

# Asymptotic derivation of high-order rod models from non-linear 3D elasticity

BASILE AUDOLY, CLAIRE LESTRINGANT

Laboratoire de Mécanique des Solides, CNRS, Institut Polytechnique de Paris, 91120 Palaiseau, France  
Structures Research Group, Department of Engineering, University of Cambridge, Cambridge CB2 1PZ, United Kingdom, casl4@cam.ac.uk

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We propose a method for deriving equivalent one-dimensional models for slender non-linear structures. The approach is designed to be broadly applicable, and can handle in principle finite strains, finite rotations, arbitrary cross-sections shapes, inhomogeneous elastic properties across the cross-section, arbitrary elastic constitutive laws (possibly with low symmetry) and arbitrary distributions of pre-strain, including finite pre-strain. It is based on a kinematic parameterization of the actual configuration that makes use of a center-line, a frame of directors, and local degrees of freedom capturing the detailed shape of cross-sections. A relaxation method is applied that holds the framed center-line fixed while relaxing the local degrees of freedom; it is asymptotically valid when the macroscopic strain and the properties of the rod vary slowly in the longitudinal direction. The outcome is a one-dimensional strain energy depending on the apparent stretching, bending and twisting strain of the framed center-line; the dependence on the strain gradients is also captured, yielding an equivalent rod model that is asymptotically exact to higher order. The method is presented in a fully non-linear setting and it is verified against linear and weakly non-linear solutions available from the literature.

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## 1. INTRODUCTION

This work aims at identifying efficient and accurate models for non-linear, slender elastic structures. Although there exists a wide range of one-dimensional beam and rod theories, understanding nonlinear effects arising during the compression of wide columns (LvHC17, CJ20), predicting the emergence of shape due to heterogeneous pre-stress generated by growth or thermal effects in slender filaments (LHS+14, TTB+20) and designing structures made of complex nonlinear materials such as nematic elastomers or active materials (ADK17, TV17) remain challenging tasks. Well-established, classical rod theories account for the stretching, bending and twisting strains in a linear way and therefore do not account for finite-strain or finite-thickness effects. Extensions have been proposed to account for some of these effects, but their justification is often patchy or relies on restrictive hypotheses on the kinematics or the constitutive behavior; their range of applicability is thus often limited and sometimes ill-defined. This leaves researchers, engineers and designers with two alternatives: either rely on full three-dimensional finite elasticity (ST98, GVD08, DDG11, CJ20) or build their own specific reduced model (LvHC17). Should they choose this latter option, however, a clear and rigorous methodology for deriving such a model is lacking.

This work builds up on a dimension reduction procedure introduced by the authors in an abstract setting (LA20b) which is applied here to the case of a hyper-elastic prismatic solid which can stretch, bend and twist arbitrarily in three dimensions. The present work extends our previous work on one-dimensional structures that can just stretch (AH16, LA20b).

The proposed method yields one-dimensional models that account for stretching, bending and twisting modes in a non-linear way. It is asymptotically correct; a scaling estimate of the error in energy with respect to the full three-dimensional theory is available in terms of the slenderness parameter. The one-dimensional model is derived based on an assumption of slow longitudinal variations, implemented by a two-scale expansion. Effectively, this approach splits the original three-dimensional problem into a set of relaxation problems formulated in the two-dimensional cross-section, and a one-dimensional variational problem at the scale of the structure, as noted in previous work (e.g., by Ber81, BV84, TV89, SS99, among others).

We improve on existing approaches to asymptotic dimension reduction in three key aspects.

- Our method is variational. While most of the existing work has started from the three-dimensional equilibrium equations (BV84, TV96), we base our reduction on the energy formulation of the three-dimensional problem. This helps keeping the derivation as simple as possible, and makes the variational structure of the one-dimensional model stand out without any effort.
- We start from finite elasticity. Most of the existing work has been limited to linear strains (TV96, YH04, Hod06) but the one-dimensional models derived using the proposed method can retain nonlinearities coming from both the geometry and from the constitutive behavior.
- Our one-dimensional model is high-order and asymptotically correct, *i.e.*, it captures the energy cost arising from the longitudinal gradients of the stretching, bending and twisting strains. Besides increasing the accuracy and expanding the range of validity of the model, gradient terms have been found to help capture localization phenomena very accurately (LA18, LA20a).

Some of the models from the literature include one or two of these ingredients. Ber81, Hod06, YHH12 use a variational approach, TV89, NPK18 introduce higher-order terms, JY16, CGL+88 handle finite strains, MLG20 work with finite elasticity in a variational setting. Yet this paper is the first attempt to combine these three aspects in a unified procedure.

The proposed approach has been designed to be as general as possible. It does not make any specific assumptions regarding the symmetry of the constitutive law, such as isotropy (CGL+88). It is not limited to small rotations, or to specific shapes of the cross-section. It can readily be applied to a variety of constitutive behaviors, and in particular it can handle inhomogeneous pre-strain as well as inhomogeneous elastic properties across the sections. By lack of space, we cannot provide detailed illustrations for all these capabilities but we shortly discuss in §2.4 how these cases can be covered. Besides, the approach is systematic: it is carried out by applying a sequence of steps, much like a cooking recipe, and it lends itself naturally to a numerical implementation.

The manuscript is organized as follows. In section 2, we introduce the center-line based representation of a prismatic hyper-elastic solid and derive the energy functional governing its elastic equilibrium in a non-linear, three-dimensional setting. In section 3, we introduce a relaxation method which achieves the one-dimensional reduction formally. In section 4, we combine this relaxation method with a two-scale expansion and derive a concrete recipe for obtaining one-dimensional models. This method is applied to the linear analysis of the twisting of a prismatic bar in section 5, and to the weakly non-linear analysis of the Euler buckling of a circular cylinder in section 6.

Our mathematical notations are as follows. We use boldface for vectors such as  $\mathbf{e}_1$  and tensors  $\mathbf{F}$ . The longitudinal and transverse coordinates in the prismatic body are denoted as  $S$  and  $\mathbf{T} = (T_1, T_2)$ , respectively. Einstein's implicit summation rules are used throughout, whereby repeated indices appearing on the same side of an equal side are implicitly summed; any index appearing once on each side of an equation is considered to be a dummy index, *i.e.*, the equation holds implicitly for any value of the index. In addition, the range of Greek indices such as  $\alpha$  is implicitly restricted to the cross-sectional directions,  $\alpha \in \{1, 2\}$  although Latin indices such as  $i$  run over the three directions of the Cartesian space,  $i \in \{1, 2, 3\}$ ; as a result,  $T_\alpha \mathbf{d}_\alpha(S)$  stands for  $\sum_{\alpha=1}^2 T_\alpha \mathbf{d}_\alpha(S)$ . The prime notation is reserved for derivatives with respect to the longitudinal coordinate  $S$  and we use the notation  $\partial_\alpha$  for partial derivatives along the cross-sectional directions,

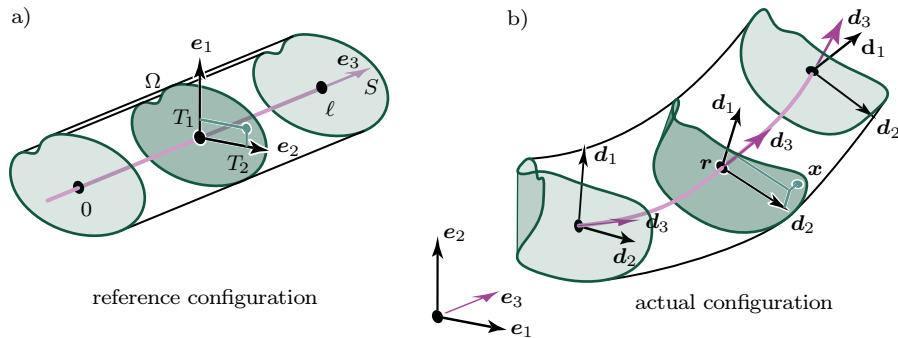
$$f'(S, \mathbf{T}) = \frac{\partial f}{\partial S}(S, \mathbf{T}) \quad \partial_\alpha f(S, \mathbf{T}) = \frac{\partial f}{\partial T_\alpha}(S, \mathbf{T}).$$

The  $\nabla$  notation is reserved for a differentiation with respect to the macroscopic strain  $\mathbf{h}$ , see equation (4.10). The notation  $\mathbf{a} \odot \mathbf{b} = \frac{\mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a}}{2}$  denotes the symmetrized outer product of two vectors. The restriction of a function  $f(S, \mathbf{T})$  to a cross-section with coordinate  $S$  is denoted as  $f|_S$ : this object is a function of the transverse coordinates  $\mathbf{T}$  only, such that  $f|_S(\mathbf{T}) = f(S, \mathbf{T})$ . Finally, functionals have their arguments inside square brackets: the notation  $\Psi[\mathbf{r}, \mathbf{d}_i]$  for the energy functional implies that the arguments of  $\Psi$  are the entire functions  $\mathbf{r}$  and  $\mathbf{d}_i$  and not just their local values.

## 2. 3D MODEL IN CENTER-LINE BASED REPRESENTATION

In this section, the non-linear equilibrium of a finitely-strained prismatic hyper-elastic solid is formulated without approximation. Attention is limited to the formulation of the elasticity problem and no attempt to solve it is made until the next sections. The formulation makes use of a center-line based representation, which sets the stage for the forthcoming dimension reduction. A similar parameterization was introduced in earlier work by Hodges (Hod06) in the framework of linear elasticity, and extended to finite elasticity by JY16, JTY16, where it is used as a basis for a numerical approach to dimension reduction, without any account for the gradient effect.

### 2.1. Center-line based representation



**Figure 2.1.** Center-line based representation of a prismatic solid in (a) reference and (b) actual configurations.

We consider a prismatic solid in reference configuration, see figure 2.1a. We denote by  $\ell$  its initial length, by  $S$  the arc-length coordinate along its axis, such that  $0 \leq S \leq \ell$ , by  $\mathbf{T} = (T_1, T_2)$  the transverse coordinates and by  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$  an orthonormal frame initially aligned with the axes  $T_1, T_2$  and  $S$ , respectively. The cross-section domain is denoted as  $\Omega \subset \mathbb{R}^2$ . Let  $dA = dT_1 dT_2$  be the area element in the domain  $\Omega$  and  $|\Omega| = \iint_{\Omega} dA$  the cross-section area. The average of a function  $f(\mathbf{T})$  over a cross-section is denoted as

$$\langle f(\mathbf{T}) \rangle = \frac{1}{|\Omega|} \iint_{\Omega} f(\mathbf{T}) dA.$$

The coordinates  $(S, \mathbf{T}) \in (0, \ell) \times \Omega$  of a material point in reference configuration are used as Lagrangian variables in the elasticity problem. The position of this material point in the actual configuration is denoted as  $\mathbf{x}(S, \mathbf{T})$ , see figure 2.1b. We do not assume that the internal stress is zero in the reference configuration, i.e., pre-stress is allowed.

In terms of the mapping  $\mathbf{x}$  from the reference to the actual configuration, we define an apparent center-line  $\mathbf{r}(S)$  passing through the centroids of the cross-sections,

$$\mathbf{r}(S) = \langle \mathbf{x}(S, \mathbf{T}) \rangle, \quad (2.1)$$

and a unit tangent to the center-line  $\mathbf{d}_3(S)$ ,

$$\mathbf{d}_3(S) = \frac{\frac{d\mathbf{r}}{dS}(S)}{\left| \frac{d\mathbf{r}}{dS}(S) \right|}. \quad (2.2)$$

The unit vector  $\mathbf{d}_3(S)$  can be complemented by two vectors  $\mathbf{d}_1(S)$  and  $\mathbf{d}_2(S)$  forming an orthonormal frame, the orientation of  $\mathbf{d}_1(S)$  and  $\mathbf{d}_2(S)$  in the plane perpendicular to  $\mathbf{d}_3(S)$  being fixed by the condition

$$\forall S \quad \langle T_\alpha \mathbf{d}_\alpha(S) \times (\mathbf{x}(S, \mathbf{T}) - \mathbf{r}(S)) \rangle \cdot \mathbf{d}_3(S) = 0. \quad (2.3)$$

By Einstein's implicit summation rule and by our convention that Greek indices are restricted to cross-sectional directions, the left-hand side in the equation above is implicitly summed over  $\alpha \in \{1, 2\}$ . By equation (2.3), the orthonormal frame  $\mathbf{d}_i(S)$  captures the average rotation of the cross-section about the tangent  $\mathbf{d}_3(S)$  at any point  $S$  along the center-line. The orthonormal vectors  $\mathbf{d}_i(S)$  are called the *directors* in the theory of rods.

The condition that the directors are orthonormal writes as

$$\mathbf{d}_i(S) \cdot \mathbf{d}_j(S) = \delta_{ij}, \quad (2.4)$$

for any  $S$  and any integers  $i$  and  $j$ , where  $\delta_{ij}$  is Kronecker's symbol, equal to 1 when  $i = j$  and to 0 otherwise.

The original transformation can be recovered as

$$\mathbf{x}(S, \mathbf{T}) = \mathbf{r}(S) + y_i(S, \mathbf{T}) \mathbf{d}_i(S), \quad (2.5)$$

where  $y_i(S, \mathbf{T}) = (\mathbf{x}(S, \mathbf{T}) - \mathbf{r}(S)) \cdot \mathbf{d}_i(S)$  for  $i = 1, 2, 3$  are the microscopic displacement functions ('displacement' is an abuse of language, since this quantity is non-zero in the reference configuration but we will use it anyway). In terms of the displacement  $y_i$ , the constraints in equations (2.1) and (2.3) write

$$\begin{aligned} \forall (S, i) \quad \langle y_i(S, \mathbf{T}) \rangle &= 0 \\ \forall S \quad \langle \eta_{\alpha\beta} T_\alpha y_\beta(S, \mathbf{T}) \rangle &= 0. \end{aligned} \quad (2.6)$$

By Einstein's conventions, the first equation with a non-repeated Latin index holds for  $i = 1, 2, 3$ , while the second equation with repeated Greek indices contains an implicit sum over  $\alpha, \beta \in \{1, 2\}$ . In the equation above,  $\eta_{\alpha\beta}$  is the skew-symmetric symbol, such that  $\eta_{11} = \eta_{22} = 0$ ,  $\eta_{12} = 1$  and  $\eta_{21} = -1$ .

Equation (2.5) shows that  $\mathbf{r}(S)$ ,  $\mathbf{d}_i(S)$  and  $y_i(S, \mathbf{T})$  can be used to parameterize the deformed configuration. Indeed, it can be checked easily that there is a one-to-one correspondence between the unknown  $\mathbf{x}(S, \mathbf{T})$  on the one hand, and the unknowns  $\mathbf{r}(S)$ ,  $\mathbf{d}_i(S)$  and  $y_i(S, \mathbf{T})$  on the other hand, provided  $\mathbf{d}_i(S)$  satisfies the orthonormality condition (2.4) and  $y_i(S, \mathbf{T})$  satisfies the four scalar kinematic constraints (2.6). We will use  $\mathbf{r}(S)$ ,  $\mathbf{d}_i(S)$  and  $y_i(S, \mathbf{T})$  as the main unknowns: we refer to this as the *center-line based parameterization*. It is natural to work with this parameterization in the context of dimension reduction as it brings in the macroscopic variables of the one-dimensional rod model,  $\mathbf{r}(S)$  and  $\mathbf{d}_i(S)$ .

Note that the apparent center-line  $\mathbf{r}(S)$  is not a material line—in the case of a hollow cylinder for instance, the curve  $\mathbf{r}(S)$  does not even lie within the material domain. Similarly, the directors  $\mathbf{d}_i(S)$  do not provide a detailed description of the microscopic displacement on their own: the only information conveyed by the directors frame  $\mathbf{d}_i(S)$  is the *average* rotation of the cross-section about the center-line, see (2.3). Neither the fact that the material frame  $\mathbf{d}_i(S)$  is orthonormal, nor the fact that  $\mathbf{d}_3(S)$  is parallel to the center-line, see equation (2.2), implies any assumption or restriction on the microscopic displacement field: as noted above, the center-line based representation can represent *any* microscopic transformation  $\mathbf{x}(S, \mathbf{T})$ .

## 2.2. Apparent stretching, twisting and bending strain

Together, the center-line  $\mathbf{r}(S)$  and the directors  $\mathbf{d}_i(S)$  define what is known as a framed curve. The standard kinematic analysis of framed curves goes as follows. First, we define the axial strain  $\varepsilon(S)$  by the relation

$$\mathbf{r}'(S) = (1 + \varepsilon(S)) \mathbf{d}_3(S), \quad (2.7)$$

which implies the condition of adaptation (2.2).

Second, we define the bending strain  $\kappa_1(S)$  and  $\kappa_2(S)$  and the twisting strain  $\kappa_3(S)$  by the relation

$$\mathbf{d}_i'(S) = -\eta_{ijk} \kappa_j(S) \mathbf{d}_k(S), \quad (2.8)$$

where  $\eta_{ijk}$  is the antisymmetric (Levi-Civita) symbol of order 3. This equation defines the quantities  $\kappa_j(S)$  uniquely since the frame of directors  $\mathbf{d}_i(S)$  is orthonormal for all  $S$ . The quantities  $\kappa_i(S)$  defined in this way are the components of the rotation gradient  $\boldsymbol{\kappa}(S) = \kappa_i(S) \mathbf{d}_i(S)$  as we have  $\mathbf{d}_i'(S) = -\eta_{ijk} \kappa_j(S) \mathbf{d}_k(S) = \kappa_j(S) \eta_{jik} \mathbf{d}_k(S) = \kappa_j(S) \mathbf{d}_j(S) \times \mathbf{d}_i(S) = \boldsymbol{\kappa}(S) \times \mathbf{d}_i(S)$  for any  $S$  and any integer  $i$ .

The strain measures are collected in a *macroscopic strain* vector

$$\mathbf{h} = (\varepsilon, \kappa_1, \kappa_2, \kappa_3).$$

They will be referred to as *apparent* strain measures as they depend on the center-line and on the directors, which are immaterial in the following sense. Consider for instance a thin cylindrical tube made up of a soft matrix and inextensible fibers initially oriented parallel to the axis of the cylinder: upon twisting, the cylinder will shorten due to the inextensibility of the fibers, making the apparent axial strain negative,  $\varepsilon < 0$ , even though the longitudinal strain along any of the material (helical) fibers is actually zero.

### 2.3. Microscopic strain

With a view of formulating an elasticity problem for the prismatic body, we derive the microscopic strain based on the center-line representation (2.5). The deformation gradient  $\mathbf{F}$  such that  $d\mathbf{x} = \mathbf{F} \cdot (d\mathbf{T}, dS)$  is first introduced as

$$\mathbf{F} = \partial_\alpha \mathbf{x} \otimes \mathbf{e}_\alpha + \mathbf{x}' \otimes \mathbf{e}_3 = \partial_\alpha y_i(S, \mathbf{T}) \mathbf{d}_i(S) \otimes \mathbf{e}_\alpha + t_i(S, \mathbf{T}) \mathbf{d}_i(S) \otimes \mathbf{e}_3, \quad (2.9)$$

where  $t_i = \mathbf{x}' \cdot \mathbf{d}_i$  is the deformed material tangent that was initially oriented parallel to the axis,

$$t_i = (1 + \varepsilon(S)) \delta_{i3} + \eta_{ijk} \kappa_j(S) y_k(S, \mathbf{T}) + y'_i(S, \mathbf{T}).$$

Next, we consider the microscopic Green–St-Venant deformation tensor  $\mathbf{E} = \frac{1}{2} (\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I})$  where  $\mathbf{I}$  is the  $3 \times 3$  identity matrix,

$$\mathbf{E} = \frac{t_i^2 - 1}{2} \mathbf{e}_3 \otimes \mathbf{e}_3 + t_i \partial_\alpha y_i \mathbf{e}_\alpha \odot \mathbf{e}_3 + \frac{\partial_\alpha y_i \partial_\beta y_i - \delta_{\alpha\beta}}{2} \mathbf{e}_\alpha \otimes \mathbf{e}_\beta. \quad (2.10)$$

We denote as  $\mathbf{Y} = (Y_1, Y_2, Y_3)$  and  $\mathbf{Y}^\dagger = (Y_1^\dagger, Y_2^\dagger, Y_3^\dagger)$  the collections of functions  $Y_i$  and  $Y_i^\dagger$  obtained by *restricting* the microscopic displacement and its longitudinal gradient to a cross-section, i.e.,

$$Y_i = y_i|_S \quad Y_i^\dagger = y'_i|_S.$$

By convention, the dagger in  $\mathbf{Y}^\dagger$  means that this cross-sectional function evaluates to a longitudinal gradient of strain, i.e.,  $Y_i^\dagger(\mathbf{T}) = y'_i(S, \mathbf{T})$ ; daggers are roughly equivalent to primes but strictly speaking the quantity  $\mathbf{Y}$  cannot bear a prime  $' = \frac{d}{dS}$  as it is a function of  $\mathbf{T}$  only and not of  $S$ .

With this notation, the strain  $\mathbf{E}$  from equation (2.10) can be written as  $\mathbf{E} = \mathbf{E}(\mathbf{T}; \mathbf{h}(S); \mathbf{y}|_S, \mathbf{y}'|_S)$  where

$$\mathbf{E}(\mathbf{T}; \mathbf{h}; \mathbf{Y}, \mathbf{Y}^\dagger) = \frac{t_i^2 - 1}{2} \mathbf{e}_3 \otimes \mathbf{e}_3 + t_i \partial_\alpha Y_i(\mathbf{T}) \mathbf{e}_\alpha \odot \mathbf{e}_3 + \frac{\partial_\alpha Y_i(\mathbf{T}) \partial_\beta Y_i(\mathbf{T}) - \delta_{\alpha\beta}}{2} \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \quad (2.11)$$

where  $t_i = (1 + \varepsilon) \delta_{i3} + \eta_{ijk} \kappa_j Y_k(\mathbf{T}) + Y_i^\dagger(\mathbf{T})$

The dependence of  $\mathbf{E}$  on  $\mathbf{h} = (\varepsilon, \kappa_1, \kappa_2, \kappa_3)$  arises through the auxiliary quantity  $t_i$ .

A couple comments on the notation  $\mathbf{E}(\mathbf{T}; \mathbf{h}(S); \mathbf{Y} = \mathbf{y}|_S, \mathbf{Y}^\dagger = \mathbf{y}'|_S)$  used in equation (2.11) are in order. The notation implies that the strain at any point  $\mathbf{T}$  of the cross-section can be calculated in terms of the local macroscopic strain  $\mathbf{h}(S)$ , and of the *restrictions* of the displacement  $\mathbf{y}|_S$  and of its longitudinal gradient  $\mathbf{y}'|_S$  to the cross-section of interest. In particular, the notation captures the fact that the strain does not depend on the higher-order longitudinal gradients of displacement, such as  $\mathbf{y}''|_S$ . Besides, the gradients of the displacement along the cross-section directions, namely  $\partial_1 y_i = \frac{\partial y_i}{\partial T_1}$  and  $\partial_2 y_i = \frac{\partial y_i}{\partial T_2}$ , are not listed as an argument to  $\mathbf{E}(\mathbf{T}; \mathbf{h}; \mathbf{Y}, \mathbf{Y}^\dagger)$  as they are reconstructed ‘internally’ from the cross-sectional restriction  $\mathbf{Y}$  as  $\partial_\alpha y_i(S, \mathbf{T}) = \partial_\alpha Y_i(\mathbf{T})$ . As a result, the dependence of the strain on longitudinal gradients of the displacement is explicit in this notation, but that on transverse gradients is not.

### 2.4. Energy formulation

In the classical elasticity theory, the strain energy  $\Phi$  is obtained by integration of a strain energy density  $w$ ,

$$\Phi[\mathbf{h}, \mathbf{y}] = \int_0^\ell \iint_\Omega w(\mathbf{T}, \mathbf{E}) dA dS, \quad (2.12)$$

where the microscopic strain  $\mathbf{E}$  appearing as an argument to  $w$  is given by equation (2.11) as

$$\mathbf{E} = \mathbf{E}(\mathbf{T}; \mathbf{h}(S); \mathbf{y}|_S, \mathbf{y}'|_S). \quad (2.13)$$

The bracket notation in  $\Phi[\mathbf{h}, \mathbf{y}]$  indicates that  $\Phi$  is a functional of its arguments.

The form of the elastic potential in equation (2.12), which serves as a starting point for our dimension reduction method, is completely general. In particular, the following situations can be handled (not all of which can be illustrated in this paper, by lack of space). The elastic properties of the body can be inhomogeneous across the section, as indicated by the explicit dependence of the density of strain energy  $w(\mathbf{T}, \mathbf{E})$  on the transverse coordinate  $\mathbf{T}$  in equation (2.12). Arbitrary hyper-elastic constitutive laws can be specified through the choice of the energy density  $w$ ; in particular, no assumption is made on the symmetries of the material. Arbitrary pre-stress distributions can be taken into account by an appropriate choice of  $w$ , the pre-stress being the quantity  $\frac{\partial w}{\partial \mathbf{E}}(\mathbf{T}, \mathbf{0})$ . It is also possible to treat the case where the elastic or geometric properties of the body vary slowly in the longitudinal direction, as discussed in the conclusion.

We assume that the prismatic solid is subjected to conservative forces, represented by a density of external potential  $V(\mathbf{r}, \mathbf{d}_i)$ . At equilibrium, the total potential energy

$$\Psi[\mathbf{r}, \mathbf{d}_i, \mathbf{y}] = \Phi[\mathbf{h}, \mathbf{y}] + \int_0^\ell V(\mathbf{r}(S), \mathbf{d}_i(S)) dS, \quad (2.14)$$

is stationary with respect to the unknowns  $\mathbf{r}$ ,  $\mathbf{d}_i$  and  $\mathbf{y}$ . The macroscopic strain  $\mathbf{h}(S) = (\varepsilon(S), \kappa_1(S), \dots, \kappa_3(S))$  is a dependent variable which can be obtained in terms of the main unknowns  $\mathbf{r}$  and  $\mathbf{d}_i$  using equations (2.7) and (2.8).

The stationarity of the total potential energy (2.14) is subject to the condition (2.4) that the directors are orthonormal, to the constraint of adaptation  $\mathbf{r}' - (1 + \varepsilon)\mathbf{d}_3 = \mathbf{0}$  in equation (2.7), and to the kinematic constraints (2.6) on the displacement. We rewrite the latter as

$$\forall S \quad \mathbf{q}(\mathbf{y}|_S) = \mathbf{0}, \quad (2.15)$$

where  $\mathbf{q}(\mathbf{Y})$  lists the constraints applicable to the cross-sectional restriction of the displacement  $\mathbf{Y} = \mathbf{y}|_S$ ,

$$\mathbf{q}(\mathbf{Y}) = (\langle Y_1(\mathbf{T}) \rangle, \langle Y_2(\mathbf{T}) \rangle, \langle Y_3(\mathbf{T}) \rangle, \langle \eta_{\alpha\beta} T_\alpha Y_\beta(\mathbf{T}) \rangle). \quad (2.16)$$

The first three constraints prevent the center-line from drifting away from the real material cross-sections—we use a redundant formulation where only the sum  $\mathbf{r} + y_i \mathbf{d}_i$  is physically meaningful, see equation (2.5).

In equation (2.14), the potential  $V(\mathbf{r}, \mathbf{d}_i)$  of the external load (per unit length  $dS$ ) depends on the macroscopic variables but not on the microscopic displacement. This is an assumption in our model. It can typically be justified by the scaling hypotheses that are introduced in the classical work on dimension reduction—typically, if the load varies on a length-scale much larger than the cross-section diameter, its potential can be derived by assuming that cross-sections are rigid, which yields an expression of the form (2.14). If, however, the external load varies quickly or induces large strain, it might become necessary to couple the potential  $V$  with the microscopic displacement  $\mathbf{y}$ . This requires an extension of our work, which entails appending the microscopic variables coupled to the external load as additional entries inside the vector  $\mathbf{h}$ . This is however beyond the scope of the present paper, where attention is limited to an external loading of the form (2.14).

## 2.5. Summary

We have completed the energy formulation of the elasticity problem. In the center-line based parameterization, the unknowns are the center-line  $\mathbf{r}(S)$ , the directors  $\mathbf{d}_i(S)$  and the microscopic displacement  $y_i(S, \mathbf{T})$ . The center-line  $\mathbf{r}(S)$  and the directors  $\mathbf{d}_i(S)$  define a framed curve which is associated with a macroscopic strain  $\mathbf{h}(S)$ , where  $\mathbf{h} = (\varepsilon, \kappa_1, \kappa_2, \kappa_3)$ ,  $\varepsilon$  is the axial strain,  $\kappa_1$  and  $\kappa_2$  are the curvature strains and  $\kappa_3$  is the twisting strain, see section 2.2. The microscopic strain is then given as  $\mathbf{E} = \mathbf{E}(\mathbf{T}; \mathbf{h}(S); \mathbf{y}|_S, \mathbf{y}'|_S)$  in equation (2.11). The total potential energy  $\Psi[\mathbf{r}, \mathbf{d}_i, \mathbf{y}]$  governing the elasticity problem is given in equation (2.14), and in particular the elastic strain energy  $\Phi[\mathbf{h}, \mathbf{y}] = \iiint w(\mathbf{T}, \mathbf{E}) dA dS$  is given in equation (2.12). The equilibrium equations can be derived variationally, taking into account the kinematic constraints (2.15) for the microscopic displacement, as well as the orthonormality and the adaptation conditions in equations (2.4) and (2.7).

## 3. IDEAL ONE-DIMENSIONAL MODEL

In this section we explore a formal method for reducing the equilibrium of the prismatic solid, which is a problem in three-dimensional elasticity, to a one-dimensional problem. The reduction is based on the relaxation of the microscopic displacement  $\mathbf{y}$ . The relaxation problem will be introduced in a formal way in this section; it will not be solved explicitly until we introduce additional assumptions in the forthcoming sections.

