Systems with a Small Number of Variables

Course: Complex Systems Modeling

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Outline

What Are Dynamical Systems?

Dynamical systems theory is the foundation of rule-based models for complex systems. It focuses on **how systems change over time**, rather than on static properties.

Definition

A dynamical system is a system whose state is uniquely specified by a set of variables and whose behavior is governed by predefined rules.

Examples of Dynamical Systems:

- Population growth
- Swinging pendulum
- Celestial motions
- Behavior of rational individuals in negotiation games

What Are Dynamical Systems?

Note: A traditional definition of dynamical systems considers deterministic systems only, but stochastic (i.e., probabilistic) behaviors can also be modeled in a dynamical system by, for example, representing the probability distribution of the system's states as a meta-level state.

Question: Can human behavior be modeled as a deterministic dynamical system?

- If individuals make decisions rationally, interactions may be modeled as deterministic.
- However, the validity of such a model depends on critical evaluation of underlying assumptions.

Discrete-Time Dynamical System

A discrete-time dynamical system is represented by the equation:

$$x_t = F(x_{t-1}, t)$$

where x_t is the state of the system at time t, and F is a function describing the system's evolution.

- Known as a **difference equation**, **recurrence equation**, or **iterative map** (when independent of t).

Continuous-Time Dynamical System

A continuous-time dynamical system is defined by:

$$\frac{dx}{dt} = F(x, t)$$

where $\frac{dx}{dt}$ describes the rate of change of the state variable x over time.

- Known as a **differential equation**.

State Variables and Function F

- In both discrete and continuous systems, x_t (or x) represents the **state variable** at time t.
- The function F determines the rules by which the system evolves over time.

The examples we are looking at are **first-order** dynamical systems (only dependent on the previous time step or first derivative), which can model a broad range of dynamics.

Example 1: Linear Difference Equation

The simplest discrete-time dynamical system:

$$x_t = ax_{t-1} + b$$

- a: growth or decay rate
- b: constant input or forcing term
- **Fixed point:** $x^* = \frac{b}{1-a}$, provided $a \neq 1$
- If $|a| < 1$, the system converges to x^* (stable attractor)

Example: Bank account balance with interest a and regular deposit b.

Example 2: Nonlinear Iterative Map — Logistic Equation

The logistic map models population growth with limited resources:

$$x_{t+1} = r\, x_t(1 - x_t)$$

- r: growth rate parameter
- For small r: converges to a fixed point
- For larger r: exhibits oscillations and chaos
- Demonstrates how simple nonlinear rules can yield complex dynamics

Typical behavior: bifurcations, period-doubling, and chaotic attractors.

Example 3: Physical Analogy — Damped Pendulum

Consider a pendulum in a dissipative (frictional) environment:

$$\theta_{t+1} = \theta_t + \dot{\theta}_t, \quad \dot{\theta}_{t+1} = \dot{\theta}_t - \gamma\dot{\theta}_t - \sin(\theta_t)$$

- γ: damping coefficient
- Without damping ($\gamma = 0$): perpetual oscillation — no attractor
- With damping: energy dissipates $\Rightarrow$ pendulum settles at bottom
- Bottom position = **stable fixed-point attractor**

Analogy: Dissipation leads to convergence toward an attractor.

Example 4: Engineering System — Temperature Control

In engineering, many systems can be modeled by discrete-time updates of a physical variable.

$$T_{t+1} = T_t + \Delta t \cdot k \,(T_{\text{set}} - T_t)$$

- T_t: current temperature
- T_{set}: desired (setpoint) temperature
- k: control gain (heat transfer or proportional control constant)
- Δt: time step

Interpretation:

- The system updates its state proportionally to the temperature error.
- If $0 < \Delta t\, k < 1$, the system converges smoothly to the setpoint.

Example 5: Mechanical Engineering System — Mass–Spring–Damper

A classic example of a dynamical system is the mass–spring–damper:

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

Discretizing in time (e.g., with time step Δt):

$$x_{t+1} = x_t + \Delta t\, \dot{x}_t$$

$$\dot{x}_{t+1} = \dot{x}_t + \frac{\Delta t}{m}\left(F_t - c\dot{x}_t - kx_t\right)$$

- m: mass, c: damping coefficient, k: stiffness
- If $F_t = 0$, the system oscillates and gradually settles due to damping
- Equilibrium $x^* = 0, \dot{x}^* = 0$ is a **stable fixed-point attractor**

Modeling Goal & Setup: Wind Turbine Tower in Tower-Shadow Excitation

Objective: Model lateral vibration of the tower (top deflection $x(t)$) excited by periodic aerodynamic loads when blades pass the tower's wind shadow.

