Parametrizing Chains in Generalized Involutions

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Introduction/Motivation

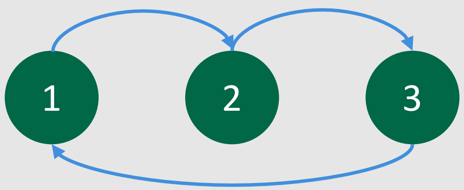
- Let **B** be the group of upper triangular invertible n by n matrices.
- Let **Q** be the manifold of non-singular quadrics in projective space, P^{n-1}.
- B partitions the manifold into subsets.
- **B-orbits of Q**, subsets of Q that can be moved to one given point by B, are parametrized by the **involutions in S_n.**

Algebraic Geometry		Combinatorics
B-orbits of **Q**	⟷	Involutions: I_n
B-orbits of **S**	⟷	Fixed-Point Free Involutions: $FPF\ I_n$
B-orbits of **M**	⟷	Generalized Fixed-Point Free Involutions: $FPF\ \mu_n$

Symmetric Group S_n

Permutations in S_3

graph



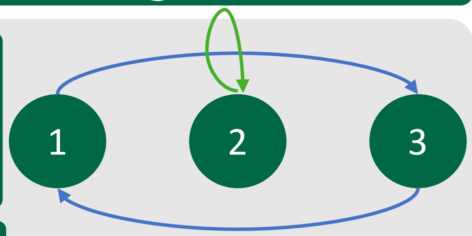
one-line notation

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

cycle notation

$$(123)$$

graph



one-line notation

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

cycle notation

$$(13)(2) = (13)$$

Definition (Permutation). A permutation is a function **from a set A to A** that is both **injective** and **surjective**.

Definition (Symmetric Group). If we let A = {1, 2, . . . , n}, then the set of all permutations of A is called the symmetric group of degree n ad is denoted by Sn.

Definition (One-Line Notation). One-line notation of a permutation is a string w, where each i, $1 \leq i \leq n$, occurs only once in w, and the permutation denoted by **w maps i to w(i) for all i.**

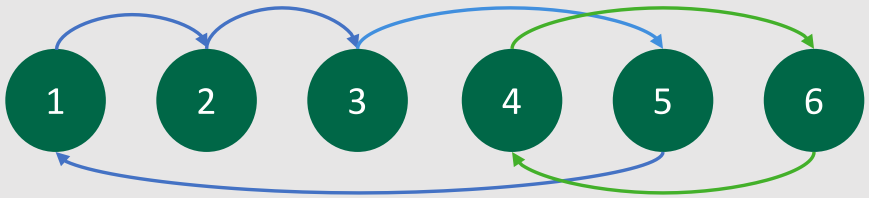
Definition (Cycle Notation). The cycle notation of a permutation w expresses w as a product of disjoint cycles.

In this permutation 2 is called a “fixed point” because it gets sent to itself

Symmetric Group S_n

Permutation in S_6

graph



one-line notation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 6 & 1 & 4 \end{pmatrix}$$

cycle notation

$(1235)(46)$

Definition (Permutation). A permutation is a function **from a set A to A** that is both **injective** and **surjective**.

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Definition (Cycle Notation). The cycle notation of a permutation w expresses w as a product of disjoint cycles.

The Group of Permutations

Function Composition

(13) (123)

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

Definition (Group of Permutations). A permutation group of set A is a set that forms a group under function composition.

Identity (id)

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

Inverse

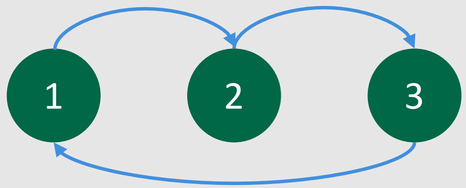
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

Involutions

Involutions in I_3

not an involution

matching



one-line notation

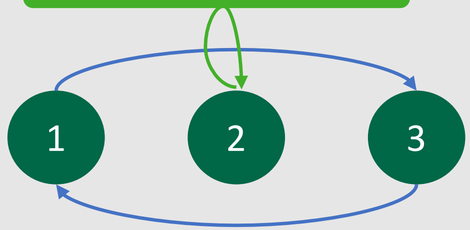
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

cycle notation

(123)

involution

matching



one-line notation

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

cycle notation

(13)(2) = (13)

Definition (Involution). An involution is a transformation whose inverse is equal to itself.

$$I_n = \{w \in S_n \mid w = w^{-1}\} = \{w \in S_n \mid w^2 = id\}$$

Condition: For all **i** in A = {1, 2, . . . , n},

(i) One-cycle:

i can be sent to **i.** In this case, we call **i** a fixed-point.

(ii) Two-cycle:

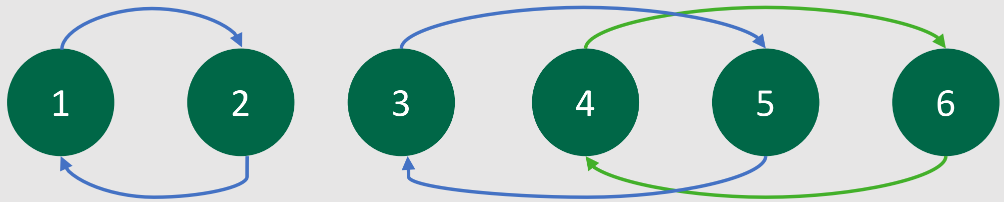
if **i** is sent to **j**, **j** should be sent back to **i** . This forms a 2-cycle

Involutions

Involution in I_6

involution

matching



one-line notation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 6 & 3 & 4 \end{pmatrix}$$

cycle notation

$(12)(35)(46)$

Definition (Involution). An involution is a transformation whose inverse is equal to itself.

$$\mathbf{I_n = \{w \in S_n \mid w = w^{-1}\} = \{w \in S_n \mid w^2 = id\}}$$

Condition: For all **i** in A = {1, 2, . . . , n},

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i can be sent to **i.** In this case, we call **i** a fixed-point.

