Postulates of Quantum Mechanics

The postulates of quantum mechanics provide a link between the mathematical formalism of quantum mechanics and reality.

Postulate 1: Each isolated physical system is a complex vector space that satisfies inner product (thus, an Hilbert space). Such space is defined as the *state space*. Such system is described by the associated *state vectors* (state space's unit vectors).

Note, quantum mechanics does not tell us the state space for a given physical system. Figuring that out is a separate problem (i.e. – QED).

Postulate 2: Each closed quantum system evolves from time t_1 to time t_2 through a *unitary transformation U*. Hence, for a state $|\psi >$ at t_1, it evolves to $|\psi' >$ at t_2 through

$$|\psi' >= U|\psi >$$

Alongside, quantum mechanics does not tell us what unitary transformation *U* describes the evolution of quantum systems.

However, to describe the evolution of a closed quantum system in continuous time, we resort to *Schrodinger equation,*

$$ih\frac{d|\psi>}{dt} = H|\psi >.$$

h above is suppose to be defined as h-bar. It's a physical constant defined as *Planck's constant*. *H* is an Hermitian operator known as the *Hamiltonian*. Again, just like with unitary operators from postulate 2, Hamiltonians are also hard to describe.

Note that given that the Hamiltonian *H* is Hermatian, it has a spectral decomposition described as

$$H = \Sigma_E E|E >< E|,$$

where |*E*> are normalized *energy eigenvector* of the system with *energy eigenvalues E*.

Postulate 3: Measurements of quantum systems are described by a collection of *measurement operators* M_m. Index *m* refers to the possible measurement outcome in an experiment. Further, for a quantum system in an initial state ψ before measurement, after measurement, the probability that index *m* occurs is

$$p(m) =< \psi|M_m^\dagger M_m|\psi >,$$

and the state of system after measurement is

$$\frac{M_m|\psi>}{\sqrt{<\psi|M_m^{\dagger}M_m|\psi>}}.$$

Note, the measurement operators satisfy the *completeness equation,*

$$\Sigma_m M_m^{\dagger} M_m = I$$

in which

$$1 = \Sigma_m p(m) = \Sigma_m < \psi|M_m^{\dagger} M_m|\psi >$$

A special case of postulate 3 is the *projective measurement* as defined below:

Projective measurements: An Hermatian observable, *M*, on a state space, with a spectral decomposition of

$$M = \Sigma_m m P_m,$$

where P_m is the set of projectors onto eigenspace of *M* with eigenvalue *m*. When measuring state $|\psi >$,

$$p(m) =< \psi|P_m|\psi >,$$

and

$$|\psi' >= \frac{P_m|\psi>}{\sqrt{p(m)}}$$

where *p(m)* is the probability of measuring *m*, and $|\psi' >$ is the quantum state of the system after measuring *m*.

Projective measurements have this following property, derived from probability theory

$$Average = E(M) = \Sigma_m m P_m$$

$$= \Sigma_m m < \psi|P_m|\psi >$$

$$=< \psi|(\Sigma_m m P_m)|\psi >$$

$$=< \psi|M|\psi >$$

in which, the variance can be constructed as

$$[\Delta(M)]^2 = <(M - <M>)^2>$$

$$= <M^2> - <M>^2.$$

where $<M> = E(M)$. This construction of the average and variance leads us to an important concept of quantum mechanics:

Heisenberg uncertainty principle

Suppose we have Hermatian operators A and B, in which we perform measurements on a large number of quantum systems in identical states, $|\psi>$, we'll show that for some operators C and D, such that A = C – <C> and B = D – <D>, we get Heisenberg's uncertainty principle

$$\Delta(C)\Delta(D) \geq \frac{|<\psi|[C,D]|\psi>|}{2}.$$

Proof: Suppose $<\psi|AB|\psi> = x + iy$, where x and y are real. We then have

$$|<\psi|[A,B]|\psi>|^2 + <\psi|A,B|\psi>|^2 = 4|<\psi|AB|\psi>|^2,$$

where $<\psi|[A,B]|\psi> = 2iy$ and $<\psi|A,B|\psi> = 2x$.

By Cauchy-Schwarz inequality

$$|<\psi|AB|\psi>|^2 \leq <\psi|A^2|\psi><\psi|B^2|\psi>,$$

which when combined with the equation

$$|<\psi|[A,B]|\psi>|^2 + <\psi|A,B|\psi>|^2 = 4|<\psi|AB|\psi>|^2,$$

we get the following true statement:

$$|<\psi|[A,B]|\psi>|^2 \leq 4<\psi|A^2|\psi><\psi|B^2|\psi>.$$

Now, suppose we have two observables C and D, such that A = C – <C> and B = D – <D>. Then by derivation, we have the Heisenberg's uncertainty principle (as stated above):

$$\Delta(C)\Delta(D) \geq \frac{|<\psi|[C,D]|\psi>|}{2}.$$

In simple English, the Heisenberg uncertainty principle states that after preparation of a large number of quantum systems, with identical states $|\psi>$, performing measurement C on some of those systems, and D on the others, will result in the multiplication of the two standard deviations for observables C and D satisfying the given inequality for the Heisenberg's uncertainty principle.

Furthermore, another formalism of measurement that focuses on just probabilities of respective measurements (and disregards post-measurement state of system), is the *POVM measurement*.

Postulate 4: A composite physical system's state space is described through the tensor product of each individual state space combined in the composite physical system. Such system is described below, where we have systems numbered 1, 2, …., n, and system i is prepared in state $|\psi_i>$, then the composite system is composed as

$$|\psi_1> \otimes |\psi_2> \otimes \ldots \otimes |\psi_n>$$

The above postulate, Postulate 4, allows for another derived phenomena of quantum mechanics, *entanglement*.