System boundary: Tower as a single bending mode (lumped at nacelle/top).

Assumptions (idealizations):

- Single DoF (first lateral bending mode dominates)
- Linear behavior near operating point
- Small vibrations, constant properties
- Harmonic external force from blade passing

Inputs/Outputs:

- Input: harmonic force $F(t) = F_0 \sin(\Omega t)$
- Output: lateral displacement $x(t)$ at tower top

From Physics to Differential Equation

Model choice: Mass–spring–damper for the first mode

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F_0 \sin(\Omega t)$$

Parameters:

$$\omega_n = \sqrt{\frac{k}{m}}, \qquad \zeta = \frac{c}{2\sqrt{km}}$$

State-space (if desired):

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -2\zeta\omega_n x_2 - \omega_n^2 x_1 + \frac{F_0}{m}\sin(\Omega t)$$

Interpretation:

- m: modal mass of first mode
- k: modal stiffness (from ω_n)
- c: modal damping (from ζ)

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Excitation Frequency from Blade Passing

Blade-passing frequency (BPF):

$$\Omega = N_b\,\omega_r, \qquad \omega_r = \frac{2\pi\, n_{\mathrm{rpm}}}{60}$$

where N_b = number of blades, n_{rpm} = rotor speed.

Example:

- $N_b = 3, \quad n_{\mathrm{rpm}} = 15$
- $\omega_r = 2\pi \cdot 15/60 = \frac{\pi}{2}\ \mathrm{rad/s}$
- $\Omega = 3 \cdot \frac{\pi}{2} = \frac{3\pi}{2} \approx 4.71\ \mathrm{rad/s}$
- $f_{\mathrm{BPF}} = \Omega/(2\pi) \approx 0.75\ \mathrm{Hz}$

Note: Yaw misalignment, shear, and tower–nacelle aerodynamics make F_0 and Ω slightly time-varying in reality — we use a constant-Ω sinusoid as a first model.

Steady-State Response & Resonance Insight

Harmonic steady-state amplitude ($x(t) = X\sin(\Omega t - \phi)$):

$$X(\Omega) = \frac{F_0/k}{\sqrt{\left(1-\left(\frac{\Omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\Omega}{\omega_n}\right)^2}}$$

$$\phi(\Omega) = \arctan\left(\frac{2\zeta\,\Omega/\omega_n}{1-(\Omega/\omega_n)^2}\right)$$

Key engineering points:

- **Resonance risk** when $\Omega \approx \omega_n$
- Higher damping (ζ) lowers peak amplitude
- Detuning: design ω_n away from dominant excitations

Quick check (numbers):

- $\omega_n = 5.5\ \mathrm{rad/s},\ \ \zeta = 0.02,\ \ \Omega = 4.71\ \mathrm{rad/s},\ \ F_0/k = 0.02\ \mathrm{m}$

Modeling Process Checklist & Next Steps

Modeling steps students should follow:

1. **Define** the question and outputs ($x(t)$ at tower top)
2. **Choose** an appropriate model (1-DoF MSD for first mode)
3. **Idealize** (linearity, small motion, harmonic input)
4. **Parameterize** (ω_n, ζ from tests or design data)
5. **Relate excitation** ($\Omega = N_b \omega_r$; estimate F_0)
6. **Analyze** $X(\Omega)$ and **check resonance margins**
7. **Decide** mitigations: detune ω_n, increase damping, or avoid critical rotor speeds (operational window)

Extensions (optional):

- Include higher modes (MDOF), stochastic wind loads, or operational speed ramps
- Compare analytical $X(\Omega)$ with time-domain simulation

Reflection 1

Question: Have you learned of any models in the natural or social sciences that are formulated as either discrete-time or continuous-time dynamical systems?

- If so, what are they?
- What assumptions do these models rely on?

Reflection 2

Question: What are some appropriate choices for state variables in the following systems?

- Population growth
- Swinging pendulum
- Motions of celestial bodies
- Behavior of *rational individuals* in a negotiation game

Phase Space

Phase space is a conceptual tool used to analyze the behavior of dynamical systems.

- It is a theoretical space where each state of the system is mapped to a unique spatial location.
- By studying the system in phase space, we gain insights into how the system evolves over time.

Definition of Phase Space

Definition

A **phase space** of a dynamical system is a theoretical space where every state of the system corresponds to a unique location.

Degrees of freedom represent the number of state variables needed to fully describe the system.

