BLM1033 - Circuit Theory and Electronics

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Energy Storage Devices

Capacitors and Inductors

Objective of Lecture

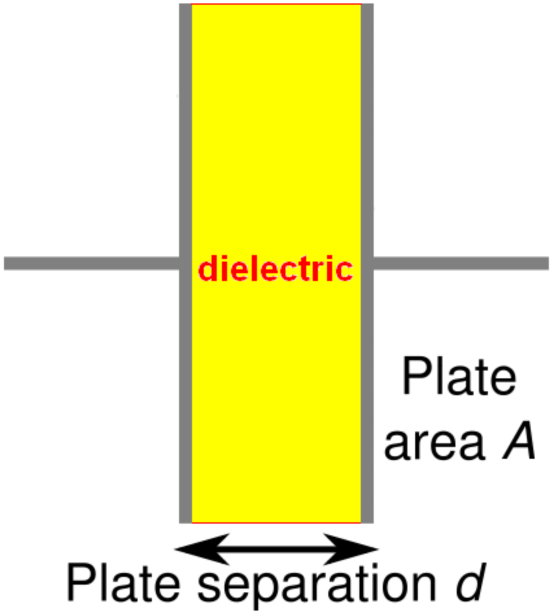
- Describe
  - the construction of a capacitor
  - how charge is stored.
  - Introduce several types of capacitors
- The electrical properties of a capacitor
  - Relationship between charge, voltage, and capacitance; power; and energy
  - Equivalent capacitance when a set of capacitors are in series and in parallel
- Describe
  - The construction of an inductor
  - How energy is stored in an inductor
- The electrical properties of an inductor
  - Relationship between voltage, current, and inductance; power; and energy
  - Equivalent inductance when a set of inductors are in series and in parallel

Capacitors

Energy Storage Devices

Capacitors

- Composed of two conductive plates separated by an insulator (or dielectric).
  - Commonly illustrated as two parallel metal plates separated by a distance, d.



$C = \varepsilon A/d$

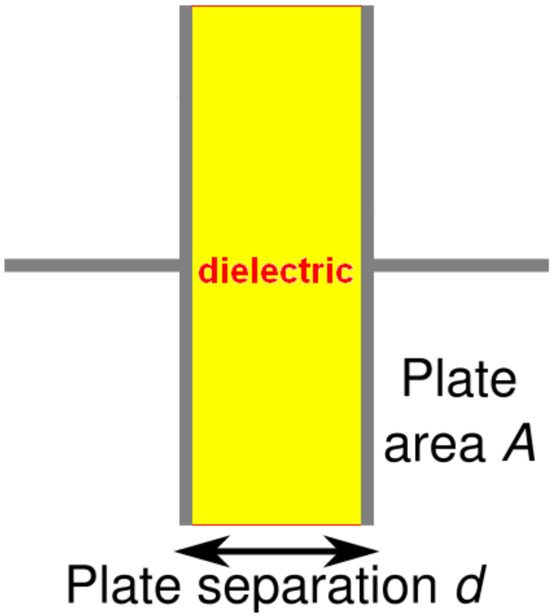
where $\varepsilon = \varepsilon_r \varepsilon_o$

ε_r is the relative dielectric constai

ε_o is the vacuum permittivity

Effect of Dimensions

- Capacitance increases with



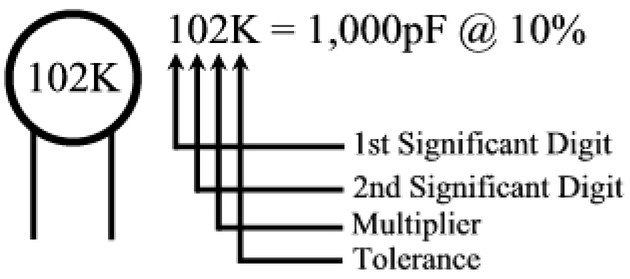
  - increasing surface area of the plates,
  - decreasing spacing between plates, and
  - increasing the relative dielectric constant of the insulator between the two plates.

Types of Capacitors

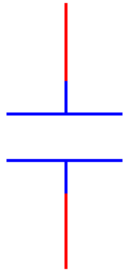
- Fixed Capacitors
  - Nonpolarized
    - May be connected into circuit with either terminal of capacitor connected to the high voltage side of the circuit.
      - Insulator: Paper, Mica, Ceramic, Polymer
  - Electrolytic
    - The negative terminal must always be at a lower voltage than the positive terminal
      - Plates or Electrodes: Aluminum, Tantalum

Nonpolarized

- Difficult to make nonpolarized capacitors that store a large amount of charge or operate at high voltages.
  - Tolerance on capacitance values is very large
    - +50%/-25% is not unusual



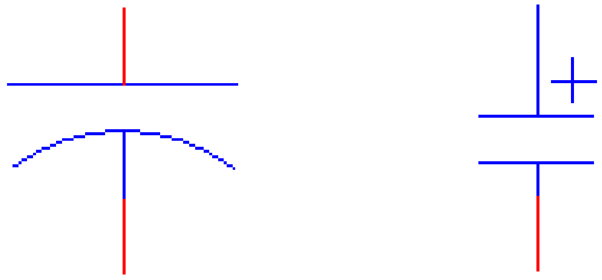
PSpice Symbol



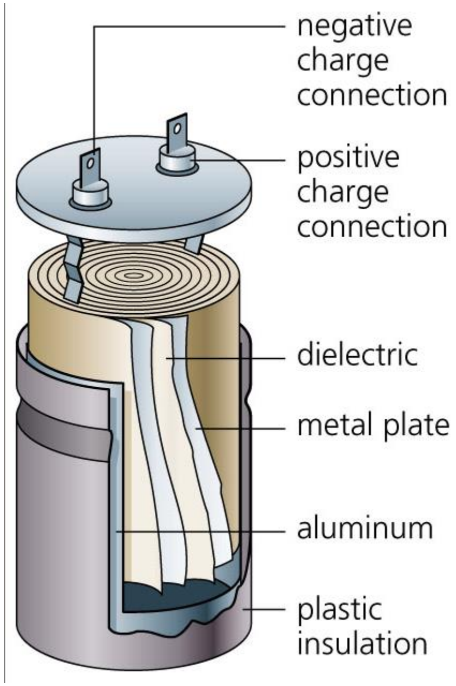
http://www.marvac.com/fun/ceramic_capacitor_codes.aspx

Electrolytic

Pspice Symbols



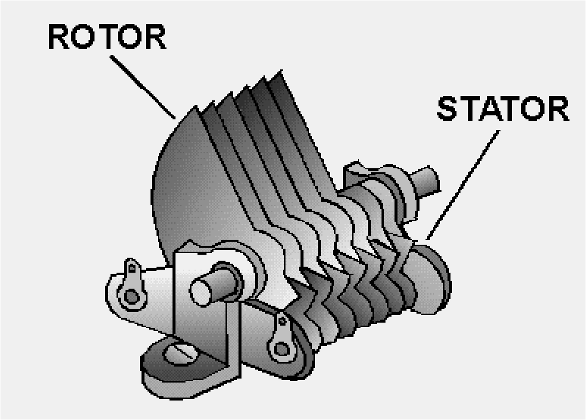
Fabrication



http://www.digitivity.com/articles/2008/11/choosing-the-right-capacitor.html

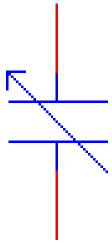
Variable Capacitors

- Cross-sectional area is changed as one set of plates are rotated with respect to the other.



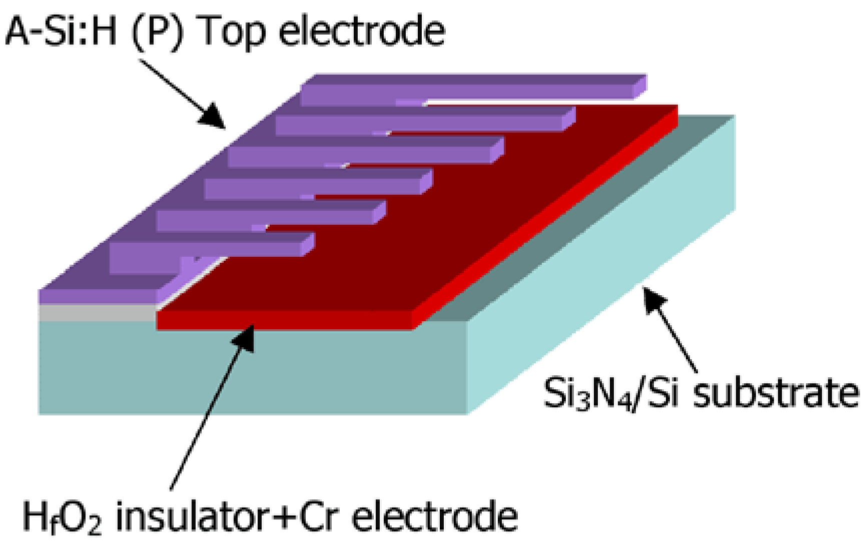
http://www.tpub.com/neets/book2/3f.htm

PSpice Symbol



MEMS Capacitor

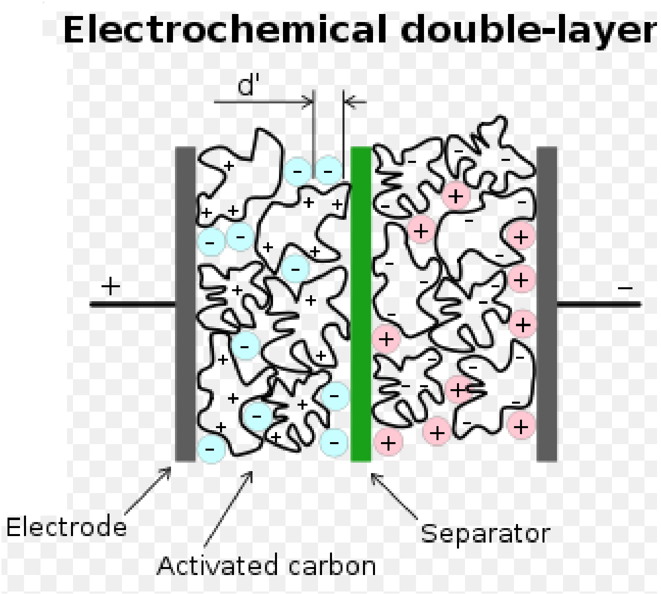
- MEMS (Microelectromechanical system)
  - Can be a variable capacitor by changing the distance between electrodes.
  - Use in sensing applications as well as in RF electronics.



http://www.silvaco.com/tech_lib_TCAD/simulationstandard/2005/aug/a3/a3.html

Electric Double Layer Capacitor

- Also known as a supercapacitor or ultracapacitor
  - Used in high voltage/high current applications.
    - Energy storage for alternate energy systems.



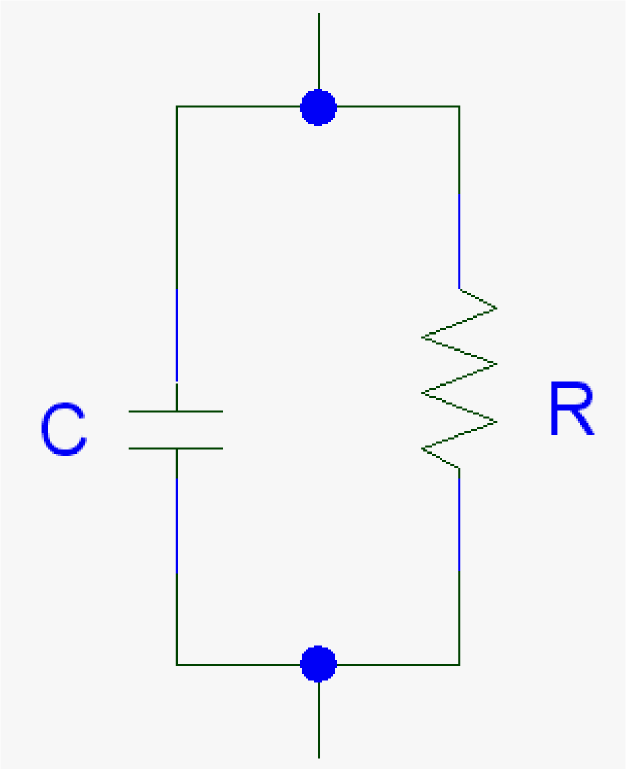
http://en.wikipedia.org/wiki/File:Supercapacitor_diagram.svg

Electrical Properties of a Capacitor

- Acts like an open circuit at steady state when connected to a d.c. voltage or current source.
- Voltage on a capacitor must be continuous
  - There are no abrupt changes to the voltage
- An ideal capacitor does not dissipate energy, it takes power when storing energy and returns it when discharging.

Properties of a Real Capacitor

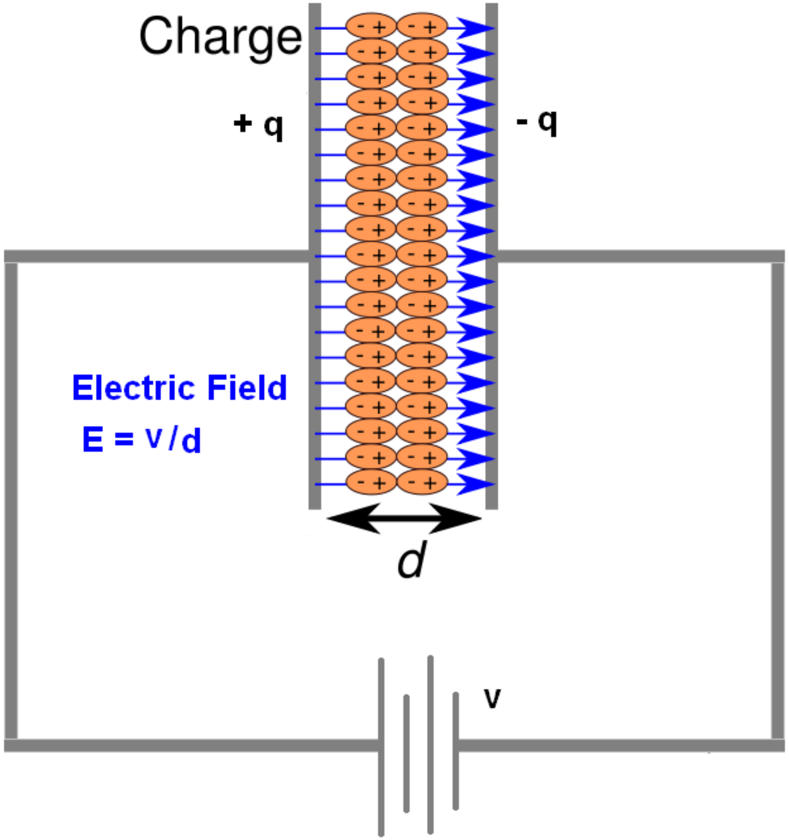
- A real capacitor does dissipate energy due to leakage of charge through its insulator.



  – This is modeled by putting a resistor in parallel with an ideal capacitor.

Energy Storage

- Charge is stored on the plates of the capacitor.



Equation:

$$Q = CV$$

Units:

Coulomb = Farad·Voltage

C = F V

Adding Charge to Capacitor

- The ability to add charge to a capacitor depends on:
  - the amount of charge already on the plates of the capacitor

  and

  - the force (voltage) driving the charge towards the plates (i.e., current)

Charging a Capacitor

- At first, it is easy to store charge in the capacitor.
- As more charge is stored on the plates of the capacitor, it becomes increasingly difficult to place additional charge on the plates.
  - Coulombic repulsion from the charge already on the plates creates an opposing force to limit the addition of more charge on the plates.
    - Voltage across a capacitor increases rapidly as charge is moved onto the plates when the initial amount of charge on the capacitor is small.
    - Voltage across the capacitor increases more slowly as it becomes difficult to add extra charge to the plates.

Discharging a Capacitor

- At first, it is easy to remove charge in the capacitor.
  - Coulombic repulsion from the charge already on the plates creates a force that pushes some of the charge out of the capacitor once the force (voltage) that placed the charge in the capacitor is removed (or decreased).
- As more charge is removed from the plates of the capacitor, it becomes increasingly difficult to get rid of the small amount of charge remaining on the plates.
  - Coulombic repulsion decreases as the charge spreads out on the plates. As the amount of charge decreases, the force needed to drive the charge off of the plates decreases.
    - Voltage across a capacitor decreases rapidly as charge is removed from the plates when the initial amount of charge on the capacitor is small.
    - Voltage across the capacitor decreases more slowly as it becomes difficult to force the remaining charge out of the capacitor.

