



Clustering in
multiway
networks

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Multiway
networks

ESS10

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Example

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References

Clustering in multiway networks

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Outline

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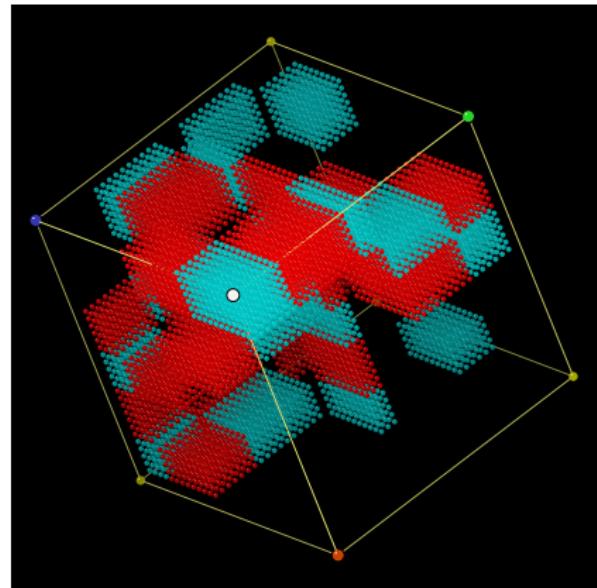
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- 1 Multiway networks
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Current version of slides (August 22, 2024 at 04:25): [slides PDF](#)

<https://github.com/bavla/ibm3m>



Multiway networks

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A *weighted multiway network* $\mathcal{N} = (\mathcal{V}, \mathcal{L}, w)$ is based on *nodes* from k finite sets (ways or dimensions) $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_k)$, the set of *links* \mathcal{L} , and the *weight* $w : \mathcal{L} \rightarrow \mathbb{R}$. The incidence function $I : \mathcal{L} \rightarrow \mathcal{V}_1 \times \mathcal{V}_2 \times \dots \times \mathcal{V}_k$ assigns to each link $e \in \mathcal{L}$ a k -tuple of its nodes $I(e) = (e(1), e(2), \dots, e(i), \dots, e(k))$, $e(i) \in \mathcal{V}_i$. If for $i \neq j$, $\mathcal{V}_i = \mathcal{V}_j$, we say that \mathcal{V}_i and \mathcal{V}_j are of the same *mode*.

In a general multiway network, different additional data can be known for nodes and/or links $\mathcal{N} = (\mathcal{V}, \mathcal{L}, \mathcal{P}, \mathcal{W})$, where \mathcal{P} is a set of node properties $p : \mathcal{V}_i \rightarrow S_p$, and \mathcal{W} is a set of link weights $w : \mathcal{L} \rightarrow S_w$.

We will illustrate the proposed approach by analyzing selected data from ESS - European Social Survey 2023. The approach is supported by the R package **MWnets**.



ESS10 – Media and social trust

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ESS10 (2020) Democracy, Digital social contacts / Media and social trust

- cntry - Country
- ppltrst - Most people can be trusted (10) or you can't be too careful (0)
- pplfair - Most people try to take advantage of you (0), or try to be fair (10)
- pplhlp - Most of the time people helpful (10) or mostly looking out for themselves (0)