(ii) Two-cycle:

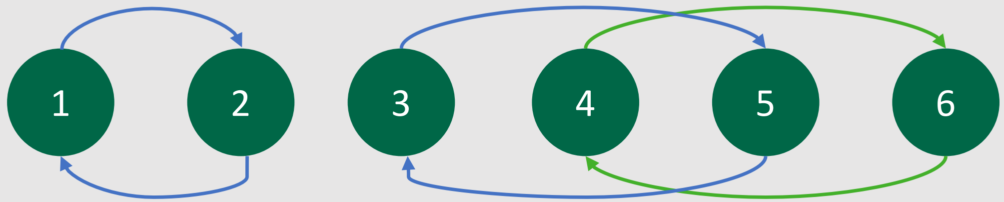
if **i** is sent to **j**, **j** should be sent back to **i** . This forms a 2-cycle

Fixed-Point Free (FPF) Involutions

Condition: For all **i** in A = {1, 2, . . . , n}, if **i** is sent to **j**, **j** should be sent back to **i** **(two-cycle).**

FPF involution

matching



one-line notation

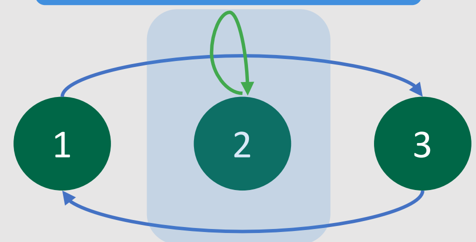
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 6 & 3 & 4 \end{pmatrix}$$

cycle notation

(12)(35)(46)

not FPF involution

matching



one-line notation

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

cycle notation

(13)(2) = (13)

Simple Transposition (Permutation)

(i , i+1)

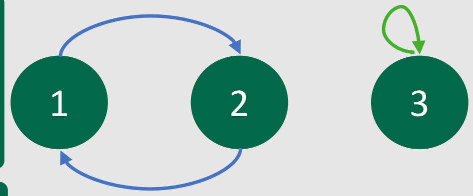
An operation s_i that switches the positions of the numbers **i** and **i + 1** in one line notation of $w \in S_n$.

$$s_1\ 231 = (12)\ 231 = 132$$
$$s_2\ 231 = (23)\ 231 = 321$$

Definition (Simple Transposition). Let w, $w' \in S_n$. The simple transposition s_i is equivalent to $w \in S_n$ with cycle notation **(i, i + 1)**. This is the simple transposition that interchanges the positions of the numbers i and i + 1 in w′ with operation $\mathbf{s_i w'}$, and interchanges the numbers at positions i and i + 1 in w′ with operation $\mathbf{w' s_i}$.

s_1

matching



one-line notation

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

cycle notation

$$(12)(3) = (12)$$

example

$$(12)\ (123)$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Simple Transposition (Involution)

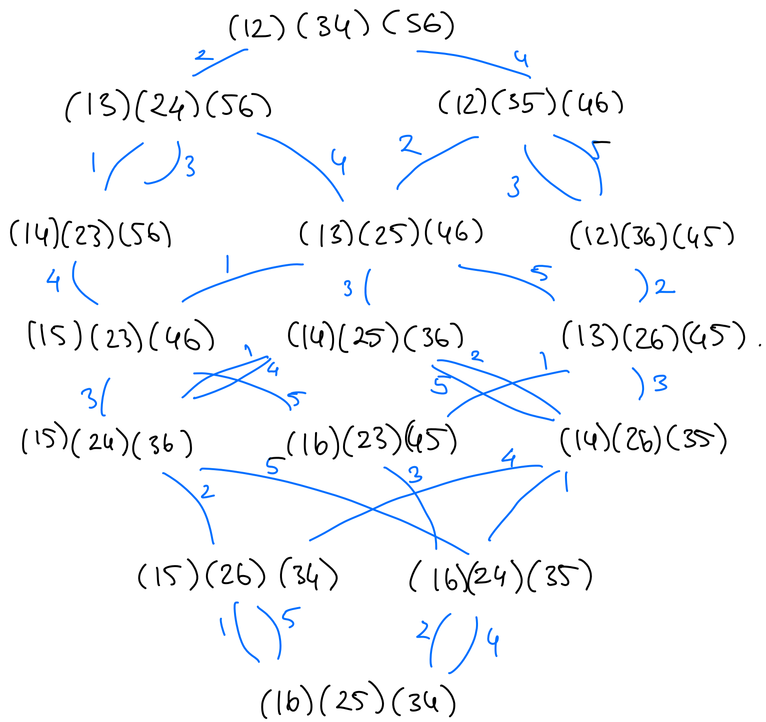
s_i

An operation $\mathbf{s_i}$ that switches the positions of the numbers **i** and **i + 1** in cycle notation of $\pi \in I_n$.

$$s_2\,(13)(24)\,s_2 = (23)\,(13)(24)\,(23) = (12)(34)$$
$$s_3\,(14)(23)\,s_3 = (34)\,(14)(23)\,(34) = (13)(24)$$
$$s_1\,(12)(34)\,s_1 = (12)\,(12)(34)\,(12) = (12)(34)$$

