Systems with a Small Number of Variables

Course: Complex Systems Modeling

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Outline

Recap: Linear Systems in One Dimension

- 1D systems: $\dot{x} = f(x)$
- Stability determined by slope $f'(x^*)$
- Phase line: arrows show motion $\rightarrow$ fixed points divide regions
- For nonlinear $f(x)$: graphical analysis suffices, but limited to 1D

Motivation

Real systems (oscillators, populations, circuits) need ≥ 2 state variables.

From 1D to 2D Systems

$$\dot{x} = ax + by, \quad \dot{y} = cx + dy$$

- Four parameters a, b, c, d define linear coupling
- In vector form:

$$\dot{\mathrm{x}} = A\mathrm{x}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- The origin $\mathrm{x}^* = (0, 0)$ is always a fixed point.

Superposition Principle

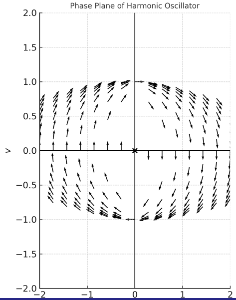
If $\mathrm{x}_1, \mathrm{x}_2$ are solutions, so is any $c_1\mathrm{x}_1 + c_2\mathrm{x}_2$.

The Phase Plane

- The pair (x, y) represents the *state* of the system.
- Trajectories evolve in the (x, y) plane $\rightarrow$ *phase plane*.
- Each point has a velocity vector given by the ODE:

$$\dot{\mathrm{x}} = A\mathrm{x}$$

- The resulting pattern of arrows is the *vector field*.



Example: The Linear Spring–Mass System

$$m\ddot{x} + kx = 0$$

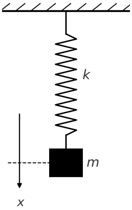
Let $v = \dot{x}$, then:

$$\dot{x} = v, \quad \dot{v} = -\frac{k}{m}x$$

Define $\omega_0^2 = \frac{k}{m}$:

$$\dot{x} = v, \quad \dot{v} = -\omega_0^2 x$$

$\Rightarrow$ state (x, v) evolves on the phase plane.

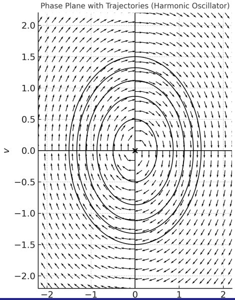


Vector Field Interpretation

- Each point (x, v) has velocity:

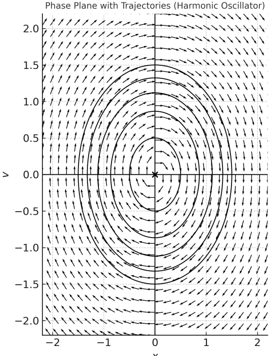
$$F(x, v) = (\dot{x}, \dot{v}) = (v, -\omega_0^2 x)$$

- On x-axis $(v = 0)$: vectors point vertically $(\dot{v} = -\omega_0^2 x)$
- On v-axis $(x = 0)$: vectors point horizontally $(\dot{x} = v)$
- Flow forms a circulating pattern about origin



Phase Portrait of the Harmonic Oscillator

- The origin $(0, 0)$ is a *fixed point* (equilibrium).
- All other trajectories form *closed orbits* (periodic motion).
- Energy is conserved: $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{const.}$
- Hence, orbits satisfy $v^2 + \omega_0^2 x^2 = C \rightarrow$ *ellipses*.



Physical Interpretation

- Fixed point = static equilibrium (mass at rest)
- Closed orbit = periodic motion (oscillation)
- Direction along orbit shows time evolution:

$x < 0, v = 0 \Rightarrow$ max compression(a)

$x = 0, v > 0 \Rightarrow$ passes equilibrium(b)

$x > 0, v = 0 \Rightarrow$ max extension(c)

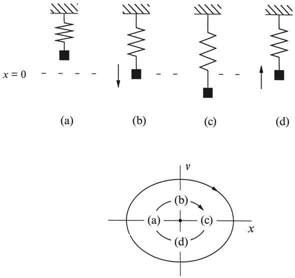


Figure 1: Credit: Strogatz (2018)

Example: System and Setup

$$\dot{\mathrm{x}} = A\mathrm{x}, \quad A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathrm{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

- Uncoupled ODEs: $\dot{x} = a\,x, \quad \dot{y} = -\,y$.
- Fixed points: always $\mathrm{x}^* = (0,0)$; special case $a = 0$ gives a whole line $\{(x,0)\}$.
- Solution with initial (x_0, y_0):

$$x(t) = x_0\,e^{at}, \qquad y(t) = y_0\,e^{-t}.$$

Trajectory slope (useful for portrait)

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{-y}{a\,x} \quad \Rightarrow \quad \ln|y| = -\frac{1}{a}\ln|x| + C \;\Rightarrow\; y = C\,x^{-1/a} \quad (a \neq 0).$$

Phase Portraits for Different Values of a

- Each panel shows the phase portrait of

$$\dot{x} = ax, \quad \dot{y} = -y$$

as the parameter a varies.

