Introduction to Digital Logic

Lecture 3

Gate Circuits and Boolean Equations

– Binary Logic and Gates

– Boolean Algebra

– Standard Forms

Binary Logic and Gates

- **Binary variables** take on one of two values.
- **Logical operators** operate on binary values and binary variables.
- Basic logical operators are the **logic functions** AND, OR and NOT.
- **Logic gates** implement logic functions.
- **Boolean Algebra**: a useful mathematical system for specifying and transforming logic functions.
- We study Boolean algebra as foundation for designing and analyzing digital systems!

Binary Variables

- **Recall that the two binary values have different names:**
  - **True/False**
  - **On/Off**
  - **Yes/No**
  - **1/0**
- **We use 1 and 0 to denote the two values.**
- **Variable identifier examples:**
  - **A, B, y, z, or X_1 for now**
  - **RESET, START_IT, or ADD1 later**

Logical Operations

- The three basic logical operations are:
  - AND
  - OR
  - NOT
- AND is denoted by a dot ($\cdot$)
- OR is denoted by a plus ($+$)
- NOT is denoted by an overbar ($\overline{\ }$), a single quote mark (') after, or (~) before the variable

Notation Examples

- **Examples:**
  - **$Y=A.B$ is read "Y is equal to A AND B."**
  - **$z=x+y$ is read "z is equal to x OR y."**
  - **$X=\bar{A}$ is read "X is equal to NOT A."**

- **Note: The statement:**

  $1 + 1 = 2$ (read "one <u>plus</u> one equals two")

  is not the same as

  $1 + 1 = 1$ (read "1 <u>or</u> 1 equals 1").

Operator Definitions

- **Operations are defined on the values "0" and "1" for each operator:**

AND	OR	NOT
$0 \cdot 0 = 0$	$0 + 0 = 0$	$\overline{0} = 1$
$0 \cdot 1 = 0$	$0 + 1 = 1$	$\overline{1} = 0$
$1 \cdot 0 = 0$	$1 + 0 = 1$	
$1 \cdot 1 = 1$	$1 + 1 = 1$	

Truth Tables

- ***Truth table*** **– a tabular listing of the values of a function for all possible combinations of values on its arguments**
- **Example: Truth tables for the basic logic operations:**

AND		
X	Y	$Z = X \cdot Y$
0	0	0
0	1	0
1	0	0
1	1	1

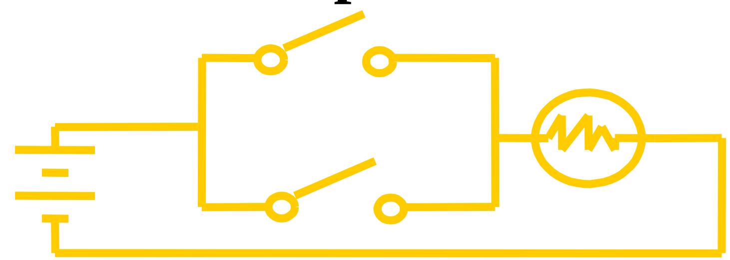
OR		
X	Y	$Z = X + Y$
0	0	0
0	1	1
1	0	1
1	1	1

NOT	
X	$Z = \overline{X}$
0	1
1	0

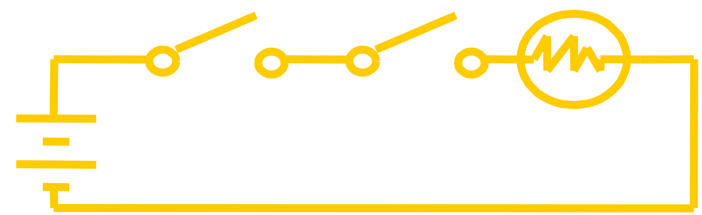
Logic Function Implementation

- **Using Switches**
  - **For inputs:**
    - **logic 1 is switch closed**
    - **logic 0 is switch open**
  - **For outputs:**
    - **logic 1 is light on**
    - **logic 0 is light off.**
  - **NOT uses a switch such that:**
    - **logic 1 is switch open**
    - **logic 0 is switch closed**