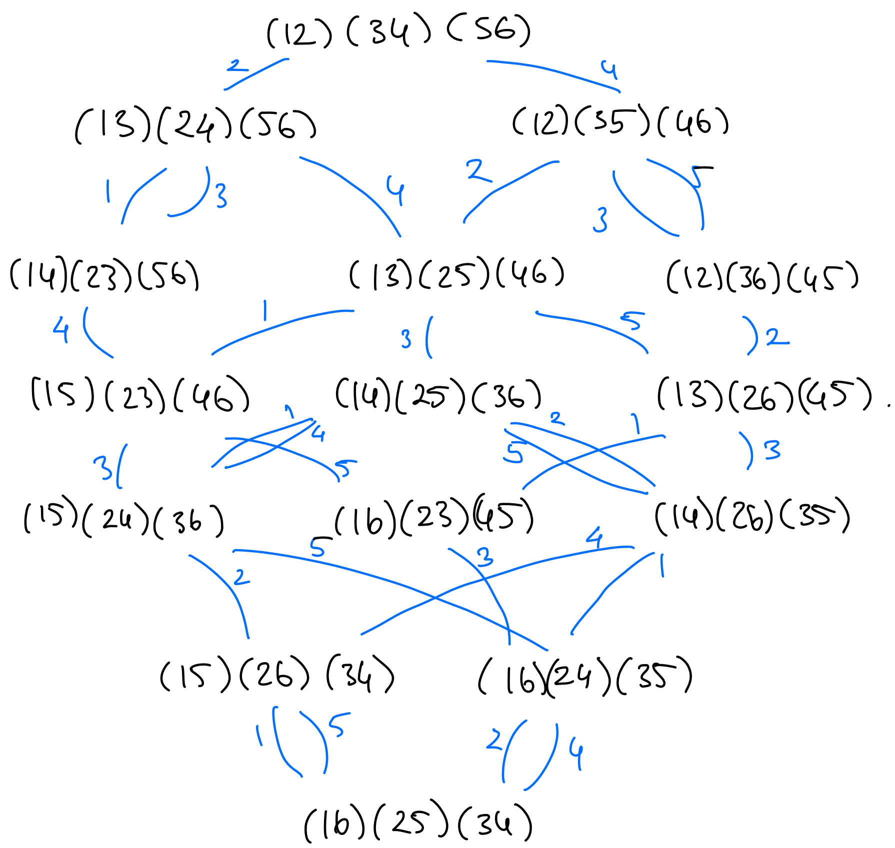
Definition (Simple Transposition in I_n).
Let $w \in S_n$ and $\pi \in I_n$. The simple transposition $\mathbf{s_i}$ is equivalent to $w \in S_n$ with cycle notation **(i, i + 1).** This is the simple transposition that interchanges the positions of the numbers i and i + 1 in π with $\mathbf{s_i}\,\pi\,\mathbf{s_i}$.

FPF Involutions (n=6)



All Simple Transpositions and All Involutions

FPF Involutions (n=6)



Notes

Each node in this tree corresponds to an involution

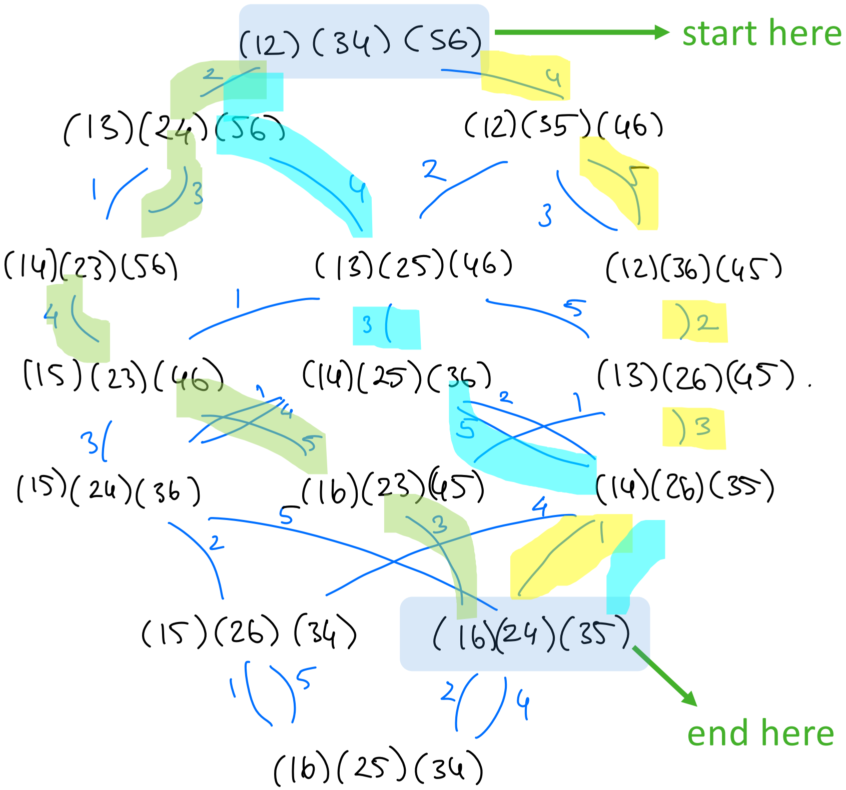
Each edge i is an operation that changes the positions of numbers i and i+1 in the cycle notation of the involution

example: edge 2 changes the positions of numbers 2 and 3

If i and i+1 are in the same cycle, the operation i does not change the involution

Tracing Paths

FPF Involutions (n=6)



What is the rule for these paths?

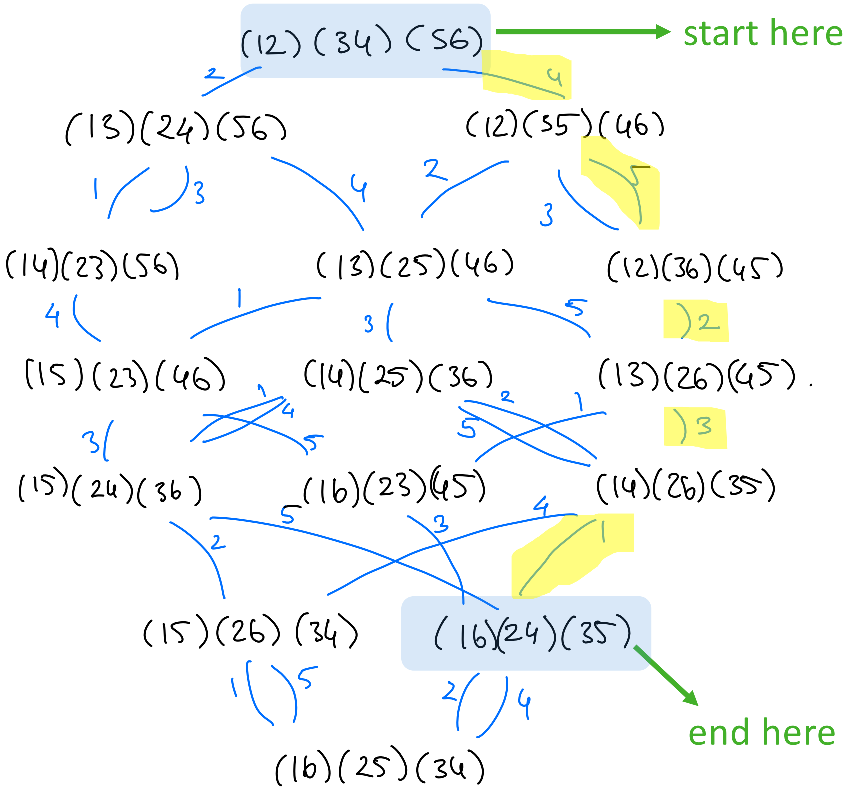
Given the end node, can we tell what all the paths are without counting them?

