Derived bibliographic networks

The impact of multi-person units and truncated networks

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Outline

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Current version of slides (November 23, 2023 at 04 : 05): PDF

https://github.com/bavla/biblio/

The multipersonality's effect on the results of bibliographic analyses

We would like to study the effect of multipersons in derived networks [5]. Let $\mathbf{M} = [m[u, v]]$ is a matrix on $U \times V$ and $\mathbf{C}_U = \{C_1, C_2, \ldots, C_k\}$ a partition of the set U, $\emptyset \subset C_i \subseteq U$, $C_i \cap C_j = \emptyset$ for $i \neq j$, and $\bigcup_i C_i = U$. The set U is the (ground truth) set of real units (persons). The partition $\mathbf{C}_U$ corresponds to units (for example authors) identified by the network construction process. A cluster $C \in \mathbf{C}_U$ with $|C| > 1$ represents a multi-unit; and for $|C| = 1$ a correctly identified unit.

We introduce the *shrinking* transformation S_r of matrix $\mathbf{M}$ by the **r**ows partition $\mathbf{C}_U$ into $S_r(\mathbf{M}, \mathbf{C}_U) = \mathbf{S} = [s[C, v]]$ on $\mathbf{C}_U \times V$ determined by the rule

$$s[C, v] = \sum_{u \in C} m[u, v]$$

The shrinking transformation can be extended to a **c**olumns partition $\mathbf{C}_V$ of the set V by

$$s[u, C] = \sum_{v \in C} m[u, v]$$

Multiunits' effect

or

$$S_c(\mathbf{M}, \mathbf{C}_V) = S_r(\mathbf{M}^T, \mathbf{C}_V)^T$$

and to partitions $\mathbf{C}_U$ and $\mathbf{C}_V$ of both sets by

$$S(\mathbf{M}, (\mathbf{C}_U, \mathbf{C}_V)) = S_c(S_r(\mathbf{M}, \mathbf{C}_U), \mathbf{C}_V)$$

Consider now the case of two compatible matrices $\mathbf{M} = [m[u,t]]$ on $U \times T$ and $\mathbf{N} = [n[t,v]]$ on $T \times V$. For a partition $\mathbf{C}_U$ of the set U it holds

$$S_r(\mathbf{M} \cdot \mathbf{N}, \mathbf{C}_U) = S_r(\mathbf{M}, \mathbf{C}_U) \cdot \mathbf{N}$$

To check this let's denote with $\mathbf{L}$ and $\mathbf{R}$ the left and right sides of this expression. We have

$$l[C,v] = \sum_{u \in C} \mathbf{M} \cdot \mathbf{N}[u,v] = \sum_{u \in C} \sum_{t \in t} m[u,t] \cdot n[t,v]$$

and

$$r[C,v] = \sum_{t \in T} S_r(\mathbf{M}, \mathbf{C}_U)[C,t] \cdot n[t,v] = \sum_{t \in T} (\sum_{u \in C} m[u,t]) \cdot n[t,v] = l[C,v]$$

Multiunits' effect

For the partition $\mathbf{C}_V$ of the set V we get

$$S_c(\mathbf{M}\cdot\mathbf{N},\mathbf{C}_V) = S_r((\mathbf{M}\cdot\mathbf{N})^T,\mathbf{C}_V)^T = S_r(\mathbf{N}^T\cdot\mathbf{M}^T,\mathbf{C}_V)^T =$$

$$= (S_r(\mathbf{N}^T,\mathbf{C}_V)\cdot\mathbf{M}^T)^T = \mathbf{M}\cdot S_r(\mathbf{N}^T,\mathbf{C}_V)^T = \mathbf{M}\cdot S_c(\mathbf{N},\mathbf{C}_V)$$

Therefore

$$S_c(\mathbf{M}\cdot\mathbf{N},\mathbf{C}_V) = \mathbf{M}\cdot S_c(\mathbf{N},\mathbf{C}_V)$$

