ASSIGNMENT 07

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Q 7.1:

$$\ddot{\phi} = -\frac{g}{l}\sin(\phi)$$

$$\omega = \frac{d\phi}{dt}$$

$$\boldsymbol{d\phi = \omega dt} \qquad (eq.1)$$

$$\frac{d^2\phi}{dt^2} = \frac{d\omega}{dt}$$

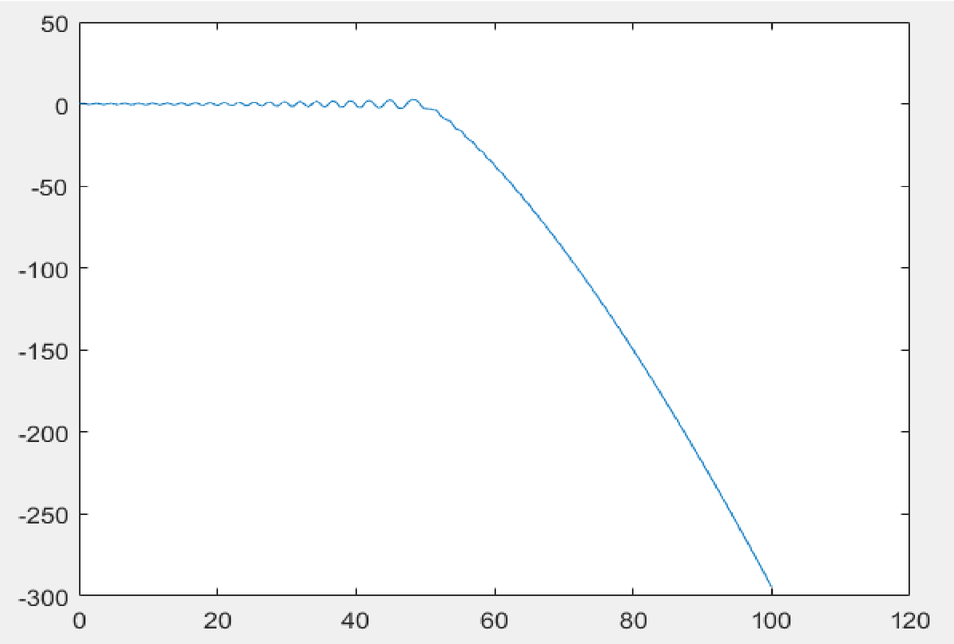
$$\ddot{\phi} = \dot{\omega}$$

$$\dot{\omega} = -\frac{g}{l}\sin(\phi)$$

$$\frac{d\omega}{dt} = -\frac{g}{l}\sin(\phi)$$

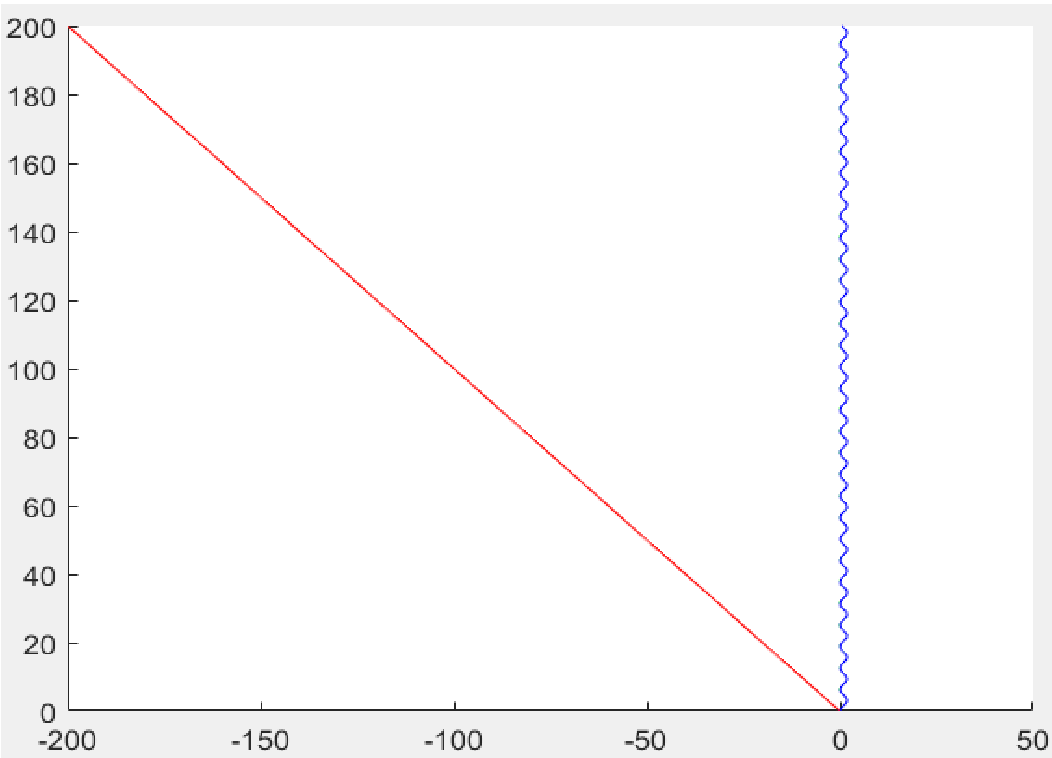
$$\boldsymbol{d\omega = -\frac{g}{l}\sin(\phi)\,dt} \qquad (eq.2)$$

```
function [ phi, omega ] = MakeStepEuler( phi, omega, dt, g, l)
%MakeStepEuler Calculate one Euler step

eq1 = inline('(-g/l)*sin(p)');
eq2 = inline('om');

om = omega + dt*eq1(g,l,phi);
p = phi + dt*eq2(omega);
omega=om;
phi=p;
end
```



Q 7.2 :

```
ds = 0.001;
dt = 0.001;
fx = @(t) sin(t);
fx1 = @(t,y) -cos(t)+cos(t-y);
a = 0; b = 200;
N = (b - a)/ds;
s = linspace(a, b, N);
t = linspace(0, b, N);
x_stream = zeros(N,1);
x_path = zeros(N, 1);
x_streak =zeros(N, 1);
y = zeros(N,1);
for i=2:N
%Euler steps on streamline eq
    x_stream(i) = x_stream(i-1) + ds*fx(3*pi/2);
%Euler steps on pathline eq