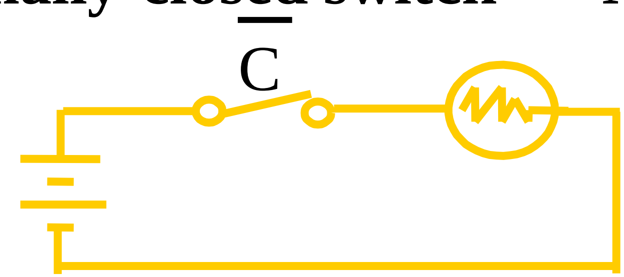
Switches in parallel => OR



Switches in series => AND

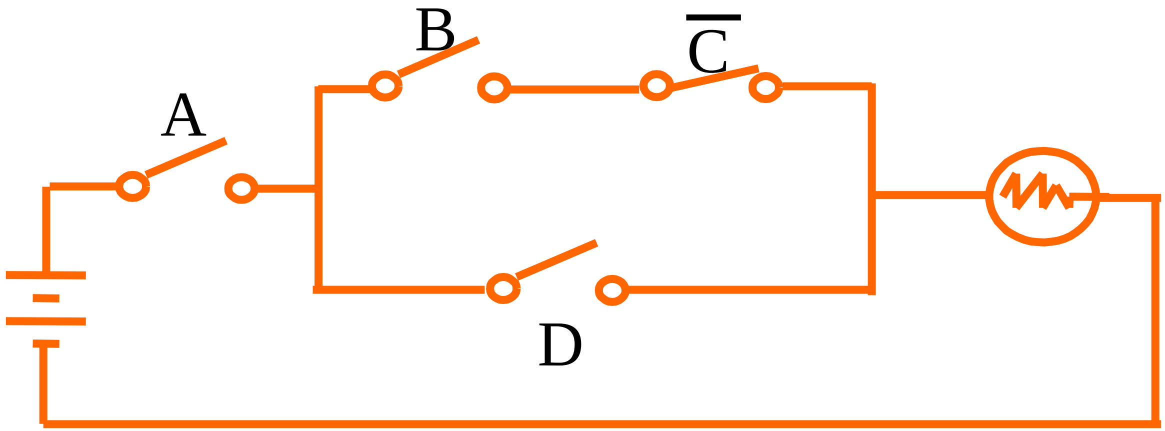


Normally-closed switch => NOT



Logic Function Implementation (Continued)

- **Example: Logic Using Switches**



- **Light is on (L = 1) for**

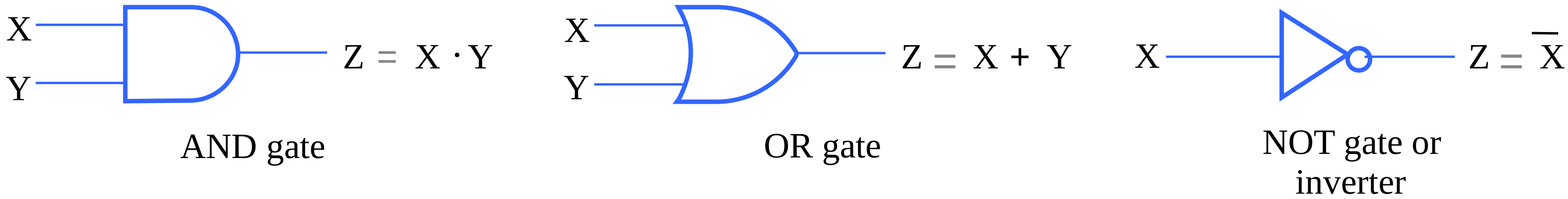
  L(A, B, C, D) = A . ((B . C')+D)

  and off (L = 0), otherwise.