- The qualitative behavior changes at $a = -1, 0$.

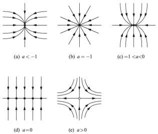


Figure 2:
(a) $a < -1$ Stable node, tangent to y-axis (b) $a = -1$ Star node
(c) $-1 < a < 0$ Stable node, tangent to x-axis (d) $a = 0$ Line of fixed points (e) $a > 0$ Saddle Credit: Strogatz (2018)

Case I: $a < -1$ (Stable node; approach tangential to y-axis)

- Both $x(t)$ and $y(t)$ decay $\Rightarrow$ stable fixed point at the origin.
- $|a| > 1 \Rightarrow x(t)$ decays *faster* than $y(t)$.
- Forward time: trajectories approach tangent to the **slower** direction (the y-axis).
- Backward time: become parallel to the **faster** direction (the x-axis).

Case II: $a = -1$ (Star/symmetric node)

- Equal decay rates in x and y.
- From solutions: $\dfrac{y(t)}{x(t)} = \dfrac{y_0}{x_0}$ is constant.
- $\Rightarrow$ All trajectories are straight lines through the origin.
- This degenerate, measure-zero case is the **star** (or symmetric) node.

Case III: $-1 < a < 0$ (Stable node; approach tangential to x-axis)

- Still stable: $x(t) \to 0$, $y(t) \to 0$.
- Now $|a| < 1 \Rightarrow x(t)$ decays *slower* than $y(t)$.
- Trajectories approach tangent to the **slower** direction (the x-axis).

Case IV: $a = 0$ (Line of fixed points)

- $\dot{x} = 0 \Rightarrow x(t) \equiv x_0$ (constant); $\dot{y} = -y \Rightarrow y(t) \to 0$.
- Every point on the $\{(x, 0)\}$ line is an equilibrium.
- Trajectories are vertical lines $x(t) = x_0$ approaching the x-axis.

Case V: $a > 0$ (Saddle)

- $x(t) = x_0 e^{at}$ grows; $y(t) = y_0 e^{-t}$ decays.
- $\Rightarrow$ The origin is an **unstable** fixed point (saddle).
- Stable manifold: y-axis $(x_0 = 0) \Rightarrow x(t) \equiv 0,\ y(t) \to 0$.
- Unstable manifold: x-axis $(y_0 = 0) \Rightarrow y(t) \equiv 0,\ x(t) \to \pm\infty$.
- Typical trajectories: asymptotic to the unstable manifold in forward time, to the stable manifold in backward time.

Summary: Classification vs. Parameter a

- $\lambda_1 = a,\ \lambda_2 = -1$ (eigenvalues of A).
- $a < -1$: stable node, approach tangent to y-axis.
- $a = -1$: star/symmetric node (straight lines).
- $-1 < a < 0$: stable node, approach tangent to x-axis.
- $a = 0$: line of equilibria (x-axis).
- $a > 0$: saddle; $W^s = \{x = 0\},\ W^u = \{y = 0\}$.

Invariant curves (for $a \neq 0$)

$y = C\,x^{-1/a}$
(level sets that trajectories follow)

Stability Language

- To describe how trajectories behave near fixed points, we introduce key stability terms.
- These notions apply to both linear and nonlinear systems.

Attracting Fixed Point

- x^* is **attracting** if trajectories starting near it satisfy

$$x(t) \to x^* \quad \text{as } t \to \infty.$$

- In Figs. 2 (a–c): the origin attracts all trajectories (globally attracting).

Liapunov Stability

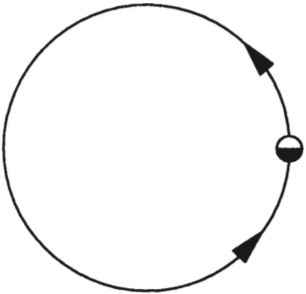
- x^* is **Liapunov stable** if trajectories that start close stay close for all $t > 0$.
- Figs. 2 (a–d): origin is Liapunov stable.

Types of Stability

- **Neutrally stable:** Liapunov stable but not attracting.
  – Nearby trajectories neither converge nor diverge.
  – Example: harmonic oscillator ($\dot{x} = v, \ \dot{v} = -x$).
- **Asymptotically stable:** both Liapunov stable and attracting.
- **Unstable:** neither attracting nor Liapunov stable.

