Introduction to the analysis of multiway networks

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Outline

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2 Multiway networks
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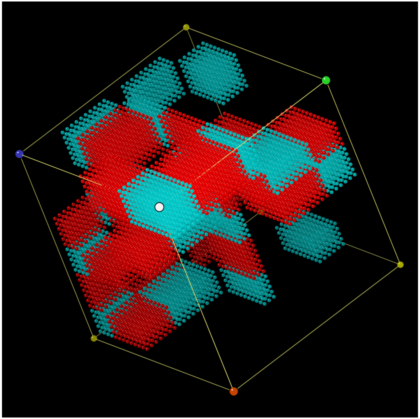

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Current version of slides (April 5, 2023 at 02 : 29): slides PDF
https://github.com/bavla/ibm3m/

IFCS 2022

My renewed interest in the analysis of multiway networks was motivated by my Italian colleagues from Napoli and Salerno. At the IFCS 2022 conference in July in Porto, they were presenting their research on Italian students' mobility. They developed their own approach and asked me for comments and alternatives.

Italian students' data

Their data consists of quadruples of *ways* (province, university, study program, year) with the corresponding value (weight) counting how many students from a given province selected in a given year to study a given program at a given university. Such data are called *weighted multiway networks*.

The approach proposed by my Italian colleagues was to analyze the data for each year separately by joining the *ways* university and study program into a single way [university, study program] thus obtaining a weighted two-mode network that can be analyzed using available methods.

I first transformed in R their data into Pajek format and applied some additional methods on them.

Afterward, I created an R representation of multiway networks and saved the data in JSON format.

Multiway networks

A *weighted multiway network* $\mathcal{N} = (\mathcal{V}, \mathcal{L}, w)$ is based on *nodes* from k finite sets (ways or dimensions) $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_k)$, the set of *links* $\mathcal{L} \subseteq \mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_k$, and the *weight* $w : \mathcal{L} \to \mathbb{R}$. If for $i \neq j$, $\mathcal{V}_i = \mathcal{V}_j$, we say that $\mathcal{V}_i$ and $\mathcal{V}_j$ are of the same *mode*.

In a general multiway network, different additional data can be known for nodes and/or links $\mathcal{N} = (\mathcal{V}, \mathcal{L}, \mathcal{P}, \mathcal{W})$, where $\mathcal{P}$ is a set of node properties $p : \mathcal{V}_i \to S_p$, and $\mathcal{W}$ is a set of link weights $w : \mathcal{L} \to S_w$.

Let $e \in \mathcal{L}$ be a link. With $e(i)$ we denote the i-th component of $e = (e(1), e(2), \ldots, e(i), \ldots, e(k))$.

For a subset of links $\mathcal{L}' \subseteq \mathcal{L}$ we denote

$$\mathcal{V}_i(\mathcal{L}') = \{e(i) : e \in \mathcal{L}'\}$$

the set of nodes from the way $\mathcal{V}_i$ that are an end-node of a link from $\mathcal{L}'$.

Simple networks

A multiway network $\mathcal{N}$ is *simple* if and only if all its links (tuples) are mutually different.

A multiway network $\mathcal{N}' = ((\mathcal{V}'_1, \mathcal{V}'_2, \ldots, \mathcal{V}'_k), \mathcal{L}', w')$ is a *subnetwork* of the multiway network $\mathcal{N} = ((\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_k), \mathcal{L}, w)$ if and only if $\mathcal{V}'_i \subseteq \mathcal{V}_i$ for all $i \in 1..k$, $\mathcal{L}' \subseteq \mathcal{L}$, $\mathcal{L}' \subseteq \mathcal{V}'_1 \times \mathcal{V}'_2 \times \ldots \times \mathcal{V}'_k$, and for all $e \in \mathcal{L}'$ holds $w'(e) = w(e)$.

The notion of a subnetwork could be extended to the omission of some ways.

For a node $u \in \mathcal{V}_i$ we call a *star* in the node u, the set of links

$$S(u) = \{e \in \mathcal{L} : e(i) = u\}.$$

Multiway network representation in R

I decided to use for multiway networks the following representation in R

```
MW <- list(
    format="MWnets",
    info= metadata,
    ways= list of ways' names,
    nodes= list of ways' dataframes,
    links= dataframe of links and weights,
    data= list of additional data
)
```