- **Useful model for relay circuits and for CMOS gate circuits, the foundation of current digital logic technology**

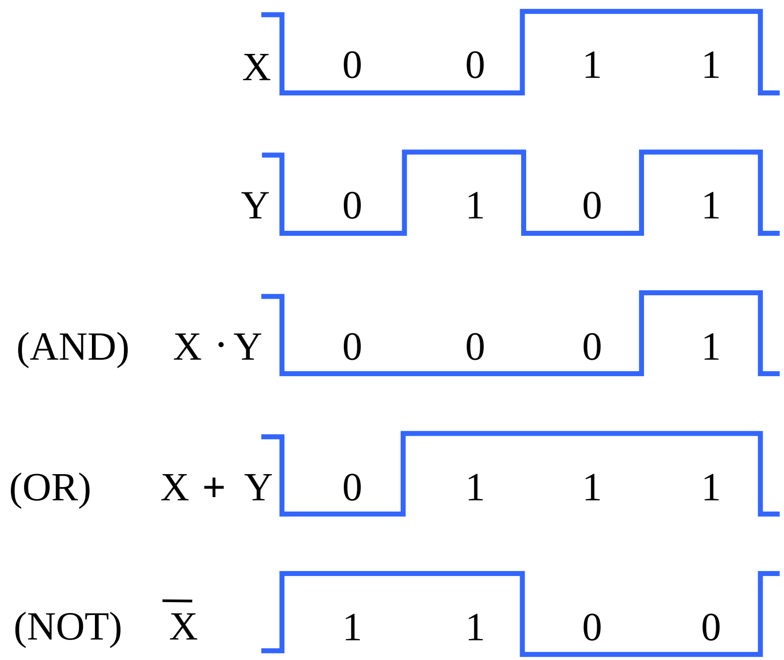
Logic Gate Symbols and Behavior

- **Logic gates have special symbols:**



(a) Graphic symbols

- **And waveform behavior in time as follows:**



(b) Timing diagram

Logic Diagrams and Expressions

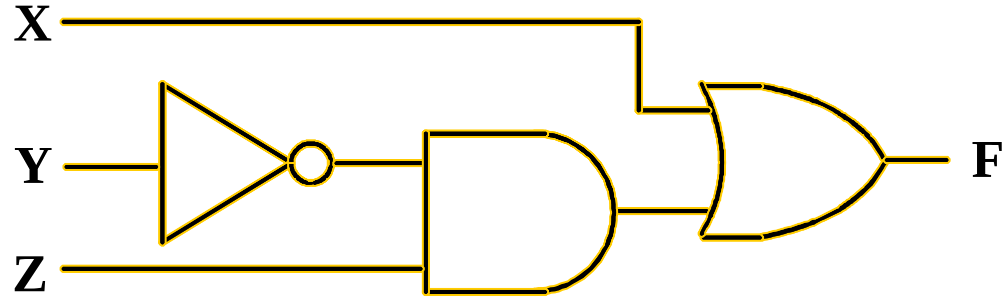
Truth Table

X Y Z	$F = X + \overline{Y} \cdot Z$
0 0 0	0
0 0 1	1
0 1 0	0
0 1 1	0
1 0 0	1
1 0 1	1
1 1 0	1
1 1 1	1

Equation

$$F = X + \overline{Y}\,Z$$

Logic Diagram



- **Boolean equations, truth tables and logic diagrams describe the same function!**
- **Truth tables are unique; expressions and logic diagrams are not. This gives flexibility in implementing functions.**

Boolean Algebra

- An algebraic structure defined on a set of at least two elements, B, together with three binary operators (denoted $+$, $\cdot$ and $\overline{}$) that satisfies the following basic identities:

1.	$X + 0 = X$	2.	$X \cdot 1 = X$	Existence of 0 and1
3.	$X + 1 = 1$	4.	$X \cdot 0 = 0$	
5.	$X + X = X$	6.	$X \cdot X = X$	Idempotence
7.	$X + \overline{X} = 1$	8.	$X \cdot \overline{X} = 0$	Existence of complement
9.	$\overline{\overline{X}} = X$			Involution
10.	$X + Y = Y + X$	11.	$XY = YX$	Commutative
12.	$(X + Y) + Z = X + (Y + Z)$	13.	$(XY)Z = X(YZ)$	Associative
14.	$X(Y + Z) = XY + XZ$	15.	$X + YZ = (X + Y)(X + Z)$	Distributive
16.	$\overline{X + Y} = \overline{X} \cdot \overline{Y}$	17.	$\overline{X \cdot Y} = \overline{X} + \overline{Y}$	DeMorgan's

