Analysis of weighted networks

3. Multiplication of networks

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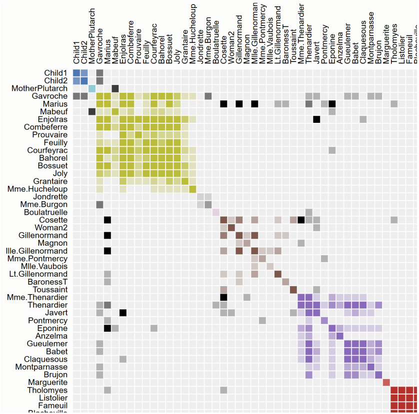
Outline



Les Misérables

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Current version of slides (June 21, 2025 at 03 : 51): slides PDF

`https://github.com/bavla/Nets`

Work in progress

Conclusions

Conclusions

Multiplication of networks

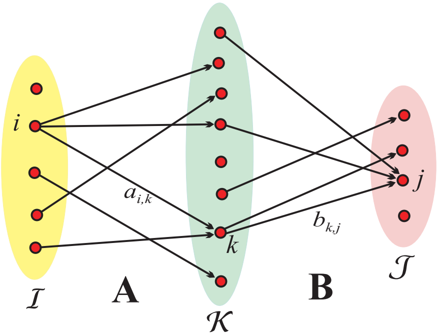
To a simple (no parallel arcs) two-mode *network* $\mathcal{N} = (\mathcal{I}, \mathcal{J}, \mathcal{A}, w)$; where $\mathcal{I}$ and $\mathcal{J}$ are sets of *nodes*, $\mathcal{A}$ is a set of *arcs* linking $\mathcal{I}$ and $\mathcal{J}$, and $w : \mathcal{A} \to \mathbb{R}$ (or some other semiring) is a *weight*; we can assign a *network matrix* $\mathbf{W} = [w_{i,j}]$ with elements: $w_{i,j} = w(i,j)$ for $(i,j) \in \mathcal{A}$ and $w_{i,j} = 0$ otherwise.

Given a pair of compatible networks $\mathcal{N}_A = (\mathcal{I}, \mathcal{K}, \mathcal{A}_A, w_A)$ and $\mathcal{N}_B = (\mathcal{K}, \mathcal{J}, \mathcal{A}_B, w_B)$ with corresponding matrices $\mathbf{A}_{\mathcal{I}\times\mathcal{K}}$ and $\mathbf{B}_{\mathcal{K}\times\mathcal{J}}$ we call a *product of networks* $\mathcal{N}_A$ and $\mathcal{N}_B$ a network $\mathcal{N}_C = (\mathcal{I}, \mathcal{J}, \mathcal{A}_C, w_C)$, where $\mathcal{A}_C = \{(i,j) : i \in \mathcal{I}, j \in \mathcal{J}, c_{i,j} \neq 0\}$ and $w_C(i,j) = c_{i,j}$ for $(i,j) \in \mathcal{A}_C$. The product matrix $\mathbf{C} = [c_{i,j}]_{\mathcal{I}\times\mathcal{J}} = \mathbf{A} * \mathbf{B}$ is defined in the standard way

$$c_{i,j} = \sum_{k \in \mathcal{K}} a_{i,k} \cdot b_{k,j}$$

In the case when $\mathcal{I} = \mathcal{K} = \mathcal{J}$ we are dealing with ordinary one-mode networks (with square matrices).

Multiplication of networks



$$c_{i,j} = \sum_{k \in N_A(i) \cap N_B^-(j)} a_{i,k} \cdot b_{k,j}$$

If all weights in networks $\mathcal{N}_A$ and $\mathcal{N}_B$ are equal to 1 the value of $c_{i,j}$ counts the number of ways we can go from $i \in \mathcal{I}$ to $j \in \mathcal{J}$ passing through $\mathcal{K}$, $c_{i,j} = |N_A(i) \cap N_B^-(j)|$.

Multiplication of networks

The standard matrix multiplication has the complexity $O(|\mathcal{I}| \cdot |\mathcal{K}| \cdot |\mathcal{J}|)$ – it is too slow to be used for large networks. For sparse large networks we can multiply much faster considering only nonzero elements.

for k **in** $\mathcal{K}$ **do**
  for (i, j) **in** $N_A^-(k) \times N_B(k)$ **do**
    if $\exists c_{i,j}$ **then** $c_{i,j} := c_{i,j} + a_{i,k} \cdot b_{k,j}$
    else new $c_{i,j} := a_{i,k} \cdot b_{k,j}$

`Networks/Multiply Networks`

In general the multiplication of large sparse networks is a 'dangerous' operation since the result can 'explode' – it is not sparse.

