Math 223 Practice Midterm 2
Fall 2018
University of Toronto

Family (last) name: ______________________________

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Instructions:

- This examination consists of 8 questions for a total of 122 points.
- You have one hour and fifty minutes to complete this examination.
- No aids are permitted. Do not use books, notes, calculators, computers, tablets or phones.
- Write legibly and darkly.
- Cross out any work that you do not wish to have scored.
- Show all of your work. Unsupported answers may not earn credit.

1. Complete the following sentences with a mathematically correct definition. No marks will be awarded for a "close" but incorrect definition.

   (a) (2 points) The *component* of the vector $\vec{v}$ in the direction of $\vec{u}$ is

   (b) (2 points) A subset $V \subseteq \mathbb{R}^n$ is a *subspace* if

   (c) (2 points) The set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a *basis* of the subspace $X \subseteq \mathbb{R}^n$ if

   (d) (2 points) The function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a *linear transformation* if

   (e) (2 points) The *range* of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is

(f) (2 points) The *image* of $X \subset \mathbb{R}^n$ under a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is

(g) (2 points) The *null space* of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is

(h) (2 points) The $n \times n$ *identity matrix* is

(i) (2 points) The *row space* of a matrix M is

(j) (2 points) The *dimension* of a subspace $V \subseteq \mathbb{R}^n$ is

2. Let $\mathcal{P} \subseteq \mathbb{R}^4$ be the plane with equations $x + y - z - w = 2$, $x + y + z + w = 0$ and let $\mathcal{Q} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

(a) (3 points) Is $\mathcal{P}$ a subspace? Prove your answer.

(b) (3 points) Is $\mathcal{Q}$ a subspace? Prove your answer.

(c) (3 points) Is $\mathcal{P} \cap \mathcal{Q}$ a subspace? Prove your answer.

(d) (3 points) Is $\mathcal{P} \cup \mathcal{Q}$ a subspace? Prove your answer.

(e) (3 points) Is there any vector $\vec{v}$ such that $\mathcal{R} = \mathcal{P} + \{\vec{v}\}$ is a subspace? If so, give an example of such a vector and a basis for $\mathcal{R}$. If not, prove that that there is no such vector.

(f) (3 points) Find the projection of the point $\begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ on $\mathcal{Q}$.

3. For each of the following, give an example if possible. Otherwise, explain why it is impossible.

(a) (2 points) A subset $X \subset \mathbb{R}^2$ such that $\text{proj}_X \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

(b) (2 points) A basis $\mathcal{B}$ such that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}_{\mathcal{B}}$, where $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$.

(c) (2 points) A 2×2 matrix whose null space equals its column space.

(d) (2 points) A 2×2 matrix whose null space equals its row space.

(e) (2 points) A 2×2 matrix whose row space equals its column space.

(f) (2 points) A linear transformation from $T : \mathbb{R}^2 \to \mathbb{R}^1$ whose null space is $\{\vec{0}\}$.

(g) (2 points) A linear transformation from $T : \mathbb{R}^2 \to \mathbb{R}^1$ whose null space equals to its range.

(h) (2 points) A non-linear transformation from $U : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $U(\vec{0}) = \vec{0}$ and whose range is a subspace.

(i) (2 points) A non-linear transformation from $U : \mathbb{R}^2 \to \mathbb{R}^2$ whose range is a single vector.

(j) (2 points) A vector $\vec{u} \in \mathbb{R}^2$ such that $\left\{ \vec{u},\, \vec{u} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is a *positively* oriented (i.e., right-handed) basis.

(k) (2 points) A linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ such that $A\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

(l) (2 points) 2×2 matrices A, B, C so that $AB = BA$ but $AC \neq CA$.

