BLM1612 - Circuit Theory

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AC Circuits

Fundamentals and relevant mathematics

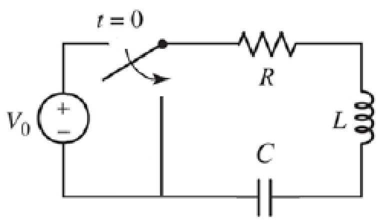
Complex numbers

Rectangular Coordinates

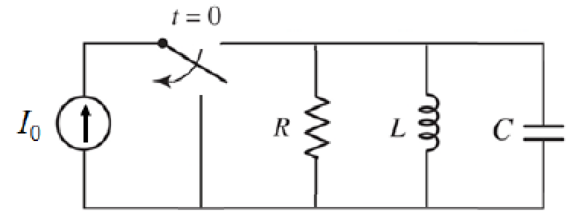
Phasor Notation

Why do we need to use Complex Numbers?

If we know that the only independent sources in our circuit are **sinusoidal**, and we know that all transients are gone (steps, switches, pulses)…



$$\alpha = \frac{R}{2L}, \quad \omega_0 = \frac{1}{\ }$$



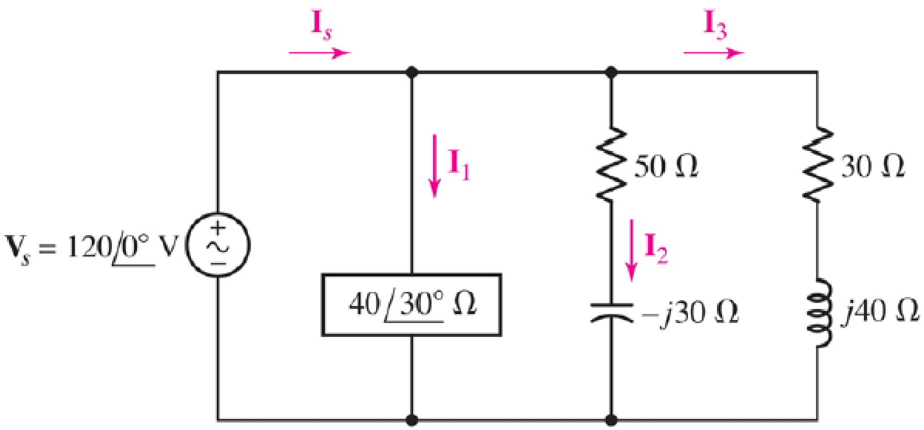
$$\alpha = \frac{1}{\ }, \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

NOT NECESSARY

$\alpha > \omega_0$		$\alpha < \omega_0$
OVERDAMPED	**CRITICALLY** DAMPED	**UNDER**DAMPED
$x(t) = X_1 e^{c_1 t} + X_2 e^{c_2 t} + X_3$	$x(t) = e^{-\alpha t}(X_1 t + X_2) + X_3$	$x(t) = e^{-\alpha t}\begin{bmatrix} X_1 \cos(\omega_d t) \\ + X_2 \sin(\omega_d t) \end{bmatrix} + X_3$

…we may instead solve the circuit **algebraically** (e.g. nodal, mesh) without determining initial conditions, final conditions, etc.

Why do we need to use Complex Numbers?



…because most analog signals consist of one or more sinusoids, by design.

60-Hz power, 900-MHz cellular telephones, 2.4-GHz wireless internet

…because **all** signals (analog *or* digital) may be **analyzed** as a sum of sinusoids.

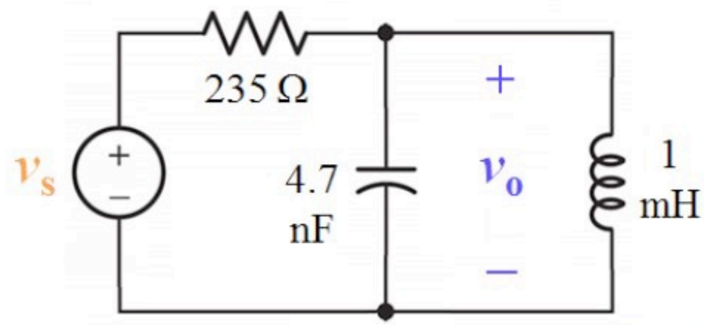
(You will see this in your Signals & Systems and Engineering Math courses.)

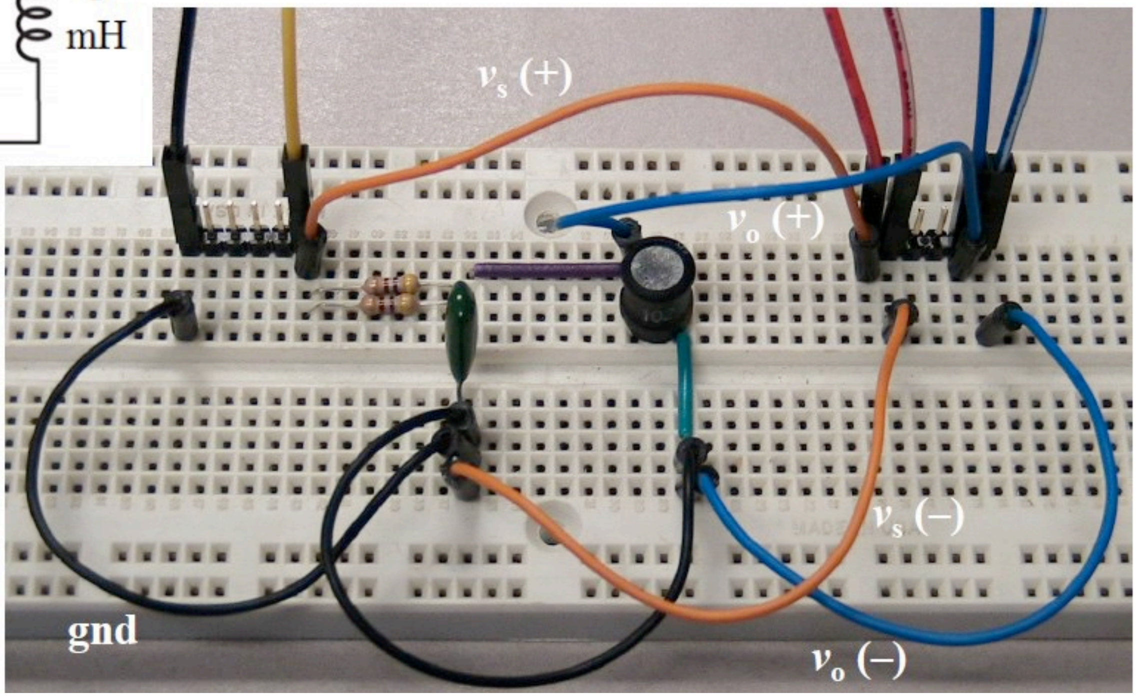
…because the **differential equations** governing **practical** systems are nearly impossible to derive and are **time-consuming** to solve.

Software tools (e.g. Matlab) can perform complex algebra *very quickly*.

A Simple Circuit

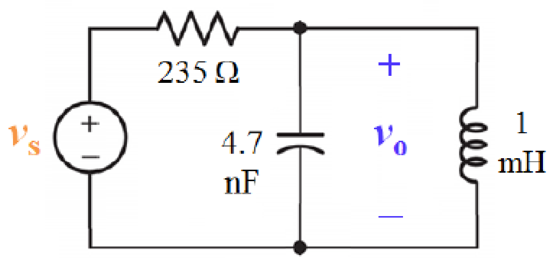
Compare $v_o(t)$ to $v_s(t)$ for $f = 5$ kHz and $f = 50$ kHz .





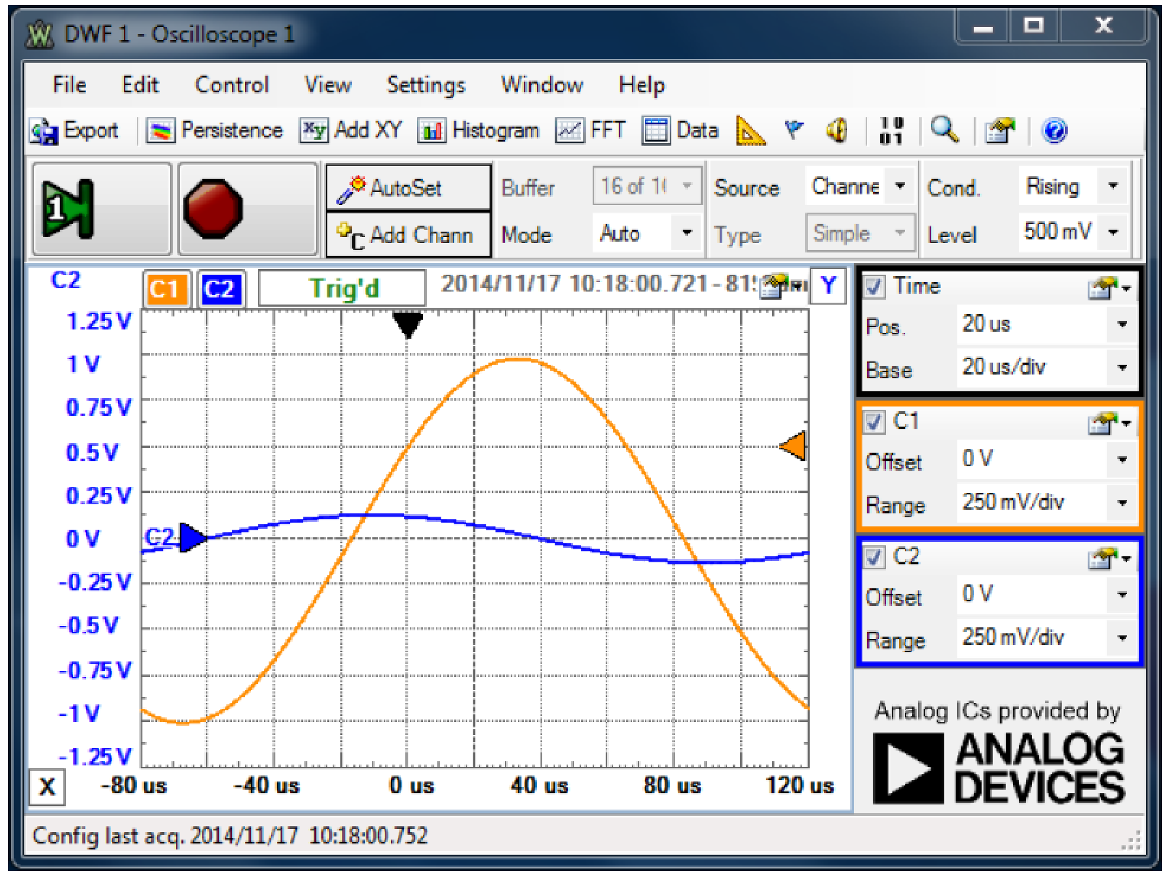
$$v_s = \sin(2\pi \cdot f \cdot t)\,\text{V}$$

Input Voltage vs Output Voltage

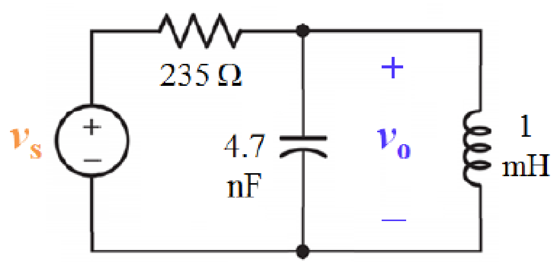


$$v_s = \sin\left(2\pi \cdot f \cdot t\right)\ \mathrm{V}$$

$f = 5$ kHz

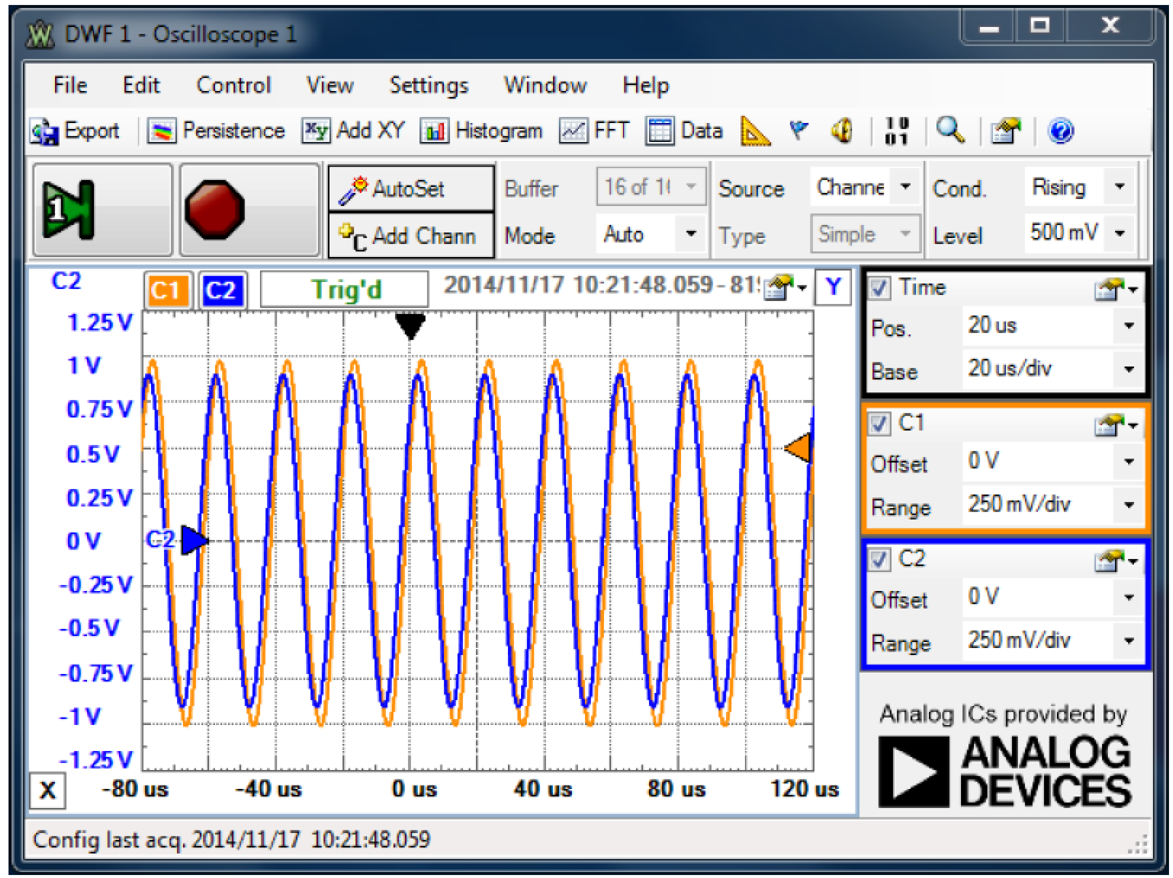


Input Voltage vs Output Voltage

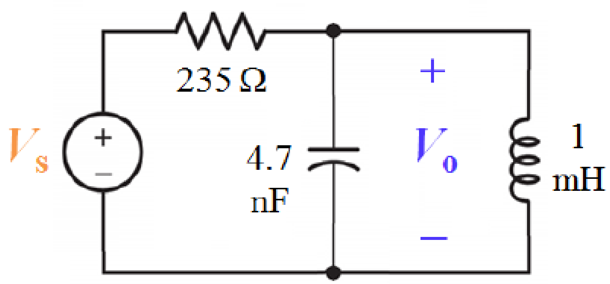


$$v_s = \sin(2\pi \cdot f \cdot t)\ \mathrm{V}$$

$f = 50$ kHz



Why do we need to use Complex Numbers?



$$v_s = \sin\left(2\pi \cdot 10^4 \cdot t\right) \text{ V}$$

Complex algebra is the math that electrical engineers use to analyze AC circuits.