For partitions of both sets U and V we have

$$S(\mathbf{M}\cdot\mathbf{N},(\mathbf{C}_U,\mathbf{C}_V)) = S_c(S_r(\mathbf{M}\cdot\mathbf{N},\mathbf{C}_U),\mathbf{C}_V) =$$

$$= S_c(S_r(\mathbf{M},\mathbf{C}_U)\cdot\mathbf{N},\mathbf{C}_V) = S_r(\mathbf{M},\mathbf{C}_U)\cdot S_c(\mathbf{N},\mathbf{C}_V)$$

and finally

$$S(\mathbf{M}\cdot\mathbf{N},(\mathbf{C}_U,\mathbf{C}_V)) = S_r(\mathbf{M},\mathbf{C}_U)\cdot S_c(\mathbf{N},\mathbf{C}_V)$$

Multiunits' effect

For $C_u \in \mathbf{C}_U$ and $C_v \in \mathbf{C}_V$ we have

$$S(\mathbf{M} \cdot \mathbf{N}, (\mathbf{C}_U, \mathbf{C}_V))[C_u, C_v] = S_c(S_r(\mathbf{M} \cdot \mathbf{N}, \mathbf{C}_U), \mathbf{C}_V)[C_u, C_v] =$$

$$\sum_{z \in C_v} S_r(\mathbf{M} \cdot \mathbf{N}, \mathbf{C}_U)[C_u, z] = \sum_{z \in C_v} \sum_{w \in C_u} \mathbf{M} \cdot \mathbf{N}[w, z] = \sum_{w \in C_u} \sum_{z \in C_v} \mathbf{M} \cdot \mathbf{N}[w, z]$$

In a special case of singelton clusters $C_u = \{u\}$ and $C_v = \{v\}$ we get

$$S(\mathbf{M} \cdot \mathbf{N}, (\mathbf{C}_U, \mathbf{C}_V))[\{u\}, \{v\}] = \sum_{w \in \{u\}} \sum_{z \in \{v\}} \mathbf{M} \cdot \mathbf{N}[w, z] = \mathbf{M} \cdot \mathbf{N}[u, v]$$

We see that **the multi-units don't affect the values of relations between singletons in the derived networks**.

Fractional approach

Assume that we have the authorship network represented by a matrix $\mathbf{WA} = [wa[w,a]]$ where $wa[w,a] = 1$ iff the author a is (co)author of the work w, and $wa[w,a] = 0$ otherwise.

We will discuss two normalizations. The *standard* normalization $n(\mathbf{WA}) = [nwa[w,a]]$ where

$$nwa[w,a] = \frac{wa[w,a]}{\max(1, \deg(w))}$$

and *strict* or *Newman*'s normalization $n'(\mathbf{WA}) = [nwa'[w,a]]$ where

$$nwa'[w,a] = \frac{wa[w,a]}{\max(1, \deg(w) - 1)}$$

We have $\sum_{a \in A} nwa[w,a] = \mathrm{sign}(\deg(w))$ and

$$\sum_{a \in A} nwa'[w,a] = \frac{\deg(w)}{\max(1, \deg(w) - 1)}.$$

. . . Fractional approach

The meaning of the weighted degree of node a in the network $n(\mathbf{WA})$, $\text{wideg}_{n(WA)}(a) = \sum_{w \in W} nwa[w,a]$, is the *fractional contribution of the author* a to all works.

Note that if $\deg(w) = 0$ then both $nwa[w,a] = 0$ and $nwa'[w,a] = 0$, and if $\deg(w) = 1$ then both $nwa[w,a] = 1$ and $nwa'[w,a] = 1$.

The standard co-appearance network matrix $\mathbf{Cn} = [cn[a,b]]$ is obtained as

$$\mathbf{Cn} = n(\mathbf{WA})^T \cdot n(\mathbf{WA})$$

and the *strict* co-appearance network matrix $\mathbf{Ct} = [ct[a,b]]$ is obtained as

$$\mathbf{Ct} = D_0(n(\mathbf{WA})^T \cdot n'(\mathbf{WA}))$$

where $D_0(\mathbf{M})$ sets the diagonal of matrix $\mathbf{M}$ to 0.