45231 = $s_4 s_5 s_2 s_3 s_1$
42531
42321
42351
21453
23453
24351

Definition. (B(π)) Let B(π) be the set of ordered lists of simple transpositions such that when we apply the transpositions in the list to the minimum element of the set of involutions we get π as an involution. For example, B = {[4,5,2,3,1], [2,3,4,5,3]}

W-set

FPF Involutions (n=6)



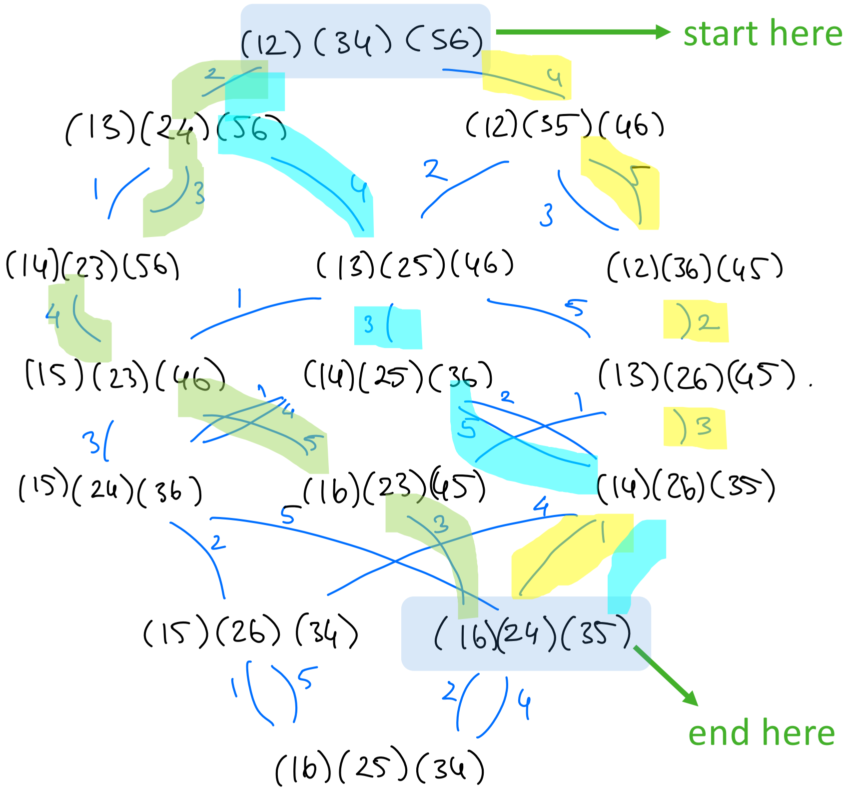
Definition (W-set) Let π be an involution. W(π) is the set obtained by applying the list of simple transpositions in B to the identity element of S_n

Get all the paths to node (16)(24)(35) → Apply the simple transpositions in the path to 123456 → Get the W-set elements for (16)(24)(35)

123456 – $s_4 s_5 s_2 s_3 s_1$
123546 – $s_5 s_2 s_3 s_1$
123645 – $s_2 s_3 s_1$
132645 – $s_3 s_1$
142635 – s_1
241635

W-set

FPF Involutions (n=6)



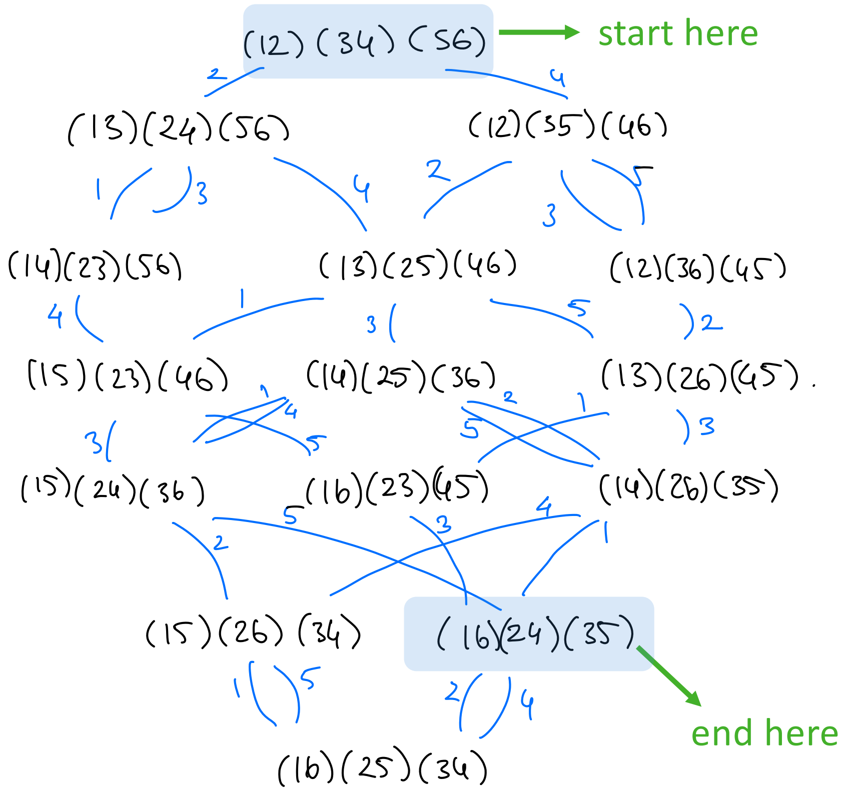
45231 = $s_4s_5s_2s_3s_1$	**45231**
42531	**123456** $s_4s_5s_2s_3s_1$
42321	= **123546** $s_5s_2s_3s_1$
42351	= **123645** $s_2s_3s_1$
21453	= **132645** s_3s_1
23453	= **142635** s_1
24351	= **241635**