- sinusoids become phasors:

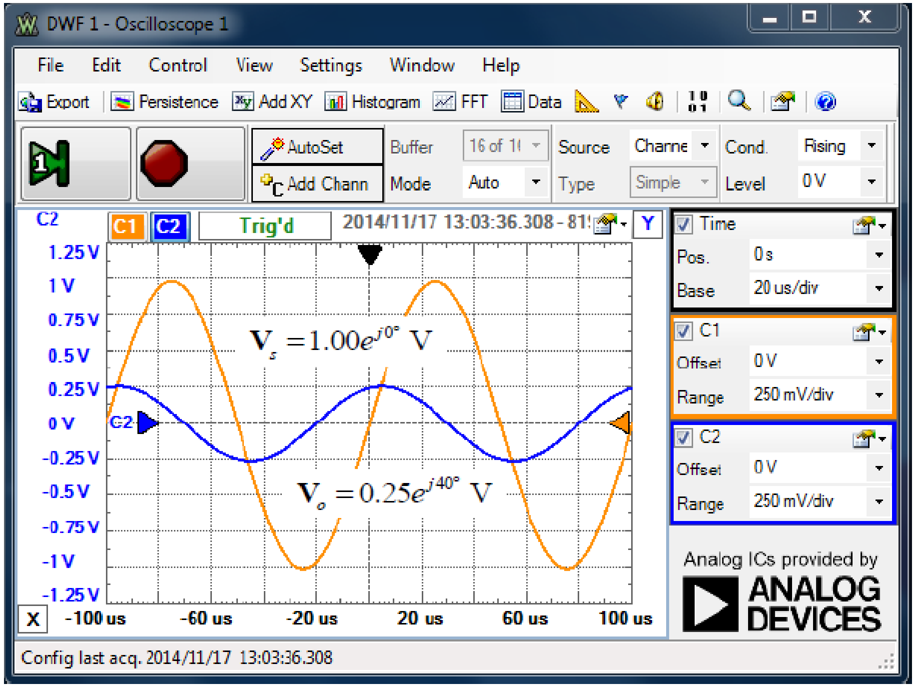
$$\mathbf{V}_s = 1.00 e^{j0^\circ} \text{ V}$$

- KCL/KVL/Ohm's Law are solved with phasors:

$$\mathbf{V}_o = 0.25 e^{j40^\circ} \text{ V}$$

- phasors turn back into sinusoids:

$$v_o = 250 \sin\left(2\pi \cdot 10^4 \cdot t + 40^\circ\right) \text{ mV}$$



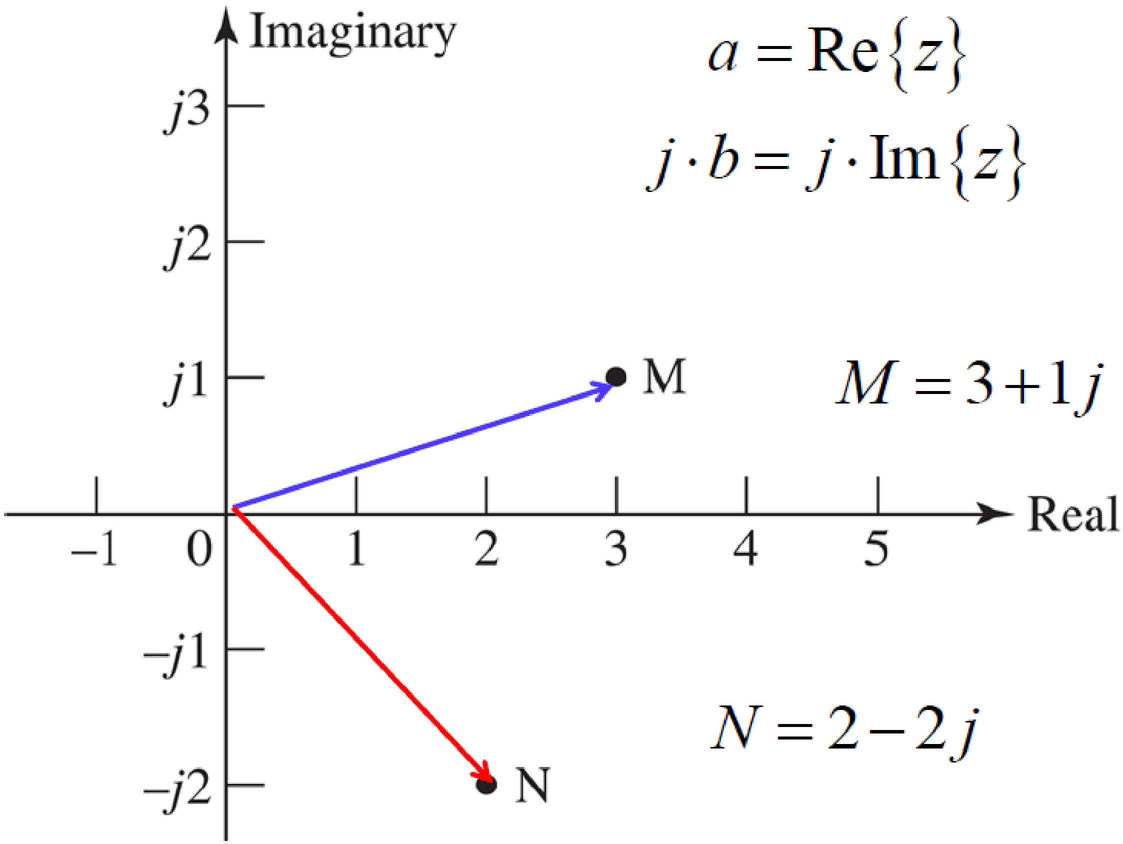
Complex numbers

Addition, subtruction, multiplication, division

Complex numbers

- A complex number z can be written in rectangular form as $z = x + jy$, where $j = \sqrt{-1}$
  - x is the real part of z
  - y is the imaginary part of z
- The complex number z can also be written in polar or exponential form as $z = \angle\phi = re^{j\emptyset}$
- You can only add or subtract
  - voltages with voltages,
  - currents with currents,
  - impedances with impedances,
  - admittances with admittances.

Complex Plane



$$a = \mathrm{Re}\{z\}$$

$$j \cdot b = j \cdot \mathrm{Im}\{z\}$$

$$M = 3 + 1j$$

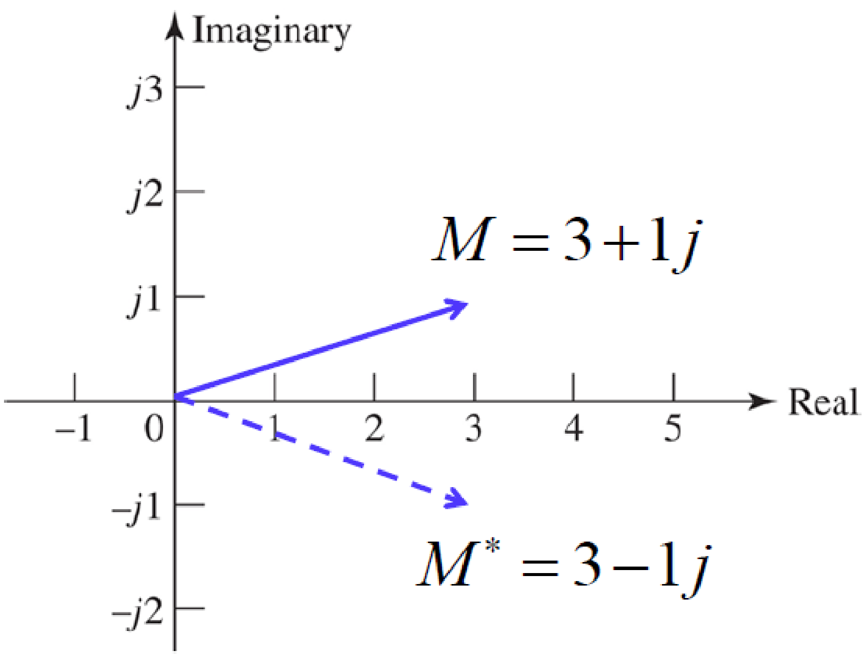
$$N = 2 - 2j$$

Complex numbers may be visualized as *vectors* in the *complex plane*.

Complex Conjugate

The **complex conjugate** of z is denoted z^*

and if $z = a + b \cdot j$ then $z^* = a - b \cdot j$



The conjugate of z is the same number, except that the imaginary part is negated.

Graphically, the complex conjugate of z is the mirror image of z across the *Real* axis.

Addition with Rectangular Coordinates

- If $Z = X + Y$ and $X = a + jb$ and $Y = c + jd$
  – Add the real component of the rectangular coordinate $(a + c)$.
  – Add the imaginary component of the rectangular coordinate $(b + d)$.
  – Combine the results into rectangular coordinates.

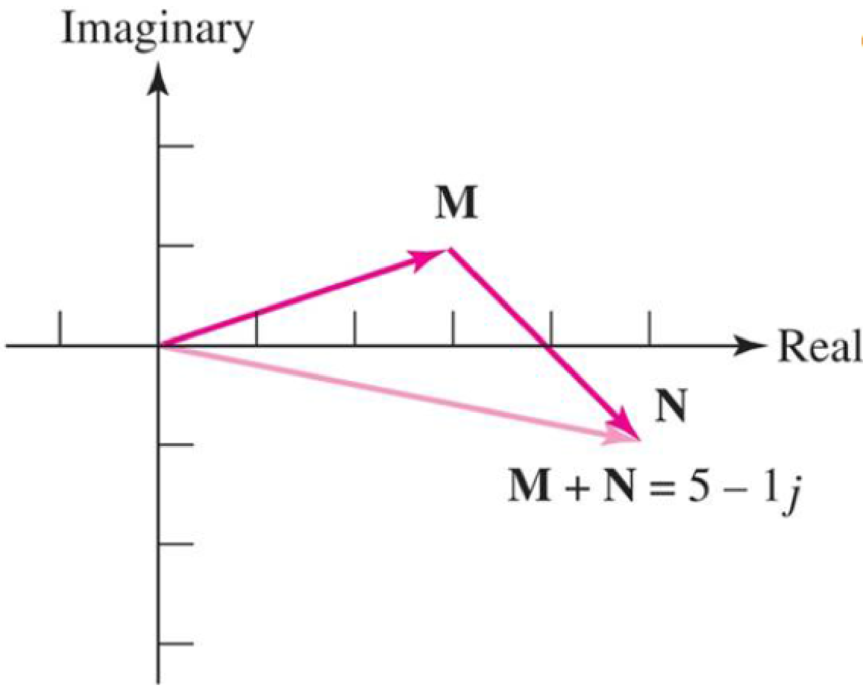
- $Z = (a + c) + j(b + d)$
  – $\mathrm{Re}\,(Z) = (a + c)$ and $\mathrm{Im}(Z) = (b + d)$

Subtraction with Rectangular Coordinates

- If $Z = X - Y$ and $X = a + jb$ and $Y = c + jd$
  - Subtract the real component of the rectangular coordinate $(a - c)$.
  - Subtract the imaginary component of the rectangular coordinate $(b - d)$.
  - Combine the results into rectangular coordinates.

- $Z = (a - c) + j(b - d)$
  - $\mathrm{Re}(Z) = (a - c)$ and $\mathrm{Im}(Z) = (b - d)$

Addition and Subtraction



$$M = 3 + 1j$$

$$N = 2 - 2j$$

$$M + N = 5 - 1j$$

Graphical addition & subtraction are performed like vector addition ("tip-to-tail").

Algebraic addition & subtraction are performed piece-wise:

$$\begin{array}{ccccc} M & = & a_1 & + & b_1 \cdot j \\ N & = & a_2 & + & b_2 \cdot j \\ M+N & = & (a_1 + a_2) & + & (b_1 + b_2) \cdot j \end{array}$$

Multiplication with Rectangular Coordinates

- If $Z = XY$ and $X = a + jb$ and $Y = c + jd$
  - Multiply the coefficients of the real components of the two numbers together ($a \cdot c$).
  - Multiply the coefficients of the imaginary components of the two numbers together ($b \cdot d$).
  - Multiply the coefficient of the real component of X with the imaginary component of the Y ($a \cdot d$).
  - Multiply the coefficient of the real component of Y with the imaginary component of the X ($c \cdot b$).

- The product $Z = [(a \cdot c)-(b \cdot d)] + j[(a \cdot d)+(c \cdot b)]$
  - $\mathrm{Re}(Z) = [(a \cdot c)-(b \cdot d)]$ and $\mathrm{Im}(Z) = [(a \cdot d)+(c \cdot b)]$

Multiplication

Multiplication may be accomplished in rectangular form…

$$z_1 = a_1 + b_1 j \qquad z_1 \cdot z_2 = \left(a_1 + b_1 j\right)\left(a_2 + b_2 j\right)$$

$$z_1 = a_2 + b_2 j \qquad = a_1 a_2 + a_1 b_2 j + a_2 b_1 j + b_1 b_2 j^2$$

$$= a_1 a_2 + \left(a_1 b_2 + a_2 b_1\right) j - b_1 b_2$$

$$= \left(a_1 a_2 - b_1 b_2\right) + \left(a_1 b_2 + a_2 b_1\right) j$$

$$M = 5 + 3j \qquad M \cdot N = \left(5 + 3j\right)\left(2 - 4j\right)$$

$$N = 2 - 4j \qquad = 10 - 20j + 6j - 12j^2$$

$$= 22 - 14j$$

…but it is more easily accomplished in *polar* form.

Division with Rectangular Coordinates

- If $Z = X/Y$ and $X = a + jb$ and $Y = c + jd$
  - Square the real component of Y (c^2).
  - Square the imaginary component of Y (d^2).
  - Multiply the coefficient of the real component of X with the real component of the Y ($a \cdot c$).
  - Multiply the coefficient of the imaginary component of X the imaginary component of the Y ($b \cdot d$).
  - Multiply the coefficient of the real component of X with the imaginary component of the Y ($a \cdot d$).
  - Multiply the coefficient of the real component of Y with the imaginary component of the X ($c \cdot b$).

- The quotient $Z = \{[(a \cdot c)+(b \cdot d)]/[c^2+d^2]\} +j\{[(b \cdot c)-(a \cdot d)]/[c^2+d^2]\}$
  - $Re(Z) = [(a \cdot c)+(b \cdot d)]/[c^2+d^2]$ and $Im(Z) = [(b \cdot c)-(a \cdot d)]/[c^2+d^2]$

Addition and Subtraction with Phasors

- To add or subtract voltages, currents, impedances, or admittances that are expressed in phasor notation, you must convert the phasor into rectangular coordinates.
  - Add or subtract the real component of the rectangular coordinate.
  - Add or subtract the imaginary component of the rectangular coordinate.
  - Combine the results into rectangular coordinates.
  - Convert rectangular coordinates to phasor notation.

Multiplication and Division with Phasors

- If $V = v\angle\phi$ and $I = i\angle\theta$
  - To multiply or divide voltages, currents, impedances, or admittances that are expressed in phasor notation,
    - Multiply or divide the magnitudes of the phasors.
    - The resulting phase angle is obtained by:
      - Add the phase angles together if the mathematical function is multiplication
      - Subtract the phase angle associated with the divisor (denominator) from the phase angle associated with the dividend (numerator) if the mathematical function is division.

- $P = VI = vi \angle(\phi+\theta)$ and $Z = V/I = v/i \angle(\phi-\theta)$

Exponential Form

$$
\begin{aligned}
e^{j\theta} &= \cos(\theta) + j\cdot\sin(\theta) \\
|z|\cdot e^{j\theta} &= |z|\cdot\cos(\theta) + j\cdot|z|\cdot\sin(\theta) \\
|z|\cdot e^{j\theta} &= a + b\cdot j
\end{aligned}
$$

assume $|z|$ is positive, real

$$a = |z|\cdot\cos(\theta)$$

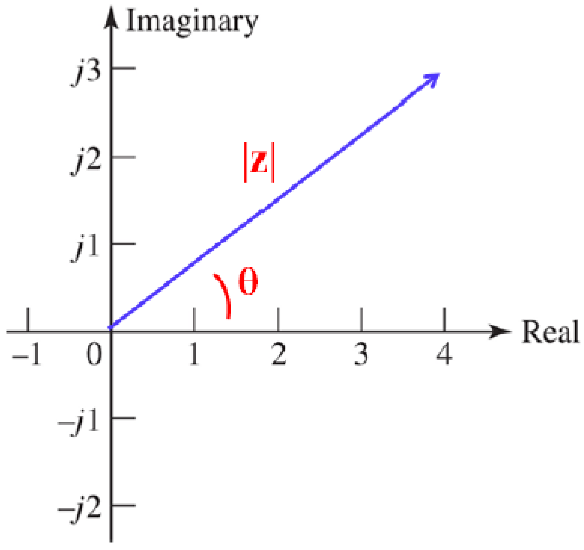
$$b = |z|\cdot\sin(\theta)$$

$$\frac{\sin(\theta)}{\cos(\theta)} = \boxed{\tan(\theta) = \frac{b}{a}}$$

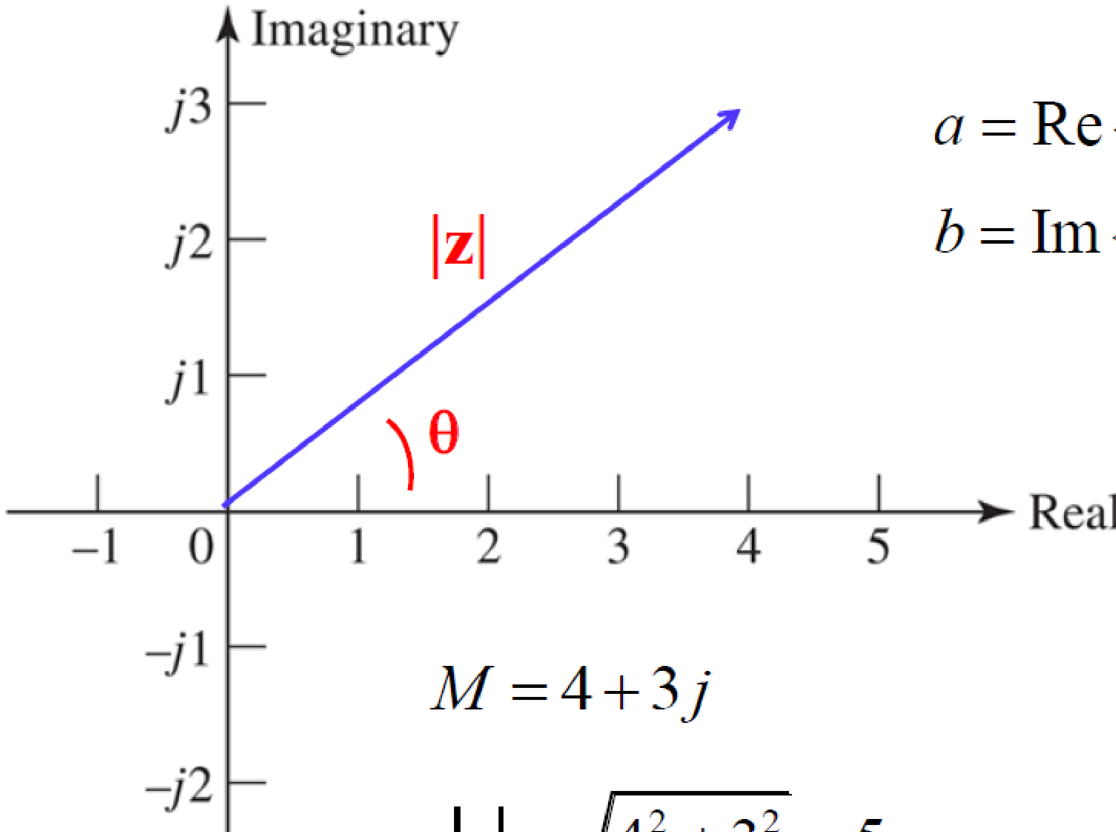
$$a^2 + b^2 = |z|^2\cdot\cos^2(\theta) + |z|^2\cdot\sin^2(\theta)$$

$$a^2 + b^2 = |z|^2\cdot\left\{\cos^2(\theta) + \sin^2(\theta)\right\} = |z|^2$$

$$\boxed{\sqrt{a^2 + b^2} = |z|}$$



Rectangular to Exponential Form



$$a = \mathrm{Re}\{z\}$$

$$b = \mathrm{Im}\{z\}$$

$$\tan(\theta) = \frac{b}{a}$$

$$|z| = \sqrt{a^2 + b^2}$$

$$M = 4 + 3j$$

$$|z| = \sqrt{4^2 + 3^2} = 5$$

$$\theta = \tan^{-1}\{3/4\} = 37°$$

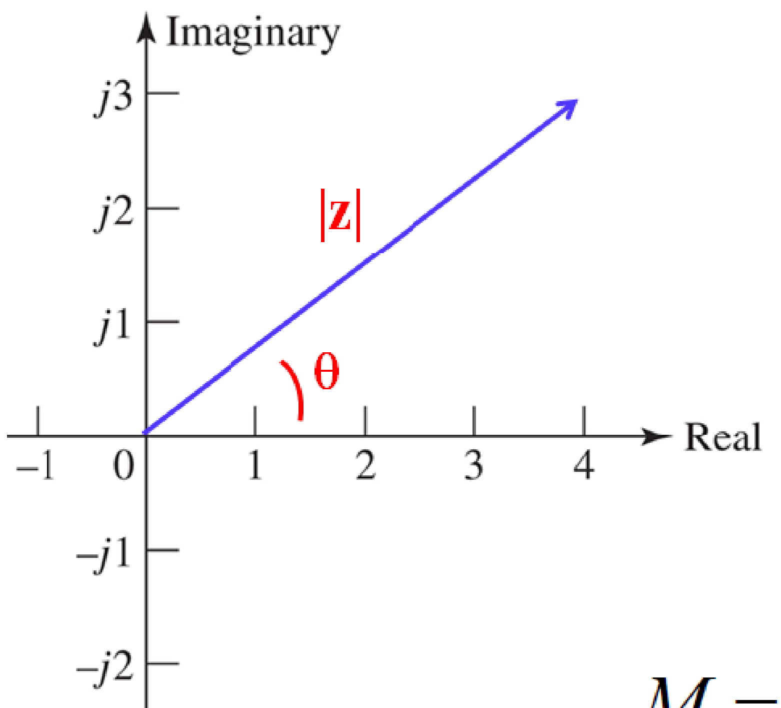
$$M = 5e^{j37°}$$

$|z|$ = **magnitude** of z
θ = **phase/angle** of z

Polar Form

$$z = a + b \cdot j = |z| \cdot e^{j\theta} = \boxed{|z| \angle \theta}$$

rectangular　　exponential　　polar



Polar form is a *shorthand* for the exponential form.

