Indirect Blockmodeling of 3-Way Networks

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Abstract. An approach to the indirect blockmodeling of 3-way network data is presented for structural equivalence. This equivalence type is defined formally and expressed in terms of an interchangeability condition that is used to construct a compatible dissimilarity. Using Ward's method, the three dimensional partitioning is obtained via hierarchical clustering and represented diagrammatically. Artificial and real data are used to illustrate these methods.

1 Introduction

One of the tasks specified under 'extending generalized blockmodeling' Doreian et al. (2005) is the blockmodeling of 3-way networks – a 3 dimensional matrix defined on 3 sets of units. If 2 sets are equal we speak about 3-way 2-mode network. Here, we present work on a sub task, the indirect approach to *structural equivalence blockmodeling in 3-way networks. Indirect* means embedding the notion of equivalence in a *compatible* dissimilarity and determining a clustering based on that dissimilarity. The idea of blockmodeling 3-way data was proposed in an *ad hoc* fashion in Baker (1986) and in Everett and Borgatti (1992). We present a more systematic and general approach.

Two units are *structurally equivalent* iff they can be interchanged without producing change in the structure – the equivalent units have the **same** connection pattern to the **same** neighbors Batagelj et al. (1992).

In a usual 2-way 1-mode network $N = (\mathbf{U}, R)$, $R \subseteq \mathbf{U} \times \mathbf{U}$, x and y are structurally equivalent iff:

s1.	$xRy \Leftrightarrow yRx$	s3.	$\forall z \in \mathbf{U} \setminus \{x, y\} : (xRz \Leftrightarrow yRz)$
s2.	$xRx \Leftrightarrow yRy$	s4.	$\forall z \in \mathbf{U} \setminus \{x, y\} : (zRx \Leftrightarrow zRy)$

The blockmodeling of 2-way 2-mode networks is discussed in Doreian et al. (2004).

2 Structural equivalence in 3-way networks

A *3-way network* N over the basic sets X, Y and Z is determined by a ternary relation $R \subseteq X \times Y \times Z$.

The relation R can be represented by a 3-dimensional binary matrix $R_{X\times Y\times Z}$

$$R[i,p,u] = \begin{cases} 1 & R(i,p,u) \\ 0 & \neg R(i,p,u) \end{cases}$$

We define the following items:

Plane: $R(i,\cdot,\cdot) = \{(i,p,u) : p \in Y \wedge u \in Z \wedge R(i,p,u)\}$

Line: $R(i,\cdot,u) = \{(i,p,u) : p \in Y \wedge R(i,p,u)\}$

Truncated line: $R(i,-T,u) = \{(i,p,u) : p \in Y \setminus T \wedge R(i,p,u)\}$

Representations of these elements by binary vectors will be indicated by replacing braces with brackets.

The subsets $X_1 \subseteq X$, $Y_1 \subseteq Y$, $Z_1 \subseteq Z$ determine a *block* $R(X_1,Y_1,Z_1) = R \cap X_1 \times Y_1 \times Z_1$. If $R(X_1,Y_1,Z_1) = \emptyset$ the block is called a *null block*; if $R(X_1,Y_1,Z_1) = X_1 \times Y_1 \times Z_1$ the block is called a *complete block*.

In the following we shall also need a dissimilarity $D(a,b)$ between vectors a and b defined in R–like notation as

$$D(a,b) = \text{sum}(\text{abs}(a-b))$$

For example, for $a = [0,1,1,0,1]$ and $b = [1,1,0,0,0]$ we have

$$D(a,b) = \text{sum}(\text{abs}([-1,0,1,0,1])) = \text{sum}([1,0,1,0,1]) = 3$$

The notion of structural equivalence depends on which of the sets X, Y and Z are (considered) the same. There are three basic cases: 1) all three sets are different – 3-mode network; 2) two sets are the same – 2-mode network; and 3) all three sets are the same – 1-mode network.

3 Case 1: All three sets are different

In this case we have a structural equivalence on each of the sets X, Y and Z. This is defined on the set X as follows:

The units $i,j \in X$ are *structurally equivalent*, $i \approx j$, iff

$$\forall p \in Y \forall u \in Z : (R(i,p,u) \Leftrightarrow R(j,p,u))$$

This is equivalent to the conditions that the 'planes' corresponding to i and j are equal $R(i,\cdot,\cdot) = R(j,\cdot,\cdot)$. The corresponding dissimilarity

$$d(i,j) = D(R[i,\cdot,\cdot], R[j,\cdot,\cdot])$$

is *compatible* with structural equivalence

$$i \approx j \Leftrightarrow d(i,j) = 0$$

The other two cases can be reduced to this one by permuting dimensions.