77 Refusal; 88 Don't know; 99 No answer

ESS10 – Media and social trust

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```
> str(MM)
List of 6
 $ format: chr "MWNets"
 $ info :List of 4
   ..$ network: chr "ESSmedia-ppl"
   ..$ title : chr "ESS media 2023"
   ..$ by   : chr "DF2MWN"
   ..$ date : chr "Sat Aug 17 02:10:10 2024"
 $ ways :List of 4
   ..$ cntry : chr "cntry"
   ..$ pplrst: chr "pplrstd"
   ..$ pplfair: chr "pplfair"
   ..$ pplhlp : chr "pplhlp"
 $ nodes :List of 4
   ..$ cntry :'data.frame': 22 obs. of 1 variable:
     ...$ ID: chr [1:22] "BE" "BG" "CH" "CZ" ...
   ..$ pplrst:'data.frame': 14 obs. of 1 variable:
     ...$ ID: chr [1:14] "0" "1" "2" "3" ...
   ..$ pplfair:'data.frame': 14 obs. of 1 variable:
     ...$ ID: chr [1:14] "0" "1" "2" "3" ...
   ..$ pplhlp :'data.frame': 14 obs. of 1 variable:
     ...$ ID: chr [1:14] "0" "1" "2" "3" ...
   ..$ links :'data.frame': 37611 obs. of 5 variables:
     ...$ one  : num [1:37611] 1 1 1 1 1 1 1 1 1 ...
     ...$ cntry : int [1:37611] 1 1 1 1 1 1 1 1 1 ...
     ...$ pplrst: int [1:37611] 7 4 7 8 4 5 2 7 8 9 ...
     ...$ pplfair: int [1:37611] 8 5 9 6 9 5 6 5 9 9 ...
     ...$ pplhlp : int [1:37611] 5 4 6 6 9 5 4 4 3 6 ...
 $ data : list()
> table(M$cntry)
   BE   BG   CH   CZ   EE   FI   FR   GB   GR   HR   HU
1341 2718 1523 2476 1542 1577 1977 1149 2799 1592 1849
   IE   IS   IT   LT   ME   MK   NL   NO   PT   SI   SK
1770  903 2640 1659 1278 1429 1470 1411 1838 1252 1418
>
```

Blockmodeling

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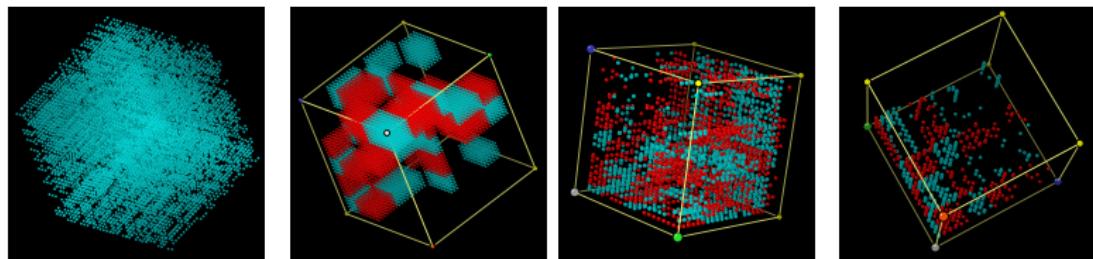
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I already worked on the indirect approach to blockmodeling of binary 3-way networks ([Github bavla/ibm3m](#)) [?]. Lazega, Krackhardt [?]

We were dealing with 3-way networks also at the INSNA 2009 Viszards session analyzing the [Bibsonomy](#) data.



Projections of two-mode networks

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Let $\mathcal{N} = ((\mathcal{U}, \mathcal{V}), \mathcal{L}, w)$ be a weighted two-mode network with a matrix $\mathbf{W} = [w[u, v]]$. A usual approach to its analysis is to project it to the set \mathcal{U} or to the set \mathcal{V} and analyze the so-obtained ordinary (one-mode) weighted network.

The *projection* to the set \mathcal{U} is determined by the matrix $\mathbf{C} = [c[u, t]] = \text{row}(\mathbf{W}) = \mathbf{W}\mathbf{W}^T$; and to the set \mathcal{V} by the matrix $\text{col}(\mathbf{W}) = \mathbf{W}^T\mathbf{W}$.