    x_path(i) = x_path(i-1) + dt*fx(t(i-1));
%Substituting values in the Streakline eq
    x_streak(i) = fx1(t(i),s(i));
    y(i) =  y(i-1) + dt;
end
hold on
plot(x_stream, y,'r');
plot(x_path, y,'g');
plot(x_streak, y,'b');
```



The stream line equations are:

$\frac{dx}{ds} = \sin t$ and $\frac{dy}{ds} = 1$

Integrating:

$x - x_0 = (s - s_0) \sin t$ and $(y - y_0) = (s - s_0)$

$\therefore x - x_0 = (y - y_0) \sin t$

$\because (x_0, y_0) = (0, 0)$

$\therefore$the stream line equation is $\boldsymbol{x = y \sin t}$

The path line equations are:

$\boldsymbol{\frac{dx}{dt} = \sin t}$ and $\boldsymbol{\frac{dy}{dt} = 1}$

Integrating,

$x = -\cos t + c_1$

and

$y = t + c_2$

Let the particles pass through (x_0, y_0) at time t'.

$\therefore x_0 = -\cos t' + c_1$

$\Rightarrow c_1 = x_0 + \cos t'$

$y_0 = t' + c_2$

$\Rightarrow c_2 = y_0 - t'$

$\therefore x = -\cos t + x_0 + \cos t'$ and $y = t + y_0 - t'$

$\therefore t' = t + y_0 - y$

$\therefore x = -\cos t + x_0 + \cos(t + y_0 - y)$

$\because (x_0, y_0) = (0,0)$

$\therefore x = -\cos t + \cos(t - y)$

Hence the equation of the streakline is $\boldsymbol{x = -\cos t + \cos(t - y)}$.

