# CS189–Spring 2013 — Solutions to Homework 2

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1. We want to pick a class, $j$ that minimizes: $\sum_k L_{kj} P(\omega_k|x)$, $L_{kj}$ being the loss incurred by choosing class $j$ when the actual class is $k$.
   $L_{kj} = 0$, if $k = j$
   $L_{kj} = \lambda_d$, if $j$ is the $c + 1$ class, the doubt class
   $L_{kj} = \lambda_s$, otherwise

   $\min(\lambda_s P(w_0|x) + \lambda_s P(w_1|x) + ... + 0 * P(\omega_j|x) + ... + \lambda_s P(w_c|x), \lambda_d)$
   When do we choose to make a decision?
   $\lambda_s P(w_0|x) + \lambda_s P(w_1|x) + ... + 0 * P(\omega_j|x) + ... + \lambda_s P(w_c|x) \leq \lambda_d$
   $\lambda_s(1 - P(\omega_j|k)) \leq \lambda_d$
   $1 - \frac{\lambda_d}{\lambda_s} \leq P(\omega_j|k)$
   And then, of course when we make a decision, we choose the best one, that is, all $\lambda_s$ being equal, we choose the maximum $P(\omega_k|x)$
   Therefore, we decide $\omega_i$ if $P(\omega_i|x) > P(\omega_j|x)$ for all $j$ and $P(\omega_i|x) \geq 1 - \frac{\lambda_d}{\lambda_s}$

2. $p(x|w_i) \sim N(\mu_i, \sigma^2)$
   $p(w_i|x) = p(x|w_i)p(w_i)$ normalized
   $p(w_i|x) = \frac{1}{2} N(\mu_i, \sigma^2)$
   Let $\mu_2 > \mu_1$. There exists some value for which we change our guess from $w_1$ to $w_2$. This value is equidistant from each of the means (this is intuitive from the symmetry in the problem because both have the same variance). We can also just set $N(\mu_1, \sigma^2) = N(\mu_2, \sigma^2)$ to find out that this value, $b$, is $\frac{\mu_2+\mu_1}{2}$. The probability of error is the probability of it being $w_2$ when $x < b$ and the probability of it being $w_1$ when $x > b$.
   $\int_{-\infty}^{b} P(w_2|x)P(x)dx + \int_{b}^{\infty} P(w_1|x)P(x)dx$
   Similarly, by symmetry, the two integrals are going to be equal (because both are normally distributed with same standard deviation).
   $2 * \int_{b}^{\infty} \frac{1}{2\sigma\sqrt{2\pi}} \exp \frac{-(x-\mu_1)^2}{2\sigma^2} dx$

Now lets substitute variables, $u = \frac{x-\mu_1}{\sigma}$, $\sigma du = dx$, and then new bound is $\alpha = \frac{\mu_2-\mu_1}{2\sigma}$

$\int\limits_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} \exp -\frac{1}{2}u^2 du$

$\int\limits_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} \exp -\frac{1}{2}u^2 du \leq \frac{1}{\sqrt{2\pi a}} e^{-(1/2)a^2}$