4. Let $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$ be the standard basis for $\mathbb{R}^2$. Let $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ and $\mathcal{D} = \{\vec{d}_1, \vec{d}_2\}$ such that $[\vec{c}_1]_{\mathcal{E}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $[\vec{c}_2]_{\mathcal{E}} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, $[\vec{d}_1]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $[\vec{d}_2]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

(a) (2 points) Is $\mathcal{C}$ a basis? Explain using the definition of basis.

(b) (2 points) Is $\mathcal{D}$ a basis? Explain using the definition of basis.

(c) (2 points) Find $[\vec{v}]_{\mathcal{E}}$ where $[\vec{v}]_{\mathcal{D}} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

(d) (2 points) Find $[\vec{w}]_{\mathcal{D}}$ where $[\vec{w}]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(e) (4 points) Is there a non-zero vector $\vec{q}$ so that $[\,\vec{c}_1 + \vec{q}\,]_{\mathcal{C}} = [\,2\vec{q}\,]_{\mathcal{D}}$? If there is, find such a $\vec{q}$, if not, prove there isn't one.

5. For each transformation below, prove whether or not it is a *linear* transformation. If the transformation is linear, find the standard matrix for the transformation. Throughout this question, $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$ is the standard basis for $\mathbb{R}^2$.

   (a) (4 points) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x + y + 1 \end{bmatrix}$

   (b) (4 points) $S : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $S(\vec{v}) = (\vec{v} \cdot \vec{e}_1)\vec{e}_2 + (\vec{v} \cdot \vec{e}_2)\vec{e}_1$.

(c) (4 points) $R : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $R(\vec{v}) = \vec{v} + \vec{w}$ for some $\vec{w} \neq 0$.

(d) (4 points) $U : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $U(\vec{v}) = \text{comp}_{\vec{w}}(\vec{v})$ for some $\vec{w} \neq 0$.

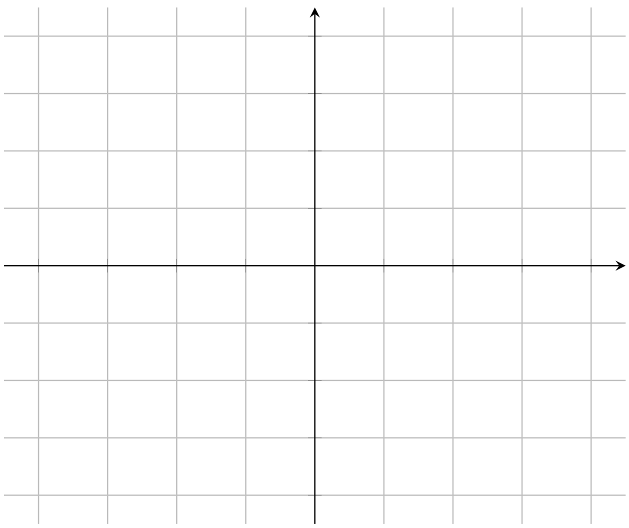
6. Let

$$S \text{ be the filled square with vertices at } \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}-2\\0\end{bmatrix}, \begin{bmatrix}-2\\-2\end{bmatrix}, \text{ and } \begin{bmatrix}0\\-2\end{bmatrix}; \qquad \vec{a} = \begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\end{bmatrix};$$

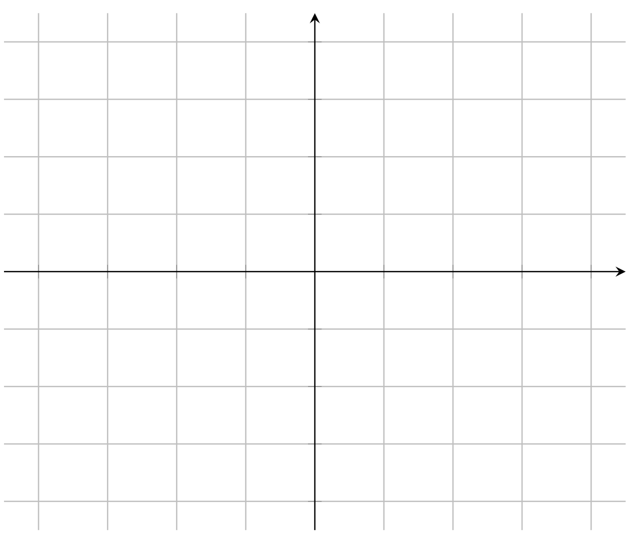
$$T : \mathbb{R}^2 \to \mathbb{R}^2 \text{ defined by } T(\vec{v}) = \begin{bmatrix}-2 & 1\\0 & 1\end{bmatrix}\vec{v} \qquad \text{and} \qquad \ell = \operatorname{span}\{\vec{a}\}$$

