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Products and Convolutions of Gaussian Probability Density Functions

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Abstract

It is well known that the product and the convolution of two Gaussian probability density functions (PDFs) are also Gaussian. This memo provides derivations for the mean and standard deviation of the resulting Gaussians in both cases. These results are useful in calculating the effects of smoothing applied as an intermediate step in various algorithms.

1 The Product of Two Gaussian PDFs

We wish to find the product of two Gaussian PDFs

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma_f} e^{-\frac{(x-\mu_f)^2}{2\sigma_f^2}} \quad \text{and} \quad g(x) = \frac{1}{\sqrt{2\pi}\sigma_g} e^{-\frac{(x-\mu_g)^2}{2\sigma_g^2}} \tag{1}$$

in the most general case i.e. non-identical means. The product gives

$$f(x)g(x) = \frac{1}{2\pi\sigma_f\sigma_g} e^{-\left(\frac{(x-\mu_f)^2}{2\sigma_f^2} + \frac{(x-\mu_g)^2}{2\sigma_g^2}\right)} \tag{2}$$

Examining the term in the exponent

$$\alpha = \frac{(x-\mu_f)^2}{2\sigma_f^2} + \frac{(x-\mu_g)^2}{2\sigma_g^2} \tag{3}$$

we can expand the two quadratics and collect terms in powers of x to give

$$\alpha = \frac{(\sigma_f^2 + \sigma_g^2)x^2 - 2(\mu_f\sigma_g^2 + \mu_g\sigma_f^2)x + \mu_f^2\sigma_g^2 + \mu_g^2\sigma_f^2}{2\sigma_f^2\sigma_g^2} \tag{4}$$

Dividing through by the coefficient of x^2 gives

$$\alpha = \frac{x^2 - 2\frac{\mu_f\sigma_g^2 + \mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2}x + \frac{\mu_f^2\sigma_g^2 + \mu_g^2\sigma_f^2}{\sigma_f^2+\sigma_g^2}}{2\frac{\sigma_f^2\sigma_g^2}{\sigma_f^2+\sigma_g^2}} \tag{5}$$

This is again a quadratic in x, and so Eq. 2 is a Gaussian function. Compare the terms in Eq. 5 to a the usual Gaussian form

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x^2-2\mu x+\mu^2)}{2\sigma^2}} \tag{6}$$

Since we can add a term γ that is independent of x to complete the square in α, this is sufficent to complete the proof in cases where the normalisation can be ignored. The product of two Gaussian PDFs is proportional to a Gaussian PDF with a mean that is half the coefficient of x in Eq. 5 and a standard deviation that is the square root of half of the denominator i.e.

$$\sigma_{fg} = \sqrt{\frac{\sigma_f^2\sigma_g^2}{\sigma_f^2+\sigma_g^2}} \quad \text{and} \quad \mu_{fg} = \frac{\mu_f\sigma_g^2 + \mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2} \tag{7}$$

In general, the product is not itself a PDF as, due to the presence of the scaling factor, it will not have the correct normalisation.

We can now either write down the product $f(x)g(x)$ in the usual Gaussian form directly, with an unknown scaling constant, or proceed from Eq. 5 to obtain the scaling constant explicitly. Taking the latter route, suppose that γ is the term required to complete the square in α i.e.

$$\gamma = \frac{\left(\frac{\mu_f\sigma_g^2+\mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2}\right)^2 - \left(\frac{\mu_f\sigma_g^2+\mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2}\right)^2}{\frac{2\sigma_f^2\sigma_g^2}{(\sigma_f^2+\sigma_g^2)}} = 0 \tag{8}$$

Adding this term to α gives

$$\alpha = \frac{x^2 - 2x\frac{\mu_f\sigma_g^2+\mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2} + \left(\frac{\mu_f\sigma_g^2+\mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2}\right)^2}{\frac{2\sigma_f^2\sigma_g^2}{(\sigma_f^2+\sigma_g^2)}} + \frac{\frac{\mu_f^2\sigma_g^2+\mu_g^2\sigma_f^2}{\sigma_f^2+\sigma_g^2} - \left(\frac{\mu_f\sigma_g^2+\mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2}\right)^2}{\frac{2\sigma_f^2\sigma_g^2}{(\sigma_f^2+\sigma_g^2)}} \tag{9}$$

After some manipulation, this reduces to

$$\alpha = \frac{\left(x - \frac{\mu_f\sigma_g^2+\mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2}\right)^2}{2\frac{\sigma_f^2\sigma_g^2}{\sigma_f^2+\sigma_g^2}} + \frac{(\mu_f-\mu_g)^2}{2(\sigma_f^2+\sigma_g^2)} = \frac{(x-\mu_{fg})^2}{2\sigma_{fg}^2} + \frac{(\mu_f-\mu_g)^2}{2(\sigma_f^2+\sigma_g^2)}$$

