SIMULACION ESTOCASTICA 2022-1

Taller 1: Probabilidad, Variables aleatorias, Generación de números aleatorios y variables aleatorias, Monte Carlo e Integración de Monte Carlo

Indicaciones: Subir al link en AVATA **un (1)** archivo pdf con el procedimiento analítico y resultados de cada punto y **un (1)** archivo en `R` con los códigos de los puntos que lo requieran.

Fecha de entrega: Domingo 6 de Marzo de 2022

Ejercicios:

1. **Events and probability (1 point)** (*Ross, Introduction to Probability Models*)

   a) A box contains three marbles: one red, one green, and one blue. Consider an experiment that consists of taking one marble from the box then replacing it in the box and drawing a second marble from the box. What is the sample space? If, at all times, each marble in the box is equally likely to be selected, what is the probability of each point in the sample space?

   b) Repeat the previous exercise when the second marble is drawn without replacing the first marble.

2. **Expectation and variance of a discrete r.v. (1 point)**

   Two fair dice are thrown and the smallest of the face values, M say, is noted.

   a) Give the probability mass function (*pmf*) of M in table form.

   b) What is the probability that M is at least 3?

   c) Calculate the expectation and variance of M.

3. **Congruential generators (1 point)** (*Jones*)

   Find all of the cycles of the following congruential generator. For each cycle identify which seeds X_0 lead to that cycle:

   $$X_{n+1} = (8X_n + 3) \; mod \; 11.$$

4. **Inverse method for a discrete r.v. (1 point)** (*Jones*)

   Consider the discrete random variable (r.v.) X with probability mass function (*pmf*) given by:

   $$P(X = 1) = 0.1, \quad P(X = 2) = 0.3, \quad P(X = 5) = 0.6.$$

   a) Calculate and plot the Cumulative Distribution Function (*CDF*) $F_X(x)$ of X.

   b) Write a program to generate n values of X with this distribution, using the function `runif` (in other words, implement the inverse method for the discrete r.v. X).

   c) Let $n = 10^3$, run the program, and plot an histogram of the output. Repeat for $n = 10^5$.

5. **Inverse method and rejection method for a continuous r.v. (3 points)** (*Jones*)

   Consider the continuous random variable X with probability density function (*pdf*) given by:

$$f_X(x) = \begin{cases} 3(x-1)^2 & \text{for} \quad 1 < x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

   a) Find the following probabilities: (i) $P(X \leq 1)$, (ii) $P(1 < X \leq 1.5)$, (iii) $P(X \geq 1.5)$.

   b) Find the *CDF*, $F_X(x)$, of X. Plot this function.

   c) Show how to simulate X with $F_X(x)$ using the inversion method.

   d) Write a program to generate n values of X using this method.

   e) Draw $n = 10^4$ samples of X from the previous program. Plot a normalized histogram of the sample along with the *pdf* of X.

   f) Write a program to generate n values of X with $f_X(x)$ using the rejection method.

   g) Repeat (e) using the program in (f).

6. **Estimating π (2 points)** (*Jones*)

   Suppose that X and Y are *iid* $U(0,1)$ random variables.

   a) What is $P((X,Y) \in [a,b] \times [c,d])$ for $0 \leq a \leq b \leq 1$ and $0 \leq c \leq d \leq 1$?

   b) Based on your previous answer, what do you think you should get for $P((X,Y) \in A)$, where A is an arbitrary subset of $[0,1] \times [0,1]$?

   c) Let $A = \{(x,y) \in [0,1] \times [0,1] : x^2 + y^2 \leq 1\}$. What is the area of A?

   d) Define the *r.v.* Z by:

$$Z = \begin{cases} 1 & \text{if} \quad X^2 + Y^2 \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

   What is $\mathbb{E}[Z]$?

   e) By simulating Z, write a program to estimate π.

7. **Monte Carlo integration (1 point)** (*Ross*)

   With respect to the following integrals: (i) Find their exact value analytically or with software (e.g., `WolframAlpha`), (ii) with Monte Carlo integration approximate the integrals and compare with the exact answer.

   a) $\int_{-2}^{2} x^2 dx$

   b) $\int_0^1 \int_0^2 e^{(x+y)^2} dy dx$