Draw the following subsets of $\mathbb{R}^2$.

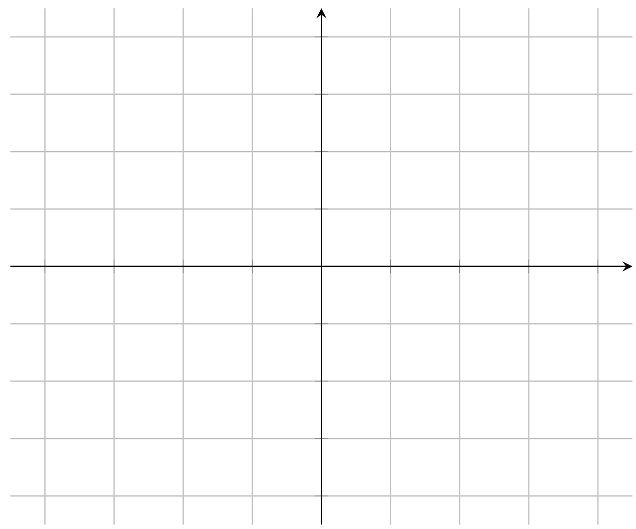
(a) (3 points) $\operatorname{proj}_S \vec{a}$



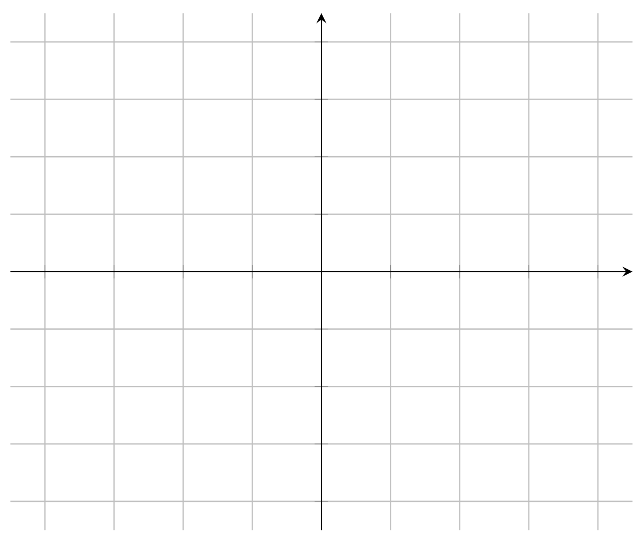
(b) (3 points) $\operatorname{proj}_\ell(S)$



(c) (3 points) $T(\ell)$



(d) (3 points) $T(S)$



7. Let $\mathcal{P} \subseteq \mathbb{R}^3$ be the plane given by the equation $2x + y - z = 1$. For each statement below, circle CORRECT if it is a correct mathematical statement or INCORRECT otherwise.