Boolean Operator Precedence

- **The order of evaluation in a Boolean expression is:**
  1. Parentheses
  2. NOT
  3. AND
  4. OR
- **Consequence: Parentheses appear around OR expressions**
- **Example:** $F = A(B + C)(C + \overline{D})$

Example 1: Boolean Algebraic Proof

- $A + A \cdot B = A$ (Absorption Theorem)

Proof Steps	Justification (identity or theorem)
$A + A \cdot B$	
$= A \cdot 1 + A \cdot B$	$X = X \cdot 1$
$= A \cdot (1 + B)$	$X \cdot Y + X \cdot Z = X \cdot (Y + Z)$ (Distributive Law)
$= A \cdot 1$	$1 + X = 1$
$= A$	$X \cdot 1 = X$

- **Our primary reason for doing proofs is to learn:**
  - Careful and efficient use of the identities and theorems of Boolean algebra, and
  - How to choose the appropriate identity or theorem to apply to make forward progress, irrespective of the application.

Example 2: Boolean Algebraic Proofs

- **$AB + \overline{A}C + BC = AB + \overline{A}C$ (Consensus Theorem)**

Proof Steps: Justification (identity or theorem)

$AB + \overline{A}C + BC$

$= AB + \overline{A}C + 1 \cdot BC$

$= AB + \overline{A}C + (A + \overline{A}) \cdot BC$

$= AB + \overline{A}C + ABC + \overline{A}BC$

$= AB\,(1+C) + \overline{A}C\,(1+B)$

$= AB \cdot 1 + \overline{A}C \cdot 1$

$= AB + \overline{A}C$

Example 3: Boolean Algebraic Proofs

- $(\overline{X+Y})Z + X\overline{Y} = \overline{Y}(X+Z)$

Proof Steps **Justification (identity or theorem)**

$(\overline{X + Y})Z + X\overline{Y}$

$=$

Useful Theorems

$$x \cdot y + \bar{x} \cdot y = y \qquad (x + y)(\bar{x} + y) = y \qquad \textbf{Minimization}$$

$$x + x \cdot y = x \qquad x \cdot (x + y) = x \qquad \textbf{Absorption}$$

$$x + \bar{x} \cdot y = x + y \qquad x \cdot (\bar{x} + y) = x \cdot y \qquad \textbf{Simplification}$$

$$x \cdot y + \bar{x} \cdot z + y \cdot z = x \cdot y + \bar{x} \cdot z \qquad \textbf{Consensus}$$

$$(x + y) \cdot (\bar{x} + z) \cdot (y + z) = (x + y) \cdot (\bar{x} + z)$$

$$\overline{x + y} = \bar{x} \cdot \bar{y} \qquad \overline{x \cdot y} = \bar{x} + \bar{y} \qquad \textbf{DeMorgan's Laws}$$

Proof of Simplification

$$x \cdot y + \bar{x} \cdot y = y$$

$$(x + y)(\bar{x} + y) = y$$

Boolean Function Evaluation

$F1 = xy\bar{z}$

$F2 = x + \bar{y}z$

$F3 = \bar{x}\bar{y}\bar{z} + \bar{x}yz + x\bar{y}$

$F4 = x\bar{y} + \bar{x}z$

x	y	z	F1	F2	F3	F4
0	0	0	0	0	1	0
0	0	1	0	1	0	1
0	1	0	0	0	0	0
0	1	1	0	0	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	1	0	0
1	1	1	0	1	0	0

Expression Simplification

- **An application of Boolean algebra**
- **Simplify to contain the smallest number of literals (complemented and uncomplemented variables):**

$$AB + \overline{A}CD + \overline{A}BD + \overline{A}C\overline{D} + ABCD$$

$$= AB + ABCD + \overline{A}CD + \overline{A}C\overline{D} + \overline{A}BD$$

$$= AB + AB(CD) + \overline{A}C(D + \overline{D}) + \overline{A}BD$$

$$= AB + \overline{A}C + \overline{A}BD = B(A + \overline{A}D) + \overline{A}C$$

$$= B(A + D) + \overline{A}C \quad \text{5 literals}$$

Complementing Functions

- **Use DeMorgan's Theorem to complement a function:**
  1. **Interchange AND and OR operators**
  2. **Complement each constant value and literal**
- **Example: Complement F = $\bar{x}y\bar{z}+x\bar{y}\bar{z}$**

