Projections of binary two-mode networks

Vladimir Batagelj

IMFM Ljubljana

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Outline

Vladimir Batagelj: vladimir.batagelj@fmf.uni-lj.si
Current version of slides (April 19, 2022 at 02 : 57): slides PDF
https://github.com/bavla/biblio/blob/master/doc

Two-mode networks

A simple directed two-mode network $\mathcal{N} = ((U, V), L, a)$ links the set of nodes U to the set of nodes V with the arcs from the set of links L. The mapping $a\colon L \to \mathbb{R}^+$ assigns to each arc (u, v) its weight $a(u, v)$. The network $\mathcal{N}$ can be represented with the corresponding matrix $\mathbf{A} = [A[u, v]]_{u \in U, v \in V}$

$$A[u, v] = \begin{cases} a(u, v) & (u, v) \in L \\ 0 & \text{otherwise} \end{cases}$$

The set $N(u)$ of *(out-)neighbors* (successors) of the node $u \in U$

$$N(u) = \{v \in V : (u, v) \in L\}$$

and the set $N^-(u)$ of *in-neighbors* (predecessors) of the node $v \in V$

$$N^-(v) = \{u \in U : (u, v) \in L\}$$

In traditional two-mode networks we usually assume that $U \cap V = \emptyset$. In the case $U = V$ we get an ordinary one-mode simple directed network.

Multiplication of networks

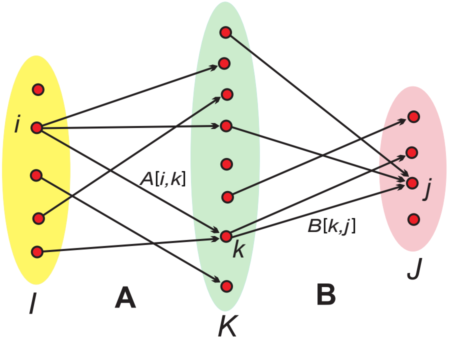
The product $\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$ of two compatible matrices $\mathbf{A}_{I \times K}$ and $\mathbf{B}_{K \times J}$ is defined in the standard way

$$C[i,j] = \sum_{k \in K} A[i,k] \cdot B[k,j]$$

(semirings !!!)

The product of two compatible networks $\mathcal{N}_A = ((I,K), L_A, a)$ and $\mathcal{N}_B = ((K,J), L_B, b)$ is the network $\mathcal{N}_C = ((I,J), L_C, c)$ where $L_C = \{(i,j) : c[i,j] \neq 0\}$ and the weight c is determined by the matrix $\mathbf{C}$, $c(i,j) = C[i,j]$.

Multiplication of networks



If all weights in networks $\mathcal{N}_A$ and $\mathcal{N}_B$ are equal to 1, the value of $C[i,j]$ of $\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$ counts the number of ways we can go from the node $i \in I$ to the node $j \in J$ passing through K, $C[i,j] = |N_A(i) \cap N_B^-(j)|$.

$$C[i,j] = \sum_{k \in N_A(i) \cap N_B^-(j)} A[i,k] \cdot B[k,j]$$

Outer product decomposition

For vectors $x = [x_1, x_2, \ldots, x_n]$ and $y = [y_1, y_2, \ldots, y_m]$ their *outer product* $x \circ y$ is defined as a matrix

$$x \circ y = [x_i \cdot y_j]_{n \times m}$$

then we can express the product $\mathbf{C}$ of two compatible matrices $\mathbf{A}$ and $\mathbf{B}$ as the *outer product decomposition*

$$\mathbf{C} = \mathbf{A} \cdot \mathbf{B} = \sum_k \mathbf{H}_k \quad \text{where} \quad \mathbf{H}_k = \mathbf{A}[\cdot, k] \circ \mathbf{B}[k, \cdot],$$

$\mathbf{A}[., k]$ is the k-th column of matrix $\mathbf{A}$, and $\mathbf{B}[k, .]$ is the k-th row of matrix $\mathbf{B}$.