$\mathcal{N}_C = (\mathcal{U}, \mathcal{L}_C, c)$, where $\mathcal{L}_C = \{(u, t) : c[u, t] \neq 0\}$ and for $(u, t) \in \mathcal{L}_C : c(u, t) = c[u, t]$.

$$c[u, t] = \sum_{v \in \mathcal{V}} w[u, v] \cdot w^T[v, t] = \sum_{v \in \mathcal{V}} w[u, v] \cdot w[t, v]$$



Projection to a selected way

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To make notation simple, we assume that we selected the way \mathcal{V}_1 . A projection to a selected way is a generalization of the projection of two-mode networks. The projection creates an ordinary weighted network $(\mathcal{V}_1, \mathcal{A}, p)$, $\mathcal{A} \subseteq \mathcal{V}_1 \times \mathcal{V}_1$ and $p : \mathcal{A} \rightarrow \mathbb{R}$. Let $u, t \in \mathcal{V}_1$ then

$$p(u, t) = \sum_{(v_2, \dots, v_k) \in \mathcal{V}_2 \times \dots \times \mathcal{V}_k} w(u, v_2, \dots, v_k) \cdot w(t, v_2, \dots, v_k)$$

This network can be analyzed using traditional methods for the analysis of weighted networks. Sometimes it is more appropriate to apply projection(s) to a normalized version of the original multi-way network.

Note that the projection network is symmetric $p(u, t) = p(t, u)$ and considering that the right side in the definition of $p(u, t)$ is a inner (scalar) product we get $p(u, t) \leq \sqrt{p(u, u) \cdot p(t, t)}$ – Cauchy-Schwarz inequality. From it it follows $p(u, u) = p(t, t) = 0 \Rightarrow p(u, t) = 0$.



Salton index

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From the projection p we can get the corresponding measure of similarity – *Salton index* $S(u, t)$ [?]

$$S(u, t) = \frac{p(u, t)}{\sqrt{p(u, u) \cdot p(t, t)}}$$

that can be used for clustering the set \mathcal{V}_1 .

The Salton index has the following properties

- ① $S(u, t) \in [-1, 1]$
- ② $S(u, t) = S(t, u)$
- ③ $S(u, u) = 1$
- ④ $w : L \rightarrow \mathbb{R}_0^+ \Rightarrow S(u, t) \in [0, 1]$
- ⑤ $S(\alpha u, \beta t) = S(u, t), \quad \alpha, \beta > 0$
- ⑥ $S(\alpha u, u) = 1, \quad \alpha > 0$

Clusterings

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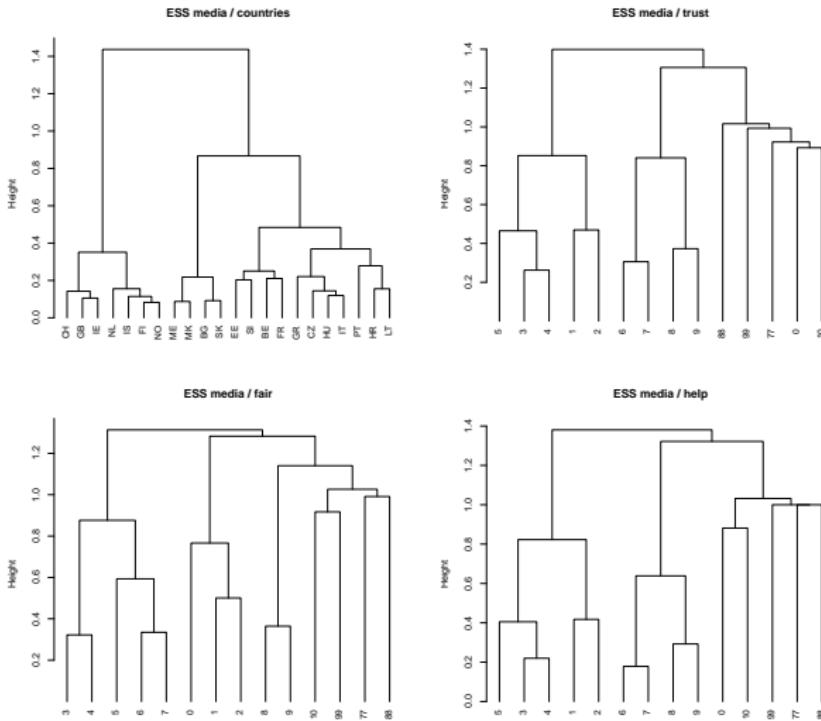
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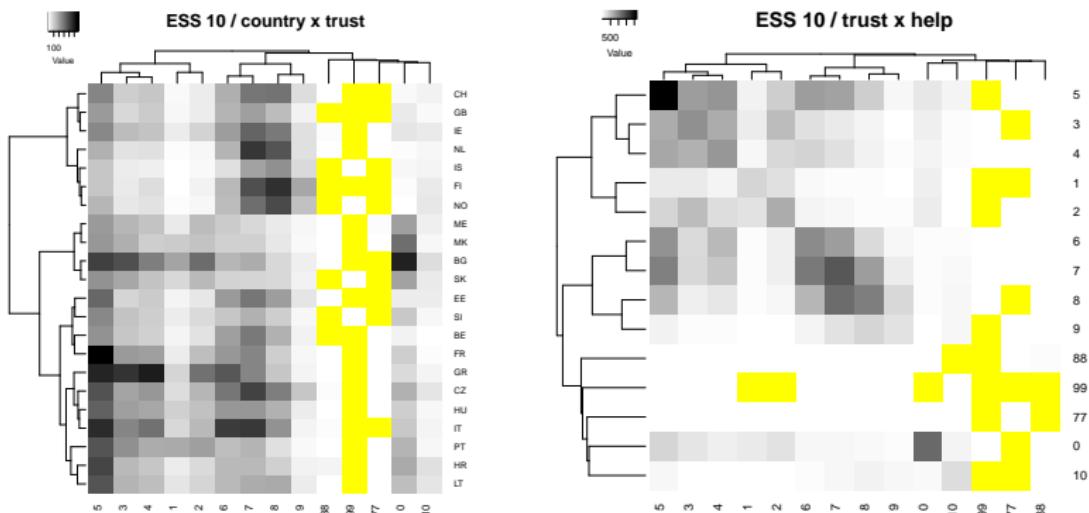
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```
MCT <- flatten(MN,"one",c("cntry","ppltrst"))
MTH <- flatten(MN,"one",c("ppltrst","pplhlp"))
```