W-set = {162435 , 241635 , 243516}

Paths are all the possible sequences of simple transpositions that have the minimum possible length such that when we apply the sequence of transpositions to the identity element of permutations to get the elements of the W-set.

W-set of FPF Involutions Theorem

FPF Involutions (n=6)



Theorem 6.1 (Parametrizing Chains in Fixed-Point Free Involutions). [2]
Let n = 2k and let $\pi = (a_1, b_1)(a_2, b_2) \cdots (a_k, b_k) \in I_n$ be written in standard form. Then $W(\pi)$ consists of all w such that

1. for each i, a_i occurs before b_i in w and no value occurs between a_i and b_i in w;
2. if $i < j$ and $b_i < b_j$, then b_i occurs before a_j in w.

π = (1,6)(2,4)(3,5)

Condition 1:
16, **24** and **35** occur as pairs in the W-set
Condition 2:
if the second elements in cycles are in increasing order, then pairs appear in the same order in the elements of the W-set

W-set = {162435 , 241635 , 243516}

Research Question 2

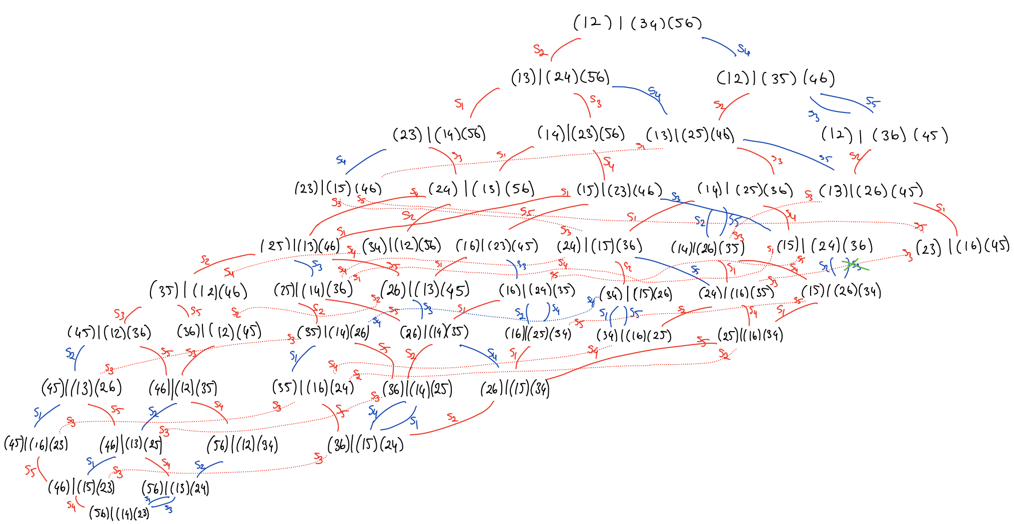
What is the rule for the W-sets of generalized involutions?

Involutions: I_n

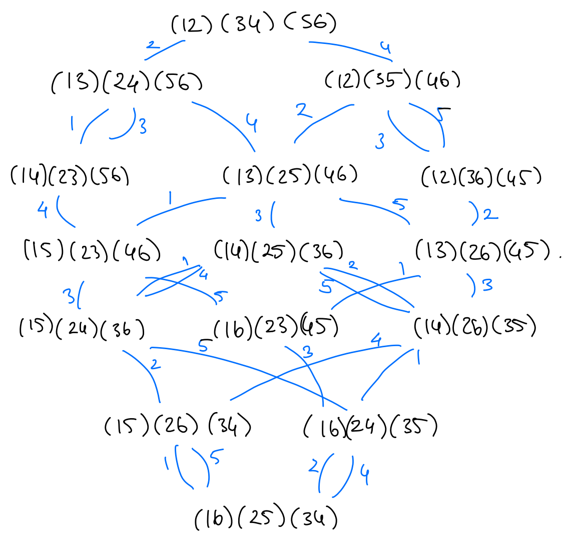
Fixed-Point Free Involutions: $FPF\ I_n$

Generalized Fixed-Point Free Involutions: $FPF\ \mu_n$

Generalized Involutions (n = 6) [2|4]