Notation

Solid black dot $\longrightarrow$ Liapunov stable
Open dot $\longrightarrow$ Unstable



Credit: adapted from Strogatz (2018, Fig. 5.1.6)

Motivation: Beyond Special Matrices

- Previously: special case with zeros in A (easy to separate variables).
- Now: general 2×2 linear system

$$\dot{\mathrm{x}} = A\mathrm{x}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

- Goal: classify all possible phase portraits by properties of A.
- Idea: find *special trajectories* analogous to the coordinate axes.

Searching for Straight-Line Trajectories

- Assume motion along a fixed direction v with exponential time dependence:

$$\mathrm{x}(t) = e^{\lambda t}\mathrm{v}, \quad \mathrm{v} \neq 0.$$

- Substitute into $\dot{\mathrm{x}} = A\mathrm{x}$:

$$\lambda e^{\lambda t}\mathrm{v} = e^{\lambda t}A\mathrm{v} \;\Rightarrow\; (A - \lambda I)\mathrm{v} = 0.$$

- Nontrivial solutions exist $\iff$ v is an **eigenvector** of A with eigenvalue λ.
- Then $\mathrm{x}(t) = e^{\lambda t}\mathrm{v}$ is called an **eigensolution**.

Eigenvalues and Eigenvectors of a 2×2 Matrix

- Characteristic equation:

$$\det(A - \lambda I) = 0.$$

- For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, expansion gives

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0.$$

- Define:

$$\tau = \mathrm{tr}(A) = a+d, \quad \Delta = \det(A) = ad-bc.$$

Eigenvalues

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}.$$

Eigenvectors

$$(A - \lambda_i I)\mathrm{v}_i = 0$$

General Solution for Distinct Eigenvalues

- If $\lambda_1 \neq \lambda_2$, the eigenvectors $\mathrm{v}_1, \mathrm{v}_2$ are linearly independent.
- Any initial condition can be written as

$$\mathrm{x}_0 = c_1 \mathrm{v}_1 + c_2 \mathrm{v}_2.$$

- Hence the general solution is

$$\mathrm{x}(t) = c_1 e^{\lambda_1 t} \mathrm{v}_1 + c_2 e^{\lambda_2 t} \mathrm{v}_2.$$

- The coefficients c_1, c_2 follow from $\mathrm{x}(0) = \mathrm{x}_0$.

Interpretation

Each eigendirection defines a straight-line trajectory with exponential growth or decay rate λ_i.

Geometric Interpretation of the Solution

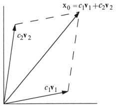
- Any initial condition x_0 can be expressed as a linear combination of eigenvectors:

$$x_0 = c_1 v_1 + c_2 v_2.$$

- Each component evolves independently along its eigendirection:

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2.$$

- The phase portrait is obtained by superimposing these exponential motions.
- The relative magnitudes of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ determine the trajectory's shape.



Example: Solving a Specific System

$$\dot{x} = x + y, \qquad \dot{y} = -4x - 2y,$$

with initial condition $(x_0, y_0) = (2, 3)$.

- Matrix form: $A = \begin{pmatrix} 1 & 1 \\ -4 & -2 \end{pmatrix}$.
- Characteristic equation: $\lambda^2 + \lambda - 6 = 0 \Rightarrow \lambda_1 = 2,\ \lambda_2 = -3$.
- Eigenvectors:

$$\lambda_1 = 2 \Rightarrow \mathrm{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \lambda_2 = -3 \Rightarrow \mathrm{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

Example (cont'd): General and Particular Solution

- **General solution:**

$$\mathrm{x}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

- Apply the initial condition $(x_0, y_0) = (2, 3)$:

$$\begin{cases} 2 = c_1 + c_2, \\ 3 = -c_1 + 4c_2, \end{cases} \qquad \Rightarrow \qquad c_1 = 1,\ c_2 = 1.$$

- Hence the **particular solution** is

$$x(t) = e^{2t} + e^{-3t}, \qquad y(t) = -\,e^{2t} + 4e^{-3t}.$$

- Interpretation:
  - Growth along eigenvector $\mathrm{v}_1 = (1, -1)^{\mathsf{T}}$ with rate $\lambda_1 = 2$,
  - Decay along eigenvector $\mathrm{v}_2 = (1, 4)^{\mathsf{T}}$ with rate $\lambda_2 = -3$.

Example (cont'd): Phase Portrait and Manifolds

- We found eigenvalues $\lambda_1 = 2$, $\lambda_2 = -3$.
- One eigendirection grows, the other decays:
  - $\lambda_1 = 2 > 0$: along $\mathrm{v}_1 = (1, -1)^\mathsf{T}$ solutions grow exponentially $\sim e^{2t}$.
  - $\lambda_2 = -3 < 0$: along $\mathrm{v}_2 = (1, 4)^\mathsf{T}$ solutions decay exponentially $\sim e^{-3t}$.
- Therefore, the origin is a **saddle point**.
  - The **stable manifold** is the line spanned by $\mathrm{v}_2 = (1, 4)^\mathsf{T}$.
  - The **unstable manifold** is the line spanned by $\mathrm{v}_1 = (1, -1)^\mathsf{T}$.
- A typical trajectory:
  - approaches the stable manifold as $t \to -\infty$,
  - and the unstable manifold as $t \to +\infty$.