The streakline should be at constant time 3pi/2, t_c is the end time, so integration from 0 to 3pi/2

Q 7.3

The given hyperbolic partial differential equation is: $\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0$, in the domain $\Omega \times \mathbb{R}$ using the Dirichlet boundary condition $u = 0$ in $\partial\Omega \times \mathbb{R}$. The initial values are the position $u(\vec{x}, t = 0) = u_0(\vec{x})$ and the velocity $\frac{\partial u}{\partial t}(\vec{x}, t = 0) = u_1(\vec{x})$.

(a)

Let $u(\vec{x}, t) = X(\vec{x}) \cdot T(t)$.

$$\therefore \frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0$$
$$\Rightarrow \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$
$$\Rightarrow X(\vec{x}) \cdot \frac{d^2 T}{dt^2} - c^2 \cdot T(t) \cdot \frac{d^2 X}{dx^2} = 0$$
$$\Rightarrow X(\vec{x}) \cdot \frac{d^2 T}{dt^2} = c^2 \cdot T(t) \cdot \frac{d^2 X}{dx^2}$$
$$\Rightarrow \frac{1}{c^2} \cdot \frac{1}{T(t)} \cdot \frac{d^2 T}{dt^2} = \frac{1}{X(\vec{x})} \cdot \frac{d^2 X}{dx^2}$$

Let $\frac{1}{c^2} \cdot \frac{1}{T(t)} \cdot \frac{d^2 T}{dt^2} = \frac{1}{X(\vec{x})} \cdot \frac{d^2 X}{dx^2} = -\lambda$

$\therefore$ the given PDE can be decomposed into two ODEs of second order:

$$\frac{1}{c^2} \cdot \frac{1}{T(t)} \cdot \frac{d^2 T}{dt^2} = -\lambda \Rightarrow \frac{d^2 T}{dt^2} = -c^2 \lambda T$$

and

$$\frac{1}{X(\vec{x})} \cdot \frac{d^2 X}{dx^2} = -\lambda \Rightarrow -\Delta X(\vec{x}) = \lambda X(\vec{x})$$

(b)

The boundary and initial conditions after the separation of variables can be written as:

$$u(0, t) = u(l, t) = X(0)T(t) = X(l)T(t) = 0$$
$$u(\vec{x}, t = 0) = X(\vec{x})T(0) = u_0(\vec{x})$$
$$\frac{\partial u}{\partial t}(\vec{x}, t = 0) = X(\vec{x}) \cdot \frac{dT(0)}{dt} = u_1(\vec{x})$$

Case I: $\lambda < 0,\ \lambda = -k^2$

$$\therefore -\Delta X(x) = -k^2 X(x)$$
$$\Rightarrow \Delta X - k^2 X = 0$$

The characteristic equation will be

$$r^2 - k^2 = 0 \Rightarrow r = \pm k$$

$\therefore$ The solution is of the form $X(x) = Ae^{kx} + Be^{-kx}$, where A and B are constants.

Putting the boundary condition,

$$X(0) = Ae^0 + Be^0 = A + B = 0 \Rightarrow A = -B$$
$$X(l) = Ae^{kl} + Be^{-kl} = 0 \Rightarrow -Be^{kl} + Be^{-kl} = 0 \Rightarrow B(e^{kl} - e^{-kl}) = 0$$

$\therefore$ either $B = 0$ or $e^{kl} = e^{-kl} \Rightarrow e^{2kl} = 1$, which is not possible since $k \neq 0$ and $l \neq 0$. Hence $A = B = 0$

Case II: $\lambda = 0$

Then the solution is of the form $X(x) = Ax + B$. Again considering the boundary conditions,

$X(0) = B = 0$

and $X(l) = Al = 0 \Rightarrow A = 0$.

Case III: $\lambda > 0$

The solution will be of the form $X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$.

From the boundary conditions,

$X(0) = A\cos 0 + B\sin 0 = A = 0$ and

$X(l) = A\cos(\sqrt{\lambda}l) + B\sin(\sqrt{\lambda}l) = B\sin(\sqrt{\lambda}l) = 0$.

This means either $B = 0$ or $\sin(\sqrt{\lambda}l) = 0 \Rightarrow \sqrt{\lambda}l = n\pi\ [n = 1,2,\ldots] \Rightarrow \lambda = \left(\frac{n\pi}{l}\right)^2$

Hence $X(x) = B_n \sin\left(\frac{n\pi x}{l}\right)$.