### 3.1. Condensing out the microscopic displacement by a thought experiment

What we refer to as a *relaxation of the microscopic displacement*  $\mathbf{y}$  is a minimization of the strain energy functional  $\Phi[\mathbf{h}, \mathbf{y}]$  for a prescribed distribution of macroscopic strain  $\mathbf{h}(S)$ ,

$$\Phi^*[\mathbf{h}] = \min_{\mathbf{y} \text{ s.t. } (\forall S) \mathbf{q}(\mathbf{y}|_S) = \mathbf{0}} \Phi[\mathbf{h}, \mathbf{y}]. \quad (3.1)$$

Note that the relaxation over  $\mathbf{y}$  is subject to the kinematic conditions  $(\forall S) \mathbf{q}(\mathbf{y}|_S) = \mathbf{0}$  ensuring that the microscopic displacement is consistent with the center-line deformation, prescribed through the macroscopic strain  $\mathbf{h}$ .

We assume that the optimization problem for  $\mathbf{y}$  is such that the minimum is attained and denote as  $\mathbf{y} = \mathbf{y}^*[\mathbf{h}]$  the optimum:

$$\Phi^*[\mathbf{h}] = \Phi[\mathbf{h}, \mathbf{y}^*[\mathbf{h}]], \quad (3.2)$$

where all quantities obtained by relaxing the microscopic displacement  $\mathbf{y}$  are marked with an asterisk.

We also assume that  $\mathbf{y}^*[\mathbf{h}]$  is the only stationary point of  $\Phi[\mathbf{h}, \mathbf{y}]$ , so that  $\mathbf{y}^*[\mathbf{h}]$  is characterized by the variational problem

$$(\forall \hat{\mathbf{y}} \text{ such that } \forall S, \mathbf{q}(\hat{\mathbf{y}}|_S) = \mathbf{0}) \quad \frac{\partial \Phi}{\partial \mathbf{y}}[\mathbf{h}, \mathbf{y}^*[\mathbf{h}]] \cdot \hat{\mathbf{y}} = 0. \quad (3.3)$$

All these assumptions are typically satisfied under appropriate convexity and compactness assumptions.

In equation (3.3), the notation  $\frac{\partial f}{\partial \mathbf{y}}[\mathbf{h}, \mathbf{y}] \cdot \hat{\mathbf{y}}$  refers to the Fréchet derivative of the functional  $f[\cdot]$  at point  $(\mathbf{h}, \mathbf{y})$  in the direction  $\hat{\mathbf{y}}$ . The problem for  $\mathbf{y}^*[\mathbf{h}]$  in (3.3) is a non-linear elasticity problem with pre-stress in three dimensions, and is typically impossible to solve in closed form.

A key remark is as follows. If we were able to solve for the optimal microscopic displacement  $\mathbf{y} = \mathbf{y}^*[\mathbf{h}]$ , we could define a one-dimensional strain energy potential  $\Phi^*$  simply by inserting  $\mathbf{y}^*[\mathbf{h}]$  into the three-dimensional strain energy,  $\Phi^*[\mathbf{h}] = \Phi[\mathbf{h}, \mathbf{y}^*[\mathbf{h}]]$ , see equation (3.2). Based on this strain energy functional, one could then build a one-dimensional rod model governed by the total potential energy functional

$$\Psi^*[\mathbf{r}, \mathbf{d}_i] = \Phi^*[\mathbf{h}] + \int_0^\ell V(\mathbf{r}(S), \mathbf{d}_i(S)) dS. \quad (3.4)$$

In this one-dimensional model,  $\mathbf{r}(S)$  and  $\mathbf{d}_i(S)$  are the unknowns, subjected to the same kinematic conditions as earlier in section 2, and the macroscopic strain  $\mathbf{h}$  is a dependent variable that can be calculated as earlier (§2.2).

We refer to this model as the *ideal one-dimensional model*. It is *one-dimensional* in the sense that it exposes the macroscopic variables only, the microscopic displacement  $\mathbf{y} = \mathbf{y}^*[\mathbf{h}]$  being reconstructed ‘under the hood’. It is *ideal* in the sense that it is rigorously equivalent to the three-dimensional elasticity problem from section 2, as shown in section A.2 from the appendix. This shows that dimension reduction is really a relaxation problem.

### 3.2. Equilibrium and constitutive laws

We derive the equilibrium equations and the constitutive laws of the ideal one-dimensional model variationally, starting from the total energy potential  $\Psi^*[\mathbf{r}, \mathbf{d}_i]$  in equation (3.4).

The densities of external force  $\mathbf{p}(S)$  and external moment  $\mathbf{m}(S)$  are first identified from the variation  $\hat{V}$  of the external potential as follows,

$$\int_0^\ell \hat{V} \, dS = - \int_0^\ell (\mathbf{p}(S) \cdot \hat{\mathbf{r}}(S) + \mathbf{m}(S) \cdot \hat{\boldsymbol{\theta}}(S)) \, dS. \quad (3.5)$$

where  $\hat{\mathbf{r}}$  is the perturbation to the center-line and  $\hat{\boldsymbol{\theta}}$  the infinitesimal rotation of the directors  $\mathbf{d}_i$ , such that  $\hat{\mathbf{d}}_i(S) = \hat{\boldsymbol{\theta}}(S) \times \mathbf{d}_i(S)$ . As usual in the principle of virtual work, we limit attention to perturbations  $\hat{\mathbf{r}}$  and  $\hat{\boldsymbol{\theta}}$  such that the incremental form of the kinematic constraint (2.7) is satisfied.

As shown in section A.1 of the appendix, the condition that the energy  $\Psi^*[\mathbf{r}, \mathbf{d}_i]$  is stationary yields the extensible variant of the classical Kirchhoff equations for the equilibrium of thin rods,

$$\begin{aligned} N(S) &= \mathbf{R}(S) \cdot \mathbf{d}_3(S) \\ \mathbf{R}'(S) + \mathbf{p}(S) &= \mathbf{0} \\ \mathbf{M}'(S) + \mathbf{r}'(S) \times \mathbf{R}(S) + \mathbf{m}(S) &= \mathbf{0} \end{aligned} \quad (3.6)$$

together with constitutive laws for the one-dimensional stress variables  $N(S)$  and  $M_i(S)$ . These constitutive laws can be identified from the first variation of the strain energy (3.2) with respect to the macroscopic strain as follows,

$$N(S) \hat{\varepsilon} + M_i(S) \hat{\kappa}_i \equiv \iint_\Omega \Sigma_{ij}(\mathbf{T}, \mathbf{E}(\mathbf{T}; \mathbf{h}(S); \mathbf{y}^*[\mathbf{h}]|_S, \mathbf{y}^*[\mathbf{h}]'|_S)) \frac{\partial E_{ij}}{\partial \mathbf{h}}(\mathbf{T}; \mathbf{h}(S); \mathbf{y}^*[\mathbf{h}]|_S, \mathbf{y}^*[\mathbf{h}]'|_S) \cdot \hat{\mathbf{h}} \, dA. \quad (3.7)$$

Here  $\hat{\mathbf{h}} = (\hat{\varepsilon}, \hat{\kappa}_1, \hat{\kappa}_2, \hat{\kappa}_3)$  is a perturbation to the macroscopic strain and  $\boldsymbol{\Sigma}$  is the microscopic Piola-Kirchhoff stress tensor,

$$\boldsymbol{\Sigma}(\mathbf{T}, \mathbf{E}) = \frac{\partial w}{\partial \mathbf{E}}(\mathbf{T}, \mathbf{E}). \quad (3.8)$$

In equations (3.6–3.7),  $\mathbf{R}(S)$  is the internal force, its component  $N(S)$  along  $\mathbf{d}_3(S)$  is called the normal force,  $\mathbf{M}(S)$  is the internal moment and  $M_i(S) = \mathbf{M}(S) \cdot \mathbf{d}_i(S)$  are its components in the directors basis. A microscopic interpretation of the internal stress  $N(S)$  and  $M_i(S)$  based on equation (3.7) is given in equation (A.5) from appendix A.3. The last two lines in equation (3.6) are the Kirchhoff equations for the equilibrium of rods; they are a balance of forces and moments on an infinitesimal segment, respectively. The equilibrium equations (3.6) must be complemented by boundary conditions which can be derived variationally and vary from one problem to another.

As discussed in section A.2 of the appendix, equations (3.6–3.7) governing the equilibrium of the ideal one-dimensional model are mathematically equivalent to those governing the original three-dimensional model from section 2. The one-dimensional model involves no approximation. It achieves the ultimate in dimension reduction: it hides the microscopic variables while preserving the solutions of the original three-dimensional problem. Incidentally, it also makes the connection with the classical Kirchhoff equations (3.6) for elastic rods. Unfortunately, the constitutive laws (3.7) are in effect useless as they depend on the optimal microscopic displacement  $\mathbf{y}^*[\mathbf{h}]$ , which is not available in closed form: the one-dimensional potential  $\Phi^*[\mathbf{h}]$  is a mathematical object that hides a daunting problem in non-linear three-dimensional elasticity.

In the following section, we construct approximations to the ideal one-dimensional model that are mathematically tractable.

## 4. ASYMPTOTICALLY EXACT ONE-DIMENSIONAL MODELS

### 4.1. Strategy

Even though it cannot be used directly, the ideal one-dimensional model from section 3 offers a natural starting point for building one-dimensional approximations to the original three-dimensional problem. A critical help towards this goal is furnished by our previous work (LA20b), in which a method to calculate the relaxed displacement  $\mathbf{y}^*[\mathbf{h}]$  in powers of the successive gradients of  $\mathbf{h}(S)$  has been obtained. In this section, we apply this asymptotic method and obtain approximations to the ideal strain energy  $\Phi^*[\mathbf{h}]$ . This leads us to *concrete* rod, models which are accurate approximations of the original three-dimensional problem when the gradients of the macroscopic strain  $\mathbf{h}(S)$  are small.

The reduction method from LA20b assumes that the macroscopic strain varies on a length-scale  $\sim \rho / \zeta$  much larger than the typical dimension of the cross-section  $\sim \rho$ , where  $\zeta \ll 1$  is a small scalar parameter that is used as an expansion parameter,

$$\mathbf{h}(S) = \mathcal{O}(1) \quad \frac{d^i \mathbf{h}}{dS^i}(S) = \mathcal{O}(\zeta^i).$$



We emphasize that  $\mathbf{h}(S)$  is allowed to vary by a finite amount across the length  $\ell$  of the structure as long as  $\mathbf{h}'(S) = \mathcal{O}(\zeta)$  remains small everywhere: unlike most (if not all) of the alternate methods from the literature, ours does not require the strain  $\mathbf{h}(S)$  to remain *uniformly* close to a specific value  $\mathbf{h}_0$ , for all values of  $S$ —this property is particularly useful for the analysis of localization, as discussed by AH16 and LA20a. Besides, the expansion has been shown to give extremely accurate results even beyond its strict conditions of mathematical validity, when the gradient  $\mathbf{h}'(S)$  is not small.

The reduction method uses as input the expressions of the strain  $\mathbf{E}(\mathbf{T}; \mathbf{h}(S); \mathbf{Y}, \mathbf{Y}^\dagger)$  and the constraint  $\mathbf{q}(\mathbf{Y})$  relevant to our particular problem from equations (2.11) and (2.16), and furnishes an approximation to the one-dimensional strain energy functional

$$\Phi^*[\mathbf{h}] \approx \Phi_{(2)}^*[\mathbf{h}] + \ell \mathcal{O}(\zeta^3) \quad (4.1)$$

of the form

$$\Phi_{(2)}^*[\mathbf{h}] = \int_0^\ell \left[ W_{\text{hom}}(\tilde{\mathbf{h}}(S)) + \mathbf{A}(\mathbf{h}(S)) \cdot \mathbf{h}'(S) + \frac{1}{2} \mathbf{h}'(S) \cdot \mathbf{D}(\mathbf{h}(S)) \cdot \mathbf{h}'(S) \right] dS, \quad (4.2)$$

where  $\tilde{\mathbf{h}}_i = h_i(S) + \xi_i(\mathbf{h}(S)) h_i''(S)$  (no implicit sum on  $i$ ) is a modified strain measure, see equation (4.17) below.

The reduction method from (LA20b) is summarized in appendix B. Explicit expressions for the potential  $W_{\text{hom}}(\mathbf{h})$ , the coefficients  $\xi_i(\mathbf{h})$  entering in the alternate strain measure  $\tilde{\mathbf{h}}$ , and for the elastic moduli  $\mathbf{A}(\mathbf{h})$  and  $\mathbf{D}(\mathbf{h})$  are available. Both geometric and material nonlinearities are accounted for, as reflected by the fact that the quantities  $\xi_i$ ,  $\mathbf{A}$  and  $\mathbf{D}$  all depend on  $\mathbf{h}$ , typically in a non-linear way.

A lower-order approximation  $\Phi^*[\mathbf{h}] \approx \Phi_{(0)}^*[\mathbf{h}] + \ell \mathcal{O}(\zeta)$  can also be obtained by discarding the gradient terms in  $\Phi_{(2)}^*[\mathbf{h}]$ , which are of order  $\zeta$  or higher,

$$\Phi_{(0)}^*[\mathbf{h}] = \int_0^\ell W_{\text{hom}}(\mathbf{h}(S)) dS. \quad (4.3)$$

Unlike  $\Phi_{(2)}^*[\mathbf{h}]$ , the strain potential  $\Phi_{(0)}^*[\mathbf{h}]$  does not capture the gradient effect: the strain energy  $W_{\text{hom}}(\mathbf{h})$  in  $\Phi_{(0)}^*[\mathbf{h}]$  is a function of the local strain  $\mathbf{h}(S)$  only.

The term  $\mathbf{A}(\mathbf{h}(S)) \cdot \mathbf{h}'(S)$  being incompatible with the most common material symmetries, see section 6.2.1, the modulus  $\mathbf{A}(\mathbf{h}(S))$  is often zero. In this case,  $\Phi_{(0)}^*[\mathbf{h}]$  is a better approximation than announced above, *i.e.*,  $\Phi^*[\mathbf{h}] \approx \Phi_{(0)}^*[\mathbf{h}] + \ell \mathcal{O}(\zeta^2)$ ; by a similar argument the other estimate can be improved as well in the presence of additional symmetries, *i.e.*,  $\Phi^*[\mathbf{h}] \approx \Phi_{(2)}^*[\mathbf{h}] + \ell \mathcal{O}(\zeta^4)$ .

In the remainder of this section, we apply the reduction method from our previous work to the prismatic solid. The potential  $W_{\text{hom}}(\mathbf{h})$  entering in the lower-order approximation  $\Phi_{(0)}^*[\mathbf{h}]$  is derived in section 4.2. The higher-order approximation  $\Phi_{(2)}^*[\mathbf{h}]$  is derived in the subsequent section 4.3.

## 4.2. Analysis of homogeneous solutions

As recalled in section B.1 from the appendix, the elastic potential  $W_{\text{hom}}(\mathbf{h})$  has to be constructed from a catalog of homogeneous solutions. By homogeneous solutions, we refer to the case where neither  $\mathbf{h}(S)$  nor the microscopic displacement  $\mathbf{y}(S, \mathbf{T})$  depend on  $S$ . Homogeneous solutions are analyzed in this section; accordingly, the macroscopic strain  $\mathbf{h} = (\varepsilon, \kappa_1, \kappa_2, \kappa_3)$  is treated as a constant. Doing so, we are temporarily limiting attention to configurations of the center-line that are a helix, an arc of circle or a straight line.

The optimal microscopic displacement  $\mathbf{y}^*[\mathbf{h}]$  being now independent of  $S$ , we denote by  $\mathbf{Y}^{\mathbf{h}} = (\mathbf{y}^*[\mathbf{h}])|_S$  its restriction to any particular cross-section  $S$ . Then,

$$\mathbf{y}^*[\mathbf{h}](S, \mathbf{T}) = \mathbf{Y}^{\mathbf{h}}(\mathbf{T}) \quad (\text{homogeneous case, } \mathbf{h} \text{ is constant}).$$

Here,  $\mathbf{Y}^{\mathbf{h}} = (Y_i^{\mathbf{h}})_{1 \leq i \leq 3}$  denotes a triple of functions defined over the cross-section, each function  $Y_i^{\mathbf{h}}(\mathbf{T})$  being a component of the local displacement in the basis of directors, see equation (2.5). The superscript  $\mathbf{h}$  in the notation  $\mathbf{Y}^{\mathbf{h}}$  serves both as an abbreviation for ‘homogeneous’, and as a container for the macroscopic strain values, on which  $\mathbf{Y}^{\mathbf{h}} = \mathbf{Y}^{(\varepsilon, \kappa_1, \kappa_2, \kappa_3)}$  depends.

From equation (2.11), the strain  $\tilde{\mathbf{E}}(\mathbf{T}, \mathbf{h}, \mathbf{Y}) = \mathbf{E}(\mathbf{T}; \mathbf{h}; \mathbf{Y}, \mathbf{0})$  relevant to homogeneous solutions writes

$$\tilde{\mathbf{E}}(\mathbf{T}, \mathbf{h}, \mathbf{Y}) = \frac{\tilde{t}_i^2 - 1}{2} \mathbf{e}_3 \otimes \mathbf{e}_3 + \tilde{t}_i \partial_\alpha Y_i(\mathbf{T}) \mathbf{e}_\alpha \odot \mathbf{e}_3 + \frac{\partial_\alpha Y_i(\mathbf{T}) \partial_\beta Y_i(\mathbf{T}) - \delta_{\alpha\beta}}{2} \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \quad (4.4)$$

where  $\tilde{t}_i = (1 + \varepsilon) \delta_{i3} + \eta_{ijk} \kappa_j Y_k(\mathbf{T})$ .

This specific expression of the strain is derived from the generic one in equation (2.11) with the gradient term  $\mathbf{Y}^\dagger$  set to zero.

For any value of the macroscopic strain  $\mathbf{h}$ , the relaxed displacement  $\mathbf{Y}^{\mathbf{h}}$  of the homogeneous solution must be found by minimizing the strain energy potential  $\Phi$  with respect to  $\mathbf{Y}$  among all the microscopic displacements satisfying the kinematic conditions  $\mathbf{q}(\mathbf{Y}) = \mathbf{0}$ , see equations (3.1) or (3.3). This leads to the following variational problem, as derived in equation (B.1) from the appendix,

$$\begin{aligned} & \iint_{\Omega} Y_i^{\mathbf{h}}(\mathbf{T}) dA = 0 \\ & \iint_{\Omega} [T_1 Y_2^{\mathbf{h}}(\mathbf{T}) - T_2 Y_1^{\mathbf{h}}(\mathbf{T})] dA = 0 \\ & \forall \hat{\mathbf{Y}} \quad \iint_{\Omega} [\Sigma(\mathbf{T}, \mathbf{E}^{\mathbf{h}}(\mathbf{T})) : \hat{\mathbf{E}}^{\mathbf{h}}(\mathbf{T}) + F_i^{\mathbf{h}} \hat{Y}_i(\mathbf{T}) + Q^{\mathbf{h}} (T_1 \hat{Y}_2(\mathbf{T}) - T_2 \hat{Y}_1(\mathbf{T}))] dA = 0. \end{aligned} \quad (4.5)$$

where  $\hat{\mathbf{E}}^h(\mathbf{T}) = \frac{d\tilde{\mathbf{E}}}{d\mathbf{Y}}(\mathbf{T}, \mathbf{h}, \mathbf{Y}^h) \cdot \hat{\mathbf{Y}}$  is the virtual change of strain, and the four scalars  $(F_1^h, F_2^h, F_3^h, Q^h)$  are Lagrange multipliers enforcing the constraints  $\mathbf{q}(\mathbf{Y}^h) = 0$  that have been spelled out in the first two lines of equation (4.5). This variational problem is a two-dimensional, non-linear problem of elasticity in the cross-section  $\Omega$  with pre-strain that depends on  $\mathbf{h}$ . For the simple examples given at the end of this paper, the solution  $\mathbf{Y}^h$  will be obtained analytically but a numerical solution might be required in more complex geometries.

Solving this variational problem repeatedly for all possible values of  $\mathbf{h}$ , one obtains a catalog of homogeneous solutions  $\mathbf{Y}^h$  indexed by the macroscopic strain  $\mathbf{h}$ . The elastic potential  $W_{\text{hom}}(\mathbf{h})$  is then defined as the strain energy per unit length of the homogeneous solution  $\mathbf{Y}^h$ ,

$$W_{\text{hom}}(\mathbf{h}) = \iint_{\Omega} w(\mathbf{T}, \mathbf{E}^h(\mathbf{T})) dA, \quad \text{where } \mathbf{E}^h(\mathbf{T}) = \tilde{\mathbf{E}}(\mathbf{T}, \mathbf{h}, \mathbf{Y}^h). \quad (4.6)$$

The lower-order one-dimensional strain energy potential  $\Phi_{(0)}^*[\mathbf{h}]$  can then be readily constructed from equation (4.3): most engineering models for slender structures make use of the energy potential  $\Phi_{(0)}^*[\mathbf{h}]$  which we have just obtained, see equation (5.11) for instance.

In terms of the catalog of homogeneous solutions  $\mathbf{Y}^h$ , we introduce the following auxiliary quantities relevant to the homogeneous solution,

$$\begin{aligned} F_{i3}^h(\mathbf{T}) &= (1 + \varepsilon) \delta_{i3} + \eta_{ijk} \kappa_j Y_k^h(\mathbf{T}) \\ F_{i\alpha}^h(\mathbf{T}) &= \partial_{\alpha} Y_i^h(\mathbf{T}) \\ \mathbf{E}^h(\mathbf{T}) &= \mathbf{E}(\mathbf{T}; \mathbf{h}; \mathbf{Y}^h, \mathbf{0}) \\ \boldsymbol{\Sigma}^h(\mathbf{T}) &= \boldsymbol{\Sigma}(\mathbf{T}, \mathbf{E}^h(\mathbf{T})) \\ \mathbf{K}^h(\mathbf{T}) &= \frac{\partial^2 w}{\partial \mathbf{E}^2}(\mathbf{T}, \mathbf{E}^h(\mathbf{T})) \end{aligned} \quad (4.7)$$

where  $F_{ij}^h(\mathbf{T})$  are the components of the deformation gradient  $\mathbf{F}^h(S, \mathbf{T}) = F_{ij}^h(\mathbf{T}) \mathbf{d}_i(S) \otimes \mathbf{e}_j$ ,  $\mathbf{E}^h$  is the microscopic strain,  $\boldsymbol{\Sigma}^h$  the microscopic stress, and  $\mathbf{K}^h$  is the matrix of tangent elastic moduli.

### 4.3. Analysis of the gradient effect

This section aims at deriving the higher-order approximation  $\Phi_{(2)}^*[\mathbf{h}]$  to the strain energy, that captures the gradient effect. We do so by following the general method from section B.2 in the appendix B.

Given a distribution of macroscopic stress  $\mathbf{h}(S)$ , the idea of the method is to seek the solution to the relaxation problem (3.1) in the form

$$\mathbf{y}^*[h](S, \mathbf{T}) = \mathbf{Y}^{h(S)}(\mathbf{T}) + \mathbf{Z}_{\text{opt}}^{h(S)}(\mathbf{h}'(S), \mathbf{T}) + \mathcal{O}(\zeta^2), \quad (4.8)$$

where  $\mathbf{Y}^{h(S)}$  is the displacement predicted by the catalog of homogeneous solutions based on the local value  $\mathbf{h}(S)$  of the macroscopic strain,  $\mathbf{Z}_{\text{opt}}^{h(S)}(\mathbf{T}, \mathbf{h}'(S))$  is a correction proportional to the local strain gradient  $\mathbf{h}'(S)$ , to be determined, and  $\mathcal{O}(\zeta^2)$  denotes higher-order terms which do not enter in the determination of  $\Phi_{(2)}^*[\mathbf{h}]$ . We proceed to show how the correction  $\mathbf{Z}_{\text{opt}}^{h(S)}(\mathbf{h}'(S), \mathbf{T})$  can be obtained, which is a first step towards constructing the functional  $\Phi_{(2)}^*[\mathbf{h}]$ .

#### 4.3.1. Optimal displacement

As shown in previous work and summarized in appendix B, the optimal correction  $\mathbf{Z}_{\text{opt}}^{h(S)}(\mathbf{T}, \mathbf{h}'(S))$  can be found by solving a variational problem on the cross-section that effectively enforces the optimality condition (3.1). This variational problem makes use of an operator  $\mathcal{B}^h(\mathbf{h}^\dagger, \mathbf{Z})$  that takes the strain  $\mathbf{h}$ , its gradient  $\mathbf{h}^\dagger$  and a generic displacement field  $\mathbf{Z}$  defined on the cross-section as arguments. We follow the step-by-step recipe from the appendix to build the operator  $\mathcal{B}^h(\mathbf{h}^\dagger, \mathbf{Z})$ , based on the knowledge of the homogeneous solutions  $\mathbf{Y}^h$ . Auxiliary operators need to be introduced in this process.

The first step is to identify the *structure operators*  $\mathbf{e}_{ij}^h$ , which are the gradients with respect to  $\mathbf{h}$ ,  $\mathbf{h}^\dagger$ ,  $\mathbf{Y}$  and  $\mathbf{Y}^\dagger$  of the microscopic strain function  $\mathbf{E}(\mathbf{T}; \mathbf{h}; \mathbf{Y}, \mathbf{Y}^\dagger)$  in equation (2.11) about a homogeneous solution. These structure operators are defined in section B.2.1 and are calculated in appendix C. They are purely geometric quantities.

Next, the linear increment of strain  $\mathcal{E}^h(\mathbf{T}, \mathbf{h}^\dagger, \mathbf{Z})$  associated with a small strain gradient  $\mathbf{h}^\dagger = (\varepsilon^\dagger, \kappa_1^\dagger, \kappa_2^\dagger, \kappa_3^\dagger)$  and with the corrective displacement  $\mathbf{Z}$  are obtained from equation (B.5) as

$$\mathcal{E}^h(\mathbf{T}, \mathbf{h}^\dagger, \mathbf{Z}) = (\mathbf{h}^\dagger \cdot \nabla \mathbf{Y}_i^h(\mathbf{T})) F_{ij}^h(\mathbf{T}) \mathbf{e}_j \odot \mathbf{e}_3 + \eta_{ijk} \kappa_k F_{jl}^h(\mathbf{T}) Z_i(\mathbf{T}) \mathbf{e}_l \odot \mathbf{e}_3 + F_{ij}^h(\mathbf{T}) \partial_{\alpha} Z_i(\mathbf{T}) \mathbf{e}_j \odot \mathbf{e}_{\alpha}. \quad (4.9)$$

The last argument  $\mathbf{Z} = (Z_1, Z_2, Z_3)$  is a triple of functions defined on the cross-section, representing the corrective displacement;  $\mathcal{E}^h$  is viewed as an operator acting on the cross-sectional functions  $(Z_1, Z_2, Z_3)$  that are not yet known.

The  $\nabla$  notation is systematically used to denote a gradient with respect to the macroscopic gradient  $\mathbf{h}$ , with the convention that the increment of  $\mathbf{h}$  is applied by a left multiplication: in equation (4.9), the first term in the right-hand side must be interpreted as

$$\begin{aligned} \mathbf{h}^\dagger \cdot \nabla \mathbf{Y}_i^h(\mathbf{T}) &= \frac{\partial \mathbf{Y}_i^h(\mathbf{T})}{\partial \mathbf{h}} \cdot \mathbf{h}^\dagger \\ &= \frac{\partial \mathbf{Y}_i^{(\varepsilon, \kappa_1, \kappa_2, \kappa_3)}(\mathbf{T})}{\partial \varepsilon} \varepsilon^\dagger + \frac{\partial \mathbf{Y}_i^{(\varepsilon, \kappa_1, \kappa_2, \kappa_3)}(\mathbf{T})}{\partial \kappa_1} \kappa_1^\dagger \end{aligned} \quad (4.10)$$



Following the general method, we introduce yet another operator  $\mathbf{C}_h^{(1)}$ . For a given value of the macroscopic strain  $\mathbf{h}$  and for a triple  $\mathbf{Z}^\dagger$  of scalar functions  $Z_i^\dagger$  defined over the cross-sections (representing the components of the longitudinal gradient of the corrective displacement, hence the dagger notation),  $\mathbf{C}_h^{(1)}$  is defined from equation (B.6) as

$$\mathbf{C}_h^{(1)} \cdot \mathbf{Z}^\dagger = \iint_{\Omega} F_{ij}^h(\mathbf{T}) \Sigma_{j3}^h(\mathbf{T}) Z_i^\dagger(\mathbf{T}) dA. \quad (4.11)$$

A related operator  $\nabla \mathbf{C}_h^{(1)}$  is defined by a formal integration by parts with respect to the  $\dagger$  symbol (formally representing the longitudinal derivative  $d/dS$ ), see equation (B.7),

$$-\mathbf{h}^\dagger \cdot \nabla \mathbf{C}_h^{(1)} \cdot \mathbf{Z} = - \iint_{\Omega} \left( \frac{d(F_{ij}^h(\mathbf{T}) \Sigma_{j3}^h(\mathbf{T}))}{d\mathbf{h}} \cdot \mathbf{h}^\dagger \right) Z_i(\mathbf{T}) dA. \quad (4.12)$$

In our previous work, we have shown that the perturbation to the strain energy per unit length caused by a strain gradient  $\mathbf{h}^\dagger$  and by the corrective displacement  $\mathbf{Z}$  is given by the operator  $\mathcal{B}^h(\mathbf{h}^\dagger, \mathbf{Z})$  defined in equation (B.6) as

$$\begin{aligned} \mathcal{B}^h(\mathbf{h}^\dagger, \mathbf{Z}) = & \iint_{\Omega} \frac{1}{2} \mathcal{E}^h(\mathbf{T}, \mathbf{h}^\dagger, \mathbf{Z}) : \mathbf{K}^h(\mathbf{T}) : \mathcal{E}^h(\mathbf{T}, \mathbf{h}^\dagger, \mathbf{Z}) dA \dots \\ & + \iint_{\Omega} \left( \left( \frac{1}{2} \sum_i (\mathbf{h}^\dagger \cdot \nabla Y_i^h(\mathbf{T}))^2 \Sigma_{33}^h(\mathbf{T}) \right) + \mathbf{h}^\dagger \cdot \nabla Y_i^h(\mathbf{T}) (\eta_{ijk} \kappa_j \Sigma_{33}^h(\mathbf{T}) Z_k(\mathbf{T}) + \Sigma_{\beta 3}^h(\mathbf{T}) \partial_\beta Z_i(\mathbf{T})) \dots \right. \\ & \quad \left. + \frac{1}{2} (\delta_{ij} \kappa_i^2 - \kappa_i \kappa_j) \Sigma_{33}^h(\mathbf{T}) Z_i(\mathbf{T}) Z_j(\mathbf{T}) + \eta_{ijk} \kappa_j \Sigma_{\alpha 3}^h(\mathbf{T}) Z_k(\mathbf{T}) \partial_\alpha Z_i(\mathbf{T}) \dots \right. \\ & \quad \left. + \frac{1}{2} \Sigma_{\alpha \beta}^h(\mathbf{T}) \partial_\alpha Z_i(\mathbf{T}) \partial_\beta Z_i(\mathbf{T}) \right) dA - \mathbf{h}^\dagger \cdot \nabla \mathbf{C}_h^{(1)} \cdot \mathbf{Z}. \end{aligned} \quad (4.13)$$

This operator  $\mathcal{B}^h(\mathbf{h}^\dagger, \mathbf{Z})$  is a quadratic form with respect to each one of its arguments  $\mathbf{h}^\dagger$  and  $\mathbf{Z}$ .