Let's look at an entry of $\mathbf{Cn}$

$$cn[a,b] = \sum_{w \in W} nwa^T[a,w] \cdot nwa[w,b] = \sum_{w \in W} nwa[w,a] \cdot nwa[w,b] = cn[b,a]$$

. . . Fractional approach

and for **Ct**

$$ct[a,b] = \sum_{w \in W} nwa^T[a,w] \cdot nwa'[w,b] = \sum_{w \in W} \frac{wa[w,a]}{\max(1,\deg(w))} \cdot \frac{wa[w,b]}{\max(1,\deg(w)-1)}$$

$$= \sum_{w \in W} \frac{wa[w,b]}{\max(1,\deg(w))} \cdot \frac{wa[w,a]}{\max(1,\deg(w)-1)} = ct[b,a]$$

The fractional co-appearance matrices **Cn** and **Ct** are symmetric.

From

$$\mathrm{wdeg}_{Cn}(a) = \sum_{b \in A} cn[a,b] = \sum_{w \in W} nwa[w,a] \cdot \sum_{b \in A} nwa[w,b] =$$

$$= \sum_{w \in W} nwa[w,a] \cdot \mathrm{sign}(\deg(w)) = \sum_{w \in W} nwa[w,a] = \mathrm{wideg}_{n(WA)}(a)$$

we see that the authors have the same weighted degree in networks $n(\mathbf{WA})$ and **Cn**.

...Fractional approach

Similarly for the network **Ct**. Because by definition, $ct[a,a] = 0$, we have

$$\mathrm{wdeg}_{Ct}(a) = \sum_{b \in A \setminus \{a\}} ct[a,b] = \sum_{w \in W} \frac{nwa[w,a]}{\max(1, \deg(w) - 1)} \sum_{b \in A \setminus \{a\}} wa[w,b] =$$

If $wa[w,a] = 0$ the term in the $\sum_{w \in W}$ has value 0. So we can assume $wa[w,a] = 1$. This means that $a \in N(w)$ and $wa[w,b] = 1$ means that also $b \in N(w)$. Therefore

$$\sum_{b \in A \setminus \{a\}} wa[w,b] = |N(w) \setminus \{a\}| = \deg(w) - 1$$

Now, we can continue ($W_2 = \{w \in W : deg(w) \geq 2\}$)

$$= \sum_{w \in W_2} \frac{nwa[w,a]}{\max(1, \deg(w) - 1)} (\deg(w) - 1) = \sum_{w \in W_2} nwa[w,a] =$$

$$= \mathrm{wideg}_{n(WA)}(a) - |S|$$

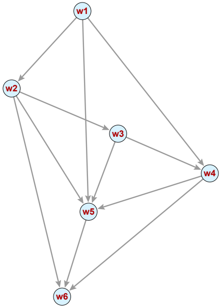
where $S = \{w \in W : \deg(w) = 1\}$ – single author works.

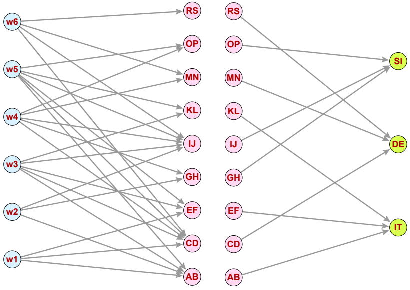
. . . Fractional approach

Package bibmat

```
> normalize <- function(M) t(apply(M,1,function(x) x/max(1,sum(x))))
> newman <- function(M) t(apply(M,1,function(x) x/max(1,sum(x)-1)))
> D0 <- function(M) {diag(M) <- 0; return(M)}
> binary <- function(M) {B <- t(apply(M,1,function(x) as.integer(x!=0)))
+   colnames(B) <- colnames(M); return(B)}
> wodeg <- function(M) apply(M,1,sum)
> wideg <- function(M) apply(M,2,sum)
> odeg <- function(M) wodeg(binary(M))
> ideg <- function(M) wideg(binary(M))
> wdeg <- wodeg
```

. . . Fractional approach





\> Ci

	w1	w2	w3	w4	w5	w6
w1	0	1	0	1	1	0
w2	0	0	1	0	1	1
w3	0	0	0	1	1	0
w4	0	0	0	0	1	1
w5	0	0	0	0	0	1
w6	0	0	0	0	0	0