The solution consists of three structural equivalences $\approx_X$, $\approx_Y$ and $\approx_Z$, corresponding to three partition functions (π,σ,τ): $i \approx_X j \Leftrightarrow \pi(i) = \pi(j)$, etc.

If R is a 3D structural equivalence the only possible blocks in R with respect to clusters determined by this solution are null and complete blocks.

4 Case 2: Two sets are the same

Assume that $Y = Z$ and X is (considered as) different. The other two cases can be reduced to this one by permuting dimensions. The solution consists of two equivalencies / partitions (π, σ). The first equivalence is defined in the same way as in Case 1.

For the second equivalence the conditions are less trivial. Conceptually two units p and q are structurally equivalent if they are interchangeable – but in our case if we swap units p and q in the set Y we have to swap them also in the set Z.

The *interchangeability condition* – definition of structural equivalence, $p \approx q$, is now

$$\begin{aligned}
\forall i \in X : (\ & \forall r \in Y \setminus \{p,q\} : (R(i,p,r) \Leftrightarrow R(i,q,r)) \\
& \wedge \forall r \in Y \setminus \{p,q\} : (R(i,r,p) \Leftrightarrow R(i,r,q)) \\
& \wedge (R(i,p,q) \Leftrightarrow R(i,q,p)) \\
& \wedge (R(i,p,p) \Leftrightarrow R(i,q,q))\ )
\end{aligned}$$

The corresponding compatible dissimilarity is

$$\begin{aligned}
d(p,q) = & D(R[\cdot,p,-\{p,q\}], R[\cdot,q,-\{p,q\}]) \\
& + D(R[\cdot,-\{p,q\},p], R[\cdot,-\{p,q\},q]) \\
& + D(R[\cdot,p,q], R[\cdot,q,p]) \\
& + D(R[\cdot,p,p], R[\cdot,q,q])
\end{aligned}$$

If R is a 3D structural equivalence the blocks in R with respect to the partitions (π, σ) are null and complete blocks, but on $Y \times Z$ diagonals can be also zero diagonal planes in complete blocks and one diagonal planes in null blocks.

5 Case 3: All three sets are the same

In this case $X = Y = Z$. The units i and j are swapped in all three sets. They are structurally equivalent, $i \approx j$, iff

$$\begin{aligned}
& \forall u,r \in X \setminus \{i,j\} : (R(i,u,r) \Leftrightarrow R(j,u,r)) \\
\wedge\ & \forall u,r \in X \setminus \{i,j\} : (R(u,i,r) \Leftrightarrow R(u,j,r)) \\
\wedge\ & \forall u,r \in X \setminus \{i,j\} : (R(u,r,i) \Leftrightarrow R(u,r,j)) \\
\wedge\ & \forall r \in X \setminus \{i,j\} : (R(i,j,r) \Leftrightarrow R(j,i,r)) \\
\wedge\ & \forall r \in X \setminus \{i,j\} : (R(i,r,j) \Leftrightarrow R(j,r,i)) \\
\wedge\ & \forall r \in X \setminus \{i,j\} : (R(r,i,j) \Leftrightarrow R(r,j,i)) \\
\wedge\ & \forall r \in X \setminus \{i,j\} : (R(i,i,r) \Leftrightarrow R(j,j,r)) \\
\wedge\ & \forall r \in X \setminus \{i,j\} : (R(i,r,i) \Leftrightarrow R(j,r,j)) \\
\wedge\ & \forall r \in X \setminus \{i,j\} : (R(r,i,i) \Leftrightarrow R(r,j,j)) \\
\wedge\ & (R(i,i,j) \Leftrightarrow R(j,j,i)) \\
\wedge\ & (R(i,j,i) \Leftrightarrow R(j,i,j)) \\
\wedge\ & (R(j,i,i) \Leftrightarrow R(i,j,j)) \\
\wedge\ & (R(i,i,i) \Leftrightarrow R(j,j,j))
\end{aligned}$$