The optimal corrective displacement  $\mathbf{Z}_{\text{opt}}^h(\mathbf{h}^\dagger)$  is characterized by the fact that it is the stationary point of the quadratic functional  $\mathcal{B}^h(\mathbf{h}^\dagger, \mathbf{Z})$  over the set of cross-sectional functions  $\mathbf{Z}$ 's satisfying the kinematic constraint  $\mathbf{q}(\mathbf{Z}) = \mathbf{0}$ ; this is fully in line with the interpretation of  $\mathcal{B}^h(\mathbf{h}^\dagger, \mathbf{Z})$  as the increment of strain energy caused by the gradient effect. This leads to the variational problem stated in equation (B.8): given  $\mathbf{h}$  and  $\mathbf{h}^\dagger$ , find the corrective cross-sectional displacement  $\mathbf{Z}_{\text{opt}}^h(\mathbf{h}^\dagger)$  and the Lagrange multipliers  $\mathbf{F}_{\text{opt}}^h(\mathbf{h}^\dagger)$  and  $\mathbf{Q}_{\text{opt}}^h(\mathbf{h}^\dagger)$  such that

$$\begin{aligned} & \iint_{\Omega} Z_{\text{opt},i}^h(\mathbf{h}^\dagger, \mathbf{T}) dA = 0 \\ & \iint_{\Omega} [T_1 Z_{\text{opt},2}^h(\mathbf{h}^\dagger, \mathbf{T}) - T_2 Z_{\text{opt},1}^h(\mathbf{h}^\dagger, \mathbf{T})] dA = 0 \\ & \forall \hat{\mathbf{Z}} \quad \frac{\partial \mathcal{B}^h}{\partial \mathbf{Z}}(\mathbf{h}^\dagger, \mathbf{Z}_{\text{opt}}^h(\mathbf{h}^\dagger)) \cdot \hat{\mathbf{Z}} + \iint_{\Omega} [\mathbf{F}_{\text{opt},i}^h(\mathbf{h}^\dagger) \hat{Z}_i(\mathbf{T}) + \mathbf{Q}_{\text{opt}}^h(\mathbf{h}^\dagger) (T_1 \hat{Z}_2(\mathbf{T}) - T_2 \hat{Z}_1(\mathbf{T}))] dA = 0, \end{aligned} \quad (4.14)$$

As  $\mathcal{B}^h(\mathbf{h}^\dagger, \mathbf{Z})$  is quadratic, this is a two-dimensional problem of linear elasticity in the cross-section, with residual strain proportional to  $\mathbf{h}^\dagger$ : its solution  $\mathbf{Z}_{\text{opt}}^h(\mathbf{T}, \mathbf{h}^\dagger)$  is linear with respect to the strain gradient  $\mathbf{h}^\dagger$ .

This completes the determination of the corrective displacement  $\mathbf{Z}_{\text{opt}}^h(\mathbf{h}^\dagger)$  as a function of the macroscopic strain  $\mathbf{h}$  and its longitudinal gradient  $\mathbf{h}^\dagger$ .

#### 4.3.2. Definition of the one-dimensional model

We have solved for the corrective displacement  $\mathbf{Z}_{\text{opt}}^{h(S)}(\mathbf{h}'(S), \mathbf{T})$ . There remains to insert the relaxed displacement  $\mathbf{y}^*[\mathbf{h}]$  from equation (4.8) into the original strain energy  $\Phi^*[\mathbf{h}] = \Phi[\mathbf{h}, \mathbf{y}^*[\mathbf{h}]]$ . This yields the reduced strain energy  $\Phi_{(2)}^*[\mathbf{h}]$  announced earlier in equation (4.2),

$$\Phi_{(2)}^*[\mathbf{h}] = \int_0^\ell \left[ W_{\text{hom}}(\tilde{\mathbf{h}}(S)) + \mathbf{A}(\mathbf{h}(S)) \cdot \mathbf{h}'(S) + \frac{1}{2} \mathbf{h}'(S) \cdot \mathbf{D}(\mathbf{h}(S)) \cdot \mathbf{h}'(S) \right] dS,$$

together with explicit expressions for the elastic moduli  $\mathbf{A}(\mathbf{h})$ ,  $\mathbf{D}(\mathbf{h})$  and for the modified strain  $\tilde{\mathbf{h}}(S)$ , which are obtained as follows—the reader is referred to appendix B for details.

The auxiliary one-dimensional moduli  $\mathbf{B}(\mathbf{h})$  and  $\mathbf{C}(\mathbf{h})$  are obtained by inserting the optimal displacement into the operators  $\mathcal{B}^h$  and  $\mathbf{C}_h^{(1)}$  introduced earlier, and by identification from the following equations,

$$\begin{aligned} \frac{1}{2} \mathbf{h}^\dagger \cdot \mathbf{B}(\mathbf{h}) \cdot \mathbf{h}^\dagger &= \mathcal{B}^h(\mathbf{h}^\dagger, \mathbf{Z}_{\text{opt}}^h(\mathbf{h}^\dagger)) \\ \mathbf{C}(\mathbf{h}) \cdot \mathbf{h}^\dagger &= \mathbf{C}_h^{(1)} \cdot \mathbf{Z}_{\text{opt}}^h(\mathbf{h}^\dagger). \end{aligned} \quad (4.15)$$

The one-dimensional moduli  $\mathbf{A}(\mathbf{h})$  and  $\mathbf{D}(\mathbf{h})$  appearing in  $\Phi_{(2)}^*[\mathbf{h}]$  write, from section B.2.3,

$$\begin{aligned} \mathbf{A}(\mathbf{h}) \cdot \mathbf{h}^\dagger &= \iint_{\Omega} \Sigma^h(\mathbf{T}) : \mathcal{E}^h(\mathbf{T}, \mathbf{h}^\dagger, \mathbf{0}) dA \\ \mathbf{D}(\mathbf{h}) &= \mathbf{B}(\mathbf{h}) + 2 \frac{d\mathbf{C}}{d\mathbf{h}}(\mathbf{h}) \end{aligned} \quad (4.16)$$

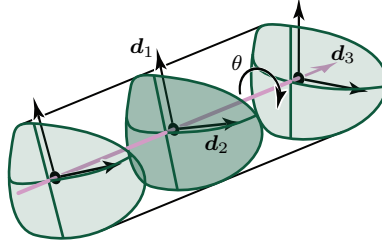
and the modified strain  $\tilde{\mathbf{h}}(S)$  is defined by

$$\begin{aligned}\tilde{h}_i(S) &= h_i(S) + \xi_i(\mathbf{h}(S)) h_i''(S) \quad (\text{no sum on } i) \\ \xi_i(\mathbf{h}) &= \frac{C_i(\mathbf{h})}{\frac{dW_{\text{hom}}}{d\mathbf{h}_i}(\mathbf{h})},\end{aligned}\tag{4.17}$$

where  $C_i(\mathbf{h})$  is the  $i$ -th component of  $\mathbf{C}(\mathbf{h})$ , *i.e.*, the coefficient in factor of  $h_i^\dagger$  in  $\mathbf{C}(\mathbf{h}) \cdot \mathbf{h}^\dagger$ , see equation (B.11).

This completes the construction of the functional  $\Phi_{(2)}^*[\mathbf{h}]$ . The process is long but straightforward. It can be turned into a fully automated procedure using symbolic calculations, something which we will explore in future work. In the remainder of this paper, we illustrate the procedure by carrying out the calculations for two problems that are tractable analytically; the first problem is linear and the other one is non-linear.

## 5. ILLUSTRATION IN A LINEAR SETTING: TWISTING OF A THICK BAR



**Figure 5.1.** Pure twisting of a thick bar: actual configuration.

A number of authors have addressed higher-order effects in beam models for prismatic solids in the limited context of linear elasticity (TV89, NPK18, BC01a, BC01b). In this section, we derive a higher-order model for the twisting of a prismatic bar using our method; we show that its predictions are consistent with prior work from the literature. This provides a first illustration of the reduction procedure described in section 4.

### 5.1. Problem setting

We consider the pure torsion of a linearly elastic bar including higher-order effect. To simplify the calculations, we make some convenience assumptions. First, the elastic material is assumed to be linear and isotropic with homogeneous properties. This corresponds to a microscopic strain energy density

$$w(\mathbf{T}, \mathbf{E}) = \frac{1}{2} (\lambda \text{tr}^2 \mathbf{E} + 2\mu \mathbf{E} : \mathbf{E}),\tag{5.1}$$

where  $\lambda$  and  $\mu$  are the Lamé elastic moduli. Second, we assume that the cross-section  $\Omega$  has two perpendicular axes of symmetry, and we set up the cross-section coordinates  $(T_1, T_2)$  along these axes; as a result, the cross-section is invariant by the two mirror symmetries  $T_\alpha \leftarrow (-T_\alpha)$ . This symmetry assumption decouples the twisting mode from the bending and stretching modes (TV89). It allows us to analyze twisting while setting  $\varepsilon = 0$  (no stretching) and  $\kappa_\alpha = 0$  (no bending). We will therefore have to deal with a single non-zero macroscopic strain  $\kappa_3$  (twisting), which we rename as  $\tau = \kappa_3$ . Accordingly, we shrink the macroscopic strain  $\mathbf{h}$  to a vector with length 1,

$$\mathbf{h} = (\tau).$$

In the context of linear elasticity, the main unknown is the (true) displacement  $v_i(S, \mathbf{T})$  from the reference to the actual configuration. It is connected to the microscopic positional unknown  $y_i(S, \mathbf{T})$  used so far by

$$\begin{aligned}y_\alpha(S, \mathbf{T}) &= T_\alpha + v_\alpha(S, \mathbf{T}) \\ y_3(S, \mathbf{T}) &= v_3(S, \mathbf{T})\end{aligned}\tag{5.2}$$

Inserting the equation above into equation (2.11), one obtains the strain  $\mathbf{E}$  as

$$\mathbf{E}(\mathbf{T}; \tau; \mathbf{V}, \mathbf{V}^\dagger) = -\tau \eta_{\alpha\beta} T_\beta \mathbf{e}_\alpha \odot \mathbf{e}_3 + \partial_\alpha V_i(\mathbf{T}) \mathbf{e}_\alpha \odot \mathbf{e}_i + V_i^\dagger(\mathbf{T}) \mathbf{e}_i \odot \mathbf{e}_3.\tag{5.3}$$

Here,  $\mathbf{V}$  and  $\mathbf{V}^\dagger$  denote the restrictions of the displacement and its longitudinal gradient to a particular cross-section,

$$V_i = v_i|_S \quad V_i^\dagger = v_i^\dagger|_S\tag{5.4}$$

In linear elasticity, the cross-sectional *displacements*  $\mathbf{V}$  and  $\mathbf{V}^\dagger$  are used as the unknowns parameterizing the microscopic displacement, instead of the cross-sectional positions  $\mathbf{Y}$  and  $\mathbf{Y}^\dagger$  relevant to the non-linear setting.

As we are working in the context of linear elasticity, we linearize all quantities with respect to the twisting strain  $\tau$ , to the displacement  $\mathbf{V}$  and to its longitudinal gradient  $\mathbf{V}^\dagger$ . Such a linearization has been carried out silently in the right-hand side of equation (5.3), in particular.

In the forthcoming sections, we apply the method from section 4 and derive a one-dimensional model describing the twisting of the linearly elastic bar that includes the gradient effect.

## 5.2. Analysis of homogeneous solutions

We first focus on homogeneous solutions, obtained by discarding the gradient of the displacement in the local basis,  $\mathbf{V}^\dagger = \mathbf{0}$ , in equation (5.3). This yields the strain of homogeneous solutions as

$$\tilde{\mathbf{E}}(\mathbf{T}, \tau, \mathbf{V}) = -\tau \eta_{\alpha\beta} T_\beta \mathbf{e}_\alpha \odot \mathbf{e}_3 + \partial_\alpha V_i(\mathbf{T}) \mathbf{e}_\alpha \odot \mathbf{e}_i.$$

Homogeneous solutions  $\mathbf{V}^{(\tau)}$  are the stationary points  $\mathbf{V} = \mathbf{V}^{(\tau)}$  of the strain energy  $\frac{1}{2} \iint_\Omega (\lambda \operatorname{tr}^2 \tilde{\mathbf{E}} + 2 \mu \tilde{\mathbf{E}} : \tilde{\mathbf{E}}) dA$  where  $\tilde{\mathbf{E}} = \tilde{\mathbf{E}}(\mathbf{T}, \tau, \mathbf{V})$ , subject to the kinematic constraint  $\mathbf{q}(\mathbf{V}) = \mathbf{0}$  in equation (2.16). With the help of equation (4.5), the variational problem satisfied by  $\mathbf{V}^{(\tau)}$  writes as

$$\begin{aligned} \iint_\Omega V_i^{(\tau)}(\mathbf{T}) dA &= \mathbf{0} \\ \iint_\Omega [T_1 V_2^{(\tau)}(\mathbf{T}) - T_2 V_1^{(\tau)}(\mathbf{T})] dA &= \mathbf{0} \\ \forall \hat{\mathbf{V}} \quad \iint_\Omega [\boldsymbol{\Sigma}(\mathbf{T}, \mathbf{E}^{(\tau)}(\mathbf{T})) : \hat{\tilde{\mathbf{E}}}^{(\tau)}(\mathbf{T}) + F_i^{(\tau)} \hat{V}_i(\mathbf{T}) + Q^{(\tau)} (T_1 \hat{V}_2(\mathbf{T}) - T_2 \hat{V}_1(\mathbf{T}))] dA &= 0. \end{aligned} \quad (5.5)$$

where  $\mathbf{E}^{(\tau)}(\mathbf{T}) = \tilde{\mathbf{E}}(\mathbf{T}, \tau, \mathbf{V}^{(\tau)})$  is the microscopic strain,  $\boldsymbol{\Sigma}(\mathbf{T}, \mathbf{E}^{(\tau)}(\mathbf{T})) = \frac{dw}{d\tilde{\mathbf{E}}}(\mathbf{T}, \mathbf{E}^{(\tau)}(\mathbf{T})) = \lambda \operatorname{tr} \mathbf{E}^{(\tau)}(\mathbf{T}) \times \mathbf{I} + 2 \mu \mathbf{E}^{(\tau)}(\mathbf{T}) = 2 \mu (-\tau \eta_{\alpha\beta} T_\beta + \partial_\alpha V_3(\mathbf{T})) \mathbf{e}_\alpha \odot \mathbf{e}_3 + (\lambda \partial_\gamma V_\gamma(\mathbf{T}) \delta_{\alpha\beta} + 2 \mu \partial_\alpha V_\beta(\mathbf{T})) \mathbf{e}_\alpha \odot \mathbf{e}_\beta + \lambda \partial_\gamma V_\gamma(\mathbf{T}) \mathbf{e}_3 \otimes \mathbf{e}_3$  is the microscopic stress,  $\mathbf{I}$  is the  $3 \times 3$  identity matrix,  $\hat{\tilde{\mathbf{E}}}^{(\tau)}(\mathbf{T}) = \partial_\alpha \hat{V}_i(\mathbf{T}) \mathbf{e}_\alpha \odot \mathbf{e}_i$  is the virtual increment of strain, and  $\mathbf{F}^{(\tau)}$  and  $Q^{(\tau)}$  are Lagrange multiplier enforcing the constraints written in the first two lines of equation (5.5).

By inserting these expressions into (5.5), one obtains two decoupled problems, namely one for the cross-sectional displacement  $V_\alpha(\mathbf{T})$  which has no source term, and one for the longitudinal displacement  $V_3(\mathbf{T})$  having a source term proportional to the kinematic strain  $\tau$ . The solution writes

$$V_\alpha^{(\tau)}(\mathbf{T}) = 0 \quad V_3^{(\tau)}(\mathbf{T}) = \tau \omega(\mathbf{T}) \quad (5.6)$$

where  $\omega(\mathbf{T})$  is the classical warping function (TV89), defined as the solution of the variational problem

$$\iint_\Omega \omega dA = 0 \quad \text{and} \quad \forall \hat{\omega} \quad \iint_\Omega \partial_\alpha \omega \partial_\alpha \hat{\omega} dA = - \iint_\Omega \eta_{\alpha\beta} T_\alpha \partial_\beta \hat{\omega} dA. \quad (5.7)$$

The function  $\omega(\mathbf{T})$  depends on the geometry of the cross-section only.

In terms of the solution (5.6), one can reconstruct the microscopic strain and the microscopic stress as

$$\begin{aligned} \mathbf{E}^{(\tau)}(\mathbf{T}) &= \tilde{\mathbf{E}}(\mathbf{T}, \tau, \mathbf{V}) = \tau (-\eta_{\alpha\beta} T_\beta + \partial_\alpha \omega(\mathbf{T})) \mathbf{e}_\alpha \odot \mathbf{e}_3 \\ \boldsymbol{\Sigma}^{(\tau)}(\mathbf{T}) &= 2 \mu \mathbf{E}^{(\tau)}(\mathbf{T}) = \mu \tau (-\eta_{\alpha\beta} T_\beta + \partial_\alpha \omega(\mathbf{T})) (\mathbf{e}_\alpha \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_\alpha) \end{aligned} \quad (5.8)$$

Next, the strain energy density  $W_{\text{hom}}(\tau)$  defined in (4.6) is found by inserting the strain  $\mathbf{E}^{(\tau)}(\mathbf{T})$  into (5.1), which yields

$$W_{\text{hom}}(\tau) = \frac{1}{2} \mu J \tau^2, \quad (5.9)$$

where  $J$  is the torsional constant, classically defined as

$$\begin{aligned} J &= \iint_\Omega \sum_\alpha (-\eta_{\alpha\beta} T_\beta + \partial_\alpha \omega(\mathbf{T}))^2 dA \\ &= \iint_\Omega (T_1^2 + T_2^2) dA - \iint_\Omega ((\partial_1 \omega)^2 + (\partial_2 \omega)^2) dA. \end{aligned} \quad (5.10)$$

The last equality can be established by using of an identity obtained by setting  $\hat{\omega} = \omega$  in (5.7).

In view of equation (4.3), the one-dimensional strain energy is

$$\Phi_{(0)}^*[ \tau ] = \int_0^\ell \frac{1}{2} \mu J \tau^2(S) dS. \quad (5.11)$$

We have recovered the classical linear model for the twisting of bars, ignoring the gradient effect for the moment. Repeating a similar reduction but for stretching and bending modes rather than for the twisting mode, one can recover the strain energy potential governing the planar Euler-Bernoulli beam model,  $\Phi_{(0)}^*[\varepsilon, \kappa] = \int \left[ \frac{YA}{2} \varepsilon^2(S) + \frac{YI}{2} \kappa^2(S) \right] dS$ , where  $\varepsilon$  and  $\kappa$  are the stretching and bending strain measures, respectively, and  $YA$  and  $YI$  are the classical traction and bending moduli, respectively. Here, we have considered rods made up of a uniform, linearly elastic, isotropic material here; extensions accounting for anisotropic or non-linear elastic materials (CGL+88), for inhomogeneous elastic properties in the cross-section (Hod06), or for a pre-strain distribution across the cross-section (CRS17, KO18, MLG20) can be obtained easily by following the same procedure.

## 5.3. Gradient effect

In appendix D, we derive the one-dimensional energy functional capturing the gradient effect associated with a non-uniform distribution of twist  $\tau(S)$ . We do so by applying the general recipe from section 4.3 to the strain function  $\mathbf{E}(\mathbf{T}; \tau; \mathbf{V}, \mathbf{V}^\dagger)$  in equation (5.3). The main results can be summarized as follows.

A second torsional constant, classically called the warping constant, is defined by

$$J_\omega = \iint_\Omega \omega^2(\mathbf{T}) \, dA. \quad (5.12)$$

The gradient of kinematic twist  $\tau'(S)$  gives rise to a corrective displacement along to the cross-section

$$Z_\alpha^{\text{opt}}(\tau^\dagger, \mathbf{T}) = \tau^\dagger u_\alpha(\mathbf{T}) \quad Z_3^{\text{opt}}(\tau^\dagger, \mathbf{T}) = 0 \quad (5.13)$$

where  $u_\alpha(\mathbf{T})$  for  $\alpha = 1, 2$  are two functions satisfying the variational problem

$$\forall \hat{u}_\alpha \iint_\Omega \left( \left\{ \lambda \partial_\rho u_\rho \delta_{\alpha\beta} + 2\mu \frac{\partial_\alpha u_\beta + \partial_\beta u_\alpha}{2} \right\} \partial_\beta \hat{u}_\alpha + (\lambda \omega \partial_\alpha \hat{u}_\alpha - \mu \partial_\alpha \omega \hat{u}_\alpha) - F_\alpha \hat{u}_\alpha - Q \eta_{\alpha\beta} T_\alpha \hat{u}_\beta \right) dA = 0 \quad (5.14)$$

and the kinematic constraints

$$\iint_\Omega u_\alpha \, dA = 0 \quad \iint_\Omega \eta_{\alpha\beta} T_\alpha u_\beta \, dA = 0. \quad (5.15)$$

In equation (5.14),  $\hat{u}_\alpha(\mathbf{T})$  for  $\alpha = 1, 2$  are test functions defined on the cross-section, and  $(F_1, F_2, Q)$  are three scalar multipliers enforcing the kinematic constraints. The solutions  $u_\alpha(\mathbf{T})$  depend on the cross-section geometry and on Poisson's ratio  $\nu = \frac{\lambda}{2(\lambda + \mu)}$ .

In terms of the corrective displacement  $u_\alpha$  and of the warping function  $\omega(\mathbf{T})$  found earlier, see equation (5.7), one can define three additional constants,

$$\begin{aligned} D_\lambda &= \lambda \iint_\Omega \omega(\mathbf{T}) \partial_\alpha u_\alpha(\mathbf{T}) \, dA \\ D_\mu &= \mu \iint_\Omega \partial_\alpha \omega(\mathbf{T}) u_\alpha(\mathbf{T}) \, dA \\ D_\omega &= (\lambda + 2\mu) J_\omega + D_\lambda \end{aligned} \quad (5.16)$$

The final expression of the one-dimensional strain energy is

$$\Phi_{(2)}^*[ \tau ] = \int_0^\ell \left( \frac{1}{2} \mu J \left( \tau(S) + \frac{D_\mu}{\mu J} \frac{d^2 \tau}{dS^2} \right)^2 + \frac{1}{2} (D_\omega + D_\mu) \left( \frac{d\tau}{dS} \right)^2 \right) dS. \quad (5.17)$$

To the best of our knowledge, this simple one-dimensional strain energy for the twisting of a linearly elastic bar is not known from the literature. It captures the gradient effect and is asymptotically correct. It underpins some of the results of TV89, as discussed below, in an accessible form.

By combining equations (2.5), (4.8), (5.2), (5.6) and (5.13), the solution in displacement is found as  $\mathbf{x}(S, \mathbf{T}) = S \mathbf{e}_3 + (T_\alpha + \tau'(S) u_\alpha(\mathbf{T})) \mathbf{d}_\alpha(S) + \tau(S) \omega(\mathbf{T}) \mathbf{e}_3(S)$  where  $\mathbf{d}_1(S) = \mathbf{e}_1 + \theta(S) \mathbf{e}_2$  and  $\mathbf{d}_2(S) = \mathbf{e}_2 - \theta(S) \mathbf{e}_1$  are the rotated directors and  $\theta(S)$  is the twisting angle, see figure 5.1 (recall that we are working in the linear setting). The usual relation defining the twisting strain  $\tau(S) = \theta'(S)$  as the gradient of the twisting angle  $\theta(S)$  is recovered in the process.

The particular case of an elliptical cross-section is worked out in section D.2 in the appendix: with  $a$  and  $b$  as the semi-major and semi-minor axes, in any order, the constants appearing in the energy  $\Phi_{(2)}^*$  are calculated as

$$J = \pi \frac{a^3 b^3}{a^2 + b^2} \quad J_\omega = \frac{1}{24} \frac{(b^2 - a^2)^2}{a^2 + b^2} J \quad D_\mu = 8 \mu J_\omega \left( \frac{a b}{a^2 + b^2} \right)^2 \quad D_\omega = Y J_\omega \quad (\text{elliptical cross-section}) \quad (5.18)$$

where  $Y$  is the Young modulus,

$$Y = \mu \frac{3\lambda + 2\mu}{\lambda + \mu}. \quad (5.19)$$

## 5.4. Equilibrium

In the presence of a distributed external twisting moment  $m_3(S)$  per unit length  $dS$ , the total potential energy of the bar is

$$\Psi^*[\theta] = \Phi_{(2)}^*[\theta'] - \int_0^\ell m_3(S) \theta(S) \, dS,$$

see equation (3.4). The equations of equilibrium of the bar can be found by making  $\Psi^*[\theta]$  stationary with respect to  $\theta$ . Upon integration by parts and after several simplifications, one obtains the equilibrium equation in the interior as

$$\frac{d}{dS} (\mu J \tau - (D_\omega - D_\mu) \tau'') + m_3(S) = 0, \quad (5.20)$$

along with the applicable boundary conditions. Note the plus sign in front of  $D_\mu$  in equation (5.17) and the minus sign in equation (5.20).

## 5.5. Comments

The equilibrium equation (5.20) underpins the analysis of the gradient effect in twisted prismatic bar done by TV89, as shown in Appendix E; however this simple and important equation did not appear explicitly in this work.

In equation (5.20), the quantity inside the derivative  $M_3 = \mu J \tau - (D_\omega - D_\mu) \tau''$  can be interpreted as the internal twisting moment in the bar; it is made up of the prediction  $M_3 = \mu J \tau$  of the classical model without gradient effect, and of a correction coming from the gradient effect; it is a hallmark of higher-order gradient models that the stress not only depends on the local strain but also on its gradients. The quantity  $M_3 = \mu J \tau - (D_\omega - D_\mu) \tau''$  is identical to that obtained by the general constitutive law in equation (3.7), as can be checked.

In the particular case of a circular cross-section,  $a = b$ , the gradient effect is absent: in equation (5.18),  $J_\omega = 0$  and therefore  $D_\mu = D_\omega = 0$ .

The gradient model for a twisted bar has been derived here in the context of linear elasticity. A non-linear extension of this model can be obtained along the same lines; in order to build the catalog of solutions having homogeneous twist, a non-linear one-dimensional boundary-value problem must be solved, which requires some numerical solution in general. In the non-linear model, the constitutive law is of the form  $M_3 = H(\tau) + Q(\tau) \tau''$ . Consistency with the linear model is warranted by the approximations  $H(\tau) \approx \mu J \tau$  and  $Q(\tau) \approx -(D_\omega - D_\mu)$  which hold when linear elasticity is applicable, *i.e.*, when the microscopic strain is small,  $|\tau| \ll 1/\max(a, b)$ . This shows that the linear model derived in this section is applicable as long as the absolute value of the twisting strain  $\tau$  remains small.

## 6. ILLUSTRATION IN A WEAKLY NON-LINEAR SETTING: BUCKLING OF A THICK BEAM

Euler buckling (*i.e.*, the buckling of an elastic cylinder subjected to an axial compressive force) can be analyzed using the classical theory of rods: this yields a prediction for the buckling load which is accurate for infinitely slender beams. With the aim to characterize the buckling load of thicker beams, several authors have derived corrections to the Euler buckling load in powers of the aspect-ratio. This requires restarting from the non-linear theory of elasticity in three dimensions, as both constitutive and geometric nonlinearities affect these corrections.

In an early and remarkable work, FS63 have carried out what is essentially a linear bifurcation analysis of a hyper-elastic cylinder having a finite length-to-radius ratio. They obtained a prediction of the buckling load that connects with Euler's prediction in the limit of a slender cylinder, thereby showing consistency of the buckling analyses based on three-dimensional versus one-dimensional models. However, their solution assumes that the internal moment in the cylinder is proportional to the local value of the center-line curvature. This is questionable for thick beams: the internal moment  $M$  given by equation (3.7) depends on higher derivatives of the curvature as well, as earlier in equation (5.20). It is therefore unclear whether their analysis is valid beyond the infinitely slender case.

In more recent work, ST98 derived the first buckling load of a stubby hyper-elastic cylinder in powers of its aspect-ratio, starting from the full theory of three-dimensional elasticity with finite strain—a similar analysis has been carried out independently by GVD08, DDG11. Here, we show that the results of ST98 can be recovered by (i) deriving a non-linear *one-dimensional* model for the stubby cylinder that captures the gradient effect, using our reduction method and (ii) by carrying out a linear bifurcation analysis of this one-dimensional model.

By doing so, our goal is twofold: we provide another illustration of our reduction method and we verify its predictions in a weakly non-linear setting.

### 6.1. Problem setting

We revisit the buckling problem of ST98 as follows. We consider a prismatic elastic body having length  $\ell$  in the undeformed configuration: in figure 6.1a, the particular case of a cylinder with initial radius  $\rho$  is shown. We use Cartesian coordinates such that the axis  $\mathbf{e}_3$  is aligned with the initial axis of the cylinder, and one of the terminal faces of the cylinder is centered on the origin  $O$  of the coordinate system. The two ends  $S=0$  and  $S=\ell$  are assumed to slide without friction on two planes perpendicular to  $\mathbf{e}_3$ , *i.e.*, the displacement along  $\mathbf{e}_3$  is zero on the terminal faces of the cylinder; in particular the longitudinal displacement is restrained on the terminal faces,

$$\mathbf{x}(0, \mathbf{T}) \cdot \mathbf{e}_3 = 0 \quad (\mathbf{x}(\ell, \mathbf{T}) - \mathbf{r}(\ell)) \cdot \mathbf{e}_3 = 0. \quad (6.1)$$

The distance between the planes is changed from  $\ell$  in the natural configuration, to  $\ell(1+\varepsilon)$  in the actual configuration with  $-1 < \varepsilon < 0$ . We seek the critical value of  $\varepsilon$  corresponding to the occurrence of the first buckling modes.