Current-Voltage Relationships

$$i_C = \frac{dq}{dt}$$

$$q = Cv_C$$

$$i_C = C\frac{dv_C}{dt}$$

$$v_C = \frac{1}{C}\int_{t_o}^{t_1} i_C dt$$

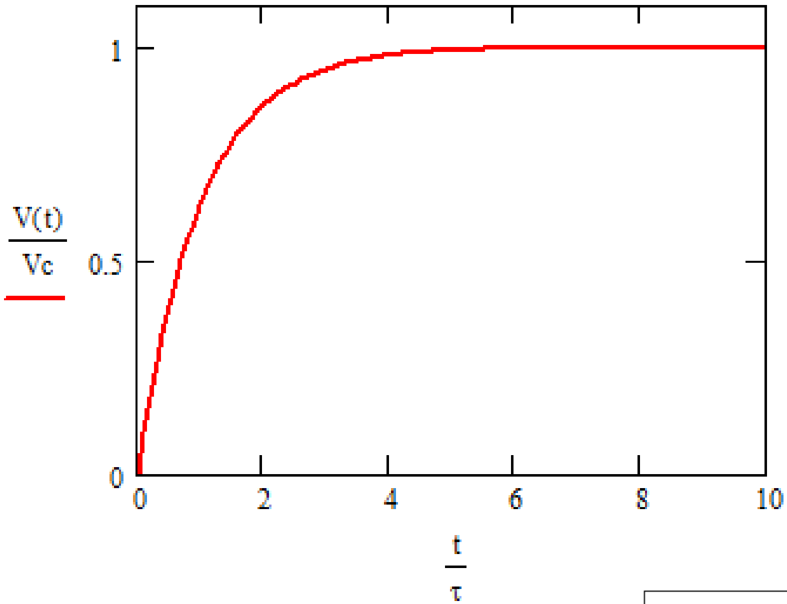
Power and Energy

$$p_C = i_C v_C$$

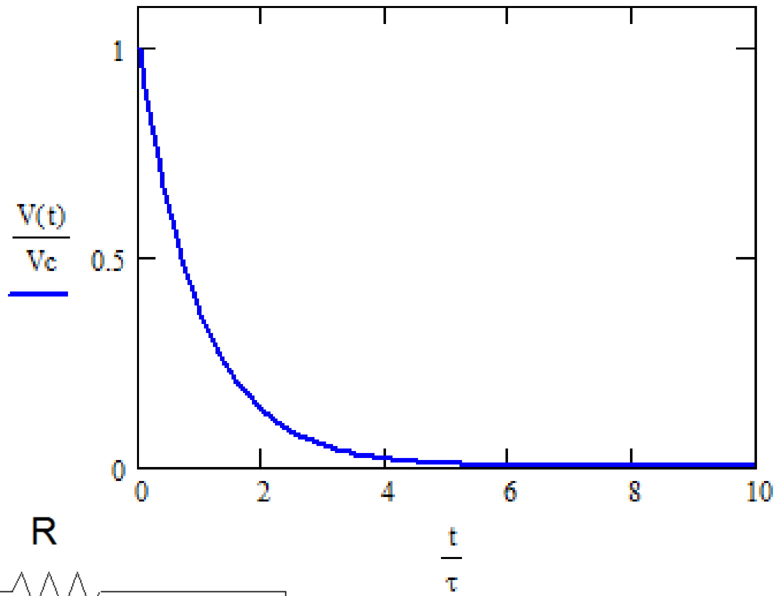
$$p_C = C v_C \frac{dv_C}{dt}$$

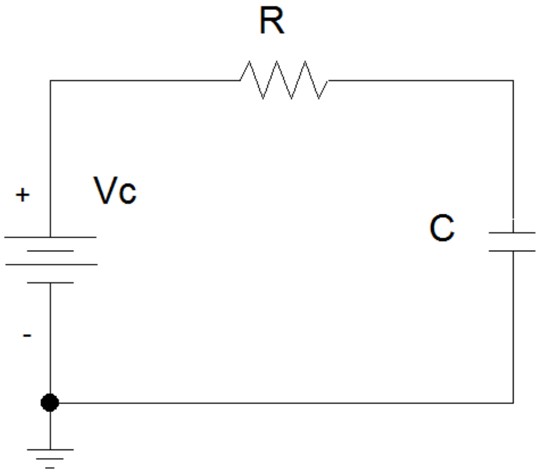
Capacitor Voltage vs. Time

d.c. voltage, Vc, is applied at t = 0s



d.c. voltage, Vc, is removed at t = 0s





Time constant, τ

- The rate at which charge can be added to or removed from the plates of a capacitor as a function of time can be fit to an exponential function.

Charging

$$V(t) = V_c\left(1 - e^{-t/\tau}\right)$$

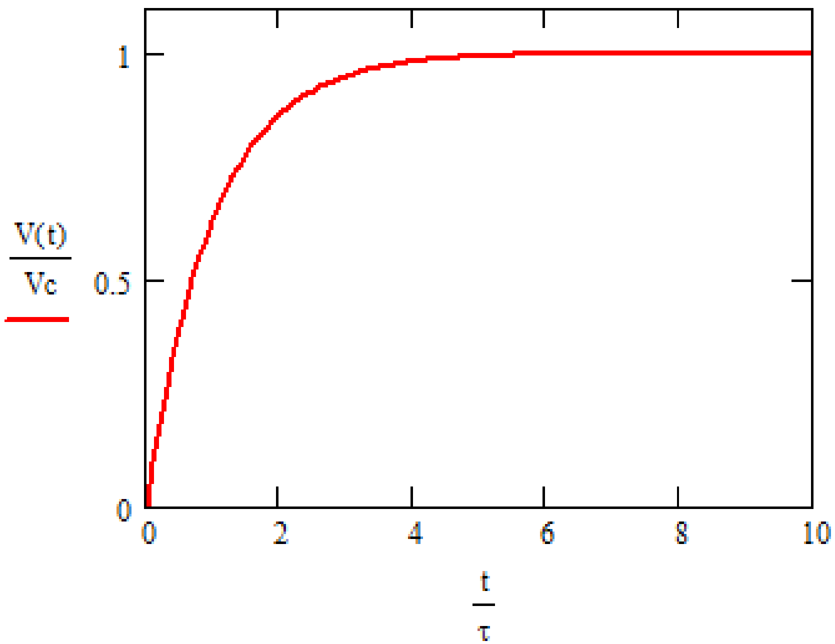
Discharging

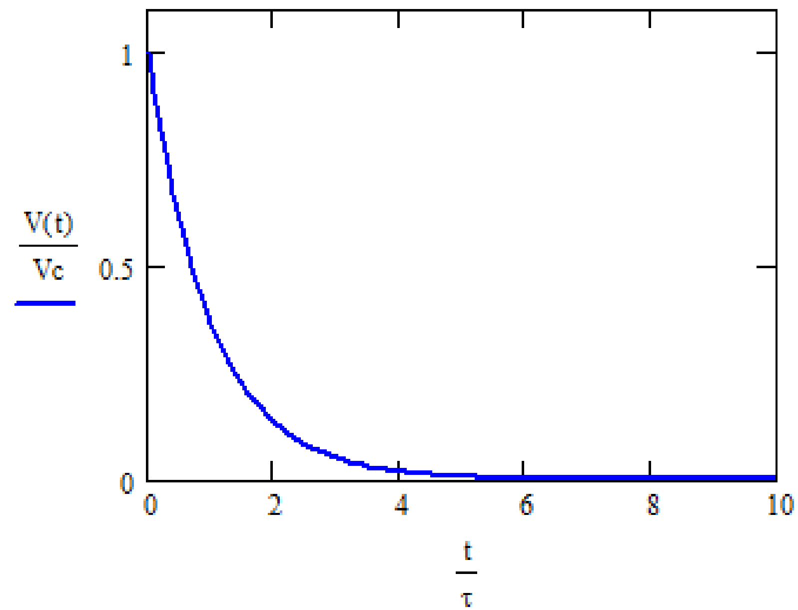
$$V(t) = V_c e^{-t/\tau}$$

$$\tau = RC$$

Transition to steady state

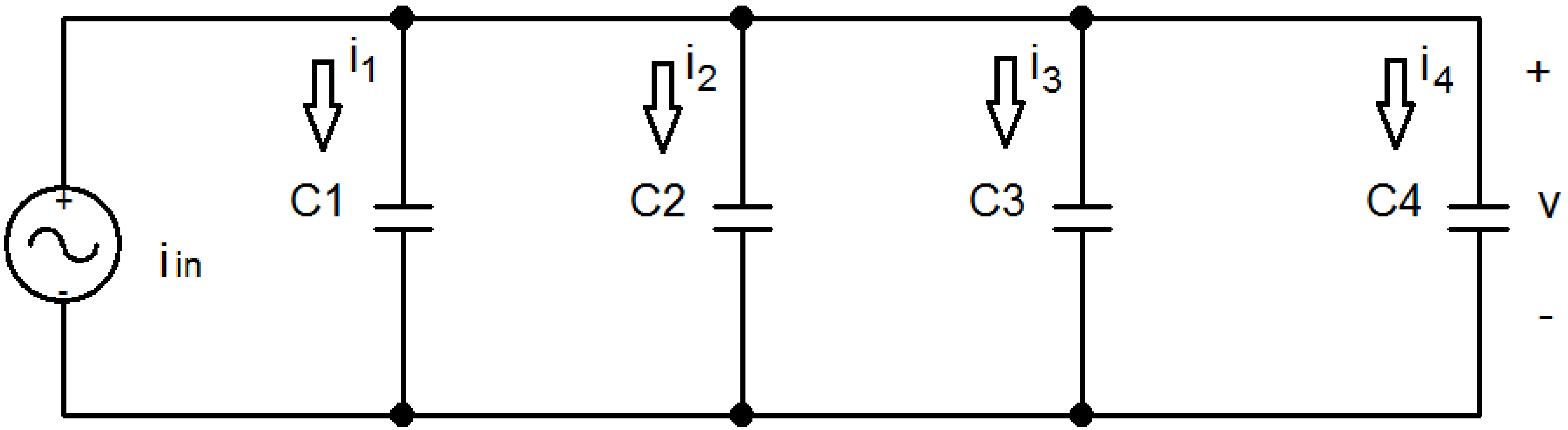
- We approximate that the exponential function reaches its final value when the charging or discharging time is equal to 5τ.





Equivalent Capacitance

- Capacitors in parallel



C_{eq} for Capacitors in Parallel

$$i_{in} = i_1 + i_2 + i_3 + i_4$$

$$i_1 = C_1 \frac{dv}{dt} \qquad\qquad i_2 = C_2 \frac{dv}{dt}$$

$$i_3 = C_3 \frac{dv}{dt} \qquad\qquad {\scriptstyle i} \quad i_4 = C_4 \frac{dv}{dt}$$

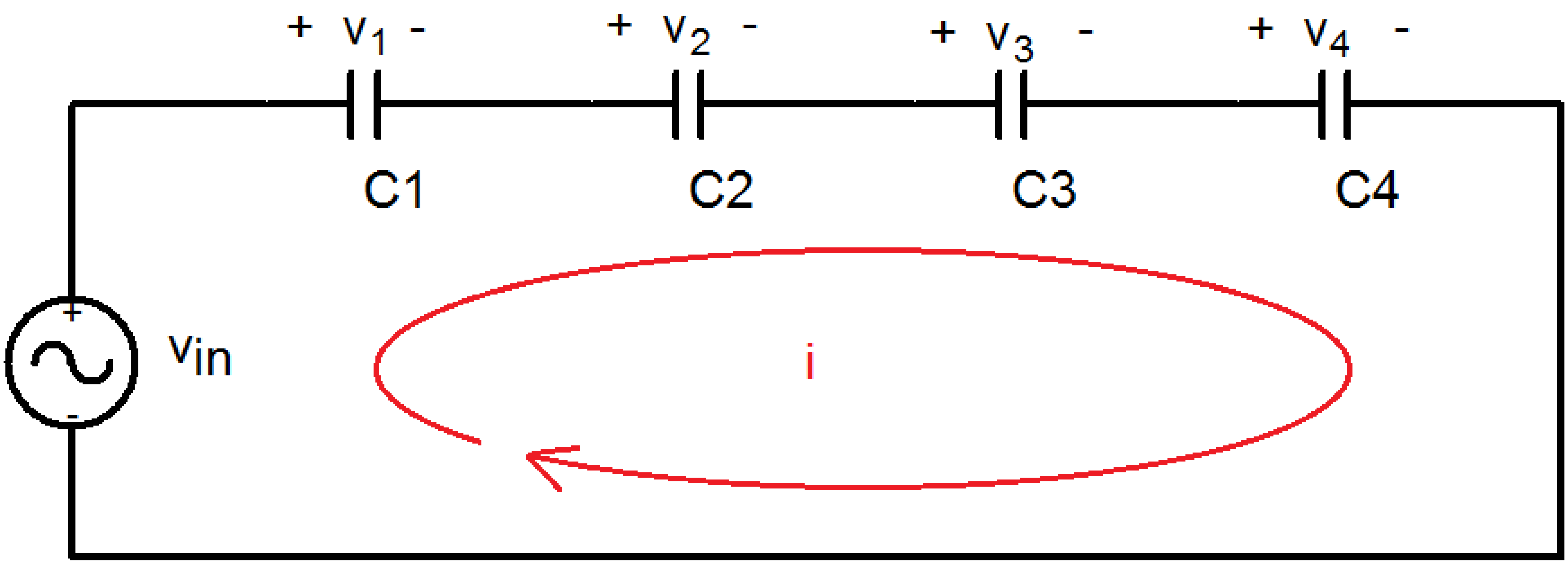
$$i_{in} = C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + C_3 \frac{dv}{dt} + C_4 \frac{dv}{dt}$$

$$i_{in} = C_{eq} \frac{dv}{dt}$$

$$\mathrm{C_{eq}} = C_1 + C_2 + C_3 + C_4$$

Equivalent Capacitance

- Capacitors in series



C_{eq} for Capacitors in Series

$$v_{in} = v_1 + v_2 + v_3 + v_4$$

$$v_1 = \frac{1}{C_1}\int_{t_o}^{t_1} \mathrm{idt} \qquad v_2 = \frac{1}{C_2}\int_{t_o}^{t_1} \mathrm{idt}$$

$$v_3 = \frac{1}{C_3}\int_{t_o}^{t_1} \mathrm{idt} \qquad i \qquad v_4 = \frac{1}{C_4}\int_{t_o}^{t_1} \mathrm{idt}$$

$$v_{in} = \frac{1}{C_1}\int_{t_o}^{t_1} \mathrm{idt} + \frac{1}{C_2}\int_{t_o}^{t_1} \mathrm{idt} + \frac{1}{C_3}\int_{t_o}^{t_1} \mathrm{idt} + \frac{1}{C_4}\int_{t_o}^{t_1} \mathrm{idt}$$

$$v_{in} = \frac{1}{C_{eq}}\int_{t_o}^{t_1} \mathrm{idt}$$

$$\mathrm{C_{eq}} = \left[\left(1/C_1\right) + \left(1/C_2\right) + \left(1/C_3\right) + \left(1/C_4\right)\right]^{-1}$$

General Equations for C_{eq}

Parallel Combination

- If P capacitors are in parallel, then

$$C_{eq} = \sum_{p=1}^{P} C_P$$

Series Combination

- If S capacitors are in series, then:

$$C_{eq} = \left[\sum_{s=1}^{S} \frac{1}{C_s} \right]^{-1}$$

Summary

- Capacitors are energy storage devices.
- An ideal capacitor act like an open circuits when a DC voltage or current has been applied for at least 5 τ.
- The voltage across a capacitor must be a continuous function; the current flowing across a capacitor can be discontinuous.
- The equation for equivalent capacitance for

capacitors in parallel

$$C_{eq} = \sum_{p=1}^{P} C_P$$

capacitors in series

$$C_{eq} = \left[\sum_{s=1}^{S} \frac{1}{C_s} \right]^{-1}$$

Inductors

Energy Storage Devices

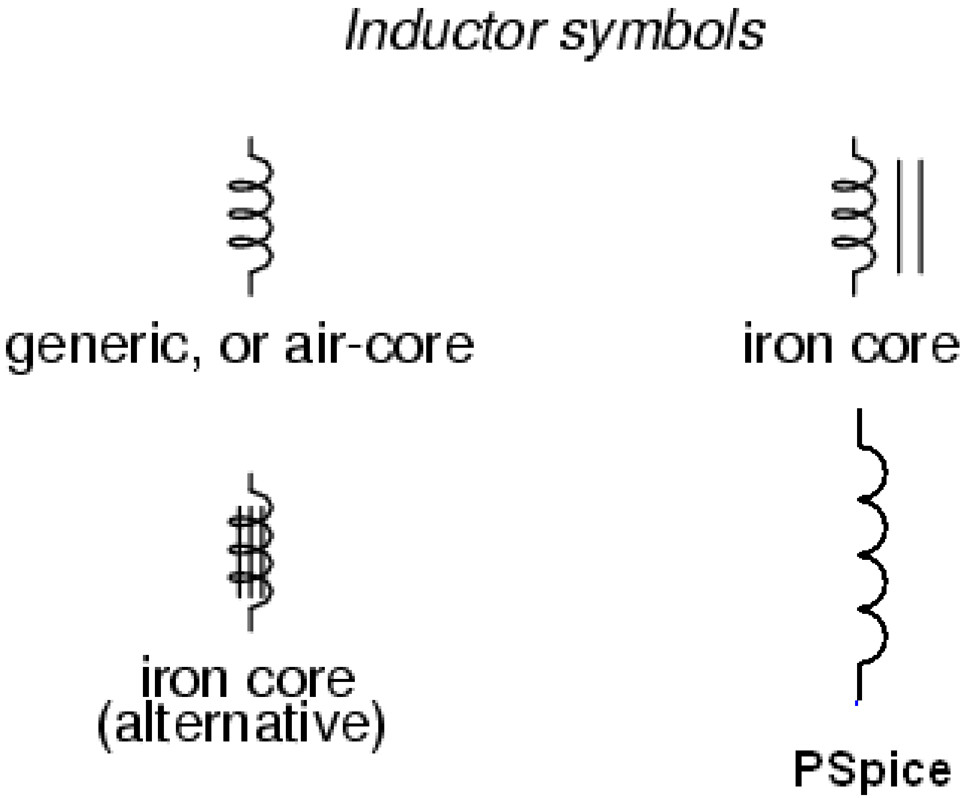
Inductors

- Generally - coil of conducting wire

  - Usually wrapped around a solid core.
  - If no core is used, then the inductor is said to have an 'air core'.