$|z|$ = **magnitude** of z
θ = **phase/angle** of z

$$M = 4 + 3j = 5e^{j37^\circ} = 5\angle 37^\circ$$

Multiplication in Polar Form

Multiplication in polar form is carried out using exponentials…

$$z_1 = a_1 + b_1 j \;\Rightarrow\; |z_1| e^{j\theta_1}$$

$$z_2 = a_2 + b_2 j \;\Rightarrow\; |z_2| e^{j\theta_2}$$

$$\begin{aligned} z_1 \cdot z_2 &= |z_1| e^{j\theta_1} \cdot |z_2| e^{j\theta_2} \\ &= |z_1||z_2| e^{j\theta_1 + j\theta_2} \\ &= |z_1||z_2| e^{j(\theta_1 + \theta_2)} \end{aligned}$$

$$z_1 = |z_1|\angle\theta_1 \;, \quad z_2 = |z_2|\angle\theta_1$$

$$z_1 \cdot z_2 = |z_1||z_2|\angle(\theta_1 + \theta_2)$$

$$M = 3 + 4j = 5\angle 53^\circ$$

$$N = \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}} j = 3\angle 45^\circ$$

$$\begin{aligned} M \cdot N &= (5\angle 53^\circ)(3\angle 45^\circ) \\ &= 15\angle 98^\circ \end{aligned}$$

Division in Polar Form

$$z_1 = |z_1|\angle\theta_1\,,\quad z_2 = |z_2|\angle\theta_1$$

$$\frac{|z_1|e^{j\theta_1}}{|z_2|e^{j\theta_2}} = \frac{|z_1|}{|z_2|}\frac{e^{j\theta_2}}{e^{j\theta_2}} = \frac{|z_1|}{|z_2|}e^{j\theta_1 - j\theta_2}$$

$$\frac{z_1\angle\theta_1}{z_2\angle\theta_2} = \frac{|z_1|}{|z_2|}\angle(\theta_1 - \theta_2)$$

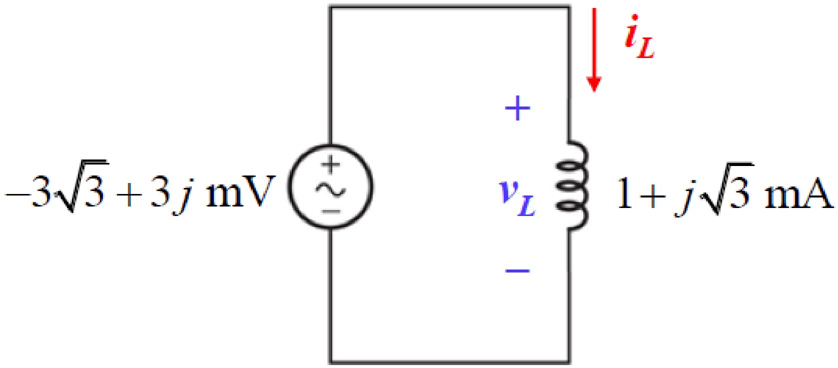
$$M = 6 + 8j = 10\angle 53°$$

$$N = \frac{5}{\sqrt{2}} + \frac{5}{\sqrt{2}}j = 5\angle 45°$$

$$M/N = (10\angle 53°)/(5\angle 45°)$$
$$= 2\angle 8°$$

Example : Ohm's Law

Determine the ratio of v_L to i_L:



$$\frac{v_L}{i_L} = \frac{|v_L|}{|i_L|}\angle\left(\theta_{VL} - \theta_{IL}\right)$$

$$\frac{v_L}{i_L} = \frac{-3\sqrt{3}+3j \ \text{ mV}}{1+j\sqrt{3} \ \text{ mA}}$$

$$= \frac{\sqrt{\left(3\sqrt{3}\right)^2+(3)^2}\angle\tan^{-1}\left\{1/-\sqrt{3}\right\}}{\sqrt{(1)^2+\left(\sqrt{3}\right)^2}\angle\tan^{-1}\left\{\sqrt{3}\right\}} = \frac{6\angle 150^\circ}{2\angle 60^\circ}$$

Example : Power Absorbed

Write the quantity V x I^* in polar form, given

$$V = 3 - 5j \text{ V}$$
$$I = 6 + 7j \text{ mA}$$

$$V \cdot I^* = (3 - 5j)(6 - 7j)$$
$$= 18 - 30j - 21j - 35$$
$$= -17 - 51j \text{ mW}$$

```
>> V = 3 - 5*j;
>> I = 6 + 7*j;
>> S = V * conj(I)

S = -17.0000 -51.0000i
```

$$V \cdot I^* = \sqrt{17^2 + 51^2} \tan^{-1}\{-51/-17\}$$
$$= 53.8\angle -108° \text{ mW}$$

```
>> abs(S)
   ans = 53.7587
>> angle(S)*180/pi
   ans = -108.4349
```

$$V \cdot I^* = \left\{\sqrt{34}\angle -59°\right\}\left\{\sqrt{85}\angle -49°\right\}$$
$$= \sqrt{34 \cdot 85}\angle -59° - 49°$$
$$= 53.8\angle -108° \text{ mW}$$

Summary

- Addition and subtraction of numbers expressed in rectangular coordinates is straight-forward.
- Multiplication and division of numbers expressed in rectangular coordinates is more complex.
  - Some students prefer to convert the number into phasor notation, perform the multiplication or division, and then convert back to rectangular coordinates.
- Multiplication and division with phasors is straight-forward.
  - Addition and subtraction requires that the phasor be rewritten in rectangular coordinates.

Alternating Current and Voltages

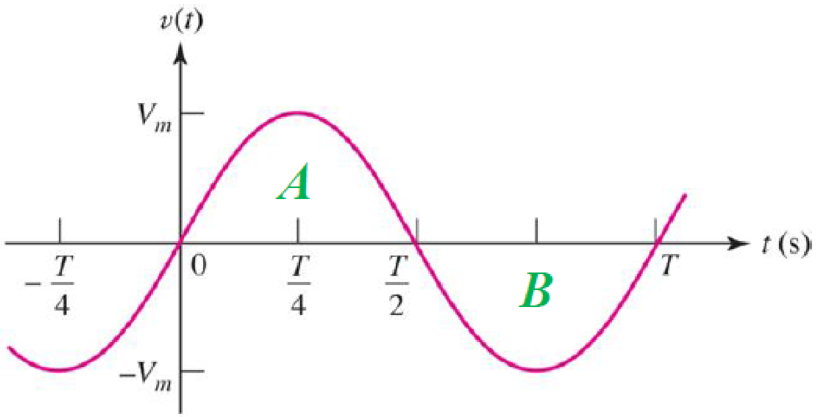
Sinusoidal Waves

Objective of Lecture

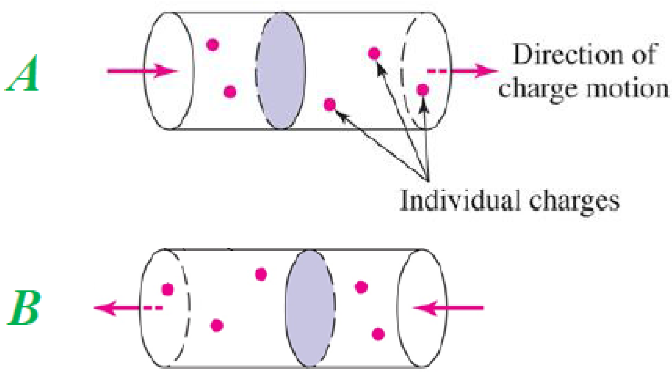
- Discuss the characteristics of a sinusoidal wave.
- Define the mathematical relationship between the period, frequency, and angular frequency of a sine wave.
- Explain how to define the amplitude of a sine wave.
- Describe what a phase angle is and the difference between lagging and leading signals.

Alternating Current (AC) -Sinusoidal

$$v(t) = V_m \cdot \sin(\omega t + \phi_0)\ , \quad \phi_0 = 0$$



$+ \quad v(t) \quad -$



V_m = amplitude (in Volts), ϕ_0 = phase (in radians)

ω = frequency (in radians/second)

T = period (in seconds)

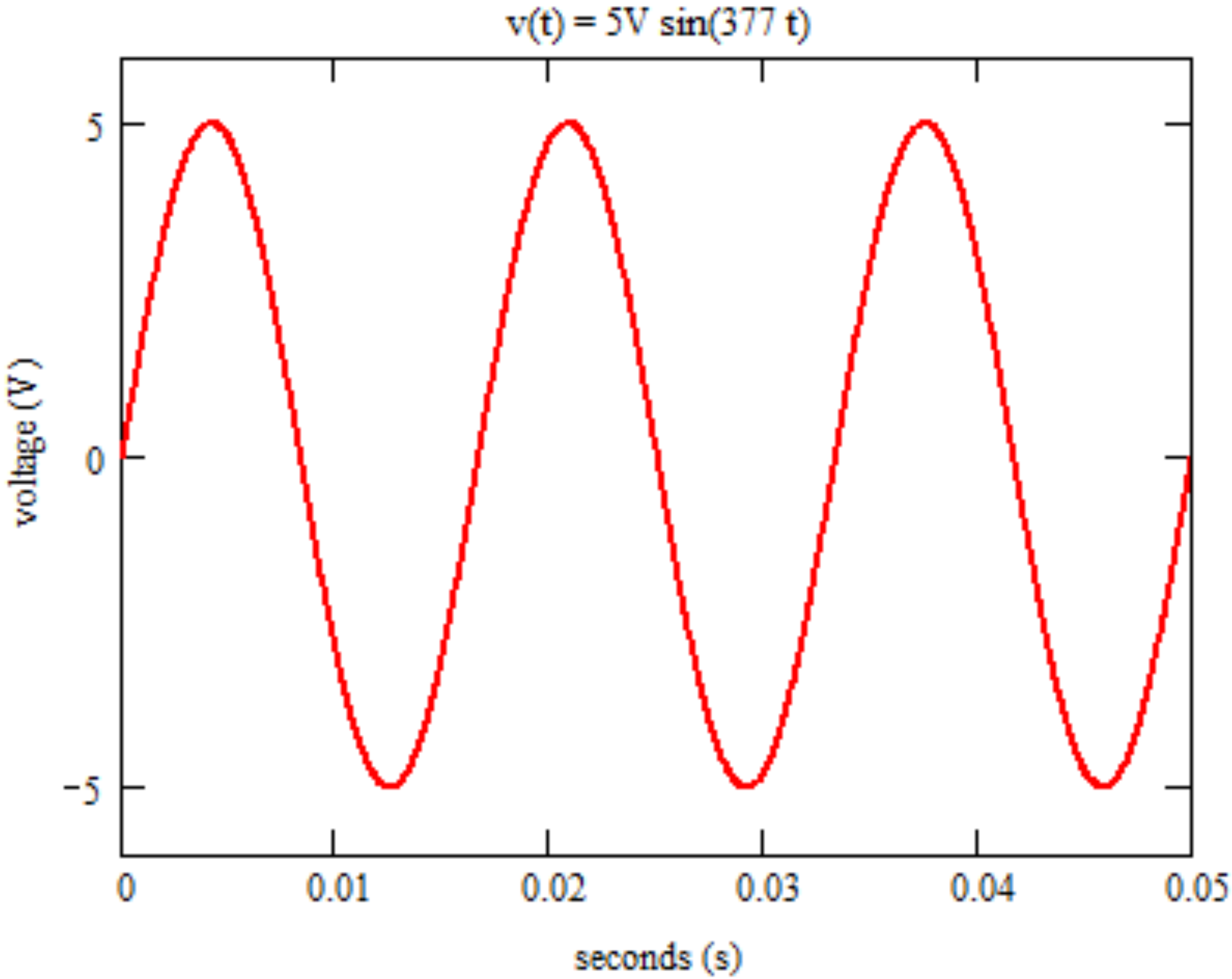
f = frequency (in cycles/second) = $1/T$ = $\omega / 2\pi$

$$V_m \cdot \sin(\omega t + \phi_0) = V_m \cdot \cos(\omega t + \phi_0 - \pi/2)$$

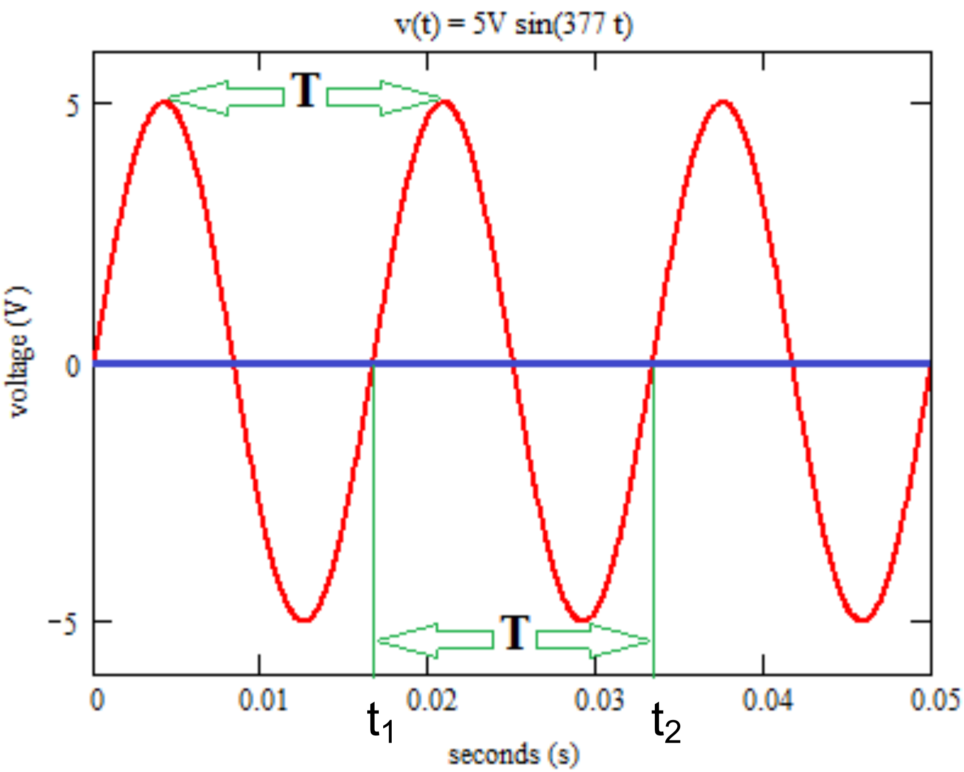
Characteristics of a Sine Wave

- The length of time it takes to complete one cycle or conversely the number of cycles that occur in one second.
  - Period
  - Frequency
  - Angular Frequency
- The maximum and minimum voltage or current swing
  - Amplitude
  - Peak-to-peak amplitude
  - Value of the root mean square (RMS)
- Average value of a sine wave
  - DC offset
- Comparison between two sine waves
  - Phase angle
  - Lagging and leading signals

Characteristics of a Sine Wave



Period, T



The time that it takes for a sine wave to complete one full cycle. This can be measured by finding the times at which the signal crosses zero (need two zero crossings). The unit usually used is seconds (s).

An alternative way to measure the period is to determine the time required for the sine wave return to the same maximum or minimum value.

$$T = t_2 - t_1 = \frac{1}{60} s = 16.7 ms$$

Frequency, f

- The number of cycles a sine wave will complete in one second.
  - The unit is cycles/second or Hertz (Hz).

$$f = \frac{1}{T}$$

  - The longer the period, the lower the frequency is.
  - The shorter the period, the higher the frequency is.

$$f = \frac{1}{T} = \frac{1}{16.7ms} = 60Hz$$

Electric Utilities

- Standardization on the frequency of the electricity distribution systems didn't occur until the mid-1900's.
  - The frequency of the ac voltage supplied by power companies in the US is 60 Hz.
  - The frequency used in much of Europe and Asia is 50 Hz.
    - While some electronic circuits function properly when connected to a power supply operating at either frequency, some are designed for a specific frequency, which is one reason why power adaptors are needed when you travel.
  - If you look at the label on the tablet 'brick', the frequency of the ac signal is specified.