Multiplication of networks

From the network multiplication algorithm we see that each intermediate node $k \in \mathcal{K}$ adds to a product network a complete two-mode subgraph $K_{N_A^-(k), N_B(k)}$ (or, in the case $\mathcal{I} = \mathcal{J}$, a complete subgraph $K_{N(k)}$). If both degrees $\deg_A(k) = |N_A^-(k)|$ and $\deg_B(k) = |N_B(k)|$ are large then already the computation of this complete subgraph has a quadratic (time and space) complexity – the result 'explodes'.

If at least one of the sparse networks $\mathcal{N}_A$ and $\mathcal{N}_B$ has small maximal degree on $\mathcal{K}$ then also the resulting product network $\mathcal{N}_C$ is sparse.

If for the sparse networks $\mathcal{N}_A$ and $\mathcal{N}_B$ there are in $\mathcal{K}$ only few nodes with large degree and no one among them with large degree in both networks then also the resulting product network $\mathcal{N}_C$ is sparse.

Projections of two-mode networks

Often we transform a two-mode network $\mathcal{N} = (\mathcal{U}, \mathcal{V}, \mathcal{E}, w)$ into an ordinary (one-mode) network $\mathcal{N}_1 = (\mathcal{U}, \mathcal{E}_1, w_1)$ or/and $\mathcal{N}_2 = (\mathcal{V}, \mathcal{E}_2, w_2)$, where $\mathcal{E}_1$ and w_1 are determined by the matrix $\mathbf{W}^{(1)} = \mathbf{W}\mathbf{W}^T$,

$$w^{(1)}_{uv} = \sum_{z \in \mathcal{V}} w_{uz} \cdot w^T_{zv} = \sum_{z \in \mathcal{V}} w_{uz} \cdot w_{vz} .$$

Evidently $w^{(1)}_{uv} = w^{(1)}_{vu}$. There is an edge $(u : v) \in \mathcal{E}_1$ in $\mathcal{N}_1$ iff $N(u) \cap N(v) \neq \emptyset$. Its weight is $w_1(u, v) = w^{(1)}_{uv}$.

The network $\mathcal{N}_2$ is determined in a similar way by the matrix $\mathbf{W}^{(2)} = \mathbf{W}^T\mathbf{W}$.
The networks $\mathcal{N}_1$ and $\mathcal{N}_2$ are analyzed using standard methods.

`Network/2-Mode Network/2-Mode to 1-Mode/Rows`

Derived networks

WA – works $\times$ authors – authorship network
WK – works $\times$ keywords
Ci – works $\times$ works – citation network
AK $=$ **WA**^{T} $*$ **WK** – authors $\times$ keywords
Co $=$ **WA**^{T} $*$ **WA** – coauthorship
ACi $=$ **WA**^{T} $*$ **Ci** $*$ **WA** – citations between authors

co_{ij} = the number of works that authors i and j wrote together
It holds: $co_{ij} = co_{ji}$. Using the weights co_{ij} we can determine the Salton's cosine similarity or Ochiai coefficient between authors i and j as

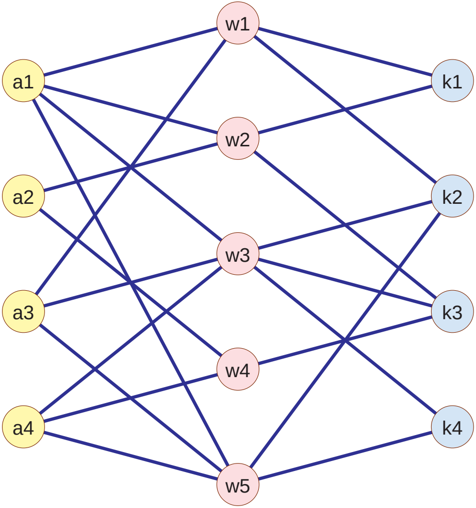
$$S(i,j) = \cos(i,j) = \frac{co_{ij}}{\sqrt{co_{ii}\, co_{jj}}}$$

