Projections of weighted two-mode networks

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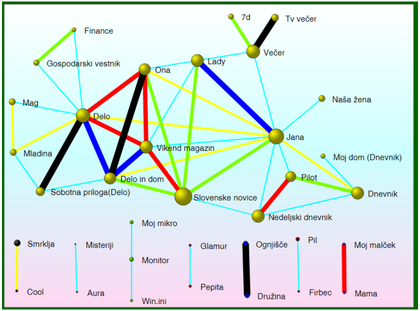
Outline



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Current version of slides (September 20, 2022 at 02 : 21): slides PDF

https://github.com/bavla/NormNet/blob/main/docs/

Two-mode networks

In a *two-mode* (affiliation or bipartite) network $\mathcal{N} = ((U, V), L, w)$ the set of nodes is split into two disjoint sets (*modes*) U and V. Each link $e \in L$ has one end-node in the set U and the other end-node in the set V. The function $w : L \to \mathbb{R}$ assigns to each link its weight.

In general, the weight can be measured on different measurement scales (counts, ratio, interval, ordinal, nominal, binary,TQ, etc.).

Names of Participants of Group I	Code Numbers and Dates of Social Events Reported in *Old City Herald*													
	(1) 6/27	(2) 3/2	(3) 4/12	(4) 9/26	(5) 2/25	(6) 5/19	(7) 3/15	(8) 9/16	(9) 4/8	(10) 6/10	(11) 2/23	(12) 4/7	(13) 11/21	(14) 8/3
1. Mrs. Evelyn Jefferson	×	×	×	×	×	×	....	×	×	....	....	....	....	....
2. Miss Laura Mandeville	×	×	×	....	×	×	×	×	....	....	....	....	....	....
3. Miss Theresa Anderson	....	×	×	×	×	×	×	×	×	....	....	....	....	....
4. Miss Brenda Rogers	×	....	×	×	×	×	×	×	....	....	....	....	....	....
5. Miss Charlotte McDowd	....	....	×	×	×	....	×	....	....	....	....	....	....	....
6. Miss Frances Anderson	....	....	×	....	×	×	....	×	....	....	....	....	....	....
7. Miss Eleanor Nye	....	....	....	....	×	×	×	×	....	....	....	....	....	....
8. Miss Pearl Oglethorpe	....	....	....	....	....	×	....	×	×	....	....	....	....	....
9. Miss Ruth DeSand	....	....	....	....	×	....	×	×	×	....	....	....	....	....
10. Miss Verne Sanderson	....	....	....	....	....	....	×	×	×	....	....	×	....	....
11. Miss Myra Liddell	....	....	....	....	....	....	....	×	×	×	....	×	....	....
12. Miss Katherine Rogers	....	....	....	....	....	....	....	×	×	×	....	×	×	×
13. Mrs. Sylvia Avondale	....	....	....	....	....	....	×	×	×	×	....	×	×	×
14. Mrs. Nora Fayette	....	....	....	....	....	×	×	....	×	×	×	×	×	×
15. Mrs. Helen Lloyd	....	....	....	....	....	....	×	×	....	×	×	×	....	....
16. Mrs. Dorothy Murchison	....	....	....	....	....	....	....	×	×	....	....	....	....	....
17. Mrs. Olivia Carleton	....	....	....	....	....	....	....	....	×	....	×	....	....	....
18. Mrs. Flora Price	....	....	....	....	....	....	....	....	×	....	×	....	....	....

Davis: Southern women, 1941 [9]

Two-mode network matrix and some notions

The network *matrix* **UV** of a two-mode network $\mathcal{UV}$ is defined as

$$UV[u,v] = \begin{cases} w(u,v) & (u,v) \in L \\ \square & \text{otherwise} \end{cases}$$

We represent number 0 with two symbols, 0 (weight 0) and $\square$ (no link) where $\square = 0$ with rules $\square + a = a$ and $\square \cdot a = \square$.

The function $\delta : \{\textbf{false}, \textbf{true}\} \to \{0,1\}$ is determined by $\delta(\textbf{false}) = 0$ and $\delta(\textbf{true}) = 1$. We will also use some additional functions:

out/in-degree $N(u)$ is the set of neighbors of node u
$\mathrm{od}_{UV}(u) = \sum_{v \in V} \delta((u,v) \in L) = |N(u)|$ and
$\mathrm{id}_{UV}(v) = \sum_{u \in U} \delta((u,v) \in L) = |N(v)|$

weighted out/in-degree (row/column sums)
$\mathrm{wod}_{UV}(u) = \sum_{v \in V} UV[u,v]$, $\quad \mathrm{wid}_{UV}(v) = \sum_{u \in U} UV[u,v]$ and
$\mathrm{wod}_{UV}(u/t) = \sum_{v \in N(u) \cap N(t)} UV[u,v]$.

