ASSIGNMENT 06

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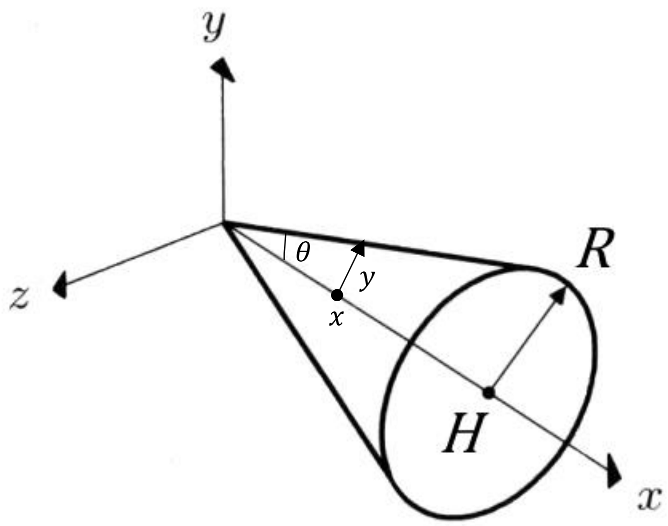
Q 6.1:

To compute the surface area of the given cone with radius R and height H, let us consider a disc centred on the X-axis with infinitesimal height dx and radius y.



Now, $\tan\theta = \frac{y}{x} = \frac{R}{H}$

$\therefore y = \dfrac{R}{H}x$

The area of the side surface of the disc will be $2\pi y dx$.

$\therefore$The surface area of the cone = $\int_0^H 2\pi y dx = \int_0^H \frac{2\pi R}{H} x dx = \frac{2\pi R}{H}\left[\frac{x^2}{2}\right]_0^H = \frac{2\pi R}{H} \times \frac{H^2}{2} = \pi RH.$

Q 6.2:

$\rho(\theta, \phi) = c\cos\theta\sin\theta$, where $\rho\colon \mathbb{R}^2 \longrightarrow \mathbb{R},\ \theta \in \left[0, \frac{\pi}{2}\right],\ \phi \in [0, 2\pi]$.

(a) Normalization constant:

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}} c\cos\theta\sin\theta\, d\theta\, d\phi = 1$$

$$or\ \frac{1}{2}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} c\sin 2\theta\, d\theta\, d\phi = 1$$

$$or\ \frac{c}{4}\int_0^{2\pi} [-\cos 2\theta]_0^{\frac{\pi}{2}}\ d\phi = 1$$

$$or\ -\frac{c}{4}\int_0^{2\pi} (-1-1) d\phi = 1$$

$$or\ \frac{c}{2}\int_0^{2\pi} d\phi = 1$$

$$or\ \frac{c}{2}\times 2\pi = 1$$

$$or\ c = \frac{1}{\pi}$$

(b) Marginal Density Function:

$$\rho(\theta) = \int_0^{2\pi} \rho(\theta,\phi)\, d\phi = \int_0^{2\pi} c\cos\theta\sin\theta\, d\phi = c\cos\theta\sin\theta \int_0^{2\pi} d\phi$$

$$= \frac{\cos\theta\sin\theta}{\pi}(2\pi) = 2\cos\theta\sin\theta = \sin 2\theta$$

Conditional Density Function:

$$\rho(\phi|\theta) = \frac{\rho(\theta,\phi)}{\rho(\theta)} = \frac{c\cos\theta\sin\theta}{\sin 2\theta} = \frac{1}{\pi}\times\frac{\cos\theta\sin\theta}{2\sin\theta\cos\theta} = \frac{1}{2\pi}$$

(c) $P(\theta) = \int_0^{\theta} \sin 2\theta' d\theta' = \left[-\frac{\cos 2\theta'}{2}\right]_0^{\theta} = \frac{1}{2}[1-\cos 2\theta]$

$$P(\phi|\theta) = \int_0^{\phi} \frac{1}{2\pi}\ d\phi' = \frac{\phi}{2\pi}$$

Sampling functions with $\xi_1, \xi_2 \in [0,1]$:

$$\xi_1 = \frac{1}{2}[1-\cos 2\theta] \Rightarrow \cos 2\theta = 1 - 2\xi_1 \Rightarrow \theta = \frac{1}{2}\mathrm{acos}(1-2\xi_1)$$

and

$$\phi = 2\pi\xi_2$$

Q 6.3:

$$y' = 2x\big(y(x)\big)^2, \qquad y(0) = y_0 > 0$$

$$\frac{dy}{dx} = 2xy^2$$

$$\Rightarrow\ \frac{dy}{y^2} = 2xdx$$

$$\Rightarrow\ \int_{y(x_0)=y_0}^{y(x)} \frac{dy}{y^2} = \int_{x_0}^{x} 2xdx$$

Let $s = y(x) \quad \therefore\ ds = dy$

$$\int_{y(x_0)=y_0}^{y(x)} \frac{ds}{s^2} = \int_{x_0}^{x} 2x dx$$

$$\Rightarrow \left[-\frac{1}{s}\right]_{y_0}^{y(x)} = [x^2]_{x_0}^{x}$$

$$\Rightarrow \frac{1}{y_0} - \frac{1}{y} = x^2$$

$$\Rightarrow y = \frac{y_0}{1 - x^2 y_0}$$

$1 - x^2 y_0 \neq 0 \Rightarrow x \neq \frac{1}{\sqrt{y_0}}$.

$\therefore x \in I_0$, where $I_0 = [0, \infty) - \left\{\frac{1}{\sqrt{y_0}}\right\}$.