The corresponding compatible dissimilarity is

$$\begin{aligned} d(i,j) = {} & D(R[i,-\{i,j\},-\{i,j\}],R[j,-\{i,j\},-\{i,j\}]) \\ & + D(R[-\{i,j\},i,-\{i,j\}],R[-\{i,j\},j,-\{i,j\}]) \\ & + D(R[-\{i,j\},-\{i,j\},i],R[-\{i,j\},-\{i,j\},j]) \\ & + D(R[i,j,-\{i,j\}],R[j,i,-\{i,j\}]) \\ & + D(R[i,-\{i,j\},j],R[j,-\{i,j\},i]) \\ & + D(R[-\{i,j\},i,j],R[-\{i,j\},j,i]) \\ & + D(R[i,i,-\{i,j\}],R[j,j,-\{i,j\}]) \\ & + D(R[i,-\{i,j\},i],R[j,-\{i,j\},j]) \\ & + D(R[-\{i,j\},i,i],R[-\{i,j\},j,j]) \\ & + D(R[i,i,j],R[j,j,i]) \\ & + D(R[i,j,i],R[j,i,j]) \\ & + D(R[j,i,i],R[i,j,j]) \\ & + D(R[i,i,i],R[j,j,j]) \end{aligned}$$

We illustrate the indirect approach to structural equivalence for 3-way data with two examples. One is an artificial data set (with known properties) and one real example drawn from the social network literature.

To support the indirect approach to 3-way blockmodeling based on structural equivalence the package `ibm3m` was developed in R (see references).

6 Example 1: Artificial dataset

The first 3-mode dataset consists of randomly generated ideal structure (5 clusters in X, 6 clusters in Y, and 4 clusters in Z; each set has 35 units) obtained using the function `rndMat3m(c(5,6,4),c(35,35,35))` from the package `ibm3m`.

Figure 1 shows the generated data with no obvious patterned structure. The right part of Figure 1 and Figure 2 show the dendrograms obtained for each of the three modes using Ward's method and Figure 3 shows the reordered three mode data. The complete three dimensional blocks are shown clearly on the upper left with the remaining three diagrams showing some slices with block structures.

7 Example 2: Krackhardt's dataset

The real example of a social network is taken from Krackhardt (1987) and takes the form of a $21 \times 21 \times 21$ cube. The dimensions X and Y correspond to individuals in the management team of a high-tech company. The X mode consists of choices made by individuals (with regard to advice getting), the Y mode has the received choices for individuals. The Z mode consists of each individual's perception of the advice getting network for the management team.

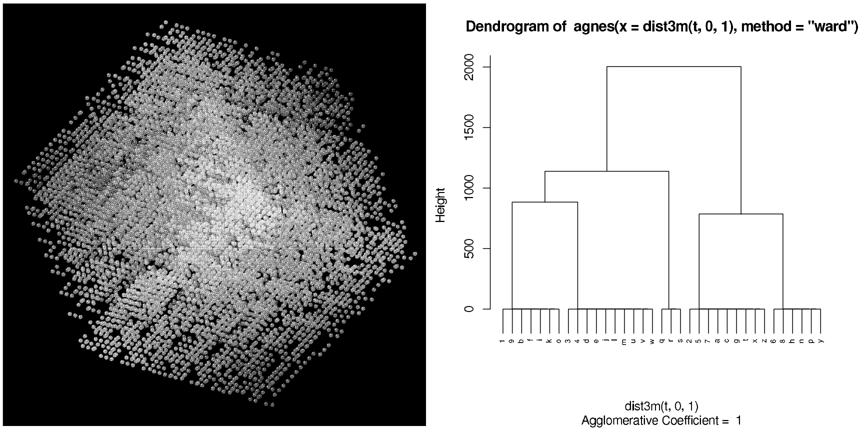

Fig. 1. Artificial dataset – original data and dendrogram on X.