(a) (1 point) $\mathcal{P} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : 2x + y - z = 1 \right\}$

CORRECT **INCORRECT**

(b) (1 point) $\mathcal{P} = \left\{ \vec{v} \in \mathbb{R}^3 : \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \cdot \vec{v} = 1 \right\}$

CORRECT **INCORRECT**

(c) (1 point) $\mathcal{P} = \left\{ \vec{x} \in \mathbb{R}^3 : \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \cdot \left(\vec{x} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = 0 \right\}$

CORRECT **INCORRECT**

(d) (1 point) $\mathcal{P} = \left\{ \vec{x} \in \mathbb{R}^3 : \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \cdot (\vec{x} - \vec{p}) = 0 \text{ for some } \vec{p} \in \mathbb{R}^3 \right\}$

CORRECT **INCORRECT**

(e) (1 point) $\mathcal{P} = \left\{ \vec{v} \in \mathbb{R}^3 : \vec{v} = s \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$

CORRECT **INCORRECT**

(f) (1 point) $\mathcal{P} = \left\{ \vec{v} \in \mathbb{R}^3 : \vec{v} = s \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \text{ for all } s,\, t \in \mathbb{R} \right\}$

CORRECT **INCORRECT**

(g) (1 point) $\mathcal{P}$ is expressed in vector form as $\mathcal{P} = s \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$.

CORRECT **INCORRECT**

(h) (1 point) $\mathcal{P}$ is expressed in vector form as $\vec{x} = s \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ for some t, s.

CORRECT **INCORRECT**

(i) (1 point) $\mathcal{P}$ is expressed in normal form as $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \cdot \left(\mathcal{P} - \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \right) = 0.$

CORRECT **INCORRECT**

(j) (1 point) $\mathcal{P} = \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \right\} + \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

CORRECT **INCORRECT**

(k) (1 point) $\mathcal{P} = \operatorname{span}\{\ell_1 \cup \ell_2\}$, where ℓ_1 is given in vector form by $\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ and ℓ_2 is given in vector form by $\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

CORRECT **INCORRECT**

(l) (1 point) $\mathcal{P} = \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} + \operatorname{span} \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \right\} + \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

CORRECT **INCORRECT**

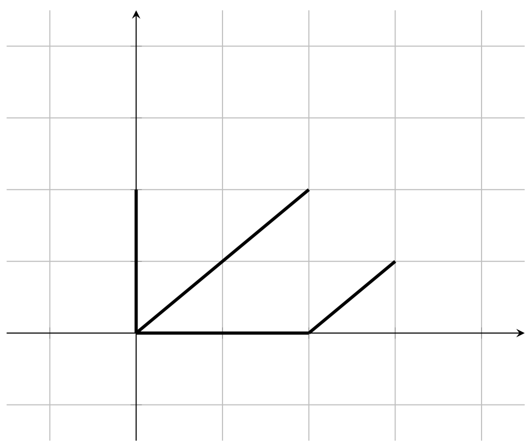
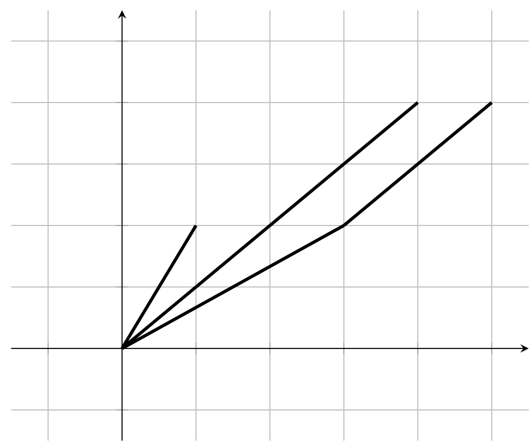
(m) (1 point) $\mathcal{P}$ is the set of vectors orthogonal to $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

CORRECT **INCORRECT**

8. (7 points) On this question, you will be graded on your *communication* as well as your *correctness*.

   Note: part of good communication is writing in complete sentences, explaining what definitions you are using, having an argument that flows logically, and arranging your work neatly on the page.

   Pat wants to find a linear transformation that will transform the figure on the left to the figure on the right.

   Either: (i) give Pat an example of such a transformation and explain how you found it **or** (ii) explain to Pat why no such linear transformation exists.

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If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

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