Due Date: April 18, 2018, Beginning of the class

How to submit: Type your HW and submit hard copy in class

4.1 Consider the optimization problem

minimize
$$x^2 + 1$$

subject to $(x-2)(x-4) \le 0$

with variable $x \in \mathbf{R}$

- (a) Analysis of primal problem. Give the feasible set, the optimal value, and the optimal solution.
- (b) Lagrangian and dual function. Plot the objective x^2+1 versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x,\lambda)$ versus x for a few positive values of λ . Verify the lower bound property ($p^* \ge \inf_x L(x,\lambda)$ for $\lambda \ge 0$). Derive and sketch the Lagrange dual function g.
- (c) Lagrange dual problem. State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution λ^* . Does strong duality hold?
- (d) Sensitivity analysis. Let $p^*(u)$ denote the optimal value of the problem

minimize
$$x^2 + 1$$

subject to $(x-2)(x-4) \le u$

as a function of the parameter u. Plot $p^*(u)$. Verify that $\frac{dp^*(0)}{du} = -\lambda^*$.

4.2 Consider the quadratic program

minimize
$$x_1^2 + 2x_2^2 - x_1x_2 - x_1$$

 $x_1 + 2x_2 \le u_1$
subject to $x_1 - 4x_2 \le u_2$
 $5x_1 + 76x_2 \le 1$

with variables x_1 , x_2 , and parameters u_1 , u_2 .

Solve this QP, for parameter values $u_1=-2$, $u_2=-3$, to find optimal primal variable values x_1^* and x_2^* , and optimal dual variable values λ_1^* , λ_2^* and λ_3^* . Let p^* denote the optimal objective value. Verify that the KKT conditions hold for the optimal primal and dual variables you found (within reasonable numerical accuracy).

Matlab hint: See the CVX users' guide to find out how to retrieve optimal dual variables. To specify the quadratic objective, use **quad_form()**.

4.3 Find the dual function of the LP

minimize
$$c^T x$$

subject to $Gx \le h, Ax = b$

Give the dual problem, and make the implicit equality constraints explicit.

4.4 The relative entropy between two vectors x, $y \in \mathbf{R}_{++}^n$ is defined as $\sum_{k=1}^n x_k \log(x_k/y_k)$. This is a convex function, jointly in x and y. In the following problem we calculate the vector x that minimizes the relative entropy with a given vector y, subject to equality constraints on x:

minimize
$$\sum_{k=1}^{n} x_k \log(x_k/y_k)$$

subject to $Ax = b, 1^T x = 1$

The optimization variable is $x \in \mathbf{R}^n$. The domain of the objective function is \mathbf{R}^n_{++} . The parameters $y \in \mathbf{R}^n_{++}$, $A \in \mathbf{R}^{m \times n}$, and $b \in \mathbf{R}^m$ are given. Derive the Lagrange dual of this problem and simplify it to get

maximize
$$b^T z - \log \sum_{k=1}^n y_k e^{a_k^T z}$$

(a_k is the kth column of A).