  $\bar{F} = (x + \bar{y} + z)(\bar{x} + y + z)$
- **Example: Complement G = $(\bar{a} + bc)\bar{d} + e$**

  $\bar{G} = ?$

Overview – Canonical Forms

- **What are Canonical Forms?**
- **Minterms and Maxterms**
- **Index Representation of Minterms and Maxterms**
- **Sum-of-Minterm (SOM) Representations**
- **Product-of-Maxterm (POM) Representations**
- **Representation of Complements of Functions**
- **Conversions between Representations**

Canonical Forms

- **It is useful to specify Boolean functions in a form that:**
  - **Allows comparison for equality.**
  - **Has a correspondence to the truth tables**
- **Canonical Forms in common usage:**
  - **Sum of Minterms (SOM)**
  - **Product of Maxterms (POM)**

Minterms

- <u>Minterms</u> are AND terms with every variable present in either true or complemented form.
- Given that each binary variable may appear normal (e.g., x) or complemented (e.g., $\bar{x}$), there are 2^n minterms for *n* variables.
- <u>Example:</u> Two variables (X and Y)produce 2 x 2 = 4 combinations:

  XY (both normal)
  $X\bar{Y}$ (X normal, Y complemented)
  $\bar{X}Y$ (X complemented, Y normal)
  $\bar{X}\bar{Y}$ (both complemented)

- Thus there are <u>four minterms</u> of two variables.

Maxterms

- **<u>Maxterms</u> are OR terms with every variable in true or complemented form.**
- **Given that each binary variable may appear normal (e.g., x) or complemented (e.g., $\overline{x}$), there are 2^n maxterms for *n* variables.**
- **<u>Example:</u> Two variables (X and Y) produce 2 x 2 = 4 combinations:**

$X+Y$	(both normal)
$X+\overline{Y}$	(X normal, Y complemented)
$\overline{X}+Y$	(X complemented, Y normal)
$\overline{X}+\overline{Y}$	(both complemented)

Maxterms and Minterms

- **Examples: Two variable minterms and maxterms.**

Index	Minterm	Maxterm
0	$\overline{x}\,\overline{y}$	$x + y$
1	$\overline{x}\,y$	$x + \overline{y}$
2	$x\,\overline{y}$	$\overline{x} + y$
3	$x\,y$	$\overline{x} + \overline{y}$

- **The index above is important for describing which variables in the terms are true and which are complemented.**

Purpose of the Index

- **The <u>index</u> for the minterm or maxterm, expressed as a binary number, is used to determine whether the variable is shown in the true form or complemented form.**
- **For Minterms:**
  - **"1" means the variable is "Not Complemented" and**
  - **"0" means the variable is "Complemented".**
- **For Maxterms:**
  - **"0" means the variable is "Not Complemented" and**
  - **"1" means the variable is "Complemented".**

Index Example in Three Variables

- **Example: (for three variables)**
- **Assume the variables are called X, Y, and Z.**
- **The standard order is X, then Y, then Z.**
- **The Index 0 (base 10) = 000 (base 2) for three variables). All three variables are complemented for minterm 0 ($\overline{X}, \overline{Y}, \overline{Z}$) and no variables are complemented for Maxterm 0 (X,Y,Z).**
  - **Minterm 0, called m_0 is $\overline{X}\,\overline{Y}\,\overline{Z}$.**
  - **Maxterm 0, called M_0 is (X + Y + Z).**
  - **Minterm 6 ?**
  - **Maxterm 6 ?**