yellow cells have value NA



Partitions

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From dendrograms we obtained partitions in 6, 5, 5, 5 clusters

Country:

BE	BG	CH	CZ	EE	FI	FR	GB	GR	HR	HU	IE	IS	IT	LT	ME	MK	NL	NO	PT	SI	SK
1	2	3	4	1	5	1	3	4	6	4	3	5	4	6	2	2	5	5	6	1	2

Trust:

0	1	2	3	4	5	6	7	8	9	10	77	88	99							
1	4	4	2	2	2	3	3	5	5	1	1	1	1							

Fair:

0	1	2	3	4	5	6	7	8	9	10	77	88	99							
1	1	1	5	5	2	2	2	3	3	4	4	4	4							

Help:

0	1	2	3	4	5	6	7	8	9	10	77	88	99							
1	5	5	2	2	2	3	3	3	3	1	4	4	4							

C_{ij} is the j -th cluster in the partition \mathcal{C}_i of the way \mathcal{V}_i .

$$C_{13} = \{CH, GB, IE\}$$



Block density

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$n_i = |\mathcal{V}_i|$ is the size of the i -th way \mathcal{V}_i .

$m = |\mathcal{L}|$ is the number of links \mathcal{L} .

$\mathbf{j} = (j_1, j_2, \dots, j_k)$ is a *selection* of clusters from partitions of all ways.

We define a *block* of the selection \mathbf{j} as $B(\mathbf{j}) = \times_i C_{ij_i}$ and its *volume* $\text{vol}(B(\mathbf{j})) = \prod_i |C_{ij_i}|$.

$B = \times_i \mathcal{V}_i$ and its *volume* $\text{vol}(B) = \prod_i |\mathcal{V}_i| = \prod_i n_i$.

$m(\mathbf{j}) = |\mathcal{L} \cap B(\mathbf{j})|$ is the number of links in the block $B(\mathbf{j})$.

The *density* $D(\mathbf{j})$ of a block $B(\mathbf{j})$ is then $m(\mathbf{j}) = D(\mathbf{j}) \frac{m}{\text{vol}(B)} \text{vol}(B(\mathbf{j}))$ or

$$D(\mathbf{j}) = \frac{m(\mathbf{j}) \cdot \text{vol}(B)}{m \cdot \text{vol}(B(\mathbf{j}))}$$

An alternative considering the weight w of links is to use $m = \sum_{e \in \mathcal{L}} w(e)$ and $m(\mathbf{j}) = \sum_{e \in \mathcal{L} \cap B(\mathbf{j})} w(e)$.