Properties of Salton index

The Salton index has the following properties

1. $S(i,j) \in [-1,1]$
2. $S(i,j) = S(j,i)$
3. $S(i,i) = 1$
4. $wa_{pi} \in \mathbb{R}_0^+ \Rightarrow S(u,t) \in [0,1]$
5. $S(\alpha i, \beta j) = S(i,j), \quad \alpha, \beta > 0$
6. $S(\alpha i, i) = 1, \quad \alpha > 0$

Outer product decomposition



$$\mathbf{WA}^T \cdot \mathbf{WK}$$

For vectors $x = [x_1, x_2, \ldots, x_n]$ and $y = [y_1, y_2, \ldots, y_m]$ their *outer*

Example A

As an example let us take the binary network matrices **WA** and **WK**:

$$\mathbf{WA} = \begin{array}{c} \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{array} \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right] \end{array}, \quad \mathbf{WK} = \begin{array}{c} \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{array} \begin{array}{c} \begin{array}{cccc} k_1 & k_2 & k_3 & k_4 \end{array} \\ \left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \end{array}$$

and compute the product $\mathbf{H} = \mathbf{WA}^T \cdot \mathbf{WK}$. We get a network matrix $\mathbf{H}$ which can be decomposed as

Conclusions

$$\begin{array}{c|cccc} \mathbf{H} & k_1 & k_2 & k_3 & k_4 \\ \hline a_1 & 2 & 3 & 2 & 2 \\ a_2 & 1 & 0 & 2 & 0 \\ a_3 & 1 & 3 & 1 & 2 \\ a_4 & 0 & 2 & 2 & 2 \end{array} = \begin{array}{c|cccc} \mathbf{H}_1 & k_1 & k_2 & k_3 & k_4 \\ \hline a_1 & 1 & 1 & 0 & 0 \\ a_2 & 0 & 0 & 0 & 0 \\ a_3 & 1 & 1 & 0 & 0 \\ a_4 & 0 & 0 & 0 & 0 \end{array} + \begin{array}{c|cccc} \mathbf{H}_2 & k_1 & k_2 & k_3 & k_4 \\ \hline a_1 & 1 & 0 & 1 & 0 \\ a_2 & 1 & 0 & 1 & 0 \\ a_3 & 0 & 0 & 0 & 0 \\ a_4 & 0 & 0 & 0 & 0 \end{array}$$

$$\begin{array}{c|cccc} \mathbf{H}_3 & k_1 & k_2 & k_3 & k_4 \\ \hline a_1 & 0 & 1 & 1 & 1 \\ a_2 & 0 & 0 & 0 & 0 \\ a_3 & 0 & 1 & 1 & 1 \\ a_4 & 0 & 1 & 1 & 1 \end{array} + \begin{array}{c|cccc} \mathbf{H}_4 & k_1 & k_2 & k_3 & k_4 \\ \hline a_1 & 0 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 1 & 0 \\ a_3 & 0 & 0 & 0 & 0 \\ a_4 & 0 & 0 & 1 & 0 \end{array} + \begin{array}{c|cccc} \mathbf{H}_5 & k_1 & k_2 & k_3 & k_4 \\ \hline a_1 & 0 & 1 & 0 & 1 \\ a_2 & 0 & 0 & 0 & 0 \\ a_3 & 0 & 1 & 0 & 1 \\ a_4 & 0 & 1 & 0 & 1 \end{array}$$

Derived networks

We can use the multiplication to obtain new networks from existing *compatible* two-mode networks. For example, from basic bibliographic networks $\mathbf{WA}$ and $\mathbf{WK}$ we get

$$\mathbf{AK} = \mathbf{WA}^T \cdot \mathbf{WK}$$

a network relating authors to keywords used in their works, and

$$\mathbf{Ca} = \mathbf{WA}^T \cdot \mathbf{Ci} \cdot \mathbf{WA}$$

is a network of citations between authors.
Networks obtained from existing networks using some operations are called *derived* networks. They are very important in analysis of collections of *linked* networks.
What is the meaning of the product network? In general we could consider weights, addition and multiplication over a selected semiring [**?**]. In this paper we will limit our attention to the traditional addition and multiplication of real numbers.
The weight $\mathbf{AK}[a, k]$ is equal to the number of times the author a used the keyword k in his/her works.