\> WA

	AB	CD	EF	GH	IJ	KL	MN	OP	RS
w1	1	1	1	0	0	0	0	0	0
w2	1	0	0	1	1	0	0	0	0
w3	1	1	1	0	1	1	0	0	0
w4	0	1	0	1	1	0	1	1	0
w5	1	1	1	0	1	1	0	1	0
w6	0	1	0	0	1	0	1	0	1

\> AC

	IT	DE	SI
AB	1	0	0
CD	0	1	0
EF	1	0	0
GH	0	0	1
IJ	0	0	1
KL	1	0	0
MN	0	1	0
OP	0	0	1
RS	0	1	0

Github/bavla/biblio

... Fractional approach

```
> wdir <- "C:/Users/vlado/test/biblio"
> setwd(wdir)
> source(
  "https://raw.githubusercontent.com/bavla/biblio/master/code/bibmat.R")
> urlEx <-
  "https://github.com/bavla/biblio/raw/master/Eu/Data/ExNets.RDS"
> download.file(url=urlEx,destfile=paste0(wdir,"/ExNets.RDS",sep=""))
> Ex <- readRDS("ExNets.RDS")
> Ci <- Ex$Ci; WA <- Ex$WA; AC <- Ex$AC

> WAn <- normalize(WA)
> Cn <- t(WAn)%*%WAn
> wideg(WAn)
    AB     CD     EF     GH     IJ     KL     MN     OP     RS
1.0333 1.1500 0.7000 0.5333 1.1500 0.3667 0.4500 0.3667 0.2500
> wideg(Cn)
    AB     CD     EF     GH     IJ     KL     MN     OP     RS
1.0333 1.1500 0.7000 0.5333 1.1500 0.3667 0.4500 0.3667 0.2500
> WAt <- newman(WA)
> Ct <- D0(t(WAn)%*%WAt)
> wideg(Ct)
    AB     CD     EF     GH     IJ     KL     MN     OP     RS
1.0333 1.1500 0.7000 0.5333 1.1500 0.3667 0.4500 0.3667 0.2500
> sum(wideg(WAn))
[1] 6
```

Totals

For a matrix $\mathbf{M}$ on $U \times V$ we define its *total* $T(\mathbf{M})$ as the sum of all its entries, $T(\mathbf{M}) = \sum_{u \in U} \sum_{v \in V} m[u, v]$. Let's compute

$$T(\mathbf{Cn}) = \sum_{a \in A} \sum_{b \in A} cn[a, b] = \sum_{w \in W} \sum_{a \in A} \sum_{b \in A} nwa[w, a] \cdot nwa[w, b] = \sum_{w \in W} T(w)$$

where

$$T(w) = \sum_{a \in A} \sum_{b \in A} nwa[w, a] \cdot nwa[w, b] = \sum_{a \in A} nwa[w, a] \cdot \sum_{b \in A} nwa[w, b] =$$

$$= \mathrm{sign}(\deg(w))^2 = \mathrm{sign}(\deg(w))$$

We see that the contribution of each work $w \in W$ with $\deg(w) > 0$ is 1. Therefore

$$T(\mathbf{Cn}) = \sum_{w \in W} \mathrm{sign}(\deg(w)) = |W_1|$$

where $W_1 = \{w \in W : \deg(w) \geq 1\}$.

Totals

For matrices $n(\mathbf{WA})$ and $\mathbf{Cn}$ we have

$$T(n(\mathbf{WA})) = \sum_{a\in A}\sum_{w\in W} nwa[w,a] = \sum_{a\in A} \mathrm{wideg}_{n(WA)}(a) =$$

$$= \sum_{a\in A} \mathrm{wdeg}_{Cn}(a) = \sum_{a\in A}\sum_{b\in A} cn[a,b] = T(\mathbf{Cn})$$

And

$$T(\mathbf{Ct}) = \sum_{a\in A}\sum_{b\in A\setminus\{a\}} ct[a,b] =$$

$$\sum_{w\in W}\sum_{a\in A}\sum_{b\in A\setminus\{a\}} nwa[w,a]\cdot nwa'[w,b] = \sum_{w\in W} T'(w)$$

where

$$T'(w) = \sum_{a\in A}\sum_{b\in A\setminus\{a\}} nwa[w,a]\cdot nwa'[w,b] =$$

$$\sum_{a\in A} \frac{nwa[w,a]}{\max(1,\deg(w)-1)} \cdot \sum_{b\in A\setminus\{a\}} wa[w,b]$$