Some notions

It holds $N(u) \cap N(t) \neq \emptyset \Rightarrow \mathrm{wod}_{UV}(u/t) \neq \square$.

We denote $U_{[d]} = \{u \in U : \mathrm{od}(u) \geq d\}$ and
$\mathcal{UV}_{[d]} = ((U_{[d]}, V), L(U_{[d]}), w|U_{[d]})$.
$\hat{U} = \{u \in U : \mathrm{wod}(u) \neq 0\}$

The *total weight* of links in the network $\mathcal{N} = (V, L, w)$

$$T(\mathbf{N}) = \sum_{(u,v)\in L} w(u, v) = \sum_{u,v} N[u, v] = \sum_{u} wod_N(u) = \sum_{v} wid_N(v)$$

Approaches

There are three main approaches to the analysis of two-mode networks:

1. treat the two-mode network as an ordinary one-mode network (degrees, components, etc.) considering a bipartition to sets U and V.
2. apply special methods developed for the analysis of two-mode networks (two-mode hubs and authorities, two-mode cores, 4-ring weights, blockmodeling, etc.).
3. transform (project) the two-mode network to a corresponding one-mode (weighted) network and use the usual methods (link cuts, cores, islands, skeletons, clustering, etc.) to analyze it.

In this talk, we will discuss the last option and limit our attention to numerical and binary scales.

Projections

$\mathcal{UV} = ((U, V), L, w)$ – two-mode network with a network matrix **UV**.

$p : \mathbf{UV} \to \mathbf{VV}$ *projection*, $\mathbf{VV} = [p(v, z)]$; and
$\mathcal{VV}$ the corresponding (ordinary, one-mode) network

1. undirected projection: $p(v, z) = p(z, v)$, resemblance
   - similarity: $p(v, z) \leq \min(p(v, v), p(z, z))$
   - dissimilarity: $p(v, z) \geq \max(p(v, v), p(z, z))$
2. directed projection: $\exists v, z : p(v, z) \neq p(z, v)$ ([2], [11])

Many projections are based on the multiplication of networks.

Multiplication of networks

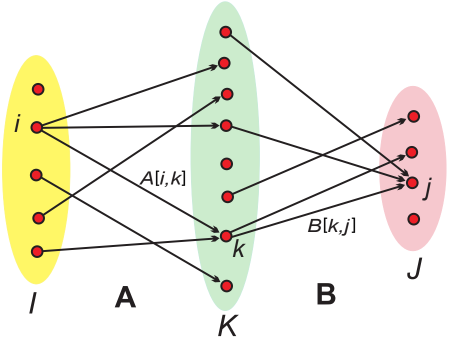
The *product* $\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$ of two compatible matrices $\mathbf{A}_{I \times K}$ and $\mathbf{B}_{K \times J}$ is defined in the standard way

$$C[i,j] = \sum_{k \in K} A[i,k] \cdot B[k,j]$$

(it can be extended to semirings !!!)

The product of two compatible networks $\mathcal{N}_A = ((I,K), L_A, a)$ and $\mathcal{N}_B = ((K,J), L_B, b)$ is the network $\mathcal{N}_C = ((I,J), L_C, c)$ where $L_C = \{(i,j) : c[i,j] \neq \square\}$ and the weight c is determined by the matrix $\mathbf{C}$, $c(i,j) = C[i,j]$.

Multiplication of networks



In binary networks $\mathcal{N}_A$ and $\mathcal{N}_B$, the value of $C[i,j]$ of $\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$ counts the number of ways we can go from the node $i \in I$ to the node $j \in J$ passing through K, $C[i,j] = |N_A(i) \cap N_B(j)|$.

$$C[i,j] = \sum_{k \in N_A(i) \cap N_B(j)} A[i,k] \cdot B[k,j]$$

Standard projections

A standard approach to the analysis of a two-mode network $\mathcal{UV}$ is to transform it into the corresponding one-mode networks determined by:

row projection to U: $\mathbf{UU} = \mathrm{row}(\mathbf{UV}) = \mathbf{UV} \cdot \mathbf{UV}^T$, or
column projection to V: $\mathbf{VV} = \mathrm{col}(\mathbf{UV}) = \mathbf{UV}^T \cdot \mathbf{UV}$

and analyze the obtained weighted network.

$$\mathrm{col}(\mathbf{UV}) = \mathbf{UV}^T \cdot \mathbf{UV} = \mathbf{UV}^T \cdot (\mathbf{UV}^T)^T = \mathrm{row}(\mathbf{UV}^T)$$

$$\mathrm{row}(\mathbf{UV}) = \mathrm{col}(\mathbf{UV}^T)$$

We will limit our discussion to column projections.