Angular frequency

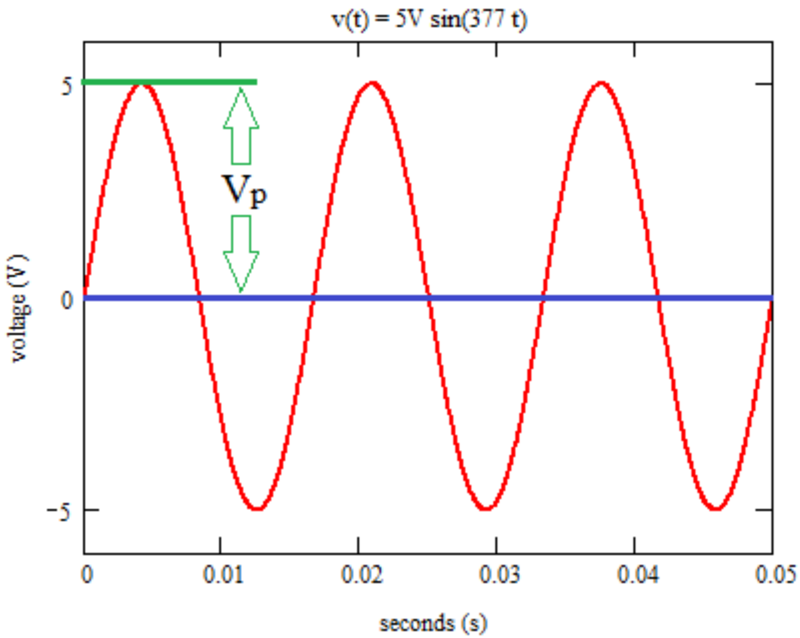
- Motors are used in the alternators in coal- and gas-powered electric generation stations.
- One full rotation of the motor shaft produces one complete cycle of the ac electricity produced.
  - Position of the motor shaft is measured in radians (rad) or degrees (^{o}).
    - 1 rad = 57.3^{o}
    - 2π rad = 360^{o}

$$\omega = 2\pi f = \frac{2\pi}{T}$$

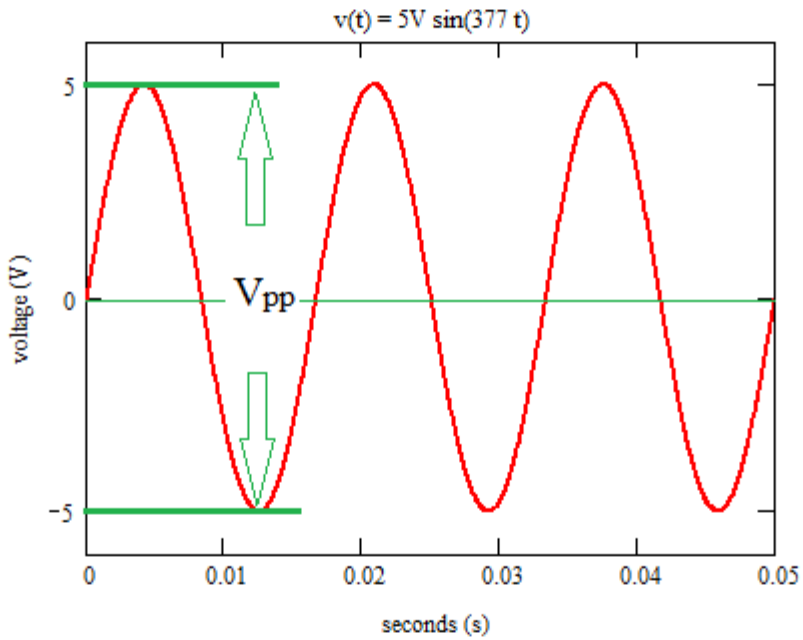
$$f = 60 Hz \qquad \omega = 377 \text{ rad/s}$$

Amplitude

Peak amplitude



Peak-to-Peak amplitude



$$V_{pp} = 2V_p$$

$$I_{pp} = 2I_p$$

Instantaneous Value

- Instantaneous value or amplitude is the magnitude of the sinusoid at a point in time.

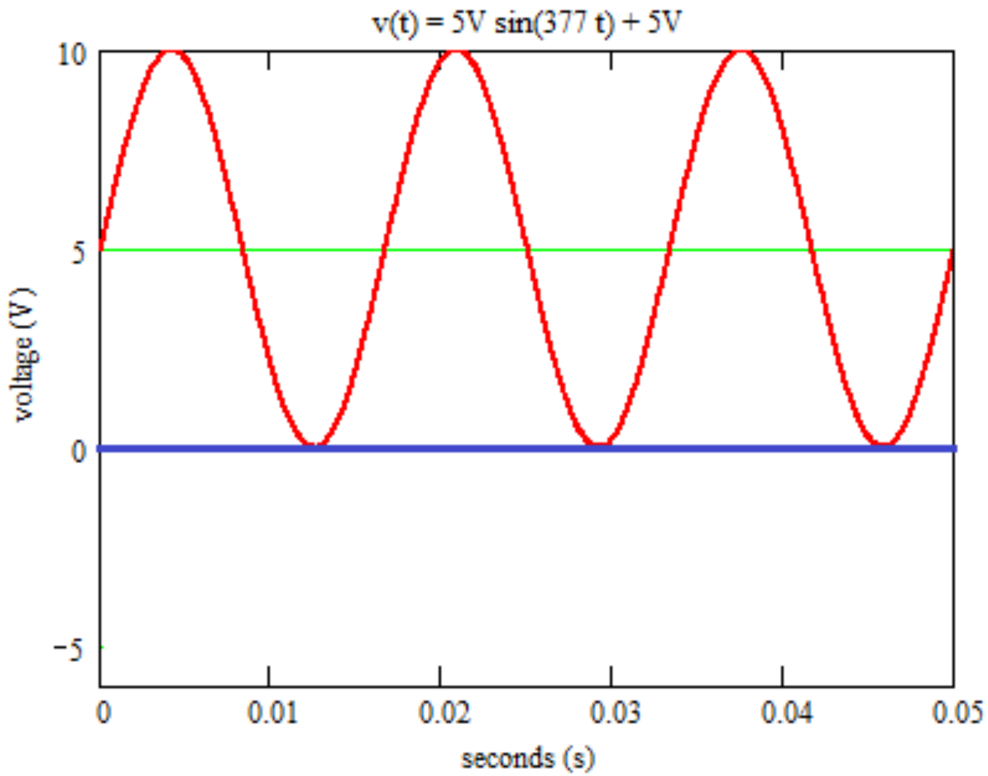
$$v(t) = 5V \sin[(377 rad/s)t]$$

$$t = 0s \qquad v(t) = 5V \sin[(377 rad/s)(0s)] = 0V$$

$$t = 10ms \qquad v(t) = 5V \sin[(377 rad/s)(0.01s)] = -2.94V$$

Average Value

- The average value of a sinusoid signal is the integral of the sine wave over one full cycle. This is always equal to zero.
  – If the average of an ac signal is not zero, then there is a dc component known as a DC offset.



Root Mean Square (RMS)

- Most equipment that measure the amplitude of a sinusoidal signal displays the results as a root mean square value. This is signified by the unit Vac or V_{RMS}.
  - RMS voltage and current are used to calculate the average power associated with the voltage or current signal in one cycle.

$$V_{RMS} = \sqrt{\frac{1}{T}\int_0^T [v(t)]^2\,dt}$$

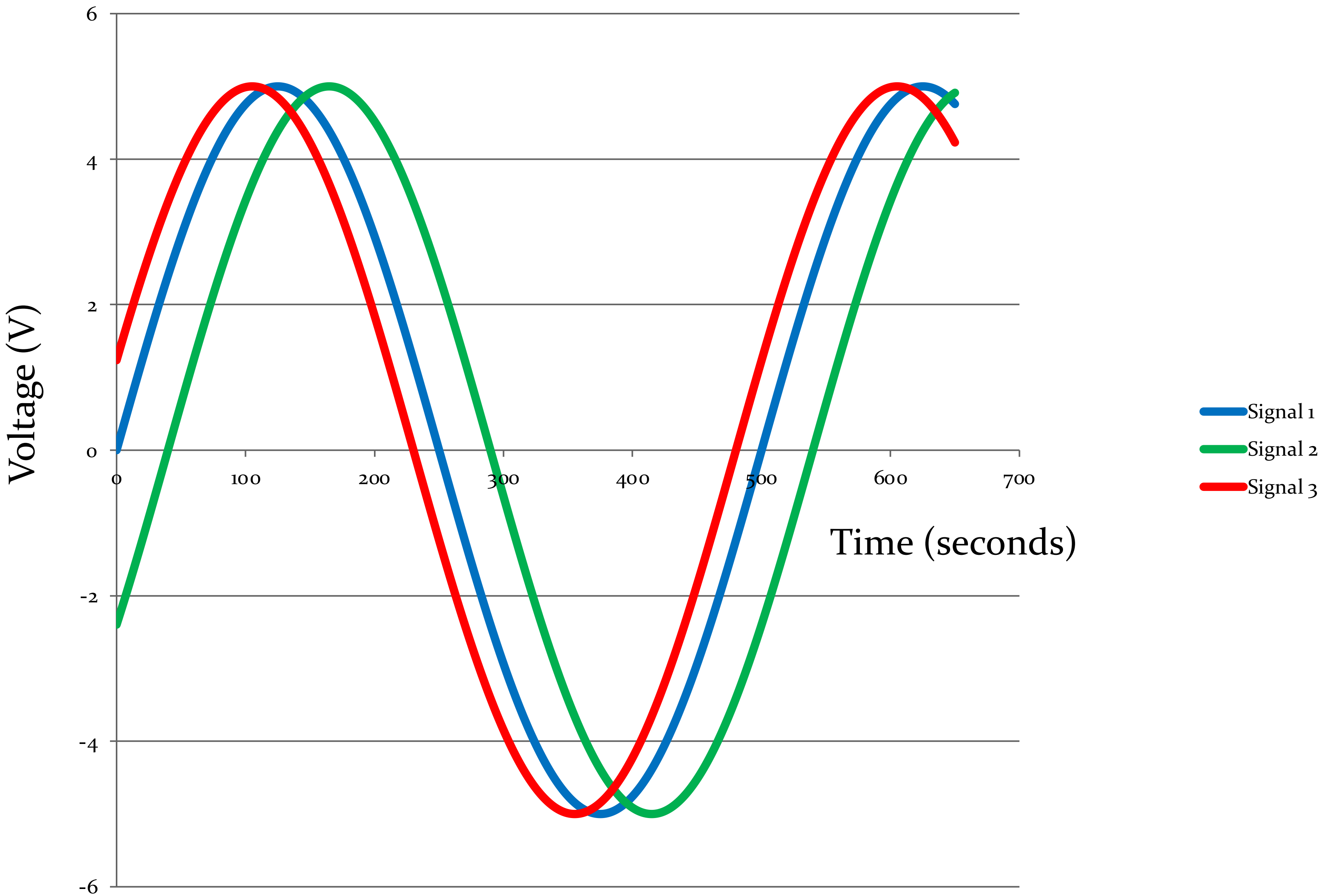
$$V_{RMS} = \frac{\sqrt{2}}{2}V_p = 0.707V_p$$

$$P_{Ave} = \left(V_{RMS}{}^2\right)/R$$

Phase Angle

- The phase angle is an angular measurement of the position of one sinusoid signal with respect to a reference.
  - The signal and reference must have the same frequency.
- Suppose there are three signals
  - One signal is the reference
    - the signal in blue on the following slide has been chosen to be the reference
  - The phase of the other two signals will be calculated with respect to the reference signal.
    - The period of each signal should be the same, which means that all signals have the same frequency.

Calculation of Phase



Example 01…

- Calculate the period, T, for the reference signal
  - This is the time for a full cycle to be completed.
    - T= 500 second for Signal 1
  - Calculate the difference in time between zero crossings of
    - Signal 2 and Signal 1:
      - Δt = 40 second – 0 seconds = 40 s
    - Signal 3 and Signal 1:
      - Δt = 480 seconds – 0 seconds= 480 s

…Example 01

- The sinusoidal function that describes Signal 1, the reference voltage, is

$$V(t) = 5V \sin(\omega t) \text{ , where } \omega = 2\pi/T = 12.6 \text{ mrad/s}$$

- To write the sinusoidal function that describes Signals 2 and 3, we need to address the fact that there is a shift in the zero crossings

$$V(t) = A \sin(\omega t + \phi) \text{ , where } \omega = 2\pi/T$$

$$\phi = -2\pi\, \Delta t/T \text{ in radians or } \phi = -360^{o}\, \Delta t/T$$

  - ϕ is called the phase shift

Lagging and Leading

- Don't get fooled by the positions of the curves on the graph!

- Signal 2: $V(t) = 5V \sin [(12.6 \text{ mrad/s})t - 28.8^{o}]$
  - ϕ is -0.502 radians or -28.8 degrees
  - Signal 2 lags Signal 1 as it reaches zero at a later time than Signal 1

- Signal 3: $V(t) = 5V \sin [(12.6 \text{ mrad/s})t + 14.4^{o}]$
  - ϕ is 0.251 radians or 14.4 degrees
  - Signal 3 leads Signal 1 as it reaches zero at an earlier time than Signal 1

Summary

- AC signals are sinusoidal functions.
  - The mathematical description of the sinusoid includes the peak amplitude and the angular frequency and may include a phase angle.

$$v(t) = V_p \sin(\omega t + \phi) \qquad \omega = 2\pi f = \frac{2\pi}{T}$$

- RMS values of a sinusoid are calculated using the formula

$$V_{RMS} = \sqrt{\frac{1}{T}\int_0^T [v(t)]^2\, dt} \qquad V_{RMS} = 0.707 V_p$$

- Phase angle for a sinusoid is calculated with respect to a reference.
  - A signal lags a reference when $\phi_{signal} - \phi_{reference} < 0^o$.
  - A signal leads a reference when $\phi_{signal} - \phi_{reference} > 0^o$.

Sinusoidal Voltage

$$v(t) = V_m \sin(\omega t + \phi) \qquad T = \frac{2\pi}{\omega} \qquad f = \frac{1}{T}$$

where

V_m is the amplitude of the sinusoid

ω is the angular frequency in radians/s

ϕ is the phase angle in degrees

$\omega t + \phi$ is the argument of the sinusoid

T is the period of a sinusoid in seconds

f is the frequency with units of Hz (cycles per second)

Phase between Cosine and Sine

- Using cosine form:

  $v_1(t) = 6V \sin(20t + 40^o)$ $v_2(t) = -4V \cos(20t + 20^o)$

  $v_1(t) = 6V \cos(20t + 40^o - 90^o) = 6V \cos(20t - 50^o)$

  $v_2(t) = 4V \cos(20t + 20^o - 180^o) = 4V \cos(20t - 160^o)$

  Phase angle between them is 110^o and v_1 leads v_2

- Using sine form:

  $v_1(t) = 6V \sin(20t + 40^o)$ $v_2(t) = -4V \cos(20t + 20^o)$

  $v_1(t) = 6V \sin(20t + 40^o)$

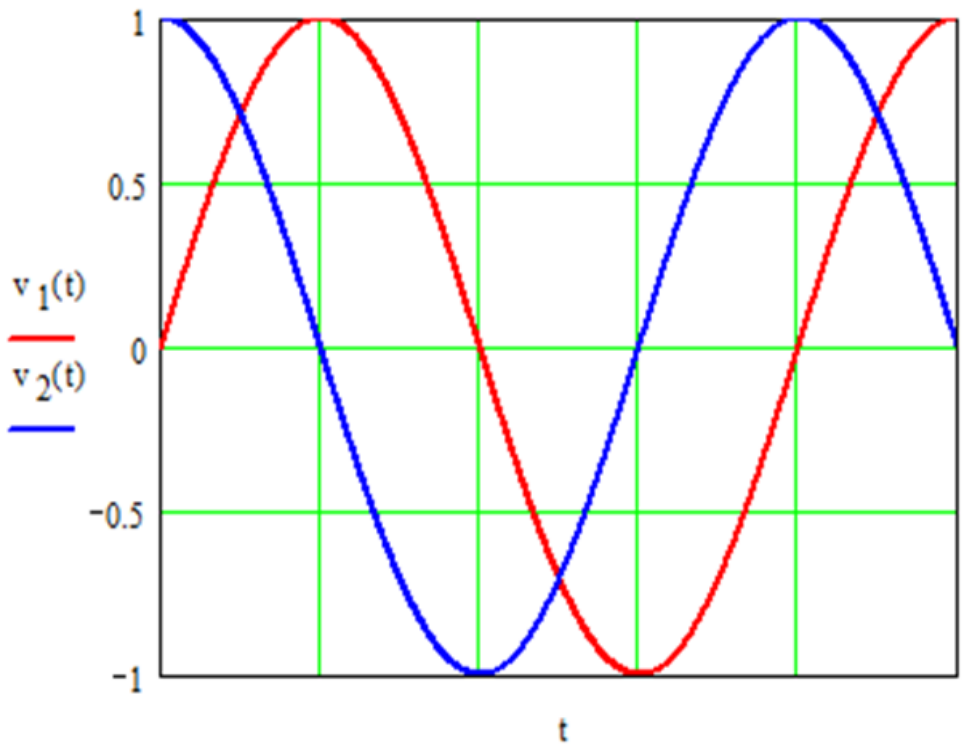
  $v_2(t) = 4V \sin(20t + 20^o - 90^o) = 4V \sin(20t - 70^o)$

  Phase angle between them is 110^o

Phase between Cosine and Sine

Suppose: $v_1(t) = 1V \sin(\omega t)$

$v_2(t) = 1V \cos(\omega t)$



The sine wave reaches its maximum when $\omega t = 90^o$, but the cosine reaches its maximum when $\omega t = 0^o$.