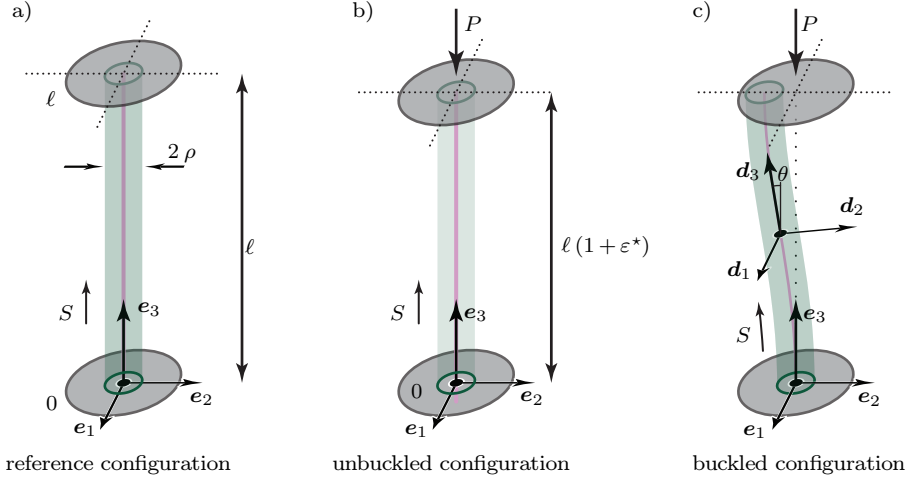
We assume that the prismatic body is made up of an isotropic material, having uniform elastic properties. We also assume that the cross-section domain  $\Omega$  is mirror-symmetric about the axes  $\mathbf{e}_1$  and  $\mathbf{e}_2$  in reference configuration, *i.e.*, it is invariant by both  $(T_1, T_2) \leftarrow (T_1, -T_2)$  and  $(T_1, T_2) \leftarrow (-T_1, T_2)$ ; this warrants

$$\langle (T_1)^i (T_2)^j \rangle = 0 \text{ whenever } i \text{ or } j \text{ is odd, or both are odd} \quad (6.2)$$

With these assumptions, the two bending modes and the stretching modes are uncoupled, and we limit attention to buckling modes such that the center-line remains in the plane perpendicular to  $\mathbf{e}_1$ . We denote by  $\theta(S)$  the rotation of the material frame about the constant vector normal  $\mathbf{d}_1(S) = \mathbf{e}_1$  to the plane of deformation.

The analysis of less symmetric cross-sections, non-isotropic materials, or elastic properties that vary in the cross-section is more involved but it does not raise any fundamental difficulty.

The prismatic body is homogeneous and made of an isotropic hyper-elastic material. We can for instance use the same constitutive model  $w(\mathbf{T}, \mathbf{E}) = w_{\text{ST}}(\mathbf{E})$  as used by ST98, which reads, after restoring a missing coefficient  $1/4$  in their equation [55],  $w_{\text{ST}}(\mathbf{E}) = A_{\text{ST}} \left( \frac{I_1}{I_3^{1/3}} - 3 \right) + B_{\text{ST}} \left( \frac{I_2}{I_3^{2/3}} - 3 \right) + \frac{1}{4} \frac{A_{\text{ST}} + B_{\text{ST}}}{24} \frac{1 + \nu_{\text{ST}}}{1 - 2\nu_{\text{ST}}} \left( I_3^2 - \frac{1}{I_3^2} \right)^2$ , where  $I_1 = \text{tr } \mathbf{C}$ ,  $I_2 = \frac{1}{2} (I_1^2 - \text{tr}(\mathbf{C}^2))$ ,  $I_3 = \det \mathbf{C}$  and  $\mathbf{C} = \mathbf{I} + 2\mathbf{E}$ . However, we do not specify the isotropic constitutive law for the moment. The only constitutive assumptions which we will use in the forthcoming analysis is that, in the unbuckled configuration, the stress is uniaxial and the incremental constitutive law is transversely isotropic: this holds for *any* isotropic constitutive law. Specifically, our analysis makes use of the three constitutive functions  $w_{\text{tr}}(\varepsilon)$ ,  $Y_{\text{t}}(\varepsilon)$ ,  $p(\varepsilon)$  that characterize the non-linear material response in simple traction, where  $\varepsilon$  is the longitudinal engineering strain:  $w_{\text{tr}}(\varepsilon)$  is the strain energy density of the material in simple traction,  $Y_{\text{t}}(\varepsilon)$  is the tangent Young modulus and  $p(\varepsilon)$  is the transverse stretch resulting from Poisson's effect, see section F.1 in the appendix for details. In terms of these material functions, we also define the initial Young modulus  $Y_0$ , the initial Poisson's ratio  $\nu_0$  and the initial curvature  $Y'_0$  of the load-displacement curve, see equation (F.10) from the appendix.



**Figure 6.1.** Buckling of a thick circular cylinder with initial radius  $\rho$ , whose terminal faces slide onto two parallel planes, as analyzed by ST98 and GVD08. Our analysis addresses the slightly more general case of a prismatic body, whose cross-section  $\Omega$  is mirror-symmetric with respect to be axes  $\mathbf{e}_1$  and  $\mathbf{e}_2$ .

## 6.2. One-dimensional reduction

In this section, we apply our reduction method to obtain the one-dimension model for the planar beam capturing the gradient effect; it describes the bending and stretching of a planar, higher-order Elasticity.

For planar deformation, there are two relevant macroscopic strain measures, namely the bending strain  $\kappa = \kappa_1$  and the stretching strain  $\varepsilon$ ; they are grouped into a macroscopic strain vector  $\mathbf{h} = (\varepsilon, \kappa)$ . In the planar twistless case, the general form (4.2) of the higher-order one-dimensional model writes

$$\begin{aligned} \Phi_{(2)}^*[ \varepsilon, \kappa ] = & \int_0^\ell \left[ W_{\text{hom}} \left( \varepsilon + \xi_0(\varepsilon, \kappa) \varepsilon'', \kappa + \xi_1(\varepsilon, \kappa) \kappa'' \right) \dots \right. \\ & \left. + A_0(\varepsilon, \kappa) \varepsilon' + A_1(\varepsilon, \kappa) \kappa' + \frac{1}{2} \begin{pmatrix} \varepsilon' \\ \kappa' \end{pmatrix} \cdot \begin{pmatrix} D_{00}(\varepsilon, \kappa) & D_{10}(\varepsilon, \kappa) \\ D_{10}(\varepsilon, \kappa) & D_{11}(\varepsilon, \kappa) \end{pmatrix} \cdot \begin{pmatrix} \varepsilon' \\ \kappa' \end{pmatrix} \right] dS. \end{aligned} \quad (6.3)$$

We now proceed to calculate the quantities  $W_{\text{hom}}$ ,  $\xi_0$ ,  $\xi_1$ ,  $A_0$ ,  $A_1$  and  $D_{ij}$ : we consider the case of a finite axial strain  $\varepsilon$  but limit attention to small values of the curvature  $\kappa$ , anticipating that this is all that is needed for the forthcoming buckling analysis.

### 6.2.1. Symmetry properties

We first characterize the symmetry properties of the functions  $W_{\text{hom}}$ ,  $\xi_0$ ,  $\xi_1$ ,  $A_0$ ,  $A_1$  and  $D_{ij}$  as they will save us from calculating some quantities that are zero by symmetry. The cylinder is invariant both by a mirror symmetry with respect to the axis  $\mathbf{e}_3$ , and by a reversal of the parameterization  $S \leftarrow (-S)$ . These symmetries correspond to changing  $(\varepsilon, \kappa, \varepsilon', \kappa', \varepsilon'', \kappa'')$  into  $(+\varepsilon, -\kappa, +\varepsilon', -\kappa', +\varepsilon'', -\kappa'')$  and  $(+\varepsilon, -\kappa, -\varepsilon', +\kappa', -\varepsilon'', +\kappa'')$ , respectively. For  $\Phi_{(2)}^*[ \varepsilon, \kappa ]$  to remain invariant by both these transformations, both  $A_0$  and  $A_1$  must be zero identically,  $W_{\text{hom}}$ ,  $\xi_0$ ,  $\xi_1$ ,  $D_{00}$  and  $D_{11}$  must be even with respect to  $\kappa$ , and  $D_{10}$  must be odd with respect to  $\kappa$ , see equation (6.3). Therefore, for any  $\varepsilon$  and  $\kappa$ , and for any set of non-negative integers  $i$  and  $j$ , we have

$$\begin{aligned} A_0(\varepsilon, \kappa) &= 0 \quad A_1(\varepsilon, \kappa) = 0 \\ \frac{\partial^{i+2j+1} f}{\partial \varepsilon^i \partial \kappa^{2j+1}}(\varepsilon, 0) &= 0 \text{ for any choice of } f \text{ in } \{W_{\text{hom}}, \xi_0, \xi_1, D_{00}, D_{11}\} \\ \frac{\partial^{i+2j} D_{10}}{\partial \varepsilon^i \partial \kappa^{2j}}(\varepsilon, 0) &= 0 \end{aligned} \quad (6.4)$$

In particular, the matrix  $D_{ij}(\varepsilon, 0)$  is diagonal, i.e.,  $D_{10}(\varepsilon, 0) = 0$ .

### 6.2.2. Analysis of homogeneous solutions

Homogeneous solutions are derived here by solving the equations from section B.1 in the appendix, using the expression of the strain from equation (4.4) relevant to the non-linear and homogeneous setting, and by specializing to the planar twistless case,  $\kappa_2 = \kappa_3 = 0$ . As we are ultimately interested in weakly bent configurations of the cylinder, close to the bifurcation threshold, we treat  $\kappa$  as a small parameter. We refrain from setting  $\kappa = 0$  from the onset, however, as some derivatives of intermediate quantities with respect to  $\kappa$  will be needed in the course of the derivation.



As shown in section F.2 of the appendix, the microscopic displacement corresponding to a homogeneous axial strain  $\varepsilon$  and curvature  $\kappa$  is

$$Y_\alpha^{(\varepsilon, \kappa)} = p(\varepsilon) \left( T_\alpha + \kappa \frac{dp}{d\varepsilon}(\varepsilon) \varphi_\alpha(\mathbf{T}) \right) + \mathcal{O}(\kappa^2) \quad Y_3^{(\varepsilon, \kappa)}(\mathbf{T}) = 0 \quad (6.5)$$

where  $\varphi_1(\mathbf{T})$  and  $\varphi_2(\mathbf{T})$  are two functions depending on the cross-section geometry, which are the solutions of the differential problem on the cross-section

$$\forall \mathbf{T} \in \Omega \quad \frac{\partial_\alpha \varphi_\beta(\mathbf{T}) + \partial_\beta \varphi_\alpha(\mathbf{T})}{2} = T_2 \delta_{\alpha\beta} \quad \langle \varphi_\alpha \rangle = 0 \quad \langle \eta_{\alpha\beta} T_\alpha \varphi_\beta(\mathbf{T}) \rangle = 0. \quad (6.6)$$

With the symmetry assumptions in (6.2), the solution is

$$\varphi_1(\mathbf{T}) = T_1 T_2 \quad \varphi_2(\mathbf{T}) = \frac{T_2^2 - T_1^2}{2} - \frac{\langle T_2^2 \rangle - \langle T_1^2 \rangle}{2}. \quad (6.7)$$

Up to a rigid-body displacement, the functions  $\varphi_1$  and  $\varphi_2$  match the functions  $\phi_{21}$  and  $\phi_{22}$  classically used in the *linear* analysis of bending, respectively—see for instance equations [2.5, 3.8, 3.9] in the work of TV89. Our analysis shows that they are relevant to the analysis of finite-stretching and infinitesimal-bending as well.

As shown in the appendix, the displacement (6.5) is such that every point in the bar is in simple traction with a local longitudinal strain  $\varepsilon + \kappa p(\varepsilon) T_2$  depending on the transverse coordinate  $T_2$ : the strain is given by  $\mathbf{E}^{(\varepsilon, \kappa)}(\mathbf{T}) = \mathbf{E}_{\text{tr}}(\varepsilon + \kappa p(\varepsilon) T_2) + \mathcal{O}(\kappa^2)$  and the stress is uniaxial,  $\boldsymbol{\Sigma}^{(\varepsilon, \kappa)}(\mathbf{T}) = \Sigma_{\text{tr}}(\varepsilon + \kappa p(\varepsilon) T_2) \mathbf{e}_3 \otimes \mathbf{e}_3 + \mathcal{O}(\kappa^2)$ , where  $\mathbf{E}_{\text{tr}}(\varepsilon)$  and  $\Sigma_{\text{tr}}(\varepsilon) \mathbf{e}_3 \otimes \mathbf{e}_3$  are the strain and the stress in simple traction for the particular material considered, see equation (F.1).

The strain energy per unit length associated with this homogeneous solution is found in the appendix as

$$W_{\text{hom}}(\varepsilon, \kappa) = A w_{\text{tr}}(\varepsilon) + \frac{1}{2} Y_t(\varepsilon) p^2(\varepsilon) I_1^0 \kappa^2 + \mathcal{O}(\kappa^4) \quad (6.8)$$

where  $A = \iint_\Omega dA$  is the initial area and  $I_1^0 = \iint_\Omega T_2^2 dA$  is the initial geometric moment of inertia. In the small-strain limit, the potential  $W_{\text{hom}}$  is consistent with the classical Euler beam model  $W_{\text{hom}}(\varepsilon, \kappa) \approx C + \frac{Y_0 A}{2} \varepsilon^2 + \frac{Y_0 I_1^0}{2} \kappa^2$  (where  $C$  is a constant) as can be shown by inserting the equivalents of  $w_{\text{tr}}$ ,  $Y_t$  and  $p$  for small  $\varepsilon$  derived in section F.1.

### 6.2.3. Gradient effect

The corrective displacement associated with longitudinal gradients of axial strain  $\varepsilon^\dagger$  and curvature  $\kappa^\dagger$  is found in the appendix as

$$Z_{\text{opt}, \alpha}^{(\varepsilon, 0)}((\varepsilon^\dagger, \kappa^\dagger), \mathbf{T}) = 0 \quad Z_{\text{opt}, 3}^{(\varepsilon, 0)}((\varepsilon^\dagger, \kappa^\dagger), \mathbf{T}) = \varepsilon^\dagger \times (\dots) + \kappa^\dagger \frac{p^2(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon)}{1 + \varepsilon} (\Theta(\mathbf{T}) + c_\Gamma(\varepsilon) \Gamma(\mathbf{T})) \quad (6.9)$$

where the contribution associated with  $\varepsilon^\dagger$  is denoted by an ellipsis and does not need to be calculated,  $c_\Gamma(\varepsilon)$  is a material parameter depending on the strain  $\varepsilon$ ,

$$c_\Gamma(\varepsilon) = \frac{Y_t(\varepsilon)}{2 G_t(\varepsilon) p(\varepsilon) \left( -\frac{dp}{d\varepsilon}(\varepsilon) \right) (1 + \varepsilon)}, \quad (6.10)$$

and  $\Theta$  and  $\Gamma$  are the cross-sectional functions satisfying the variational problems

$$\begin{aligned} \forall \hat{Z}_3 \quad \iint_\Omega [\partial_\alpha \Theta(\mathbf{T}) \partial_\alpha \hat{Z}_3(\mathbf{T}) + \varphi_\alpha(\mathbf{T}) \partial_\alpha \hat{Z}_3(\mathbf{T})] dA &= 0 & \langle \Theta \rangle &= 0 \\ \forall \hat{Z}_3 \quad \iint_\Omega [\partial_\alpha \Gamma(\mathbf{T}) \partial_\alpha \hat{Z}_3(\mathbf{T}) + 2 T_2 \hat{Z}_3(\mathbf{T})] dA &= 0 & \langle \Gamma \rangle &= 0 \end{aligned} \quad (6.11)$$

The functions  $\Theta$  and  $\Gamma$  are denoted as  $\theta_2$  and  $\eta_2$  in the work of TV89, see their equations [2.23] and [2.17].

Finally, we define four constants depending on the cross-section shape,

$$M = \sum_\alpha \iint_\Omega \varphi_\alpha^2(\mathbf{T}) dA \quad J_{\Theta\Theta} = \iint_\Omega \partial_\alpha \Theta \partial_\alpha \Theta dA \quad J_{\Theta\Gamma} = \iint_\Omega \partial_\alpha \Theta \partial_\alpha \Gamma dA \quad J_{\Gamma\Gamma} = \iint_\Omega \partial_\alpha \Gamma \partial_\alpha \Gamma dA. \quad (6.12)$$

As shown in appendix F.3, the reduction method from section 4 yields the following expressions for the quantities entering in the one-dimensional model (6.3),

$$\begin{aligned} \xi_0(\varepsilon, 0) &= 0 \\ \xi_1(\varepsilon, 0) &= A \frac{p(\varepsilon) \left( -\frac{dp}{d\varepsilon}(\varepsilon) \right)}{2(1 + \varepsilon)} \frac{1}{I_1^0 / A^2} \left( \frac{J_{\Theta\Gamma}}{A^3} + \frac{J_{\Gamma\Gamma}}{A^3} c_\Gamma(\varepsilon) \right) \\ D_{11}(\varepsilon, 0) &= A^3 \left( p(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon) \right)^2 \left[ \frac{w'_{\text{tr}}(\varepsilon)}{1 + \varepsilon} \frac{M}{A^3} + p^2(\varepsilon) G_t(\varepsilon) \left( \frac{M - J_{\Theta\Theta}}{A^3} + \frac{J_{\Gamma\Gamma}}{A^3} c_\Gamma^2(\varepsilon) \right) \right]. \end{aligned} \quad (6.13)$$

These are the only properties of the one-dimensional model which are needed in the linear buckling analysis, as we will show.

For reference, the microscopic solution in displacement is found by combining equations (2.5), (4.8), (6.5) and (6.9) as

$$\mathbf{x}(S, \mathbf{T}) = \mathbf{r}(S) + p(\varepsilon) \left( T_\alpha + \kappa \frac{dp}{d\varepsilon}(\varepsilon) \varphi_\alpha(\mathbf{T}) \right) \mathbf{d}_\alpha(S) + \left( \varepsilon'(\dots) + \kappa' \frac{p^2(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon)}{1 + \varepsilon} (\Theta(\mathbf{T}) + c_\Gamma(\varepsilon) \Gamma(\mathbf{T})) \right) \mathbf{d}_3(S) + \dots \quad (6.14)$$

#### 6.2.4. Case of a circular cross-section

When the cross-section is a disk with radius  $\rho$ , as shown in figure 6.1, the initial area is  $A = \pi \rho^2$ , the initial moment of inertia is  $I_1^0 = \iint T_2^2 dA = \frac{\pi \rho^4}{4} = \frac{A^2}{4\pi}$ , the functions  $\varphi_\alpha$  in equation (6.7) take the slightly simpler form

$$\varphi_1(\mathbf{T}) = T_1 T_2 \quad \varphi_2(\mathbf{T}) = \frac{T_2^2 - T_1^2}{2}, \quad (6.15)$$

and the solutions  $\Theta$  and  $\Gamma$  to equation (6.11) are

$$\begin{aligned} \Theta(\mathbf{T}) &= -\frac{1}{4}(T_1^2 + T_2^2 - \rho^2) T_2 \\ \Gamma(\mathbf{T}) &= +\frac{1}{4}(T_1^2 + T_2^2 - 3\rho^2) T_2. \end{aligned} \quad (6.16)$$

After factoring out the cube area  $A^3 = (\pi \rho^2)^3$ , the constants appearing in equation (6.12) can then be expressed as

$$M = \frac{A^3}{12\pi^2} \quad (J_{\Theta\Theta}, J_{\Theta\Gamma}, J_{\Gamma\Gamma}) = \frac{A^3}{24\pi^2} (+1, -1, +7).$$

Finally, the quantities defining the one-dimensional model in equation (6.13) are calculated as

$$\begin{aligned} \xi_0(\varepsilon, 0) &= 0 \\ \xi_1(\varepsilon, 0) &= \rho^2 p(\varepsilon) \left( -\frac{dp}{d\varepsilon}(\varepsilon) \right) \frac{-1 + 7 c_\Gamma(\varepsilon)}{12(1 + \varepsilon)} \\ D_{11}(\varepsilon, 0) &= \left( p(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon) \right)^2 \frac{(\pi \rho^2)^3}{24\pi^2} \left( \frac{2 w_{tr}'(\varepsilon)}{1 + \varepsilon} + p^2(\varepsilon) G_t(\varepsilon) (1 + 7 c_\Gamma^2(\varepsilon)) \right), \end{aligned} \quad (6.17)$$

### 6.3. Buckling analysis of the one-dimensional model

We turn to the analysis of the buckling problem based on the one-dimensional model just derived. We denote by  $\varepsilon^*$  the axial strain in the unbuckled configuration, and by  $u(S)$  and  $v(S)$  the longitudinal and transverse displacement associated with the buckling mode. The center-line position therefore writes, in the buckled configuration of figure 6.1c,  $\mathbf{r}(S) = v(S) \mathbf{e}_2 + (S(1 + \varepsilon^*) + u(S)) \mathbf{e}_1$ .

The axial strain  $\varepsilon(S)$  and the rotation  $\theta(S)$  are defined by  $\mathbf{r}'(S) = (1 + \varepsilon(S)) \mathbf{d}_3(\theta(S))$  where  $(\mathbf{d}_2(\theta), \mathbf{d}_3(\theta)) = (\cos \theta \mathbf{e}_2 + \sin \theta \mathbf{e}_3, -\sin \theta \mathbf{e}_2 + \cos \theta \mathbf{e}_3)$  is the rotated basis, see equation (2.7). The bending strain is  $\kappa(S) = \theta'(S)$  from equation (2.8). This yields the strain in terms of the displacement as

$$\begin{aligned} \varepsilon(S) &= -1 + \sqrt{(1 + \varepsilon^* + u'(S))^2 + v'^2(S)} \\ \kappa(S) &= \frac{d}{dS} \left( \tan^{-1} \frac{v'(S)}{1 + \varepsilon^* + u'(S)} \right). \end{aligned} \quad (6.18)$$

The buckling problem is governed by the total potential energy

$$\Psi^*[u, v] = \Phi_{(2)}^*[\varepsilon, \kappa] - P u(\ell), \quad (6.19)$$

where  $\Phi_{(2)}^*$  is the one-dimensional strain energy obtained in section 6.2 and  $P$  is the buckling load applied on the plane in sliding contact with the endpoint of the cylinder. For the sake of definiteness, we analyze buckling under force control rather than displacement control; this makes no difference for the calculation of the critical loads.

We proceed to identify the boundary conditions applicable to the one-dimensional model. By inserting the microscopic displacement (6.14) into the sliding conditions (6.1), we find that the following boundary conditions must hold on both ends:  $\mathbf{d}_\alpha \cdot \mathbf{e}_3 = 0$  (which is equivalent to  $\theta = 0$ ),  $\varepsilon' = 0$  and  $\kappa' = 0$ . In addition, the bottom support is fixed, which yields  $u(0) = 0$ . The following kinematic boundary conditions are therefore applicable,

$$u(0) = 0 \quad \theta(0) = 0 \quad \theta(\ell) = 0 \quad \theta''(0) = 0 \quad \theta''(\ell) = 0 \quad \varepsilon'(0) = 0 \quad \varepsilon'(\ell) = 0. \quad (6.20)$$

The high-order boundary conditions on  $\kappa' = \theta''$  are legal in the variational problem of equilibrium as the energy (6.3) depends on  $\kappa'' = \theta'''$  when  $\xi_1 \neq 0$ . The high-order boundary conditions on  $\varepsilon'$  are normally not legal since  $\xi_0 = 0$  and the energy depends on  $\varepsilon'$  but not on  $\varepsilon''$ ; this points to the fact that boundary layers occur generically near the boundaries as is known since the work of St-Venant. Such layers are nevertheless absent for the particular choice of sliding boundaries made here; this will enable us to ultimately satisfy all boundary conditions, even if the problem looks ill-posed from a variational standpoint.

A principle of virtual work is obtained by inserting the strain (6.18) into the total potential energy  $\Psi^*$ , and by calculating the first variation with respect to the unknowns  $u$  and  $v$ .

To characterize the fundamental solution, we require that  $\Psi^*[u, v]$  is stationary at  $u \equiv 0$  and  $v \equiv 0$ : this yields the condition  $\int_0^\ell \left( \frac{\partial W_{\text{hom}}}{\partial \varepsilon}(\varepsilon^*, 0) - P \right) \hat{u}'(S) dS = 0$  for any  $\hat{u}(S)$  such that  $\hat{u}(0) = 0$ , after taking into account the identity  $\xi_0(\varepsilon^*, 0) = 0$  in equation (6.17) and the symmetry properties (6.4). Therefore, the fundamental solution selects the axial strain  $\varepsilon^*$  such that

$$P = \frac{\partial W_{\text{hom}}}{\partial \varepsilon}(\varepsilon^*, 0). \quad (6.21)$$

We have just recovered the force-displacement relation of our particular material in simple traction.

The bifurcation condition is found by setting to zero the second variation of  $\Psi^*[u, v]$  about the fundamental solution  $u \equiv 0$  and  $v \equiv 0$ . With the help of a symbolic calculation language, we obtain the following variational problem for the critical strain  $\varepsilon^* = \varepsilon_{\text{cr}}$  and the buckling mode  $\bar{v}(S)$ : for any  $\forall \hat{v}$  such that  $\hat{v}'(0) = \hat{v}'''(0) = \hat{v}'(\ell) = \hat{v}'''(\ell) = 0$ ,

$$\int_0^\ell \left[ (1 + \varepsilon_{\text{cr}}) \left( \frac{\partial W_{\text{hom}}}{\partial \varepsilon} \right)_{\text{cr}} \bar{v}' \hat{v}' + \left( \frac{\partial^2 W_{\text{hom}}}{\partial \kappa^2} \right)_{\text{cr}} (\bar{v}'' + (\xi_1)_{\text{cr}} \bar{v}''''') (\hat{v}'' + (\xi_1)_{\text{cr}} \hat{v}''''') + (D_{11})_{\text{cr}} \bar{v}''' \hat{v}''' \right] dS = 0. \quad (6.22)$$

A decoupled eigenvalue problem is obtained for the longitudinal displacement  $u(S)$  as well but it is not reported here as it characterizes necking instabilities, which we ignore. In equation above, all quantities bearing the subscript 'cr' are evaluated in the fundamental solution, i.e.,  $(f)_{\text{cr}} = f(\varepsilon_{\text{cr}}, 0)$ .

It is interesting to contrast equation (6.22) with the bifurcation equation predicted by a classical beam model, which ignores the gradient effect. The latter can be recovered by setting  $\Phi_{(2)}^*[\varepsilon, \kappa] = \int_0^\ell W_{\text{hom}}(\varepsilon, \kappa) dS$  in equation (6.3) and hence corresponds to  $(\xi_1)_{\text{cr}} = 0$  and  $(D_{11})_{\text{cr}} = 0$ ; this yields a different bifurcation equation, namely

$$\int_0^\ell \left[ (1 + \varepsilon_{\text{cr}}) \left( \frac{\partial W_{\text{hom}}}{\partial \varepsilon} \right)_{\text{cr}} \bar{v}' \hat{v}' + \left( \frac{\partial^2 W_{\text{hom}}}{\partial \kappa^2} \right)_{\text{cr}} \bar{v}'' \hat{v}'' \right] dS = 0 \quad (\text{classical beam model}).$$

Here,  $\left( \frac{\partial W_{\text{hom}}}{\partial \varepsilon} \right)_{\text{cr}} = P$  is the applied load, see equation (6.21), and  $\left( \frac{\partial^2 W_{\text{hom}}}{\partial \kappa^2} \right)_{\text{cr}}$  is the incremental bending modulus. Comparison with equation (6.22) shows that our asymptotic one-dimensional model corrects the classical buckling analysis of beams in two ways, which are important for thick cylinders: it makes use of the modified bending strain,  $\bar{\kappa} = \bar{v}'' + (\xi_1)_{\text{cr}} \bar{v}'''''$  instead of the standard bending strain  $\bar{v}''$ , and it takes into account the energy cost associated with the *gradient* of curvature  $\bar{v}'''$ , through the term proportional to  $D_{11}$ .

