Theoretical and Methodological Foundations of Visual Computing

Assignment 1

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Exercise 1.1:

A: According to the question,

$\mathbb{A} = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ and $\vec{\mathbb{V}} = \vec{\mathbb{R}}$. The composition operation is defined as $(x, y) \oplus (\alpha) = (xcos\alpha - ysin\alpha,\ xsin\alpha + ycos\alpha)$.

$(\mathbb{A}, \vec{\mathbb{V}}, \oplus)$ is an affine space as it satisfies the conditions to be an affine space, as follows:

1. **For every $\boldsymbol{a} \in \mathbb{A}$: $\boldsymbol{a} \oplus \vec{0} = a$:**

   When $\alpha = 0$, $(x, y) \oplus (\alpha) = (xcos0 - ysin0, xsin0 + ycos0) = (x, y)$

   $$[\because cos0 = 1 \; and \; sin0 = 0]$$

   Hence, the first condition is satisfied

2. **For every ordered pair $(\boldsymbol{p}, \boldsymbol{q}) \in \mathbb{A}$, there is a unique $\vec{\boldsymbol{v}} = \overrightarrow{\boldsymbol{pq}} \in \vec{\mathbb{V}}$ such that $\boldsymbol{p} \oplus \vec{\boldsymbol{v}} = \boldsymbol{q}$:**

   Let $p = (x, y) \in \mathbb{A}$

   $\therefore\ x^2 + y^2 = 1$

   Let $q = (x', y') = (x, y) \oplus \alpha = (xcos\alpha - ysin\alpha, xsin\alpha + ycos\alpha)$, for any $\vec{\alpha} \in \vec{\mathbb{V}}$

   To be in $\mathbb{A}$, $x'^2 + y'^2$ should be 1.

   $$\begin{aligned} x'^2 + y'^2 &= (xcos\alpha - ysin\alpha)^2 + (xsin\alpha + ycos\alpha)^2 \\ &= x^2cos^2\alpha + y^2sin^2\alpha - 2xysin\alpha cos\alpha + x^2sin^2\alpha \\ &\quad + y^2cos^2\alpha + 2xysin\alpha cos\alpha \\ &= x^2 + y^2 \\ &= 1 \end{aligned}$$

   $\therefore q \in \mathbb{A}$

$\therefore$ For every ordered pair $(p, q) \in \mathbb{A}$, $\vec{v}$ is the unique vector such that $p \oplus \vec{v} = q$

3. **For every $\boldsymbol{a} \in \mathbb{A}$ and every $\vec{u}, \vec{v} \in \vec{\mathbb{V}}$: $(a \oplus \vec{u}) \oplus \vec{v} = a \oplus (\vec{u} + \vec{v})$:**

   Let $a = (x, y) \in \mathbb{A}$ and $\overrightarrow{\alpha_1}, \overrightarrow{\alpha_2} \in \vec{\mathbb{V}}$

   $\therefore (a \oplus \overrightarrow{\alpha_1}) \oplus \overrightarrow{\alpha_2}$

   $= ((x, y) \oplus \overrightarrow{\alpha_1}) \oplus \overrightarrow{\alpha_2}$

   $= (xcos\alpha_1 - ysin\alpha_1, xsin\alpha_1 + ycos\alpha_1) \oplus \overrightarrow{\alpha_2}$

   $$= \big((xcos\alpha_1 - ysin\alpha_1)cos\alpha_2 - (xsin\alpha_1 + ycos\alpha_1)sin\alpha_2, (xcos\alpha_1 - ysin\alpha_1)sin\alpha_2 + (xsin\alpha_1 + ycos\alpha_1)cos\alpha_2\big)$$

   $$= \big((xcos\alpha_1cos\alpha_2 - ysin\alpha_1cos\alpha_2 - xsin\alpha_1sin\alpha_2 - ycos\alpha_1sin\alpha_2), (xcos\alpha_1sin\alpha_2 - ysin\alpha_1sin\alpha_2 + xsin\alpha_1cos\alpha_2 + ycos\alpha_1cos\alpha_2)\big)$$

   $$= \big(x(cos\alpha_1cos\alpha_2 - sin\alpha_1sin\alpha_2) - y(sin\alpha_1cos\alpha_2 + cos\alpha_1sin\alpha_2), x(cos\alpha_1sin\alpha_2 + sin\alpha_1cos\alpha_2) + y(cos\alpha_1cos\alpha_2 - sin\alpha_1sin\alpha_2)\big)$$

   $$= (xcos(\alpha_1 + \alpha_2) - ysin(\alpha_1 + \alpha_2), xsin(\alpha_1 + \alpha_2) + ycos(\alpha_1 + \alpha_2))$$

   Now,

   $a \oplus (\overrightarrow{\alpha_1} + \overrightarrow{\alpha_2})$

   $= (x, y) \oplus (\overrightarrow{\alpha_1} + \overrightarrow{\alpha_2})$

   $= \big(xcos(\alpha_1 + \alpha_2) - ysin(\alpha_1 + \alpha_2), xsin(\alpha_1 + \alpha_2) + ycos(\alpha_1 + \alpha_2)\big)$

   $= (a \oplus \overrightarrow{\alpha_1}) \oplus \overrightarrow{\alpha_2}$

Hence, $(\mathbb{A}, \vec{\mathbb{V}}, \oplus)$ satisfies all the three conditions of an affine space and hence is an affine space.