Totals

If $wa[w,a] = 0$ or $wa[w,b] \leq 1$ the term in $\sum_{a \in A}$ has value 0. For $wa[w,a] = 1$ and $wa[w,b] \geq 2$ (or $deg(w) \geq 2$) we have $\sum_{b \in A \setminus \{a\}} wa[w,b] = \deg(w) - 1$. Therefore

$$T'(w) = \begin{cases} 1 & deg(w) \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

and finally

$$T(\mathbf{Ct}) = \sum_{w \in W} T'(w) = |W_2|$$

where $W_2 = \{w \in W : deg(w) \geq 2\}$.

Note also that

$$\sum_{a \in A} \text{wideg}_{n(WA)}(a) = T(n(\mathbf{WA}))$$

... Totals

```
> sum(Cn)
[1] 6
> sum(Ct)
[1] 6
> empty <- rep(0,length(A)); WA1 <- rbind(WA,empty,empty)
> rownames(WA1)[7:8] <- c("w7","w8"); WA1["w8","CD"] <- 1
> WA1
   AB CD EF GH IJ KL MN OP RS
w1  1  1  1  0  0  0  0  0  0
w2  1  0  0  1  1  0  0  0  0
w3  1  1  1  0  1  1  0  0  0
w4  0  1  0  1  1  0  1  1  0
w5  1  1  1  0  1  1  0  1  0
w6  0  1  0  0  1  0  1  0  1
w7  0  0  0  0  0  0  0  0  0
w8  0  1  0  0  0  0  0  0  0
> WAn1 <- normalize(WA1); Cn1 <- t(WAn1)%*%WAn1
> sum(Cn1)
[1] 7
> WAt1 <- newman(WA1); Ct1 <- DO(t(WAn1)%*%WAt1)
> sum(Ct1)
[1] 6
> wideg(WAn1)
    AB     CD     EF     GH     IJ     KL     MN     OP     RS
1.0333 2.1500 0.7000 0.5333 1.1500 0.3667 0.4500 0.3667 0.2500
> wdeg(Cn1)
    AB     CD     EF     GH     IJ     KL     MN     OP     RS
1.0333 2.1500 0.7000 0.5333 1.1500 0.3667 0.4500 0.3667 0.2500
> wdeg(Ct1)
    AB     CD     EF     GH     IJ     KL     MN     OP     RS
1.0333 1.1500 0.7000 0.5333 1.1500 0.3667 0.4500 0.3667 0.2500
```

Truncated fractional co-appearance networks

A usual approach to the analysis of a two-mode network is its projection to its selected mode. The obtained weighted one-mode network is afterward analyzed using standard methods. Most bibliographic two-mode networks are sparse – they have a small average degree. If the other mode has some nodes of very large degree the projection can "explode" – it is not a sparse network (increased time and space complexity) [3]. In fractional projections, the contribution of the nodes of large degree is very small and mostly doesn't affect the resulting important subnetworks – the important part of the result can be obtained by projection of a two-mode subnetwork on a subset of important nodes – a truncated projection. This idea is elaborated in the following.

Truncated standard fractional network

Let's split the set of authors A into two sets A_1 (selected authors) and A_0 (remaining authors), $A_1 \cup A_0 = A$ and $A_1 \cap A_0 = \emptyset$. We call a *truncated (standard) fractional network* the network

$$\mathbf{Cn}_{11} = \mathbf{Cn}[A_1, A_1] = n(\mathbf{WA})[W, A_1]^T \cdot n(\mathbf{WA})[W, A_1]$$

For a selected author $a \in A_1$, we denote with $\alpha(a) = \mathrm{wdeg}_{Cn_{11}}(a)$ her/his *internal* fractional contribution, and with $\beta(a) = \mathrm{wdeg}_{n(WA)}(a) - \alpha(a)$ her/his *external* fractional contribution.