Outer product decomposition

For vectors $x = [x_1, x_2, \ldots, x_n]$ and $y = [y_1, y_2, \ldots, y_m]$ their *outer product* $x \circ y$ is defined as a matrix

$$x \circ y = [x_i \cdot y_j]_{n \times m}$$

then we can express the product **C** of two compatible matrices **A** and **B** as the *outer product decomposition*

$$\mathbf{C} = \mathbf{A} \cdot \mathbf{B} = \sum_k \mathbf{H}_k \quad \text{where} \quad \mathbf{H}_k = \mathbf{A}[\cdot, k] \circ \mathbf{B}[k, \cdot],$$

$\mathbf{A}[\cdot, k]$ is the k-th column of matrix **A**, and $\mathbf{B}[k, \cdot]$ is the k-th row of matrix **B**.
On the basis of outer product decomposition we have

$$T(\mathbf{C}) = T(\sum_k \mathbf{H}_k) = \sum_k T(\mathbf{H}_k) \ \text{ and } \ T(\mathbf{H}_k) = \mathrm{wid}_A(k) \cdot \mathrm{wod}_B(k)$$

Structure of projection

In words

1. the product network is a sum of complete subgraphs;
2. the contribution of a node $k \in K$ to the total T is $T(\mathbf{H}_k)$.

This means that the nodes with different weighted degrees in K are not equally represented in the projection.

For a column projection $\mathbf{VV} = \text{col}(\mathbf{UV})$, a real-life network $\mathcal{UV}$ can contain nodes $u \in U$ of degree 0 (in WA, works with no author) and 1 (in WA, single author works). Nodes from U of degree 0 do not contribute to the matrix $\mathbf{VV}$, and nodes of degree 1 contribute only to its neighbor's diagonal entry.

Fractional approach / "stochastic" normalization

$$n(\mathbf{UV}) = [n(UV)[u,v]]$$

$$n(UV)[u,v] = \begin{cases} \dfrac{UV[u,v]}{\mathrm{wod}_{UV}(u)} & u \in \hat{U} \\ 0 & u \notin \hat{U} \end{cases}$$

$$T(n(\mathbf{UV})) = \sum_{u \in U} \mathrm{wod}_{n(UV)}(u) = \sum_{u \in \hat{U}} 1 = |\hat{U}|$$

Interpretation: probabilistic co-linkage.

Fractional co-appearance: $\mathbf{Cn} = n(\mathbf{UV})^T \cdot n(\mathbf{UV})$,
$Cn[v,z] = \sum_{u \in \hat{U}} \frac{UV[u,v] \cdot UV[u,z]}{\mathrm{wod}(u)^2}$
$Cn[v,z] = Cn[z,v]$, $\quad T(u) = T(\mathbf{H}_u) = \mathrm{wod}_{n(UV)}(u)^2$

$$T(\mathbf{Cn}) = \sum_{u \in U} \mathrm{wod}_{n(UV)}(u)^2 = \sum_{u \in \hat{U}} 1 = |\hat{U}|$$

Total weight preserving normalization

$$s(UV)[u,v] = \begin{cases} \dfrac{UV[u,v]}{\sqrt{\mathrm{wod}_{UV}(u)}} & u \in \hat{U} \\ 0 & u \notin \hat{U} \end{cases}$$

$$\mathrm{wod}_{n(UV)}(u) = \begin{cases} 1 & u \in \hat{U} \\ 0 & u \notin \hat{U} \end{cases}, \quad wod_{s(UV)}(u) = \begin{cases} \sqrt{\mathrm{wod}_{UV}(u)} & u \in \hat{U} \\ 0 & u \notin \hat{U} \end{cases}$$

Total weight preserving projection: $\mathbf{Cs} = s(\mathbf{UV})^T \cdot s(\mathbf{UV})$

$$Cs[v,z] = \sum_{u \in \hat{U}} \frac{UV[u,v] \cdot UV[u,z]}{\mathrm{wod}(u)}$$

$$Cs[v,z] = Cs[z,v], \ T(u) = \mathrm{wod}_{s(UV)}(u)^2 = \mathrm{wod}_{UV}(u)$$

$$T(\mathbf{Cs}) = \sum_{u \in U} \mathrm{wod}_{s(UV)}(u)^2 = \sum_{u \in \hat{U}} \mathrm{wod}_{UV}(u) = T(\mathbf{UV})$$