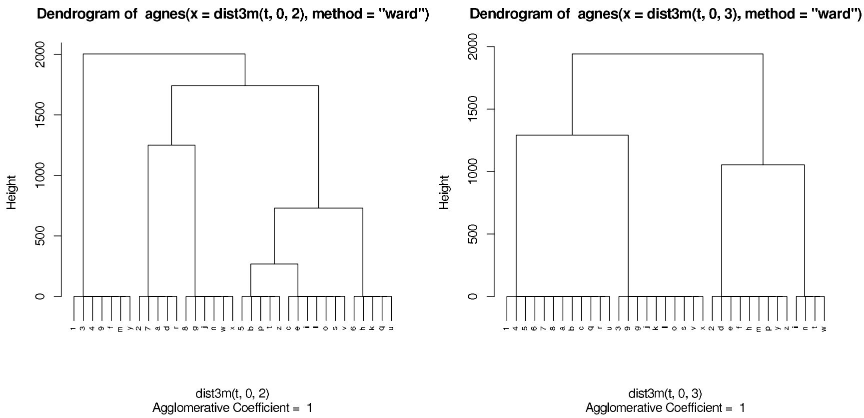

Fig. 2. Artificial dataset – dendrograms on Y and Z.

Figure 4 shows the dendrogram for each of the three dimensions. The dendrogram on the left depicts (approximate) structural equivalence for the sending of help seeking choices and the middle dendrogram depicts structural equivalence for the receipt of these choices. The right hand dendrogram shows structural equivalence for the perceptions of the help seeking relation. The full structural equivalence partition is shown in Figure 5 together with a slice. We can notice that the solution is very far from an ideal structural solution, but some structure can be seen.

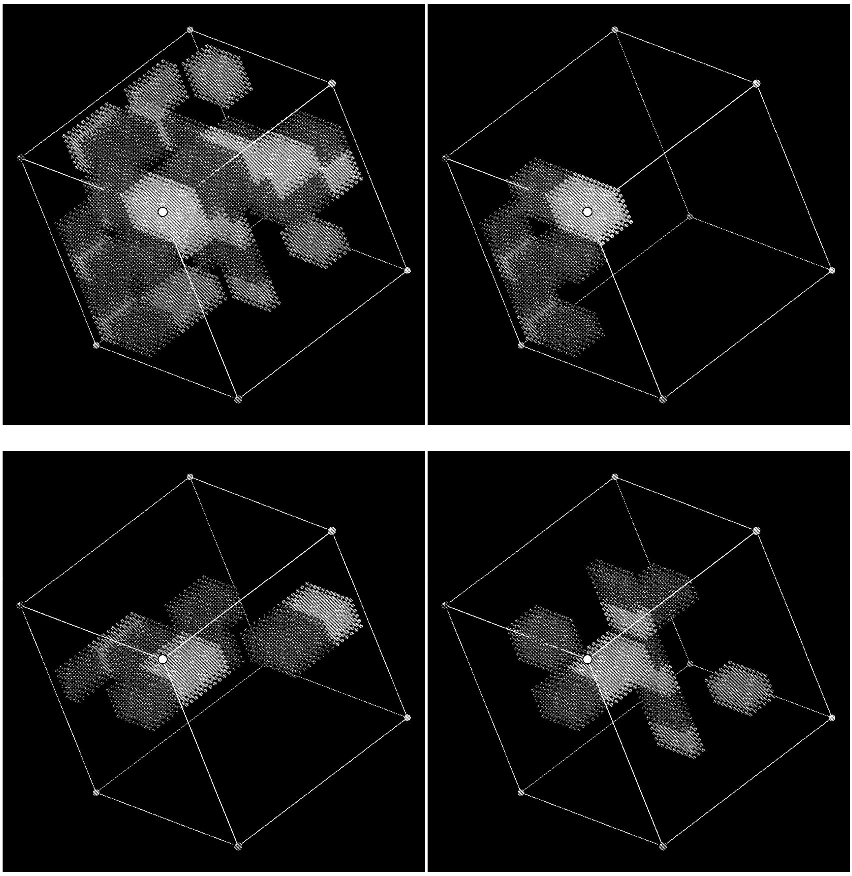

Fig. 3. Artificial dataset – reordered data; complete and some slices.

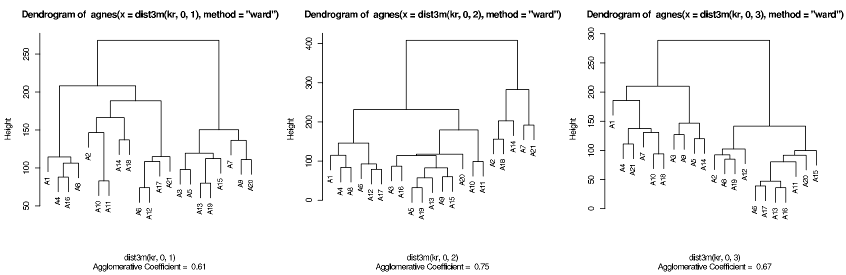

Fig. 4. Krackhardt – dendrograms (X, Y, Z).

