

# Systems with a Small Number of Variables

Course: Complex Systems Modeling

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# Outline

1 Continuous-Time Models: Analysis

2 Exercise

3 Asymptotic Analysis

# Equilibrium Points

## Definition

Equilibrium points of a continuous-time model  $\frac{dx}{dt} = F(x)$  are found by solving:

$$0 = F(x_{\text{eq}})$$

- At equilibrium, the system's state does not change over time.
- Example: Logistic growth model:

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{K}\right)$$

Equilibrium points:

$$x_{\text{eq}} = 0, \quad x_{\text{eq}} = K$$

# Phase Space Analysis

## Phase Space

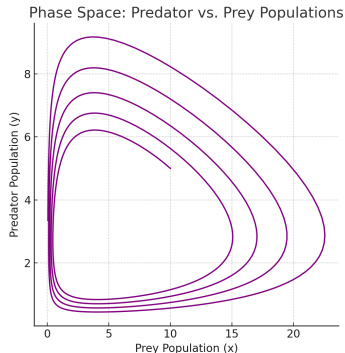
The phase space represents the state of the system as a trajectory in a coordinate space.

- Continuous-time models have smooth trajectories in phase space.
- Key features:
  - Equilibrium points: Points where  $\frac{dx}{dt} = 0$ .
  - Trajectory directions: Show how the state evolves over time.
- Example: A predator-prey model:

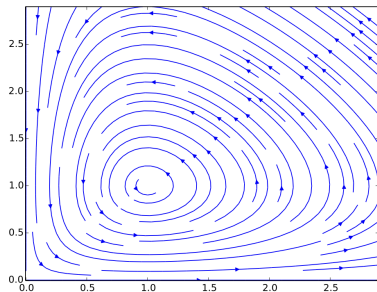
$$\frac{dx}{dt} = x - xy, \quad \frac{dy}{dt} = -y + xy$$

The trajectories rotate around equilibrium points.

# Example: Phase Space Analysis



**Figure:** Phase space representation of the Lotka–Volterra system.



**Figure:** Alternative phase space trajectory visualization. Credit: H. Sayama 2015

# Nullclines

## Definition

A nullcline is a set of points where one of the time derivatives becomes zero.

- For  $\frac{dx}{dt} = 0$ : Horizontal movement stops, trajectories flow vertically.
- For  $\frac{dy}{dt} = 0$ : Vertical movement stops, trajectories flow horizontally.
- Intersection points of nullclines are equilibrium points.

# Nullclines

## Example: Lotka-Volterra Model

$$\frac{dx}{dt} = ax - bxy, \quad \frac{dy}{dt} = -cy + dxy$$

### ■ Nullclines:

$$x = 0 \text{ or } y = \frac{a}{b}, \quad y = 0 \text{ or } x = \frac{c}{d}$$

# Nullclines

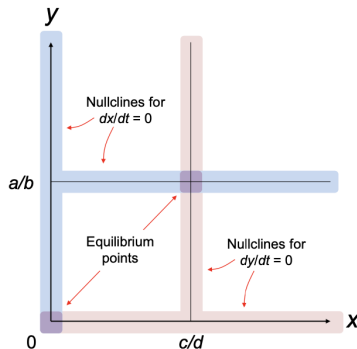


Figure: Nullclines according to the above equations (conditions). Credit: H. Sayama 2015



# Nullclines

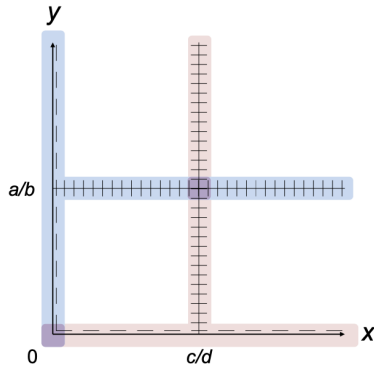


Figure: Nullclines with added directions. Credit: H. Sayama 2015

# Phase Space Regions and Trajectory Directions

## Phase Space Division

The phase space is divided into four regions, with trajectories in each region flowing in one of four directions:

- Northeast:  $\frac{dx}{dt} > 0, \frac{dy}{dt} > 0$
  - Northwest:  $\frac{dx}{dt} < 0, \frac{dy}{dt} > 0$
  - Southeast:  $\frac{dx}{dt} > 0, \frac{dy}{dt} < 0$
  - Southwest:  $\frac{dx}{dt} < 0, \frac{dy}{dt} < 0$
- 
- Trajectories in a region never switch between these directional categories.
  - Any switching would appear as part of the nullclines.

# Nullclines

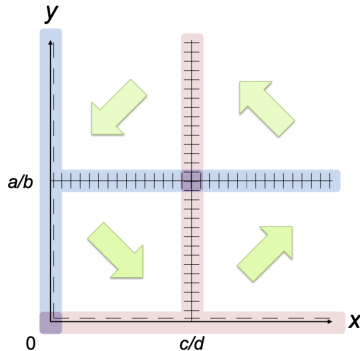


Figure: Directions inside the nullclines. Credit: H. Sayama 2015

# Testing Directions in Regions

- Sample a point from each region and test its trajectory direction using the model equations.
- Example: Upper-right region  $((x, y) = (\frac{2c}{d}, \frac{2a}{b}))$ .