Index Examples – Four Variables

Index i	Binary Pattern	Minterm m_i	Maxterm M_i
0	0000	$\bar{a}\bar{b}\bar{c}\bar{d}$	$a+b+c+d$
1	0001	$\bar{a}\bar{b}\bar{c}d$	?
3	0011	?	$a+b+\bar{c}+\bar{d}$
5	0101	$\bar{a}b\bar{c}d$	$a+\bar{b}+c+\bar{d}$
7	0111	?	$a+\bar{b}+\bar{c}+\bar{d}$
10	1010	$a\bar{b}c\bar{d}$	$\bar{a}+b+\bar{c}+d$
13	1101	$ab\bar{c}d$	?
15	1111	$abcd$	$\bar{a}+\bar{b}+\bar{c}+\bar{d}$

Minterm and Maxterm Relationship

- **Review: DeMorgan's Theorem**
  $\overline{x \cdot y} = \overline{x} + \overline{y}$ **and** $\overline{x + y} = \overline{x} \cdot \overline{y}$
- **Two-variable example:**
  $M_2 = \overline{x} + y$ **and** $m_2 = x \cdot \overline{y}$
  Thus M_2 is the complement of m_2 and vice-versa.
- **Since DeMorgan's Theorem holds for *n* variables, the above holds for terms of *n* variables**
- **giving:**

$$M_i = \overline{m_i} \text{ and } m_i = \overline{M_i}$$

Thus M_i is the complement of m_i.

Minterm Function Example

- Example: Find $F_1 = m_1 + m_4 + m_7$
- $F1 = \overline{x}\ \overline{y}\ z + x\ \overline{y}\ \overline{z} + x\ y\ z$

x y z	index	m_1	+	m_4	+	m_7	$= F_1$
0 0 0	0	0	+	0	+	0	= 0
0 0 1	1	1	+	0	+	0	= 1
0 1 0	2	0	+	0	+	0	= 0
0 1 1	3	0	+	0	+	0	= 0
1 0 0	4	0	+	1	+	0	= 1
1 0 1	5	0	+	0	+	0	= 0
1 1 0	6	0	+	0	+	0	= 0
1 1 1	7	0	+	0	+	1	= 1

Minterm Function Example

- $F(A, B, C, D, E) = m_2 + m_9 + m_{17} + m_{23}$
- $F(A, B, C, D, E) =$

Maxterm Function Example

- **Example: Implement F1 in maxterms:**

$$F_1 = M_0 \cdot M_2 \cdot M_3 \cdot M_5 \cdot M_6$$

$$F_1 = (x + y + z) \cdot (x + \overline{y} + z) \cdot (x + \overline{y} + \overline{z}) \cdot (\overline{x} + y + \overline{z}) \cdot (\overline{x} + \overline{y} + z)$$

x y z	i	$M_0 \cdot M_2 \cdot M_3 \cdot M_5 \cdot M_6$ = F1
0 0 0	0	0 · 1 · 1 · 1 · 1 = 0
0 0 1	1	1 · 1 · 1 · 1 · 1 = 1
0 1 0	2	1 · 0 · 1 · 1 · 1 = 0
0 1 1	3	1 · 1 · 0 · 1 · 1 = 0
1 0 0	4	1 · 1 · 1 · 1 · 1 = 1
1 0 1	5	1 · 1 · 1 · 0 · 1 = 0
1 1 0	6	1 · 1 · 1 · 1 · 0 = 0
1 1 1	7	1 · 1 · 1 · 1 · 1 = 1

Maxterm Function Example

- $F(A,B,C,D) = M_3 \cdot M_8 \cdot M_{11} \cdot M_{14}$
- $F(A, B,C,D) =$

Canonical Sum of Minterms

- **Any Boolean function can be expressed as a <u>Sum of Minterms</u>.**
  - **For the function table, the <u>minterms</u> used are the terms corresponding to the 1's**
  - **For expressions, <u>expand</u> all terms first to explicitly list all minterms. Do this by "ANDing" any term missing a variable v with a term $(v + \bar{v})$.**
- **Example: Implement $f = x + \bar{x}\,\bar{y}$ as a sum of minterms.**

  First expand terms: $f = x(y + \bar{y}) + \bar{x}\,\bar{y}$

  Then distribute terms: $f = xy + x\bar{y} + \bar{x}\,\bar{y}$

  Express as sum of minterms: $f = m_3 + m_2 + m_0$

Another SOM Example

- **Example:** $F = A + \overline{B}\,C$
- **There are three variables, A, B, and C which we take to be the standard order.**
- **Expanding the terms with missing variables:**
- **Collect terms (removing all but one of duplicate terms):**
- **Express as SOM:**

