1 Basic Impedance Control

Basic Impedance Controller is a cascaded controller architecture for series compliant actuators. The cascaded control architecture is given in the Figure 1. One port analysis is applied on this controller[**calanca**] in order to derive the conditions to passively render pure stiffness and voigt model environments.

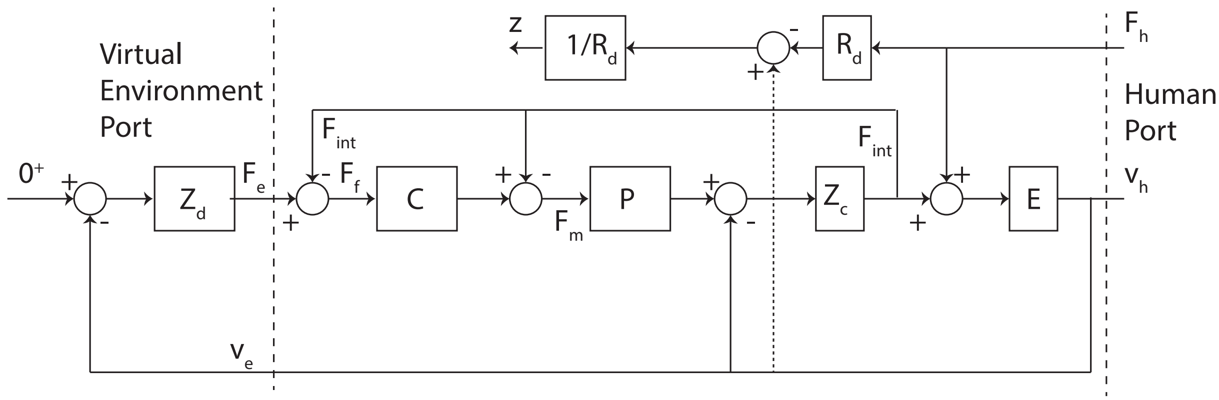

Figure 1: Control diagram of Basic Impedance Control

Driven equations for this control diagram are represented below

$$\left(\begin{array}{c} F_e = Z_d \, v_e \\ F_f = F_e - F_{\text{int}} \\ F_c = C \, F_f \\ v_m = F_m \, P \\ F_m = F_c - F_{\text{int}} \\ F_{\text{int}} = -Z_c \, (v_h - v_m) \end{array} \right) \tag{1}$$

2 Velocity Sourced Impedance Control

Velocity sourced impedance controller is a cascaded controller architecture for series compliant actuator. The cascaded control architecture is given in the Figure 2

This controller is made of an inner velocity controlled plant and outer impedance control. The inner velocity control provides a robust motion to plant to compensate the unmodeled dynamics of friction, stiction and slip. The intermediate loop provides force feedback to ensure good force tracking. Effective impedance of the system is controlled by the outer loop.

Remark 1. *If a block diagram reduction applied on the controller architecture as given in the Figure 2, one can see that velocity sourced impedance controller is nothing but basic impedance controller with motion controlled plant.*

Remark 2. *VSIC can be analyzed with the same algorithm by defining the plant as motion controlled plant*

$$P_2(s) = \frac{P_1}{P_1 C_v + 1} = \frac{1}{C_v + \frac{1}{P_1}} = \frac{1}{P_c + \frac{I_c}{s} + Ms + B} = \frac{s}{Ms^2 + (B + P_c)s + I_c} \tag{2}$$

and the controller will be defined as

$$C_f C_v = (P_t + \frac{I_t}{s})(P_m + \frac{I_m}{s}) = P_t P_m + \frac{P_t I_m + P_m I_t}{s} + \frac{I_t I_m}{s^2} \tag{3}$$