Italian students mobility

```
> str(MN)
List of 6
 $ format: chr "MWnets"
 $ info  :List of 5
  ..$ network: chr "students"
  ..$ title  : chr "Student mobility data in Italian universities 2008, 2011, 2014, 2017"
  ..$ by     : chr [1:4] "VG Genova" "G Giordano" "G Ragozini" "MP Vitale"
  ..$ creator: chr "V. Batagelj"
  ..$ date   : chr "Sun Nov  6 22:31:11 2022"
 $ ways  :List of 4
  ..$ prov: chr "province"
  ..$ univ: chr "university"
  ..$ prog: chr "programme"
  ..$ year: chr "year"
 $ nodes :List of 4
  ..$ prov:'data.frame':        107 obs. of  9 variables:
  .. ..$ ID        : chr [1:107] "AG" "AL" "AN" "AO" ...
  .. ..$ type      : chr [1:107] "F" "O" "O" "D" ...
  .. ..$ province  : chr [1:107] "Agrigento" "Alessandria" "Ancona" "Aosta" ...
  .. ..$ capital   : chr [1:107] "Agrigento" "Alessandria" "Ancona" "Aosta" ...
  .. ..$ region    : chr [1:107] "Sicily" "Piedmont" "Marche" "Aosta Valley" ...
  .. ..$ IDreg     : chr [1:107] "SIC" "PIE" "MAR" "VAL" ...
  .. ..$ macreg    : chr [1:107] "Insular" "North-West" "Centre" "North-West" ...
  .. ..$ population: int [1:107] 416181 409392 464419 124089 203425 290811 336501 209390 402929
  .. ..$ area      : num [1:107] 3053 3559 1963 3261 1228 ...
  ..$ univ:'data.frame':        79 obs. of  2 variables:
  .. ..$ ID  : chr [1:79] "Bicocca" "Bocconi" "Foscari" "Biomedico" ...
  .. ..$ long: chr [1:79] "Bicocca" "Bocconi" "Cà Foscari" "Campus Biomedico" ...
  ..$ prog:'data.frame':        11 obs. of  2 variables:
  .. ..$ ID  : chr [1:11] "Q" "AFFV" "AH" "BAL" ...
  .. ..$ long: chr [1:11] "?" "Agriculture, forestry, fisheries and veterinary" "Arts and humani
  ..$ year:'data.frame':        4 obs. of  1 variable:
  .. ..$ ID: chr [1:4] "2008" "2011" "2014" "2017"
```

Italian students mobility

```
$ links :'data.frame': 37205 obs. of  5 variables:
 ..$ prov: int [1:37205] 1 1 1 1 1 1 1 1 1 1 ...
 ..$ univ: int [1:37205] 1 1 1 1 2 3 4 5 5 5 ...
 ..$ prog: int [1:37205] 4 5 9 11 4 3 9 3 5 9 ...
 ..$ year: int [1:37205] 1 1 1 1 1 1 1 1 1 1 ...
 ..$ w   : int [1:37205] 4 1 1 1 11 1 1 1 1 1 ...
$ data  :List of 1
 ..$ regs:'data.frame':        20 obs. of  2 variables:
 .. ..$ ID  : chr [1:20] "ABR" "BAS" "CAL" "CAM" ...
 .. ..$ long: chr [1:20] "Abruzzo" "Basilicata" "Calabria" "Campania" ...
```

Multi-relational networks

A special kind of multiway networks are multi-relational (multiplex) networks.

In a *multi-relational* network

$$\mathcal{N} = (\mathcal{V}, (\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_k), \mathcal{P}, \mathcal{W})$$

the set of links $\mathcal{L}$ is partitioned into different relations (sets of links) $(\mathcal{L}_i)_{i \in I}$ over the same set of nodes. Also the weights from $\mathcal{W}$ are defined on different relations or their union.

The corresponding multiway network $\mathcal{M} = (\mathcal{V}_M, \mathcal{L}_M, \mathcal{W}_M)$ is constructed as follows:

- $\mathcal{V}_M = (\mathcal{V}, \mathcal{V}, 1..k)$
- the link $(u, v) \in \mathcal{L}_i$ with a weight $w_i(u, v)$ produces a link $e = (u, v, i) \in \mathcal{V}_M$ and $w_M(e) = w_i(u, v)$

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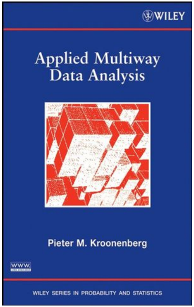
References

Sources of multiway networks

- Manlio De Domenico: Datasets Released for Reproducibility
- Alephsys NETWORK DATASETS HIGGS TWITTER, CS-AARHUS, VICKERS CHAN 7TH GRADERS
- Multiplex networks StarWars
- zalmquist/networkdata Krackhardt
- sciMAG2015 data access???
- ICON - The Colorado Index of Complex Networks
- UCI: Cars, Mushrooms
- PEW: GitHub, Data lab, Tools and resources, Data, Belief, Global religion
- European social survey