We inspect the blocks with the largest density.



Top 40 blocks

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1 30.22031 1205	11 13.21204 889	21 8.777685 175	31 6.95527 208
1 cntry FI+IS+NL+NO	1 cntry BE+EE+FR+SI	1 cntry BG+ME+MK+SK	1 cntry BG+ME+MK+SK
2 ppltrst 8+9	2 ppltrst 3+4+5	2 ppltrst 1+2	2 ppltrst 1+2
3 pplfair 8+9	3 pplfair 5+6+7	3 pplfair 3+4	3 pplfair 3+4
4 pplhlp 6+7+8+9	4 pplhlp 3+4+5	4 pplhlp 1+2	4 pplhlp 3+4+5
2 25.61412 1149	12 13.10801 784	22 8.530609 287	32 6.925547 233
1 cntry CZ+GR+HU+IT	1 cntry FI+IS+NL+NO	1 cntry HR+LT+PT	1 cntry CH+GB+IE
2 ppltrst 3+4+5	2 ppltrst 6+7	2 ppltrst 3+4+5	2 ppltrst 3+4+5
3 pplfair 3+4	3 pplfair 5+6+7	3 pplfair 3+4	3 pplfair 3+4
4 pplhlp 3+4+5	4 pplhlp 6+7+8+9	4 pplhlp 3+4+5	4 pplhlp 3+4+5
3 22.75324 1531	13 12.46153 559	23 8.322546 420	33 6.696119 267
1 cntry CZ+GR+HU+IT	1 cntry BE+EE+FR+SI	1 cntry CH+GB+IE	1 cntry BE+EE+FR+SI
2 ppltrst 3+4+5	2 ppltrst 6+7	2 ppltrst 3+4+5	2 ppltrst 8+9
3 pplfair 5+6+7	3 pplfair 5+6+7	3 pplfair 5+6+7	3 pplfair 8+9
4 pplhlp 3+4+5	4 pplhlp 3+4+5	4 pplhlp 3+4+5	4 pplhlp 6+7+8+9
4 20.83237 1246	14 12.08813 482	24 8.292822 248	34 6.568866 442
1 cntry CZ+GR+HU+IT	1 cntry CZ+GR+HU+IT	1 cntry CH+GB+IE	1 cntry CH+GB+IE
2 ppltrst 6+7	2 ppltrst 8+9	2 ppltrst 6+7	2 ppltrst 3+4+5
3 pplfair 5+6+7	3 pplfair 8+9	3 pplfair 8+9	3 pplfair 5+6+7
4 pplhlp 6+7+8+9	4 pplhlp 6+7+8+9	4 pplhlp 6+7+8+9	4 pplhlp 6+7+8+9
5 17.82288 533	15 11.23544 336	25 7.985185 597	35 6.453688 193
1 cntry CH+GB+IE	1 cntry BG+ME+MK+SK	1 cntry BG+ME+MK+SK	1 cntry CZ+GR+HU+IT
2 ppltrst 8+9	2 ppltrst 1+2	2 ppltrst 0+10+77+88+99	2 ppltrst 6+7
3 pplfair 8+9	3 pplfair 0+1+2	3 pplfair 0+1+2	3 pplfair 3+4
4 pplhlp 6+7+8+9	4 pplhlp 1+2	4 pplhlp 0+10	4 pplhlp 3+4+5