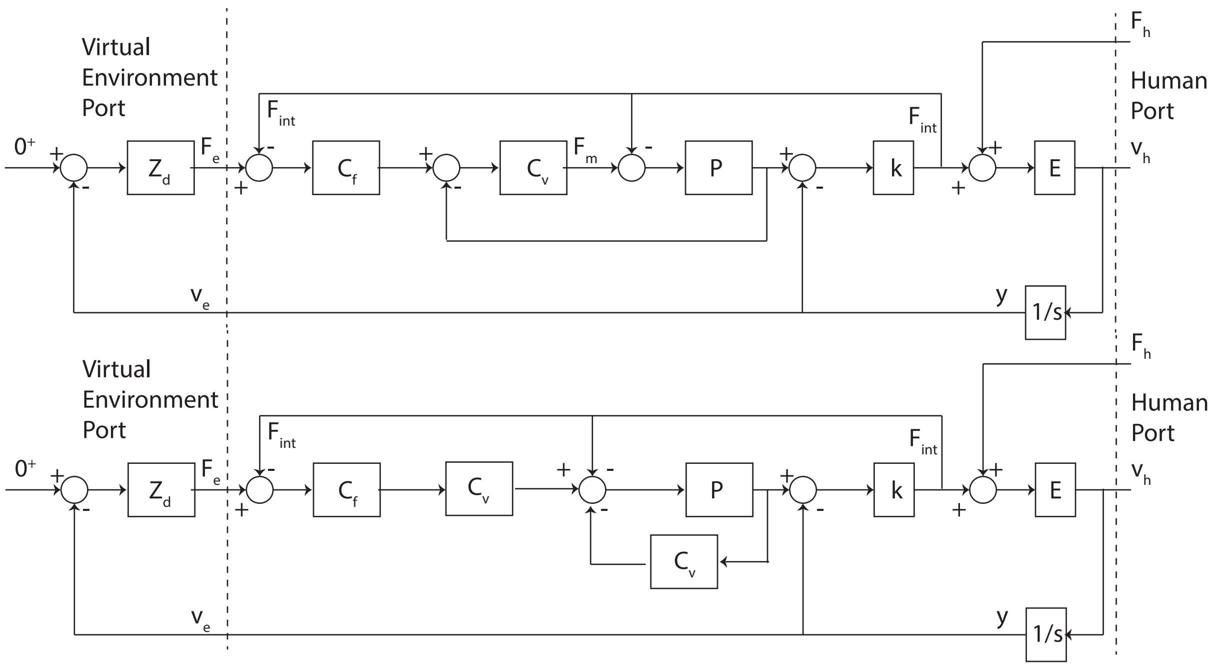

Figure 2: Control diagram of Velocity Sourced Impedance Control(above), equivalent control diagram after block diagram reduction(below)

2.1 Routh-Hurwitz

2.1.1 BIC

Routh-hurwitz analysis of BIC gives the routh array below

$$\begin{pmatrix} J & k_f + P_t\, k_f \\ B & I_t\, k_f \\ \frac{B\, k_f + B\, P_t\, k_f - I_t\, J\, k_f}{B} & 0 \\ I_t\, k_f & 0 \end{pmatrix} \tag{4}$$

this is stable for all B, where

$$B \geq \frac{I_t J}{P_t + 1} \tag{5}$$

2.1.2 VSIC

Routh-hurwitz analysis of VSIC gives the routh array below

$$\begin{pmatrix} J & I_m + k_f + P_m\,P_t\,k_f & I_m\,I_t\,k_f \\ B + P_m & I_m\,P_t\,k_f + I_t\,P_m\,k_f & 0 \\ \frac{\sigma}{B+P_m} & I_m\,I_t\,k_f & 0 \\ \frac{\beta}{\sigma} & 0 & 0 \\ I_m\,I_t\,k_f & 0 & 0 \end{pmatrix} \tag{6}$$

where

$$\sigma = B\,k_f + P_m\,k_f + B\,I_m + I_m\,P_m + {P_m}^2\,P_t\,k_f + B\,P_m\,P_t\,k_f - I_m\,J\,P_t\,k_f - I_t\,J\,P_m\,k_f$$

$$\begin{aligned}\beta = -B^2\,I_m\,I_t\,k_f + B\,{I_m}^2\,P_t\,k_f - B\,I_m\,I_t\,P_m\,k_f + B\,I_m\,P_m\,P_t^2\,{k_f}^2 + B\,I_m\,P_t\,{k_f}^2 + B\,I_t\,{P_m}^2\,P_t\,{k_f}^2 \\ + B\,I_t\,P_m\,{k_f}^2 + {I_m}^2\,P_m\,P_t\,k_f - J\,{I_m}^2\,P_t^2\,{k_f}^2 - 2\,J\,I_m\,I_t\,P_m\,P_t\,{k_f}^2 + I_m\,{P_m}^2\,P_t^2\,{k_f}^2 + I_m\,P_m\,P_t\,{k_f}^2 \\ - J\,I_t^2\,{P_m}^2\,{k_f}^2 + I_t\,{P_m}^3\,P_t\,{k_f}^2 + I_t\,{P_m}^2\,{k_f}^2\end{aligned}$$

σ and β is positive definite where

$$k_f \geq \frac{den_{k_f}}{num_{k_f}} \tag{7}$$

where

$$\begin{aligned}den_{k_f} = -J\,{I_m}^2\,P_t^2 - 2\,J\,I_m\,I_t\,P_m\,P_t + I_m\,{P_m}^2\,P_t^2 + B\,I_m\,P_m\,P_t^2 + I_m\,P_m\,P_t + B\,I_m\,P_t \\ - J\,I_t^2\,{P_m}^2 + I_t\,{P_m}^3\,P_t + B\,I_t\,{P_m}^2\,P_t + I_t\,{P_m}^2 + B\,I_t\,P_m\end{aligned}$$

$$num_{k_f} = I_t\,B^2\,I_m - P_t\,B\,{I_m}^2 + I_t\,P_m\,B\,I_m - P_m\,P_t\,{I_m}^2$$

and

$$k_f \geq -\frac{B\,I_m + I_m\,P_m}{B + P_m + {P_m}^2\,P_t + B\,P_m\,P_t - I_m\,J\,P_t - I_t\,J\,P_m} \tag{8}$$

3 Passivity analysis

The environment impedance is defined as

$$F_{int}(s) = -Z_e v_h \tag{9}$$

$$Z_e = \frac{F_{int}}{-v_h} \tag{10}$$

This linear time-invariant 1-port system is passive if and only if

- $Z_e(s)$ has no poles in the right hand plane
- Any imaginary poles of $Z_e(s)$ are simple, and have positive real residues.
- Re(Z_e(jw))≥0

