$x = 0 \text{ or } y = \frac{a}{b}, \quad y = 0 \text{ or } x = \frac{c}{d}$$

Nullclines

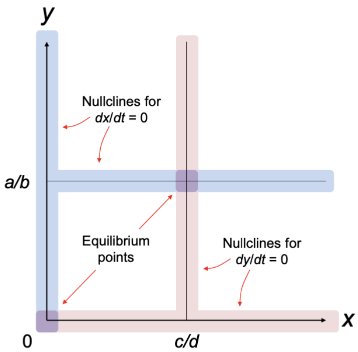

Figure: Nullclines according to the above equations (conditions). Credit: H. Sayama 2015

Nullclines

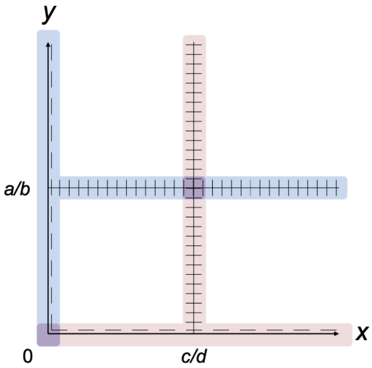

Figure: Nullclines with added directions. Credit: H. Sayama 2015

Phase Space Regions and Trajectory Directions

Phase Space Division

The phase space is divided into four regions, with trajectories in each region flowing in one of four directions:

- Northeast: $\frac{dx}{dt} > 0, \frac{dy}{dt} > 0$
- Northwest: $\frac{dx}{dt} < 0, \frac{dy}{dt} > 0$
- Southeast: $\frac{dx}{dt} > 0, \frac{dy}{dt} < 0$
- Southwest: $\frac{dx}{dt} < 0, \frac{dy}{dt} < 0$

- Trajectories in a region never switch between these directional categories.
- Any switching would appear as part of the nullclines.

Nullclines

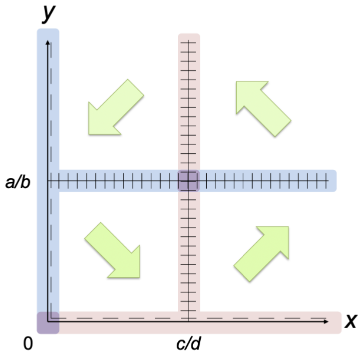

Figure: Directions inside the nullclines. Credit: H. Sayama 2015

Testing Directions in Regions

- Sample a point from each region and test its trajectory direction using the model equations.
- Example: Upper-right region $((x, y) = (\frac{2c}{d}, \frac{2a}{b}))$.

Trajectory Calculation

$$\frac{dx}{dt}\Big|_{(x,y)=(\frac{2c}{d},\frac{2a}{b})} = a\frac{2c}{d} - b\frac{2c}{d}\frac{2a}{b} = -\frac{2ac}{d} < 0$$

$$\frac{dy}{dt}\Big|_{(x,y)=(\frac{2c}{d},\frac{2a}{b})} = -c\frac{2a}{b} + d\frac{2c}{d}\frac{2a}{b} = \frac{2ac}{b} > 0$$

- Direction: Northwest ($\frac{dx}{dt} < 0$, $\frac{dy}{dt} > 0$).
- Repeat for other regions to outline the full phase space behavior.

Predator-Prey Model Phase Space

Equations

$$\frac{dx}{dt} = x - xy, \quad \frac{dy}{dt} = -y + xy$$

- Nullclines:

$$\frac{dx}{dt} = 0 \implies y = 1, \quad \frac{dy}{dt} = 0 \implies x = 1$$

- Equilibrium point: $(x, y) = (1, 1)$
- Trajectories rotate around the equilibrium point in a cyclic manner.

Summary

- Equilibrium Points:
  - Found by solving $\frac{dx}{dt} = 0$.
  - Indicate steady-state solutions.
- Phase Space:
  - Visualizes system behavior over time.
  - Shows trajectories and flow of the state.
- Nullclines:
  - Identify where movement stops in one direction.
  - Separate the phase space into regions.

Practical Use

Combine these tools to understand and predict the dynamics of continuous-time systems.

Equilibrium Points of $\frac{dx}{dt} = x^2 - rx + 1$

Given Model

$$\frac{dx}{dt} = x^2 - rx + 1$$

- Step 1: Set $\frac{dx}{dt} = 0$:

$$x^{*2} - rx^* + 1 = 0$$

- Step 2: Solve the quadratic equation:

$$x^*_{\pm} = \frac{r \pm \sqrt{r^2 - 4}}{2}$$

- Real solutions exist only if $r^2 \geq 4$.

Equilibrium Points

$$x^*_{+} = \frac{r + \sqrt{r^2 - 4}}{2}, \quad x^*_{-} = \frac{r - \sqrt{r^2 - 4}}{2}$$

Equilibrium Points of a Simple Pendulum

Model

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

Step 1: First-Order System

$$\omega = \frac{d\theta}{dt}, \qquad \frac{d\omega}{dt} = -\frac{g}{L}\sin\theta$$

Step 2: Equilibrium Conditions

$$\frac{d\theta}{dt} = 0, \qquad \frac{d\omega}{dt} = 0$$

Step 3: Solve

$$\omega^* = 0, \quad \sin\theta^* = 0 \implies \theta^* = n\pi, \; n \in \mathbb{Z}$$

Equilibria

$$(\theta^*, \omega^*) = (n\pi, 0), \quad n \in \mathbb{Z}$$

Equilibrium Points of the SIR Model (1/2)

Model

$$\frac{dS}{dt} = -aSI, \qquad \frac{dI}{dt} = aSI - bI, \qquad \frac{dR}{dt} = bI$$

Step 1: Equilibrium Conditions

$$\frac{dS}{dt} = 0, \quad \frac{dI}{dt} = 0, \quad \frac{dR}{dt} = 0$$

Step 2: From $\frac{dS}{dt} = 0$

$$-aSI = 0 \implies S = 0 \text{ or } I = 0$$

From $\frac{dI}{dt} = 0$

$$aSI - bI = 0 \implies I(aS - b) = 0$$

$$\Rightarrow I = 0 \text{ or } S = \frac{b}{a}$$

From $\frac{dR}{dt} = 0$

$$bI = 0 \implies I = 0$$

(no new constraint)

Equilibrium Points of the SIR Model (2/2)

Equilibria

$$(1)\quad (S^*, I^*, R^*) = (S_0, 0, R_0) \qquad \text{(Disease-free equilibrium)}$$

$$(2)\quad (S^*, I^*, R^*) = \left(\frac{b}{a}, 0, R_0 + S_0 - \frac{b}{a}\right) \qquad \text{(Endemic equilibrium, if } S_0 > \tfrac{b}{a}\text{)}$$

- Both equilibria satisfy $I^* = 0$.
- The second case exists only if the infection can spread, i.e. $R_0 = \frac{aS_0}{b} > 1$.

Asymptotic Stability Analysis next lectures...