We return to the asymptotically correct model, and proceed to solve the bifurcation equation (6.22). An ordinary differential equation with constant coefficients can be obtained by integration by parts and elimination of the virtual quantity  $\hat{v}$ . In view of the kinematic boundary conditions  $v'(0) = v'''(0) = v'(\ell) = v'''(\ell) = 0$ , a simple calculation shows that the first buckling mode is  $\bar{v}(S) = \frac{1 - \cos(kS)}{2}$  where  $k = \frac{\pi}{\ell}$ , and that the critical strain  $\varepsilon_{\text{cr}}$  is selected by the dispersion equation

$$(1 + \varepsilon_{\text{cr}}) \frac{\partial W_{\text{hom}}}{\partial \varepsilon}(\varepsilon_{\text{cr}}, 0) + \frac{\partial^2 W_{\text{hom}}}{\partial \kappa^2}(\varepsilon_{\text{cr}}, 0) (1 - \xi_1(\varepsilon_{\text{cr}}, 0) k^2)^2 k^2 + D_{11}(\varepsilon_{\text{cr}}, 0) k^4 = 0. \quad (6.23)$$

This implicit equation for the first buckling load  $\varepsilon_{\text{cr}}$  is valid for finite  $\varepsilon_{\text{cr}}$ . For a long beam, i.e., when  $\ell/\rho$  is large and  $(k\rho)$  is small, we can seek an expansion of the critical strain  $\varepsilon_{\text{cr}}$  in powers of the aspect-ratio parameter  $e = \frac{k\rho}{\sqrt{\pi}} = \frac{\sqrt{\pi}\rho}{\ell}$ , i.e.,

$$e^2 = \frac{\pi \rho^2}{\ell^2}. \quad (6.24)$$

With the help of a symbolic calculation language, the series  $\varepsilon_{\text{cr}}$  satisfying the dispersion equation (6.23) is found as

$$\varepsilon_{\text{cr}} = -\frac{\pi \chi_0}{4} e^2 - \pi^2 \left( \chi_1 - (\chi_2 + \chi_4) \chi_0 + \frac{2 + \chi_3}{32} \chi_0^2 \right) e^4 + \mathcal{O}(e^6) \quad (6.25)$$

where the  $\chi_i$ 's are dimensionless parameters from the one-dimensional model,

$$\chi_0 = \frac{4}{\rho^2 \tilde{\chi}} \left( \frac{\partial^2 W_{\text{hom}}}{\partial \kappa^2} \right)_0 \quad \chi_1 = \frac{1}{\rho^4 \tilde{\chi}} (D_{11})_0 \quad \chi_2 = \frac{\xi_1(0, 0)}{2 \rho^2} \quad \chi_3 = \frac{1}{\tilde{\chi}} \left( \frac{\partial^2 W_{\text{hom}}}{\partial \varepsilon^2} \right)_0 \quad \chi_4 = \frac{1}{4 \rho^2 \tilde{\chi}} \left( \frac{\partial^3 W_{\text{hom}}}{\partial \varepsilon \partial \kappa^2} \right)_0, \quad (6.26)$$

where  $\tilde{\chi} = \left( \frac{\partial^2 W_{\text{hom}}}{\partial \varepsilon^2} \right)_0$ . Here the '0' is subscript means that the quantity inside the corresponding parentheses must be evaluated in the undeformed configuration,  $(f)_0 = f(0, 0)$ . To derive the expansion (6.25), we have used the fact that there is no pre-stress in the reference configuration,  $\frac{\partial W_{\text{hom}}}{\partial \varepsilon}(0, 0) = 0$ , as shown by combining equations (6.8) and (F.7): this warrants that  $\varepsilon_{\text{cr}} \rightarrow 0$  for  $e \rightarrow 0$ .

#### 6.4. Expansion of the critical load

With the help of equations (F.7–F.10) from the appendix, equations (6.10) and (6.17) for a circular cross-section become, in the limit  $\varepsilon \rightarrow 0$ ,

$$c_{\text{r}}(0) = \frac{1 + \nu_0}{\nu_0} \quad \xi_1(0, 0) = \frac{7 + 6\nu_0}{12} \rho^2 \quad D_{11}(0, 0) = Y_0 \frac{(\pi \rho^2)^3}{48 \pi^2} \frac{7 + 14\nu_0 + 8\nu_0^2}{1 + \nu_0}.$$

Using this, the relations (F.1–F.10) and the expression of  $W_{\text{hom}}$  found in (6.8), we can calculate the coefficients appearing in (6.26) as

$$\chi_0 = 1 \quad \chi_1 = \frac{7 + 14\nu_0 + 8\nu_0^2}{48(1 + \nu_0)} \quad \chi_2 = \frac{7 + 6\nu_0}{24} \quad \chi_3 = \frac{Y_0'}{Y_0} \quad \chi_4 = \frac{1}{16} \left( \frac{Y_0'}{Y_0} - 2\nu_0 \right). \quad (6.27)$$

Inserting into equation (6.25), we obtain our final expression for the first critical load of the cylinder as a function of the aspect-ratio parameter,

$$\varepsilon_{\text{cr}} = -\frac{\pi e^2}{4} + \frac{\pi^2 e^4}{48} \left( \frac{3 Y_0'}{2 Y_0} + 4 - \frac{\nu_0 (1 + 2 \nu_0)}{1 + \nu_0} \right) + \mathcal{O}(e^6). \quad (6.28)$$

This is identical to the result of ST98; we refer the reader to their paper for a comparison of this expansion with finite-element simulations for a finite aspect-ratio  $e$ . Note that the correction to the classical Euler prediction  $\varepsilon_{\text{cr}} = -\frac{\pi e^2}{4}$ , *i.e.*, the term proportional to  $e^4$  in equation above, depends on both material nonlinearity (through the non-linear elastic modulus  $Y_0'$ ) and on geometric nonlinearity (through the other terms in the parentheses).

ST98 observed that classical models such as the Euler beam model fail to capture the correction of order  $e^4$  in equation (6.28) and are therefore inappropriate for the analysis of stubby or thick structures; we concur with this statement. It has apparently gone unnoticed that this difficulty can be overcome by using a refined one-dimensional model capturing the gradient effect, as we just did: when this is done in an asymptotically correct way, the expansion of the critical buckling strain is correctly predicted in terms of the aspect-ratio.

Unlike in earlier work, we have split the analysis of this buckling problem into two distinct tasks: deriving a one-dimensional model on the one hand, and carrying out a bifurcation analysis on the other hand. Keeping these two tasks separate is not only arguably more elegant, it also avoids the need to reinvent the wheel for every buckling problem: if one were to study the buckling of a stubby circular *ring* or the *post-buckling* of a stubby *Elastica* in compression, for instance, one could reuse the one-dimensional structural model from section 6.2 and simply update the buckling analysis to reflect the geometry of interest.

## 7. DISCUSSION

We have proposed an asymptotic method for constructing one-dimensional models, see section 4. The method achieves dimension reduction by relaxing the microscopic displacement. Concretely, it is implemented as a straightforward (albeit lengthy) series of steps, as described in appendix B. It builds up on the general recipe for dimension reduction published in our previous work (LA20b). The method yields an asymptotically exact, variational model that accounts for the gradient effect. It also accounts for geometric and material nonlinearities, and thereby may help broaden the range of applicability of rod theories significantly. With a view to illustrating the method, we have treated the linear twisting of an elastic cylinder, including higher-order effects; in equation (5.17) we have derived a simple one-dimensional strain energy potential that governs the equilibrium, which is new to the best of our knowledge. We have also applied our reduction method to the Euler buckling of a beam having a moderate aspect-ratio: the expansion of the critical load in powers of the aspect-ratio from earlier work has been recovered based on a high-order rod model. The capabilities of the method go much beyond these two simple examples; it can be readily applied to structures involving finite strain, arbitrary pre-stress distributions or low material symmetries, which will be the subject of future work.

For more complex geometries, the analytical approach adopted here may no longer be tractable, and the quantities  $W_{\text{hom}}(\mathbf{h})$ ,  $\mathbf{B}(\mathbf{h})$ , etc. defining the one-dimensional model may have to be found by solving variational problems on the cross-section numerically; the finite-element method is perfectly suited to this task. In this approach, the 3D elasticity problem for the slender elastic body is split into 2D+1D problems, where the 2D problems are microscopic and are formulated in the cross-section while the 1D one is a macroscopic structural problem. This splitting approach makes the solution considerably easier, which is precisely the point of dimension reduction.

The rod models that are derived by our method include a kinematic constraint which ensures that the tangent to the center-line  $\mathbf{r}'(S)$  stays aligned with the director  $\mathbf{d}_3(S)$ , see equation (2.7). One-dimensional model of this kind are referred to as *unshearable* but this qualifier is misleading: shear can take place at the microscopic scale in our approach, even if it is not exposed in the one-dimensional model. As discussed at the very end of §2.1, the directors capture the deformed cross-sections in an average sense only: the cross-sections are by no means constrained to remain aligned with the directors  $\mathbf{d}_1(S)$  and  $\mathbf{d}_2(S)$ , *i.e.*, to remain perpendicular to the center-line, not even in an average sense. For example, in a rod made up of an anisotropic material that is very stiff in a direction making a  $45^\circ$  angle with the axis of the rod, the cross-sections rotate about  $\mathbf{d}_2$  (and therefore tilt along the axis) by an angle  $a \varepsilon$  proportional to axial stretch  $\varepsilon$ . This microscopic shear is accounted for in our approach but it is not reflected in the directors  $\mathbf{d}_i(S)$ : their assigned role is to keep track of the twisting of the cross-sections about the tangent, *not* to provide a faithful representation of the microscopic solution. To some extent, our approach therefore has the same capabilities as Timoshenko models, except that the microscopic shear is dealt with *internally*. The benefit is that a minimal set of degrees of freedom are presented to the user. The only minor complication is that, in order to block the rotation at the endpoints, one has to look up the average orientation of the terminal cross-sections from the microscopic solution, as the vectors  $\mathbf{d}_1(S)$  and  $\mathbf{d}_2(S)$  cannot be used directly.

In this paper, we have carried out dimension reduction without making scaling assumptions on the intensity of the loading. This is not a standard way of proceeding. It is the special form of the external force potential in equation (2.14) that plays the role of the standard scaling assumptions, as discussed at the end of §2.4. Let us briefly expose how our method can be extended to handle non-standard scaling assumptions for the loading. Consider for instance the case where the distributed applied torque is so large that it can induce shear, by tilting the cross-sections towards the center-line. Such a load cannot be represented by equation (2.14) since by design the directors  $\mathbf{d}_1$  and  $\mathbf{d}_2$  remain perpendicular to the center-line. The solution is to introduce an additional kinematic variable, similar to Timoshenko's shear angle, and to modify the procedure as follows. The new internal degree of freedom is appended to the set of macroscopic strain  $\mathbf{h}$ , meaning that it is fixed during the relaxation procedure and is a variable of the one-dimensional model; it is coupled to the large applied torque in the potential energy  $\Psi$ . The result of this modified relaxation procedure is an *asymptotic* Timoshenko-like model, in which the average microscopic shear is explicitly represented.

It is also possible to extend the method to the case where the geometric or elastic properties of the body vary slowly in the longitudinal direction—such as the case of rods having non-uniform cross-sectional dimensions—, with little additional work. This extension is discussed at the very end of our previous paper (LA20b): in this case the operator  $C_h^{(1)}$  gets an explicit dependence on the axial variable  $S$ , and an additional term proportional to  $\left. \frac{\partial C_h^{(1)}(S)}{\partial S} \right|_{h=h(S)}$  appears in the one-dimensional potential  $\Phi_{(2)}^*$ . This, along with other extensions, will be described in follow-up work.

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## APPENDIX A. EQUILIBRIUM OF THE ORIGINAL (3D) AND IDEAL (1D) MODELS

### A.1. Equilibrium of the ideal model

In this section, we derive the equations of equilibrium for the ideal one-dimensional model from section 3 using a variational method. Doing so, we prove the results announced in section 3.2: the ideal model is governed by the classical Kirchhoff equations for the equilibrium thin rods and by the constitutive laws (3.7).

Combining equations (3.4) for the total potential  $\Psi^*$ , and (3.2) for the condensed strain energy, we obtain

$$\Psi^*[\mathbf{r}, \mathbf{d}_i] = \Phi[\mathbf{h}, \mathbf{y}^*[\mathbf{h}]] + \int_0^\ell V(\mathbf{r}(S), \mathbf{d}_i(S)) \, dS.$$

We introduce perturbations  $\hat{\mathbf{r}}$  and  $\hat{\boldsymbol{\theta}}$  to the macroscopic fields  $\mathbf{r}$  and  $\mathbf{d}_i$ , as in section 3.2, that satisfy the incremental form of the kinematic constraint (2.7),

$$\hat{\mathbf{r}}'(S) = \hat{\varepsilon}(S) \mathbf{d}_3(S) + \hat{\boldsymbol{\theta}}(S) \times \mathbf{r}'(S). \quad (\text{A.1})$$

Using the incremental form of the equations defining the macroscopic strain from section 2.2, one can define the strain perturbations  $\hat{\mathbf{h}} = (\hat{\varepsilon}, \hat{\kappa}_1, \hat{\kappa}_2, \hat{\kappa}_3)$  in terms of  $\hat{\mathbf{r}}$  and  $\hat{\boldsymbol{\theta}}$ . This leads to a classical result from rod theory,

$$\hat{\kappa}_i(S) = \hat{\boldsymbol{\theta}}(S) \cdot \mathbf{d}_i(S), \quad (\text{A.2})$$

see for instance AP10.

Using the first variation of the potential of external load in (3.5), one can write the perturbation to the total potential as

$$\hat{\Psi}^* = \frac{\partial \Phi}{\partial \mathbf{y}}[\mathbf{h}, \mathbf{y}^*[\mathbf{h}]] \cdot \left( \frac{d\mathbf{y}^*}{d\mathbf{h}}[\mathbf{h}] \cdot \hat{\mathbf{h}} \right) + \frac{\partial \Phi}{\partial \mathbf{h}}[\mathbf{h}, \mathbf{y}^*[\mathbf{h}]] \cdot \hat{\mathbf{h}} - \int_0^\ell (\mathbf{p}(S) \cdot \hat{\mathbf{r}}(S) + \mathbf{m}(S) \cdot \hat{\boldsymbol{\theta}}(S)) \, dS.$$

In the first term, the perturbation  $\left( \frac{d\mathbf{y}^*}{d\mathbf{h}}[\mathbf{h}] \cdot \hat{\mathbf{h}} \right)$  to the microscopic solution satisfies the kinematic constraint  $\mathbf{q}(\mathbf{Y}) = \mathbf{0}$ , as can be shown by differentiating  $\mathbf{q}(\mathbf{y}^*[\mathbf{h}]) = \mathbf{0}$  with respect to  $\mathbf{h}$ , recalling that  $\mathbf{q}$  is linear. In view of equation (3.3), we conclude that this first term is zero, so that

$$\hat{\Psi}^* = \frac{\partial \Phi}{\partial \mathbf{h}}[\mathbf{h}, \mathbf{y}^*[\mathbf{h}]] \cdot \hat{\mathbf{h}} - \int_0^\ell (\mathbf{p}(S) \cdot \hat{\mathbf{r}}(S) + \mathbf{m}(S) \cdot \hat{\boldsymbol{\theta}}(S)) \, dS. \quad (\text{A.3})$$

The first term in the right-hand side of equation (A.3) can be rewritten as follows, using equations (2.12), (3.8), (3.7) and (A.2) successively,

$$\begin{aligned} \frac{\partial \Phi}{\partial \mathbf{h}}[\mathbf{h}, \mathbf{y}^*[\mathbf{h}]] &= \frac{\partial}{\partial \mathbf{h}} \left( \int_0^\ell \iint_{\Omega} w(\mathbf{T}, \mathbf{E}(\mathbf{T}; \mathbf{h}(S); \mathbf{y}^*[\mathbf{h}]|_S, \mathbf{y}^*[\mathbf{h}]'|_S)) \, dA \, dS \right) \cdot \hat{\mathbf{h}} \\ &= \int_0^\ell \iint_{\Omega} \Sigma_{ij}(\mathbf{T}, \mathbf{E}(\mathbf{T}; \mathbf{h}(S); \mathbf{y}^*[\mathbf{h}]|_S, \mathbf{y}^*[\mathbf{h}]'|_S)) \frac{\partial E_{ij}}{\partial \mathbf{h}}(\mathbf{T}; \mathbf{h}(S); \mathbf{y}^*[\mathbf{h}]|_S, \mathbf{y}^*[\mathbf{h}]'|_S) \cdot \hat{\mathbf{h}}(S) \, dA \, dS \\ &= \int_0^\ell [N(S) \hat{\varepsilon} + M_i(S) \hat{\kappa}_i] \, dS \\ &= \int_0^\ell [N(S) \hat{\varepsilon} + \mathbf{M}(S) \cdot \hat{\boldsymbol{\theta}}'(S)] \, dS \end{aligned}$$

where  $\mathbf{M}(S) = M_i(S) \mathbf{d}_i(S)$  is the internal moment.

Treating the kinematic constraint (A.1) using a Lagrange multiplier  $\mathbf{R}(S)$ , we obtain by setting  $\hat{\Psi}^* = 0$  in equation (A.3) the principle of virtual work of the ideal one-dimensional model as

$$\begin{aligned} \forall(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}) \quad & \int_0^\ell (N(S) \hat{\varepsilon}(S) + \mathbf{M}(S) \cdot \hat{\boldsymbol{\theta}}'(S)) \, dS - \int_0^\ell (\mathbf{p}(S) \cdot \hat{\mathbf{r}}(S) + \mathbf{m}(S) \cdot \hat{\boldsymbol{\theta}}(S)) \, dS \dots \\ & + \int_0^\ell \mathbf{R}(S) \cdot (\hat{\mathbf{r}}'(S) - \hat{\varepsilon}(S) \mathbf{d}_3(S) - \hat{\boldsymbol{\theta}}(S) \times \mathbf{r}'(S)) \, dS = 0 \end{aligned} \quad (\text{A.4})$$

We have recovered the classical form of the principle of virtual for thin elastic rods. Upon integration by parts, one recovers the Kirchhoff equations (3.6) governing the equilibrium of rods, together with the relevant equilibrium conditions.

### A.2. Equivalence with the equilibrium of the three-dimensional model

In this section, we derive the equilibrium equations of the three-dimensional model in center-line representation, as formulated in section 2. We show that these equilibrium equations are mathematically equivalent to those of the ideal one-dimensional model.

In the three-dimensional model, the microscopic displacement  $\mathbf{y}(S, \mathbf{T})$  and the macroscopic variables  $\mathbf{r}(S)$  and  $\mathbf{d}_i(S)$  are treated as independent unknowns.



The principle of virtual work for the prismatic solid is obtained by setting to zero the first variation of the total potential energy  $\Psi[\mathbf{r}, \mathbf{d}_i, \mathbf{y}]$  in equation (2.14), with respect to both the microscopic and the macroscopic variables. The variation with respect to the microscopic displacement  $\mathbf{y}$ , which is subject to the constraint (2.15) yields

$$(\forall \hat{\mathbf{y}} \text{ such that } \forall S, \mathbf{q}(\hat{\mathbf{y}}|_S) = \mathbf{0}) \quad \frac{\partial \Phi}{\partial \mathbf{y}}[\mathbf{h}, \mathbf{y}] \cdot \mathbf{y} = 0.$$

We have recovered the relaxation problem from equation (3.3), whose solution is

$$\mathbf{y} = \mathbf{y}^*[\mathbf{h}].$$

The variation of the total potential energy (2.14) with respect to the macroscopic variables writes

$$\hat{\Psi}[\mathbf{r}, \mathbf{d}_i, \mathbf{y}] = \frac{\partial \Phi}{\partial \mathbf{h}}[\mathbf{h}, \mathbf{y}] \cdot \hat{\mathbf{h}} - \int_0^\ell (\mathbf{p}(S) \cdot \hat{\mathbf{r}}(S) + \mathbf{m}(S) \cdot \hat{\boldsymbol{\theta}}(S)) \, dS$$

after identifying the external load from equation (3.5). Inserting  $\mathbf{y} = \mathbf{y}^*[\mathbf{h}]$  in this equation, and taking into account the constraints in equation (2.4) and (2.7), we obtain the same variational problem as that of the ideal one-dimensional model, see equation (A.3). By the same reasoning as earlier, one can show that the equilibrium of the three-dimensional model is governed by the Kirchhoff equations for thin rods, and by the same constitutive laws as those identified earlier in equation (3.7).

We conclude that the original three-dimensional model and its ideal reduction have the same equations of equilibrium.

### A.3. Microscopic interpretation of the one-dimensional stress

In this section, we show that the one-dimensional stress quantities  $N(S)$  and  $\mathbf{M}(S) = M_i(S) \mathbf{d}_i(S)$  that appear in equation (3.7) can be interpreted microscopically as the normal force (*i.e.*, the tangent component of the internal force  $\mathbf{R}(S)$ ) and the internal moment, respectively.

The integrand in the right hand side of (3.7) can be rewritten as  $\Sigma_{ij} \hat{E}_{ij}$ , where  $\hat{E}_{ij} = \frac{\partial E_{ij}}{\partial \mathbf{h}}(\mathbf{T}; \mathbf{h}(S); \mathbf{y}|_S, \mathbf{y}'|_S) \cdot \hat{\mathbf{h}}(S)$  is the perturbation to the microscopic strain due to a change  $\hat{\mathbf{h}}$  in the macroscopic strain  $\mathbf{h}$ . By taking the first variation of the general strain in equation (2.11) with respect to  $\mathbf{h}$ , we obtain

$$\Sigma_{ij} \hat{E}_{ij} = (t_k \Sigma_{33} + \partial_\alpha Y_k(\mathbf{T}) \Sigma_{\alpha 3}) \hat{t}_k$$

where  $\hat{t}_k = \hat{\varepsilon} \delta_{i3} + \eta_{ijk} \hat{\kappa}_j y_k(\mathbf{T})$ . In the equation above, we can identify the components of the transformation gradient from equation (2.9) as  $t_k = F_{k3}$  and  $\partial_\alpha Y_k(\mathbf{T}) = F_{k\alpha}$ , so that

$$\begin{aligned} \Sigma_{ij} \hat{E}_{ij} &= (F_{k3} \Sigma_{33} + F_{k\alpha} \Sigma_{\alpha 3}) \hat{t}_k \\ &= (\mathbf{F} \cdot \boldsymbol{\Sigma} \cdot \mathbf{e}_3)_k (\hat{\varepsilon} \delta_{i3} + \eta_{kij} \hat{\kappa}_i y_j(\mathbf{T})) \end{aligned}$$

According to the standard interpretation of the Piola-Kirchhoff stress  $\boldsymbol{\Sigma}$ , the quantity  $\mathbf{F} \cdot \boldsymbol{\Sigma} \cdot \mathbf{e}_3$  is the microscopic internal force  $d\boldsymbol{\sigma}$  per unit area  $dA$  transmitted across a cross-section (the unit normal to the cross-section is  $\mathbf{e}_3$  in reference configuration). Denoting as  $d\sigma_k$  the component  $d\sigma_k = d\boldsymbol{\sigma} \cdot \mathbf{d}_k$  of the microscopic internal force on the orthonormal basis of directors, we introduce the density of internal force across a cross-section as

$$\frac{d\sigma_k}{dA} = \mathbf{d}_k \cdot \mathbf{F} \cdot \boldsymbol{\Sigma} \cdot \mathbf{e}_3 = t_k \Sigma_{33} + \partial_\alpha Y_k(\mathbf{T}) \Sigma_{\alpha 3}.$$

Inserting this into the expression of the integrand  $\Sigma_{ij} \hat{E}_{ij}$ , we can rewrite equation (3.7) as

$$N(S) \hat{\varepsilon} + M_i(S) \hat{\kappa}_i \equiv \iint_\Omega (\hat{\varepsilon} \delta_{i3} + \eta_{kij} \hat{\kappa}_i y_j(\mathbf{T})) \frac{d\sigma_k}{dA} \, dA.$$

Identifying the coefficients of  $\hat{\varepsilon}$  and  $\hat{\kappa}_i$  on both sides, we obtain

$$\begin{aligned} N &= \mathbf{d}_3 \cdot \iint_\Omega d\boldsymbol{\sigma} \\ \mathbf{M} &= \iint_\Omega (y_j(\mathbf{T}) \mathbf{d}_j(S)) \times d\boldsymbol{\sigma} = \iint_\Omega (\mathbf{x} - \mathbf{r}(S)) \times d\boldsymbol{\sigma}, \end{aligned} \tag{A.5}$$

which is the classical microscopic interpretation of the internal stress  $N$  and  $\mathbf{M}$  from the one-dimensional model: in these equations  $\iint_\Omega d\boldsymbol{\sigma} = \mathbf{R}$  is the internal force resultant,  $\mathbf{r}(S)$  is the point of the center-line that is formally associated with the cross-section with coordinate  $S$ , and  $\mathbf{x}$  is a current point in the deformed cross-section.

## APPENDIX B. A COMPENDIUM OF THE REDUCTION METHOD

This section is a self-contained summary of the reduction method from our previous paper, with (i) small changes of notation, (ii) a new method for removing boundary terms, see the end of §B.2.3, (iii) some additional simplifications arising when the equations are specialized to the case relevant to three-dimensional elasticity where the strain  $\mathbf{E}$  depends on the displacement  $\mathbf{Y}$ , on the longitudinal gradient of displacement  $\mathbf{Y}^\dagger$  and on the macroscopic strain  $\mathbf{h}$  only but not on higher-order derivatives. The reader is referred to the original paper (LA20b) for a detailed justification of the method.

The reduction procedure uses the definition of the full model in equations (2.12–2.15) as a starting point.

### B.1. Analysis of homogeneous solutions

We first focus on homogeneous solutions, such that  $\mathbf{y}' \equiv \mathbf{0}$ . The strain is then  $\tilde{\mathbf{E}}(\mathbf{T}, \mathbf{h}, \mathbf{Y}) = \mathbf{E}(\mathbf{T}; \mathbf{h}; \mathbf{Y}, \mathbf{0})$ , where  $\mathbf{Y} = \mathbf{y}|_S$  denotes the microscopic displacement restricted to a particular cross-section, which is independent of  $S$  in the homogeneous case.

As explained earlier in equation (3.3), we identify the relaxed displacement of the homogeneous solutions by making stationary the strain energy per unit length  $\iint_{\Omega} w(\mathbf{T}, \tilde{\mathbf{E}}(\mathbf{T}, \mathbf{h}, \mathbf{Y})) dA$  with respect to  $\mathbf{Y}$  for prescribed  $\mathbf{h}$ , taking the kinematic constraint (2.15) into account. We denote as  $\mathbf{Y}^h(\mathbf{T})$  this solution. This variational problem can be formulated using Lagrange multipliers as follows. Given  $\mathbf{h}$ , we seek a cross-sectional function  $\mathbf{Y}^h$  and Lagrange multipliers  $\mathbf{M}^h = (F_1^h, F_2^h, F_3^h, Q^h)$  dual to the constraints  $\mathbf{q}$ , such that

$$\begin{aligned} \mathbf{q}(\mathbf{Y}^h) &= \mathbf{0} \\ \forall \hat{\mathbf{Y}} \quad \iint_{\Omega} \Sigma(\mathbf{T}, \mathbf{E}^h(\mathbf{T})) : \hat{\mathbf{E}}^h(\mathbf{T}) dA + \mathbf{M}^h \cdot \mathbf{q}(\hat{\mathbf{Y}}^h) &= 0. \end{aligned} \quad (\text{B.1})$$

Here,  $\Sigma(\mathbf{T}, \mathbf{E}) = \frac{\partial w}{\partial \mathbf{E}}(\mathbf{T}, \mathbf{E})$  denotes the microscopic stress,  $\mathbf{E}^h(\mathbf{T}) = \tilde{\mathbf{E}}(\mathbf{T}; \mathbf{h}, \mathbf{Y}^h)$  is the distribution of microscopic strain in the homogeneous solution having macroscopic strain  $\mathbf{h}$ , and  $\hat{\mathbf{E}}^h(\mathbf{T}) = \frac{\partial \tilde{\mathbf{E}}}{\partial \mathbf{Y}}(\mathbf{T}, \mathbf{h}, \mathbf{Y}^h) \cdot \hat{\mathbf{Y}}$  is the virtual increment of microscopic strain.

Having solved for the microscopic displacement, one can define the strain energy per unit length  $W_{\text{hom}}(\mathbf{h})$  from equation (4.6), as well as the homogeneous strain  $\mathbf{E}^h$ , stress  $\Sigma^h$  and tangent elastic moduli  $\mathbf{K}^h$  from equation (4.7).

### B.2. Gradient effect

The microscopic displacement solving the relaxation problem (3.1) is sought in the form

$$\mathbf{y}^*[h](S, \mathbf{T}) = \mathbf{Y}^{h(S)}(\mathbf{T}) + \mathbf{z}(S, \mathbf{T}), \quad (\text{B.2})$$

i.e., as the solution  $\mathbf{Y}^{h(S)}$  predicted by the catalog of homogeneous solutions based on the local value  $\mathbf{h}(S)$  of the macroscopic strain, plus a small correction  $\mathbf{z}(S, \mathbf{T})$  to be calculated by solving the equilibrium equations.

#### B.2.1. Structure operators

The restrictions of the displacement and its gradient in equation (B.2) can be calculated as  $\mathbf{y}^*[h]|_S = \mathbf{Y}^h + \mathbf{Z}$  and  $\mathbf{y}^*[h]'|_S = \mathbf{h}^\dagger \cdot \nabla \mathbf{Y}^h + \mathbf{Z}^\dagger$ , respectively, where

$$\mathbf{h}^\dagger = \mathbf{h}'(S) \quad \mathbf{Z} = \mathbf{z}|_S \quad \mathbf{Z}^\dagger = \mathbf{z}'|_S, \quad (\text{B.3})$$

and we use the  $\nabla$  notation for gradients with respect to  $\mathbf{h}$ , i.e.,  $\mathbf{h}^\dagger \cdot \nabla \mathbf{Y}^h = \frac{d(\mathbf{Y}^h)}{d\mathbf{h}} \cdot \mathbf{h}^\dagger$  as earlier in equation (4.10).

We define the structure operators  $e_{jk}^i(\mathbf{T}, \mathbf{h})$  as the coefficients entering in the expansion of the strain  $\mathbf{E}(\mathbf{T}; \mathbf{h}; \mathbf{Y}^h + \mathbf{Z}, \mathbf{h}^\dagger \cdot \nabla \mathbf{Y}^h + \mathbf{Z}^\dagger)$  associated with the solution (B.2), in powers of  $\mathbf{h}^\dagger$ ,  $\mathbf{Z}$  and  $\mathbf{Z}^\dagger$

$$\begin{aligned} \mathbf{E}(\mathbf{T}; \mathbf{h}; \mathbf{Y}^h + \mathbf{Z}, \mathbf{h}^\dagger \cdot \nabla \mathbf{Y}^h + \mathbf{Z}^\dagger) &= \mathbf{E}^h(\mathbf{T}) + e_{00}^1(\mathbf{T}, \mathbf{h}) \cdot \mathbf{h}^\dagger + e_{10}^0(\mathbf{T}, \mathbf{h}) \cdot \mathbf{Z} + e_{01}^0(\mathbf{T}, \mathbf{h}) \cdot \mathbf{Z}^\dagger \dots \\ &+ \frac{1}{2} \left( \mathbf{h}^\dagger \cdot e_{00}^2(\mathbf{T}, \mathbf{h}) \cdot \mathbf{h}^\dagger + 2 \mathbf{h}^\dagger \cdot e_{10}^1(\mathbf{T}, \mathbf{h}) \cdot \mathbf{Z} + \mathbf{Z} \cdot e_{20}^0(\mathbf{T}, \mathbf{h}) \cdot \mathbf{Z} + \dots \right) \end{aligned} \quad (\text{B.4})$$

The structure operators  $e_{jk}^i(\mathbf{T}, \mathbf{h})$  are calculated explicitly in appendix C.