Proposition 1. *For a given impedance function Z_e, condition of positive realness of $Re(Z_e)$, can be expressed as*

$$Re(num_{Z_e}(jw)den_{Z_e}(-jw)) \geq 0 \equiv Re(Z_e) \geq 0 \tag{11}$$

Proof.

$$Z_e(j\omega) = \frac{num(j\omega)}{den(j\omega)} = \frac{num(j\omega)}{den(j\omega)}\frac{den(-j\omega)}{den(-j\omega)} \tag{12}$$

Condition given in the Equation 31 is satisfied if

$$P(w) = Re(\frac{num_{Z_e}(j\omega)}{den_{Z_e}(j\omega)}\frac{den_{Z_e}(-j\omega)}{den_{Z_e}(-j\omega)}) \geq 0 \tag{13}$$

Denominator of the equation above is positive definite, however numerator of that equation can be represented as

$$num_{Z_e}(jw)den_{Z_e}(-jw) = A\omega^n + B\omega^{n-1}j + C\omega^{n-2}\cdots + D\omega^0 \tag{14}$$

□

Proposition 2. *Necessary condition of positive realness of $Re(Z_e)$,*

$$P(w) = a_m\omega^m + a_{m-2}\omega^{m-2} + \cdots + a_2\omega^2 + a_0\omega^0 \tag{15}$$

where $\forall\ a_i \geq 0$ for $i = 0, 2, 4, \ldots m$, m is the highest order power.

3.1 Passivity analysis of BIC

$$Z_2 = \frac{Z_c\ (C_f\ P\ Z_d + 1)}{P\ Z_c + C_f\ P\ Z_c + 1} \tag{16}$$

Residue of the impedance function is checked

$$\lim_{s\to 0} sZ_e(s) = k_c \geq 0 \tag{17}$$

3.1.1 Pure spring rendering

Proposition 3. *Rendering pure spring in BIC of SEA with PI controller is not passive.*

$$P(\omega) = \left(B\,{k_f}^2 + B\,P_t\,{k_f}^2 - I_t\,J\,{k_f}^2 - B\,P_t\,k_c\,k_f + I_t\,J\,k_c\,k_f\right)\,w^4 - \left(I_t\,k_c\,{k_f}^2\right)\,w^2 \tag{18}$$

Proposition 4. *Cancelling the integrator in the controller improves passivity of the system*

$$P(\omega) = \left(B\,{k_f}^2 + B\,P_t\,{k_f}^2 - B\,P_t\,k_c\,k_f\right)\,w^2 \tag{19}$$

this is satisfied where

$$k_f \geq k_c \frac{P_t}{P_t + 1} \tag{20}$$

3.1.2 Voigt model rendering

Proposition 5. *Rendering voigt model in BIC of SEA with PI controller is not passive.*

$$\begin{aligned} P(\omega) &= \left(-J\,P_t\,b_c\,k_f\right)\,w^6 \\ &+ \left(B\,{k_f}^2 + {P_t}^2\,b_c\,{k_f}^2 + B\,P_t\,{k_f}^2 - I_t\,J\,{k_f}^2 + P_t\,b_c\,{k_f}^2 - B\,I_t\,b_c\,k_f - B\,P_t\,k_c\,k_f + I_t\,J\,k_c\,k_f\right)\,w^4 \\ &+ \left({I_t}^2\,b_c\,{k_f}^2 - I_t\,k_c\,{k_f}^2\right)\,w^2 \end{aligned} \tag{21}$$

Proposition 6. *Cancelling the integrator in the controller does not ensures passivity of the system*

$$P(\omega) = \left(-J\,P_t\,b_c\,k_f\right)\,w^6 + \left(B\,{k_f}^2 + {P_t}^2\,b_c\,{k_f}^2 + B\,P_t\,{k_f}^2 + P_t\,b_c\,{k_f}^2 - B\,P_t\,k_c\,k_f\right)\,w^4 \tag{22}$$

3.1.3 Inertance rendering

$$\begin{aligned} P(\omega) &= \left(B\,P_t\,k_f\,m_c - J\,P_t\,b_c\,k_f - I_t\,J\,k_f\,m_c\right)\,w^6 \\ &+ \left(B\,{k_f}^2 + {P_t}^2\,b_c\,{k_f}^2 + B\,P_t\,{k_f}^2 - I_t\,J\,{k_f}^2 + P_t\,b_c\,{k_f}^2 + I_t\,{k_f}^2\,m_c - B\,I_t\,b_c\,k_f - B\,P_t\,k_c\,k_f + I_t\,J\,k_c\,k_f\right)\,w^4 \\ &+ \left({I_t}^2\,b_c\,{k_f}^2 - I_t\,k_c\,{k_f}^2\right)\,w^2 \end{aligned} \tag{23}$$

Proposition 7. *Necessary conditions of passivity of rendering voigt model in BIC of SEA with PI controller are*

$$b_c \geq \frac{k_c}{I_t} \tag{24}$$

$$m_c \geq \frac{J\,P_t\,b_c}{B\,P_t - I_t\,J} \tag{25}$$

$$k_f \geq k_c \frac{B\,P_t - I_t\,J + \frac{B\,I_t\,b_c}{k_c}}{B + P_t\,b_c + I_t\,m_c + {P_t}^2\,b_c + B\,P_t - I_t\,J} \tag{26}$$

