Let's get *deranged!*
John C. Baez, December 14, 2003

A **derangement** *of a set S is a bijection* $f\colon S \to S$ *such that no element* $x \in S$ *has* $f(x) = x$. *The challenge of finding the number of derangements of an n-element set was first posed by Pierre de Montmort in 1708, and it was solved by him in 1713. Sometimes this number is called the* **subfactorial** $!n$. *It is easy to compute* $!n$ *for low values of n by explicitly listing all derangements of an n-element set, but it gets harder as n increases:*

$!0 = 1:$ ()
$!1 = 0:$
$!2 = 1:$ (21)
$!3 = 2:$ (231), (312)
$!4 = 9:$ (2143), (2341), (2413), (3142), (3412), (3421), (4123), (4312), (4321)

Here we will tackle the challenge of counting derangements in two different ways.

Nicholas Bernoulli solved this problem at about the same de Montmort did, using the **inclusion-exclusion principle**. *If you don't know this principle, I should start by explaining it. Suppose there's a weird country where the provinces overlap, so that some people live in one province, but some live in two provinces, and some live in three, and so on. And suppose you want to count the number of people in this country. You could do it as follows:*

- First count the number of people in each province and add up the results. This is an overestimate because a bunch of people live in two or more provinces, and you've counted them more than once. But, you'd be done if nobody lived in more than one province!

- Then, for each pair of provinces, find out the number of people who live in both. Sum up the results for all (unordered) pairs of (distinct) provinces. Then subtract this from the overestimate you had before! Now you have an *underestimate*, because a bunch of people live in three or more provinces, and you've subtracted them off more than once. But, you'd be done if nobody lived in more than two provinces!

- Then, for each triple of provinces, find out the number of people who live in all three. Sum up the results for all (unordered) triples of (distinct) provinces. Then add this to the underestimate you had before! Now you have an *overestimate*, because a bunch of people live in four or more provinces, and you've added them back in more than once. But, you'd be done if nobody lived in more than three provinces.

And so on. If there are only finitely many provinces, you'll get the right answer eventually! In short, given a collection of not-necessarily-disjoint sets $\{A_1, \ldots, A_n\}$, *the cardinality of their union is given by*

$$\left| \bigcup_i A_i \right| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots$$

where the alternating sum stops at the nth term. This is the inclusion-exclusion principle.

As an immediate consequence, if X is a set and $\{A_1, \ldots, A_n\}$ *is a collect of not-necessarily-disjoint subsets of X, we have*

$$\left| X - \bigcup_i A_i \right| = |X| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \cdots$$

Now let X be the set of all permutations of the n-element set, and let A_i *($1 \le i \le n$) be the set of all permutations that map the ith element to itself.* $X - \bigcup_i A_i$ *is the set of derangements of n, so*

$$!n = n! - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \cdots$$

1. Prove that

$$!n = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n}(n-n)!$$

2. Prove that

$$!n = n! \left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!} \right)$$

3. At the banquet of a large conference, n mathematicians hang their coats on the coat rack as they enter. At the end of the night they leave in a drunken stupor, each one randomly putting on a coat without checking that it's their own. Show that in the limit as $n \to \infty$, the probability that *none* of the mathematicians staggers home in their own coat approaches $1/e$. *Moral: if you see a mathematician wearing an ill-fitting coat, there's a good chance he's deranged.*

4. Show that if $n > 0$ then $!n$ is the integer closest to $n!/e$.

Now let's count derangements another way, using generating functions:

5. Let D be the structure type such that a D-structure on the finite set S is a derangement of S. Let P be the structure type such that a P-structure on S is a permutation of S. Recall that the structure type E^Z, whose generating function is e^z, is the structure of 'being a finite set'. Construct an isomorphism

$$P \cong E^Z D.$$

Hint: show that any permutation $f\colon S \to S$ gives a way to chop S into two subsets: one on which f is quite dull, the other on which f acts as a derangement.

6. Using part 5, show that

$$|D|(z) = \frac{e^{-z}}{1-z}.$$

7. Using part 6, show that

$$(1-z)\frac{d}{dz}|D|(z) = |D|(z) - e^{-z}.$$

8. Since the number of derangements of the n-element set is $!n$, we have

$$|D|(z) = \sum_{n=0}^{\infty} \frac{!n}{n!} z^n$$

Using this and part 7, show that

$$!(n+1) = (n+1)\,!n + (-1)^{n+1}$$

This recurrence is the subfactorial analogue of the more famous

$$(n+1)! = (n+1)\,n!$$

9. To convince yourself that this stuff is really cool, calculate $!n$ for $1 \le n \le 6$ using both the formula in part 4 and the recurrence in part 8.