So, the shape of the sine wave is 90^o out of phase with the cosine wave.

Since the cosine reaches zero before the sine, the sine lags the cosine by 90^o.

To force the sine wave to overlap the cosine wave, $v_1(t)$ must be rewritten as

$v_1(t) = 1V \cos(\omega t - 90^o)$.

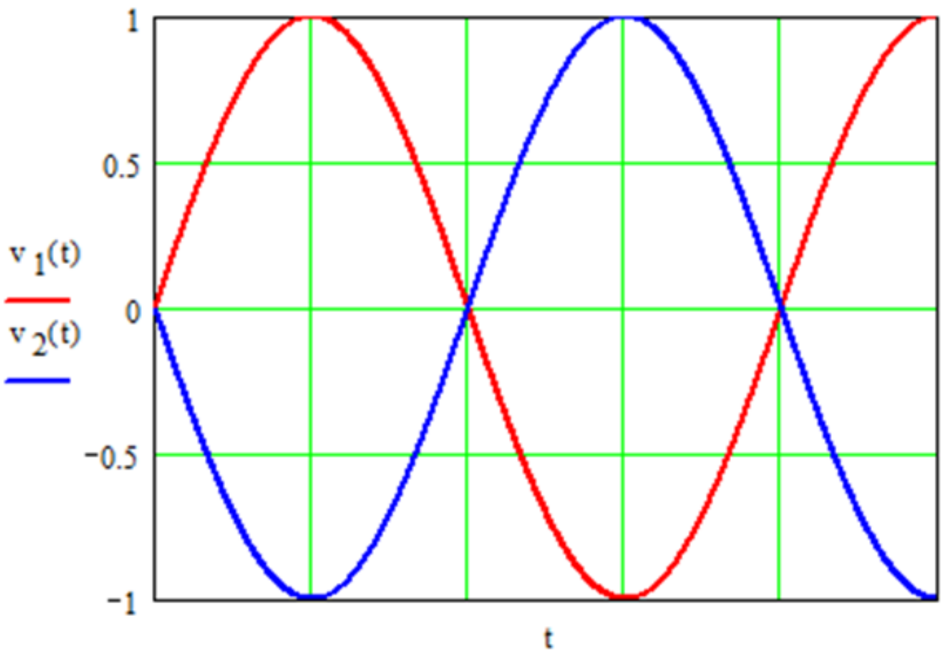
To force the cosine wave to overlap the sine wave, $v_2(t)$ must be rewritten as

$v_2(t) = 1V \sin(\omega t + 90^o)$.

Phase between Cosine and Sine

Suppose: $v_1(t) = 1\text{V} \sin(\omega t)$

$v_2(t) = -1\text{V} \sin(\omega t)$



The $v_1(t)$ reaches its maximum when $\omega t = 90^o$, but the $v_2(t)$ reaches its maximum when $\omega t = 270^o$.

So, the two sine waves are 180^o out of phase with the each other.

To force the negative sine function to have the same magnitude as the positive sine function, a phase angle of 180^o must be added or subtracted (it doesn't matter which is done as both move the sinusoid the same fraction of a cycle).

So, $v_2(t)$ can be rewritten as

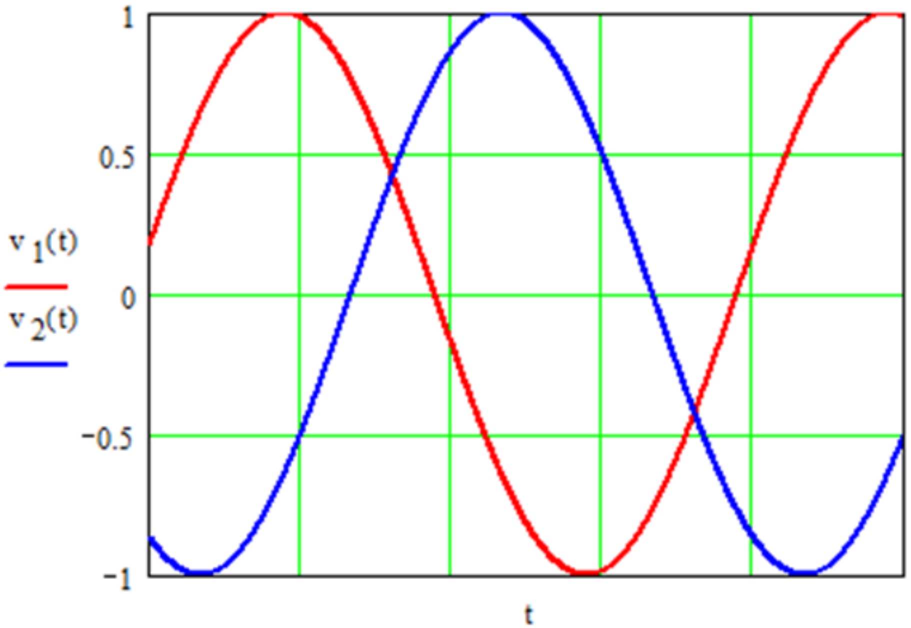
$v_2(t) = 1\text{V} \sin(\omega t - 180^o)$ or

$v_2(t) = 1\text{V} \sin(\omega t + 180^o)$.

Phase between Cosine and Sine

Suppose: $v_1(t) = 1V \sin(\omega t + 10^o)$

$v_2(t) = -1V \cos(\omega t - 30^o)$



Change the sine to a cosine, since that is preferable when writing voltages in phasor notation.

$v_1(t) = 1V \cos(\omega t + 10^o - 90^o)$

$v_1(t) = 1V \cos(\omega t - 80^o)$

Change the magnitude of the $v_2(t)$ to a positive number, subtract a phase angle of 180^o (adding 180^o will cause the phase angle difference to be more than 180^o, which is not desirable).

$v_2(t) = 1V \cos(\omega t - 30^o - 180^o)$

$v_2(t) = 1V \cos(\omega t - 210^o)$

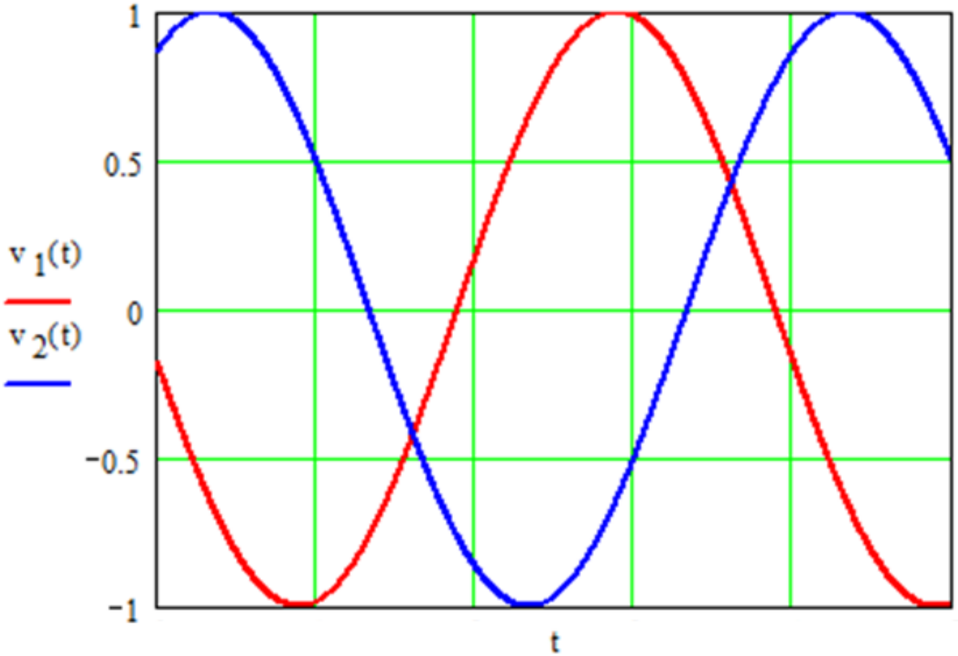
The phase angle between the two voltages is $[-80^o - (-210^o)]$ or $+130^o$

Phase between Cosine and Sine

Suppose:

$$v_1(t) = -1V \sin(\omega t + 10^o)$$

$$v_2(t) = 1V \cos(\omega t - 30^o)$$



Change the sine to a cosine, since that is preferable when writing voltages in phasor notation.

$v_1(t) = -1V \cos(\omega t + 10^o - 90^o)$

$v_1(t) = -1V \cos(\omega t - 80^o)$

Change the magnitude of the $v_1(t)$ to a positive number, add a phase angle of 180^o (substracting 180^o will cause the phase angle difference to be more than 180^o, which is not desirable).

$v_1(t) = 1V \cos(\omega t - 80^o + 180^o)$

$v_1(t) = 1V \cos(\omega t + 100^o)$

The phase angle between the two voltages is $[100^o – (-30^o)]$ or $+130^o$

Conversions for Sinusoids

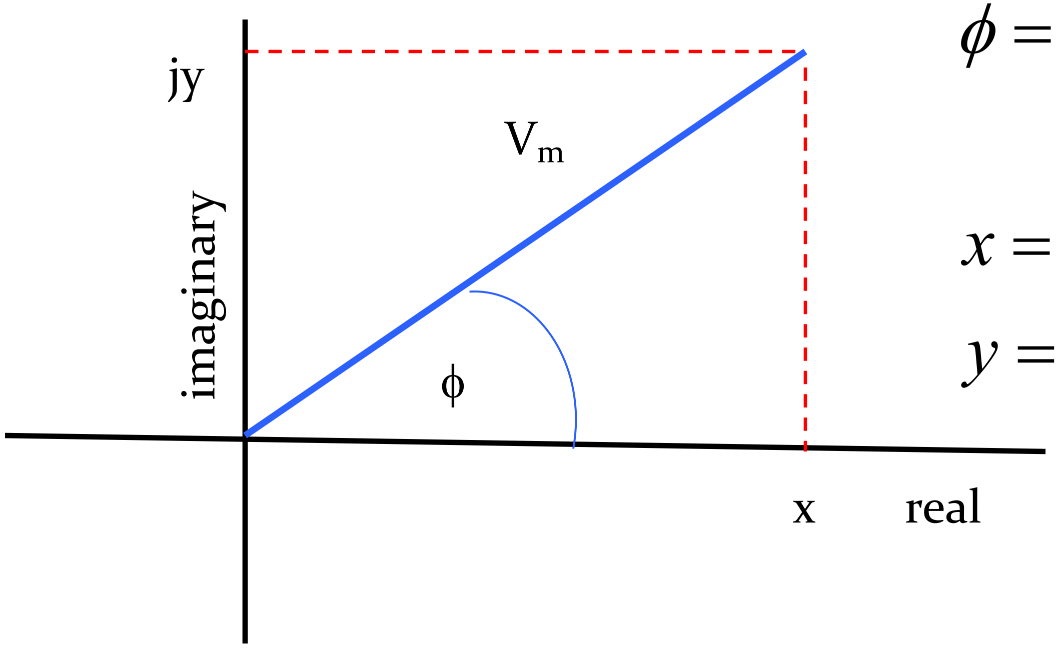
$A \sin(\omega t + \phi)$	$A \cos(\omega t + \phi - 90^o)$
$- A \sin(\omega t + \phi)$	$A \sin(\omega t + \phi + 180^o)$ Or $A \sin(\omega t + \phi - 180^o)$
$- A \cos(\omega t + \phi)$	$A \cos(\omega t + \phi + 180^o)$ Or $A \cos(\omega t + \phi - 180^o)$
$A \sin(\omega t + \phi)$	$A \sin(\omega t + \phi - 360^o)$ Or $A \sin(\omega t + \phi + 360^o)$
$A \cos(\omega t + \phi)$	$A \cos(\omega t + \phi - 360^o)$ Or $A \cos(\omega t + \phi + 360^o)$

Steps to Perform Before Comparing Angles between Signals

- The comparison can only be done if the angular frequency of both signals are equal.
- Express the sinusoidal signals as the same trigonometric function (either all sines or cosines).
- If the magnitude is negative, modify the angle in the trigonometric function so that the magnitude becomes positive.
- If there is more than 180^{o} difference between the two signals that you are comparing, rewrite one of the trigonometric functions
- Subtract the two angles to determine the phase angle.

Phasor

- A complex number that represents the amplitude and phase of a sinusoid



$$V_m = \sqrt{x^2 + y^2}$$

$$\phi = \tan^{-1}(y/x) = \arctan(y/x)$$

$$x = V_m \cos(\phi)$$

$$y = V_m \sin(\phi)$$

Real Number Line

- If there is no imaginary component to the phasor, then the phasor lies on the real number line (x-axis).
  - Positive real numbers are written as:
    - Phasor notation $P_m \angle 0^o$
    - Rectangular coordinates P_m
  - Negative real numbers are written as:
    - Phasor notation $P_m \angle -180^o$
    - Rectangular coordinates $-P_m$

Imaginary Number Line

- If there is no real component to the phasor, then the phasor lies on the imaginary number line (y-axis).
  - Positive imaginary numbers are written as:
    - Phasor notation $P_m \angle 90^o$
    - Rectangular coordinates jP_m
  - Negative imaginary numbers are written as:
    - Phasor notation $P_m \angle -90^o$
    - Rectangular coordinates $-jP_m$

Phasor Representation

- Polar coordinates:

$$V = V_m \angle \phi$$

- Rectangular coordinates
  - Sum of sines and cosines

$$V = V_m\left[\cos(\phi) + j\sin(\phi)\right]$$

$$x = V_m\cos(\phi) \quad y = V_m\sin(\phi)$$

- Exponential form:

$$V = V_m e^{j\phi}$$

Where the sinusoidal function is:

$$v(t) = V_m\cos(\omega t + \phi)$$

Sinusoid to Phasor Conversion

- The sinusoid should be written as a cosine.
- Amplitude or magnitude of the cosine should be positive.
  - This becomes the magnitude of the phasor
- Angle should be between $+180^o$ and -180^o.
  - This becomes the phase angle of the phasor.
- Note that the frequency of the sinusoid is not included in the phasor notation.
  - It must be provided elsewhere.
    - Phasors are commonly used in power systems, where the frequency is understood to be 60 Hz in the United States.

Sinusoid-Phasor Transformations

Time Domain	Phasor Domain
$V_m \cos(\omega t + \phi)$	$V_m \angle \phi$
$V_m \sin(\omega t + \phi)$	$V_m \angle \left(\phi - 90^o\right)$
$I_m \cos(\omega t + \theta)$	$I_m \angle \theta$
$I_m \sin(\omega t + \theta)$	$I_m \angle \left(\theta - 90^o\right)$

Assumes V_m is positive and $-180^o \leq \phi \leq 180^o$

Phasor Notation

- Phasor notation is used when there are one or more ac power sources in a circuit. All of these power sources operate at the same single frequency.
- Used extensively in power systems because almost all of these systems operate at 50 Hz in Tutkey.
- Bold **V** and **I** are used to show that phasor notation is being used.

Examples

Sinusoidal Function :

$$3V\sin(100t+20^0)=3V\cos(100t-70^0)$$

Converting to phasor notation : $\quad 3V\angle -70^0$

Sinusoidal Function :

$$7A\sin(350t-100^0)=7A\cos(350t-190^0)$$

$$=-7A\cos(350t-10^0)=7A\cos(350t+170^0)$$

Converting to phasor notation : $\quad 7A\angle 170^0$

Examples

Rectangular Coordinates	Phasor Notation
$(5+3j)V$	$5.83V \ \angle 31.0^0$
$(-30+j100)\,A$	$104\text{A} \ \angle -73.3^0$
$(-0.4-0.25j)\Omega$	$0.472\Omega \ \angle 32.0^0$
$(75-j150)\,\text{A}$	$168\text{A} \ \angle -63.4^0$

Summary

- Phasor notation is used in circuits that have only ac power sources that operate at one frequency.
  - The frequency of operation is not included in the notation, but must be stated somewhere in the circuit description or schematic.
  - The steps to convert between sinusoidal functions and rectangular coordinates were described.
  - To express a phasor $P_m = \angle\phi$ in rectangular coordinates $(\mathrm{Re} + j\mathrm{Im})$ can be performed using the following equations:

$$P_m = \sqrt{\mathrm{Re}^2 + \mathrm{Im}^2} \qquad \mathrm{Re} = P_m \cos(\phi)$$

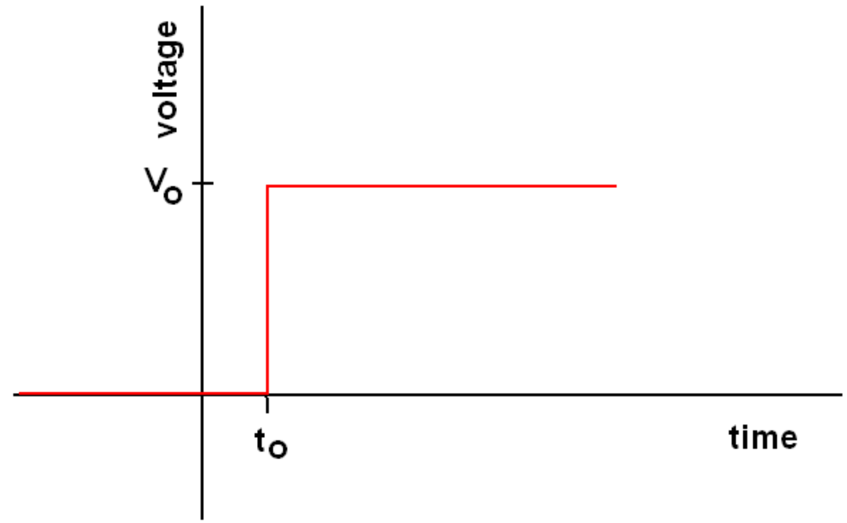
$$\phi = \tan^{-1}(\mathrm{Im}/\mathrm{Re}) \qquad \mathrm{Im} = P_m \sin(\phi)$$

Singularity Functions

- Are discontinuous or have discontinuous derivatives.
  - Also known as switching functions.
  - They are:
    - Impulse function
    - Unit function
    - Pulse function
    - Square wave
    - Ramp functions (triangular and sawtooth)
  - Combinations of these functions can be used to describe complex composite waveforms.