Proposition 8. *Necessary conditions of passivity of rendering voigt model in BIC of SEA with P controller are*

$$\frac{m_c}{P_t} \geq \frac{J}{B} \tag{27}$$

$$k_f \geq k_c \frac{B P_t}{(P_t + 1)(B + P_t b_c)} \tag{28}$$

3.2 Passivity analysis of VSIC

$$Z_1 = \frac{Z_c\,\left(C_m\,P + C_f\,C_m\,P\,Z_d + 1\right)}{P\,Z_c + C_m\,P + C_f\,C_m\,P\,Z_c + 1} \tag{29}$$

Residue of the impedance function is checked

$$\lim_{s \to 0} s Z_e(s) = k_c \geq 0 \tag{30}$$

Real part of the impedance function at frequency domain should greater or equal to the zero.

$$Re(Z_e) \geq 0 \tag{31}$$

3.2.1 Pure spring rendering

Proposition 9. *Rendering pure spring in VSIC of SEA with PI controller is passive if* $P(w) \geq 0$ *where*

$$\begin{aligned} P(\omega) = &\left(B\,{k_f}^2 + P_m\,{k_f}^2 + {P_m}^2\,P_t\,{k_f}^2 + B\,P_m\,P_t\,{k_f}^2 - I_m\,J\,P_t\,{k_f}^2 - I_t\,J\,P_m\,{k_f}^2\right)\,w^6 \\ &\left(-{P_m}^2\,P_t\,k_c\,k_f - B\,P_m\,P_t\,k_c\,k_f + I_m\,J\,P_t\,k_c\,k_f + I_t\,J\,P_m\,k_c\,k_f\right)\,w^6 \\ &+ \left({I_m}^2\,P_t\,{k_f}^2 - I_m\,P_t\,k_c\,{k_f}^2 - I_t\,P_m\,k_c\,{k_f}^2 - {I_m}^2\,P_t\,k_c\,k_f - B\,I_m\,I_t\,{k_f}^2 + B\,I_m\,I_t\,k_c\,k_f\right)\,w^4 \end{aligned} \tag{32}$$

This is satisfied where

$$k_f \geq k_c \frac{\left(B\,I_m\,I_t - {I_m}^2\,P_t\right)}{\left(I_m\,P_t + I_t\,P_m\right)\,k_c + B\,I_m\,I_t - {I_m}^2\,P_t} \tag{33}$$

and

$$k_f \geq k_c \frac{{P_m}^2\,P_t + B\,P_m\,P_t - I_m\,J\,P_t - I_t\,J\,P_m}{B + P_m + {P_m}^2\,P_t + B\,P_m\,P_t - I_m\,J\,P_t - I_t\,J\,P_m} \tag{34}$$

This inequality can be written as

$$J \geq \frac{B\,{k_f}^2 + P_m\,{k_f}^2 + {P_m}^2\,P_t\,{k_f}^2 + B\,P_m\,P_t\,{k_f}^2 - {P_m}^2\,P_t\,k_c\,k_f - B\,P_m\,P_t\,k_c\,k_f}{I_m\,P_t\,{k_f}^2 + I_t\,P_m\,{k_f}^2 - I_m\,P_t\,k_c\,k_f - I_t\,P_m\,k_f\,k_c} \tag{35}$$

$$J \geq k_c \frac{(B+P_m)(P_m P_t \Delta K + k_f)}{\alpha_3 \Delta K} \tag{36}$$

Proposition 10. *Rendering pure spring in VSIC of SEA with P controller is passive if $P(w) \geq 0$ where*

$$P(\omega) \quad = \quad \left(B\, {k_f}^2 + P_m\, {k_f}^2 + {P_m}^2\, P_t\, {k_f}^2 + B\, P_m\, P_t\, {k_f}^2 - {P_m}^2\, P_t\, k_c\, k_f - B\, P_m\, P_t\, k_c\, k_f\right)\, w^6 \tag{37}$$

This is satisfied where

$$k_f \geq k_c \frac{P_m\, P_t}{P_m\, P_t + 1} \tag{38}$$

3.2.2 Voigt model rendering

Proposition 11. *Rendering voigt model in VSIC of SEA with PI controller is not passive.*

$$P(\omega) = \left(-J\, P_m\, P_t\, b_c\, k_f\right)\, w^8 + \left(B\, {k_f}^2 + P_m\, {k_f}^2 + {P_m}^2\, P_t\, {k_f}^2 + B\, P_m\, P_t\, {k_f}^2 - I_m\, J\, P_t\, {k_f}^2 - I_t\, J\, P_m\, {k_f}^2\right.$$

Proposition 12. *Cancelling the both of integrators in the controllers does not ensures passivity of the system*

Proposition 13. *Cancelling one of the proportional gains of controllers provides set of conditions for passivity.*

Proposition 14. *Cancelling proportional gain of torque controller($P_t = 0$) provides set of conditions for passivity.*

$$\begin{aligned} P(\omega) = &\left(B\, {k_f}^2 + P_m\, {k_f}^2 - I_t\, J\, P_m\, {k_f}^2 - I_t\, {P_m}^2\, b_c\, k_f - B\, I_t\, P_m\, b_c\, k_f + I_m\, I_t\, J\, b_c\, k_f + I_t\, J\, P_m\, k_c\, k_f\right)\, w^6 \\ &+ \left({I_t}^2\, {P_m}^2\, b_c\, {k_f}^2 - {I_m}^2\, I_t\, b_c\, k_f - I_t\, P_m\, k_c\, {k_f}^2 - I_m\, I_t\, b_c\, {k_f}^2 - B\, I_m\, I_t\, {k_f}^2 + B\, I_m\, I_t\, k_c\, k_f\right)\, w^4 \\ &+ \left({I_m}^2\, {I_t}^2\, b_c\, {k_f}^2\right)\, w^2 \end{aligned} \tag{40}$$