$P_e \leq \lim_{a\to\infty} \frac{1}{\sqrt{2\pi a}} e^{-(1/2)a^2} = 0$

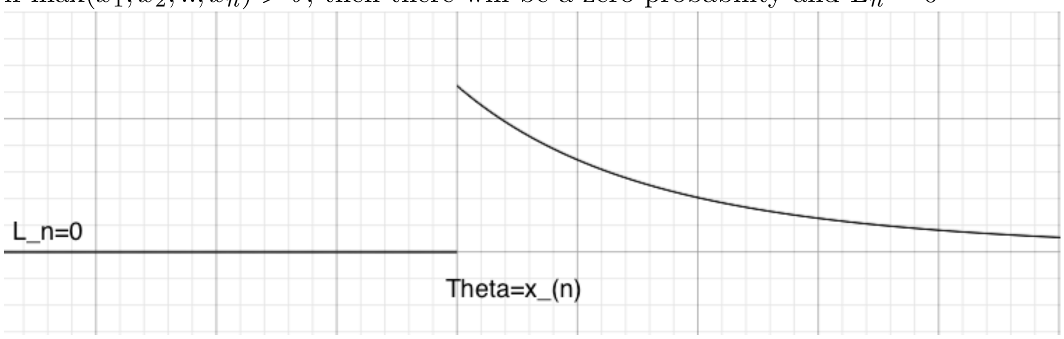
3. Let Y be the discrete r.v. corresponding to score of a shot
   $E(Y) = 4 * P(Y = 4) + 3 * P(Y = 3) + 2 * P(Y = 2) + 0$
   $E(Y) = 4 * P(X \leq \frac{1}{\sqrt{3}}) + 3 * P(\frac{1}{\sqrt{3}} \leq X \leq 1) + 2 * P(1 \leq X \leq \sqrt{3})$
   $E(Y) = 4 * \int\limits_{0}^{\frac{1}{\sqrt{3}}} f(x)dx + 3 * \int\limits_{\frac{1}{\sqrt{3}}}^{1} f(x)dx + 2 * \int\limits_{1}^{\sqrt{3}} f(x)dx$
   $E(Y) = 4 * \frac{1}{3} + 3 * \frac{1}{6} + 2 * \frac{1}{6}$
   $E(Y) = 13/6 \approx 2.167$

4. $f(x, y) = x + y$
   $g(x) = \int\limits_{0}^{1} (x + y)dy$
   $g(x) = xy + \frac{1}{2}y^2\Big|_{y=0}^{1}$
   $g(x) = x + \frac{1}{2}$
   $E(X) = \int\limits_{0}^{1} xg(x)dx$
   $E(X) = \int\limits_{0}^{1} x^2 + \frac{1}{2}xdx$
   $E(X) = \frac{1}{3}x^3 + \frac{1}{4}x^2\Big|_{x=0}^{1}$
   $E(X) = 7/12$
   $\mathrm{Var}(X) = E(X^2) - (7/12)^2$
   $\mathrm{Var}(X) = \int\limits_{0}^{1} x^2(x + \frac{1}{2})dx$
   $\mathrm{Var}(X) = 5/12$

   $h(y) = \int\limits_{0}^{1} (x + y)dx$
   $h(y) = y + \frac{1}{2}$
   $E(Y) = \int\limits_{0}^{1} y^2 + \frac{1}{2}ydy$
   $E(Y) = 7/12$

$\text{Var}(Y) = \int\limits_0^1 y^2(y + \frac{1}{2})dy$
$\text{Var}(Y) = 5/12$
$\text{CoVar}(X, Y) = E(XY) - (7/12)^2$
$\text{CoVar}(X, Y) = \int\limits_0^1\int\limits_0^1 xy(x + y)dxdy - (7/12)^2$
$\text{CoVar}(X, Y) = 1/3$

5. (a) The likelihood of of getting samples $x_1..x_n$, $L_n$ is equal to $p(x_1; \theta)p(x_2; \theta)..p(x_n; \theta)$
   As long as $\theta > x_i$, $p(x_i; \theta) = \frac{1}{\theta}$
   $L_n = \left(\frac{1}{\theta}\right)^n$
   if $\max(x_1, x_2, .., x_n) > \theta$, then there will be a zero probability and $L_n = 0$




   (b) $x_{(n)}$ is the maximum likelihood estimate – $\theta = x_{(n)}$ maximizes $L_n$, and is therefore the most likely value of *theta* given the observations $x_1, .., x_n$.
   (c) Our estimate is most likely not the expected value of $\theta$. If we have a reading of $x_n$ then it is likely that $\theta$ is larger than $x_n$. Therefore, our estimate is biased.