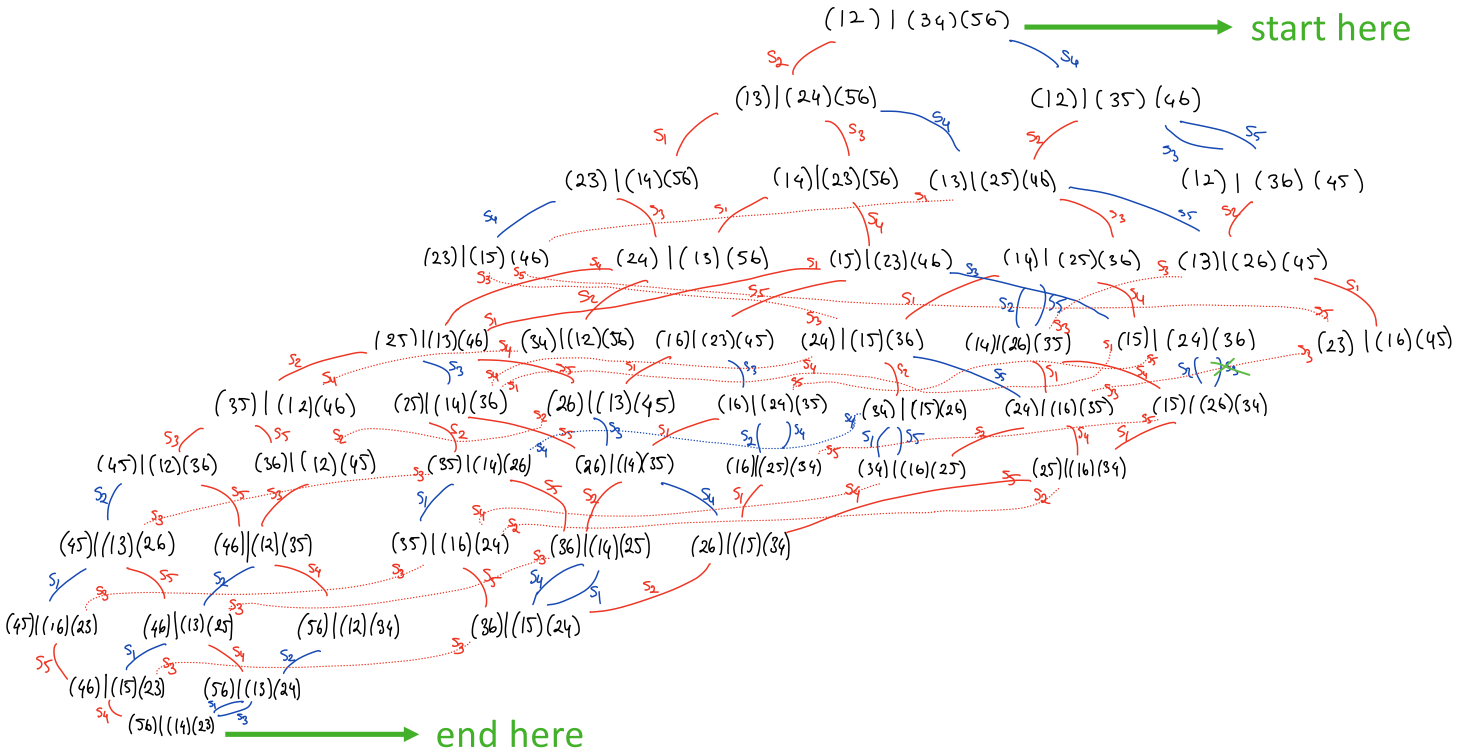


FPF Involutions (n=6)



Research Question

Generalized Involutions (n = 6) [2|4]



Methodology: 3 steps

Computational Experiments

- Code that generates W-sets for generalized involutions
- Improving the code to be more efficient
- In its more efficient version the code can calculate the W-sets

Conjecture/Theorems

- An Original Theorem for the rule for Generalized FPF involutions

Proofs

- A self-contained proof for the rule for FPF involutions
- A proof for the rule for Generalized Involutions

Methodology

Computational Experiments

N = 6

```
[1 2 3 4 5 6]
number of levels: 6
current_level: 1 prev_level_list length: 1
current_level: 2 prev_level_list length: 2
current_level: 3 prev_level_list length: 3
current_level: 4 prev_level_list length: 3
current_level: 5 prev_level_list length: 3
current_level: 6 prev_level_list length: 2

{(1, 6, 2, 5, 3, 4),
 (1, 6, 3, 4, 2, 5),
 (2, 5, 1, 6, 3, 4),
 (2, 5, 3, 4, 1, 6),
 (3, 4, 1, 6, 2, 5),
 (3, 4, 2, 5, 1, 6)}
```

```
[1 2|3 4 5 6]
number of levels: 10
current_level: 1 prev_level_list length: 1
current_level: 2 prev_level_list length: 2
current_level: 3 prev_level_list length: 4
current_level: 4 prev_level_list length: 5
current_level: 5 prev_level_list length: 7
current_level: 6 prev_level_list length: 7
current_level: 7 prev_level_list length: 7
current_level: 8 prev_level_list length: 5
current_level: 9 prev_level_list length: 4
current_level: 10 prev_level_list length: 2

{(5, 6, 1, 4, 2, 3), (5, 6, 2, 3, 1, 4)}
```

```
[1 2|3 4|5 6]
number of levels: 12
current_level: 1 prev_level_list length: 1
current_level: 2 prev_level_list length: 2
current_level: 3 prev_level_list length: 5
current_level: 4 prev_level_list length: 7
current_level: 5 prev_level_list length: 11
current_level: 6 prev_level_list length: 12
current_level: 7 prev_level_list length: 14
current_level: 8 prev_level_list length: 12
current_level: 9 prev_level_list length: 11
current_level: 10 prev_level_list length: 7
current_level: 11 prev_level_list length: 5
current_level: 12 prev_level_list length: 2

{(5, 6, 3, 4, 1, 2)}
```