Proposition 15. *Cancelling proportional gain of motion controller($P_m = 0$) provides set of conditions for passivity.*

$$\begin{aligned} P(\omega) = &\left(B\, {k_f}^2 - I_m\, J\, P_t\, {k_f}^2 - B\, I_m\, P_t\, b_c\, k_f + I_m\, I_t\, J\, b_c\, k_f + I_m\, J\, P_t\, k_c\, k_f\right)\, w^6 \\ &+ \left({I_m}^2\, P_t\, {k_f}^2 - I_m\, I_t\, b_c\, {k_f}^2 - {I_m}^2\, I_t\, b_c\, k_f - I_m\, P_t\, k_c\, {k_f}^2 - {I_m}^2\, P_t\, k_c\, k_f\right)\, w^4 \\ &+ \left({I_m}^2\, {P_t}^2\, b_c\, {k_f}^2 - B\, I_m\, I_t\, {k_f}^2 + B\, I_m\, I_t\, k_c\, k_f\right)\, w^4 \\ &+ \left({I_m}^2\, {I_t}^2\, b_c\, {k_f}^2\right)\, w^2 \end{aligned} \tag{41}$$

3.2.3 Inertance rendering

$$
\begin{aligned}
P(\omega) = \big(&{P_m}^2\, P_t\, k_f\, m_c - J\, P_m\, P_t\, b_c\, k_f + B\, P_m\, P_t\, k_f\, m_c - I_m\, J\, P_t\, k_f\, m_c - I_t\, J\, P_m\, k_f\, m_c\big)\; w^8 \\
+\big(&B\, {k_f}^2 + P_m\, {k_f}^2 + {P_m}^2\, P_t\, {k_f}^2 + B\, P_m\, P_t\, {k_f}^2 - I_m\, J\, P_t\, {k_f}^2 - I_t\, J\, P_m\, {k_f}^2 - I_t\, {P_m}^2\, b_c\, k_f + P_m\, P_t\, b_c\, {k_f}^2\big)\; w \\
+\big(&I_m\, P_t\, {k_f}^2\, m_c + I_t\, P_m\, {k_f}^2\, m_c + {I_m}^2\, P_t\, k_f\, m_c - {P_m}^2\, P_t\, k_c\, k_f + {P_m}^2\, {P_t}^2\, b_c\, {k_f}^2 - B\, I_m\, P_t\, b_c\, k_f - B\, I_t\, P_m\, b_c \\
+\big(&{I_m}^2\, P_t\, {k_f}^2 - I_m\, I_t\, b_c\, {k_f}^2 - {I_m}^2\, I_t\, b_c\, k_f - I_m\, P_t\, k_c\, {k_f}^2 - I_t\, P_m\, k_c\, {k_f}^2 - {I_m}^2\, P_t\, k_c\, k_f + {I_m}^2\, {P_t}^2\, b_c\, {k_f}^2 + {I_t}^2 \\
&+ \big({I_m}^2\, {I_t}^2\, b_c\, {k_f}^2\big)\; w^2 \qquad (42)
\end{aligned}
$$

This is satisfied where

$$m_c \geq \frac{J\, P_m\, P_t\, b_c}{{P_m}^2\, P_t + B\, P_m\, P_t - I_m\, J\, P_t - I_t\, J\, P_m} \tag{43}$$

$$k_f \geq \frac{num_{k_f}}{den_{k_f}} \tag{44}$$

where

$$
\begin{aligned}
num_{k_f} = {}& I_t\, {P_m}^2\, b_c - {I_m}^2\, P_t\, m_c + {P_m}^2\, P_t\, k_c + B\, I_m\, P_t\, b_c + B\, I_t\, P_m\, b_c - I_m\, I_t\, J\, b_c + B\, I_m\, I_t\, m_c \\
&+ B\, P_m\, P_t\, k_c - I_m\, J\, P_t\, k_c - I_t\, J\, P_m\, k_c
\end{aligned}
$$

and

$$den_{k_f} = B + P_m + {P_m}^2\, P_t + {P_m}^2\, {P_t}^2\, b_c + B\, P_m\, P_t - I_m\, J\, P_t - I_t\, J\, P_m + P_m\, P_t\, b_c + I_m\, P_t\, m_c + I_t\, P_m\, m_c$$