http://bzupages.com/f231/energy-stored-inductor-uzma-noreen-group6-part2-1464/

Symbols



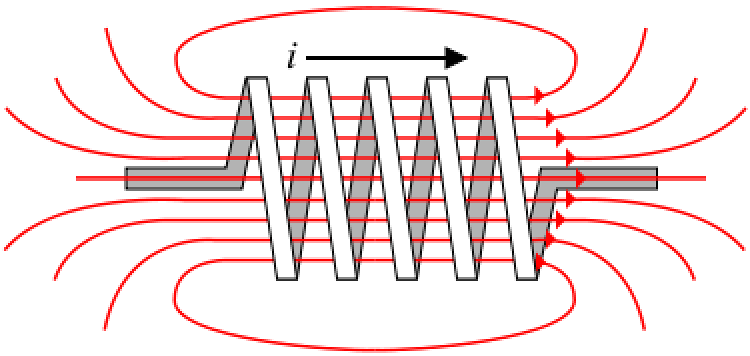
http://www.allaboutcircuits.com/vol_1/chpt_15/1.html

Alternative Names for Inductors

- Reactor
  – inductor in a power grid
- Choke
  – designed to block a particular frequency while allowing currents at lower frequencies or d.c. currents through
    - Commonly used in RF (radio frequency) circuitry
- Coil
  – often coated with varnish and/or wrapped with insulating tape to provide additional insulation and secure them in place
    - A winding is a coil with taps (terminals).
- Solenoid
  – a three dimensional coil.
    - Also used to denote an electromagnet where the magnetic field is generated by current flowing through a toroidal inductor.

Energy Storage

- The flow of current through an inductor creates a magnetic field (right hand rule).



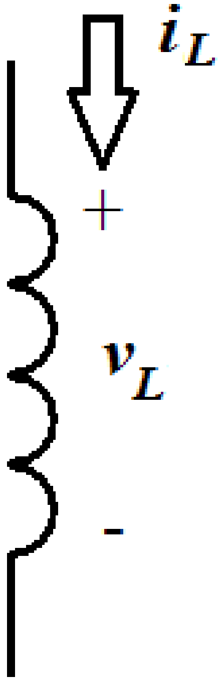
B field

http://en.wikibooks.org/wiki/Circuit_Theory/Mutual_Inductance

- If the current flowing through the inductor drops, the magnetic field will also decrease and energy is released through the generation of a current.

Sign Convention

- The sign convention used with an inductor is the same as for a power dissipating device.



  - When current flows into the positive side of the voltage across the inductor, it is positive and the inductor is dissipating power.
  - When the inductor releases energy back into the circuit, the sign of the current will be negative.

Current and Voltage Relationships

- L , inductance, has the units of Henries (H)

  1 H = 1 V-s/A

$$v_L = L\frac{di}{dt}$$

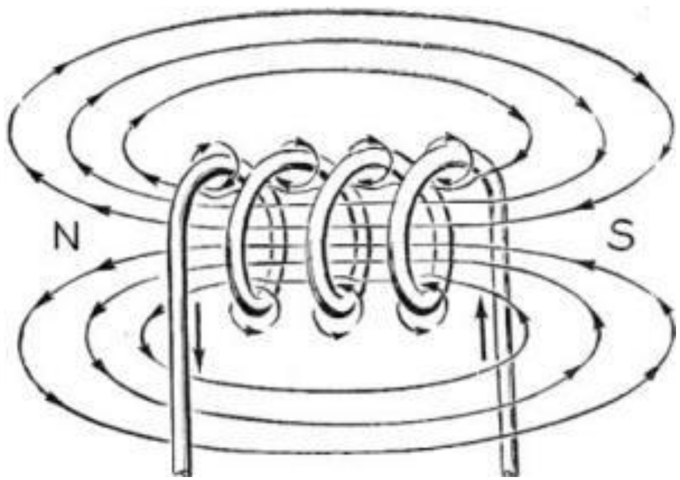
$$i_L = \frac{1}{L}\int_{t_o}^{t_1} v_L\,dt$$

Power

$$p_L = v_L i_L = L i_L \int_{t_o}^{t_1} i_L dt$$

Inductors

- Stores energy in a magnetic field created by the electric current flowing through it.
  - Inductor opposes change in current flowing through it.
    - Current through an inductor is continuous; voltage can be discontinuous.



http://www.rfcafe.com/references/electrical/Electricity%20-%20Basic%20Navy%20Training%20Courses/electricity%20-%20basic%20navy%20training%20courses%20-%20chapter%2012.htm

Calculations of L

- For a solenoid (toroidal inductor)

$$L = \frac{N^2 \mu A}{\ell} = \frac{N^2 \mu_r \mu_o A}{\ell}$$

N is the number of turns of wire

A is the cross-sectional area of the toroid in m^2.

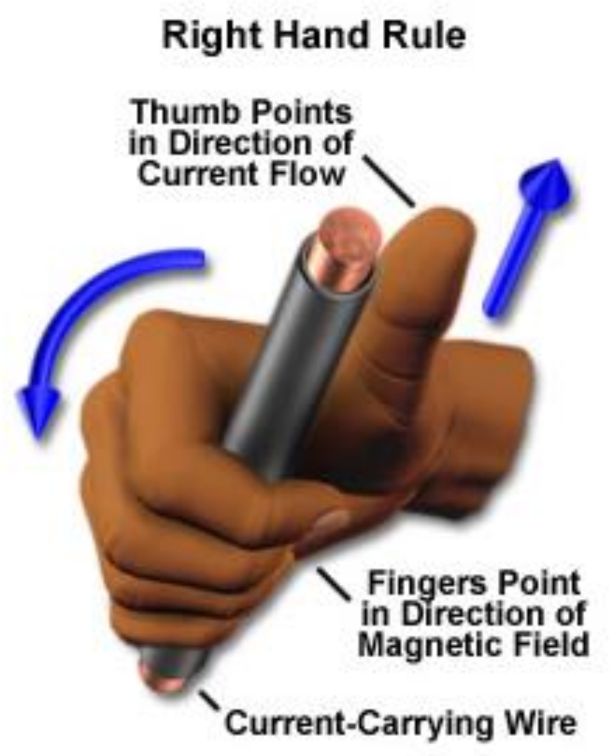
μ_r is the relative permeability of the core material

μ_o is the vacuum permeability ($4\pi \times 10^{-7}$ H/m)

ℓ is the length of the wire used to wrap the toroid in meters

Wire

- Unfortunately, even bare wire has inductance.



$$L = \ell\left[\ln\left(4\frac{\ell}{d}\right) - 1\right]\left(2x10^{-7}\right)H$$

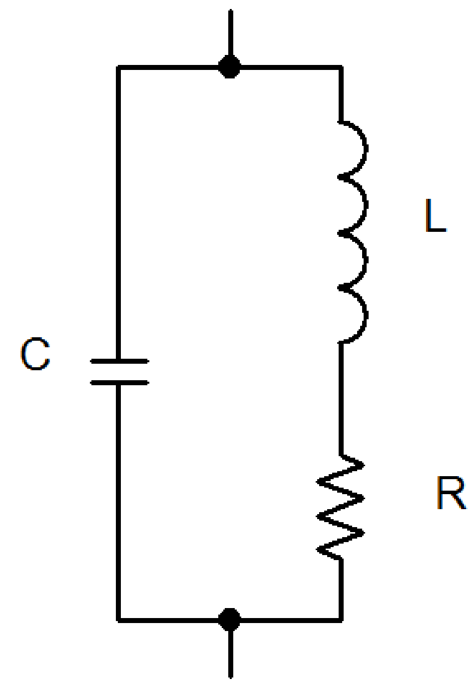
- d is the diameter of the wire in meters.

Properties of an Inductor

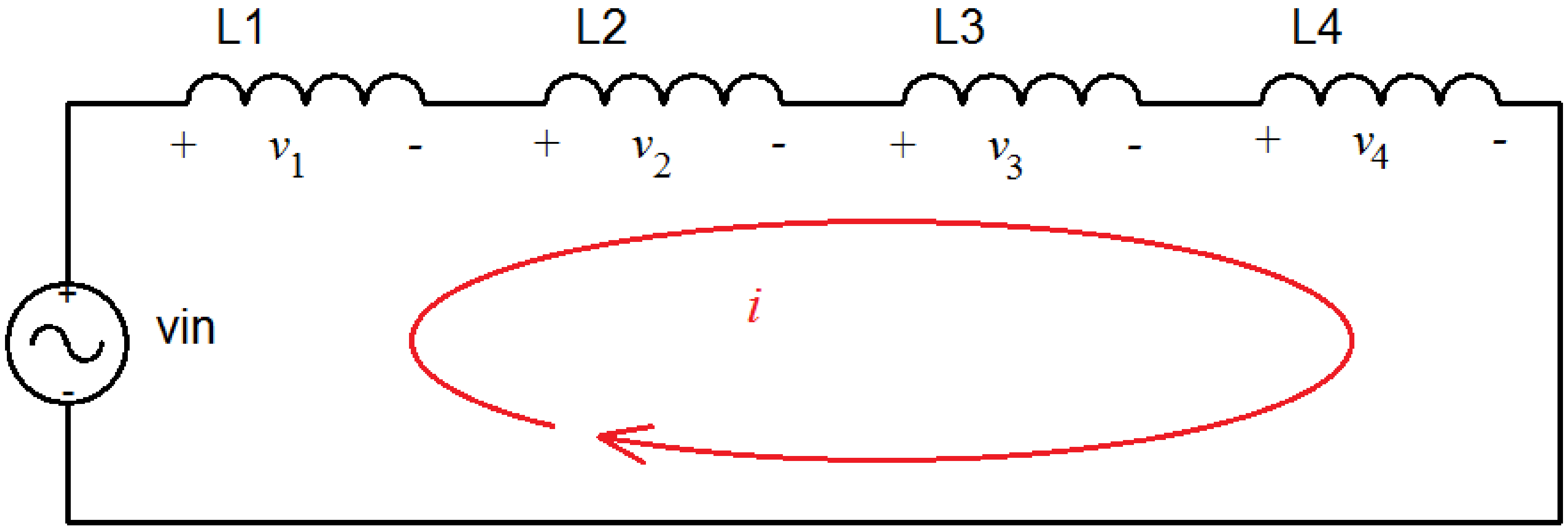
- Acts like a short circuit at steady state when connected to a d.c. voltage or current source.
- Current through an inductor must be continuous
  - There are no abrupt changes to the current, but there can be abrupt changes in the voltage across an inductor.
- An ideal inductor does not dissipate energy, it takes power from the circuit when storing energy and returns it when discharging.

Properties of a Real Inductor

- Real inductors do dissipate energy due to resistive losses in the length of wire and capacitive coupling between turns of the wire.



Inductors in Series



L_{eq} for Inductors in Series

$$v_{in} = v_1 + v_2 + v_3 + v_4$$

$$v_1 = L_1 \frac{\mathrm{di}}{\mathrm{dt}} \qquad v_2 = L_2 \frac{\mathrm{di}}{\mathrm{dt}}$$

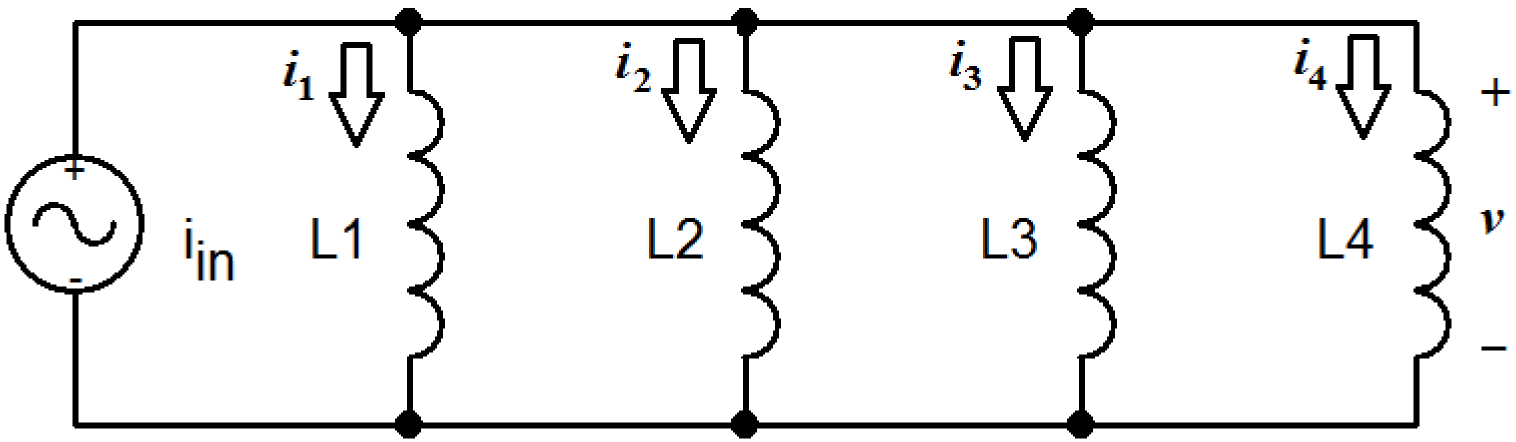
$$v_3 = L_3 \frac{\mathrm{di}}{\mathrm{dt}} \qquad i \qquad v_4 = L_4 \frac{\mathrm{di}}{\mathrm{dt}}$$

$$v_{in} = L_1 \frac{\mathrm{di}}{\mathrm{dt}} + L_2 \frac{\mathrm{di}}{\mathrm{dt}} + L_3 \frac{\mathrm{di}}{\mathrm{dt}} + L_4 \frac{\mathrm{di}}{\mathrm{dt}}$$

$$v_{in} = L_{eq} \frac{\mathrm{di}}{\mathrm{dt}}$$

$$\mathrm{L}_{\mathrm{eq}} = L_1 + L_2 + L_3 + L_4$$

Inductors in Parallel



L_{eq} for Inductors in Parallel

$$i_{in} = i_1 + i_2 + i_3 + i_4$$

$$i_1 = \frac{1}{L_1}\int_{t_o}^{t_1} \mathrm{vdt} \qquad i_2 = \frac{1}{L_2}\int_{t_o}^{t_1} \mathrm{vdt}$$

$$i_3 = \frac{1}{L_3}\int_{t_o}^{t_1} \mathrm{vdt} \qquad {}_{i}\, i_4 = \frac{1}{L_4}\int_{t_o}^{t_1} \mathrm{vdt}$$

$$i_{in} = \frac{1}{L_1}\int_{t_o}^{t_1} \mathrm{vdt} + \frac{1}{L_2}\int_{t_o}^{t_1} \mathrm{vdt} + \frac{1}{L_3}\int_{t_o}^{t_1} \mathrm{vdt} + \frac{1}{L_4}\int_{t_o}^{t_1} \mathrm{vdt}$$

$$i_{in} = \frac{1}{L_{eq}}\int_{t_o}^{t_1} \mathrm{vdt}$$

$$\mathrm{L}_{\mathrm{eq}} = \left[\left(1/L_1\right) + \left(1/L_2\right) + \left(1/L_3\right) + \left(1/L_4\right)\right]^{-1}$$

General Equations for L_{eq}

Series Combination

- If S inductors are in series, then

$$L_{eq} = \sum_{s=1}^{S} L_s$$

Parallel Combination

- If P inductors are in parallel, then:

$$L_{eq} = \left[\sum_{p=1}^{P} \frac{1}{L_p} \right]^{-1}$$

Summary

- Inductors are energy storage devices.
- An ideal inductor act like a short circuit at steady state when a DC voltage or current has been applied.
- The current through an inductor must be a continuous function; the voltage across an inductor can be discontinuous.
- The equation for equivalent inductance for

inductors in series

$$L_{eq} = \sum_{s=1}^{S} L_s$$

inductors in parallel

$$L_{eq} = \left[\sum_{p=1}^{P} \frac{1}{L_p} \right]^{-1}$$

RC and RL Circuits

First Order Circuits

Objectives of Lecture

- Explain the operation of a RC circuit in dc circuits
  - As the capacitor stores energy when voltage is first applied to the circuit or the voltage applied across the capacitor is increased during the circuit operation.
  - As the capacitor releases energy when voltage is removed from the circuit or the voltage applied across the capacitor is decreased during the circuit operation.
- Explain the operation of a RL circuit in dc circuit
  - As the inductor stores energy when current begins to flow in the circuit or the current flowing through the inductor is increased during the circuit operation.
  - As the inductor releases energy when current stops flowing in the circuit or the current flowing through the inductor is decreased during the circuit operation.