Projections

A standard approach to the analysis of a two-mode network $\mathcal{N}$ is to transform it into the corresponding one-mode networks determined by:

row projection to U: $\mathrm{row}(\mathbf{A}) = \mathbf{A} \cdot \mathbf{A}^T$, or
column projection to V: $\mathrm{col}(\mathbf{A}) = \mathbf{A}^T \cdot \mathbf{A}$

and analyze the obtained weighted network.

$$\mathrm{col}(\mathbf{A}) = \mathbf{A}^T \cdot \mathbf{A} = \mathbf{A}^T \cdot (\mathbf{A}^T)^T = \mathrm{row}(\mathbf{A}^\mathbf{T}), \quad \mathrm{row}(\mathbf{A}) = \mathrm{col}(\mathbf{A}^\mathbf{T})$$

Binary projections

In the following, we will consider a two-mode network $\mathcal{N} = ((W, A), L, wa)$ described with the corresponding binary matrix $\mathbf{WA}$. To support our intuition, we can interpret it as an authorship network linking a work $w \in W$ to its authors from A.

Its column projection $\mathbf{Co} = col(\mathbf{WA}) = \mathbf{WA}^T \cdot \mathbf{WA}$ has entries

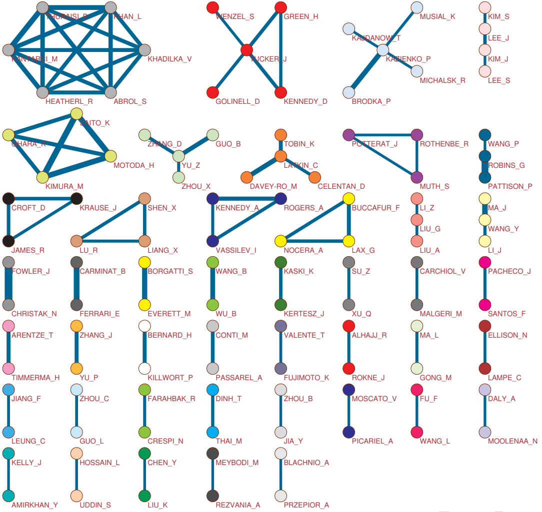
$Co[a, b] = |N(a) \cap N(b)| = Co[b, a] = \#$ of works that authors a and b coauthored

$\mathbf{Co}$ is a *co-appearance* / coauthorship matrix.

$Co[a, a] = |N(a)| = \deg(a) = \#$ of works (co)authored by the author a.

Example: SNA18[8] /edge cut at level 15

$|W| = 70792,\ |A| = 93011$



... nodes

A real-life network $\mathcal{N}$ can contain nodes $w \in W$ of degree 0 (works with no author) and 1 (single author works). Works with no author do not contribute to the matrix **Co**. Single author works contribute only to the author's diagonal entry in the matrix **Co**.

We denote $W_{[d]} = \{w \in W : \deg(w) \geq d\}$ and
$\mathcal{N}_{[d]} = ((W_{[d]}, A), L(W_{[d]}), wa/W_{[d]})$.

Using the entries of the matrix **Co** we can express the *Salton's cosine similarity* between nodes a_i and a_j as

$$Cos[a_i, a_j] = \frac{Co[a_i, a_j]}{\sqrt{Co[a_i, a_i] \cdot Co[a_j, a_j]}}$$

Total weight

The *total weight* of links in the network $\mathcal{N} = (V, L, w)$, $w \colon L \to \mathbb{R}$

$$T(\mathcal{N}) = \sum_{(u,v)\in L} w(u,v)$$

On the basis of outer product decomposition we have

$$T(\mathbf{C}) = T(\sum_k \mathbf{H}_k) = \sum_k T(\mathbf{H}_k)$$

$$T(\mathbf{H}_k) = (\sum_i A[i,k]) \cdot (\sum_j B[k,j])$$

Therefore for **Co** we have $\mathbf{H}_w = \mathbf{K}_{N(w)}$ and

$$T(\mathbf{Co}) = \sum_{w\in W} \deg(w)^2$$

Structure of projection

In words

- a projection network is a sum of complete subgraphs;
- the contribution of a node w from the other set W to the total is $\deg(w)^2$.

This means that the nodes with a large degree in W are over-represented in the projection.

Fractional approach

To make in a projection the contributions of all nodes from the set W equal we apply the *fractional approach* by normalizing the weights [2].

The normalized 2-mode network $n(\mathbf{WA})$ has weights

$$n(WA)[w,a] = \frac{WA[w,a]}{\max(1, \deg(w))}$$

Normalized projection $\mathbf{Cn} = n(\mathbf{WA})^T \cdot n(\mathbf{WA})$

$$T(\mathbf{Cn}) = |W_{[1]}|$$

Newman's normalization – strict coauthorship

Mark Newman proposed an alternative normalization that considers only coauthorship between different authors – single-author works and self coauthorship are excluded.