Conclusions

Using network multiplication we can also transform a given two-mode network, for example $\mathbf{WA}$, into corresponding ordinary one-mode networks (*projections*)

$$\mathbf{WW} = \mathbf{WA} \cdot \mathbf{WA}^T \qquad \text{and} \qquad \mathbf{AA} = \mathbf{WA}^T \cdot \mathbf{WA}$$

The obtained projections can be analyzed using standard network analysis methods. This is a traditional recipe how to analyze two-mode networks. Often the weights are not considered in the analysis; and when they are considered we have to be very careful about their meaning.

The weight $\mathbf{WW}[p, q]$ is equal to the number of common authors of works p and q.

The weight $\mathbf{AA}[a, b]$ is equal to the number of works that author a and b coauthored. In a special case when $a = b$ it is equal to the number of works that the author a wrote. The network $\mathbf{AA}$ is describing the *coauthorship* (collaboration) between authors and is also denoted as $\mathbf{Co}$ – the "first" coauthorship network.

Conclusions

In the paper [?] it was shown that there can be problems with the network **Co** when we try to use it for identifying the most collaborative authors. By the outer product decomposition the coauthorship network **Co** is composed of complete subgraphs on the set of work's coauthors. Works with many authors produce large complete subgraphs, thus bluring the collaboration structure, and are over-represented by its total weight. To see this, let $S_x = \sum_i x_i$ and $S_y = \sum_j y_j$ then the *contribution* of the outer product $x \circ y$ is equal

$$T = \sum_{i,j} (x \circ y)_{ij} = \sum_i \sum_j x_i \cdot y_j = \sum_i x_i \cdot \sum_j y_j = S_x \cdot S_y$$

In general each term $\mathbf{H}_w$ in the outer product decomposition of the product **C** has different total weight $T(\mathbf{H}_w) = \sum_{a,k} (\mathbf{H}_w)_{ak}$ leading to over-representation of works with large values. In the case of coautorship network **Co** we have $S(\mathbf{WA}[w, .]) = \text{outdeg}_{\mathbf{WA}}(w)$ and therefore $T(\mathbf{H}_w) = \text{outdeg}_{\mathbf{WA}}(w)^2$. To resolve the problem we apply the fractional approach.

Fractional approach

To make the contributions of all works equal we can apply the *fractional* approach by normalizing the weights: setting $x' = x/S_x$ and $y' = y/S_y$ we get $S_{x'} = S_{y'} = 1$ and therefore $T(\mathbf{H}'_w) = 1$ for all works w.

In the case of two-mode networks **WA** and **WK** we denote

$$S_w^{\mathbf{WA}} = \begin{cases} \sum_a \mathbf{WA}[w,a] & \text{outdeg}_{\mathbf{WA}}(w) > 0 \\ 1 & \text{outdeg}_{\mathbf{WA}}(w) = 0 \end{cases}$$

(and similarly $S_w^{\mathbf{WK}}$) and define the *normalized* matrices

$$\mathbf{WAn} = \text{diag}(\frac{1}{S_w^{\mathbf{WA}}}) \cdot \mathbf{WA}, \quad \mathbf{WKn} = \text{diag}(\frac{1}{S_w^{\mathbf{WK}}}) \cdot \mathbf{WK}$$

In real life networks **WA** (or **WK**) it can happen that some work has no author. In such a case $S_w^{\mathbf{WA}} = \sum_a \mathbf{WA}[w,a] = 0$ which makes problems in the definition of the normalized network **WAn**. We can bypass the problem by setting $S_w^{\mathbf{WA}} = 1$, as we did in the above definition.

Normalized product

Then the *normalized product* matrix is $\mathbf{AKt} = \mathbf{WAn}^T \cdot \mathbf{WKn}$.

Denoting $\mathbf{F}_w = \dfrac{1}{S_w^{\mathbf{WA}} S_w^{\mathbf{WK}}} \mathbf{H}_w$ the outer product decomposition gets form

$$\mathbf{AKt} = \sum_w \mathbf{F}_w$$

Since

$$T(\mathbf{F}_w) = \begin{cases} 1 & (\mathrm{outdeg}_{\mathbf{WA}}(w) > 0) \wedge (\mathrm{outdeg}_{\mathbf{WK}}(w) > 0) \\ 0 & \text{otherwise} \end{cases}$$

we have further

$$\sum_{a,k} \mathbf{F}[a,k] = \sum_{a,k} \sum_w \mathbf{F}_w[a,k] = \sum_w T(\mathbf{F}_w) = |W^+|$$

where $W^+ = \{w \in W : (\mathrm{outdeg}_{\mathbf{WA}}(w) > 0) \wedge (\mathrm{outdeg}_{\mathbf{WK}}(w) > 0)\}$.
In the network **AKt**, the contribution of each work to the bibliography is 1. These contributions are redistributed to arcs from authors to keywords.