(Phase portrait: saddle with stable and unstable manifolds.)

Phase Portraits: Stable and Unstable Nodes

- Consider a linear system $\dot{x} = Ax$ with two **real, distinct** eigenvalues λ_1, λ_2 and eigenvectors v_1, v_2.
- **Stable node:** $\lambda_1 < 0$ and $\lambda_2 < 0$.
  - Both eigensolutions decay exponentially.
  - The origin is a **stable node**.
  - Trajectories approach the origin **tangent to the slow eigendirection**, i.e. the eigenvector with smaller $|\lambda|$.
- **Unstable node:** $\lambda_1 > 0$ and $\lambda_2 > 0$.
  - Both eigensolutions grow exponentially.
  - The origin is an **unstable node**.
  - If we reverse all arrows in a stable-node phase portrait, we obtain a typical phase portrait for an unstable node.

(Typical phase portrait: node with fast and slow eigendirections.)

Phase Portraits: Complex Eigenvalues

- Eigenvalues:

$$\lambda_{1,2} = \frac{\tau}{2} \pm \sqrt{\left(\frac{\tau}{2}\right)^2 - \Delta}, \qquad \tau = \operatorname{tr} A,\ \Delta = \det A.$$

- Complex case: $(\frac{\tau}{2})^2 - \Delta < 0$

$$\lambda_{1,2} = \alpha \pm i\omega, \quad \omega \neq 0.$$

- **Center** $(\alpha = 0)$:
  - Purely imaginary eigenvalues.
  - Closed orbits; fixed point is neutrally stable.
- **Spiral / focus** $(\alpha \neq 0)$:
  - $\alpha < 0$: stable spiral (damped oscillations).
  - $\alpha > 0$: unstable spiral (growing oscillations).
- Rotation direction determined by checking the vector field.

Phase Portraits: Repeated Eigenvalues

- Repeated eigenvalue: $\lambda_1 = \lambda_2 = \lambda$.
- **Case 1: two independent eigenvectors**
  - Eigenspace is 2D $\Rightarrow A = \lambda I$.
  - $\lambda < 0$: straight lines *into* origin (stable star node).
  - $\lambda > 0$: straight lines *out of* origin (unstable star node).
  - $\lambda = 0$: every point is fixed ($\dot{x} = 0$).
- **Case 2: one eigenvector**
  - Eigenspace is 1D $\Rightarrow$ **degenerate node**.
  - Trajectories align with the single eigendirection as $t \to \pm\infty$.
  - Lies on the boundary between a node and a spiral.

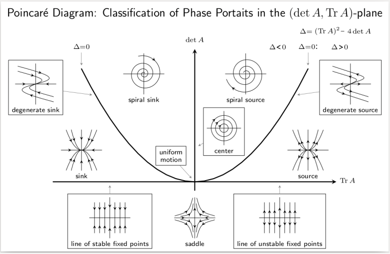
(Phase portrait: degenerate node with one eigendirection.)

Trace–Determinant Classification for $\dot{x} = Ax$

- $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,
- Trace: $\tau = a + d$,
  Determinant: $\Delta = ad - bc$.
- Eigenvalues:

$$\lambda_{1,2} = \frac{\tau}{2} \pm \sqrt{\left(\frac{\tau}{2}\right)^2 - \Delta}.$$

- Discriminant: $D = \tau^2 - 4\Delta$.
- $\Rightarrow$ Regions in the (τ, Δ)-plane give: nodes, saddles, spirals, centers.



Trace–determinant diagram (qualitative regions).

Example: Classify the Fixed Point

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

- $\tau = 1 + 4 = 5, \quad \Delta = 1 \cdot 4 - 2 \cdot 3 = -2.$
- Since $\Delta < 0$:

$$\text{One eigenvalue } > 0, \quad \text{one } < 0.$$

Fixed point is a saddle.

Example: Classify the Fixed Point (Node Case)

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$$

- $\tau = 2 + 4 = 6, \quad \Delta = 2 \cdot 4 - 1 \cdot 3 = 5.$
- $\Delta > 0$ and

$$D = \tau^2 - 4\Delta = 36 - 20 = 16 > 0.$$

  So the eigenvalues are real with the same sign.
- Since $\tau > 0$:

$$\text{Both eigenvalues } > 0.$$

  Unstable node.