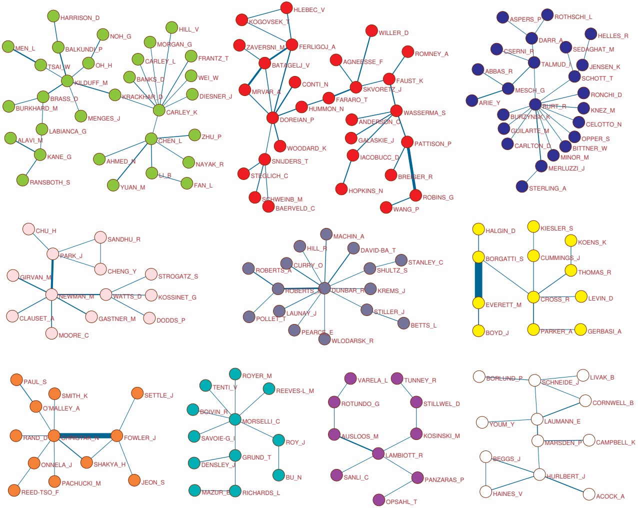
The *Newman's normalized* 2-mode network $n'(\mathbf{WA})$ has weights

$$n'(WA)[w,a] = \begin{cases} \dfrac{WA[w,a]}{\deg(w)-1} & w \in W_{[2]} \\ 0 & \text{otherwise} \end{cases}$$

Newman's projection $\mathbf{Cn}' = n(\mathbf{WA})^T \cdot n'(\mathbf{WA})$.
Network: symmetrize with sum and remove loops.

Example: SNA18[8] /selected Newman islands 10-30

$|W| = 70792$, $|A| = 93011$



Binary similarity measures

For analysis of two-mode networks, besides Salton's cosine similarity, the use of some other similarities between binary vectors was already proposed in SNA literature [5, p.420-424]. In principle, we could consider any similarity measure between binary vectors [1, 6].

For all these similarities the corresponding matrices can be computed from the events co-affiliation or co-appearances matrix (ordinary column projection) **Co**. The similarities between vectors $v = WA[., a_i]$ and $z = WA[., a_j]$ are expressed in terms of the quantities a, b, c, and d

$$
\begin{array}{cc|c|c|c}
 & & & z & \\
 & & 1 & 0 & \\
\hline
\multirow{2}{*}{v} & 1 & a & b & a+b \\
\cline{2-5}
 & 0 & c & d & c+d \\
\hline
 & & a+c & b+d & |W|
\end{array}
$$

The quantity a counts the number of cases (indices) for which both vectors v and z have value 1, etc.

Association coefficients

Association Coefficients

measure		definition	range	class
Russel and Rao (1940)	s_1	$\frac{a}{m}$	$[1,0]$	
Kendall, Sokal-Michener (1958)	s_2	$\frac{a+d}{m}$	$[1,0]$	S
Rogers and Tanimoto (1960)	s_3	$\frac{a+d}{m+b+c}$	$[1,0]$	S
Hamann (1961)	s_4	$\frac{a+d-b-c}{m}$	$[1,-1]$	S
Sokal & Sneath (1963), un_3^{-1}, S	s_5	$\frac{b+c}{a+d}$	$[0,\infty]$	S
Jaccard (1900)	s_6	$\frac{a}{a+b+c}$	$[1,0]$	T
Kulczynski (1927), T^{-1}	s_7	$\frac{a}{b+c}$	$[\infty,0]$	T
Dice (1945), Czekanowski (1913)	s_8	$\frac{a}{a+\frac{1}{2}(b+c)}$	$[1,0]$	T
Sokal and Sneath	s_9	$\frac{a}{a+2(b+c)}$	$[1,0]$	T
Kulczynski	s_{10}	$\frac{1}{2}(\frac{a}{a+b}+\frac{a}{a+c})$	$[1,0]$	
Sokal & Sneath (1963), un_4	s_{11}	$\frac{1}{4}(\frac{a}{a+b}+\frac{a}{a+c}+\frac{d}{d+b}+\frac{d}{d+c})$	$[1,0]$	
Q_0	s_{12}	$\frac{bc}{ad}$	$[0,\infty]$	Q
Yule (1912), ω	s_{13}	$\frac{\sqrt{ad}-\sqrt{bc}}{\sqrt{ad}+\sqrt{bc}}$	$[1,-1]$	Q
Yule (1927), Q	s_{14}	$\frac{ad-bc}{ad+bc}$	$[1,-1]$	Q
– bc –	s_{15}	$\frac{4bc}{m^2}$	$[0,1]$	
Driver & Kroeber (1932), Ochiai (1957)	s_{16}	$\frac{a}{\sqrt{(a+b)(a+c)}}$	$[1,0]$	
Sokal & Sneath (1963), un_5	s_{17}	$\frac{ad}{\sqrt{(a+b)(a+c)(d+b)(d+c)}}$	$[1,0]$	
Pearson, ϕ	s_{18}	$\frac{ad-bc}{\sqrt{(a+b)(a+c)(d+b)(d+c)}}$	$[1,-1]$	
Baroni-Urbani, Buser (1976), S^{**}	s_{19}	$\frac{a+\sqrt{ad}}{a+b+c+\sqrt{ad}}$	$[1,0]$	
Braun-Blanquet (1932)	s_{20}	$\frac{a}{\max(a+b,a+c)}$	$[1,0]$	
Simpson (1943)	s_{21}	$\frac{a}{\min(a+b,a+c)}$	$[1,0]$	
Michael (1920)	s_{22}	$\frac{4(ad-bc)}{(a+d)^2+(b+c)^2}$	$[1,-1]$	