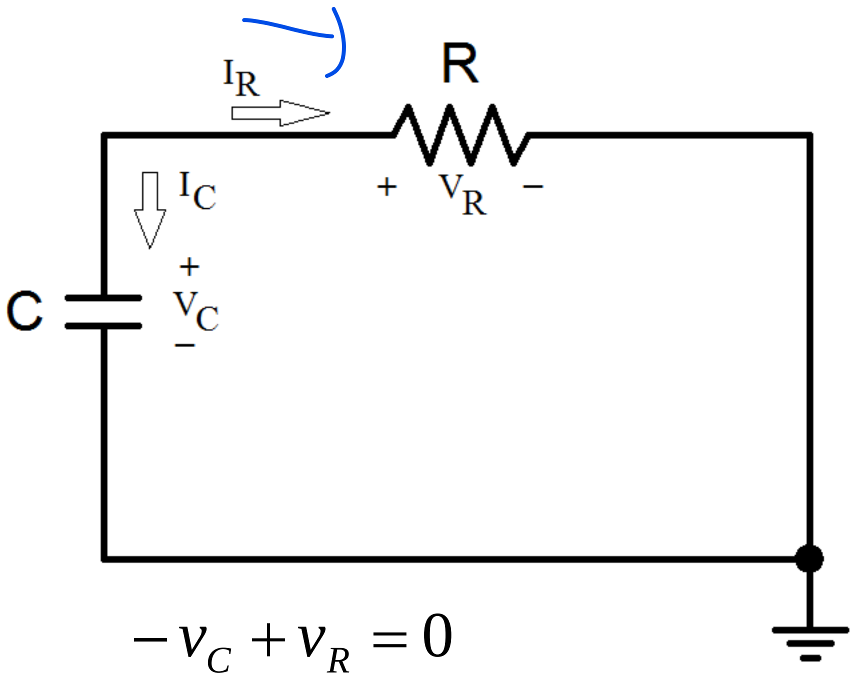
Natural Response

- The behavior of the circuit with no external sources of excitation.
  - There is stored energy in the capacitor or inductor at *time* = 0 s.
  - For $t > 0$ s, the stored energy is released
    - Current flows through the circuit and voltages exist across components in the circuit as the stored energy is released.
    - The stored energy will decays to zero as time approaches infinite, at which point the currents and voltages in the circuit become zero.

RC Circuit

- Suppose there is some charge on a capacitor at time $t = 0$ s.
  - This charge could have been stored because a voltage or current source had been in the circuit at $t < 0$ s, but was switched off at $t = 0$ s.
- We can use the equations relating voltage and current to determine how the charge on the capacitor is removed as a function of time.
  - The charge flows from one plate of the capacitor through the resistor R to the other plate to neutralize the charge on the opposite plate of the capacitor.

Equations for RC Circuit



$$-v_C + v_R = 0$$

$$i_C = -i_R$$

$$i_C = C\frac{dv_C}{dt}$$

$$i_R = \frac{v_R}{R}$$

$$C\frac{dV_C}{dt} + \frac{V_R}{R} = 0$$

$$V_R = V_C$$

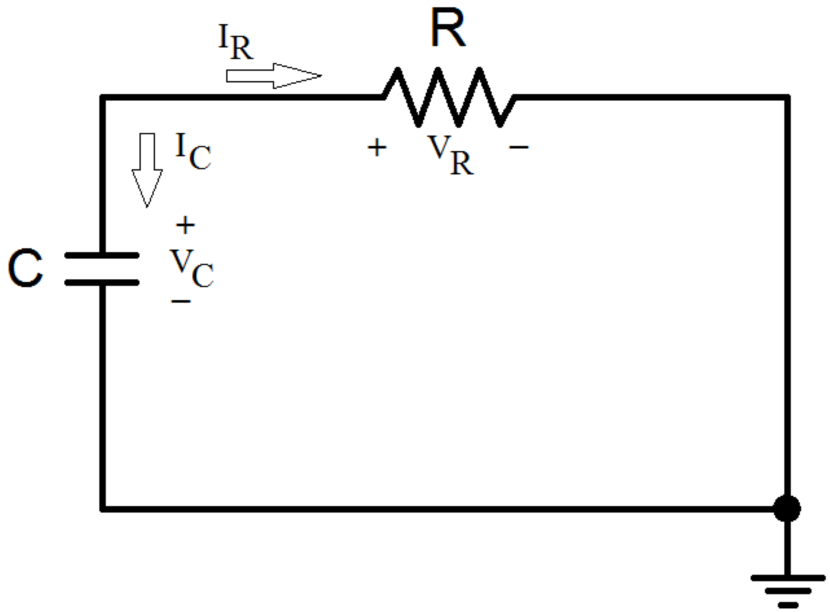
$$\frac{dV_C}{dt} + \frac{V_C}{RC} = 0$$

$$\frac{1}{V_C}\frac{dV_C}{dt} + \frac{1}{RC} = 0$$

$$\frac{dV_C}{V_C} = -\frac{1}{RC}dt$$

$$\ln\left(V_C\right) = -\frac{t}{RC} + \ln\left(V_C\big|_{t=t_o}\right)$$

Equations for RC Circuit



If $V_o = V_C\big|_{t=0s}$ and $\tau = RC$

$$V_C(t) = V_o e^{-\frac{t}{\tau}} \quad \text{when } t \geq 0s$$

$$I_R(t) = -I_C(t) = \frac{V_o}{R} e^{-\frac{t}{\tau}}$$

$$p_R(t) = V_R I_R = \frac{{V_o}^2}{R} e^{-\frac{2t}{\tau}}$$

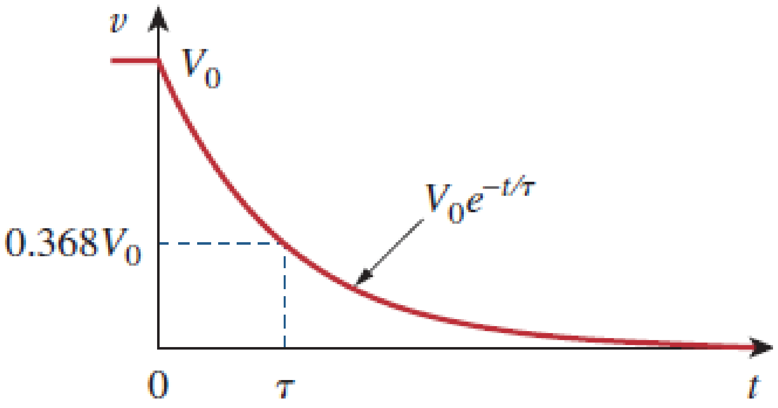
Since the voltages are equal and the currents have the opposite sign, the power that is dissipated by the resistor is the power that is being released by the capacitor.

The Key to Working with a Source-Free RC Circuit Is Finding:

- The initial voltage $v(0) = V_0$ across the capacitor.
  - Can be obtained by inserting a d.c. source to the circuit for a time much longer than τ (at least t = -5τ) and then removing it at $t = 0$.
    - Capacitor
      - Open Circuit Voltage
- The time constant τ.
  - In finding the time constant $\tau = RC$, R is often the Thevenin equivalent resistance at the terminals of the capacitor;
    - that is, we take out the capacitor C and find $R = R_{Th}$ at its terminals

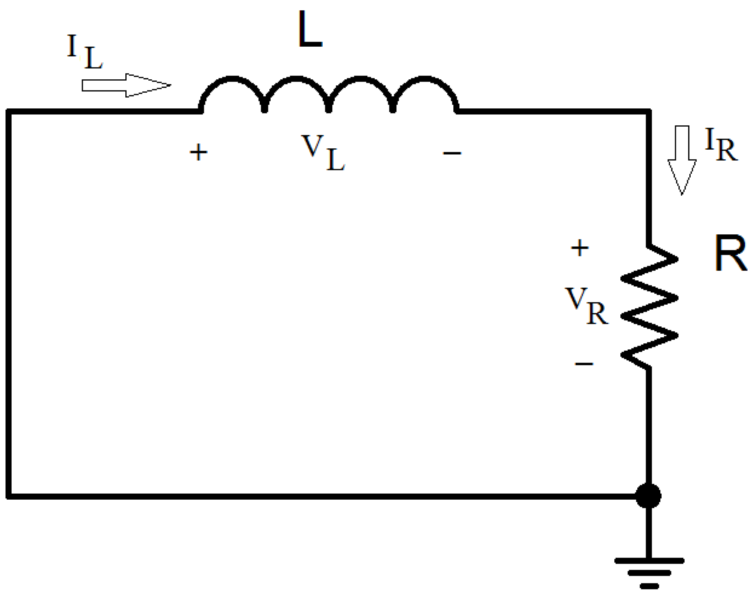
Time constant

- The natural response of a capacitive circuit refers to the behavior (in terms of voltages) of the circuit itself, with no external sources of excitation.
  - The natural response depends on the nature of the circuit alone, with no external sources.
    - In fact, the circuit has a response only because of the energy initially stored in the capacitor.
- The voltage response of the *RC* circuit



  - Time constant, $\tau = RC$
    - The time required for the voltage across the capacitor to decay by a factor of 1/e or 36.8% of its initial value.

Equations for RL Circuits



$$L\frac{dI_L}{dt} + RI_R = 0$$

$$\frac{dI_L}{dt} + \frac{RI_L}{L} = 0$$

$$\frac{1}{I_L}\frac{dI_L}{dt} + \frac{R}{L} = 0$$

$$V_L + V_R = 0$$

$$I_L = I_R$$

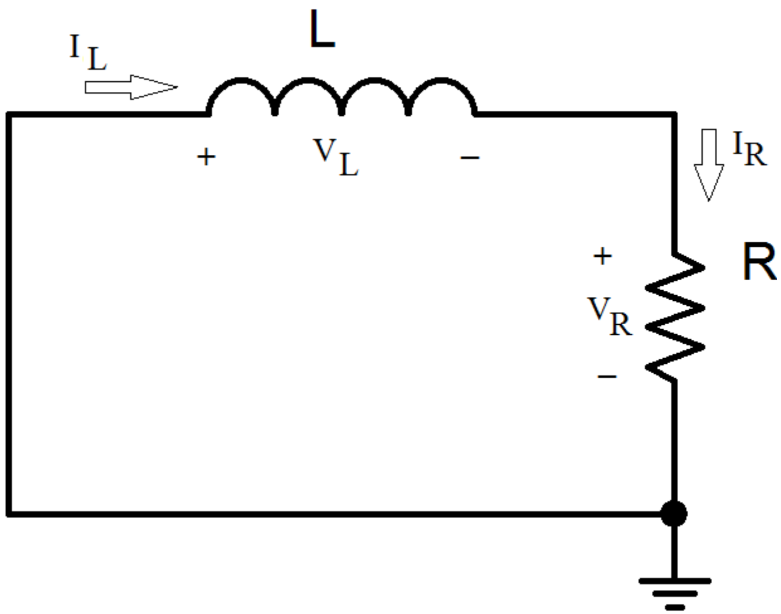
$$\frac{dI_L}{I_L} = -\frac{R}{L}dt$$

$$V_L = L\frac{dI_L}{dt}$$

$$\ln\left(I_L\right) = -\frac{R}{L}t + \ln\left(I_L\big|_{t=0s}\right)$$

$$I_R = V_R / R$$

Equations for RL Circuit



If $I_o = I_L\big|_{t=0s}$ and $\tau = \dfrac{L}{R}$

$$I_L(t) = I_o e^{-\frac{t}{\tau}} \quad \text{when } t \geq 0s$$

$$V_R(t) = -V_L(t) = RI_o e^{-\frac{t}{\tau}}$$

$$p_R(t) = V_R I_R = RI_o{}^2 e^{-\frac{2t}{\tau}}$$

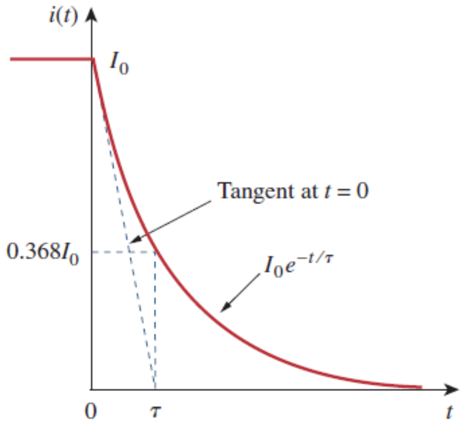
Since the voltages are equal and opposite and the currents have the same sign, the power that is dissipated by the resistor is the power that is being released by the inductor.

The Key to Working with a Source-Free RL Circuit Is Finding:

- The initial current $i(0) = I_0$ through the inductor.
  - Can be obtained by inserting a d.c. source to the circuit for a time much longer than τ (at least t = -5τ) and then removing it at $t = 0$.
    - Inductor
      - Short Circuit Current
- The time constant τ.
  - In finding the time constant $\tau = L/R$, R is often the Thevenin equivalent resistance at the terminals of the inductor;
    - that is, we take out the inductor L and find $R = R_{Th}$ at its terminals

Time constant

- The natural response of an inductive circuit refers to the behavior (in terms of currents) of the circuit itself, with no external sources of excitation.
  - The natural response depends on the nature of the circuit alone, with no external sources.
    - In fact, the circuit has a response only because of the energy initially stored in the inductor.
- The current response of the *RL* circuit



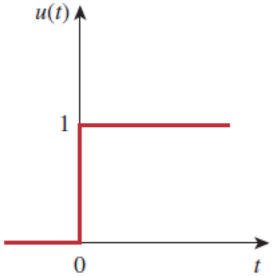
  - Time constant, $\tau = L/R$
    - The time required for the current in the inductor to decay by a factor of 1/e or 36.8% of its initial value.

Singularity Functions

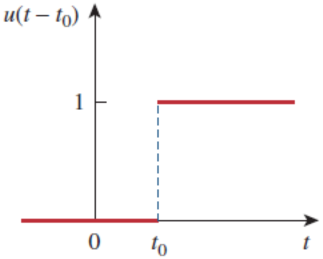
- Singularity functions (also called switching functions) are very useful in circuit analysis.
- They serve as good approximations to the switching signals that arise in circuits with switching operations.
- They are helpful in the neat, compact description of some circuit phenomena,
  - especially the step response of *RC* or *RL* circuits
- Singularity functions are functions that either are discontinuous or have discontinuous derivatives.

Unit Step Function

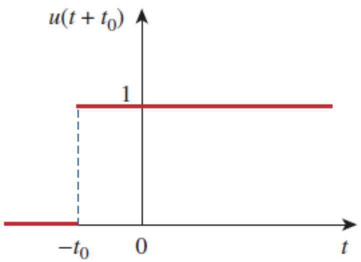
- The unit step function ($u(t)$) is 0 for negative values of t and 1 for positive values of t.