This inequality can be written as

$$J \geq \frac{(B + P_m)(P_m P_t \Delta K + k_f) + \eta_1}{\frac{\alpha_3}{k_c}\Delta K - I_m I_t b_c} \tag{45}$$

where

$$
\begin{aligned}
\eta_1 = {}& b_c\, \left(-k_f\, {P_m}^2\, {P_t}^2 + I_t\, {P_m}^2 - k_f\, P_m\, P_t + B\, I_t\, P_m + B\, I_m\, P_t\right) \\
&- m_c\, \left({I_m}^2\, P_t - B\, I_m\, I_t + I_m\, P_t\, k_f + I_t\, P_m\, k_f\right) \qquad (46)
\end{aligned}
$$

and

$$\eta_1 = b_c(\frac{B}{k_c}\alpha_3 + I_t P_m^2 - \frac{k_f}{k_c}\alpha_4) + m_c(\alpha_5 - \frac{k_f}{k_c}\alpha_3) \tag{47}$$

more expanded

$$J \geq \frac{(B+P_m)(P_m P_t \Delta K + k_f) + b_c(\frac{B}{k_c}\alpha_3 + I_t P_m^2 - \frac{k_f}{k_c}\alpha_4) + m_c(\alpha_5 - \frac{k_f}{k_c}\alpha_3)}{\frac{\alpha_3}{k_c}\Delta K + I_m I_t b_c} \tag{48}$$

$$k_f \geq -\frac{{I_m}^2\, I_t\, b_c + {I_m}^2\, P_t\, k_c - B\, I_m\, I_t\, k_c}{B\, I_m\, I_t - {I_m}^2\, {P_t}^2\, b_c - {I_t}^2\, {P_m}^2\, b_c - {I_m}^2\, P_t + I_m\, I_t\, b_c + I_m\, P_t\, k_c + I_t\, P_m\, k_c} \tag{49}$$

this equation can be written as

$$k_f \geq \frac{\left(B\, I_m\, I_t - {I_m}^2\, P_t\right)\, k_c - {I_m}^2\, I_t\, b_c}{\left(I_m\, P_t + I_t\, P_m\right)\, k_c + B\, I_m\, I_t - {I_m}^2\, P_t + \eta_2} \tag{50}$$

where

$$\eta_2 \quad = \quad \left(-{I_m}^2\, {P_t}^2 + I_m\, I_t - {I_t}^2\, {P_m}^2\right)\, b_c \tag{51}$$

Proposition 16. *Cancelling both of the integral gains provides*

$$\begin{aligned} P(\omega) = &\left({P_m}^2\, P_t\, k_f\, m_c - J\, P_m\, P_t\, b_c\, k_f + B\, P_m\, P_t\, k_f\, m_c\right)\, w^8 \\ &+ \left(B\, {k_f}^2 + P_m\, {k_f}^2 + {P_m}^2\, P_t\, {k_f}^2 + B\, P_m\, P_t\, {k_f}^2 + P_m\, P_t\, b_c\, {k_f}^2 - {P_m}^2\, P_t\, k_c\, k_f + {P_m}^2\, {P_t}^2\, b_c\, {k_f}^2\right)\, w^6 \\ &- \left(B\, P_m\, P_t\, k_c\, k_f\right)\, w^6 \end{aligned} \tag{52}$$

This is satisfied where

$$k_f \geq k_c \frac{P_t\, {P_m}^2 + B\, P_t\, P_m}{B + P_m + {P_m}^2\, P_t + {P_m}^2\, {P_t}^2\, b_c + B\, P_m\, P_t + P_m\, P_t\, b_c} \tag{53}$$

$$m_c \geq \frac{J\, P_m\, P_t\, b_c\, k_f}{P_t\, k_f\, {P_m}^2 + B\, P_t\, k_f\, P_m} \tag{54}$$

Remark 3. *To validate the system model, null impedance is rendered and Fatih Emre's results are obtained.*

$$\begin{aligned} P(\omega) = &{I_m}^2\, P_t\, {k_f}^2\, w^4 - B\, I_m\, I_t\, {k_f}^2\, w^4 + \\ &B\, {k_f}^2\, w^6 + P_m\, {k_f}^2\, w^6 + {P_m}^2\, P_t\, {k_f}^2\, w^6 + B\, P_m\, P_t\, {k_f}^2\, w^6 - I_m\, J\, P_t\, {k_f}^2\, w^6 - I_t\, J\, P_m\, {k_f}^2\, w^6 \end{aligned} \tag{55}$$

Impedance Controller	Controller	Pure Spring V.E	Voigt Model V.E	Inertance V.E
BIC	P	Equation 20	Not passive	Equation 27 $\wedge$ Equation 28
	PI	Not passive	Not passive	Equation 24 $\wedge$ Equation 25 $\wedge$ Equation 26
VSIC	P-P	Equation 38	Not passive	Equation 53 $\wedge$ Equation 54
	PI-PI	Equation 50 $\wedge$ Equation 34	Not passive	Equation 43 $\wedge$ Equation 44 $\wedge$ Equation 49

Property	Symbol	Value	Units
Effective inertia	J	0.2	[N-m/(s^2rad)]
Effective damping	B	3	[N-ms/rad]
Physical spring stiffness	K	250	[N-m/rad]
Integral motion controller	I_m	300	[rad/(s^2N-m)]
Proportional torque controller	P_t	0.01	[rad/(sN-m)]
Time delay	τ	50	[ms]

4 General coupler

For a generalized coupler whose impedance transfer function is given by

$$Z(s) = \frac{k\ (a_0\, s^2 + a_1\, s + 1)}{s\ (d_0\, s^2 + d_1\, s + 1)} \tag{56}$$