Degrees of Freedom and Dimensions of Phase Space

- Each degree of freedom in a system corresponds to an axis in its phase space.
- The dimensionality of phase space is equal to the number of degrees of freedom.

For instance, a system with two degrees of freedom will have a 2-dimensional phase space.

Example: Ball Thrown Upward 1/2

Consider a ball thrown upward in a frictionless vertical tube:

- This system has **two degrees of freedom**: position and velocity.
- We can represent the state of the ball in a 2-dimensional phase space with:
  - Position on one axis
  - Velocity on the other axis

This phase space describes the system's behavior until the ball hits the bottom again.

Example: Ball Thrown Upward 2/2

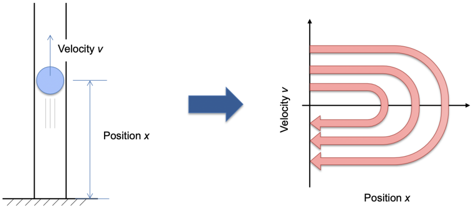


Figure: A ball thrown upward in a vertical tube (left) and a schematic illustration of its phase space (right). The dynamic behavior of the ball can be visualized as a static trajectory in the phase space (red arrows). (Source: H. Sayama)

Benefits of Drawing a Phase Space

Advantages of Phase Space:

- A phase space diagram represents the dynamic behavior of a system as a **static trajectory**.
- This visualization provides intuitive, geometrical insights into the system's dynamics.
- Allows for easier interpretation compared to purely algebraic equations.

What Can We Learn from Phase Space?

Long-Term Behavior

- In a deterministic dynamical system, the future state is uniquely determined by its current state.
- Phase space trajectories do not branch (only merge), ensuring that each initial state has a unique trajectory.
- By observing trajectories, we can determine if they:
  - Diverge to infinity
  - Converge to a point
  - Remain confined within a region
- **Attractors:** Points or regions where trajectories converge, representing stable long-term behaviors in complex systems.

Basins of Attraction and Initial Conditions

Basins of Attraction

- For each attractor, there exists a set of initial states that lead to it, called the **basin of attraction**.
- Multiple attractors in phase space allow us to divide it into regions, showing different possible outcomes.

Sensitivity to Initial Conditions

- A dominant region in phase space means that initial conditions have little impact on the system's fate.
- Multiple regions indicate **sensitivity** to initial conditions, where small changes can lead to different outcomes.

Attractor

Definition

- An attractor A in a dynamical system is a set in the phase space.
- For a neighborhood of initial states x_0 around A, the trajectory $x(t)$ as $t \to \infty$ tends toward A.

$$\lim_{t \to \infty} d(x(t), A) = 0$$

where $d(x(t), A)$ is the distance between $x(t)$ and A.

Intuitive Explanation of Attractor

- An attractor A is a point or set of points in phase space that represents a stable state the system tends toward over time.
- Once the system is near the attractor, it will stay near or move closer to it as time progresses.

Example: Imagine a pendulum swinging in air. Because of friction, it gradually loses energy and eventually stops at the lowest point. This bottom position is a *stable attractor*, since the system tends toward it over time.

Basin of Attraction

Definition

- The **basin of attraction** for an attractor A is the set of all initial states x_0 that will ultimately move towards A over time.

Mathematically:

$$B(A) = \{x_0 \in \mathbb{R}^n : \lim_{t \to \infty} d(x(t), A) = 0\}$$

where $d(x(t), A)$ is the distance between $x(t)$ and A.

Intuitive Explanation of Basin of Attraction

- The basin of attraction is like a "zone of influence" around an attractor.
- Any starting point within this zone will lead the system to the attractor as time goes on.

Example: For a ball in a bowl, the basin of attraction includes all points in the bowl, as no matter where you start the ball, it will eventually settle at the bottom (the attractor).

Stability of System States

System Stability

- Converging trajectories in phase space indicate **stable states**.
- Diverging trajectories indicate **unstable states**.

Stability analysis is crucial for understanding, designing, and controlling real-world systems.

Reflection

For each of the phase spaces shown below, identify the following:

- attractor(s)
- basin of attraction for each attractor
- stability of the system's state at several locations in the phase space

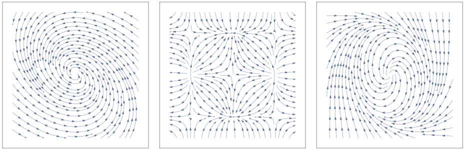


Figure: Source: H. Sayama

Market Competition Between Products A and B

Consider a market with two equally good products, A and B, competing for market share.

There is a customer review website where users submit ratings. Although the average ratings are similar, customers can see the **total number of reviews**, influencing their choice due to popularity.