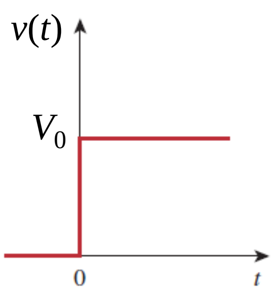
$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



$$u(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases}$$



$$u(t + t_0) = \begin{cases} 0, & t < -t_0 \\ 1, & t > -t_0 \end{cases}$$

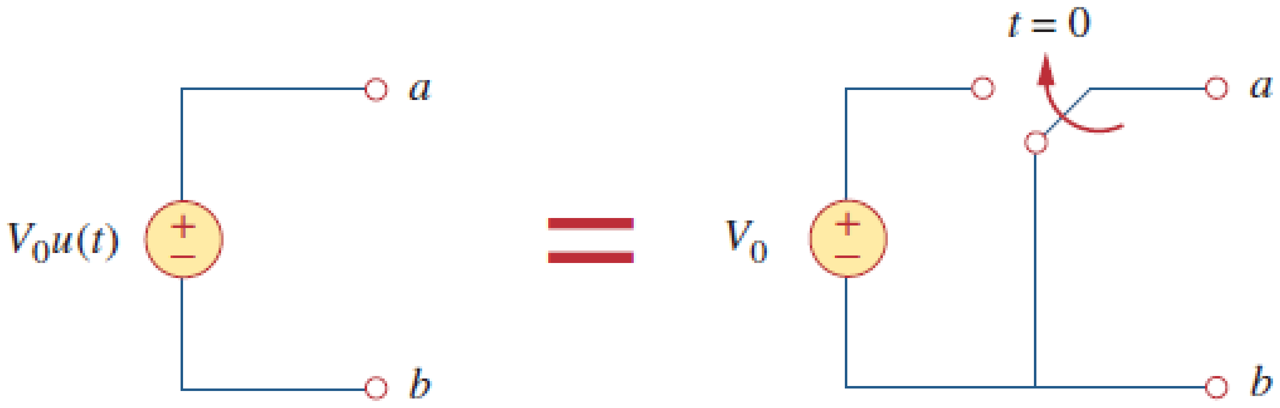


$$v(t) = \begin{cases} 0, & t < t_0 \\ V_0, & t > t_0 \end{cases}$$

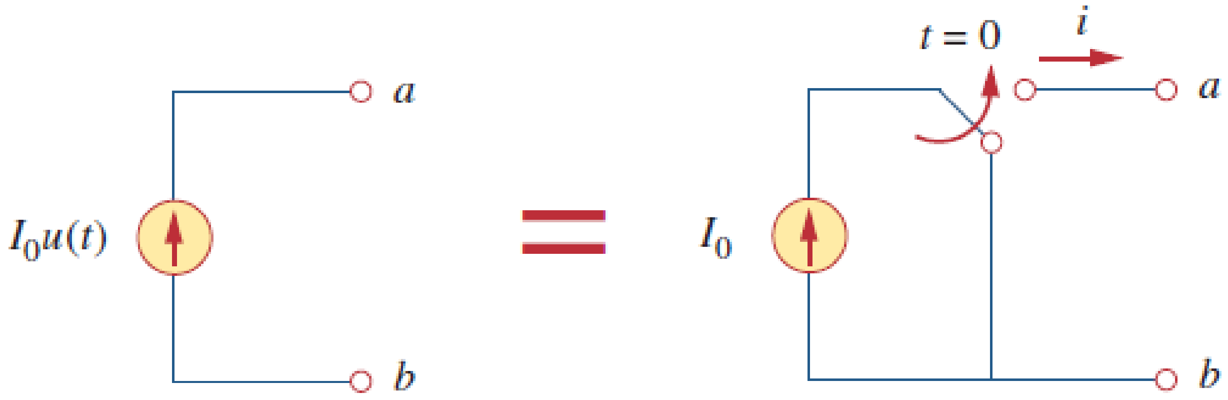
$$v(t) = V_0 u(t - t_0)$$

Unit Step Function

- Voltage source of $V_0u(t)$ and its equivalent circuit.

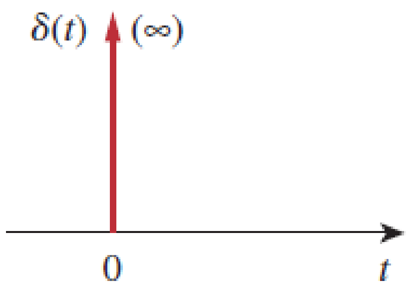


- Current source of $I_0u(t)$ and its equivalent circuit.



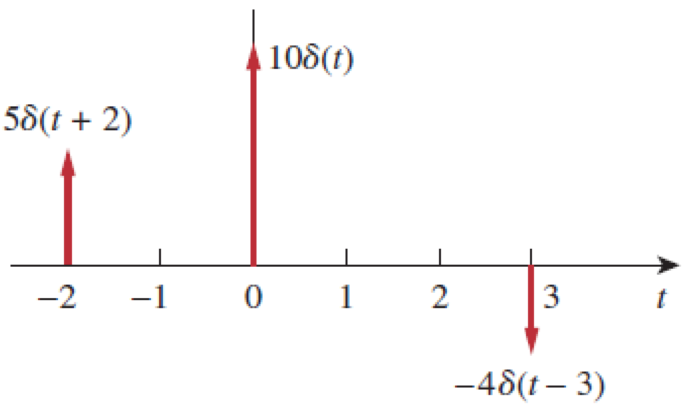
Unit Impulse Function

- The derivative of the unit step function $u(t)$ is the unit impulse function ($\delta(t)$)



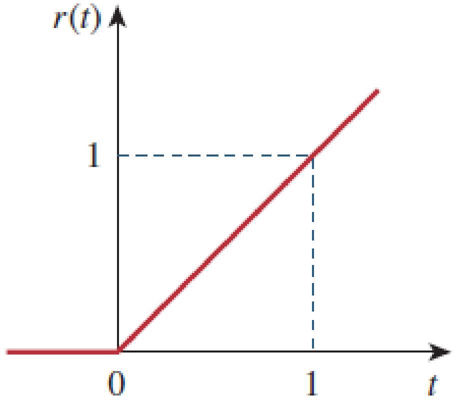
$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0, & t < 0 \\ \text{Undefined}, & t = 0 \\ 0, & t > 0 \end{cases}$$

$$\int_{0^-}^{0^+} \delta(t)\, dt = 1$$



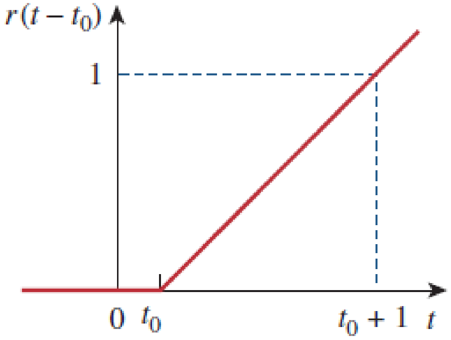
$x(t) = 5\delta(t+2) + 10\delta(t) - 4\delta(t-3)$

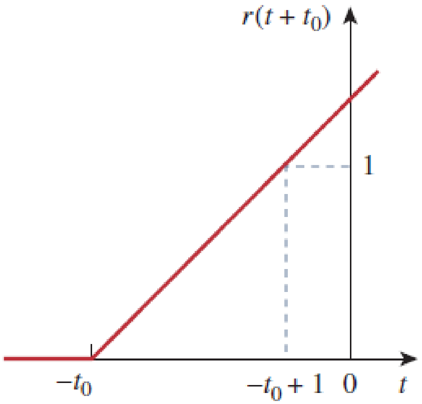
Unit Ramp Function



$$r(t) = \int_{-\infty}^{t} u(\lambda)\,d\lambda = tu(t)$$

$$\boxed{r(t) = \begin{cases} 0, & t \leq 0 \\ t, & t \geq 0 \end{cases}}$$





$$r(t - t_0) = \begin{cases} 0, & t \leq t_0 \\ t - t_0, & t \geq t_0 \end{cases}$$

$$r(t + t_0) = \begin{cases} 0, & t \leq -t_0 \\ t + t_0, & t \geq -t_0 \end{cases}$$

Relationships of singularity functions

- The three singularity functions (impulse, step, and ramp) are related by differentiation as

$$\delta(t) = \frac{du(t)}{dt}, \qquad u(t) = \frac{dr(t)}{dt}$$

- or by integration as

$$u(t) = \int_{-\infty}^{t} \delta(\lambda)\, d\lambda, \qquad r(t) = \int_{-\infty}^{t} u(\lambda)\, d\lambda$$

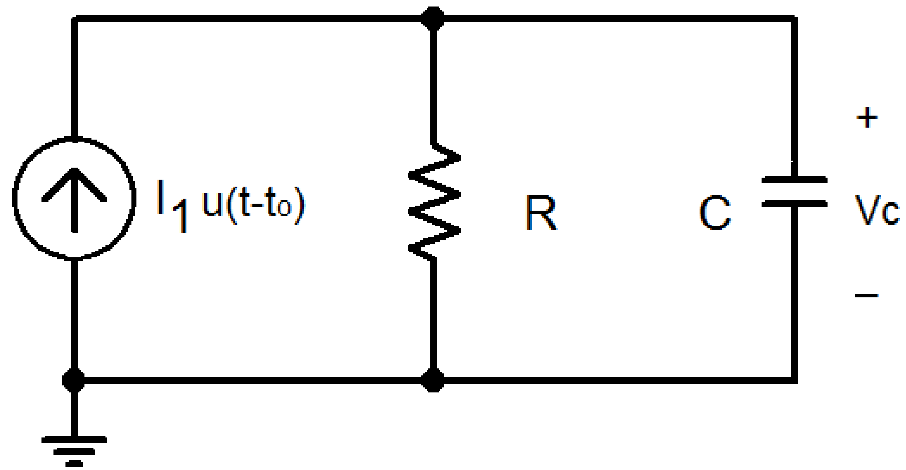
Transient responses of RC and RL circuits

- AKA a forced response to an independent source
- Capacitor and inductor store energy when there is:
  - a transition in a unit step function source, $u(t-t_o)$
  - a voltage or current source is switched into the circuit.

RC Circuit

I_C = 0A when $t < t_o$

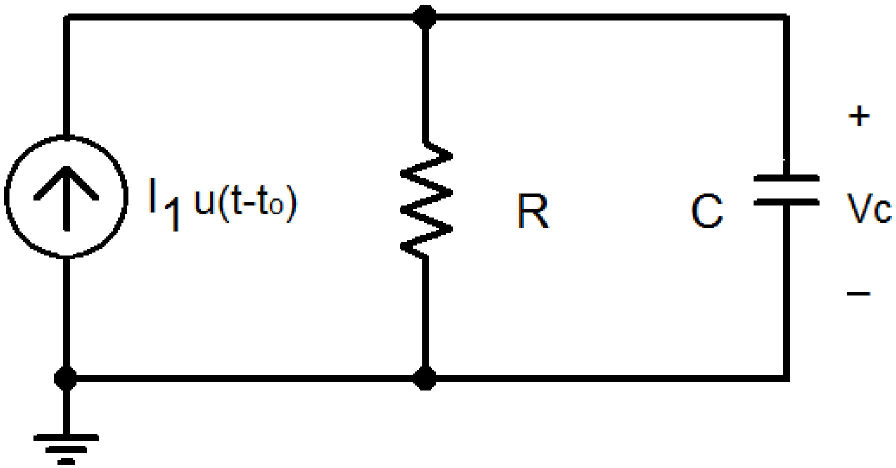
V_C = 0V when $t < t_o$



Because I_1 = 0A (replace it with an open circuit).

RC Circuit

- Find the final condition of the voltage across the capacitor.



  - Replace C with an open circuit and determine the voltage across the terminal.

$I_C = 0A$ when $t \sim \infty$ s

$V_C = V_R = I_1R$ when $t \sim \infty$ s

RC Circuit

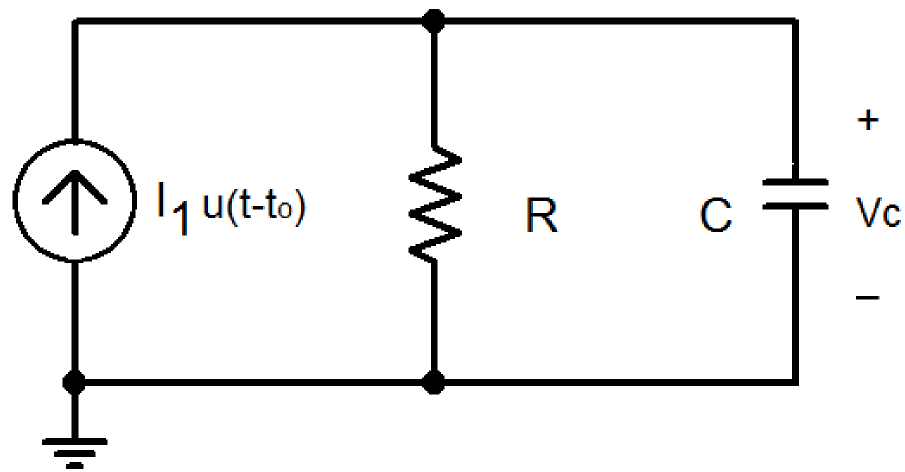
- In the time between t_o and $t = \infty$ s, the capacitor stores energy and currents flow through R and C.

$$V_C = V_R$$

$$I_C = C\frac{dV_C}{dt}$$

$$I_R = \frac{V_R}{R}$$

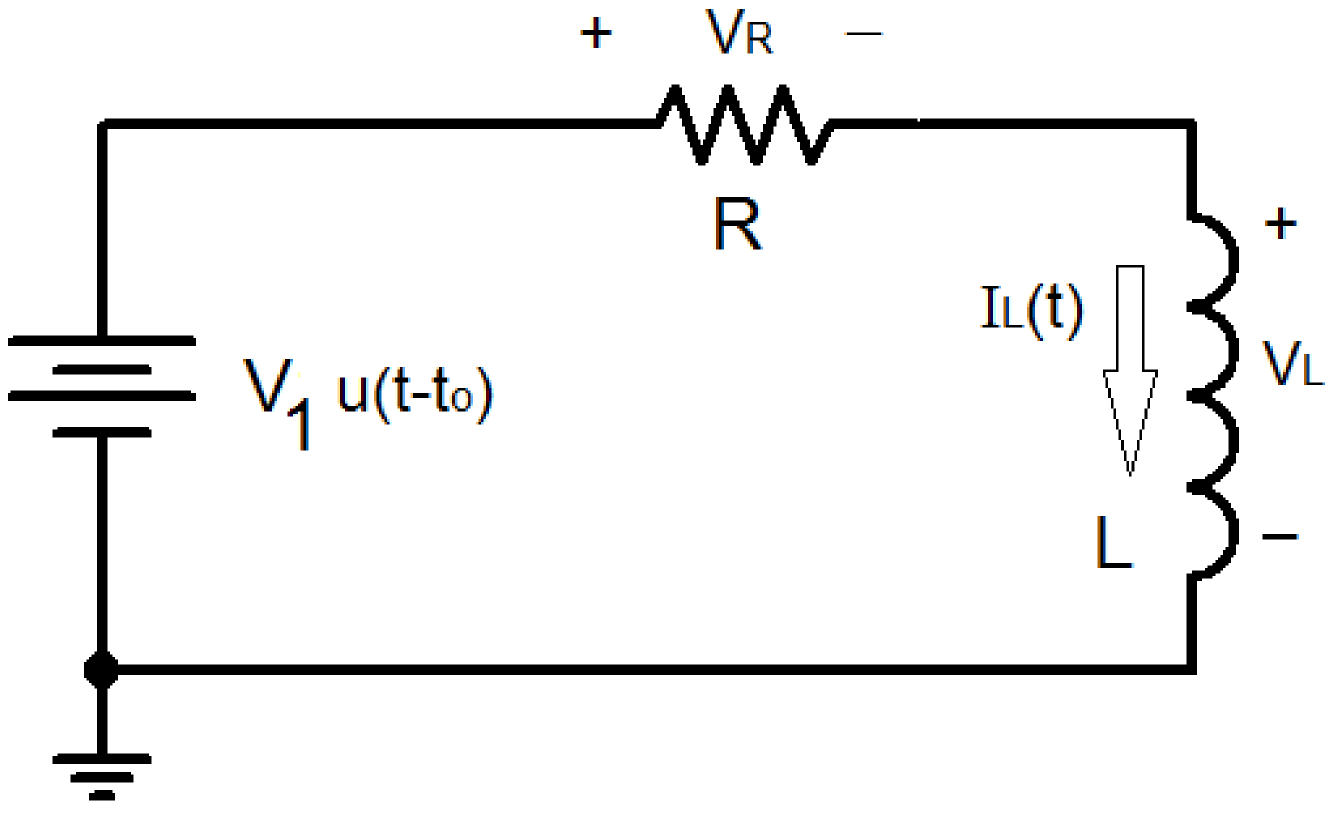
$$I_R + I_C - I_1 = 0$$



$$\frac{V_C}{R} + C\frac{dV_C}{dt} - I_1 = 0$$

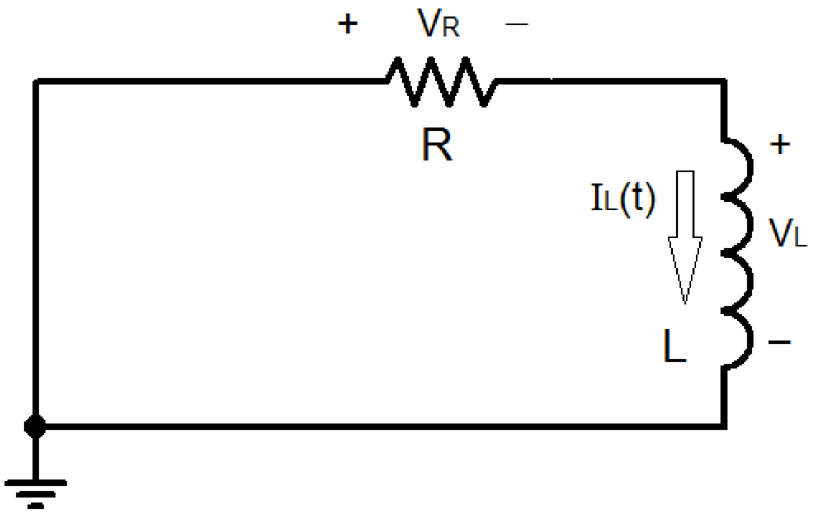
$$V_C(t) = RI_1\left[1 - e^{-\frac{t-t_0}{\tau}}\right] \qquad \tau = RC$$

RL Circuit



RL Circuit

- Initial condition is not important as the magnitude of the voltage source in the circuit is equal to 0V when $t \leq t_o$.