Embedding primary node values

Node values $c : U \to \mathbb{R}_0^+$ – impact factor, number of citations,...

$$x(UV)[u,v] = \begin{cases} \frac{\sqrt{c(u)}}{\mathrm{wod}_{UV}(u)} UV[u,v] & u \in \hat{U} \\ 0 & u \notin \hat{U} \end{cases}$$

$$\mathrm{wod}_{x(UV)}(u) = \begin{cases} \sqrt{c(u)} & u \in \hat{U} \\ 0 & u \notin \hat{U} \end{cases}$$

Embedded primary node values: $\mathbf{Cx} = x(\mathbf{UV})^T \cdot x(\mathbf{UV})$

$$Cx[v,z] = \sum_{u \in \hat{U}} \frac{c(u)}{\mathrm{wod}(u)^2} UV[u,v] \cdot UV[u,z], \quad Cx[v,z] = Cx[z,v]$$

$$T(u) = \sum_{v \in V} \sum_{z \in V} Cx[v,z] = \frac{c(u)}{\mathrm{wod}(u)^2} \sum_{v \in V} UV[u,v] \cdot \sum_{z \in V} UV[u,z] = c(u)$$

$$T(\mathbf{Cx}) = \sum_{u \in U} \mathrm{wod}_{x(UV)}(u)^2 = \sum_{u \in \hat{U}} c(u)$$

Binarization and left and right (fractional) contribution

Binarization: $b(\mathbf{UV})$: $b(UV)[u,v] = \delta(UV[u,v] \neq \square)$
Left contribution: $L(\mathbf{UV}) = \mathbf{UV} \cdot b(\mathbf{UV})^T$

$$L(UV)[u,t] = \sum_{v \in V} UV[u,v] \cdot b(UV)[t,v] = \mathrm{wod}_{UV}(u/t)$$

Left fractional contribution: $\ell(\mathbf{UV}) = n(\mathbf{UV}) \cdot b(\mathbf{UV})^T$

$$\ell(UV)[u,t] = \frac{1}{\mathrm{wod}_{UV}(u)} \sum_{v \in V} UV[u,v] \cdot b(UV)[t,v] = \frac{\mathrm{wod}_{UV}(u/t)}{\mathrm{wod}_{UV}(u)} \leq 1$$

Right fractional contribution: $r(\mathbf{UV}) = b(\mathbf{UV}) \cdot n(\mathbf{UV})^T$

$$r(UV)[u,t] = \frac{1}{\mathrm{wod}_{UV}(t)} \sum_{v \in V} b(UV)[u,v] \cdot UV[t,v] = \frac{\mathrm{wod}_{UV}(t/u)}{\mathrm{wod}_{UV}(t)}$$

$$r(UV)[u,t] = \ell(UV)[t,u]$$

Mean value similarities

$$
\begin{aligned}
VV_X[u,t] &= meanX(\ell(UV)[u,t], r(UV)[u,t]) \\
&= meanX(\ell(UV)[u,t], \ell(UV)[t,u])
\end{aligned}
$$

$VV_A[u,t] = \frac{1}{2}(\ell(UV)[u,t] + \ell(UV)[t,u])$ – arithmetic mean
$VV_m[u,t] = \min(\ell(UV)[u,t], \ell(UV)[t,u])$ – minimum
$VV_M[u,t] = \max(\ell(UV)[u,t], \ell(UV)[t,u])$ – maximum
$VV_G[u,t] = \sqrt{\ell(UV)[u,t] \cdot \ell(UV)[t,u]}$ – geometric, Salton
$VV_H[u,t] = 2(\ell(UV)[u,t]^{-1} + \ell(UV)[t,u]^{-1})^{-1}$ – harmonic, Dice
$VV_J[u,t] = (\ell(UV)[u,t]^{-1} + \ell(UV)[t,u]^{-1} - 1)^{-1}$ – Jaccard

Note: $\ell(\mathbf{UV})$ can be computed from $L(\mathbf{UV})$
$L(UV)[u,u] = \mathrm{wod}_{UV}(u/u) = \mathrm{wod}_{UV}(u)$.

It holds: $VV_X[u,t] = VV_X[t,u]$, $VV_X[u,t] \in [0,1]$ and $VV_J[u,t] \le VV_m[u,t] \le VV_H[u,t] \le VV_G[u,t] \le VV_A[u,t] \le VV_M[u,t]$.

Conclusions

- some projections (Salton, Jaccard, etc.) can be approached also considering a projection matrix as a table of the inner products of rows/columns.
- in principle we can base a projection on any resemblance measure between rows/columns (related to our question(s))
- https://github.com/bavla/NormNet/

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