Methodology

Computational Experiments

N = 8

```
[1 2|3 4 5 6 7 8]
number of levels: 18
current_level: 1 prev_level_list length: 1
current_level: 2 prev_level_list length: 3
current_level: 3 prev_level_list length: 7
current_level: 4 prev_level_list length: 12
current_level: 5 prev_level_list length: 19
current_level: 6 prev_level_list length: 26
current_level: 7 prev_level_list length: 34
current_level: 8 prev_level_list length: 40
current_level: 9 prev_level_list length: 45
current_level: 10 prev_level_list length: 46
current_level: 11 prev_level_list length: 45
current_level: 12 prev_level_list length: 40
current_level: 13 prev_level_list length: 34
current_level: 14 prev_level_list length: 26
current_level: 15 prev_level_list length: 19
current_level: 16 prev_level_list length: 12
current_level: 17 prev_level_list length: 7
current_level: 18 prev_level_list length: 3

{(7, 8, 1, 6, 2, 5, 3, 4),
 (7, 8, 1, 6, 3, 4, 2, 5),
 (7, 8, 2, 5, 1, 6, 3, 4),
 (7, 8, 2, 5, 3, 4, 1, 6),
 (7, 8, 3, 4, 1, 6, 2, 5),
 (7, 8, 3, 4, 2, 5, 1, 6)}
```

```
[1 2|3 4|5 6|7 8]
number of levels: 24
current_level: 1 prev_level_list length: 1
current_level: 2 prev_level_list length: 3
current_level: 3 prev_level_list length: 9
current_level: 4 prev_level_list length: 18
current_level: 5 prev_level_list length: 35
current_level: 6 prev_level_list length: 56
current_level: 7 prev_level_list length: 87
current_level: 8 prev_level_list length: 119
current_level: 9 prev_level_list length: 158
current_level: 10 prev_level_list length: 190
current_level: 11 prev_level_list length: 222
current_level: 12 prev_level_list length: 238
current_level: 13 prev_level_list length: 248
current_level: 14 prev_level_list length: 238
current_level: 15 prev_level_list length: 222
current_level: 16 prev_level_list length: 190
current_level: 17 prev_level_list length: 158
current_level: 18 prev_level_list length: 119
current_level: 19 prev_level_list length: 87
current_level: 20 prev_level_list length: 56
current_level: 21 prev_level_list length: 35
current_level: 22 prev_level_list length: 18
current_level: 23 prev_level_list length: 9
current_level: 24 prev_level_list length: 3

{(7, 8, 5, 6, 3, 4, 1, 2)}
```

N = 10

```
[1 2|3 4|5 6|7 8|9 10]
number of levels: 40
current_level: 1 prev_level_list length: 1
current_level: 2 prev_level_list length: 4
current_level: 3 prev_level_list length: 14
current_level: 4 prev_level_list length: 35
current_level: 5 prev_level_list length: 80
current_level: 6 prev_level_list length: 157
current_level: 7 prev_level_list length: 289
current_level: 8 prev_level_list length: 485
current_level: 9 prev_level_list length: 775
current_level: 10 prev_level_list length: 1160
current_level: 11 prev_level_list length: 1668
current_level: 12 prev_level_list length: 2279
current_level: 13 prev_level_list length: 3008
current_level: 14 prev_level_list length: 3804
current_level: 15 prev_level_list length: 4664
current_level: 16 prev_level_list length: 5507
current_level: 17 prev_level_list length: 6319
current_level: 18 prev_level_list length: 7004
current_level: 19 prev_level_list length: 7555
current_level: 20 prev_level_list length: 7885
current_level: 21 prev_level_list length: 8014
current_level: 22 prev_level_list length: 7885
current_level: 23 prev_level_list length: 7555
current_level: 24 prev_level_list length: 7004
current_level: 25 prev_level_list length: 6319
current_level: 26 prev_level_list length: 5507
current_level: 27 prev_level_list length: 4664
current_level: 28 prev_level_list length: 3804
current_level: 29 prev_level_list length: 3008
current_level: 30 prev_level_list length: 2279
current_level: 31 prev_level_list length: 1668
current_level: 32 prev_level_list length: 1160
current_level: 33 prev_level_list length: 775
current_level: 34 prev_level_list length: 485
current_level: 35 prev_level_list length: 289
current_level: 36 prev_level_list length: 157
current_level: 37 prev_level_list length: 80
current_level: 38 prev_level_list length: 35
current_level: 39 prev_level_list length: 14
current_level: 40 prev_level_list length: 4

{(9, 10, 7, 8, 5, 6, 3, 4, 1, 2)}
```

Methodology

Formulating a Conjecture

N = 6

(16)(25)(34)

```
{(1, 6, 2, 5, 3, 4),
 (1, 6, 3, 4, 2, 5),
 (2, 5, 1, 6, 3, 4),
 (2, 5, 3, 4, 1, 6),
 (3, 4, 1, 6, 2, 5),
 (3, 4, 2, 5, 1, 6)}
```

(16)(25)(34)

```
{(1, 6, 2, 5, 3, 4),
 (1, 6, 3, 4, 2, 5),
 (2, 5, 1, 6, 3, 4),
 (2, 5, 3, 4, 1, 6),
 (3, 4, 1, 6, 2, 5),
 (3, 4, 2, 5, 1, 6)}
```

(16)(25)(34)

1,6	2,5	3,4
1,6	3,4	2,5
2,5	1,6	3,4
2,5	3,4	1,6
3,4	2,5	1,6
3,4	1,6	2,5

(56)|(14)(23)

```
{(5, 6, 1, 4, 2, 3), (5, 6, 2, 3, 1, 4)}
```

(56)|(14)(23)

```
{(5, 6, 1, 4, 2, 3), (5, 6, 2, 3, 1, 4)}
```

(56)|(14)(23)

5,6	1,4	2,3
5,6	2,3	1,4

(56)|(34)|(12)

```
{(5, 6, 3, 4, 1, 2)}
```

(56)|(34)|(12)

```
{(5, 6, 3, 4, 1, 2)}
```

(56)|(34)|(12)

5,6	3,4	1,2

Results

Theorem

N = 6

Let $\pi = (a_{11}, b_{11})(a_{12}, b_{12})...(a_{1n_1}, b_{1n_1})|...|(a_{m1}, b_{m1})(a_{m2}, b_{m2})...(a_{mn_m}, b_{mn_m}) \in \mu_n$ be a fixed point free μ-involution where $i < j \implies a_{in_p} < a_{jn_p}$ and $\forall i, pa_{in_p} < a_{jn_p}$. Let $n = \sum_{i=1}^{m} n_i$.

Theorem 7.1 (Parametrizing Chains in Fixed-Point Free μ-Involutions). Let $A(\pi) := \{w \in S_n$ such that w satisfies conditions 1, 2, and 3 below$\}$.

1. $a_{jp}b_{jp}$ pairs in π also appear as a single block in w:
$$w(2i-1) = a_{jp} \implies w(2i) = b_{jp}$$
2. if $b_{jp} < b_{jq}$ are ordered in π, then they are also ordered in w:
$$p < q \text{ and } b_{jp} < b_{jq} \implies w^{-1}(b_{jp}) < w^{-1}(b_{jq}) \forall i, j$$
3. parts do not permute with each other:
$$i < j \implies w^{-1}(a_{ip}) < w^{-1}(a_{jq})$$

Then, $A(\pi) = W(\pi) \forall \pi \in \mu_n$.