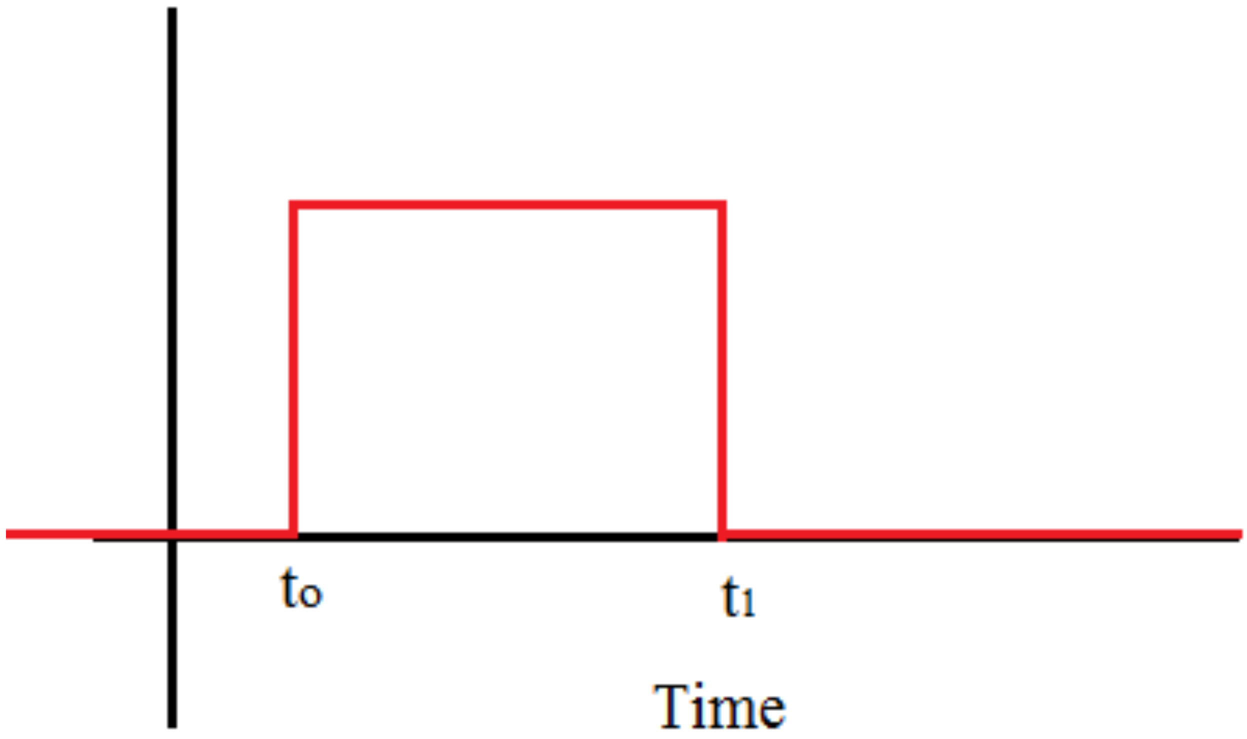
Unit Step Function

$$u(t-t_o)=\begin{cases}0 & t<t_0\\ 1 & t>t_0\end{cases}$$

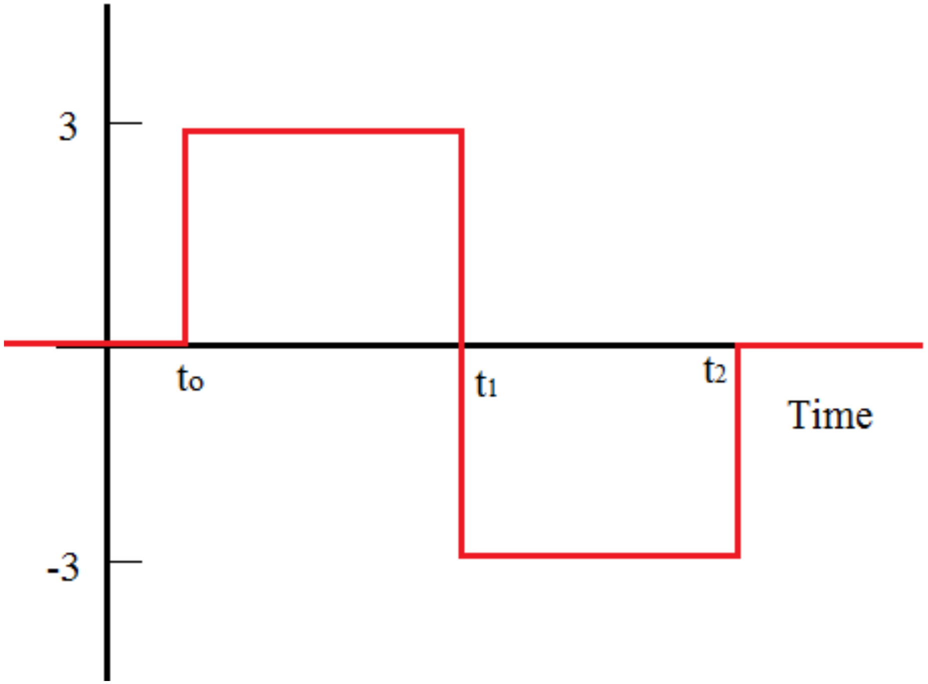


$$V(t)=V_o u(t-t_o)$$

Pulse Function

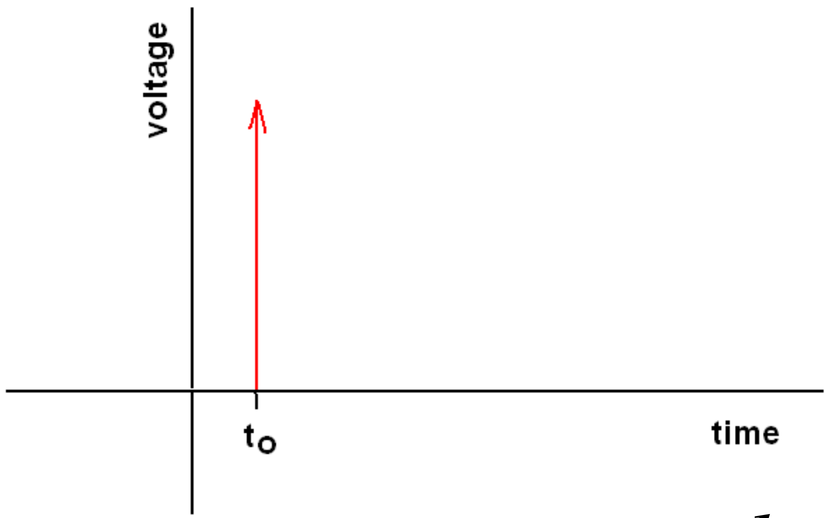


Square Wave



$$3\left[u(t-t_o)-2u(t-t_1)+u(t-t_2)\right]$$

Unit Impulse Function



- An impulse function is the derivative of a unit step function.

$$\delta(t - t_o) = \frac{du(t_o)}{dt}$$

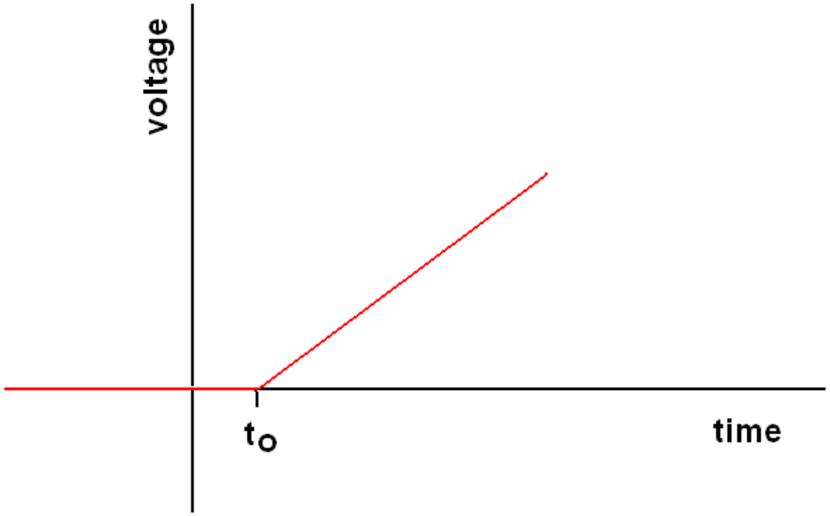
$$\int_{-\infty}^{+\infty} \delta(t - t_0)dt = \int_{t_o^-}^{t_o^+} \delta(t - t_0)dt = 1$$

Unit Ramp Function

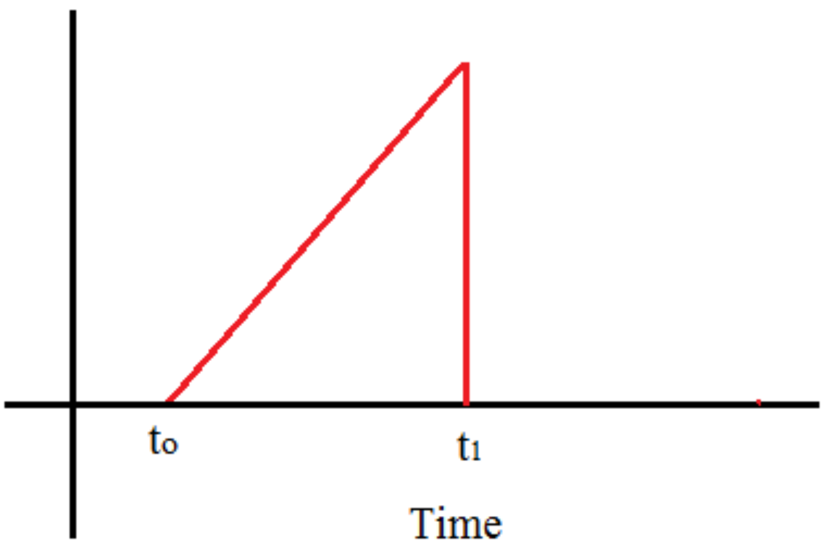
- A ramp function is the result of the integration of a unit function.

$$r(t) = \int_{-\infty}^{t} u(t - t_o)dt = (t - t_o)u(t - t_o)$$

$$r(t) = \begin{cases} 0 & t < t_o \\ t - t_o & t > t_o \end{cases}$$



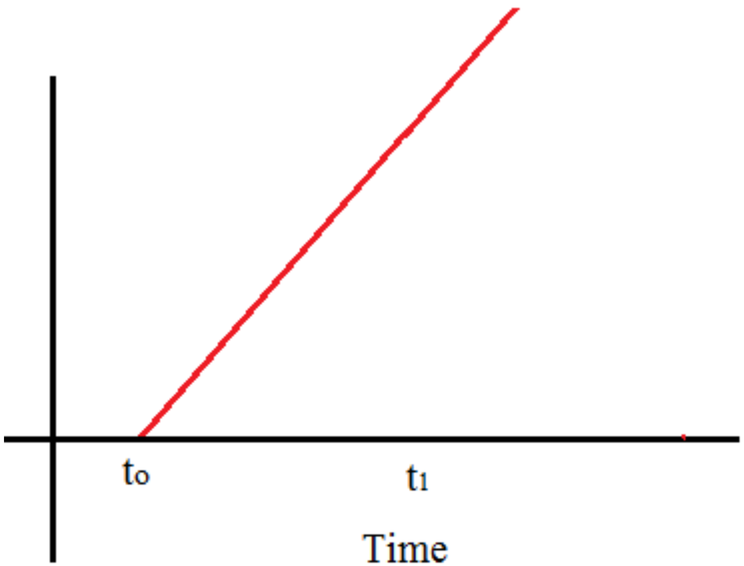
Sawtooth Waveform

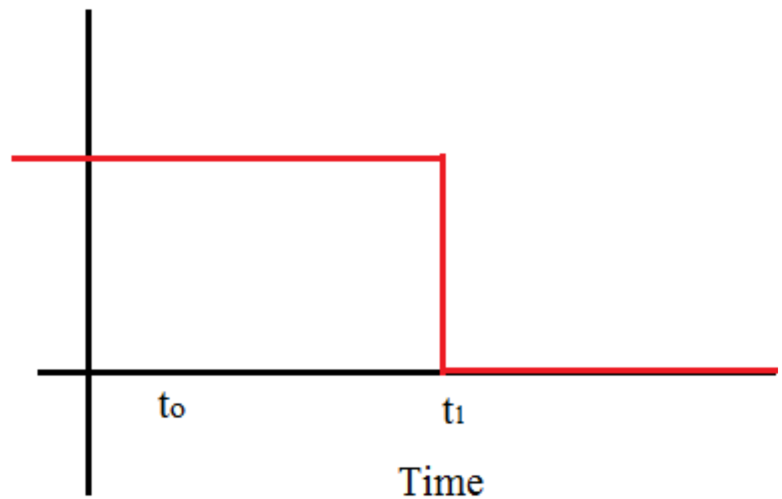


- This is a ramp function that is discontinued at time $t = t_1$.
- To discontinue a ramp function, the function must be multiplied by a second function that doesn't disturb the ramp when $t < t_1$, but the result of the multiplication is zero for $t \geq t_1$.

Mathematical Formula

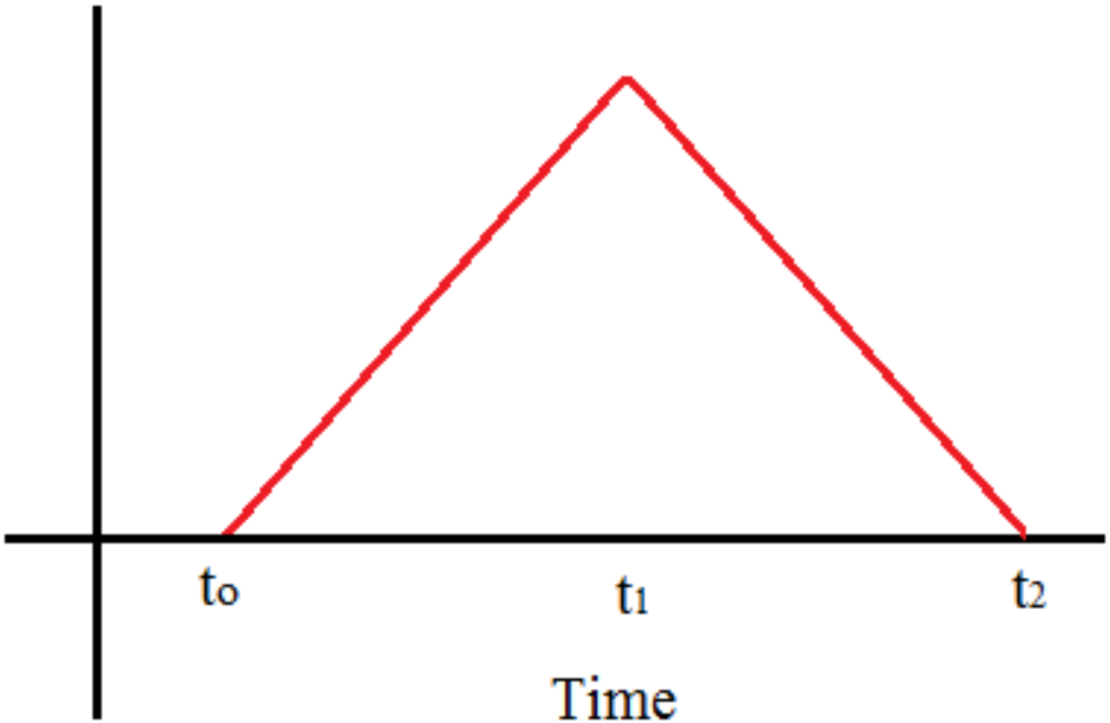
Sawtooth Waveform is formed by multiplying the Ramp Function and the Unit Step Function





$$r(t-t_o)\cdot u(t_1-t)$$

Triangular Waveform



$$r(t-t_o)-2r(t-t_1)+r(t-t_2)$$

Composite Waveforms

- Any nonsinusoidal function can be expressed as a sum (or composite) of multiple sinusoidal functions.
  - The sinusoids are related as the frequencies of the sinusoids are integral multiples of some base frequency, known as the fundamental frequency.
  - The higher frequencies are known as harmonics.
    - For even and odd harmonics, the multiplier is an even and odd integer, respectively.
    - The multiplier for 2^{nd} and 3^{rd} harmonics is 2 and 3, respectively.