... Top 40 blocks

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6 17.54422 787 1 cntry CZ+GR+HU+IT 2 ppltrst 6+7 3 pplfair 5+6+7 4 pplhlp 3+4+5	16 11.18528 446 1 cntry FI+IS+NL+NO 2 ppltrst 6+7 3 pplfair 8+9 4 pplhlp 6+7+8+9	26 7.824679 234 1 cntry CZ+GR+HU+IT 2 ppltrst 1+2 3 pplfair 3+4 4 pplhlp 3+4+5	36 6.397957 287 1 cntry BG+ME+MK+SK 2 ppltrst 6+7 3 pplfair 5+6+7 4 pplhlp 3+4+5
7 16.09025 812 1 cntry HR+LT+PT 2 ppltrst 3+4+5 3 pplfair 5+6+7 4 pplhlp 3+4+5	17 10.85089 649 1 cntry BE+EE+FR+SI 2 ppltrst 6+7 3 pplfair 5+6+7 4 pplhlp 6+7+8+9	27 7.79124 233 1 cntry CZ+GR+HU+IT 2 ppltrst 3+4+5 3 pplfair 3+4 4 pplhlp 1+2	37 6.397957 287 1 cntry HR+LT+PT 2 ppltrst 6+7 3 pplfair 5+6+7 4 pplhlp 6+7+8+9
8 16.07292 721 1 cntry CH+GB+IE 2 ppltrst 6+7 3 pplfair 5+6+7 4 pplhlp 6+7+8+9	18 9.897884 296 1 cntry CZ+GR+HU+IT 2 ppltrst 1+2 3 pplfair 0+1+2 4 pplhlp 1+2	28 7.520014 253 1 cntry CH+GB+IE 2 ppltrst 6+7 3 pplfair 5+6+7 4 pplhlp 3+4+5	38 6.375664 286 1 cntry BE+EE+FR+SI 2 ppltrst 3+4+5 3 pplfair 3+4 4 pplhlp 3+4+5
9 14.06659 631 1 cntry BG+ME+MK+SK 2 ppltrst 3+4+5 3 pplfair 3+4 4 pplhlp 3+4+5	19 9.749268 328 1 cntry HR+LT+PT 2 ppltrst 6+7 3 pplfair 5+6+7 4 pplhlp 3+4+5	29 7.423413 222 1 cntry BG+ME+MK+SK 2 ppltrst 3+4+5 3 pplfair 3+4 4 pplhlp 1+2	39 6.319933 189 1 cntry FI+IS+NL+NO 2 ppltrst 8+9 3 pplfair 8+9 4 pplhlp 3+4+5
10 13.98485 941 1 cntry BG+ME+MK+SK 2 ppltrst 3+4+5 3 pplfair 5+6+7 4 pplhlp 3+4+5	20 9.318279 418 1 cntry FI+IS+NL+NO 2 ppltrst 6+7 3 pplfair 5+6+7 4 pplhlp 3+4+5	30 7.272939 145 1 cntry CZ+GR+HU+IT 2 ppltrst 1+2 3 pplfair 3+4 4 pplhlp 1+2	40 6.202897 371 1 cntry FI+IS+NL+NO 2 ppltrst 8+9 3 pplfair 5+6+7 4 pplhlp 6+7+8+9

Blocks / Country clusters

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	1	30.22031	12	13.10801	16	11.18528	20	9.318279	39	6.319933	40	6.202897
		1205		784		446		418		189		371
Multiway networks	1	cntry	FI+IS+NL+NO									
	2	pplrst	8+9	6+7	6+7		6+7					
	3	pplfair	8+9	5+6+7			5+6+7				5+6+7	
	4	pplhlp	6+7+8+9				3+4+5		3+4+5			
ESS10				5	8	23	24	28	32	34		
				17.82288	16.07292	8.322546	8.292822	7.520014	6.925547	6.568866		
Clustering Example				533	721	420	248	253	233	442		
Conclusions	1	cntry	CH+GB+IE									
	2	pplrst	8+9	6+7	3+4+5		6+7	6+7	3+4+5	3+4+5		
	3	pplfair	8+9	5+6+7	5+6+7		5+6+7	5+6+7	3+4	3+4	5+6+7	
	4	pplhlp	6+7+8+9		3+4+5		3+4+5	3+4+5	3+4+5	3+4+5		
References				11	13	17	33	38				
				13.21204	12.46153	10.85089	6.696119	6.375664				
				889	559	649	267	286				
	1	cntry	BE+EE+FR+SI									
	2	pplrst	3+4+5	6+7	6+7		8+9					
	3	pplfair	5+6+7				8+9		3+4			
	4	pplhlp	3+4+5		6+7+8+9		6+7+8+9					