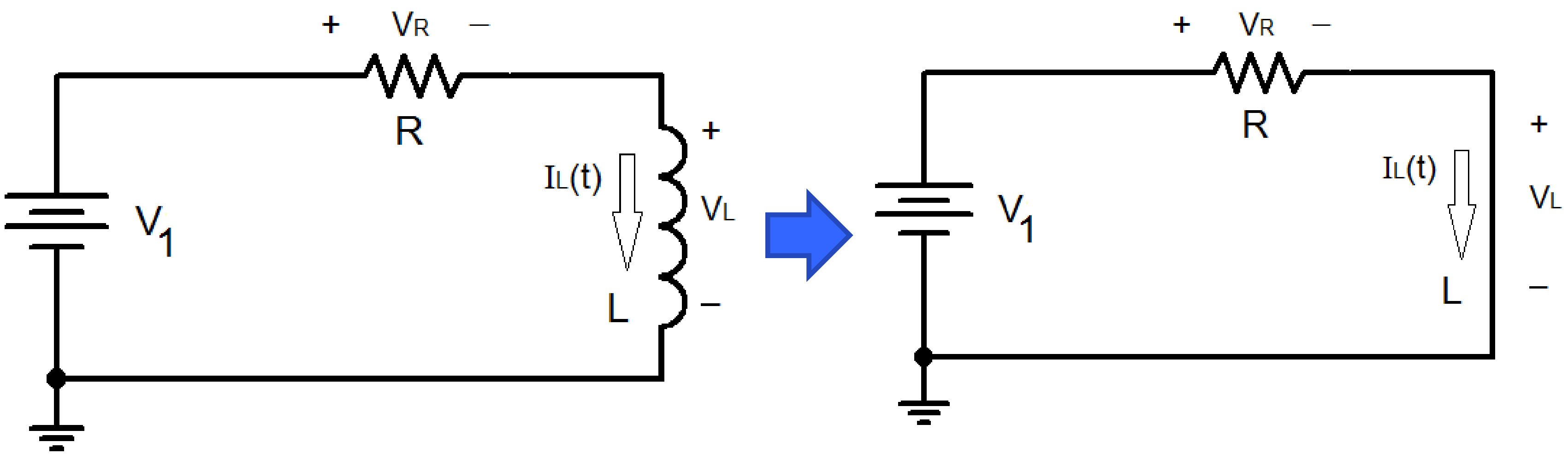


– Since the voltage source has only been turned on at $t = t_o$, the circuit at $t \leq t_o$ is as shown on the left.

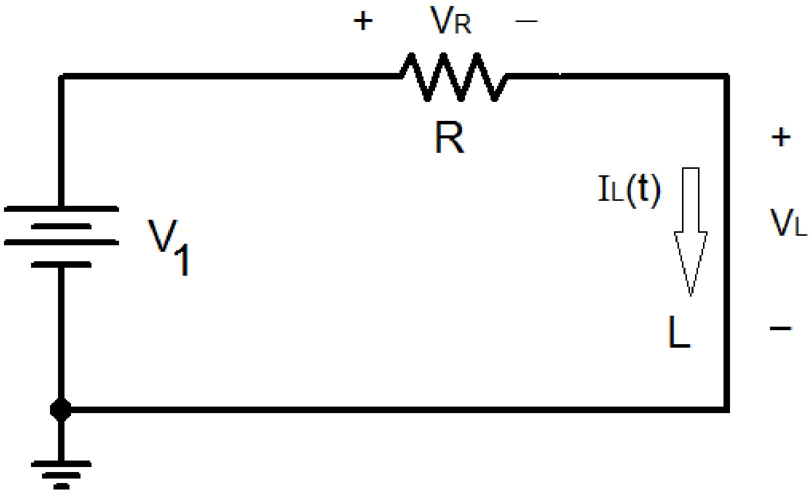
- As the inductor has not stored any energy because no power source has been connected to the circuit as of yet, all voltages and currents are equal to zero.

RL Circuit

- So, the final condition of the inductor current needs to be calculated after the voltage source has switched on.
  - Replace L with a short circuit and calculate $I_L(\infty)$.



Final Condition

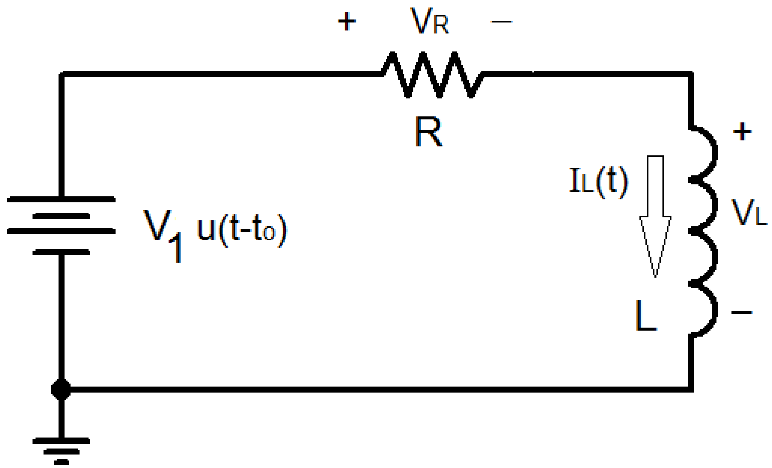


$$V_L(\infty) = 0V$$

$$I_L(\infty) = I_R$$

$$I_R = \frac{V_1}{R}$$

RL Circuit



$$-V_1 + V_L + V_R = 0$$

$$I_L = I_R = V_R / R$$

$$V_L = L\frac{dI_L}{dt}$$

$$\frac{dI_L}{dt} + RI_R - V_1 = 0$$

$$\frac{dI_L}{dt} + \frac{R}{L}I_L - \frac{V_1}{L} = 0$$

$$I_L(t) = \frac{V_1}{R}\left[1 - e^{-(t-t_o)/\tau}\right]$$

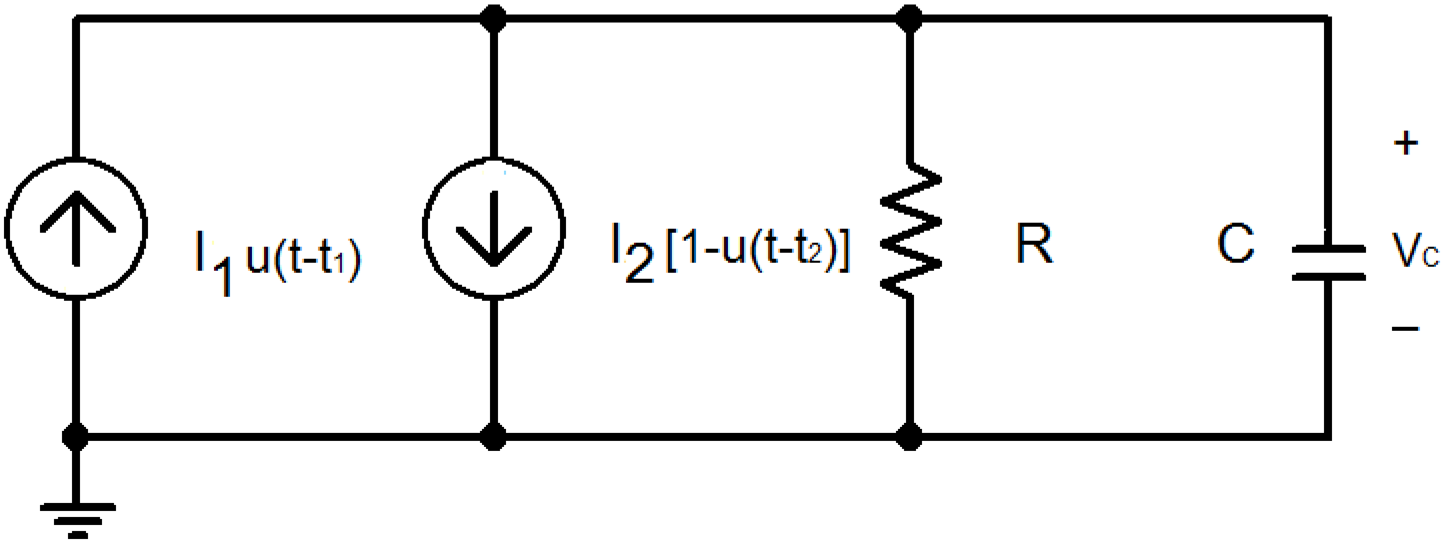
$$\tau = \frac{L}{R}$$

Complete Response

- Is equal to the natural response of the circuit plus the forced response
  - Use superposition to determine the final equations for voltage across components and the currents flowing through them.
- Typically, it is assumed that the currents and voltages in a circuit have reached steady-state once 5τ have passed after a change has been made to the value of a current or voltage source in the circuit.

Example 01…

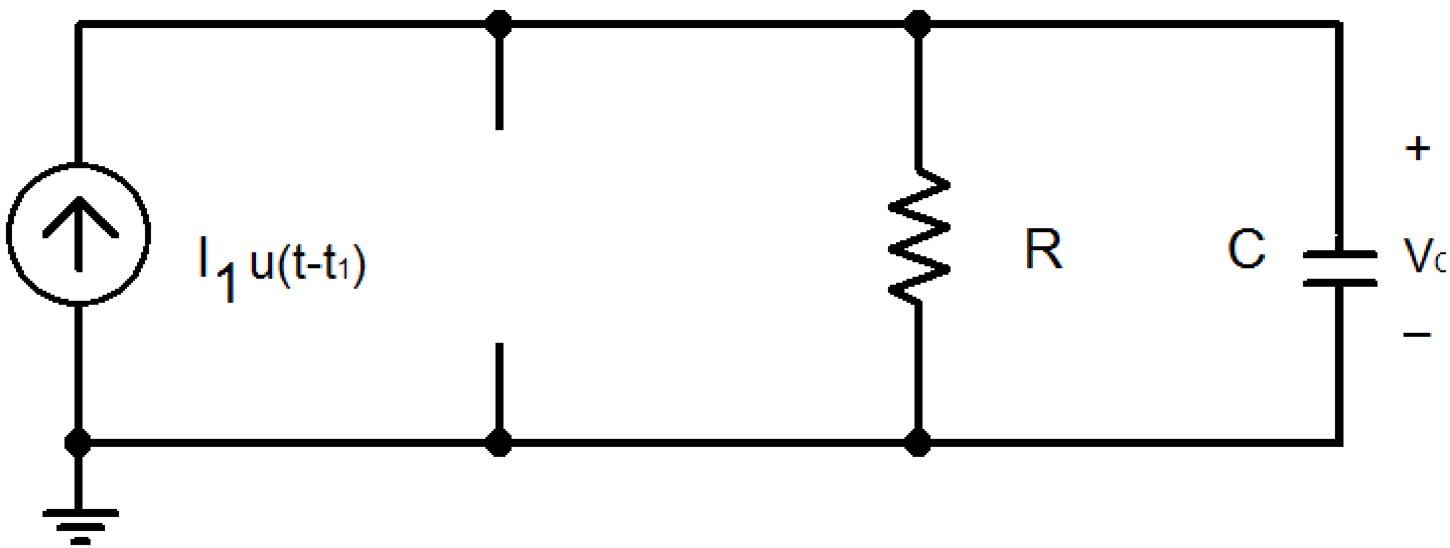
- Suppose there were two unit step function sources in the circuit.



…Example 01…

- The solution for Vc would be the result of superposition where:
  - $I_2 = 0A$, I_1 is left on
    - The solution is a forced response since I_1 turns on at $t = t_1$
  - $I_1 = 0A$, I_2 is left on
    - The solution is a natural response since I_2 turns off at $t = t_2$

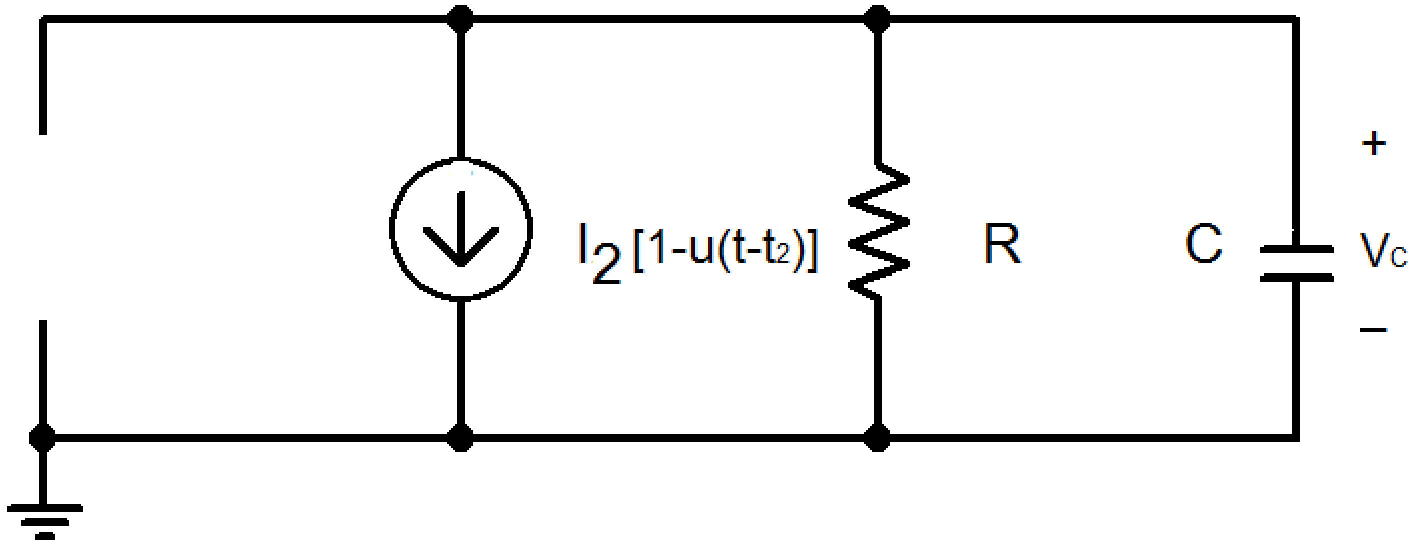
…Example 01…



$$V_C(t) = 0V \quad \text{when } t < t_1$$

$$V_C(t) = RI_1\left[1 - e^{-\frac{(t-t_1)}{RC}}\right] \quad \text{when } t > t_1$$

…Example 01…



$$V_C(t) = -RI_2 \quad \text{when } t < t_2$$

$$V_C(t) = -RI_2 e^{-\frac{(t-t_2)}{RC}} \quad \text{when } t > t_2$$

…Example 01…

- If $t_1 < t_2$

$$V_C(t) = 0V - RI_2 \qquad \text{when } t < t_1$$

$$V_C(t) = RI_1\left[1 - e^{-\frac{(t-t_1)}{RC}}\right] - RI_2 \qquad \text{when } t_1 < t < t_2$$

$$V_C(t) = RI_1\left[1 - e^{-\frac{(t-t_1)}{RC}}\right] - RI_2 e^{-\frac{(t-t_2)}{RC}} \qquad \text{when } t > t_2$$