Questions:

1. What would the phase space of this system look like?
2. Are there any attractors? Are there any basins of attraction?
3. How does the system's fate depend on its initial state?
4. If you were in charge of marketing product A, what would you do?

Hint for Question 1: Phase Space

Hint:

- Consider the market share of each product as state variables.
- Think of a two-dimensional phase space where each axis represents the market share of products A and B.
- The sum of the two market shares should be constant (e.g., 100% of the market), which could create a constraint in phase space.

Hint for Question 2: Attractors and Basins of Attraction

Hint:

- An attractor in this context could represent a stable market share distribution between the two products.
- Think about whether one product could end up dominating the market or if there could be a balance.
- The basin of attraction could indicate initial conditions (market share) that lead to one product becoming more popular.

Hint for Question 3: System's Dependence on Initial State

Hint:

- Consider if a slight advantage in initial popularity could drive a product to dominate due to customer preference for the more popular choice.
- Think about how *path dependency* might play a role in the evolution of the system.

Hint for Question 4: Marketing Strategy for Product A

Hint:

- If popularity influences customer choice, what strategies could you implement to boost the perceived popularity of Product A?
- Consider incentivizing early adopters to submit reviews or focusing on visibility in initial stages.

Solution: Phase Space of Market Competition

Phase Space:

- The phase space can be represented with the market share of Product A on one axis and Product B on the other.
- Since the total market share is 100%, this constrains the phase space to a line where:

$$\text{Market Share of A} + \text{Market Share of B} = 100\%$$

- As a result, this phase space is one-dimensional, showing how the market share shifts between products.

Solution: Attractors and Basins of Attraction

Attractors:

- Potential attractors in this system are the points where one product fully dominates the market.
- These attractors represent stable states: either 100% market share for Product A or for Product B.

Basins of Attraction:

- Each attractor has a basin of attraction defined by the initial market share.
- If Product A starts with a higher initial market share, it is more likely to attract more customers, making it the basin of attraction.
- Similarly, a larger initial share for Product B makes it the basin for B's dominance.

Solution: Dependence on Initial State

System's Fate and Initial State:

- The system's outcome is highly sensitive to initial conditions due to customer preference for the more popular product.
- A small initial advantage in market share for one product can lead to that product ultimately dominating the market.
- This sensitivity illustrates **path dependency**, where early advantages strongly influence the final outcome.

Solution: Marketing Strategy for Product A

Suggested Strategy for Product A:

- **Boost initial popularity:** Increase initial customer engagement and reviews to create an early advantage.
- **Incentivize reviews:** Encourage existing customers to submit positive reviews to boost visibility and perceived popularity.
- **Increase visibility:** Promote Product A aggressively at launch to attract early adopters, enhancing its perceived popularity.

By focusing on these strategies, Product A can gain a competitive edge, which may lead to capturing a larger market share in the long run.

Exercise: The Lotka-Volterra System

The **Lotka-Volterra equations** describe the dynamics between two species: a **predator** and its **prey**. This system of differential equations is given by:

$$\frac{dx}{dt} = \alpha x - \beta xy$$
$$\frac{dy}{dt} = \delta xy - \gamma y$$

where:

- x represents the prey population.
- y represents the predator population.
- $\alpha, \beta, \delta, \gamma$ are positive constants representing interaction rates.

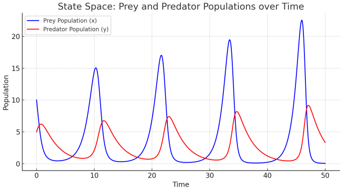
Questions for Analysis

1. Draw the **state space** (populations of prey and predator over time) for the Lotka-Volterra system.
2. Draw the **phase space** (predator population versus prey population).
3. Are there any **attractors** in this system? If so, where?
4. Describe the type of dynamics you observe (e.g., cycles, convergence, or divergence).
5. What happens to the populations if we increase the interaction rate β (impact of predators on prey)?

State and Phase Space of Lotka-Volterra System

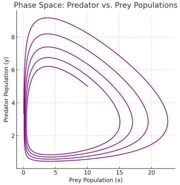
State Space Plot:

- Shows populations of prey (x) and predator (y) over time.



Phase Space Plot:

- Shows the relationship between predator and prey populations.



Hints for Analysis

- For **state space**, plot prey and predator populations as functions of time.
- For **phase space**, consider how predator and prey populations interact with each other, often forming closed cycles.
- Check for any points where the populations remain constant or stabilize over time?these may be attractors.
- Increasing β intensifies predation. Consider the impact on prey and predator populations over time.