Shorthand SOM Form

- From the previous example, we started with:

$$F = A + \overline{B}\,C$$

- We ended up with:

$$F = m_1 + m_4 + m_5 + m_6 + m_7$$

- This can be denoted in the formal shorthand:

$$F(A, B, C) = \Sigma_m(1,4,5,6,7)$$

- Note that we explicitly show the standard variables in order and drop the "m" designators.

Canonical Product of Maxterms

- **Any Boolean Function can be expressed as a <u>Product of Maxterms (POM)</u>.**
  - **For the function table, the maxterms used are the terms corresponding to the 0's.**
  - **For an expression, expand all terms first to explicitly list all maxterms. Do this by first applying the second distributive law , "ORing" terms missing variable V with a term equal to $V \cdot \overline{V}$ and then applying the distributive law again.**
- **Example: Convert to product of maxterms:**

$$f(x, y, z) = x + \overline{x}\,\overline{y}$$

Apply the distributive law:

$$x + \overline{x}\,\overline{y} = (x + \overline{x})(x + \overline{y}) = 1 \cdot (x + \overline{y}) = x + \overline{y}$$

Add missing variable z:

$$x + \overline{y} + z \cdot \overline{z} = (x + \overline{y} + z)(x + \overline{y} + \overline{z})$$

Express as POM: $f = M_2 \cdot M_3$

Another POM Example

- **Convert to Product of Maxterms:**

$$f(A,B,C) = A\overline{C} + BC + \overline{A}\,\overline{B}$$

- **Use** $x + yz = (x+y)\cdot(x+z)$ **with** $x = (A\overline{C} + BC)$, $y = \overline{A}$, **and** $z = \overline{B}$ **to get:**

$$f = (A\overline{C} + BC + \overline{A})(A\overline{C} + BC + \overline{B})$$

- **Then use** $x + \overline{x}y = x + y$ **to get:**

$$f = (\overline{C} + BC + \overline{A})(A\overline{C} + C + \overline{B})$$

and a second time to get:

$$f = (\overline{C} + B + \overline{A})(A + C + \overline{B})$$

- **Rearrange to standard order,**

$$f = (\overline{A} + B + \overline{C})(A + \overline{B} + C)$$ **to give** $f = M_5 \cdot M_2$

Function Complements

- The complement of a function expressed as a sum of minterms is constructed by selecting the minterms missing in the sum-of-minterms canonical forms.
- Alternatively, the complement of a function expressed by a Sum of Minterms form is simply the Product of Maxterms with the same indices.
- Example: Given $F(x, y, z) = \Sigma_m(1,3,5,7)$

$$\overline{F}(x, y, z) = \Sigma_m(0,2,4,6)$$
$$\overline{F}(x, y, z) = \Pi_M(1,3,5,7)$$

Conversion Between Forms

- **To convert between sum-of-minterms and product-of-maxterms form (or vice-versa) we follow these steps:**
  - **Find the function complement by swapping terms in the list with terms not in the list.**
  - **Change from products to sums, or vice versa.**
- **Example: Given F as before:** $F(x, y, z) = \Sigma_m(1,3,5,7)$
- **Form the Complement:** $\overline{F}(x, y, z) = \Sigma_m(0,2,4,6)$
- **Then use the other form with the same indices – this forms the complement again, giving the other form of the original function:** $F(x, y, z) = \Pi_M(0,2,4,6)$

Standard Forms

- **Standard Sum-of-Products (SOP) form:** equations are written as an OR of AND terms
- **Standard Product-of-Sums (POS) form:** equations are written as an AND of OR terms
- **Examples:**
  - SOP: $A\,B\,C + \overline{A}\,\overline{B}\,C + B$
  - POS: $(A+B)\cdot(A+\overline{B}+\overline{C})\cdot C$
- **These "mixed" forms are neither SOP nor POS**
  - $(A\,B + C)\,(A + C)$
  - $A\,B\,\overline{C} + A\,C\,(A+B)$