... Blocks / Country clusters

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Multiway networks

7	16.09025	19	9.749268	22	8.530609	37	6.397957
	812		328		287		287

1 cntry HR+LT+PT

2 pplrst 3+4+5

3 pplfair 5+6+7

4 pplhlp 3+4+5

6+7

6+7

3+4

6+7+8+9

ESS10

2	25.61412	3	22.75324	4	20.83237	6	17.54422	14	12.08813	18	9.897884	26	7.824679	27	7.79124	30	
	1149		1531		1246		787		482		296		234		233		145

1 cntry CZ+GR+HU+IT

2 pplrst 3+4+5

3 pplfair 3+4

4 pplhlp 3+4+5

6+7

6+7

8+9

1+2

1+2

1+2

Conclusions

9	14.06659	10	13.98485	15	11.23544	21	8.777685	29	7.423413	31	6.95527	36	6.397957	25	
	631		941		336		175		222		208		287		597

1 cntry BG+ME+MK+SK

2 pplrst 3+4+5

3 pplfair 3+4

4 pplhlp 3+4+5

1+2

1+2

1+2

1+2

References

9	14.06659	10	13.98485	15	11.23544	21	8.777685	29	7.423413	31	6.95527	36	6.397957	25	
	631		941		336		175		222		208		287		597

1 cntry BG+ME+MK+SK

2 pplrst 3+4+5

3 pplfair 3+4

4 pplhlp 3+4+5

0+1+2

0+1+2

1+2

1+2

1+2

1+2

1+2

1+2

Flags

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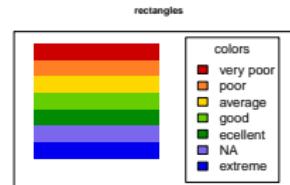
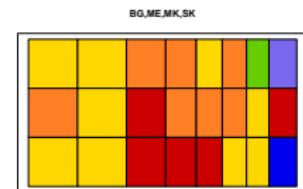
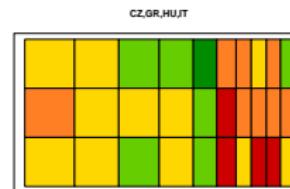
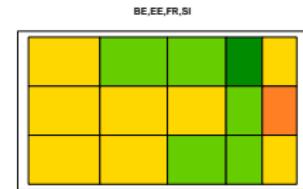
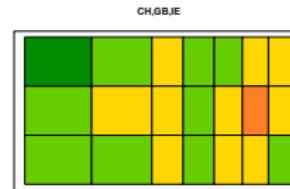
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Work in progress.

- how to consider the weights in ESS?
- more readable presentation/visualization of results
- sparse networks with many ways
- *Jaccard index*

$$J(u, t) = \frac{p(u, t)}{p(u, u) + p(t, t) - p(u, t)}$$



Acknowledgments

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References

The computational work reported in this presentation was performed in R using the multiway network analysis library *MwNets*. The code and data are available at <https://github.com/bavla/ibm3m>.

This work is supported in part by the Slovenian Research Agency (research program P1-0294 and research projects J5-2557, J1-2481, and J5-4596), and prepared within the framework of the COST action CA21163 (HiTEc).



References I

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References

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Partial projections

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In multiway networks with many ways, the probability of co-appearance can be very small - almost all non-diagonal projections $p(u, t)$ are zero.

An option is to use for a way a “partial projections” to “planes” $(\mathcal{V}_a, \mathcal{V}_b), b \in I_a$ where $I_a = 1 : k \setminus \{a\}$

$$\mathbf{M}_{ab} = \text{flatten}(M, w, (\mathcal{V}_a, \mathcal{V}_b))$$

we define a projection

$$\mathbf{Q}_a = \sum_{b \in I_a} \mathbf{M}_{ab} \cdot \mathbf{M}_{ab}^T$$

or collecting the planes in a matrix $\mathbf{K}_a = [\mathbf{M}_{a1}, \mathbf{M}_{a2}, \dots, \mathbf{M}_{ak}]$ we get

$$\mathbf{Q}_a = \mathbf{K}_a \cdot \mathbf{K}_a^T$$

In the package MwNets it is implemented with functions `projector2` and `projector`.