Note that the arguments  $\mathbf{Z}$  and  $\mathbf{Z}^\dagger$  are unknown so far: both of them is a triple of functions  $Z_i$  and  $Z_i^\dagger$  defined on the cross-section.

#### B.2.2. Optimal displacement

In terms of the structure operators  $e_{ij}^k(\mathbf{T}, \mathbf{h})$  just identified, we define the operator

$$\mathcal{E}^h(\mathbf{T}, \mathbf{h}^\dagger, \mathbf{Z}) = e_{00}^1(\mathbf{T}, \mathbf{h}) \cdot \mathbf{h}^\dagger + e_{10}^0(\mathbf{T}, \mathbf{h}) \cdot \mathbf{Z}, \quad (\text{B.5})$$

which corresponds to the difference between the actual microscopic strain associated with the displacement (B.2) and the crude prediction  $\mathbf{E}^{h(S)}(\mathbf{T})$  obtained by looking up the catalog of homogeneous solutions with the local value of the macroscopic strain  $\mathbf{h}(S)$ . We also define the operators

$$\begin{aligned} \mathbf{C}_h^{(1)} \cdot \mathbf{Z}^\dagger &= \iint_{\Omega} \Sigma^h(\mathbf{T}) : (e_{01}^0(\mathbf{T}, \mathbf{h}) \cdot \mathbf{Z}^\dagger) dA \\ \mathcal{B}^h(\mathbf{h}^\dagger, \mathbf{Z}) &= \iint_{\Omega} \frac{1}{2} \mathcal{E}^h(\mathbf{T}, \mathbf{h}^\dagger, \mathbf{Z}) : \mathbf{K}^h(\mathbf{T}) : \mathcal{E}^h(\mathbf{T}, \mathbf{h}^\dagger, \mathbf{Z}) dA \dots \\ &+ \iint_{\Omega} \frac{1}{2} \Sigma^h(\mathbf{T}) : (\mathbf{h}^\dagger \cdot e_{00}^2(\mathbf{T}, \mathbf{h}) \cdot \mathbf{h}^\dagger + 2 \mathbf{h}^\dagger \cdot e_{10}^1(\mathbf{T}, \mathbf{h}) \cdot \mathbf{Z} + \mathbf{Z} \cdot e_{20}^0(\mathbf{T}, \mathbf{h}) \cdot \mathbf{Z}) dA \dots \\ &- \mathbf{h}^\dagger \cdot \nabla \mathbf{C}_h^{(1)} \cdot \mathbf{Z} \end{aligned} \quad (\text{B.6})$$

Note that the operator  $\mathbf{C}_h^{(0)}$  defined in (LA20b) is zero here, as the strain  $\mathbf{E}$  does not depend on the second gradient of the macroscopic strain  $\mathbf{h}''$  in the theory of elasticity of bulk materials.

In equation (B.6), the notation  $-\mathbf{h}^\dagger \cdot \nabla \mathbf{C}_h^{(1)} \cdot \mathbf{Z}$  in the right-hand side of  $\mathcal{B}^h(\mathbf{h}^\dagger, \mathbf{Z})$  refers to the quantity obtained by integrating by parts  $\mathbf{C}_{h(S)}^{(1)} \cdot \mathbf{z}'(S)$  with respect to the variable  $S$ ,

$$-\mathbf{h}^\dagger \cdot \nabla \mathbf{C}_h^{(1)} \cdot \mathbf{Z} = -\left( \frac{d\mathbf{C}_h^{(1)}}{d\mathbf{h}} \cdot \mathbf{h}^\dagger \right) \cdot \mathbf{Z}. \quad (\text{B.7})$$

As shown in our previous work, the optimal displacement  $\mathbf{Z}_{\text{opt}}^h(\mathbf{h}^\dagger)$  is a stationary point of  $\mathcal{B}^h(\mathbf{h}^\dagger, \mathbf{Z})$  over all  $\mathbf{Z}$ 's satisfying the kinematic constraint,  $\mathbf{q}(\mathbf{Z}) = \mathbf{0}$ , for fixed  $\mathbf{h}^\dagger$ . It is therefore the solution of the variational problem

$$(\mathbf{Z}, \mathbf{f}) = (\mathbf{Z}_{\text{opt}}^h(\mathbf{h}^\dagger), \mathbf{f}_{\text{opt}}^h(\mathbf{h}^\dagger)) \text{ is the solution of } \begin{cases} \mathbf{q}(\mathbf{Z}) = \mathbf{0} \\ \forall \hat{\mathbf{Z}} \quad \frac{\partial \mathcal{B}^h}{\partial \mathbf{Z}}(\mathbf{h}^\dagger, \mathbf{Z}) \cdot \hat{\mathbf{Z}} - \mathbf{f} \cdot \mathbf{q}(\hat{\mathbf{Z}}) = 0 \end{cases} \quad (\text{B.8})$$

As the operator  $\mathcal{B}^h(\mathbf{h}^\dagger, \mathbf{Z})$  is quadratic with respect to both  $\mathbf{h}^\dagger$  and  $\mathbf{Z}$ , this is a linear problem of elasticity in the cross-section with residual strain proportional to  $\mathbf{h}^\dagger$ .

### B.2.3. Equivalent one-dimensional model

The microscopic solution in equation (B.2) is now available as

$$\mathbf{y}^*[\mathbf{h}](S, \mathbf{T}) = \mathbf{Y}^h(S)(\mathbf{T}) + \mathbf{Z}_{\text{opt}}^h(S)(\mathbf{h}'(S), \mathbf{T}) + \dots$$

where  $\mathbf{Y}^h$  is the catalog of homogeneous solutions,  $\mathbf{Z}_{\text{opt}}^h(\mathbf{h}^\dagger)$  is the corrective displacement just found, and the ellipsis stands for higher order terms that do not enter in the approximation  $\Phi_{(2)}^*[\mathbf{h}]$ .

The one-dimensional strain energy  $\Phi_{(2)}^*[\mathbf{h}]$  can then be obtained by inserting the expansion for  $\mathbf{y}^*[\mathbf{h}](S, \mathbf{T})$  above into  $\Phi^*[\mathbf{h}] = \Phi[\mathbf{h}, \mathbf{y}^*[\mathbf{h}]]$ . As shown in previous work, the result of this calculation is

$$\Phi_{(2)}^*[\mathbf{h}] = \int_0^\ell W_{\text{hom}}(\mathbf{h}(S)) \, dS + \int_0^\ell \mathbf{A}(\mathbf{h}(S)) \cdot \mathbf{h}'(S) \, dS + \left[ \mathbf{C}(\mathbf{h}(S)) \cdot \mathbf{h}'(S) \right]_0^\ell + \frac{1}{2} \int_0^\ell \mathbf{h}'(S) \cdot \mathbf{B}(\mathbf{h}(S)) \cdot \mathbf{h}'(S) \, dS,$$

where  $\mathbf{A}(\mathbf{h})$ ,  $\mathbf{B}(\mathbf{h})$  and  $\mathbf{C}(\mathbf{h})$  are the one-dimensional operators

$$\begin{aligned} \mathbf{A}(\mathbf{h}) \cdot \mathbf{h}^\dagger &= \iint_{\Omega} \boldsymbol{\Sigma}^h(\mathbf{T}) : \mathcal{E}^h(\mathbf{T}, \mathbf{h}^\dagger, \mathbf{0}) \, dA, \\ \frac{1}{2} \mathbf{h}^\dagger \cdot \mathbf{B}(\mathbf{h}) \cdot \mathbf{h}^\dagger &= \mathcal{B}^h(\mathbf{h}^\dagger, \mathbf{Z}_{\text{opt}}^h(\mathbf{T}, \mathbf{h}^\dagger)), \\ \mathbf{C}(\mathbf{h}) \cdot \mathbf{h}^\dagger &= \mathbf{C}_h^{(1)} \cdot \mathbf{Z}_{\text{opt}}^h(\mathbf{T}, \mathbf{h}^\dagger). \end{aligned} \quad (\text{B.9})$$

We complement these known results with an original method that eliminates the boundary terms in the expression of  $\Phi_{(2)}^*[\mathbf{h}]$  above, as follows. We rewrite

$$\left[ \mathbf{C}(\mathbf{h}(S)) \cdot \mathbf{h}'(S) \right]_0^\ell = \int_0^\ell \left[ \mathbf{h}'(S) \cdot \frac{d\mathbf{C}}{d\mathbf{h}}(\mathbf{h}(S)) \cdot \mathbf{h}'(S) + \mathbf{C}(\mathbf{h}(S)) \cdot \mathbf{h}''(S) \right] dS$$

so that

$$\Phi_{(2)}^*[\mathbf{h}] = \int_0^\ell \left[ \left\{ W_{\text{hom}}(\mathbf{h}(S)) + \mathbf{C}(\mathbf{h}(S)) \cdot \mathbf{h}''(S) \right\} + \mathbf{A}(\mathbf{h}(S)) \cdot \mathbf{h}'(S) + \frac{1}{2} \mathbf{h}'(S) \cdot \mathbf{D}(\mathbf{h}(S)) \cdot \mathbf{h}'(S) \right] dS \quad (\text{B.10})$$

where  $\mathbf{D}(\mathbf{h}) = \mathbf{B}(\mathbf{h}) + 2 \frac{d\mathbf{C}}{d\mathbf{h}}(\mathbf{h})$  as announced in equation (4.16). The terms in curly braces can be rewritten as

$$\begin{aligned} W_{\text{hom}}(\mathbf{h}) + \mathbf{C}(\mathbf{h}) \cdot \mathbf{h}'' &= W_{\text{hom}}(h_0, h_1, \dots, h_{n-1}) + \sum_{i=0}^{n-1} C_i(\mathbf{h}) h_i''(S) \\ &= W_{\text{hom}} \left( h_0 + \frac{C_0(\mathbf{h})}{\frac{dW_{\text{hom}}}{dh_0}(\mathbf{h})} h_0'', \dots, h_{n-1} + \frac{C_{n-1}(\mathbf{h})}{\frac{dW_{\text{hom}}}{dh_{n-1}}(\mathbf{h})} h_{n-1}'' \right) + \mathcal{O}(\zeta^4), \end{aligned} \quad (\text{B.11})$$

where  $n=4$  is the number of macroscopic strain measures in the vector  $\mathbf{h}$ . The term  $\mathcal{O}(\zeta^4)$  is beyond the order  $\zeta^2$  of approximation which  $\Phi_{(2)}^*[\mathbf{h}]$  can resolve, and it will be ignored. Inserting the above result into the expression (B.10) of  $\Phi_{(2)}^*[\mathbf{h}]$ , we obtain the expression of  $\Phi_{(2)}^*[\mathbf{h}]$  announced in equation (4.2). In addition, the modified strain is identified as  $\tilde{h}_i(S) = h_i(S) + \xi_i(\mathbf{h}(S)) h_i''(S)$  (no implicit sum on  $i$ ) where

$$\xi_i(\mathbf{h}) = \frac{C_i(\mathbf{h})}{\frac{dW_{\text{hom}}}{dh_i}(\mathbf{h})}, \quad (\text{B.12})$$

as announced in equation (4.17).

## APPENDIX C. CALCULATION OF THE STRUCTURE OPERATORS

We derive the structure operators  $\mathbf{e}_{jk}^i(\mathbf{T}, \mathbf{h})$  by identifying the two sides of equation (B.4). To do so, we expand  $\mathbf{E}(\mathbf{T}; \mathbf{h}; \mathbf{Y}^h + \mathbf{Z}, \mathbf{h}^\dagger \cdot \nabla \mathbf{Y}^h + \mathbf{Z}^\dagger)$  in powers of  $\mathbf{h}^\dagger$ ,  $\mathbf{Z}$  and  $\mathbf{Z}^\dagger$ . We start by setting  $\mathbf{Y} = \mathbf{Y}^h + \mathbf{Z}$  and  $\mathbf{Y}^\dagger = \mathbf{h}^\dagger \cdot \nabla \mathbf{Y}^h + \mathbf{Z}^\dagger$  in the auxiliary quantity  $t_i$  in (2.11),

$$\begin{aligned} t_i &= (1 + \varepsilon) \delta_{i3} + \eta_{ijk} \kappa_j Y_k^h(\mathbf{T}) + \mathbf{h}^\dagger \cdot \nabla Y_i^h(\mathbf{T}) + \eta_{ijk} \kappa_j Z_k(\mathbf{T}) + Z_i^\dagger(\mathbf{T}) \\ &= F_{i3}^h(\mathbf{T}) + \mathbf{h}^\dagger \cdot \nabla Y_i^h(\mathbf{T}) + \eta_{ijk} \kappa_j Z_k(\mathbf{T}) + Z_i^\dagger(\mathbf{T}) \end{aligned}$$

where  $F_{i3}^h(\mathbf{T})$  has been defined earlier in (4.7).

Inserting this into the expression of  $\mathbf{E}(\mathbf{T}; \mathbf{h}; \mathbf{Y}, \mathbf{Y}^\dagger)$  in (2.11), we obtain

$$\begin{aligned} \mathbf{E}(\mathbf{T}; \mathbf{h}; \mathbf{Y}^h + \mathbf{Z}, \mathbf{h}^\dagger \cdot \nabla \mathbf{Y}^h + \mathbf{Z}^\dagger) = & \\ & \frac{1}{2} (-1 + [F_{i3}^h(\mathbf{T}) + \mathbf{h}^\dagger \cdot \nabla Y_i^h(\mathbf{T}) + \eta_{ijk} \kappa_j Z_k(\mathbf{T}) + Z_i^\dagger(\mathbf{T})]^2) \mathbf{e}_3 \otimes \mathbf{e}_3 \dots \\ & + [F_{i3}^h(\mathbf{T}) + \mathbf{h}^\dagger \cdot \nabla Y_i^h(\mathbf{T}) + \eta_{ijk} \kappa_j Z_k(\mathbf{T}) + Z_i^\dagger(\mathbf{T})] [F_{i\alpha}^h(\mathbf{T}) + \partial_\alpha Z_i(\mathbf{T})] \mathbf{e}_\alpha \odot \mathbf{e}_3 \dots \\ & + \frac{[F_{i\alpha}^h(\mathbf{T}) + \partial_\alpha Z_i(\mathbf{T})] [F_{i\beta}^h(\mathbf{T}) + \partial_\beta Z_i(\mathbf{T})] - \delta_{\alpha\beta}}{2} \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \end{aligned}$$

Expanding and identifying with (B.4), we obtain the explicit expression of the operators as

$$\begin{aligned} \mathbf{e}_{00}^1(\mathbf{T}, \mathbf{h}) \cdot \mathbf{h}^\dagger &= (\mathbf{h}^\dagger \cdot \nabla Y_i^h(\mathbf{T})) F_{ij}^h(\mathbf{T}) \mathbf{e}_j \odot \mathbf{e}_3 \\ \mathbf{e}_{10}^0(\mathbf{T}, \mathbf{h}) \cdot \mathbf{Z} &= \eta_{ijk} \kappa_k F_{jl}^h(\mathbf{T}) Z_i(\mathbf{T}) \mathbf{e}_l \odot \mathbf{e}_3 + F_{ij}^h(\mathbf{T}) \partial_\alpha Z_i(\mathbf{T}) \mathbf{e}_j \odot \mathbf{e}_\alpha \\ \mathbf{e}_{01}^0(\mathbf{T}, \mathbf{h}) \cdot \mathbf{Z}^\dagger &= F_{ij}^h(\mathbf{T}) Z_i^\dagger(\mathbf{T}) \mathbf{e}_j \odot \mathbf{e}_3 \\ \frac{1}{2} \mathbf{h}^\dagger \cdot \mathbf{e}_{00}^2(\mathbf{T}, \mathbf{h}) \cdot \mathbf{h}^\dagger &= \frac{1}{2} (\mathbf{h}^\dagger \cdot \nabla Y_i^h(\mathbf{T}))^2 \mathbf{e}_3 \otimes \mathbf{e}_3 \\ \mathbf{h}^\dagger \cdot \mathbf{e}_{10}^1(\mathbf{T}, \mathbf{h}) \cdot \mathbf{Z} &= \mathbf{h}^\dagger \cdot \nabla Y_i^h(\mathbf{T}) (\eta_{ijk} \kappa_j Z_k(\mathbf{T}) \mathbf{e}_3 + \partial_\beta Z_i(\mathbf{T}) \mathbf{e}_\beta) \odot \mathbf{e}_3 \\ \frac{1}{2} \mathbf{Z} \cdot \mathbf{e}_{20}^0(\mathbf{T}, \mathbf{h}) \cdot \mathbf{Z} &= \frac{1}{2} \left( (\delta_{ij} \kappa_i \kappa_j - \kappa_i \kappa_j) Z_i(\mathbf{T}) Z_j(\mathbf{T}) \mathbf{e}_3 \otimes \mathbf{e}_3 \dots \right. \\ &\quad \left. + 2 \eta_{ijk} \kappa_j Z_k(\mathbf{T}) \partial_\alpha Z_i(\mathbf{T}) \mathbf{e}_\alpha \odot \mathbf{e}_3 + \partial_\alpha Z_i(\mathbf{T}) \partial_\beta Z_i(\mathbf{T}) \mathbf{e}_\alpha \odot \mathbf{e}_\beta \right) \end{aligned} \quad (\text{C.1})$$

## APPENDIX D. GRADIENT EFFECT FOR A TWISTED BAR

### D.1. Derivation of the higher-model for a twisted bar

In this section, we apply the general method from section 4.3 to derive the higher-order model  $\Phi_{(2)}^*[\tau]$  for a twisted bar.

In view of the strain definition in equation (5.3) and of homogeneous solution in equation (5.6), the strain reads

$$\mathbf{E}(\mathbf{T}; \tau; \mathbf{V}^{(\tau)} + \mathbf{Z}, \tau^\dagger \nabla \mathbf{V}^{(\tau)} + \mathbf{Z}^\dagger) = \mathbf{E}^{(\tau)}(\mathbf{T}) + \partial_\alpha Z_i(\mathbf{T}) \mathbf{e}_\alpha \odot \mathbf{e}_i + (\tau^\dagger \nabla V_i^{(\tau)}(\mathbf{T}) + Z_i^\dagger(\mathbf{T})) \mathbf{e}_i \odot \mathbf{e}_3 \quad (\text{D.1})$$

where  $\nabla V_i^{(\tau)}(\mathbf{T}) = \frac{dV_i^{(\tau)}(\mathbf{T})}{d\tau}$  is the gradient of the homogeneous solution with respect to the homogeneous strain  $\tau$ ,

$$\nabla V_\alpha^{(\tau)}(\mathbf{T}) = 0, \quad \nabla V_3^{(\tau)}(\mathbf{T}) = \omega(\mathbf{T}).$$

By identifying equation (D.1) with the generic expansion (B.4), we obtain the structure operators as

$$\begin{aligned} \mathbf{e}_{00}^1(\mathbf{T}, \tau) \cdot (\tau^\dagger) &= \tau^\dagger \omega(\mathbf{T}) \mathbf{e}_3 \otimes \mathbf{e}_3 \\ \mathbf{e}_{10}^0(\mathbf{T}, \tau) \cdot \mathbf{Z} &= \partial_\alpha Z_i(\mathbf{T}) \mathbf{e}_\alpha \odot \mathbf{e}_i \\ \mathbf{e}_{01}^0(\mathbf{T}, \tau) \cdot \mathbf{Z}^\dagger &= Z_i^\dagger(\mathbf{T}) \mathbf{e}_i \odot \mathbf{e}_3 \\ \frac{1}{2} (\tau^\dagger) \cdot \mathbf{e}_{00}^2(\mathbf{T}, \tau) \cdot (\tau^\dagger) &= 0 \\ (\tau^\dagger) \cdot \mathbf{e}_{10}^1(\mathbf{T}, \tau) \cdot \mathbf{Z} &= 0 \\ \frac{1}{2} \mathbf{Z} \cdot \mathbf{e}_{20}^0(\mathbf{T}, \tau) \cdot \mathbf{Z} &= 0 \end{aligned} \quad (\text{D.2})$$

This is a special form of the non-linear expressions derived in appendix C, relevant to linear elasticity.

The operators defined in equations (4.9) and (4.11) read, respectively,

$$\begin{aligned} \mathcal{E}^{(\tau)}(\mathbf{T}, \tau^\dagger, \mathbf{Z}) &= \mathbf{e}_{00}^1(\mathbf{T}, \tau) \cdot (\tau^\dagger) + \mathbf{e}_{10}^0(\mathbf{T}, \tau) \cdot \mathbf{Z} \\ &= \tau^\dagger \omega(\mathbf{T}) \mathbf{e}_3 \otimes \mathbf{e}_3 + \partial_\alpha Z_i(\mathbf{T}) \mathbf{e}_\alpha \odot \mathbf{e}_i \end{aligned}$$

and

$$\begin{aligned} \mathbf{C}_{(\tau)}^{(1)} \cdot \mathbf{Z}^\dagger &= \iint_{\Omega} \boldsymbol{\Sigma}^{(\tau)}(\mathbf{T}) : (\mathbf{e}_{01}^0(\mathbf{T}, \tau) \cdot \mathbf{Z}^\dagger) dA \\ &= \iint_{\Omega} 2 \mu \tau (-\eta_{\alpha\beta} T_\beta + \partial_\alpha \omega(\mathbf{T})) (\mathbf{e}_\alpha \otimes \mathbf{e}_3) : (Z_\beta^\dagger(\mathbf{T}) \mathbf{e}_\beta \odot \mathbf{e}_3) dA \\ &= \mu \tau \iint_{\Omega} (-\eta_{\alpha\beta} T_\beta + \partial_\alpha \omega(\mathbf{T})) Z_\alpha^\dagger(\mathbf{T}) dA. \end{aligned}$$

The operator  $-\tau^\dagger \nabla \mathbf{C}_h^{(1)} \cdot \mathbf{Z}$  is obtained by an integration by parts, see equation (4.12),

$$-\tau^\dagger \nabla \mathbf{C}_h^{(1)} \cdot \mathbf{Z} = -\mu \tau^\dagger \iint_{\Omega} (-\eta_{\alpha\beta} T_\beta + \partial_\alpha \omega) Z_\alpha dA.$$

We proceed to calculate the operator  $\mathcal{B}^{(\tau)}(\tau^\dagger, \mathbf{Z})$  from equation (B.6). Dropping the operators  $\mathbf{e}_{jk}^i$  with  $i + j + k \geq 2$  which are zero in the linear setting, see equation (D.2), we have

$$\mathcal{B}^{(\tau)}(\tau^\dagger, \mathbf{Z}) = \iint_{\Omega} \frac{1}{2} \mathcal{E}^{(\tau)}(\mathbf{T}, \tau^\dagger, \mathbf{Z}) : \mathbf{K}^{(\tau)}(\mathbf{T}) : \mathcal{E}^{(\tau)}(\mathbf{T}, \tau^\dagger, \mathbf{Z}) dA - \tau^\dagger \nabla \mathbf{C}_h^{(1)} \cdot \mathbf{Z}.$$

In linear elasticity,  $\frac{1}{2} \mathcal{E} : \mathbf{K}^{(\tau)}(\mathbf{T}) : \mathcal{E} = w(\mathbf{T}, \mathcal{E})$  is the (quadratic) strain energy density, as defined in equation (5.1). This yields

$$\begin{aligned} \mathcal{B}^{(\tau)}(\tau^\dagger, \mathbf{Z}) &= \frac{1}{2} \iint_{\Omega} (\lambda \operatorname{tr}^2 \mathcal{E}^{(\tau)}(\mathbf{T}, \tau^\dagger, \mathbf{Z}) + 2 \mu \mathcal{E}^{(\tau)}(\mathbf{T}, \tau^\dagger, \mathbf{Z}) : \mathcal{E}^{(\tau)}(\mathbf{T}, \tau^\dagger, \mathbf{Z})) \, dA \dots \\ &\quad - \mu \tau^\dagger \iint_{\Omega} (-\eta_{\alpha\beta} T_\beta + \partial_\alpha \omega(\mathbf{T})) Z_\alpha(\mathbf{T}) \, dA \\ &= \frac{1}{2} \iint_{\Omega} \left( \lambda (\tau^\dagger \omega + \partial_\alpha Z_\alpha)^2 + 2 \mu \left( \tau^{\dagger 2} \omega^2 + \sum_{\alpha} \frac{(\partial_\alpha Z_3)^2}{2} + \sum_{\alpha\beta} \left( \frac{\partial_\alpha Z_\beta + \partial_\beta Z_\alpha}{2} \right)^2 \right) \right) dA \dots \\ &\quad - \mu \tau^\dagger \iint_{\Omega} (-\eta_{\alpha\beta} T_\beta + \partial_\alpha \omega) Z_\alpha \, dA \\ &= \frac{1}{2} \iint_{\Omega} \left( (2\mu + \lambda) \tau^{\dagger 2} \omega^2 + 2 \tau^\dagger (\lambda \omega \partial_\alpha Z_\alpha(\mathbf{T}) - \mu \partial_\alpha \omega Z_\alpha + \mu \eta_{\alpha\beta} T_\beta Z_\alpha) \dots \right. \\ &\quad \left. + \left\{ \lambda \left( \sum_{\alpha} \partial_\alpha Z_\alpha \right)^2 + \mu \sum_{\alpha} (\partial_\alpha Z_3)^2 + 2 \mu \sum_{\alpha\beta} \left( \frac{\partial_\alpha Z_\beta + \partial_\beta Z_\alpha}{2} \right)^2 \right\} \right) dA \end{aligned}$$

The next step is to find the stationary point  $\mathbf{Z}$  of  $\mathcal{B}^{(\tau)}(\tau^\dagger, \mathbf{Z})$  subject to the constraint  $\mathbf{q}(\mathbf{Z}) = \mathbf{0}$ , see equation (4.14). We can drop the term  $\mu \eta_{\alpha\beta} T_\beta Z_\alpha(\mathbf{T})$  which integrates to zero thanks to the second constraint in (4.14). In addition, we observe that the longitudinal correction  $Z_3$  appears only in the term  $\mu \sum_{\alpha} (\partial_\alpha Z_3)^2$ : the stationarity condition with respect to  $Z_3$  is that  $Z_3(\mathbf{T})$  is constant, this constant being zero by the first constraint in (4.14),

$$Z_3^{\text{opt}}(\mathbf{T}) = 0. \quad (\text{D.3})$$

We are left with

$$\begin{aligned} \mathcal{B}^{(\tau)}(\tau^\dagger, Z_\alpha) &= \frac{2\mu + \lambda}{2} J_\omega \tau^{\dagger 2} \dots \\ &+ \frac{1}{2} \iint_{\Omega} \left( \left\{ \lambda \left( \sum_{\alpha} \partial_\alpha Z_\alpha(\mathbf{T}) \right)^2 + 2 \mu \sum_{\alpha\beta} \left( \frac{\partial_\alpha Z_\beta + \partial_\beta Z_\alpha}{2} \right)^2 \right\} + 2 \tau^\dagger (\lambda \omega(\mathbf{T}) \partial_\alpha Z_\alpha(\mathbf{T}) - \mu \partial_\alpha \omega(\mathbf{T}) Z_\alpha(\mathbf{T})) \right) dA \end{aligned}$$

where the curly bracket is the quadratic strain energy operator in two-dimensional elasticity, and  $J_\omega$  is the warping constant defined in equation (5.12).

In view of the linearity of the problem with respect to the strain gradient  $\tau^\dagger$ , the solution  $Z_\alpha(\mathbf{T}) = Z_\alpha^{\text{opt}}(\tau^\dagger, \mathbf{T})$  that renders  $\mathcal{B}^{(\tau)}(\tau^\dagger, Z_\alpha)$  stationary subject to the conditions  $\mathbf{q}(Z_\alpha) = \mathbf{0}$  is of the form

$$Z_\alpha^{\text{opt}}(\tau^\dagger, \mathbf{T}) = \tau^\dagger u_\alpha(\mathbf{T}). \quad (\text{D.4})$$

This leads to the variational problem (5.14–5.15) stated in the main text.

Inserting the optimal displacement given by (D.3) and (D.4) into the expression of  $\mathcal{B}^{(\tau)}$ , we find from equation (4.15)

$$\begin{aligned} \frac{1}{2} B(\tau) \tau^{\dagger 2} &= \mathcal{B}^{(\tau)}(\tau^\dagger, \mathbf{Z}_{\text{opt}}^{(\tau)}(\mathbf{T}, \tau^\dagger)) \\ &= \frac{2\mu + \lambda}{2} J_\omega \tau^{\dagger 2} + \frac{\tau^{\dagger 2}}{2} \left( \iint_{\Omega} \left\{ \lambda \left( \sum_{\alpha} \partial_\alpha u_\alpha \right)^2 + 2 \mu \sum_{\alpha\beta} \left( \frac{\partial_\alpha u_\beta + \partial_\beta u_\alpha}{2} \right)^2 \right\} dA + 2(D_\lambda - D_\mu) \right) \end{aligned}$$

where  $D_\lambda$  and  $D_\mu$  are the quantities defined in equation (5.16).

This expression can be simplified using an identity for  $u_\alpha$  found by inserting  $\hat{u}_\alpha = u_\alpha$  into (5.14), namely

$$\iint_{\Omega} \left\{ \lambda \left( \sum_{\alpha} \partial_\alpha u_\alpha \right)^2 + 2 \mu \sum_{\alpha\beta} (\partial_\alpha u_\beta + \partial_\beta u_\alpha)^2 \right\} dA = -(D_\lambda - D_\mu),$$

which yields

$$B(\tau) = D_\omega - D_\mu,$$

where  $D_\omega$  is the quantity defined in (5.16).

The second-gradient modulus  $B(\tau)$  appears to be independent of  $\tau$ , which is always the case in the framework of linearized elasticity.