Substituting back into Eq. 2 gives

$$f(x)g(x) = \frac{1}{2\pi\sigma_f\sigma_g}\exp\left[-\frac{(x-\mu_{fg})^2}{2\sigma_{fg}^2}\right]\exp\left[-\frac{(\mu_f-\mu_g)^2}{2(\sigma_f^2+\sigma_g^2)}\right]$$

Multiplying by σ_{fg}/σ_{fg} and rearranging gives

$$= \frac{1}{\sqrt{2\pi}\sigma_{fg}}\exp\left[-\frac{(x-\mu_{fg})^2}{2\sigma_{fg}^2}\right]\frac{1}{\sqrt{2\pi(\sigma_f^2+\sigma_g^2)}}\exp\left[-\frac{(\mu_f-\mu_g)^2}{2(\sigma_f^2+\sigma_g^2)}\right] \tag{10}$$

Therefore, the product of two Gaussians PDFs $f(x)$ and $g(x)$ is a scaled Gaussian PDF

$$f(x)g(x) = \frac{S}{\sqrt{2\pi}\sigma_{fg}}\exp\left[-\frac{(x-\mu_{fg})^2}{2\sigma_{fg}^2}\right] \tag{11}$$

where

$$\sigma_{fg} = \sqrt{\frac{\sigma_f^2\sigma_g^2}{\sigma_f^2+\sigma_g^2}} \quad \text{and} \quad \mu_{fg} = \frac{\mu_f\sigma_g^2+\mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2} \tag{12}$$

and the scaling factor S is itself a Gaussian PDF on both μ_f and μ_g with standard deviation $\sqrt{\sigma_f^2+\sigma_g^2}$

$$S = \frac{1}{\sqrt{2\pi(\sigma_f^2+\sigma_g^2)}}\exp\left[-\frac{(\mu_f-\mu_g)^2}{2(\sigma_f^2+\sigma_g^2)}\right]$$

2 The Convolution of Two Gaussian PDFs

We wish to find the convolution of two Gaussian PDFs

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma_f}e^{-\frac{(x-\mu_f)^2}{2\sigma_f^2}} \quad \text{and} \quad g(x) = \frac{1}{\sqrt{2\pi}\sigma_g}e^{-\frac{(x-\mu_g)^2}{2\sigma_g^2}} \tag{13}$$

in the most general case i.e. non-identical means. The convolution of two functions $f(t)$ and $g(t)$ over a finite range[1] is defined as

$$\int_0^x f(x-\tau)g(\tau)d\tau = f\otimes g \tag{14}$$

[1] In practice, convolutions are more often performed over an infinite range

$$\int_{-\infty}^{\infty} f(x-\tau)g(\tau)d\tau = f\otimes g$$

However, the usual approach is to use the convolution theorem [2],

$$F^{-1}[F(f(x))F(g(x))] = f(x) \otimes g(x) \tag{15}$$

where F is the Fourier transform

$$F(f(x)) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ikx}\ dx \tag{16}$$

and F^{-1} is the inverse Fourier transform

$$F^{-1}(F(k)) = \int_{-\infty}^{\infty} F(k)e^{2\pi ikx}\ dk \tag{17}$$

Using the transformation

$$x' = x - \mu_f \tag{18}$$

the Fourier transform of $f(x)$ is given by

$$F(f(x)) = \frac{1}{\sqrt{2\pi}\sigma_f}\int_{-\infty}^{\infty} e^{-\frac{x'^2}{2\sigma_f^2}} e^{-2\pi ik(x'-\mu_f)}\ dx' = \frac{e^{-2\pi ik\mu_f}}{\sqrt{2\pi}\sigma_f}\int_{-\infty}^{\infty} e^{-\frac{x'^2}{2\sigma_f^2}} e^{-2\pi ikx'}\ dx' \tag{19}$$

Using Euler's formula [2],

$$e^{-i\theta} = \cos\theta - i\sin\theta \tag{20}$$

we can split the term in $e^{x'}$ to give

$$F(f(x)) = \frac{e^{-2\pi ik\mu_f}}{\sqrt{2\pi}\sigma_f}\int_{-\infty}^{\infty} e^{-\frac{x'^2}{2\sigma_f^2}}[\cos(2\pi kx') - i\sin(2\pi kx')]\ dx' \tag{21}$$