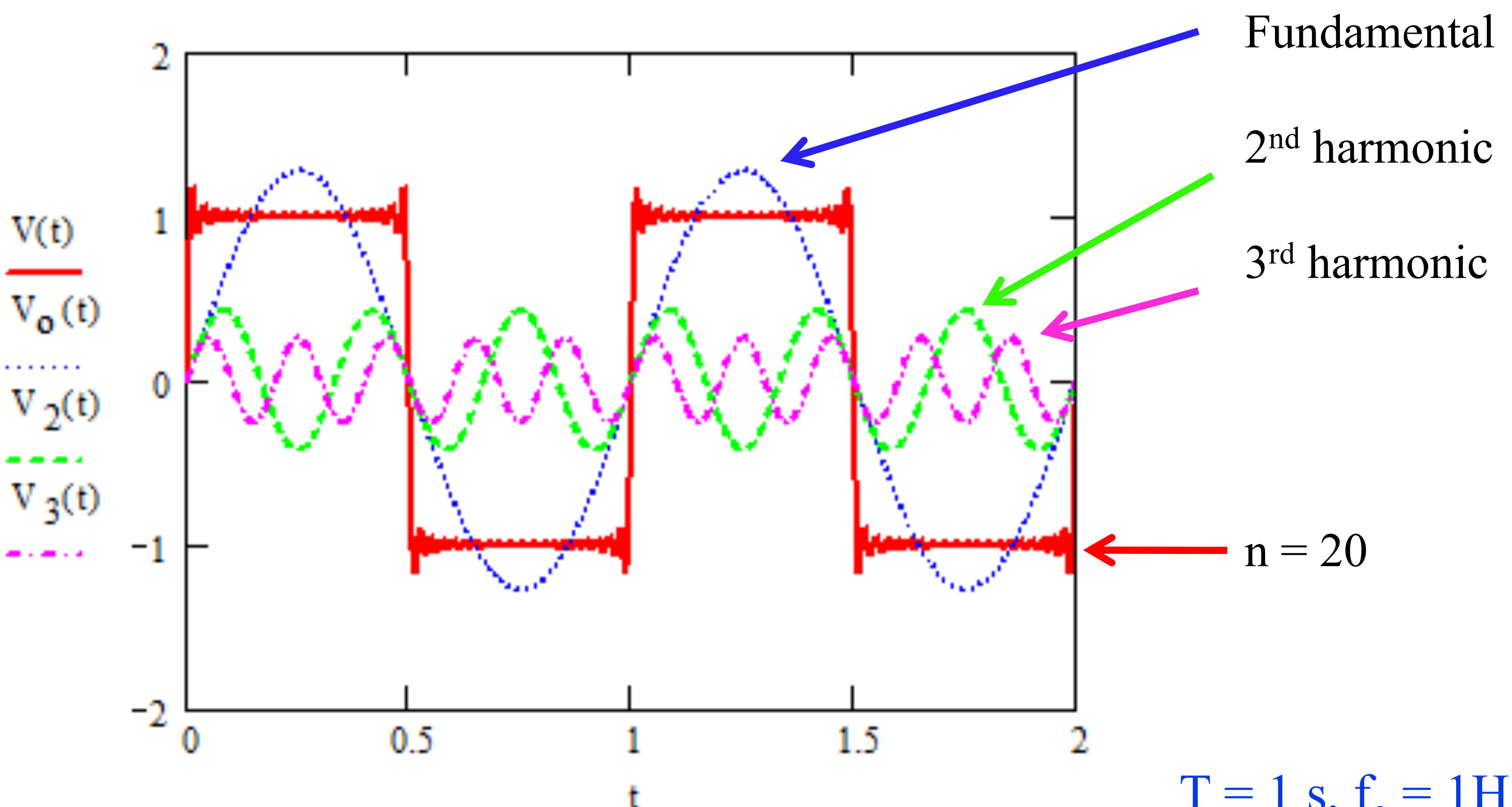
Infinite Square Wave…

- An infinite square wave is a weighted sum of a fundamental sinusoid and its odd harmonics.

$$V(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{2n-1}\sin\left[\frac{(2n-1)\pi t}{T}\right]\right\}$$

- T is the period of the square waveform.
- The fundamental frequency, $f_o = 1/T$.

…Infinite Square Wave

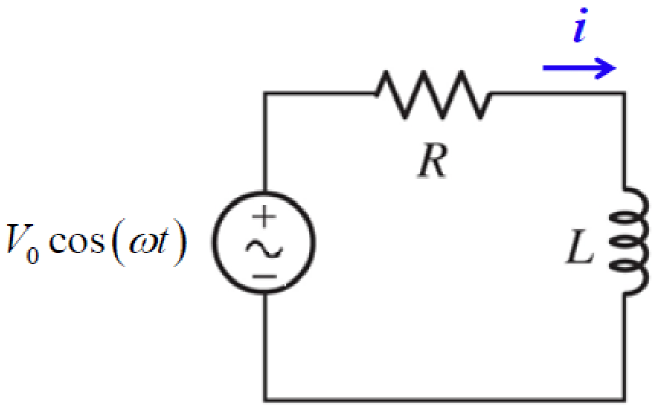


T = 1 s, f_o = 1Hz

Summary

- Several unit functions were described and their mathematical functions were given.
  - The convolution of the unit functions can be used to form more complex functions.
- Composite waveforms are summations of sinusoidal waves, which is an alternative method to describe complex functions.

RL Circuit with a Sinusoidal Source



$$\frac{d}{dt}i(t)+\frac{R}{L}\cdot i(t)=\frac{V_0}{L}\cos(\omega t)$$

-- oscillates forever

-- never settles to a DC value (e.g. zero)

It's possible that the solution is of the form $i(t)=I_0\cos(\omega t+\theta)$

Substituting $i(t)$ into the differential equation…

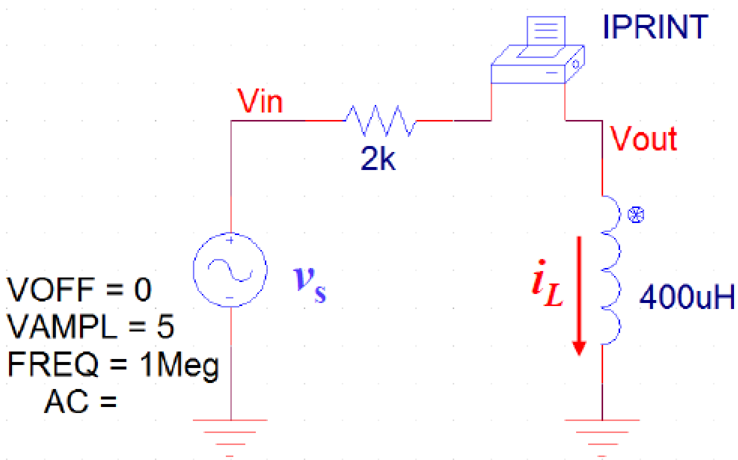
$$-\omega I_0\sin(\omega t+\theta)+\frac{R}{L}I_0\cos(\omega t+\theta)=\frac{V_0}{L}\cos(\omega t)$$

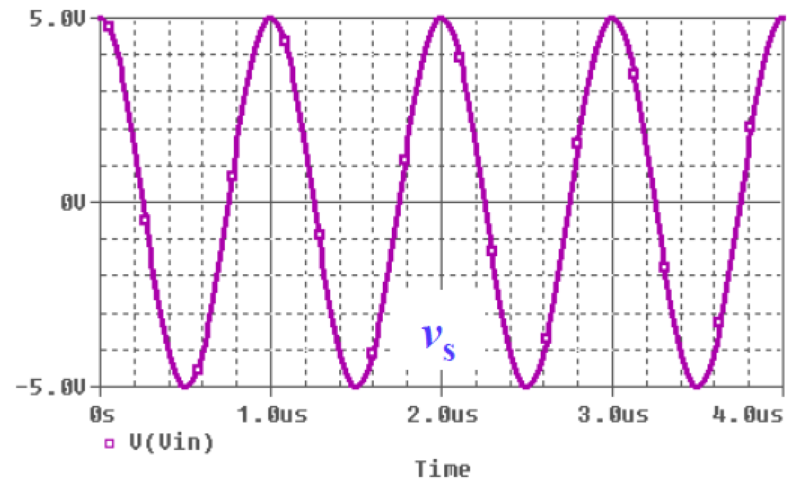
Solving for I_0 and substituting back into $i(t)$ yields

$$i(t)=\boxed{\frac{V_0}{\sqrt{R^2+\omega^2L^2}}}\cos\left\{\omega t-\boxed{\tan^{-1}\left(\frac{\omega L}{R}\right)}\right\}$$

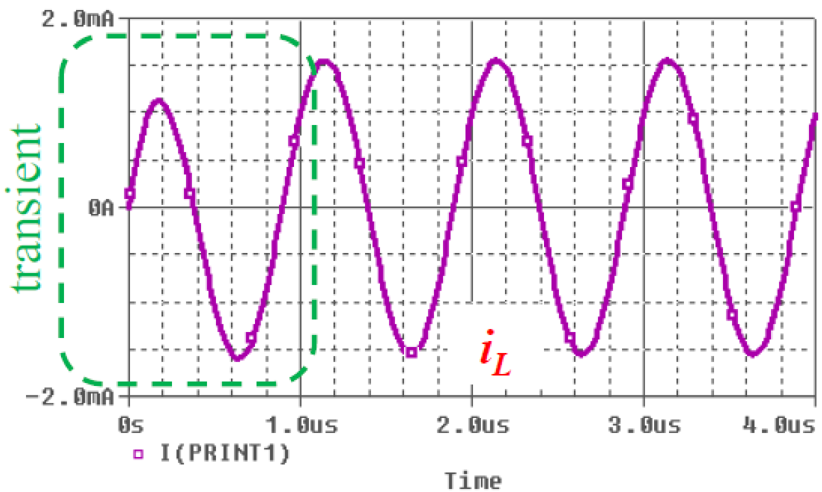
amplitude scaling, phase shift

RL Circuit with a Sinusoidal Source





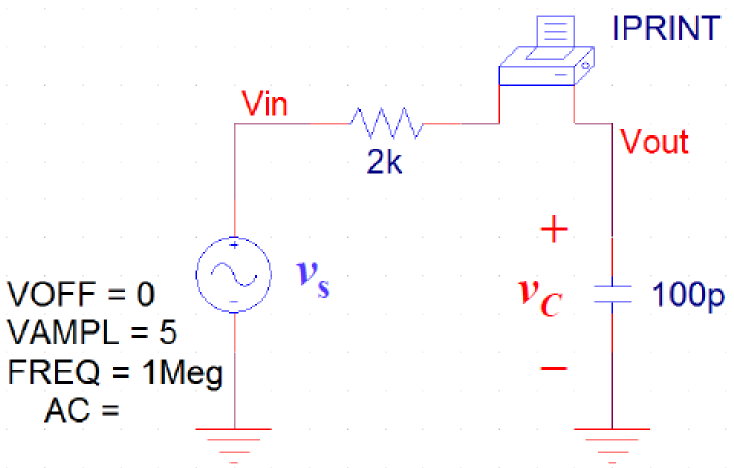
$$5 \cdot \tau = 5 \cdot \frac{L}{R} = (5)\frac{400\ \mu\text{H}}{2\ \text{k}\Omega} = 1\ \mu\text{s}$$

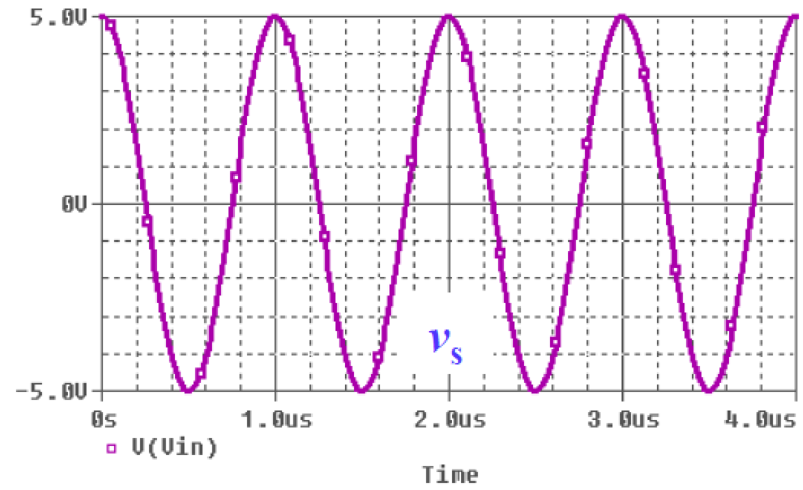


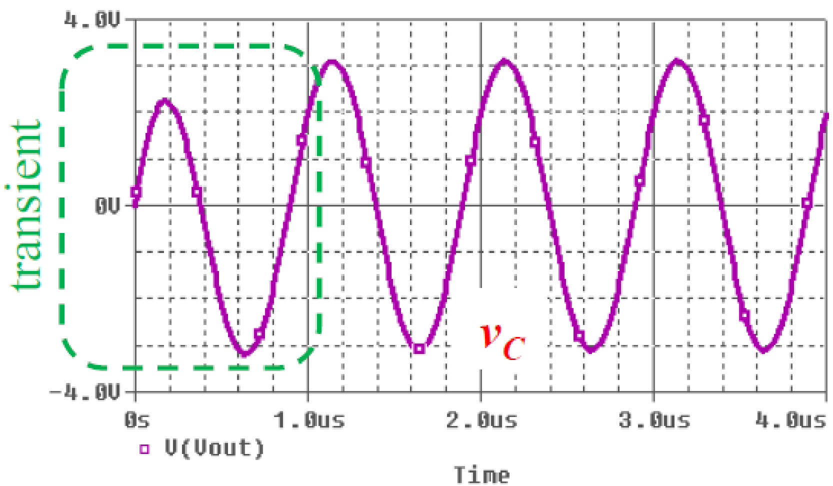
The *RL* circuit's *transient* response is negligible after $\approx 5\tau$.

The remaining response is sinusoidal.

RC Circuit with a Sinusoidal Source





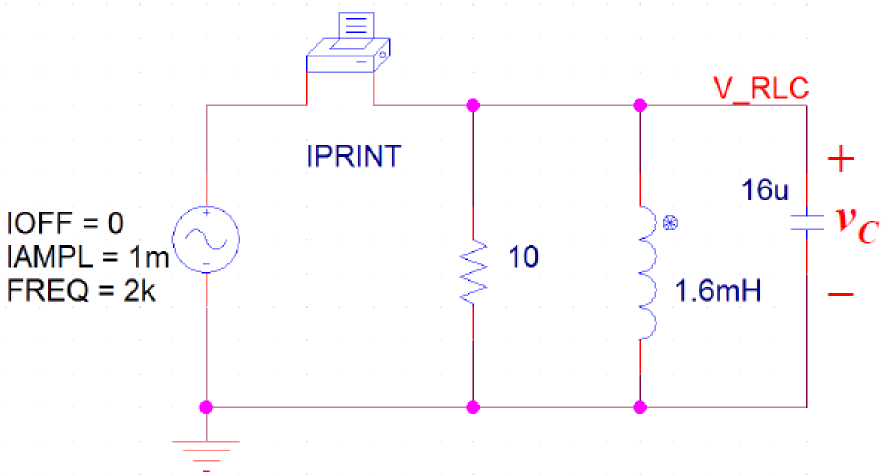


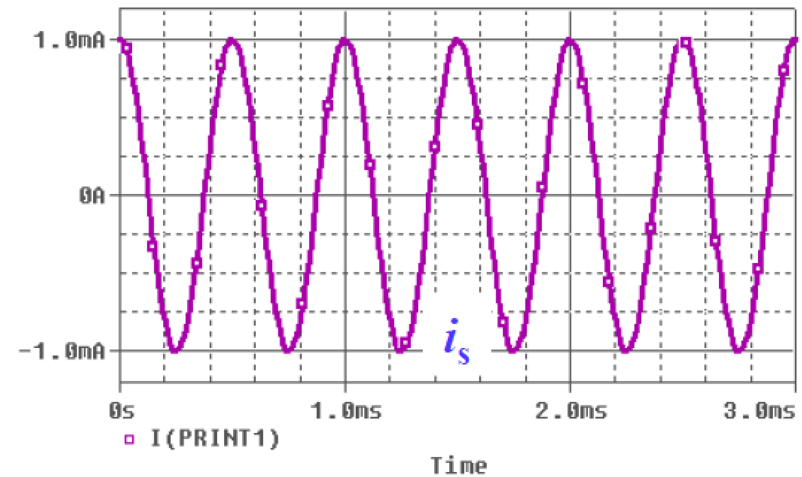
$$5 \cdot \tau = 5 \cdot RC$$
$$= (5)(2\ \text{k}\Omega)(100\ \text{pF}) = 1\ \mu\text{s}$$

The *RC* circuit's *transient* response is negligible after $\approx 5\tau$.

The remaining response is sinusoidal.

RLC Circuit with a Sinusoidal Source





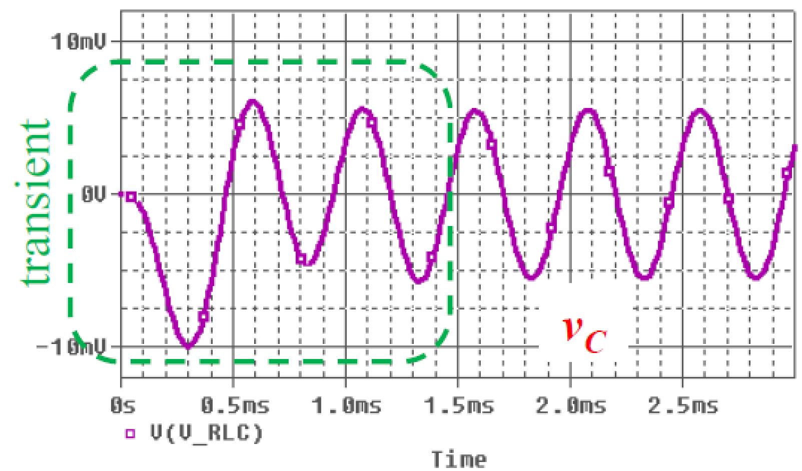
$$v_c(t) = e^{-\alpha t}\left[V_1 \cos(\omega_d t) + V_2 \sin(\omega_d t)\right] + V_3$$

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \cdot 10\,\Omega \cdot 16\,\mu\text{F}} = 3125\,\frac{\text{rad}}{\text{s}}$$

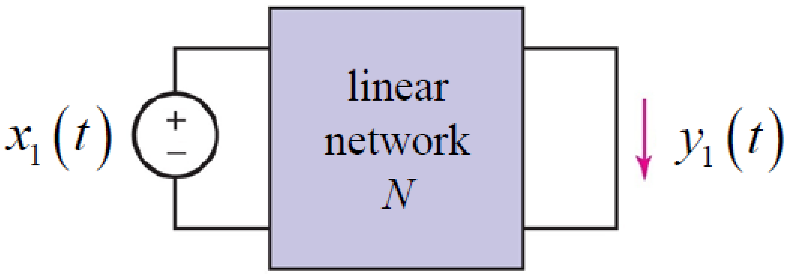
$$t_s \approx 5/\alpha \approx 1.6\text{ ms}$$

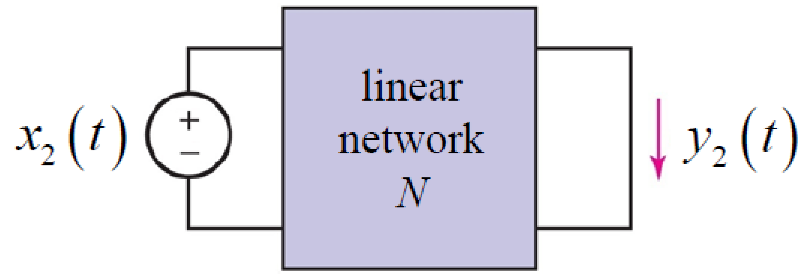
The *RLC* circuit's *transient* response is negligible after $\approx t_s$.