6. $\log L(\theta) = \log \prod\limits_i \theta e^{-\theta x_i}$
   $\log L(\theta) = \sum\limits_i \log \theta e^{-\theta x_i}$
   $\log L(\theta) = \sum\limits_i \log \theta - \theta x_i$
   $\frac{dL(\theta)}{d\theta} = \sum\limits_i \frac{1}{\theta} - x_i = 0$
   $\frac{5}{\theta} = (x_1 + x_2 + .. + x_5) = 5.7$
   $\theta = .877$
   $\frac{d^2L(\theta)}{d\theta^2 e} = -\sum\limits_i \frac{1}{\theta^2} < 0$
   $\theta = .877$

7. Let $x_m$ be the middle $x$ value, for which we change our guess (there shouldn't be more than one)

$P(\omega_1|x) = P(x|\omega_1)P(\omega_1)$
$P(\omega_1|x) = \frac{1}{2}e^{-\lambda_1}\frac{\lambda_1^x}{x!}$
$P(\omega_2|x) = \frac{1}{2}e^{-\lambda_2}\frac{\lambda_2^x}{x!}$
$P(\omega_1|x_m) = P(\omega_2|x_m)$
$e^{-\lambda_1}\frac{\lambda_1^{x_m}}{x_m!} = e^{-\lambda_2}\frac{\lambda_2^{x_m}}{x_m!}$
$-\lambda_1 + x_m \ln(\lambda_1) = -\lambda_2 + x_m \ln(\lambda_2)$
$x_m(\ln(\lambda_1) - \ln(\lambda_2)) = -\lambda_2 + \lambda_1$
$x_m = \frac{\lambda_2 - \lambda_1}{\ln(\lambda_2) - \ln(\lambda_1)} = 12.33$
For $x < x_m$, we pick the lower value of $\lambda$, lets say $\lambda_1$. For $x \geq x_m$ we pick $\lambda_2$
Probability of correct classification for $\omega_1$ is
$\sum_0^{\lfloor x_m \rfloor} P(\omega_1, x)$
$\sum_0^{12} \frac{1}{2}e^{-10}\frac{10^x}{x!}$
Probability of correct classification for $\omega_1$ is 39.6%
$\sum_{13}^{\infty} \frac{1}{2}e^{-15}\frac{15^x}{x!}$
Probability of correct classification if its of class $\omega_2$ is 36.6%
Probability of error:
$P(error) = \sum_0^{12} P(\omega_2, x) + \sum_{13}^{\infty} P(\omega_1, x)$
$P(error) = \sum_0^{12} \frac{1}{2}e^{-15}\frac{15^x}{x!} + \sum_{13}^{\infty} \frac{1}{2}e^{-10}\frac{10^x}{x!}$
$= .24$

Now lets do it using two independent trials. Let $y = x_1 + x_2$
$P(y|\omega_1) \sim Poisson(2\omega_1)$
$P(y|\omega_2) \sim Poisson(2\omega_2)$
$P(\omega_1|y) = \frac{1}{2}e^{-2\lambda_1}\frac{(2\lambda_1)^x}{x!}$
$P(\omega_2|y) = \frac{1}{2}e^{-2\lambda_2}\frac{(2\lambda_2)^x}{x!}$
$P(\omega_1|y) = P(\omega_2|y)$
$-2\lambda_1 + y_m \ln(2\lambda_1) = -2\lambda_2 + y_m \ln(2\lambda_2)$
$y_m = \frac{2(\lambda_2 - \lambda_1)}{\ln(2\lambda_2) - \ln(2\lambda_1)}$
$y_m = 24.66$
Probability of correct classification for $\omega_1$
$\sum_0^{\lfloor y_m \rfloor} \frac{1}{2}e^{-2\lambda_1}\frac{(2\lambda_1)^x}{x!}$
$= 42.2\%$
Probability of correct classification for $\omega_2$

$$\sum_{\lceil y_m \rceil}^{\infty} \tfrac{1}{2} e^{-2\lambda_2} \tfrac{(2\lambda_2)^x}{x!}$$
$= 42.1\%$
$$P(error) = \sum_{0}^{24} P(\omega_2, y) + \sum_{25}^{\infty} P(\omega_1, y)$$
$$P(error) = \sum_{0}^{24} \tfrac{1}{2} e^{-30} \tfrac{30^x}{x!} + \sum_{25}^{\infty} \tfrac{1}{2} e^{-20} \tfrac{20^x}{x!}$$
$= 15.7\%$