PHYS 849: Assignment 4

Jake Bauer

April 12, 2017

1 Sampling the Distributions

We are first asked to devise a method to sample from an exponential distribution. Since the exponential distribution is monotonic, we can sample by inverting the CDF. We should really consider a truncated exponential since the domain is the positive half-space, but we will consider a large enough range such that the deviation will be negligible. Our sampling algorithm then considers the exponential CDF,

$$F(t) = -e^{-t/\tau} + 1 \tag{1}$$

where τ is the decay constant. The inverse is then,

$$t = -\frac{\ln[1 - U(0,1)]}{\tau} \tag{2}$$

where $U(0,1)$ is a uniform sample between 0 and 1.

We are then asked to sample a Gaussian distribution of mean 1 and standard deviation 0.1. To sample the requested normal distribution, we use R's built-in function which implements Box-Muller.

2 The Maximum Likelihood Estimator

We construct the unbinned log-likelihood,

$$\begin{aligned} \mathcal{L} &= \frac{e^{-\mu(\tau)}\mu(\tau)^n}{n!}\prod_{i=1}^{n} f(t_i;\tau) && (3) \\ \ln\mathcal{L} &= -\mu(\tau) + n\ln\mu(\tau) - \ln n! + \sum_{i=1}^{n} f(t_i;\tau) && (4) \\ & && (5) \end{aligned}$$

where $\mu(\tau)$ is the expected number of counts, n is the observed number of counts, and $f(t;\tau)$ is an evaluation of the PDF at t. We are given that the number of counts is known and fixed, meaning that $n = \mu$ is constant w.r.t. τ. Then, we maximize the likelihood by,

$$\begin{aligned} \frac{\partial \ln\mathcal{L}}{\partial\tau} &= -0 + 0 - 0 + \frac{\partial}{\partial\tau}\sum_{i=1}^{n}\ln f(t_i;\tau) && (6) \\ &= \sum_{i=1}^{n} n\ln\left[\frac{1}{\tau}\right] - \sum_{i=1}^{n}\frac{t_i}{\tau} && (7) \\ &= 0 && (8) \end{aligned}$$

Solving gives the familiar,

$$\hat{\tau} = \frac{1}{n}\sum_{i=1}^{n} t_i, \tag{9}$$

or the mean of the observations. That is to say, it does not matter whether or not the data are binned for an estimate of the decay constant.

3 Constructing an Estimator from the Chi-Squared

We could also derive an estimator for τ from the χ^2 statistic,

$$\chi^2 = \sum_{i=1}^{n_b} \frac{(n_i - \tau^{-1}e^{-t_i/\tau}n\delta t)^2}{\tau^{-1}e^{-t_i/\tau}n\delta t} \tag{10}$$

where n_b is the number of bins, a free parameter, n_i are the observed counts per bin, t_i is the position of the bin center, $n = \sum_i n_i$, and δt is the bin spacing. This quantity is minimized if,

$$n_i = \tau^{-1}e^{-t_i/\tau}n\delta t. \tag{11}$$

This gives us the condition,

$$\sum_{i=1}^{n_b}\left[\ln\frac{n_i}{n\delta t} + \ln\tau + \frac{t_i}{\tau}\right] = 0 \tag{12}$$

$$n\ln\tau + \frac{1}{\tau}\sum_{t_i} = 0 \tag{13}$$