The term in $\sin(x')$ is odd and so its integral over all space will be zero, leaving

$$F(f(x)) = \frac{e^{-2\pi ik\mu_f}}{\sqrt{2\pi}\sigma_f}\int_{-\infty}^{\infty} e^{-\frac{x'^2}{2\sigma_f^2}}\cos(2\pi kx')\ dx' \tag{22}$$

This integral is given in standard form in [1]

$$\int_0^{\infty} e^{-at^2}\cos(2xt)\ dt = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-\frac{x^2}{a}} \tag{23}$$

and so

$$F(f(x)) = e^{-2\pi ik\mu_f}e^{-2\pi^2\sigma_f^2k^2} \tag{24}$$

The second term in this expression is a Gaussian PDF in k: the Fourier transform of a Gaussian PDF is another Gaussian PDF. The first term is a phase term accounting for the mean of $f(x)$ i.e. its offset from zero. The Fourier transform of $g(x)$ will give a similar expression, and so

$$F(f(x))F(g(x)) = e^{-2\pi ik\mu_f}e^{-2\pi^2\sigma_f^2k^2}e^{-2\pi ik\mu_g}e^{-2\pi^2\sigma_g^2k^2} = e^{-2\pi ik(\mu_f+\mu_g)}e^{-2\pi^2(\sigma_f^2+\sigma_g^2)k^2} \tag{25}$$

Comparing Eq. 25 to Eq. 24, we can see that it is the Fourier transform of a Gaussian PDF with mean and standard deviation

$$\mu_{f\otimes g} = \mu_f + \mu_g \quad \text{and} \quad \sigma_{f\otimes g} = \sqrt{\sigma_f^2 + \sigma_g^2} \tag{26}$$

and therefore, since the Fourier transform is invertible,

$$P_{f\otimes g}(x) = F^{-1}[F(f(x))F(g(x))] = \frac{1}{\sqrt{2\pi(\sigma_f^2+\sigma_g^2)}}e^{-\frac{(x-(\mu_f+\mu_g))^2}{2(\sigma_f^2+\sigma_g^2)}} \tag{27}$$

It may be worth noting a general result at this point; the area under a convolution is equal to the product of the areas under the factors

$$\begin{aligned}\int_{-\infty}^{\infty}(f\otimes g)dt &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)g(t-u)du\right]dt \\ &= \int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} g(t-u)dt\right]du \\ &\left[\int_{-\infty}^{\infty} f(u)du\right]\left[\int_{-\infty}^{\infty} g(t)dt\right]\end{aligned}$$

Therefore, the preservation of the normalisation when convolving PDFs i.e. the fact that the convolution is also a PDF, normalised such that the area under the function is equal to unity, is a special case rather than being true in general.

3 Summary

It is well known that the product and the convolution of a pair of Gaussian PDFs are also Gaussian. However, the derivations are not commonly seen in the literature, particularly in the case of Gaussian PDFs with non-identical means. This document has provided both derivations. In the case of the product of two Gaussian PDFs, the result is proportional to a Gaussian PDF with mean and standard deviation

$$\mu_{fg} = \frac{\mu_f \sigma_g^2 + \mu_g \sigma_f^2}{\sigma_f^2 + \sigma_g^2} \quad \text{and} \quad \sigma_{fg} = \sqrt{\frac{\sigma_f^2 \sigma_g^2}{\sigma_f^2 + \sigma_g^2}}$$

where μ_f and μ_g are the means of the two original Gaussians and σ_f and σ_g are their standard deviations. The constant of proportionality is itself a Gaussian PDF on both μ_f and μ_g with standard deviation $\sqrt{\sigma_f^2 + \sigma_g^2}$

$$S = \frac{1}{\sqrt{2\pi(\sigma_f^2 + \sigma_g^2)}} \exp\left[-\frac{(\mu_f - \mu_g)^2}{2(\sigma_f^2 + \sigma_g^2)}\right]$$

It should be noted that this result is not the PDF of the product of two Gaussian random variates; in that case, the product normal distribution applies.

In the case of the convolution of two Gaussian PDFs, the result is again a Gaussian PDF with mean and standard deviation

$$\mu_{f \otimes g} = \mu_f + \mu_g \quad \text{and} \quad \sigma_{f \otimes g} = \sqrt{\sigma_f^2 + \sigma_g^2}$$

These results, particularly the second result, can be useful in calculating the effects of Gaussian smoothing applied as an intermadiate step in various machine vision algorithms.

References

[1] M Abramowitz and I A Stegun. *Handbook of Mathematical Functions.* National Bureau of Standards, Washington DC, 1972.

[2] M L Boas. *Mathematical Methods in the Physical Sciences.* John Wiley and Sons Ltd., 1983.