General Equations

- When a voltage or current source changes its magnitude at t = 0s in a simple RC or RL circuit.
  - Equations for a simple RC circuit

$$V_C(t) = V_C(\infty) + [V_C(0) - V_C(\infty)]e^{-t/\tau}$$

$$I_C(t) = \frac{C}{\tau}[V_C(\infty) - V_C(0)]e^{-t/\tau}$$

$$\tau = RC$$

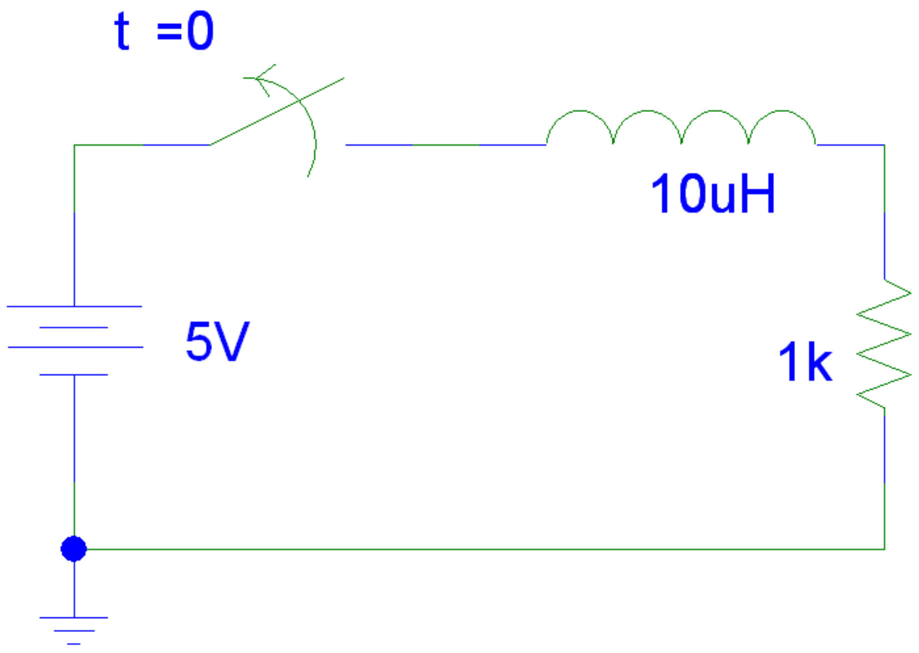
  - Equations for a simple RL circuit

$$I_L(t) = I_L(\infty) + [I_L(0) - I_L(\infty)]e^{-t/\tau}$$

$$V_L(t) = \frac{L}{\tau}[I_L(\infty) - I_L(0)]e^{-t/\tau}$$

$$\tau = L/R$$

Example 02…

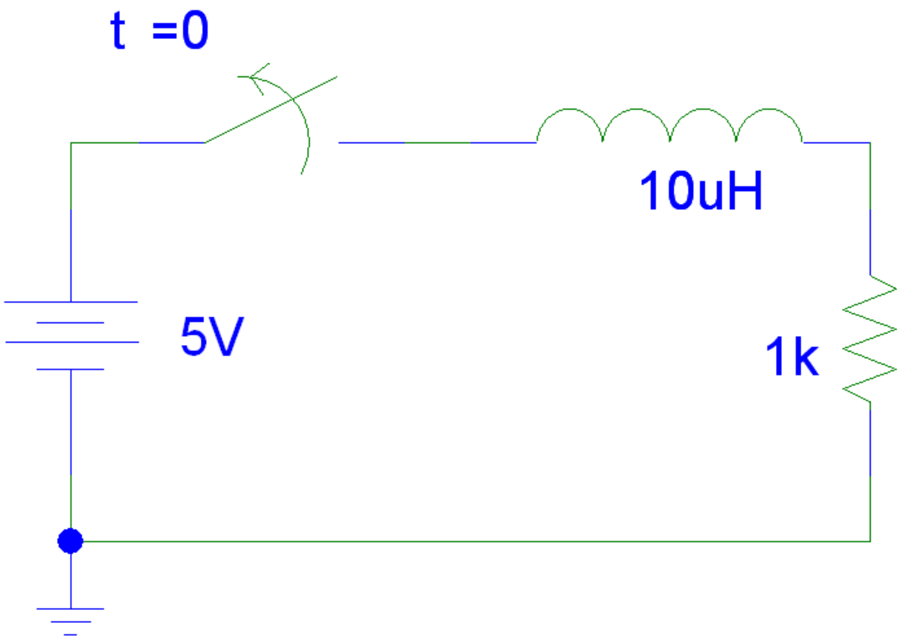


t < 0

$V_L = 0V$
$V_R = 5V$
$I_L = I_R = 5mA$

$V(t) = 5V\,[1 - u(t)]$

…Example 02



$t > 0$

$\tau = L/R = 10\text{mH}/1\text{k}\Omega = 10 \text{ ns}$

$I_L = I_R = i(0)e^{-t/\tau} = 5\text{mA } e^{-t/10\text{ns}}$

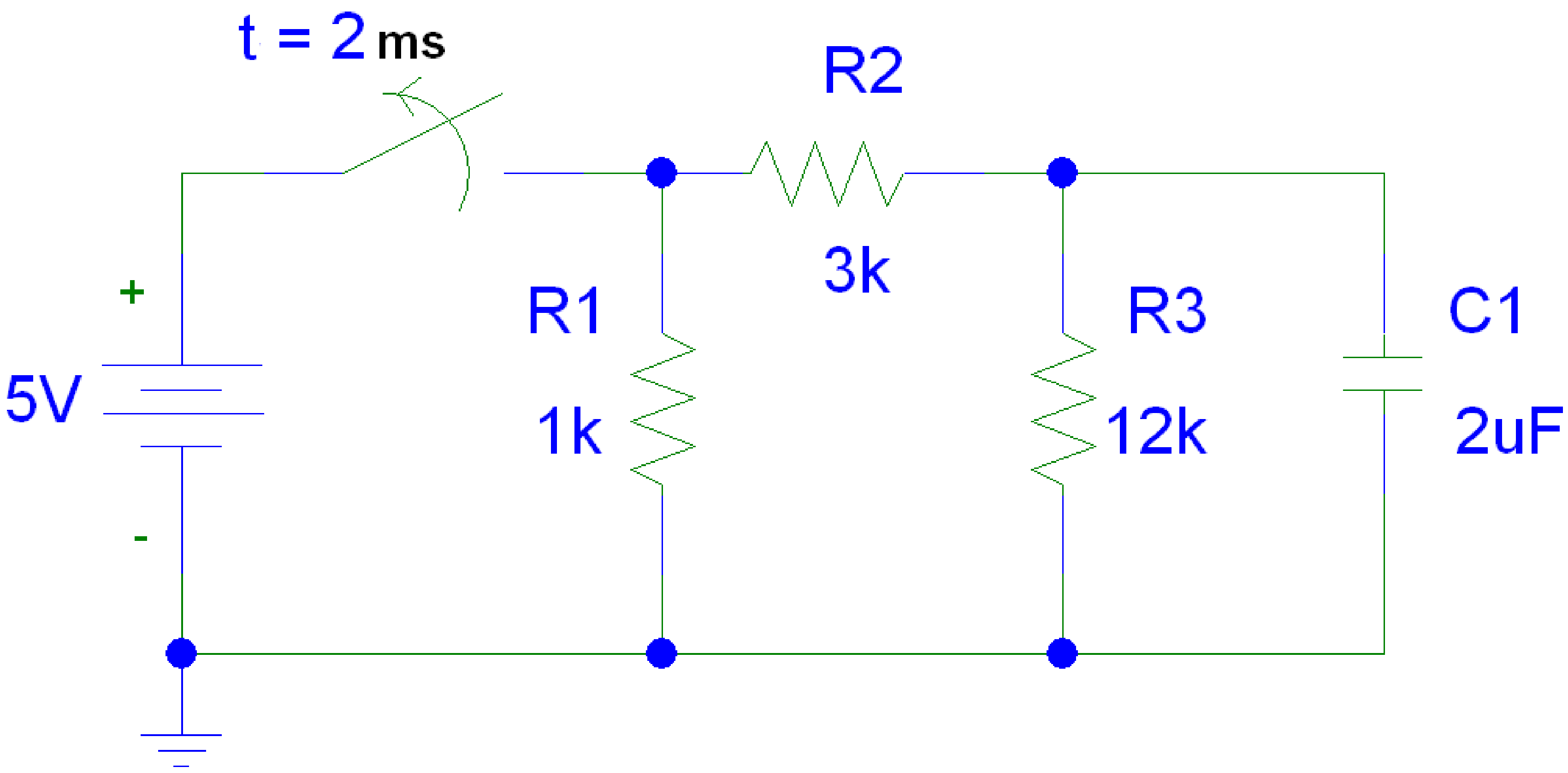
$V_R = 1\text{k}\Omega\ I_R = 5\text{V } e^{-t/10\text{ns}}$

$V_L = L(dI_L/dt) = -5\text{V } e^{-t/10\text{ns}}$

$V(t) = 5\text{V } [1 - u(t)]$

Note $V_R + V_L = 0$ V

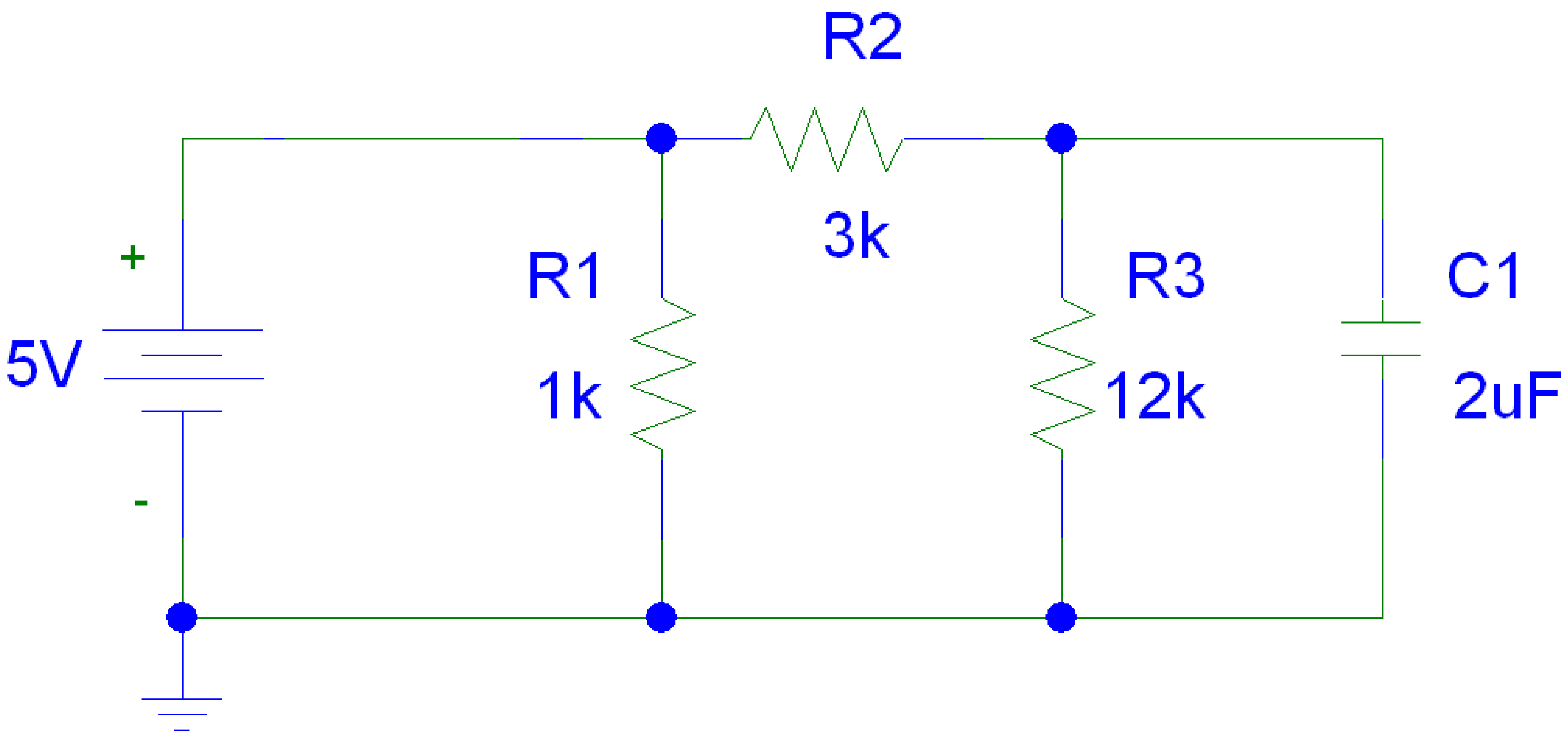
Example 03…



$V(t) = 5V\,[1 - u(t - 2ms)]$

…Example 03…

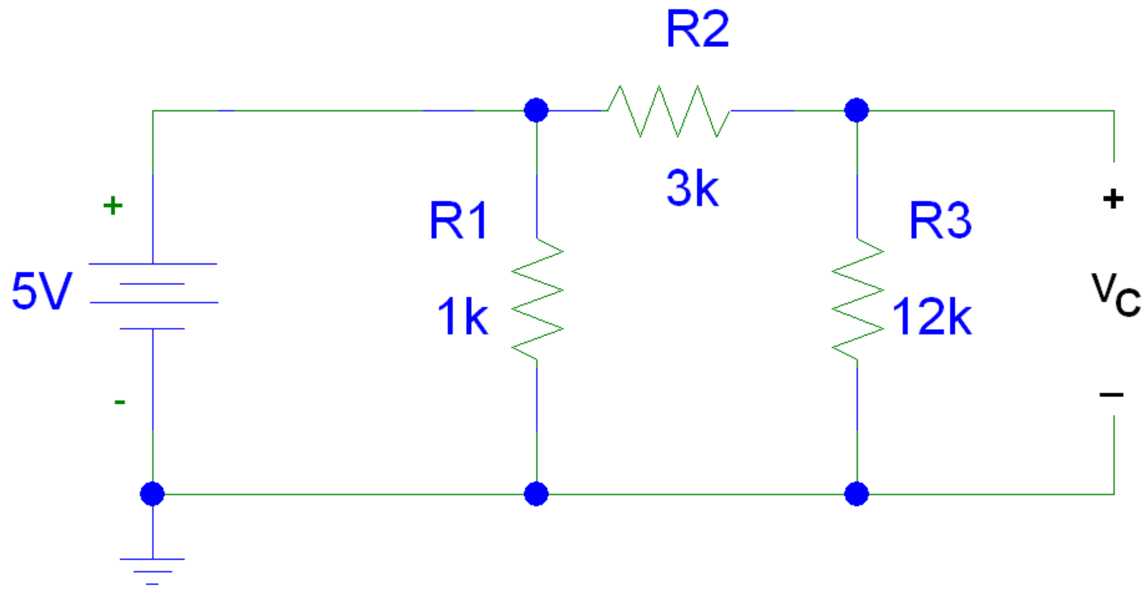
t < 2ms



…Example 03…

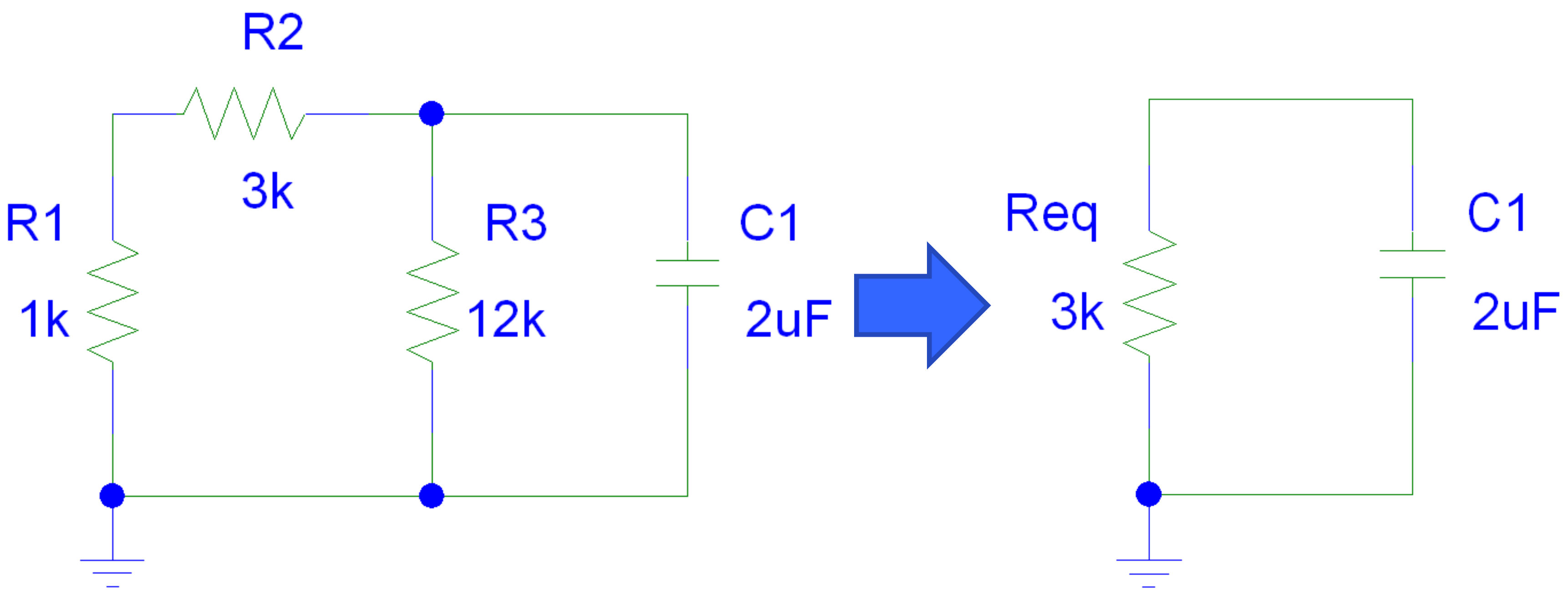
- **t < 2ms**
  - C1 is an open.
    - The voltage across the capacitor is equal to the voltage across the 12kΩ resistor.