In addition, we have from equation (4.15)

$$\begin{aligned} C(\tau) \tau^\dagger &= \mathbf{C}_{(\tau)}^{(1)} \cdot \mathbf{Z}_{\text{opt}}^{(\tau)}(\mathbf{T}, \tau^\dagger) \\ &= \mu \tau \iint_{\Omega} (-\eta_{\alpha\beta} T_\beta + \partial_\alpha \omega) Z_\alpha^{\text{opt}} \, dA \\ &= \mu \tau \tau^\dagger \iint_{\Omega} (-\eta_{\alpha\beta} T_\beta + \partial_\alpha \omega) u_\alpha \, dA \\ &= \mu \tau \tau^\dagger \iint_{\Omega} \partial_\alpha \omega u_\alpha \, dA \\ &= D_\mu \tau \tau^\dagger \end{aligned}$$

and from equation (4.16), and using the expression for the homogeneous stress  $\Sigma^{(\tau)}$  in (5.8),

$$\begin{aligned} \mathbf{A}(\tau) \cdot (\tau^\dagger) &= \iint_{\Omega} \Sigma^{(\tau)}(\mathbf{T}) : \mathcal{E}^{(\tau)}(\mathbf{T}, \tau^\dagger, \mathbf{0}) \, dA \\ &= \iint_{\Omega} \tau^\dagger \omega(\mathbf{T}) \Sigma^{(\tau)}(\mathbf{T}) : \mathbf{e}_3 \otimes \mathbf{e}_3 \, dA \\ &= 0 \end{aligned}$$

Using equation (4.17), we obtain

$$D(\tau) = B(\tau) + 2 \frac{dC(\tau)}{d\tau} = (D_\omega - D_\mu) + 2 D_\mu = D_\omega + D_\mu.$$

In view of equation (4.17), the modified strain measure reads  $\hat{\tau} = \tau + \xi_0(\tau) \frac{d^2\tau}{dS^2}$  where

$$\xi_0(\tau) = \frac{C(\tau)}{\frac{dW_{\text{hom}}}{d\tau}} = \frac{D_\mu \tau}{\mu J \tau} = \frac{D_\mu}{\mu J}.$$

Inserting into (4.2), we arrive at the expression of  $\Phi_2^*[h]$  announced in equation (5.17).

## D.2. Case of an elliptical cross-section

We consider the case of an elliptical cross-section with semi-minor and major axes  $a$  and  $b$ , in any order. The elliptical cross-section satisfies the symmetry assumptions listed at the beginning of section 5.

Analytical solutions for the various quantities defined in section 5 can be obtained as follows. First, the warping function is found as  $\omega(\mathbf{T}) = \frac{b^2 - a^2}{a^2 + b^2} T_1 T_2$ , which yields the expressions of  $J$  and  $J_\omega$  announced in equation (5.18). The corrective displacement is found by solving the variational problem in equation (5.14–5.15) as

$$\begin{aligned} u_1(\mathbf{T}) &= \frac{1}{24} \frac{b^2 - a^2}{a^2 + b^2} T_2 \left( 4a^2 \frac{(5a^2b^2 + b^4 - 2(a^2 + b^2)T_2^2)}{(a^2 + b^2)^2} + \frac{\lambda}{\mu + \lambda} (a^2 - b^2 - 6T_1^2 + 2T_2^2) \right) \\ u_2(\mathbf{T}) &= \frac{1}{24} \frac{b^2 - a^2}{a^2 + b^2} T_1 \left( 4b^2 \frac{(a^4 + 5a^2b^2 - 2(a^2 + b^2)T_1^2)}{(a^2 + b^2)^2} + \frac{\lambda}{\mu + \lambda} (b^2 - a^2 + 2T_1^2 - 6T_2^2) \right) \\ D_\mu &= 8\mu \left( \frac{ab}{a^2 + b^2} \right)^2 J_\omega \\ D_\omega &= YJ_\omega, \end{aligned}$$

where  $Y$  is the Young modulus, see equation (5.19). The Lagrange multipliers are found as  $F_\alpha = 0$  and  $Q = \left( \frac{b^2 - a^2}{a^2 + b^2} \right)^2 \mu$ .

The quantities defined in equation (5.16) are obtained as  $D_\lambda = -\frac{\lambda^2}{\lambda + \mu} J_\omega$ , and as stated in equation (5.18).

## APPENDIX E. HIGHER-ORDER MODEL FOR A TWISTING BAR: CONSISTENCY WITH PRIOR WORK

In this appendix, we verify that our equation (5.20) for the twisting of a bar accounting for the gradient effect is equivalent to the equations derived by Trabucho and Viaño [TV89]. Mathematical quantities defined in Trabucho and Viaño's paper such as  $[\underline{u}_\alpha^4]$  are enclosed in square brackets; equation number in square brackets refer to equations in their paper.

We focus on the case where the external load is of order  $\varepsilon^2$ , where  $\varepsilon$  is the small aspect-ratio, which they use as an expansion parameter: in their notation, the external load is represented by the force  $\varepsilon^2 [f_i^2]$  per unit volume in the bulk, and the force  $\varepsilon^2 [g_i^2]$  per unit area applied on the lateral surface.

Trabucho and Viaño seek the average rotation of the cross-section with coordinate  $S = [x_3]$  as a power series of the aspect-ratio parameter  $\varepsilon$ ,  $[-v(S)] = -\varepsilon^2 [v^2(S)] - \varepsilon^4 [v^4(S)] - \dots$  where the minus signs arise because of their non-standard sign conventions. This corresponds to a kinematic twist  $\tau(S) = -\varepsilon^2 \frac{d[v^2]}{dS} - \varepsilon^4 \frac{d[v^4]}{dS} - \dots$

They obtain the principle of virtual work for the twisting of the bar order by order: summing up their equation [3.11] at order  $\varepsilon^2$  and their equation [5.11] at order  $\varepsilon^4$ , their principle of virtual work reads

$$\forall \hat{\theta} \quad - \int \mu J \tau(S) \hat{\theta}'(S) \, dS = \int (-Y J_\omega \tau''(S) + [M_4^3]) \hat{\theta}'(S) \, dS - \int m_3(S) \hat{\theta}(S) \, dS, \quad (\text{E.1})$$

where  $m_3 = \iint_{\Omega} ([x_1] [f_2^2] - [x_2] [f_1^2]) \, dA + \int_{\partial\Omega} ([x_1] [g_2^2] - [x_2] [g_1^2]) \, d\ell$  is the external twisting moment applied per unit length,  $[M_4^3]$  is the auxiliary quantity defined in [5.12] as

$$[M_4^3] = -\nu \frac{d}{dS} \left( \iint_{\Omega} 2(\lambda + \mu) \partial_\alpha [\underline{u}_\alpha^4] \omega \, dA \right) - \mu \tau''(S) \iint_{\Omega} (\partial_2 u_1 - \partial_1 u_2) [\Psi] \, dA, \quad (\text{E.2})$$

and  $[\Psi(\mathbf{T})]$  is the auxiliary function satisfying, see [2.11–12],

$$\partial_1 [\Psi] = -\partial_2 \omega - T_1 \quad \partial_2 [\Psi] = \partial_1 \omega - T_2$$

as well as

$$[\Psi] = 0 \text{ on } \partial\Omega.$$

The second integrand in equation (E.2) is a rewriting of that given by Trabucho and Viaño that uses the identity

$$[\underline{u}_\alpha^4] + [\underline{\tilde{u}}_\alpha^4] = \tau' u_\alpha(\mathbf{T}). \quad (\text{E.3})$$



This identity can be established by noting that our variational problem (5.14) for  $u_\alpha$  is a linear combination of the variational problems [3.23] for  $[u_\alpha^4]$  and [5.8] for  $[\tilde{u}_\alpha^4]$ .

To obtain equation (E.2), we have also used the fact that the quantities related to the stretching and bending modes  $[u_\alpha^0]$ ,  $[\underline{u}_3^0]$ ,  $[z_2^0]$  as well as the quantity  $[w^0]$  are all zero as a consequence of our symmetry assumptions [TV89].

Using the properties of  $[\Psi]$  just listed, the second term in the right-hand side of (E.2) can be integrated by parts as

$$\begin{aligned} -\mu \tau'' \iint_{\Omega} (\partial_2 u_1 - \partial_1 u_2) [\Psi] dA &= \mu \tau'' \iint_{\Omega} (u_1 \partial_2 [\Psi] - u_2 \partial_1 [\Psi]) dA \\ &= \mu \tau'' \iint_{\Omega} u_\alpha \partial_\alpha \omega dA + \mu \tau'' \iint_{\Omega} \eta_{\alpha\beta} T_\alpha u_\beta dA \\ &= \tau'' D_\mu \end{aligned}$$

To obtain the last equality, we have identified the quantity  $D_\mu$  defined in (5.16), and have observed that the integrand of the second term is zero by equation (5.15). Inserting this into (E.2), and using the identity  $\nu = \frac{\lambda}{2(\lambda + \mu)}$ , we have

$$[M_4^3] = -\lambda \frac{d}{dS} \left( \iint_{\Omega} \partial_\alpha [\underline{u}_\alpha^4] \omega dA \right) - \tau'' D_\mu$$

Using equation (E.3) one more time, one can eliminate  $[\underline{u}_\alpha^4]$  in favor of  $[\tilde{u}_\alpha^4]$ ,

$$\begin{aligned} [M_4^3] &= \lambda \frac{d}{dS} \left( \iint_{\Omega} \partial_\alpha [\tilde{u}_\alpha^4] \omega dA \right) - \lambda \frac{d}{dS} \left( \tau'(S) \iint_{\Omega} \partial_\alpha u_\alpha \omega dA \right) + \tau'' D_\mu \\ &= \lambda \frac{d}{dS} \left( \iint_{\Omega} \partial_\alpha [\tilde{u}_\alpha^4] \omega dA \right) - \tau'' (D_\lambda - D_\mu) \end{aligned}$$

At this point, we introduce two auxiliary functions  $\tilde{u}_\alpha(\mathbf{T})$  such that

$$\partial_1 \tilde{u}_1(\mathbf{T}) = \omega(\mathbf{T}) \quad \partial_2 \tilde{u}_2(\mathbf{T}) = \omega(\mathbf{T}) \quad \partial_2 \tilde{u}_1(\mathbf{T}) + \partial_1 \tilde{u}_2(\mathbf{T}) = 0 \quad (\text{E.4})$$

The existence of the functions  $\tilde{u}_\alpha(\mathbf{T})$  is warranted by the kinematic compatibility condition

$$\partial_{12}(\partial_2 \tilde{u}_1 + \partial_1 \tilde{u}_2) - \partial_{22}(\partial_1 \tilde{u}_1) - \partial_{11}(\partial_2 \tilde{u}_2) = \partial_{12}(0) - \partial_{22}\omega - \partial_{11}\omega = -\partial_{\alpha\alpha}\omega = 0$$

as can be checked by applying the Euler-Lagrange method to the weak form of the variational problem for  $\omega(\mathbf{T})$  in (5.7).

The functions  $\tilde{u}_\alpha(\mathbf{T})$  are defined up to a rigid-body motion that can be set by requiring

$$\iint_{\Omega} \tilde{u}_\alpha dA = 0 \quad \iint_{\Omega} \eta_{\alpha\beta} T_\alpha \tilde{u}_\beta dA = 0$$

The variational problem [5.8] for  $[\tilde{u}_\alpha^4]$  writes

$$\forall \hat{v}_\alpha \quad \iint_{\Omega} \left\{ (\lambda \partial_\rho [\tilde{u}_\rho^4] \delta_{\alpha\beta} + \mu (\partial_\alpha [\tilde{u}_\beta^4] + \partial_\beta [\tilde{u}_\alpha^4])) \partial_\beta \hat{v}_\alpha + \lambda \tau'(S) \omega \partial_\alpha \hat{v}_\alpha \right\} dA = 0$$

Setting  $\hat{v}_\alpha = \tilde{u}_\alpha$  in this variational problem, we obtain, after making use of the identities in equation (E.4),

$$2(\lambda + \mu) \iint_{\Omega} \partial_\alpha [\tilde{u}_\alpha^4] \omega dA + 2\lambda \tau' J_\omega = 0.$$

This identity can be used to rewrite  $[M_3^4]$  as

$$[M_4^3] = -\left( \frac{\lambda^2}{\lambda + \mu} J_\omega + D_\lambda - D_\mu \right) \tau'' = -((\lambda + 2\mu) - Y) J_\omega + D_\lambda - D_\mu \tau'' = -(-Y J_\omega + D_\omega - D_\mu) \tau''$$

after making use successively of the identity  $\frac{\lambda^2}{\lambda + \mu} = (\lambda + 2\mu) - Y$ , see the definition of the Young modulus in (5.19), and of the definition of  $D_\omega$  in (5.16).

The principle of virtual work (E.1) derived by Trabuco and Viaño can therefore be rewritten as

$$\forall \hat{\theta} \quad \int \left\{ \mu J \tau(S) - (D_\omega - D_\mu) \tau''(S) \right\} \hat{\theta}'(S) dS = \int m_3(S) \hat{\theta}(S) dS.$$

Integrating by parts and eliminating the virtual rotation  $\hat{\theta}$ , one recovers the equation of equilibrium derived by our method, see equation (5.20).

## APPENDIX F. A NON-LINEAR CYLINDER THAT BENDS AND STRETCHES

### F.1. Simple traction of an isotropic hyper-elastic material

Here, we characterize the finite-strain material model introduced in section 6.1. We are particularly interested in simple traction along the material axis  $e_3$ , since this is the state of stress corresponding to the unbuckled solution. We denote by  $\varepsilon$  the axial strain, such that the axial stretch ratio is  $1 + \varepsilon$ . The simple traction is such that the strain  $\mathbf{E}_{\text{tr}}(\varepsilon)$  is equi-biaxial and the stress  $\boldsymbol{\Sigma}(\mathbf{E}_{\text{tr}}(\varepsilon))$  is uniaxial:

$$\begin{aligned} \mathbf{E}_{\text{tr}}(\varepsilon) &= \frac{(1 + \varepsilon)^2 - 1}{2} \mathbf{e}_3 \otimes \mathbf{e}_3 + \frac{p^2(\varepsilon) - 1}{2} \mathbf{e}_\alpha \otimes \mathbf{e}_\alpha \\ \boldsymbol{\Sigma}(\mathbf{E}_{\text{tr}}(\varepsilon)) &= \Sigma_{\text{tr}}(\varepsilon) \mathbf{e}_3 \otimes \mathbf{e}_3, \end{aligned} \quad (\text{F.1})$$

where  $p(\varepsilon)$  is the transverse stretch ratio due to Poisson's effect.

The first derivative of the strain energy density in simple traction

$$w_{\text{tr}}(\varepsilon) = w(\mathbf{E}_{\text{tr}}(\varepsilon)) \quad (\text{F.2})$$

is related to the stress in simple traction by

$$(1 + \varepsilon) \Sigma_{\text{tr}}(\varepsilon) = \frac{dw_{\text{tr}}(\varepsilon)}{d\varepsilon}, \quad (\text{F.3})$$

while the second derivative of  $w_{\text{tr}}(\varepsilon)$  defines the tangent Young modulus  $Y_t(\varepsilon)$ ,

$$Y_t(\varepsilon) = \frac{d^2 w_{\text{tr}}(\varepsilon)}{d\varepsilon^2}. \quad (\text{F.4})$$

When in simple traction, an isotropic material has an incremental elastic behavior that is transversely isotropic. As a result, the tangent elastic moduli

$$\mathbf{K}^{\text{tr}}(\varepsilon) = \mathbf{K}(\mathbf{E}_{\text{tr}}(\varepsilon)) = \frac{d^2 w}{d\mathbf{E}^2}(\mathbf{E}_{\text{tr}}(\varepsilon)) \quad (\text{F.5})$$

are such that

$$\begin{aligned} (\mathbf{e}_\alpha \odot \mathbf{e}_\beta) : \mathbf{K}^{\text{tr}}(\varepsilon) : (\mathbf{e}_\beta \otimes \mathbf{e}_\gamma) &= (\mathbf{e}_\beta \otimes \mathbf{e}_\gamma) : \mathbf{K}^{\text{tr}}(\varepsilon) : (\mathbf{e}_\alpha \odot \mathbf{e}_\beta) = 0 \\ (\mathbf{e}_\alpha \odot \mathbf{e}_\beta) : \mathbf{K}^{\text{tr}}(\varepsilon) : (\mathbf{e}_\beta \odot \mathbf{e}_\beta) &= \delta_{\alpha\beta} G_t(\varepsilon), \end{aligned} \quad (\text{F.6})$$

where  $G_t(\varepsilon)$  is the tangent shear modulus. In equation (6.10), the ratio of the shear to the Young modulus is defined in terms of a dimensionless constitutive parameter  $c_T(\varepsilon) = Y_t(\varepsilon) / \left[ 2 G_t(\varepsilon) p(\varepsilon) \left( -\frac{dp}{d\varepsilon}(\varepsilon) \right) (1 + \varepsilon) \right]$ .

Let us now focus on the natural configuration,  $\varepsilon = 0$ . This configuration being free of stress, we have

$$\frac{dw_{\text{tr}}}{d\varepsilon}(0) = 0. \quad (\text{F.7})$$

The following relations warrant that the configuration  $\varepsilon = 0$  is undeformed, and define the initial Poisson's ratio  $\nu_0$ ,

$$p(0) = 1 \quad \frac{dp}{d\varepsilon}(0) = -\nu_0. \quad (\text{F.8})$$

In this initial configuration, Hookean elasticity applies and the *initial* shear modulus is given by

$$G_t(0) = \frac{Y_0}{2(1 + \nu_0)}. \quad (\text{F.9})$$

We denote by  $Y_0 = Y_t(0)$  the initial Young modulus and by  $Y'_0 = \frac{dY_t}{d\varepsilon}(0)$  the initial curvature of the traction curve that yields the Kirchhoff stress  $w'_{\text{tr}}(\varepsilon)$  as a function of the strain  $\varepsilon$ ,

$$Y_0 = \frac{d^2 w_{\text{tr}}}{d\varepsilon^2}(0) \quad Y'_0 = \frac{d^3 w_{\text{tr}}}{d\varepsilon^3}(0) \quad \nu_0 = -\frac{dp}{d\varepsilon}(0). \quad (\text{F.10})$$

For the particular constitutive law  $w(\mathbf{T}, \mathbf{E}) = w_{\text{ST}}(\mathbf{E})$  used by ST98 and given in section 6.1, these material constants read

$$\begin{aligned} Y_0 &= 4(A_{\text{ST}} + B_{\text{ST}})(1 + \nu_{\text{ST}}) \\ \nu_0 &= \nu_{\text{ST}} \\ Y'_0 &= \frac{4(1 + \nu_{\text{ST}})}{9}(A_{\text{ST}}(-23 + 8\nu_{\text{ST}} + 4\nu_{\text{ST}}^2) - B_{\text{ST}}(31 + 8\nu_{\text{ST}} + 4\nu_{\text{ST}}^2)). \end{aligned}$$

## F.2. Analysis of homogeneous solutions

In this section, we check that the solution proposed in equation (6.5) satisfies all the equations applicable to the homogeneous solution, which have been listed in section 4.2. As the displacement (6.5) satisfies  $\mathbf{Y}_3^{(\varepsilon, \kappa)}(\mathbf{T}) = 0$ , the non-linear expression (4.4) of the strain simplifies to  $\mathbf{E}^h(\mathbf{T}) = \mathbf{E}^{(\varepsilon, \kappa)}(\mathbf{T}) = \tilde{\mathbf{E}}(\mathbf{T}, \mathbf{h}, \mathbf{Y}^{(\varepsilon, \kappa)})$  where

$$\mathbf{E}^{(\varepsilon, \kappa)}(\mathbf{T}) = \frac{(1 + \varepsilon + \kappa Y_2^{(\varepsilon, \kappa)}(\mathbf{T}))^2 - 1}{2} \mathbf{e}_3 \otimes \mathbf{e}_3 + \frac{\partial_\alpha Y_\gamma^{(\varepsilon, \kappa)}(\mathbf{T}) \partial_\beta Y_\gamma^{(\varepsilon, \kappa)}(\mathbf{T}) - \delta_{\alpha\beta}}{2} \mathbf{e}_\alpha \odot \mathbf{e}_\beta. \quad (\text{F.11})$$

To derive this equality, we have used the fact that the shear term  $\tilde{t}_i \partial_\alpha Y_i^{(\varepsilon, \kappa)}(\mathbf{T}) \mathbf{e}_\alpha \odot \mathbf{e}_3$  in equation (4.4) is zero, as  $\tilde{t}_i = (1 + \varepsilon) \delta_{i3} + \eta_{ijk} \kappa_j Y_k^{(\varepsilon, \kappa)} = (1 + \varepsilon) \delta_{i3} + \eta_{i1\alpha} \kappa Y_\alpha^{(\varepsilon, \kappa)} = (1 + \varepsilon + Y_2^{(\varepsilon, \kappa)}) \delta_{i3}$  and so  $\tilde{t}_i \partial_\alpha Y_i^{(\varepsilon, \kappa)}(\mathbf{T}) \mathbf{e}_\alpha \odot \mathbf{e}_3 = \tilde{t}_3 \partial_\alpha Y_3^{(\varepsilon, \kappa)}(\mathbf{T}) \mathbf{e}_\alpha \odot \mathbf{e}_3 = 0$ .

Inserting the expansion of  $Y_\alpha^{(\varepsilon, \kappa)}$  in powers of  $\kappa$  from equation (6.5), we have

$$\begin{aligned} \mathbf{E}^{(\varepsilon, \kappa)}(\mathbf{T}) &= \frac{(1 + \varepsilon + \kappa p(\varepsilon) T_2)^2 - 1}{2} \mathbf{e}_3 \otimes \mathbf{e}_3 \\ &\quad + \frac{1}{2} \left( p^2(\varepsilon) \left( \delta_{\alpha\gamma} + \kappa \frac{dp}{d\varepsilon}(\varepsilon) \partial_\alpha \varphi_\gamma \right) \left( \delta_{\beta\gamma} + \kappa \frac{dp}{d\varepsilon}(\varepsilon) \partial_\beta \varphi_\gamma \right) - \delta_{\alpha\beta} \right) \mathbf{e}_\alpha \odot \mathbf{e}_\beta + \mathcal{O}(\kappa^2) \\ &= \frac{(1 + \varepsilon + \kappa p(\varepsilon) T_2)^2 - 1}{2} \mathbf{e}_3 \otimes \mathbf{e}_3 + \frac{1}{2} \left( p^2(\varepsilon) \left( 1 + 2\kappa \frac{dp}{d\varepsilon}(\varepsilon) T_2 \right) - 1 \right) \mathbf{e}_\alpha \otimes \mathbf{e}_\alpha + \mathcal{O}(\kappa^2) \\ &= \frac{(1 + \varepsilon + \kappa p(\varepsilon) T_2)^2 - 1}{2} \mathbf{e}_3 \otimes \mathbf{e}_3 + \frac{1}{2} \left( \left( p(\varepsilon) + \frac{dp}{d\varepsilon}(\varepsilon) \times [\kappa p(\varepsilon) T_2] \right)^2 - 1 \right) \mathbf{e}_\alpha \otimes \mathbf{e}_\alpha + \mathcal{O}(\kappa^2) \\ &= \frac{(1 + \varepsilon + \kappa p(\varepsilon) T_2)^2 - 1}{2} \mathbf{e}_3 \otimes \mathbf{e}_3 + \frac{(p(\varepsilon + [\kappa p(\varepsilon) T_2]))^2 - 1}{2} \mathbf{e}_\alpha \otimes \mathbf{e}_\alpha + \mathcal{O}(\kappa^2) \\ &= \mathbf{E}_{\text{tr}}(\varepsilon + \kappa p(\varepsilon) T_2) + \mathcal{O}(\kappa^2) \end{aligned} \quad (\text{F.12})$$

Here, we have used the identity  $\partial_\alpha \varphi_\beta + \partial_\beta \varphi_\alpha = 2 T_2 \delta_{\alpha\beta}$  from equation (6.6), as well as the identity  $(1 + 2x) = (1 + x)^2 + \mathcal{O}(x^2)$  with  $x = \kappa \frac{dp}{d\varepsilon}(\varepsilon) T_2$ , and we have identified the strain in simple traction  $\mathbf{E}_{\text{tr}}$  from equation (F.1).

By the constitutive law, the stress writes  $\Sigma^{(\varepsilon, \kappa)} = \Sigma_{\text{tr}}(\varepsilon + \kappa p(\varepsilon) T_2) \mathbf{e}_3 \otimes \mathbf{e}_3 + \mathcal{O}(\kappa^2)$ , see equation (F.1).

We proceed to check that  $\mathbf{Y}^{(\varepsilon, \kappa)}$  satisfies the variational problem (4.5).

By design from equation (6.6), the quantities  $\varphi_\alpha(\mathbf{T})$  and  $T_1 \varphi_2(\mathbf{T}) - T_2 \varphi_1(\mathbf{T})$  average out to zero on the cross-section: the kinematic constraints on the first two lines of (4.5) are verified up to terms of order  $\kappa^2$ .

Given that the stress is uniaxial to order  $\kappa$ , we only need the projection along  $\mathbf{e}_3 \otimes \mathbf{e}_3$  of the virtual change of strain  $\hat{\mathbf{E}}^h(\mathbf{T}) = \frac{d\hat{\mathbf{E}}}{d\mathbf{Y}}(\mathbf{T}, \mathbf{h}, \mathbf{Y}^h) \cdot \hat{\mathbf{Y}}$  appearing in the principle of virtual work (4.5). Using equation (4.4), we find  $\hat{\mathbf{E}}^{(\varepsilon, \kappa)}(\mathbf{T}) : (\mathbf{e}_3 \otimes \mathbf{e}_3) = (1 + \varepsilon) \kappa \hat{Y}_2(\mathbf{T}) + \mathcal{O}(\kappa^2)$ , hence  $\Sigma(\mathbf{T}, \mathbf{E}^h(\mathbf{T})) : \hat{\mathbf{E}}^h(\mathbf{T}) = (1 + \varepsilon) \kappa \Sigma_{\text{tr}}(\varepsilon + \kappa p(\varepsilon) T_2) \hat{Y}_2(\mathbf{T}) + \mathcal{O}(\kappa^2) = (1 + \varepsilon) \kappa \Sigma_{\text{tr}}(\varepsilon) \hat{Y}_2(\mathbf{T}) + \mathcal{O}(\kappa^2)$ . Therefore, the variational problem (4.5) writes

$$\forall \hat{Y}_i(\mathbf{T}) \quad \iint_{\Omega} [(1 + \varepsilon) \kappa \Sigma_{\text{tr}}(\varepsilon) \hat{Y}_2(\mathbf{T}) + F_i^{(\varepsilon, \kappa)} \hat{Y}_i(\mathbf{T}) + Q^{(\varepsilon, \kappa)} (T_1 \hat{Y}_2(\mathbf{T}) - T_2 \hat{Y}_1(\mathbf{T})) + \mathcal{O}(\kappa^2)] dA = 0, \quad (\text{F.13})$$

where  $F_i^{(\varepsilon, \kappa)}$  and  $Q^{(\varepsilon, \kappa)}$  are Lagrange multipliers. This variational problem is indeed satisfied with the choice  $F_i^{(\varepsilon, \kappa)} = -\delta_{i2} (1 + \varepsilon) \kappa \Sigma_{\text{tr}}(\varepsilon)$  and  $Q^{(\varepsilon, \kappa)} = 0$ , as can be checked.

There remains to calculate the strain energy  $W_{\text{hom}}(\varepsilon, \kappa) = \iint_{\Omega} w(\mathbf{E}^{(\varepsilon, \kappa)}(\mathbf{T})) dA$  associated with the homogeneous solution  $\mathbf{Y}^{(\varepsilon, \kappa)}$ , see equation (4.6). Let  $\mathbf{O}(\mathbf{T}) = \mathbf{E}^{(\varepsilon, \kappa)}(\mathbf{T}) - \mathbf{E}_{\text{tr}}(\varepsilon + \kappa p(\varepsilon) T_2)$  denote the deviation of the strain from the simple traction estimate:  $\mathbf{O}(\mathbf{T}) = \mathcal{O}(\kappa^2)$  from equation (F.12). One can then expand  $W_{\text{hom}}(\varepsilon, \kappa)$  as

$$\begin{aligned} W_{\text{hom}}(\varepsilon, \kappa) &= \iint_{\Omega} w(\mathbf{E}_{\text{tr}}(\varepsilon + \kappa p(\varepsilon) T_2) + \mathbf{O}(\mathbf{T})) dA \\ &= \iint_{\Omega} [w(\mathbf{E}_{\text{tr}}(\varepsilon + \kappa p(\varepsilon) T_2)) + \Sigma_{\text{tr}}(\varepsilon + \kappa p(\varepsilon) T_2) : \mathbf{O}(\mathbf{T}) + \mathcal{O}(\kappa^4)] dA \end{aligned}$$

after identifying the stress  $\Sigma_{\text{tr}}(e) = \frac{dw}{d\mathbf{E}}(\mathbf{E}_{\text{tr}}(e))$  in simple traction, see equations (4.7) and (F.1). Now, the second term in the integrand can be evaluated as follows by using equations (F.1), (F.11), (F.1) again, (6.5) and (6.6) sequentially,

$$\begin{aligned} \iint_{\Omega} \Sigma_{\text{tr}}(\varepsilon + \kappa p(\varepsilon) T_2) : \mathbf{O}(\mathbf{T}) dA &= \iint_{\Omega} \Sigma_{\text{tr}}(\varepsilon + \kappa p(\varepsilon) T_2) \{ \mathbf{e}_3 \otimes \mathbf{e}_3 : (\mathbf{E}^{(\varepsilon, \kappa)}(\mathbf{T}) - \mathbf{E}_{\text{tr}}(\varepsilon + \kappa p(\varepsilon) T_2)) \} dA \\ &= \iint_{\Omega} (\Sigma_{\text{tr}}(\varepsilon) + \mathcal{O}(\kappa)) \left\{ \frac{(1 + \varepsilon + \kappa Y_2^{(\varepsilon, \kappa)}(\mathbf{T}))^2 - 1}{2} - \frac{(1 + \varepsilon + \kappa p(\varepsilon) T_2)^2 - 1}{2} \right\} dA \\ &= \iint_{\Omega} (\Sigma_{\text{tr}}(\varepsilon) + \mathcal{O}(\kappa)) \{ (1 + \varepsilon) (\kappa Y_2^{(\varepsilon, \kappa)}(\mathbf{T}) - \kappa p(\varepsilon) T_2) + \mathcal{O}(\kappa^2) \} dA \\ &= \iint_{\Omega} (1 + \varepsilon) (\Sigma_{\text{tr}}(\varepsilon) + \mathcal{O}(\kappa)) \kappa^2 \left( p(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon) \varphi_2(\mathbf{T}) + \mathcal{O}(\kappa) \right) dA \\ &= (1 + \varepsilon) \Sigma_{\text{tr}}(\varepsilon) p(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon) \kappa^2 \iint_{\Omega} \varphi_2(\mathbf{T}) dA + \mathcal{O}(\kappa^3) \\ &= \mathcal{O}(\kappa^3) \end{aligned}$$

Returning to the expression of  $W_{\text{hom}}(\varepsilon, \kappa)$ , we are therefore left with

$$\begin{aligned} W_{\text{hom}}(\varepsilon, \kappa) &= \iint_{\Omega} [w(\mathbf{E}_{\text{tr}}(\varepsilon + \kappa p(\varepsilon) T_2)) + \mathcal{O}(\kappa^3)] dA \\ &= \iint_{\Omega} \left[ w_{\text{tr}}(\varepsilon) + \frac{dw_{\text{tr}}}{d\varepsilon}(\varepsilon) \kappa p(\varepsilon) T_2 + \frac{\kappa^2}{2} \frac{d^2 w_{\text{tr}}}{d\varepsilon^2}(\varepsilon) p^2(\varepsilon) T_2^2 + \mathcal{O}(\kappa^3) \right] dA, \end{aligned}$$

where  $w_{\text{tr}}(\varepsilon) = w(\mathbf{E}_{\text{tr}}(\varepsilon))$  is the strain energy density for simple traction.