Standard Sum-of-Products (SOP)

- **A sum of minterms form for *n* variables can be written down directly from a truth table.**
  - **Implementation of this form is a two-level network of gates such that:**
  - **The first level consists of *n*-input AND gates, and**
  - **The second level is a single OR gate (with fewer than 2^n inputs).**
- **This form often can be simplified so that the corresponding circuit is simpler.**

Standard Sum-of-Products (SOP)

- **A Simplification Example:**
- $F(A, B, C) = \Sigma m(1,4,5,6,7)$
- **Writing the minterm expression:**

$$F = \overline{A}\,\overline{B}\,C + A\,\overline{B}\,\overline{C} + A\,\overline{B}\,C + AB\overline{C} + ABC$$

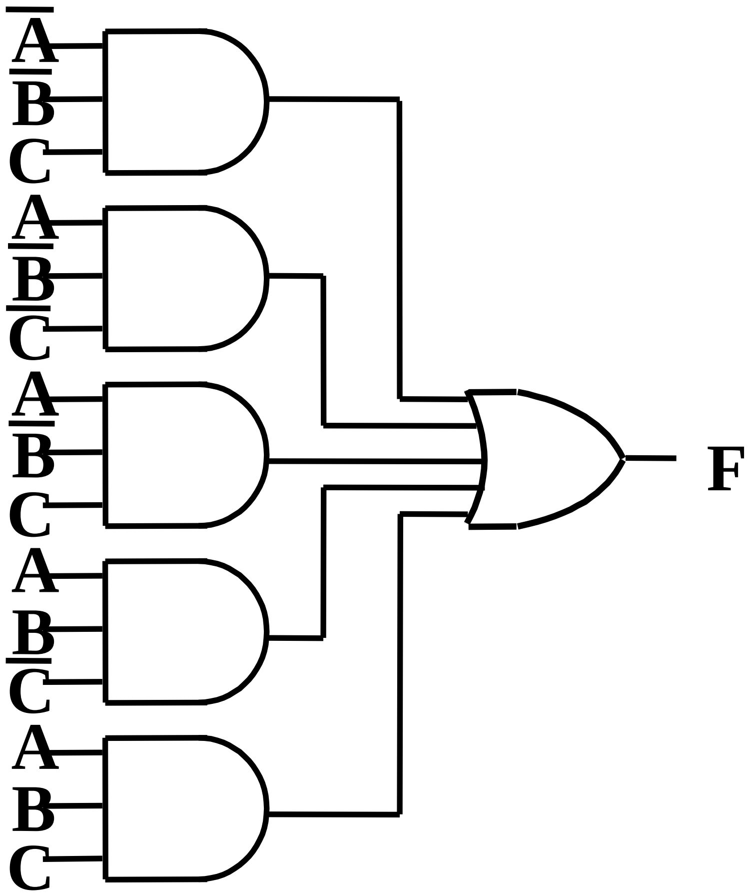
- **Simplifying:**

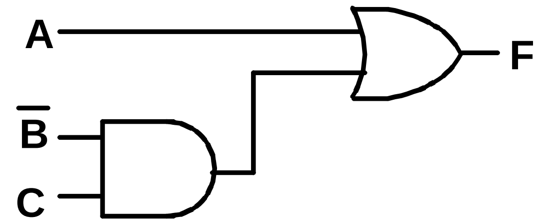
$$F = A + \overline{B}C$$

- **Simplified F contains 3 literals compared to 15 in minterm F**

AND/OR Two-level Implementation of SOP Expression

- **The two implementations for F are shown below – it is quite apparent which is simpler!**





- **The previous examples show that:**
  - **Canonical Forms (Sum-of-minterms, Product-of-Maxterms), or other standard forms (SOP, POS) differ in complexity**
  - **Boolean algebra can be used to manipulate equations into simpler forms.**
  - **Simpler equations lead to simpler two-level implementations**
- **Questions:**
  - **How can we attain a “simplest” expression?**
  - **Is there only one minimum cost circuit?**
  - **The next part will deal with these issues.**