Association coefficients and co-appearance

For example, the *Jaccard similarity*

$$J = \frac{|N(v) \cap N(z)|}{|N(v) \cup N(z)|} = \frac{a}{a+b+c}.$$

The following equalities hold

$$\begin{aligned}
a &= |N(v) \cap N(z)| = Co[v,z] \\
a+b &= |N(v)| = \deg(v) = Co[v,v] \\
a+c &= |N(z)| = \deg(z) = Co[z,z] \\
a+b+c+d &= |W|.
\end{aligned}$$

From them we get

$$\begin{aligned}
b &= |N(v) \setminus N(z)| = Co[v,v] - Co[v,z] \\
c &= |N(z) \setminus N(v)| = Co[z,z] - Co[v,z] \\
d &= |W| + Co[v,z] - Co[v,v] - Co[z,z] \\
a+b+c &= |N(v) \cup N(z)| = Co[v,v] + Co[z,z] - Co[v,z]
\end{aligned}$$

Jaccard

For Jaccard similarity we get [3]

$$J[v,z] = \frac{Co[v,z]}{Co[v,v] + Co[z,z] - Co[v,z]}$$

In this sense, the similarity measures (matrices) can be seen as a kind of compatibility normalization of the weights obtained with the standard projection [4].

The corresponding Jaccard network $\mathcal{J} = (V, L_J, J)$ is undirected with loops removed. Note that $L_J = L_{Co}$.

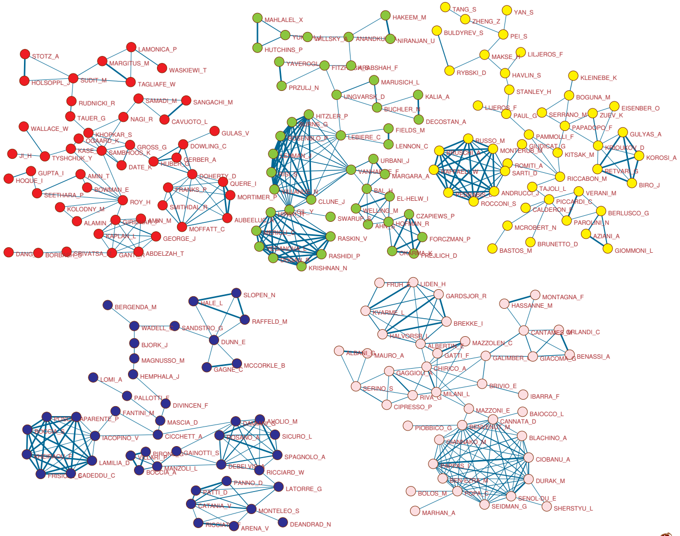
The Jaccard dissimilarity

$$d_J(A,B) = 1 - J(A,B) = \frac{|A \oplus B|}{|A \cup B|}$$

is also a distance. Note: $b + c = |N(v) \oplus N(z)|$.

Example: SNA18[8] /some Jaccard islands 30-50

$|W| = 70792$, $|A| = 93011$



Acknowledgments

The computational work reported on in this presentation was performed using R and Pajek. The code and data are available at https://github.com/bavla/NormNet/

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