## Trajectory Calculation

$$\begin{aligned}\frac{dx}{dt} \Big|_{(x,y)=(\frac{2c}{d}, \frac{2a}{b})} &= a \frac{2c}{d} - b \frac{2c}{d} \frac{2a}{b} = -\frac{2ac}{d} < 0 \\ \frac{dy}{dt} \Big|_{(x,y)=(\frac{2c}{d}, \frac{2a}{b})} &= -c \frac{2a}{b} + d \frac{2c}{d} \frac{2a}{b} = \frac{2ac}{b} > 0\end{aligned}$$

- Direction: Northwest ( $\frac{dx}{dt} < 0, \frac{dy}{dt} > 0$ ).
- Repeat for other regions to outline the full phase space behavior.

# Predator-Prey Model Phase Space

## Equations

$$\frac{dx}{dt} = x - xy, \quad \frac{dy}{dt} = -y + xy$$

- Nullclines:

$$\frac{dx}{dt} = 0 \implies y = 1, \quad \frac{dy}{dt} = 0 \implies x = 1$$

- Equilibrium point:  $(x, y) = (1, 1)$
- Trajectories rotate around the equilibrium point in a cyclic manner.

# Summary

- Equilibrium Points:
  - Found by solving  $\frac{dx}{dt} = 0$ .
  - Indicate steady-state solutions.
- Phase Space:
  - Visualizes system behavior over time.
  - Shows trajectories and flow of the state.
- Nullclines:
  - Identify where movement stops in one direction.
  - Separate the phase space into regions.

## Practical Use

Combine these tools to understand and predict the dynamics of continuous-time systems.

# Equilibrium Points of $\frac{dx}{dt} = x^2 - rx + 1$

## Given Model

$$\frac{dx}{dt} = x^2 - rx + 1$$

- Step 1: Set  $\frac{dx}{dt} = 0$ :

$$x^{*2} - rx^* + 1 = 0$$

- Step 2: Solve the quadratic equation:

$$x_{\pm}^* = \frac{r \pm \sqrt{r^2 - 4}}{2}$$

- Real solutions exist only if  $r^2 \geq 4$ .

## Equilibrium Points

$$x_+^* = \frac{r + \sqrt{r^2 - 4}}{2}, \quad x_-^* = \frac{r - \sqrt{r^2 - 4}}{2}$$

# Equilibrium Points of a Simple Pendulum

## Model

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L} \sin \theta$$

### Step 1: First-Order System

$$\omega = \frac{d\theta}{dt}, \quad \frac{d\omega}{dt} = -\frac{g}{L} \sin \theta$$

### Step 3: Solve

$$\omega^* = 0, \quad \sin \theta^* = 0 \implies \theta^* = n\pi, \quad n \in \mathbb{Z}$$

### Step 2: Equilibrium Conditions

$$\frac{d\theta}{dt} = 0, \quad \frac{d\omega}{dt} = 0$$

## Equilibria

$$(\theta^*, \omega^*) = (n\pi, 0), \quad n \in \mathbb{Z}$$



# Equilibrium Points of the SIR Model (1/2)

## Model

$$\frac{dS}{dt} = -aSI, \quad \frac{dI}{dt} = aSI - bI, \quad \frac{dR}{dt} = bI$$

### Step 1: Equilibrium Conditions

From  $\frac{dI}{dt} = 0$

$$\frac{dS}{dt} = 0, \quad \frac{dI}{dt} = 0, \quad \frac{dR}{dt} = 0$$

$$aSI - bI = 0 \implies I(aS - b) = 0$$

Step 2: From  $\frac{dS}{dt} = 0$

$$\implies I = 0 \text{ or } S = \frac{b}{a}$$

$$-aSI = 0 \implies S = 0 \text{ or } I = 0$$

From  $\frac{dR}{dt} = 0$

$$bI = 0 \implies I = 0$$

(no new constraint)

# Equilibrium Points of the SIR Model (2/2)

## Equilibria

(1)  $(S^*, I^*, R^*) = (S_0, 0, R_0)$  (Disease-free equilibrium)

(2)  $(S^*, I^*, R^*) = \left(\frac{b}{a}, 0, R_0 + S_0 - \frac{b}{a}\right)$  (Endemic equilibrium, if  $S_0 > \frac{b}{a}$ )

- Both equilibria satisfy  $I^* = 0$ .
- The second case exists only if the infection can spread, i.e.  $R_0 = \frac{aS_0}{b} > 1$ .

# Exercise: Nullclines of the SIR Model

Draw an outline of the phase space of the following SIR model (variable  $R$  omitted) by studying the nullclines and estimating the directions of trajectories within each region separated by those nullclines.

$$\frac{dS}{dt} = -aSI$$

$$\frac{dI}{dt} = aSI - bI$$

$$S \geq 0, \quad I \geq 0, \quad a > 0, \quad b > 0$$

## Nullclines

$$\frac{dS}{dt} = 0 \Rightarrow S = 0 \text{ or } I = 0,$$

$$\frac{dI}{dt} = 0 \Rightarrow I = 0 \text{ or } S = \frac{b}{a}.$$

# Exercise: Nullclines of a Nonlinear Oscillator

Draw an outline of the phase space of the following nonlinear system by studying nullclines and estimating the directions of trajectories within each region separated by those nullclines.

$$\frac{d^2x}{dt^2} - x \frac{dx}{dt} + x^2 = 0$$

Let  $y = \frac{dx}{dt}$ , then

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = xy - x^2.$$

## Nullclines

$$\frac{dx}{dt} = 0 \Rightarrow y = 0,$$

$$\frac{dy}{dt} = 0 \Rightarrow x(y - x) = 0 \Rightarrow x = 0 \text{ or } y = x.$$

*Asymptotic Stability Analysis* next lectures...