Example B

For matrices from Example A we get the corresponding diagonal normalization matrices

$$\mathrm{diag}\left(\frac{1}{S_w^{\mathbf{WA}}}\right) = \begin{array}{c} \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{array} \begin{array}{c} \begin{array}{ccccc} w_1 & w_2 & w_3 & w_4 & w_5 \end{array} \\ \begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/3 \end{bmatrix} \end{array}$$

$$\mathrm{diag}\left(\frac{1}{S_w^{\mathbf{WK}}}\right) = \begin{array}{c} \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{array} \begin{array}{c} \begin{array}{ccccc} w_1 & w_2 & w_3 & w_4 & w_5 \end{array} \\ \begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1/2 \end{bmatrix} \end{array}$$

compute the normalized matrices

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outer products such as

$$\mathbf{F}_1 = \begin{array}{c c} & \begin{array}{cccc} k_1 & k_2 & k_3 & k_4 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} & \left[\begin{array}{cccc} 1/4 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \qquad \mathbf{F}_5 = \begin{array}{c c} & \begin{array}{cccc} k_1 & k_2 & k_3 & k_4 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} & \left[\begin{array}{cccc} 0 & 1/6 & 0 & 1/6 \\ 0 & 0 & 0 & 0 \\ 0 & 1/6 & 0 & 1/6 \\ 0 & 1/6 & 0 & 1/6 \end{array}\right] \end{array}$$

and finally the product matrix $\mathbf{AKt} = \mathbf{WAn}^T \cdot \mathbf{WKn} =$

$$\sum_{w=1}^{5} \mathbf{F}_w = \begin{array}{c c} & \begin{array}{cccc} k_1 & k_2 & k_3 & k_4 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} & \left[\begin{array}{cccc} 0.50000 & 0.52778 & 0.36111 & 0.27778 \\ 0.25000 & 0.00000 & 0.75000 & 0.00000 \\ 0.25000 & 0.52778 & 0.11111 & 0.27778 \\ 0.00000 & 0.27778 & 0.61111 & 0.27778 \end{array}\right] \end{array}$$

Conclusions

Normalized co-authorship network

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Matrices

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Let $\mathbf{N}$ be the normalized version $\forall p \in W : \sum_{i\in A} n_{pi} \in \{0,1\}$ obtained from $\mathbf{WA}$ by $n_{pi} = wa_{pi} / \max(1, \mathrm{outdeg}_{WA}(p))$, or by some other rule determining the author's contribution – the *fractional* approach. Then the *normalized co-authorship network* is

$$\mathbf{Cn} = \mathbf{N}^T \cdot \mathbf{N}$$

cn_{ij} = the total contribution of 'collaboration' of authors i and j to works.

It holds $cn_{ij} = cn_{ji}$ and $\sum_{i\in A}\sum_{j\in A} n_{pi} n_{pj} = 1$.

The total contribution of a complete subgraph corresponding to the authors of a work p is 1.

$$\sum_{i\in A}\sum_{j\in A} cn_{ij} = |W|$$

Normalized strict co-authorship network

Newman defined a *strict normalization* $\mathbf{N}'$ obtained from $\mathbf{WA}$ by $n'_{pi} = wa_{pi} / \max(1, \mathrm{outdeg}_{WA}(p) - 1)$. Then the *normalized strict co-authorship network* is

$$\mathbf{Cn}' = \mathbf{N}^T \cdot \mathbf{N}'$$

The diagonal (loops) of the so-obtained network $\mathbf{Cn}'$ is set to 0.

The network $\mathbf{Cn}'$ doesn't consider the contribution of single-author works.

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Conclusions

1. Why is Spain the most attractive country?
2. How can the blue between less and high developed countries be reduced?
3. This is exploratory network analysis. Collect and use additional data (neighbors relation, population size, GDP, etc.).
4. Temporal version of the network.

Acknowledgments

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