Exercise 1.2:

A: According to the given problem,

$\mathbb{A}_1 = \mathbb{A}_2 = \mathbb{A}, \mathbb{V}_1 = \mathbb{V}_2 = \mathbb{V}, \ \overrightarrow{v_0} \in \mathbb{V}$ is fixed. $F(a) := a \oplus \overrightarrow{v_0}$

Let $a, b \in \mathbb{A}$ and $\overrightarrow{ab} = \vec{v}$ be the vector from a to b

$\therefore b = a \oplus \vec{v}$

Let $F(a) = a \oplus \overrightarrow{v_0} = a'$ and $F(b) = b \oplus \overrightarrow{v_0} = b'$

$\therefore F(b) = b \oplus \overrightarrow{v_0}$

$$= a \oplus \vec{v} \oplus \overrightarrow{v_0}$$

$$= a \oplus (\vec{v} + \overrightarrow{v_0})$$

$$= a \oplus (\overrightarrow{v_0} + \vec{v})$$

$$= a \oplus (\overrightarrow{v_0} + \vec{v})$$

$$= a \oplus \overrightarrow{v_0} \oplus \vec{v}$$

$$= (a \oplus \overrightarrow{v_0}) \oplus \vec{v}$$

$$= a' \oplus \vec{v} = b'$$

$\therefore \overrightarrow{a'b'} = \overrightarrow{ab} = \vec{v}$, i.e. the vector $\vec{v}$ remains unchanged.

$\therefore f\left(\overrightarrow{ab}\right) = \overrightarrow{F(a)F(b)} = \overrightarrow{a'b'}$ and for any $\vec{v} \in \mathbb{V}, f(\vec{v}) = \vec{v}$

$\therefore F: \mathbb{A}_1 \to \mathbb{A}_2$ is an affine map

To prove that the given affine map is one-to-one:

Let a and b be two points in $\mathbb{A}$ such that $F(a) = F(b)$

$\therefore F(a) = F(b)$

$or\ a \oplus \overrightarrow{v_0} = b \oplus \overrightarrow{v_0}$

$or\ a \oplus \overrightarrow{v_0} \oplus (-\overrightarrow{v_0}) = b \oplus \overrightarrow{v_0} \oplus (-\overrightarrow{v_0})$

$or\ a \oplus \{\overrightarrow{v_0} + (-\overrightarrow{v_0})\} = b \oplus \{\overrightarrow{v_0} + (-\overrightarrow{v_0})\}$

$or\ a \oplus \vec{0} = b \oplus \vec{0}$

$or\ a = b,$

Which indicates that the affine map is injective.

$\forall a \in \mathbb{A}, \exists\ a' \in \mathbb{A}, such\ that$:

$$a' = a \oplus \overrightarrow{v_0}$$

$\Rightarrow\ a' \oplus (-\overrightarrow{v_0}) = \ a \oplus \overrightarrow{v_0} \oplus (-\overrightarrow{v_0})$

$\Rightarrow \; a' \oplus (-\overrightarrow{v_0}) = \; a \oplus (\overrightarrow{v_0} + -\overrightarrow{v_0})$

$\Rightarrow \; a' \oplus (-\overrightarrow{v_0}) = \; a \oplus \vec{0}$

$\Rightarrow \; a' \oplus (-\overrightarrow{v_0}) = \; a$

$\therefore \; \forall \, a' \in \mathbb{A}$, there exists a pre-image a

Also, since $\mathbb{A}_1 = \mathbb{A}_2 = \mathbb{A}, n(\mathbb{A}_1) - n(\mathbb{A}_2) = 0$

$\therefore$ All elements in $\mathbb{A}$ have a pre-image as there are no additional elements in set $\mathbb{A}_2$, and the function is surjective.

$\therefore$the affine map is both injective and surjective and hence bijective.

$\therefore$the given affine map is an affinity

Exercise 1.3:

1. **A:**

```
>> A(1:3, 2:3)

ans =

     2     3
     6     7
    10    11
```

2. **A:**

```
>> A(2:3,2)

ans =

     6
    10
```

3. **A:**

```
>> a=0;
while a<10
a=a+3;
end
disp(a)
    12
```

In order to display 4 7 10 13, the code will be:

```
>> a=1;
while a<11
a=a+3;
disp(a);
end
     4

     7

    10

    13
```

4. **A:**

```
num = randi([1,6]);
if num == 1 | num == 2
   disp('Try again!');
elseif num == 5 | num == 6
      disp('Great!');
else
   disp('Good!');
end
```

5. **A:**

```
fid = fopen('test.txt', 'r');
x = fscanf(fid, '%f');
disp(x);
```

6. **A:**

```
t1 = linspace(0, 1, 1000);
t2 = linspace(1, 6, 1000);
y1 = t1.^2;
y2 = t2.^3;
t = [t1, t2];
y = [y1, y2];
figure
plot(t,y);
```