The remaining response is sinusoidal.

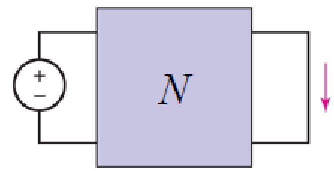


Consequences of Linearity



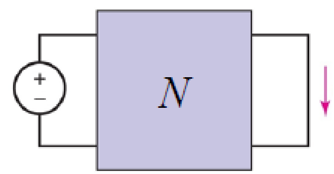


$A_1 x_1(t-t_1) + j \cdot A_2 x_2(t-t_2)$



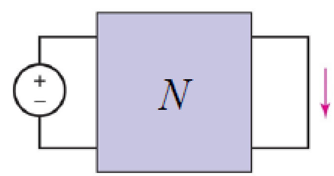
$A_1 y_1(t-t_1) + j \cdot A_2 y_2(t-t_2)$

$$\mathrm{Re}\{A_1 x_1(t-t_1) + j \cdot A_2 x_2(t-t_2)\} = A_1 x_1(t-t_1)$$



$$\mathrm{Re}\{A_1 y_1(t-t_1) + j \cdot A_2 y_2(t-t_2)\} = A_1 y_1(t-t_1)$$

$$\mathrm{Im}\{A_1 x_1(t-t_1) + j \cdot A_2 x_2(t-t_2)\} = A_2 x_2(t-t_2)$$

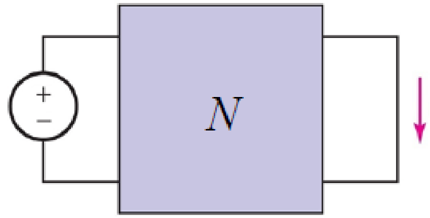


$$\mathrm{Im}\{A_1 y_1(t-t_1) + j \cdot A_2 y_2(t-t_2)\} = A_2 y_2(t-t_2)$$

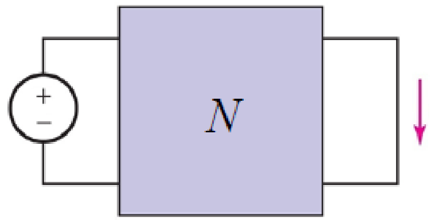
The real part of the response is caused by the real part of the source(s).
The imaginary part of the response is caused by the imaginary part of the source(s).

Consequences of Linearity

$$V_m \cos(\omega t - \theta_1)$$

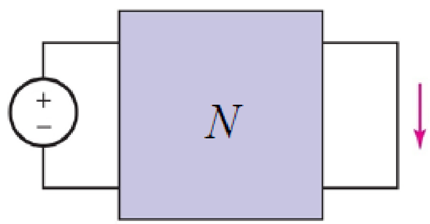


$$j \cdot V_m \sin(\omega t - \theta_1)$$

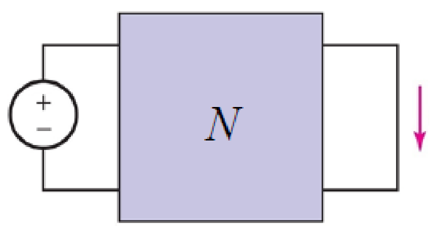


$$e^{j\theta} = \cos(\theta) + j \cdot \sin(\theta)$$

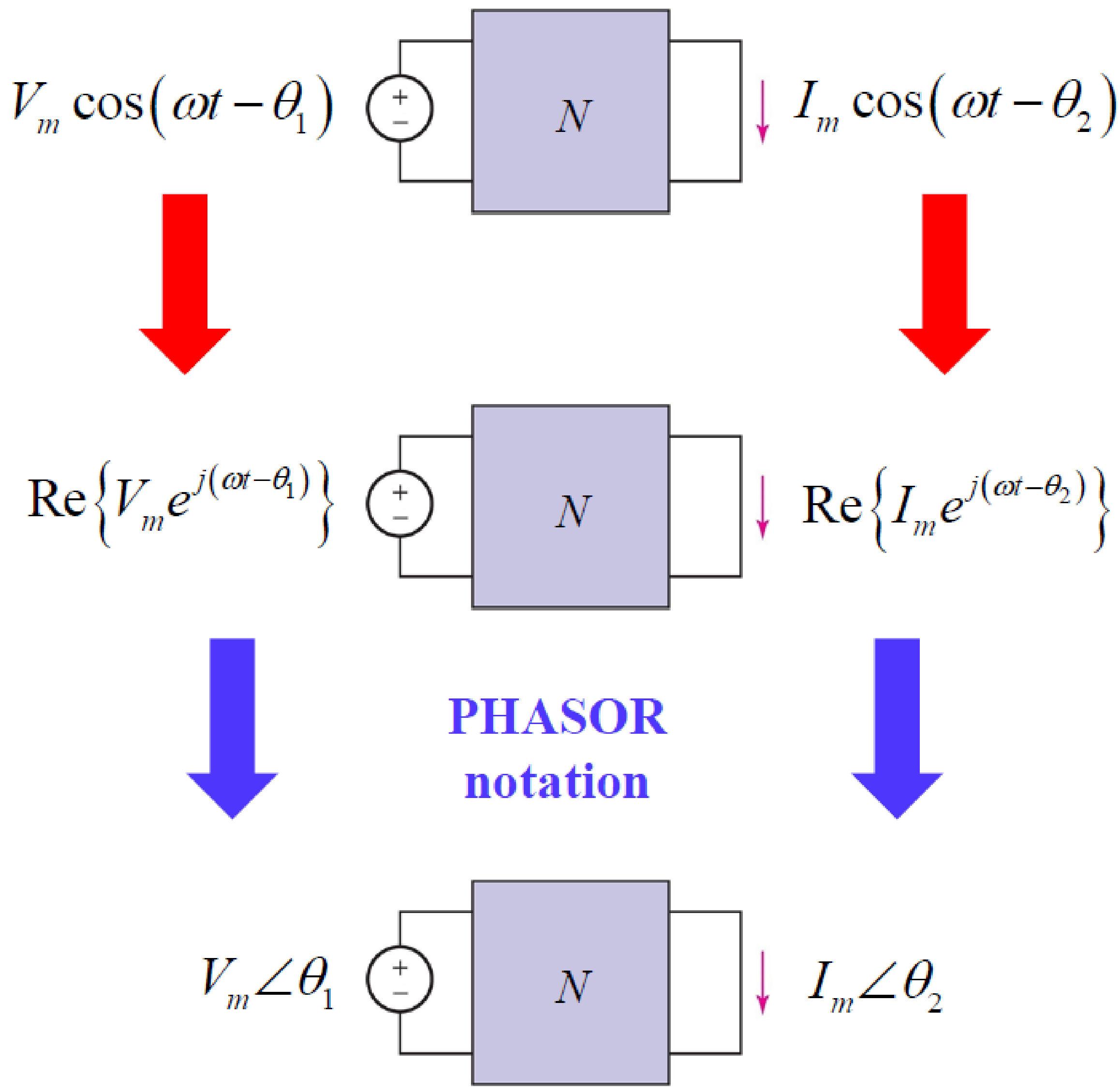
$$V_m e^{j(\omega t - \theta_1)}$$



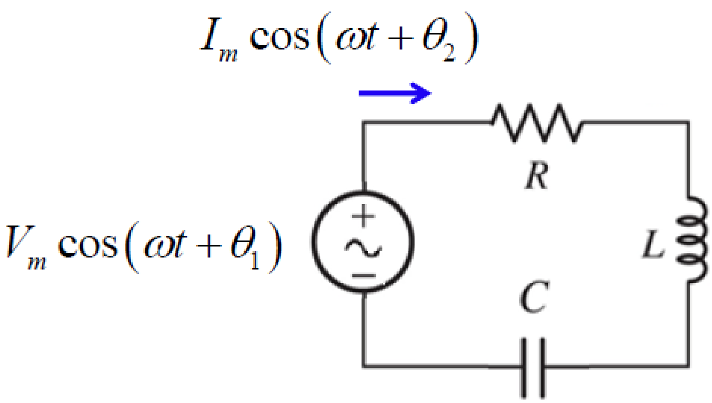
$$\mathrm{Re}\left\{V_m e^{j(\omega t - \theta_1)}\right\}$$

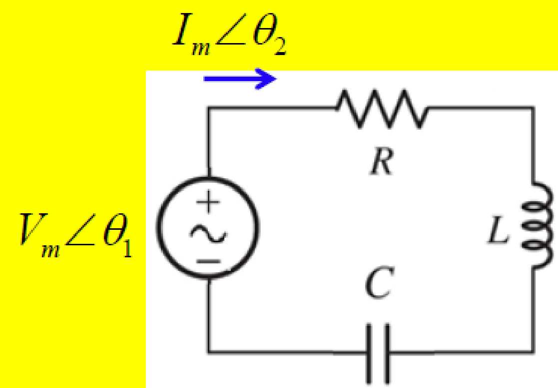


Sinusoidal vs. Complex Representation



Phasor Notation





$$V_m \cos(\omega t + \theta_1) = \mathrm{Re}\left\{ V_m e^{j(\omega t + \theta_1)} \right\} = \mathrm{Re}\left\{ V_m e^{j\omega t} e^{j\theta_1} \right\} \Rightarrow V_m e^{j\theta_1} = V_m \angle \theta_1$$

$$I_m \cos(\omega t + \theta_2) = \mathrm{Re}\left\{ I_m e^{j(\omega t + \theta_2)} \right\} = \mathrm{Re}\left\{ I_m e^{j\omega t} e^{j\theta_2} \right\} \Rightarrow I_m e^{j\theta_2} = I_m \angle \theta_2$$

Assume all voltages & currents oscillate with frequency $\omega = 2\pi f$ ...

Pick off the the amplitude & phase for each v/i ; write each in polar form.

→ **phasor** notation

Phasor Voltage vs. Current: R, L, C

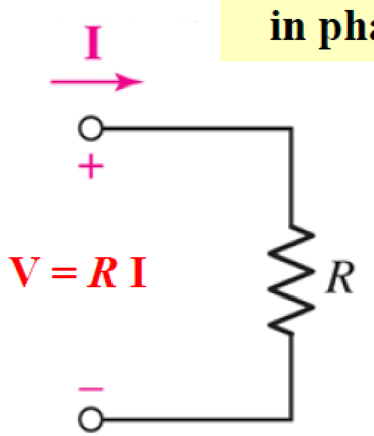
$$v(t) = V_m \cos(\omega t + \theta_1) \qquad i(t) = I_m \cos(\omega t + \theta_2)$$

$$v(t) = R \cdot i(t)$$

For this equation to be true,

$$V_m = I_m \cdot R \quad \text{and} \quad \theta_1 = \theta_2$$

v and *i* are **in phase**

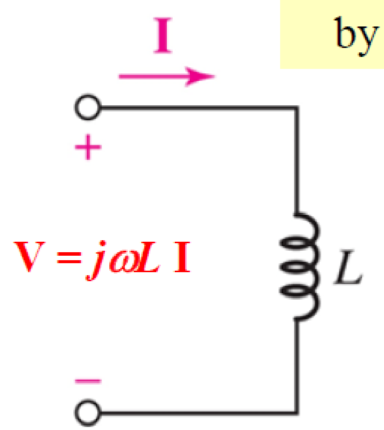


$$v_L(t) = L \cdot \frac{d}{dt} i_L(t)$$

For this equation to be true,

$$\theta_1 = \theta_2 + 90° \ , \quad \frac{V_m}{I_m} = \omega L$$

v* leads *i
by 90°

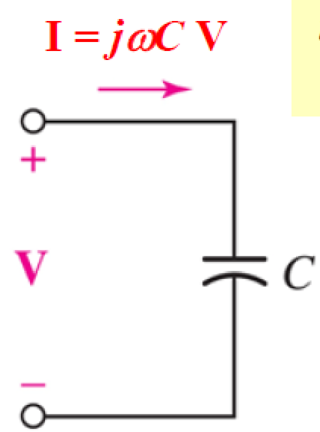


$$i_C(t) = C \cdot \frac{d}{dt} v_C(t)$$

For this equation to be true,

$$\theta_2 = \theta_1 + 90° \ , \quad \frac{I_m}{V_m} = \omega C$$

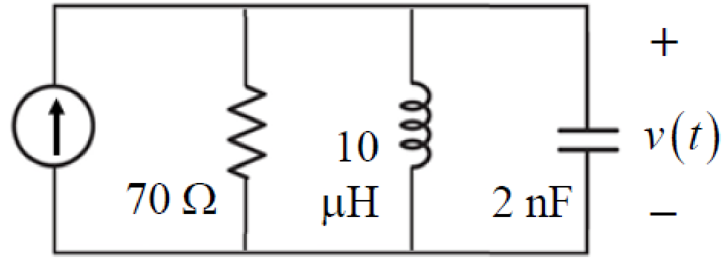
i* leads *v
by 90°



Example 02…

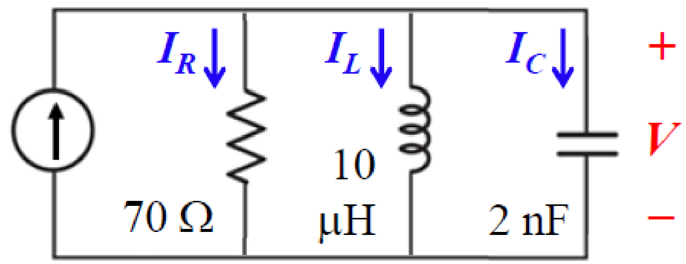
Determine $v(t)$.

$$i_s(t) = 8\cos\left(2\pi \cdot 10^6 t + 30^\circ\right) \text{mA}$$



- Convert to phasor form…

$$\mathbf{I}_s = 8\angle 30^\circ \text{ mA}$$



- Employ the appropriate Kirchhoff Law(s)…

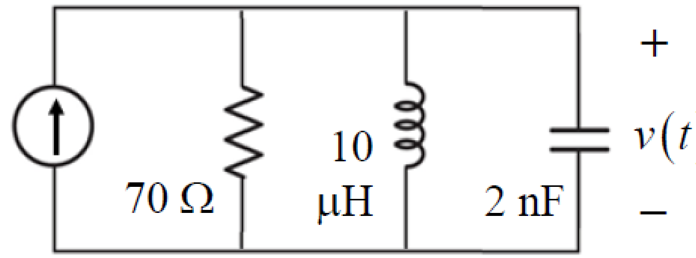
$$\mathbf{I}_s - \frac{\mathbf{V}}{R} - \frac{\mathbf{V}}{j\omega L} - j\omega C \cdot \mathbf{V} = 0$$

$$8\angle 30^\circ - \frac{\mathbf{V}}{70} - \frac{\mathbf{V}}{j\left(2\pi \cdot 10^6\right)\left(10 \cdot 10^{-6}\right)} - j\left(2\pi \cdot 10^6\right)\left(2 \cdot 10^{-9}\right) \cdot \mathbf{V} = 0$$

…Example 02…

Determine *v*(*t*).

$$i_s(t) = 8\cos\left(2\pi\cdot 10^6 t + 30°\right)\text{ mA}$$



- Convert between rectangular & polar forms as necessary…

$$8\angle 30° - \frac{\mathbf{V}}{70} - \frac{\mathbf{V}}{j\left(2\pi\cdot 10^6\right)\left(10\cdot 10^{-6}\right)} - j\left(2\pi\cdot 10^6\right)\left(2\cdot 10^{-9}\right)\cdot\mathbf{V} = 0$$

$$\mathbf{V}\left\{\frac{1}{70} + \frac{1}{j(62.8)} + j(0.0126)\right\} = 8\angle 30°$$

$$\mathbf{V}\cdot(0.0143 - 0.0159j + 0.0126j) = 8\angle 30°$$

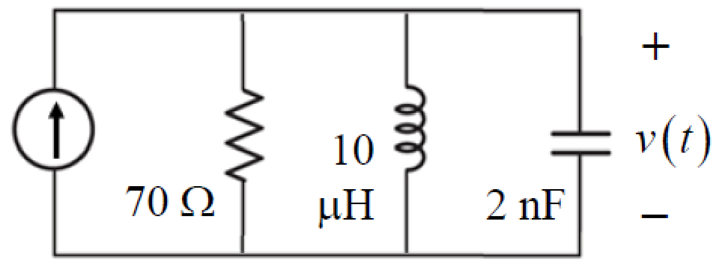
$$\mathbf{V}\cdot(0.0143 - 0.0033j) = 8\angle 30°$$

$$\mathbf{V}\cdot\{0.0147\angle -13°\} = 8\angle 30°$$

…Example 02

Determine $v(t)$.

$$i_s(t) = 8\cos\left(2\pi \cdot 10^6 t + 30°\right)\,\text{mA}$$



- Convert between rectangular & polar forms as necessary…

$$\mathbf{V} \cdot \{0.0147\angle -13°\} = 8\angle 30°$$

$$\mathbf{V} = \frac{8\angle 30° \;\; \text{mA}}{0.0147\angle -13° \;\; \Omega} = 544\angle 43°\,\text{mV}$$

- Convert from phasors to time domain…

```
omega = 2*pi*10^6;
I = 8*exp(j*30*pi/180);

R = 70;
L = 10e-6;
C = 2e-9;

Y = (1/R + 1/(j*omega*L) + j*omega*C);
V = I / Y;

abs(V)
                        ans = 545.2174
angle(V)*180/pi
                        ans = 43.1941
```