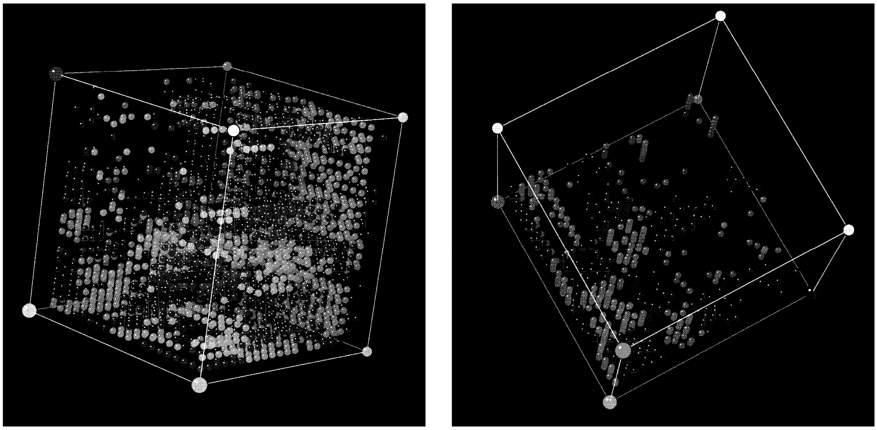

Fig. 5. Krackhardt – Structural Equivalence Partition.

8 R code for the analyses (ibm3m)

Here is a test procedure using the package ibm3m to generate and cluster random ideal datasets. For hierarchical clustering of the obtained dissimilarities and dendrograms drawing the functions agnes and plot from the R-package cluster (Kaufman and Rousseeuw (1990)) are used. The 3D pictures of the data matrix and its slices are exported into kinimages format (see references).

```
rndTest <- function(m=c(3,3,3),n=c(30,30,30),p=0.35){
  t <- rndMat3m(m,n,p)
  saveTriplets3m('test.tri',t,tit="random test")
  rx <- agnes(dist3m(t,0,1),method='ward')
  ry <- agnes(dist3m(t,0,2),method='ward')
  rz <- agnes(dist3m(t,0,3),method='ward')
  pdf('testXD.pdf')
  plot(rx,which.plots=2,nmax.lab=50,cex=0.6); dev.off()
  pdf('testYD.pdf')
  plot(ry,which.plots=2,nmax.lab=50,cex=0.6); dev.off()
  pdf('testZD.pdf')
  plot(rz,which.plots=2,nmax.lab=50,cex=0.6); dev.off()
  kin3m('testOrg.kin',"test - original",t,
    seq(n[1]),seq(n[2]),seq(n[3]))
  kinBlocks3m('testXYZ.kin',"test - all different",t,rx,ry,rz,m)
  if (n[2]==n[3]){
    rb <- agnes(dist3m(t,1,1),method='ward'); pdf('testBD.pdf')
    plot(rb,which.plots=2,nmax.lab=50,cex=0.6); dev.off()
    kinBlocks3m('testXYY.kin',"test - two equal",t,rx,rb,rb,m)
  }
  if ((n[1]==n[2])&(n[2]==n[3])){
    ra <- agnes(dist3m(t,2,0),method='ward'); pdf('testAD.pdf')
    plot(ra,which.plots=2,nmax.lab=50,cex=0.6); dev.off()
    kinBlocks3m('testXXX.kin',"test - all equal",t,ra,ra,ra,m)
  }
}
```

The procedure for analysis of Krackhardt's data is similar – only we have to read the data files:

```
kr <- readDL3m('krack.dat')
la <- paste('A',1:21,sep=''); dimnames(kr) <- list(la,la,la)
```

The computation of dissimilarities is quite time consuming – it is of order $O(n^4)$. But for small networks as in the above examples it takes only some seconds.

9 Discussion

We have presented an approach to blockmodeling 3-way network data using indirect methods. The indirect approach is feasible only when an appropriate compatible dissimilarity can be defined, as is the case for structural equivalence. When a compatible dissimilarity cannot be defined, direct and graph theoretical methods are appropriate. Both approaches to blockmodeling 3-way data are the focus of ongoing work.

Acknowledgments

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