The integral of  $T_2$  is zero in a disk. Identifying the Young modulus  $Y_t(\varepsilon) = \frac{d^2 w_{\text{tr}}}{d\varepsilon^2}(\varepsilon)$  from (F.4) and the initial geometric moment of inertia  $I_1^0 = \iint_{\Omega} T_2^2 dA$ , we find

$$W_{\text{hom}}(\varepsilon, \kappa) = A w_{\text{tr}}(\varepsilon) + \frac{\kappa^2}{2} Y_t(\varepsilon) p^2(\varepsilon) I_1^0 + \mathcal{O}(\kappa^3).$$

The expression announced in equation (6.8) follows by observing that  $W_{\text{hom}}(\varepsilon, \kappa)$  is an even function of  $\kappa$  by symmetry, implying that the remainder  $\mathcal{O}(\kappa^3)$  in equation above is actually of order  $\mathcal{O}(\kappa^4)$ .

### F.3. Analysis of the gradient effect

This section derives the gradient effect, following closely the outline given in section 4.3.

From equation (4.7), we find the transformation gradient in the homogeneous configuration as  $F_{33}^{(\varepsilon, \kappa)}(\mathbf{T}) = 1 + \varepsilon + \kappa p(\varepsilon) T_2 + \mathcal{O}(\kappa^2)$ . Using the identity  $\Sigma_{\text{tr}}(e) = \frac{dw_{\text{tr}}}{de}(e) / (1 + e)$  from equation (F.3), the operator  $\mathbf{C}_h^{(1)}$  defined in (4.11) is found as

$$\begin{aligned} \mathbf{C}_{(\varepsilon, \kappa)}^{(1)} \cdot \mathbf{Z}^\dagger &= \iint_{\Omega} (1 + \varepsilon + \kappa p(\varepsilon) T_2) \Sigma_{\text{tr}}(\varepsilon + \kappa p(\varepsilon) T_2) Z_3^\dagger(\mathbf{T}) dA + \mathcal{O}(\kappa^2) \\ &= \iint_{\Omega} w'_{\text{tr}}(\varepsilon + \kappa p(\varepsilon) T_2) Z_3^\dagger(\mathbf{T}) dA + \mathcal{O}(\kappa^2) \\ &= w'_{\text{tr}}(\varepsilon) \iint_{\Omega} Z_3^\dagger(\mathbf{T}) dA + \kappa Y_t(\varepsilon) p(\varepsilon) \iint_{\Omega} T_2 Z_3^\dagger(\mathbf{T}) dA + \mathcal{O}(\kappa^2) \\ &= \kappa Y_t(\varepsilon) p(\varepsilon) \iint_{\Omega} T_2 Z_3^\dagger(\mathbf{T}) dA + \mathcal{O}(\kappa^2) \end{aligned} \quad (\text{F.14})$$

where  $Y_t(\varepsilon)$  is the tangent Young modulus from equation (F.4). The operator  $\mathbf{C}_h^{(1)}$  is applied exclusively to cross-sectional functions  $\mathbf{Z}^\dagger$  satisfying the constraints  $\mathbf{q}(\mathbf{0}) = \mathbf{0}$  which implies in particular  $\iint_{\Omega} Z_3^\dagger(\mathbf{T}) \, dA = 0$ , hence the last equality above.

In the remainder of this derivation, we limit attention to  $\kappa = 0$ .

For  $\kappa = 0$ , the transformation gradient (4.7) is  $F_{ij}^{(\varepsilon,0)}(\mathbf{T}) = (1 + \varepsilon) \delta_{i3} \delta_{j3} + p(\varepsilon) \delta_{i\alpha} \delta_{j\alpha}$ . The linearized strain  $\mathcal{E}$  in equation (4.9) writes

$$\mathcal{E}^{(\varepsilon,0)}(\mathbf{T}, \mathbf{h}^\dagger, \mathbf{Z}) = \left( \mathbf{h}^\dagger \cdot \left( \nabla Y_\alpha^{(\varepsilon,0)}(\mathbf{T}) \right)_{\kappa=0} p(\varepsilon) + (1 + \varepsilon) \partial_\alpha Z_3(\mathbf{T}) \right) \mathbf{e}_\alpha \odot \mathbf{e}_3 + p(\varepsilon) \partial_\alpha Z_\beta(\mathbf{T}) \mathbf{e}_\alpha \odot \mathbf{e}_\beta \quad (\text{F.15})$$

where  $\mathbf{h}^\dagger = (\varepsilon^\dagger, \kappa^\dagger)$  denotes the strain gradient. Recalling that the  $\nabla$  operator stands for a derivative with respect to the macroscopic strain  $\mathbf{h} = (\varepsilon, \kappa)$ , see equation (4.10),

$$\begin{aligned} \mathbf{h}^\dagger \cdot \left( \nabla Y_\alpha^{(\varepsilon,0)}(\mathbf{T}) \right)_{\kappa=0} &= \left( \left( \varepsilon^\dagger \frac{d}{d\varepsilon} + \kappa^\dagger \frac{d}{d\kappa} \right) \left( p(\varepsilon) \left( T_\alpha + \kappa \frac{dp}{d\varepsilon}(\varepsilon) \varphi_\alpha(\mathbf{T}) \right) + \mathcal{O}(\kappa^2) \right) \right)_{\kappa=0} \\ &= \varepsilon^\dagger \frac{dp}{d\varepsilon}(\varepsilon) T_\alpha + \kappa^\dagger p(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon) \varphi_\alpha(\mathbf{T}) \end{aligned}$$

and so

$$\begin{aligned} \mathcal{E}^{(\varepsilon,0)}(\mathbf{T}, \mathbf{h}^\dagger, \mathbf{Z}) &= \\ p(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon) \left( T_\alpha \varepsilon^\dagger + p(\varepsilon) \varphi_\alpha(\mathbf{T}) \kappa^\dagger \right) \mathbf{e}_\alpha \odot \mathbf{e}_3 + \left( (1 + \varepsilon) \partial_\alpha Z_3(\mathbf{T}) \mathbf{e}_3 + p(\varepsilon) \partial_\alpha Z_\beta(\mathbf{T}) \mathbf{e}_\beta \right) \odot \mathbf{e}_\alpha. \end{aligned} \quad (\text{F.16})$$

For  $\kappa = 0$ , the operator  $\nabla \mathbf{C}_{(\varepsilon,\kappa)}^{(1)}$  introduced in equation (4.12) writes, with the help of equation (F.14),

$$\begin{aligned} -(\varepsilon^\dagger, \kappa^\dagger) \cdot \nabla \mathbf{C}_{(\varepsilon,0)}^{(1)} \cdot \mathbf{Z} &= - \iint_{\Omega} \left( \frac{\partial(\kappa Y_t(\varepsilon) p(\varepsilon) T_2)}{\partial \varepsilon} \varepsilon^\dagger + \frac{\partial(\kappa Y_t(\varepsilon) p(\varepsilon) T_2)}{\partial \kappa} \kappa^\dagger \right)_{\kappa=0} Z_3(\mathbf{T}) \, dA \\ &= - \iint_{\Omega} (0 + Y_t(\varepsilon) p(\varepsilon) T_2 \kappa^\dagger) Z_3(\mathbf{T}) \, dA \\ &= -\kappa^\dagger Y_t(\varepsilon) p(\varepsilon) \iint_{\Omega} T_2 Z_3(\mathbf{T}) \, dA. \end{aligned} \quad (\text{F.17})$$

We proceed to calculate the bilinear operator  $\mathcal{B}^h(\mathbf{h}^\dagger, \mathbf{Z})$  from equation (4.13):

$$\begin{aligned} \mathcal{B}^{(\varepsilon,0)}(\mathbf{h}^\dagger, \mathbf{Z}) &= \iint_{\Omega} \frac{1}{2} \mathcal{E}^{(\varepsilon,0)}(\mathbf{T}, \mathbf{h}^\dagger, \mathbf{Z}) : \mathbf{K}^{(\varepsilon,0)}(\mathbf{T}) : \mathcal{E}^{(\varepsilon,0)}(\mathbf{T}, \mathbf{h}^\dagger, \mathbf{Z}) \, dA \\ &\quad \dots + \iint_{\Omega} \left( \frac{1}{2} \sum_{\alpha} \left( \mathbf{h}^\dagger \cdot \left( \nabla Y_\alpha^{(\varepsilon,0)}(\mathbf{T}) \right)_{\kappa=0} \right)^2 \Sigma_{\text{tr}}(\varepsilon) \right) \, dA - \mathbf{h}^\dagger \cdot \nabla \mathbf{C}_{(\varepsilon,0)}^{(1)} \cdot \mathbf{Z}, \end{aligned}$$

where  $\nabla \mathbf{C}_{(\varepsilon,0)}^{(1)}$  has just been obtained in equation (F.17).

Using the transverse isotropy of  $\mathbf{K}^{(\varepsilon,0)}(\mathbf{T}) = \mathbf{K}_{\text{tr}}(\varepsilon)$  from equation (F.6), and the expression of  $\mathcal{E}^{(\varepsilon,0)}(\mathbf{T}, \mathbf{h}^\dagger, \mathbf{Z})$  in equation (F.15), one can calculate the elastic stiffness term appearing in  $\mathcal{B}^{(\varepsilon,0)}(\mathbf{h}^\dagger, \mathbf{Z})$  as

$$\frac{1}{2} \mathcal{E}^{(\varepsilon,0)}(\mathbf{T}, \mathbf{h}^\dagger, \mathbf{Z}) : \mathbf{K}^{(\varepsilon,0)}(\mathbf{T}) : \mathcal{E}^{(\varepsilon,0)}(\mathbf{T}, \mathbf{h}^\dagger, \mathbf{Z}) = \frac{1}{2} G_t(\varepsilon) \left( \sum_{\alpha} 2^2 (\mathcal{E}_{\alpha 3}^{(\varepsilon,0)})^2 \right) + \frac{1}{2} \mathcal{E}_{\alpha \beta}^{(\varepsilon,0)} K_{\alpha \beta \gamma \delta}^{(\varepsilon,0)}(\mathbf{T}) \mathcal{E}_{\gamma \delta}^{(\varepsilon,0)},$$

where  $\mathcal{E}_{\alpha 3}^{(\varepsilon,0)} = \mathcal{E}^{(\varepsilon,0)}(\mathbf{T}, \mathbf{h}^\dagger, \mathbf{Z}) : (\mathbf{e}_\alpha \otimes \mathbf{e}_3) = \frac{1}{2} \left( \mathbf{h}^\dagger \cdot \left( \nabla Y_\alpha^{(\varepsilon,0)}(\mathbf{T}) \right)_{\kappa=0} p(\varepsilon) + (1 + \varepsilon) \partial_\alpha Z_3(\mathbf{T}) \right)$  from equation (F.15).

By equation (F.16),  $\mathcal{E}_{\alpha 3}^{(\varepsilon,0)}$  depends on the longitudinal corrective displacement  $Z_3$  while  $\mathcal{E}_{\alpha \beta}^{(\varepsilon,0)}$  depends on the transverse one. This shows that the quantities  $Z_\alpha(\mathbf{T})$  are uncoupled from  $Z_3(\mathbf{T})$  in  $\mathcal{B}^{(\varepsilon,0)}(\mathbf{h}^\dagger, \mathbf{Z})$ . The source terms  $\varepsilon^\dagger$  and  $\kappa^\dagger$  in  $\mathcal{B}^{(\varepsilon,0)}(\mathbf{h}^\dagger, \mathbf{Z})$  being coupled to  $Z_3(\mathbf{T})$ , see equation (F.17), we conclude that the optimal transverse displacement is zero,

$$Z_{\text{opt},\alpha}^{(\varepsilon,0)}((\varepsilon^\dagger, \kappa^\dagger), \mathbf{T}) = 0.$$

We are left with

$$\begin{aligned} \mathcal{B}^{(\varepsilon,0)}(\mathbf{h}^\dagger, \mathbf{Z}) &= \frac{1}{2} \iint_{\Omega} G_t(\varepsilon) \sum_{\alpha} \left( p(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon) T_\alpha \varepsilon^\dagger + p^2(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon) \varphi_\alpha(\mathbf{T}) \kappa^\dagger + (1 + \varepsilon) \partial_\alpha Z_3(\mathbf{T}) \right)^2 \, dA \\ &\quad \dots + \iint_{\Omega} \left( \frac{1}{2} \sum_{\alpha} \left( \frac{dp}{d\varepsilon}(\varepsilon) T_\alpha \varepsilon^\dagger + p(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon) \varphi_\alpha(\mathbf{T}) \kappa^\dagger \right)^2 \Sigma_{\text{tr}}(\varepsilon) \right) \, dA - \kappa^\dagger p(\varepsilon) Y_t(\varepsilon) \iint_{\Omega} T_2 Z_3(\mathbf{T}) \, dA. \end{aligned}$$

Since we are only interested in capturing the gradient effect associated with bending, we focus attention to the case where the argument of  $\mathcal{B}^{(\varepsilon,0)}$  has  $\varepsilon^\dagger = 0$ , i.e., we set  $\mathbf{h}^\dagger = (0, \kappa^\dagger)$ . Expanding the square in equation above, we have

$$\begin{aligned} \mathcal{B}^{(\varepsilon,0)}((0, \kappa^\dagger), \mathbf{Z}) &= \frac{(\kappa^\dagger)^2}{2} \left( p(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon) \right)^2 \Sigma_{\text{tr}}(\varepsilon) M \dots \\ &\quad \dots + \frac{1}{2} G_t(\varepsilon) \iint_{\Omega} \left( p^2(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon) \varphi_\alpha(\mathbf{T}) \kappa^\dagger + (1 + \varepsilon) \partial_\alpha Z_3(\mathbf{T}) \right)^2 \, dA - \kappa^\dagger p(\varepsilon) Y_t(\varepsilon) \iint_{\Omega} T_2 Z_3(\mathbf{T}) \, dA \end{aligned}$$

where  $M$  is the constant defined in equation (6.12).

The optimal correction  $Z_3 = Z_{\text{opt},3}^{(\varepsilon,0)}((0, \kappa^\dagger), \mathbf{T})$  associated with a bending gradient  $\kappa^\dagger$  makes stationary  $\mathcal{B}^{(\varepsilon,0)}((0, \kappa^\dagger), \mathbf{Z})$  among all functions  $Z_3$  that have a zero average on the cross-section, see equation (4.14):  $Z_3(\mathbf{T})$  is such that, for any virtual displacement  $\hat{Z}_3(\mathbf{T})$ ,

$$\begin{aligned} \iint_{\Omega} G_t(\varepsilon) (1 + \varepsilon) \left( p^2(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon) \varphi_\alpha(\mathbf{T}) \kappa^\dagger + (1 + \varepsilon) \partial_\alpha Z_3(\mathbf{T}) \right) \partial_\alpha \hat{Z}_3(\mathbf{T}) dA \\ \dots - \kappa^\dagger p(\varepsilon) Y_t(\varepsilon) \iint_{\Omega} T_2 \hat{Z}_3(\mathbf{T}) dA + \iint_{\Omega} F_3^\dagger(\varepsilon) \hat{Z}_3(\mathbf{T}) dA = 0, \end{aligned} \quad (\text{F.18})$$

where  $F_3^\dagger(\varepsilon)$  is a Lagrange multiplier enforcing the constraint  $\iint_{\Omega} Z_3 dA = 0$ . Taking a constant virtual field  $\hat{Z}_3(\mathbf{T}) = 1$  and using  $\iint_{\Omega} T_2 dA = A \langle T_2 \rangle = 0$  by equation (6.2), we find  $F_3^\dagger(\varepsilon) = 0$ .

With  $c_\Gamma(\varepsilon)$  as the material constant defined in equation (6.10) the equation above can be rewritten as

$$\forall \hat{Z}_3 \quad \iint_{\Omega} \left[ \partial_\alpha Z_3(\mathbf{T}) \partial_\alpha \hat{Z}_3(\mathbf{T}) + \kappa^\dagger \frac{p^2(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon)}{1 + \varepsilon} (\varphi_\alpha(\mathbf{T}) \partial_\alpha \hat{Z}_3(\mathbf{T}) + c_\Gamma(\varepsilon) \times 2 T_2 \hat{Z}_3(\mathbf{T})) \right] dA = 0.$$

By linearity, the solution  $Z_3(\mathbf{T})$  of this variational problem is the function  $Z_{\text{opt},3}^{(\varepsilon,0)}((0, \kappa^\dagger), \mathbf{T})$  given in equation (6.9) in terms of the cross-sectional functions  $\Theta$  and  $\Gamma$  introduced in equation (6.11).

Inserting this solution  $Z_{\text{opt},3}^{(\varepsilon,0)}((0, \kappa^\dagger), \mathbf{T})$  into the expression of  $\mathcal{B}^{(\varepsilon,0)}((0, \kappa^\dagger), \mathbf{Z})$  above, and expanding the  $\frac{1}{2} G_t(\varepsilon) \iint_{\Omega} (\dots)^2 dA$  term, we find the operator  $\mathbf{B}$  defined in equation (4.15) as

$$\begin{aligned} \frac{1}{2} (0, \kappa^\dagger) \cdot \mathbf{B}(\varepsilon, 0) \cdot (0, \kappa^\dagger) &= \mathcal{B}^{(\varepsilon,0)}((0, \kappa^\dagger), \mathbf{Z}_{\text{opt}}^{(\varepsilon,0)}((0, \kappa^\dagger))) \\ &= \frac{(\kappa^\dagger)^2}{2} \left( p(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon) \right)^2 (\Sigma_{\text{tr}}(\varepsilon) + p^2(\varepsilon) G_t(\varepsilon)) M - \kappa^\dagger p(\varepsilon) Y_t(\varepsilon) \iint_{\Omega} T_2 Z_{\text{opt},3}^{(\varepsilon,0)}(\mathbf{T}) dA \dots \\ &\quad + G_t(\varepsilon) (1 + \varepsilon) \iint_{\Omega} \left( p^2(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon) \varphi_\alpha(\mathbf{T}) \kappa^\dagger + \frac{1 + \varepsilon}{2} \partial_\alpha Z_{\text{opt},3}^{(\varepsilon,0)}(\mathbf{T}) \right) \partial_\alpha Z_{\text{opt},3}^{(\varepsilon,0)}(\mathbf{T}) dA. \end{aligned}$$

This expression can be simplified using the following identity, obtained by choosing the virtual motion in (F.18) to be the actual solution,  $\hat{Z}_3 = Z_{\text{opt},3}^{(\varepsilon,0)}$ ,

$$\begin{aligned} \iint_{\Omega} G_t(\varepsilon) (1 + \varepsilon) \left( p^2(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon) \varphi_\alpha(\mathbf{T}) \kappa^\dagger + (1 + \varepsilon) \partial_\alpha Z_{\text{opt},3}^{(\varepsilon,0)}(\mathbf{T}) \right) \partial_\alpha Z_{\text{opt},3}^{(\varepsilon,0)}(\mathbf{T}) dA \\ \dots - \kappa^\dagger p(\varepsilon) Y_t(\varepsilon) \iint_{\Omega} T_2 Z_{\text{opt},3}^{(\varepsilon,0)}(\mathbf{T}) dA = 0. \end{aligned}$$

This yields

$$\begin{aligned} \frac{1}{2} (0, \kappa^\dagger) \cdot \mathbf{B}(\varepsilon, 0) \cdot (0, \kappa^\dagger) &= \frac{(\kappa^\dagger)^2}{2} \left( p(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon) \right)^2 (\Sigma_{\text{tr}}(\varepsilon) + p^2(\varepsilon) G_t(\varepsilon)) M \dots \\ &\quad - \frac{1}{2} G_t(\varepsilon) (1 + \varepsilon)^2 \iint_{\Omega} \sum_{\alpha=1}^2 (\partial_\alpha Z_{\text{opt},3}^{(\varepsilon,0)}(\mathbf{T}))^2 dA \\ &= \frac{(\kappa^\dagger)^2}{2} \left( p(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon) \right)^2 \left[ (\Sigma_{\text{tr}}(\varepsilon) + p^2(\varepsilon) G_t(\varepsilon)) M \dots \right. \\ &\quad \left. - G_t(\varepsilon) p^2(\varepsilon) \iint_{\Omega} \sum_{\alpha=1}^2 (\partial_\alpha \Theta(\mathbf{T}) + c_\Gamma(\varepsilon) \partial_\alpha \Gamma(\mathbf{T}))^2 dA \right]. \end{aligned}$$

Using the constants from equation (6.12) and eliminating  $\Sigma_{\text{tr}}(\varepsilon)$  using equation (F.3), one obtains an following identity concerning the operator  $\mathbf{B}(\varepsilon, 0)$  introduced in equation (4.15):

$$\frac{1}{2} (0, \kappa^\dagger) \cdot \mathbf{B}(\varepsilon, 0) \cdot (0, \kappa^\dagger) = \frac{(\kappa^\dagger)^2}{2} B_{11}(\varepsilon, 0)$$

where

$$B_{11}(\varepsilon, 0) = \left( p(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon) \right)^2 \left( \frac{1}{1 + \varepsilon} \frac{dw_{\text{tr}}}{d\varepsilon}(\varepsilon) M + p^2(\varepsilon) G_t(\varepsilon) (M - J_{\Theta\Theta} - 2 c_\Gamma(\varepsilon) J_{\Theta\Gamma} - c_\Gamma^2(\varepsilon) J_{\Gamma\Gamma}) \right). \quad (\text{F.19})$$

In addition, the operator  $\mathbf{C}(\varepsilon, \kappa)$  defined in equation (4.15) as  $\mathbf{C}(\varepsilon, \kappa) \cdot (\varepsilon^\dagger, \kappa^\dagger) = \mathbf{C}_{(\varepsilon, \kappa)}^{(1)} \cdot \mathbf{Z}_{\text{opt}}^{(\varepsilon, \kappa)}(\varepsilon^\dagger, \kappa^\dagger)$  is found from equations (F.14) and (6.9) as

$$\begin{aligned} \mathbf{C}(\varepsilon, \kappa) \cdot (\varepsilon^\dagger, \kappa^\dagger) &= \kappa Y_t(\varepsilon) p(\varepsilon) \iint_{\Omega} T_2 Z_3^{(\varepsilon,0)}((\varepsilon^\dagger, \kappa^\dagger), \mathbf{T}) dA + \mathcal{O}(\kappa^2) \\ &= \kappa Y_t(\varepsilon) p(\varepsilon) \iint_{\Omega} T_2 \left( \varepsilon^\dagger \times (\dots) + \kappa^\dagger \frac{p^2(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon)}{1 + \varepsilon} (\Theta(\mathbf{T}) + c_\Gamma(\varepsilon) \Gamma(\mathbf{T})) \right) dA + \mathcal{O}(\kappa^2) \\ &= \varepsilon^\dagger \kappa \times (\dots) + \kappa^\dagger \kappa Y_t(\varepsilon) \frac{p^3(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon)}{1 + \varepsilon} \iint_{\Omega} T_2 (\Theta(\mathbf{T}) + c_\Gamma(\varepsilon) \Gamma(\mathbf{T})) dA + \mathcal{O}(\kappa^2) \\ &= \varepsilon^\dagger C_0(\varepsilon, \kappa) + \kappa^\dagger C_1(\varepsilon, \kappa) \end{aligned}$$

where the ellipsis stands for an expression that does not need to be calculated,

$$C_0(\varepsilon, \kappa) = \kappa \times (\dots) + \mathcal{O}(\kappa^2) = \mathcal{O}(\kappa) \quad (\text{F.20})$$

and

$$C_1(\varepsilon, \kappa) = \kappa Y_t(\varepsilon) \frac{p^3(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon)}{1 + \varepsilon} \left( \iint_{\Omega} T_2 \Theta(\mathbf{T}) dA + c_{\Gamma}(\varepsilon) \iint_{\Omega} T_2 \Gamma(\mathbf{T}) dA \right) + \mathcal{O}(\kappa^2). \quad (\text{F.21})$$

To evaluate the integrals in the right-hand side, we set  $\hat{Z}_3(\mathbf{T}) = g(\mathbf{T})$  in the variational problem for  $\Gamma$  in equation (6.11),

$$\begin{aligned} \iint_{\Omega} T_2 g(\mathbf{T}) dA &= -\frac{1}{2} \iint_{\Omega} \partial_{\alpha} \Gamma(\mathbf{T}) \partial_{\alpha} g(\mathbf{T}) dA \\ &= \begin{cases} -J_{\Theta\Gamma}/2 & \text{if } g = \Theta \\ -J_{\Gamma\Gamma}/2 & \text{if } g = \Gamma. \end{cases} \end{aligned} \quad (\text{F.22})$$

This yields, after using (6.10),

$$\begin{aligned} C_1(\varepsilon, \kappa) &= \kappa Y_t(\varepsilon) \frac{p^3(\varepsilon) \left( -\frac{dp}{d\varepsilon}(\varepsilon) \right)}{2(1 + \varepsilon)} (J_{\Theta\Gamma} + J_{\Gamma\Gamma} c_{\Gamma}(\varepsilon)) + \mathcal{O}(\kappa^2) \\ &= \kappa G_t(\varepsilon) \left( p^2(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon) \right)^2 (J_{\Theta\Gamma} + J_{\Gamma\Gamma} c_{\Gamma}(\varepsilon)) c_{\Gamma}(\varepsilon) + \mathcal{O}(\kappa^2). \end{aligned} \quad (\text{F.23})$$

The elastic modulus  $D_{11}$  associated with the gradient effect in bending and defined in equation (4.16) is found by combining equations (F.19) and (F.23) as

$$\begin{aligned} D_{11}(\varepsilon, 0) &= B_{11}(\varepsilon, 0) + 2 \frac{\partial C_1}{\partial \kappa}(\varepsilon, 0) \\ &= \left( p(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon) \right)^2 \left( \frac{1}{1 + \varepsilon} \frac{dw_{\text{tr}}}{d\varepsilon}(\varepsilon) M + p^2(\varepsilon) G_t(\varepsilon) (M - J_{\Theta\Theta} + J_{\Gamma\Gamma} c_{\Gamma}^2(\varepsilon)) \right) \end{aligned}$$

as announced in the third equation of equation (6.13).

By a symmetry argument similar to that presented in section 6.2.1, one can show that  $C_0(\varepsilon, \kappa)$  is even with respect to  $\kappa$ , so that equation (F.20) implies the stronger estimate

$$C_0(\varepsilon, \kappa) = \mathcal{O}(\kappa^2). \quad (\text{F.24})$$

The modified strains are then found from equation (4.17) and (6.8) as  $\tilde{\varepsilon} = \varepsilon + \xi_0(\varepsilon, \kappa) \varepsilon''$  and  $\tilde{\kappa} = \kappa + \xi_1(\varepsilon, \kappa) \kappa''$ , where

$$\begin{aligned} \xi_0(\varepsilon, 0) &= \lim_{\kappa \rightarrow 0} \frac{C_0(\varepsilon, \kappa)}{\frac{\partial W_{\text{hom}}}{\partial \varepsilon}(\varepsilon, \kappa)} \\ &= \lim_{\kappa \rightarrow 0} \frac{\mathcal{O}(\kappa^2)}{\mathcal{O}(\kappa)} \\ &= 0 \end{aligned}$$

and, recalling equations (F.23) and (6.10),

$$\begin{aligned} \xi_1(\varepsilon, 0) &= \lim_{\kappa \rightarrow 0} \frac{C_1(\varepsilon, \kappa)}{\frac{\partial W_{\text{hom}}}{\partial \kappa}(\varepsilon, \kappa)} \\ &= \left( \frac{\kappa G_t(\varepsilon) \left( p^2(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon) \right)^2 (J_{\Theta\Gamma} + J_{\Gamma\Gamma} c_{\Gamma}(\varepsilon)) c_{\Gamma}(\varepsilon) + \mathcal{O}(\kappa^2)}{\kappa Y_t(\varepsilon) p^2(\varepsilon) I_1^0 + \mathcal{O}(\kappa^3)} \right)_{\kappa=0} \\ &= \frac{G_t(\varepsilon) \left( p^2(\varepsilon) \frac{dp}{d\varepsilon}(\varepsilon) \right)^2 (J_{\Theta\Gamma} + J_{\Gamma\Gamma} c_{\Gamma}(\varepsilon)) c_{\Gamma}(\varepsilon)}{Y_t(\varepsilon) p^2(\varepsilon) I_1^0} \\ &= p(\varepsilon) \left( -\frac{dp}{d\varepsilon}(\varepsilon) \right) \frac{(J_{\Theta\Gamma} + J_{\Gamma\Gamma} c_{\Gamma}(\varepsilon))}{2(1 + \varepsilon) I_1^0} \end{aligned}$$

This completes the proof of the results announced in equation (6.13).