We reorder the nodes of the network **Cn** according to the A_1, A_0 split (see next slide). The network matrix is split into four submatrices $\mathbf{Cn}_{ij}, i, j \in \{0, 1\}$. We denote their totals $T_{ij} = T(\mathbf{Cn}_{ij})$. Because the matrix **Cn** is symmetric we have $T_{01} = T_{10}$. T_{11} is the fractional contribution of collaboration among selected authors, $T_{10} + T_{01} = 2T_{10}$ is the fractional contribution of collaboration of selected authors with remaining authors, and T_{00} is the fractional contribution of collaboration among remaining authors.

. . . Truncated standard fractional network

	A_1	A_0
A_1	$\mathbf{Cn}_{11}$	$\mathbf{Cn}_{10}$
A_0	$\mathbf{Cn}_{01}$	$\mathbf{Cn}_{00}$

... Truncated standard fractional network

We can compute $T_{11} = T(\mathbf{Cn}_{11})$ and $T_{10} = \sum_{a \in A_1} \beta(a)$, and finally $T_{00} = T(n(\mathbf{WA})) - T_{11} - 2T_{10}$. Note that we used only information from $n(\mathbf{WA})$ and $\mathbf{Cn}_{11}$.

A primary application of the truncated standard fractional network scheme is for the set of the most active authors

$$A_1 = \{a \in A : \text{wideg}_{n(WA)}(a) \geq t\}$$

where t is a selected threshold value.

Note that the computation of the vector $\mathbf{wideg}_{n(WA)}$ is cheap.

Truncated strict fractional network

Again we have $\text{wdeg}_{Ct}(a) = \text{wideg}_{n(WA)}(a)$. Therefore the scheme used for the truncated standard fractional network can be applied also for *truncated strict fractional network*

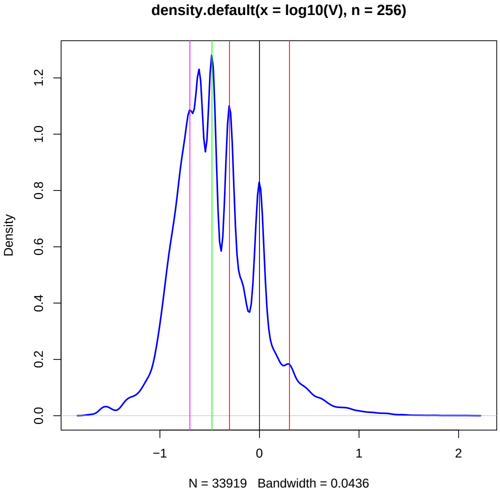
$$\mathbf{Ct}_{11} = \mathbf{Ct}[A_1, A_1] = D_0(n(\mathbf{WA})[W, A_1]^T \cdot n'(\mathbf{WA})[W, A_1])$$

Sizes of truncated networks

iMetrics network [2] and Nataliya's HKUST1 network.

	iMetrics			HKUST1		
interval	n	$m = \|A\|$	*avdeg*	n	$m = \|A\|$	*avdeg*
≥ 0	33919	225931	13.32	28108	45365272	3227.92
$\geq 1/10$	32418	191888	11.84	17656	3216796	364.39
$\geq 1/5$	26247	134049	10.21	10213	86529	16.94
$\geq 1/3$	14381	71967	10.01	5171	45845	17.73
$\geq 1/2$	12781	60587	9.48	4032	32806	16.27
≥ 1	6211	32395	10.43	1799	13723	15.26
≥ 2	1832	14900	16.27	689	4195	12.18
≥ 3	964	9306	19.31	369	1743	9.45
≥ 5	446	4646	20.83	172	646	7.51
≥ 10	162	1450	17.90	55	125	4.55

Density of $\log_{10}(\mathrm{wdeg}_{WAn}(a))$



Linking through a network

Assume that another network $\mathbf{S}$ on $W \times W$ is given. The network

$$\mathbf{Q} = n(\mathbf{WA})^T \cdot \mathbf{S} \cdot n(\mathbf{WA})$$

links authors to authors *through* the network $\mathbf{S}$.
From [1] we know that

- If $\mathbf{S}$ is symmetric, $\mathbf{S}^T = \mathbf{S}$, then also $\mathbf{Q}$ is symmetric, $\mathbf{Q}^T = \mathbf{Q}$.
- $T(\mathbf{Q}) = T(\mathbf{S})$.