Z(s) is positive real if and only if

$$Re(Z(j\omega) \geq 0 \tag{57}$$

and

$$Z(j\omega) = \frac{num(j\omega)}{den(j\omega)} = \frac{num(j\omega)}{den(j\omega)} \frac{den(-j\omega)}{den(-j\omega)} \tag{58}$$

Proposition 17. *For a given impedance transfer function Z(s), condition of positive realness of $Re(Z(j\omega))$, can be expressed as*

$$Re(num_{Z(j\omega)}(jw)den_{Z(j\omega)}(-jw)) \geq 0 \equiv Re(Z(j\omega)) \geq 0 \tag{59}$$

Proof. Any complex number multiplied by its complex conjugate is a real number, equal to the square of the modulus of the complex numbers. Square of the modulus of any complex number is positive real. □

So the condition of positive realness of given impedance transfer function can be derived by using following equation

$$Re(num_{Z(j\omega)}(jw)den_{Z(j\omega)}(-jw)) = a_1\, k\, w^2 - d_1\, k\, w^2 + a_0\, d_1\, k\, w^4 - a_1\, d_0\, k\, w^4 \tag{60}$$

This equation is positive definite if and only if

$$\begin{aligned} a_1\, k - d_1\, k &\geq 0 \\ a_0\, d_1\, k - a_1\, d_0\, k &\geq 0 \end{aligned} \tag{61}$$

This set of conditions implies

$$\beta_1 := a_0 d_1 - a_1 d_0 \geq 0 \tag{62}$$

$$\beta_3 := a_1 - d_1 \geq 0 \tag{63}$$

Finally, if we multiply β_3 with d_0

$$\beta_3' := a_1 d_0 - d_1 d_0 \geq 0 \tag{64}$$

and then

$$\beta_1 + \beta_3' := a_0 d_1 - d_0 d_1 \geq 0 \tag{65}$$

The last condition can be derived as:

$$\beta_2 := a_0 - d_0 \geq 0 \tag{66}$$

additionally, we define another abbreviations as:

$$\beta_4 := a_0 - a_1 d_1 + d_0 \tag{67}$$

$$\beta_5 := 2d_0 - d_1^2 \tag{68}$$

4.0.1 General Coupler - Inertance model relation

the following abbreviations are used through analysis

$$
\begin{aligned}
\alpha_1 :=& I_m^2 P_t - B I_m I_t \\
\alpha_2 :=& (B + P_m) P_m P_t - J \alpha_3 \\
\alpha_3 :=& I_t P_m + I_m P_t \\
\alpha_4 :=& I_m^2 P_t^2 - I_m I_t + I_t^2 P_m^2 \\
\alpha_5 :=& I_t P_m^2 - J I_m I_t \\
\alpha_6 :=& P_m P_t + P_m^2 P_t^2 \\
\beta_1 :=& a_0 d_1 - a_1 d_0 \\
\beta_2 :=& a_0 - d_0 \\
\beta_3 :=& a_1 - d_1 \\
\beta_4 :=& a_0 - a_1 d_1 + d_0 \\
\beta_5 :=& d_1^2 - 2 d_0 \\
\sigma_1 :=& B P_t - I_t J \\
\sigma_2 :=& K P_t (P_t + 1) - B I_t
\end{aligned}
$$

For VSIC, with virtual general coupler

$$P(\omega) = d_2\omega^2 + d_4\omega^4 + d_6\omega^6 + d_8\omega^8 + d_10\omega^{10} \tag{69}$$

$$\begin{aligned}
&d_2 = K^2[{I_m}^2\,{I_t}^2\,k_e\beta_3]\\
&d_4 = K[\alpha_1\Delta K - \alpha_3 K k_e + k_e\beta_3(\alpha_4 K - I_m^2 I_t) + k_e\beta_1 I_m^2 I_t^2]\\
&d_4 = K[\alpha_2\Delta K + K(P_m + B) + k_e\beta_4(\alpha_1 + \alpha_3 K) + k_e\beta_3(\alpha_6 K - \alpha_5 - \alpha_3 B) + k_e\beta_1(\alpha_4 K - I_m^2 I_t)\\
&+ K\beta_5\alpha_1]\\
&d_8 = K[\alpha_2 k_e\beta_4 - \beta_3 k_e J P_m P_t + d_0^2 K\alpha_1 - a_0 d_0 k_e(\alpha_1 + \alpha_3 K) + k_e\beta_1(\alpha_6 K - \alpha_5 - \alpha_3 B)\\
&+ \beta_5(K\alpha_2 + K(B + P_m)]\\
&d_{10} = K[d_0^2(K(B + P_m) + K\alpha_2) - a_0 d_0 k_e\alpha_2 - \beta_1 k_e J P_m P_t]
\end{aligned}$$