$$V_C = [12k\Omega / 15k\Omega]\ 5V = 4V$$



…Example 03…

t > 2ms



…Example 03

t > 2ms

$\tau = R_{eq}C = 3k\Omega(2\mu F) = 6\text{ ms}$

$V_C = V_C(2ms)e^{-(t-2ms)/\tau} = 4V\, e^{-(t-2ms)/6ms}$

$V_R = V_C$

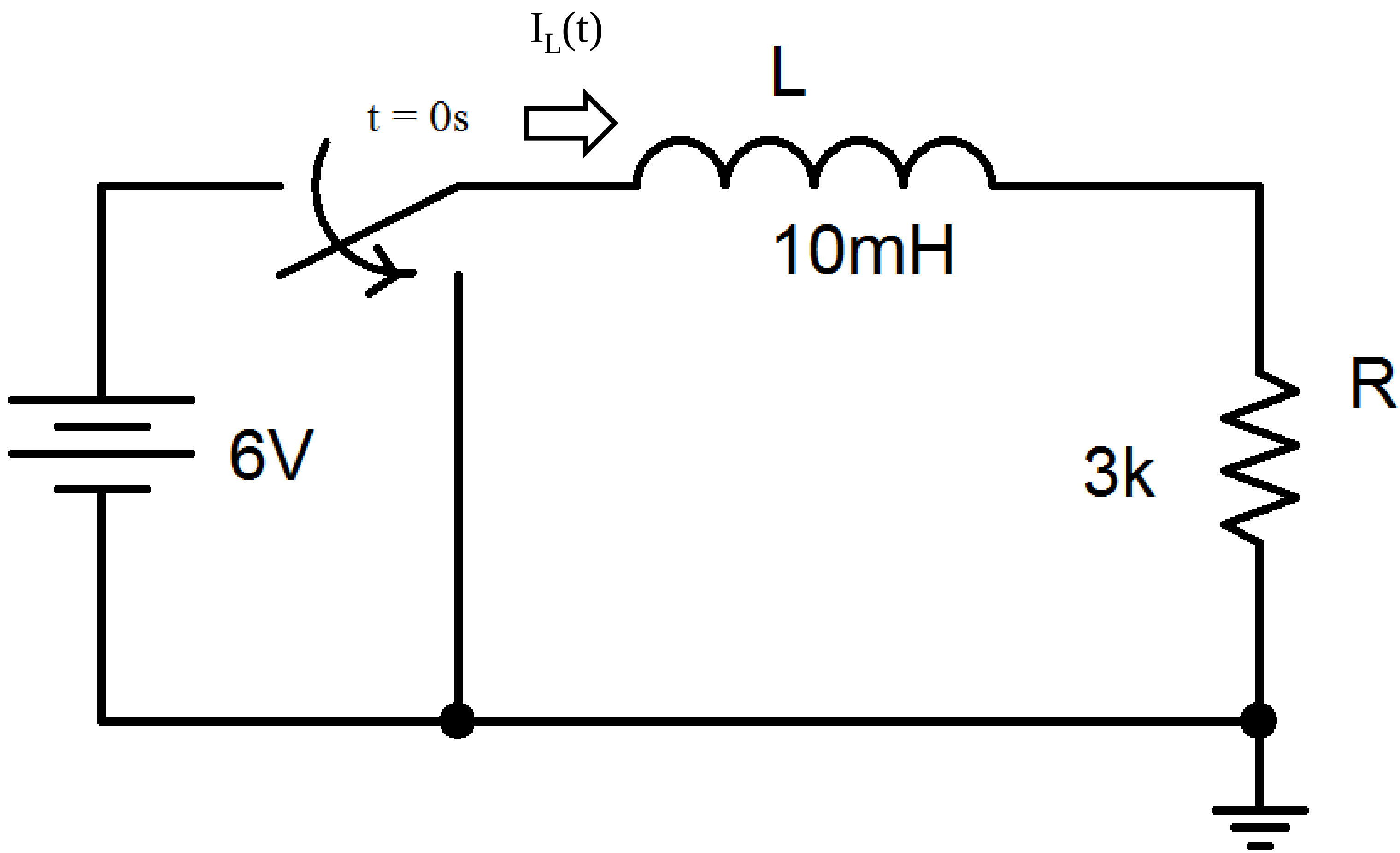
$I_C = C\, dV_c/dt = 2\mu F(-4V/6ms)\, e^{-(t-2ms)/6ms}$

$= -1.33\, e^{-(t-2ms)/6ms}\text{ mA}$

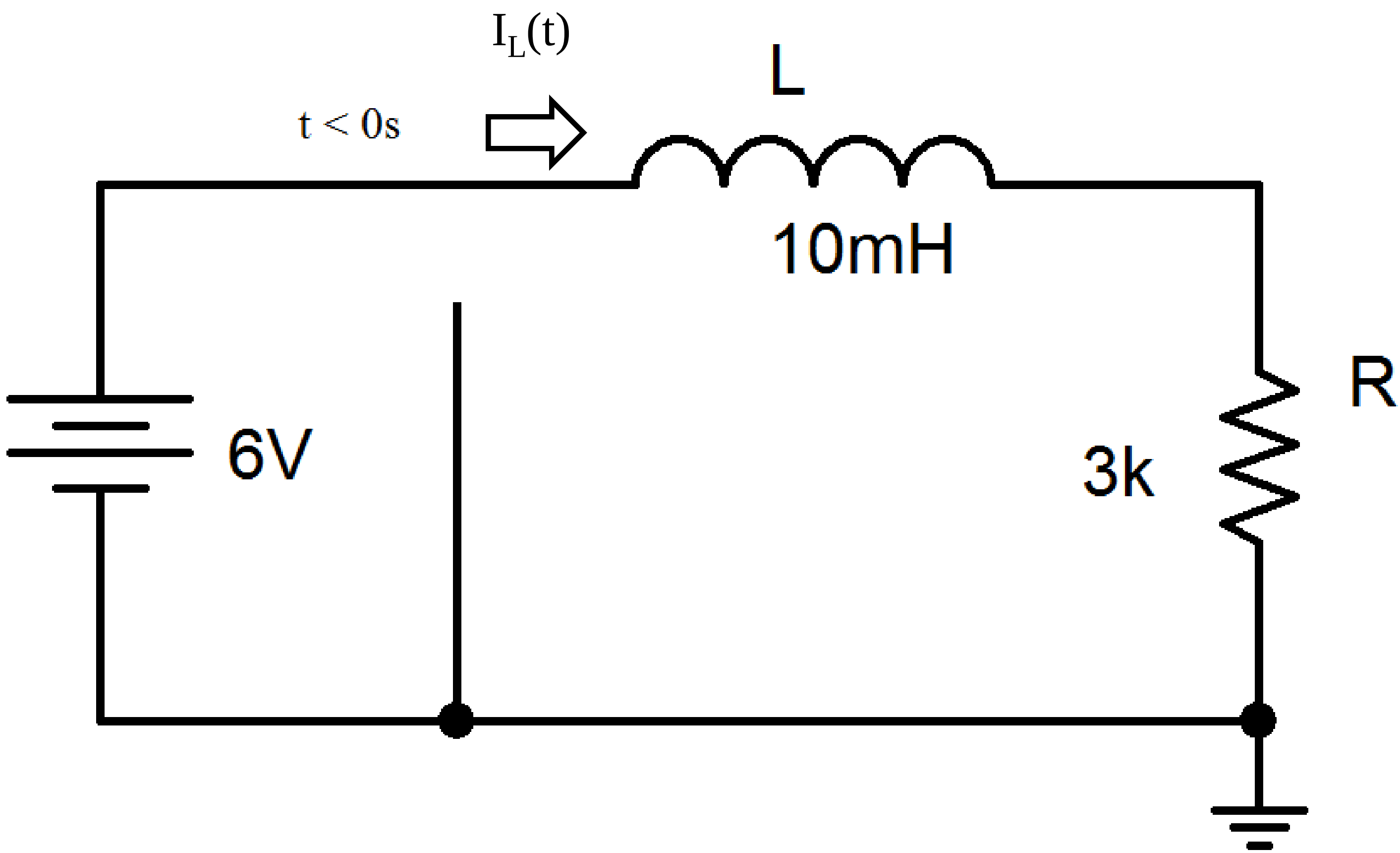
$I_R = -\, I_C = 1.33\, e^{-(t-2ms)/6ms}\text{ mA}$

Note $I_R + I_L = 0$ mA

Example 04…

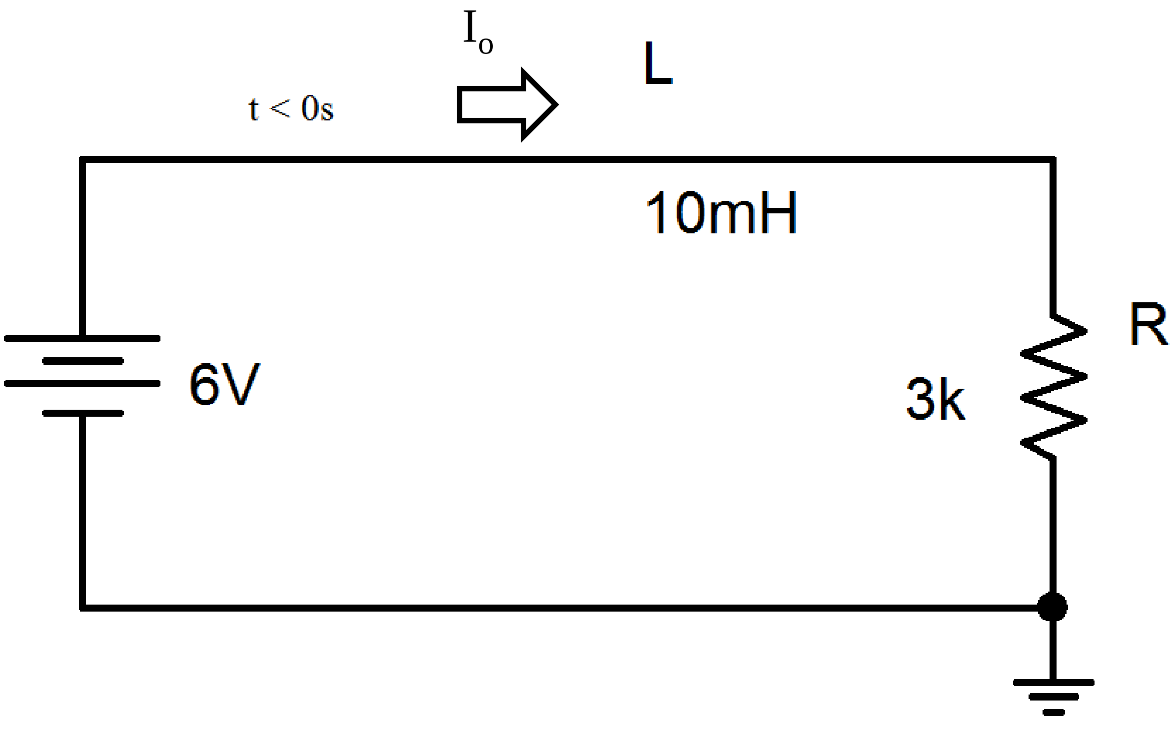


…Example 04…



…Example 04…

Find the initial condition.



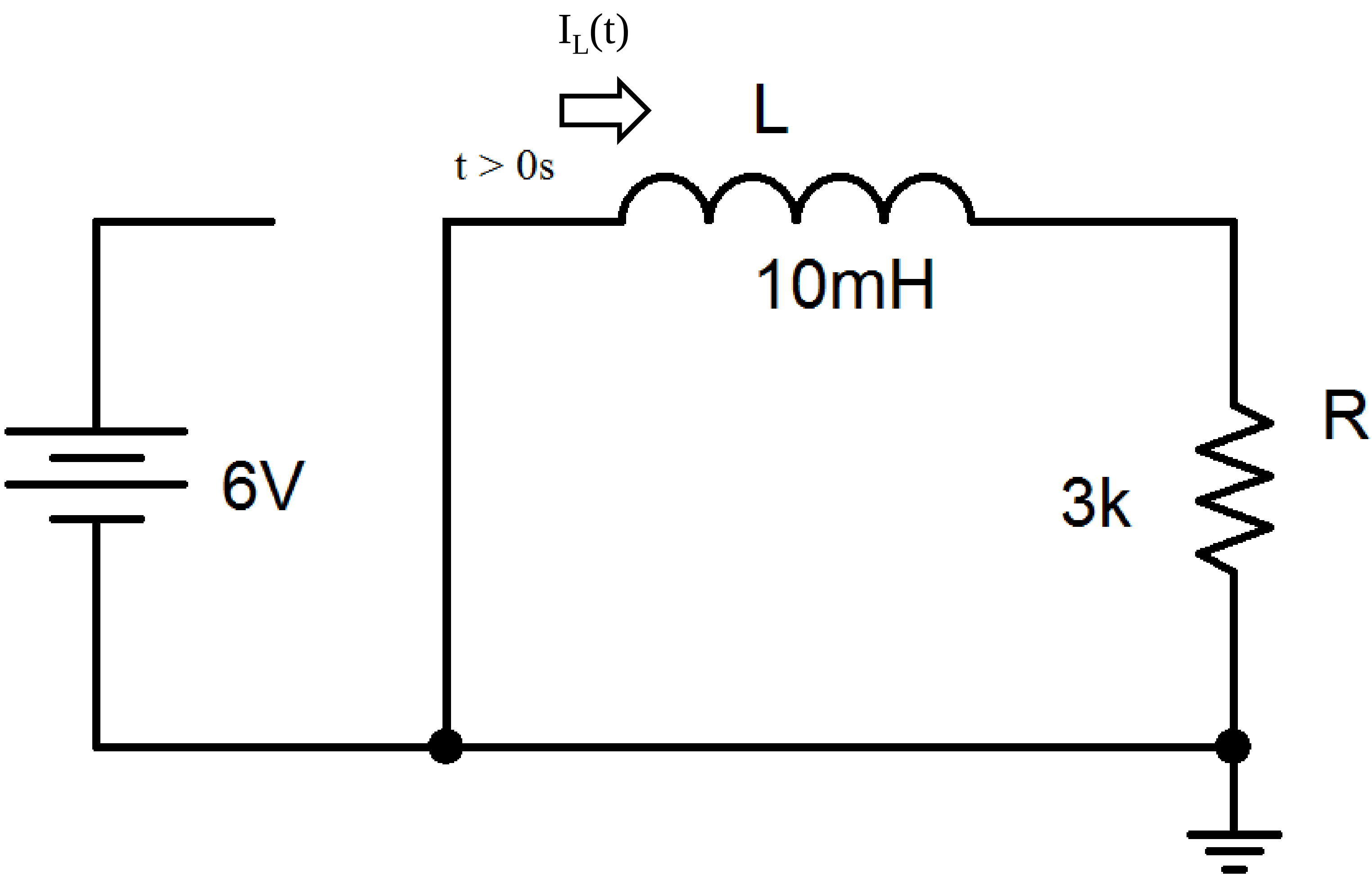
$t < 0s$

$V_L = 0V$
$V_R = 6V$
$I_L = I_R = 2mA$

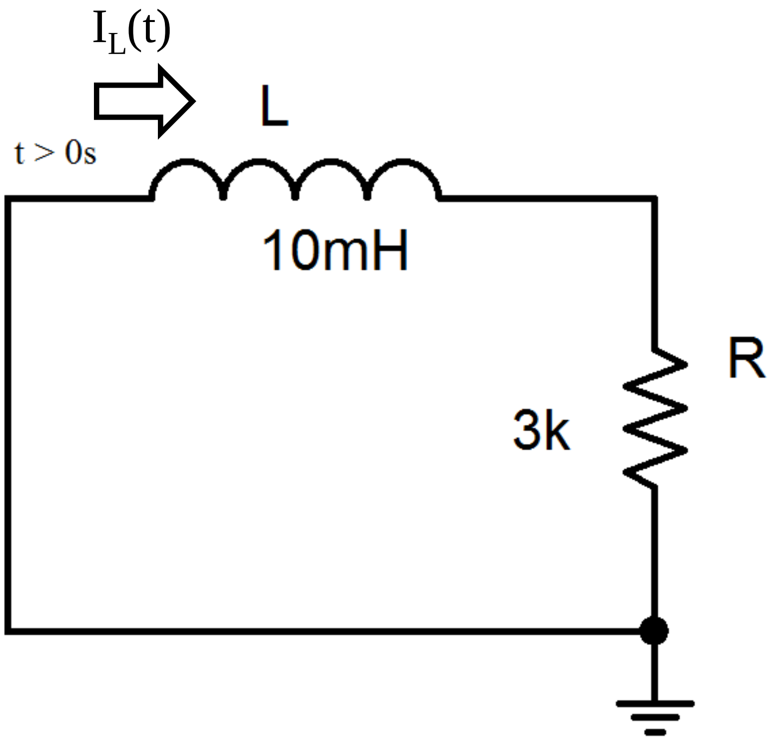
Therefore,

$I_o = 2mA$

…Example 04…



…Example 04



t > 0s

t = L/R = 10mH/3kΩ/ = 3.33μs

$I_L = I_R = I_o e^{-t/\tau} = 2mA\ e^{-(t/3.33\mu s)}$

$V_R = 3k\Omega\ I_R = 6V\ e^{-(t/3.33\mu s)}$

$V_L = L\ dI_L/dt = -6V\ e^{-(t/3.33\mu s)}$

Note $V_R + V_L = 0\ V$