...Linking through a network

Let's look at

$$\mathrm{wdeg}_Q(a) = \sum_{b\in A} q[a,b] = \sum_{b\in A}\sum_{w\in W}\sum_{z\in W} nwa[w,a]\cdot s[w,z]\cdot nwa[z,b] =$$

$$= \sum_{w\in W}\sum_{z\in W} nwa[w,a]\cdot s[w,z]\cdot \mathrm{sign}(\deg(w)) =$$

$$\sum_{w\in W} nwa[w,a]\cdot \sum_{z\in W} s[w,z] = \sum_{w\in W} nwa[w,a]\cdot \mathrm{wdeg}_S(w)$$

or in a vector form

$$\mathbf{wdeg}_Q = n(\mathbf{WA})^T \cdot \mathbf{wdeg}_S$$

The most active authors are

$$A_1 = \{a \in A : \mathrm{wdeg}_Q(a) \geq t\}$$

Again the truncation scheme can be applied.

Authors co-citation

The co-citation network is defined as $\mathbf{CoCi} = \mathbf{Ci}^T \cdot \mathbf{Ci}$ and the fractional co-citation network as $\mathbf{CoCin} = \mathbf{Cin}^T \cdot \mathbf{Cin}$. Both $\mathbf{CoCi}$ and $\mathbf{CoCin}$ are symmetric. Authors fractional co-citation network is obtained by linking authors through the fractional co-citation network

$$\mathbf{CoCan} = n(\mathbf{WA})^T \cdot \mathbf{CoCin} \cdot n(\mathbf{WA}) =$$

$$= n(\mathbf{WA})^T \cdot \mathbf{Cin}^T \cdot \mathbf{Cin} \cdot n(\mathbf{WA}) = \mathbf{Can}^T \cdot \mathbf{Can}$$

where $\mathbf{Can} = \mathbf{Cin} \cdot n(\mathbf{WA})$.

As in the case of $\mathbf{WA}$ we have $\mathrm{wdeg}_{Cn}(a) = \mathrm{wdeg}_{n(WA)}(a)$, for $\mathbf{Ci}$ it holds also $\mathrm{wdeg}_{CoCin}(w) = \mathrm{wideg}_{n(Ci)}(w)$. Therefore

$$\mathbf{wdeg}_{CoCan} = n(\mathbf{WA})^T \cdot \mathbf{wdeg}_{CoCin} = n(\mathbf{WA})^T \cdot \mathbf{wideg}_{n(Ci)}$$

and it is easy to see that also

$$\mathbf{wideg}_{Can} = n(\mathbf{WA})^T \cdot \mathbf{wideg}_{n(Ci)} = \mathbf{wdeg}_{CoCan}$$

. . . Authors co-citation

```
> Cin <- normalize(Ci)
> CoCin <- t(Cin)%*%Cin
> CoCan <- t(WAn)%*%CoCin%*%WAn
> wdeg_CoCan <- (t(WAn)%*%wideg(Cin))[,1]
> wdeg_CoCan
    AB     CD     EF     GH     IJ     KL     MN     OP     RS
0.4556 0.9694 0.3444 0.2778 1.0806 0.3444 0.6250 0.4444 0.4583
> wodeg(CoCan)
    AB     CD     EF     GH     IJ     KL     MN     OP     RS
0.4556 0.9694 0.3444 0.2778 1.0806 0.3444 0.6250 0.4444 0.4583
> Can <- Cin%*%WAn
> wideg(Can)
    AB     CD     EF     GH     IJ     KL     MN     OP     RS
0.4556 0.9694 0.3444 0.2778 1.0806 0.3444 0.6250 0.4444 0.4583
```

Conclusions

- the fractional version(s) of bibliographic coupling $\mathbf{biCo} = \mathbf{Ci} \cdot \mathbf{Ci}^T$ are more complicated [1]. The corresponding "through" constructions are still on the "to do" list.
- efficient implementation as Pajek macros and in Python Nets package.

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