If we arrange the following equation set for inertance model

$$\begin{aligned}
&d_2 = K^2[{I_m}^2\,{I_t}^2\,b_e]\\
&d_4 = K[\alpha_1\Delta K - \alpha_3 K k_e + b_e(\alpha_4 K - I_m^2 I_t)]\\
&d_6 = K[\alpha_2\Delta K + K(P_m + B) + m_e(\alpha_1 + \alpha_3 K)\\
&+ b_e(\alpha_6 K - \alpha_5 - \alpha_3 B)]\\
&d_8 = K[\alpha_2 m_e - b_e J P_m P_t]
\end{aligned}$$

where $\gamma_1 = \alpha_1\Delta K - \alpha_3 K k_e$ and $\gamma_2 = \alpha_2\Delta K + K(P_m + B)$

$$\begin{aligned}
&d_2 = K^2[{I_m}^2\,{I_t}^2\,b_e]\\
&d_4 = K[\gamma_1 + b_e(\alpha_4 K - I_m^2 I_t)]\\
&d_6 = K[\gamma_2 + m_e(\alpha_1 + \alpha_3 K)\\
&+ b_e(\alpha_6 K - \alpha_5 - \alpha_3 B)]\\
&d_8 = K[\alpha_2 m_e - b_e J P_m P_t]
\end{aligned}$$

and the equation set for inertance model is related with the general coupler with following substitutions

$$\begin{aligned}
m_e &\rightarrow k_e(\beta_4 - a_0 d_0\omega^2)\\
b_e &\rightarrow k_e(\beta_3 + \beta_1\omega^2)\\
\gamma_1 &\rightarrow \gamma_1 + \omega^2\beta_5(\lim_{k_e\to 0}\gamma_1)\\
\gamma_2 &\rightarrow \gamma_2 + \omega^2 d_0^2(\lim_{k_e\to 0}\gamma_2)
\end{aligned}$$

Equations for BIC of inertance model rendering

$$P_1(\omega) = d_2\omega^2 + d_4\omega^4 + d_6\omega^6 \tag{70}$$

$$\begin{aligned}
d_2 &= K^2 I_t[I_t b_e - k_e] \\
d_4 &= K[(BP_t - I_t J)\Delta K + BK + m_e I_t K + b_e(KP_t(P_t + 1) - BI_t)] \\
d_6 &= K[(BP_t - I_t J)m_e - JP_t b_e]
\end{aligned}$$

if we arrange this where $\gamma = \sigma_1 \Delta K + BK =$, $\sigma_1 = BP_t - I_t J$ and $\sigma_2 = KP_t(P_t + 1) - BI_t$

$$\begin{aligned}
d_2 &= K^2 I_t[I_t b_e - k_e] \\
d_4 &= K[(\sigma_1)\Delta K + BK + m_e I_t K + b_e(\sigma_2)] \\
d_6 &= K[(\sigma_1)m_e - JP_t b_e]
\end{aligned}$$

$$\begin{aligned}
d_2 &= K^2 I_t[I_t b_e - k_e] \\
d_4 &= K[\gamma + m_e I_t K + b_e(\sigma_2)] \\
d_6 &= K[(\sigma_1)m_e - JP_t b_e]
\end{aligned}$$

For BIC with virtual general coupler

$$P(\omega) = d_2\omega^2 + d_4\omega^4 + d_6\omega^6 + d_8\omega^8 \tag{71}$$

$$\begin{aligned}
d_2 &= K^2 I_t[I_t k_e \beta_3 - k_e] \\
d_4 &= K[(\sigma_1 \Delta K + BK) + k_e\beta_3(\sigma_2) + I_t K k_e(I_t\beta_1 + \beta_4)] \\
d_6 &= K[\beta_5(\sigma_1 \Delta K + BK) - k_e\beta_3 JP_t + k_e\beta_1(\sigma_2) + (\beta_4 + \beta_5)k_e\sigma_1 - I_t a_0 d_0 k_e K] \\
d_8 &= K[d_0^2(\sigma_1 K + BK) - JP_t k_e \beta_1 - a_0 d_0 k_e(\sigma_1)]
\end{aligned}$$

and the equation set for inertance model is related with the general coupler with following substitutions

$$\begin{aligned}
m_e &\rightarrow k_e(\beta_4 - a_0 d_0 \omega^2) \\
b_e &\rightarrow k_e(\beta_3 + \beta_1 \omega^2) \\
\gamma &\rightarrow \gamma + \omega^2 \beta_5(\lim_{k_e \to 0} \gamma) + \omega^4 d_0^2(\lim_{k_e \to 0} \gamma)
\end{aligned}$$

Proposition 18. *General coupler is passive if $d_0, d_1, k_e, \beta_1, \beta_2, \beta_3 \geq 0$ and $a_0 < 0$.*

In terms of passive impedance control conditions, effect of mass element in general coupler is driven by k_e times $\beta_4 (= a_0 - a_1 d_1 + d_0)$ at low frequencies and $-a_0 d_0$ at high frequencies.

Damping's effect is k_e times $\beta_3 (= a_1 - d_1)$ at low frequencies and $\beta_1 (= a_0 d_1 - a_1 d_0)$ at high frequencies.

Behaviour of stiffness element of general coupler is driven by $\beta_5 (= d_1^2 - 2d_0)$ at low frequencies and d_0^2 at high frequencies.

5 Sturm on BIC

$$\begin{pmatrix} 0 & + & 0 & - & 0 & + \\ + & + & \begin{pmatrix} - \Leftrightarrow d_4 > 0 \\ + \Leftrightarrow d_4 < 0 \end{pmatrix} & -sign(s_1) & -sign(s_2) & + \end{pmatrix} \tag{72}$$

where

$$s_1 = 4d_4 - 12\frac{d_2 d_6}{d_4}$$
$$s_2 = \frac{2d_2 d_4}{3s_1} - \frac{2}{3}d_2$$