π = (5,6)(1,4)(2,3)

Condition 1:
56, **14** and **23** occur as pairs in the W-set

Condition 2:
if the second elements in cycles are in increasing order, then pairs appear in the same order in the elements of the W-set

Condition 3:
Pairs do not permute across parts (or across the vertical line in the notation).

(56)|(14)(23)

`{(5, 6, 1, 4, 2, 3), (5, 6, 2, 3, 1, 4)}`

(56)|(14)(23)

`{(5, 6, 1, 4, 2, 3), (5, 6, 2, 3, 1, 4)}`

(56)|(14)(23)

5,6	1,4	2,3
5,6	2,3	1,4

Results

1. **Computational experiments that generate W-sets of Generalized FPF involutions**

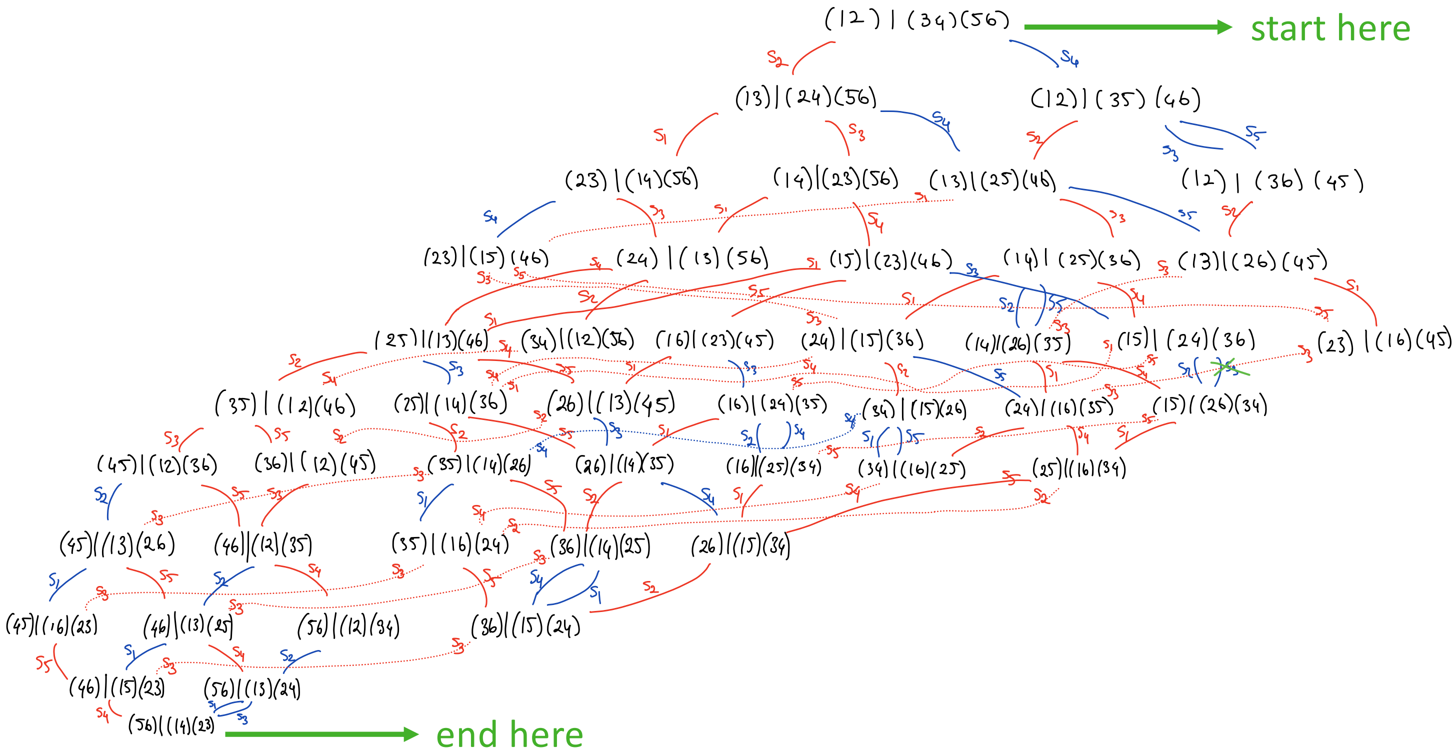
2. **Conjecture on W-sets of Generalized FPF involutions (Theorem 7.1)**

3. **Self-contained proof of W-sets of FPF involutions (Theorem 6.1), using induction**

4. **Proof of Theorem 7.1, using induction**

Proof by induction

Generalized Involutions (n = 6) [2|4]



Conclusion

Previous Research

Permutation Group: S_n $\supset$ Involutions: I_n $\supset$ Fixed-Point Free Involutions: $FPF\ I_n$

a new Proof to an existing Theorem

$\cup$ $\cap$

Generalized Involutions: μ_n $\supset$ Generalized Fixed-Point Free Involutions: $FPF\ \mu_n$

Future Research

An Original Theorem

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Appendix: Motivation

Let B be the group of upper triangular invertible n by n matrices.
Let Q be the manifold of non-singular quadrics in projective space, P^{n-1}.

There is a group action of the group B that partitions the manifold into subsets such that any element can be moved to any other point on the manifold by a group element. The B-orbits of Q, subsets of Q that can be moved to one given point by B, are parametrized by the involutions in S_n. That is, there is a **bijective map between the involutions and the B-orbits**. Each involution corresponds to a B-orbit.

Let S be a manifold consisting of nonsingular, skew symmetric n by n matrices. The group B also acts on this manifold and its B-orbits are parametrized by the fixed-point free involutions in S_n.

Let M be another manifold, which contains S as an open subset, and also has an action of the group B, such that its B-orbits are parametrized by generalized fixed point free involutions.