Multiway analysis



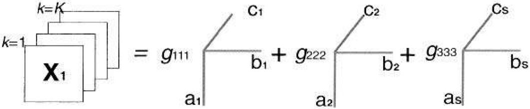
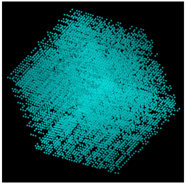
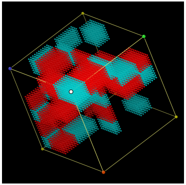
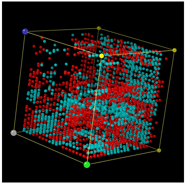
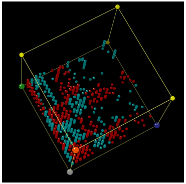


Figure 4.9 The Parafac model.

In data analysis already exists an approach called *multiway analysis* which deals with generalizations of ideas from the multivariate analysis to multiway arrays. They are known as Tucker, Parafac, SimPCA, . . . models [14, 17].

We will continue in another, "combinatorial" direction.

Blockmodeling

I already worked (with Anuška and Pat, in 2006) on the indirect approach to blockmodeling of binary 3-way networks (Github bavla/ibm3m) [4]. Lazega, Krackhardt [6]

We were dealing with 3-way networks also at the INSNA 2009 Viszards session analyzing the Bibsonomy data.

Drilling into the data of my Italian friends I got some ideas for possible approaches.

MWnets

To support the analysis of multiway networks I started to develop an R package `MWnets`. It is available at GitHub/Bavla

https://github.com/bavla/ibm3m

```
> source("https://raw.githubusercontent.com/bavla/ibm3m/master/
multiway/MWnets.R")
```

Multiway network data sets

https://github.com/bavla/ibm3m/tree/master/data

Some functions

`reorderways(MN,ord)` – Reorder ways of the multiway network MN in the order determined in the vector ord.

`slice(MN,P)` – Extract from the multiway network MN a subnetwork of links satisfying a predicate P on links columns.

`extract(MN,ways,clus)` – Extract from the multiway network MN a subnetwork determined by selected ways and corresponding clusters (vectors containing list of nodes).

`flatten(MN,col,by,FUN="sum")` – Removes ways that are not listed in the vector by and applies FUN on the weights col for so obtained duplicates.

$$\mathcal{V}' = (\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_{i-1}, \mathcal{V}_{i+1}, \ldots, \mathcal{V}_k)$$

$$w'(v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k) = \sum_{v \in \mathcal{V}_i} w(v_1, v_2, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_k)$$

Some functions

It can be used, by contains a complete list of ways, to aggregate duplicate links in the original data. A selected way is removed from the network.

`reorderlinks(MN,per=NULL)` – Reorders the links (rows) by the given permutation per.
per = NULL - random, per = permutation vector, per = expression

`joinways(MN,way1,way2,way3,sep="÷")` – Replaces a given pair of ways way_1 and way_2 with a new way way_3:
$way_3 = \{a \div b : \exists e \in \mathcal{L} : e(way_1) = a \wedge e(way_2) = b\}$

Robins Bank

Data from the paper Pattison, Wasserman, Robins, Kanfer: Statistical Evaluation of Algebraic Constraints for Social Networks. J of Math Psychology, 44(2000)4, 536-568.
I copied the matrices from the paper, put them into Pajek format, and saved them as a list of arcs to "bankArc.csv".

```
> wdir <- "C:/Users/vlado/docs/papers/2022/ifcs2022/genova/MWnets/Bank"
> setwd(wdir)
> library(jsonlite)
> V <- read.table("https://raw.githubusercontent.com/bavla/ibm3m/master/data/bankArc.csv")
> names(V) <- c("P1","P2","R")
> V$w <- rep(1,nrow(V))
> S <- c("BM","DM","A1","A2","A3","T1","T2","T3","T4","T5","T6")
> P <- c("branch manager", "deputy manager", "service adviser 1",
+  "service adviser 2", "service adviser 3", "teller 1", "teller 2",
+  "teller 3", "teller 4", "teller 5", "teller 6" )
> R <- c("Advice seeking","Close friendship","Satisfying interaction","Confiding")
> Q <- c("advice","friend","satisfy","confide")
> info <- list(network="Robins bank",
+   title="Robins: organizational structure in a bank branch",
+   by="Garry Robins",
+   ref="Pattison, etal: Statistical Evaluation of Algebraic Constraints for Social Networks. J
+   href="https://www.sciencedirect.com/science/article/pii/S0022249699912610",
+   creator="V. Batagelj",
+   date=date() )
> ways=list(P1="first person",P2="second person",R="relation")
> nodes=list(P1=data.frame(ID=S,long=P),P2=data.frame(ID=S),R=data.frame(ID=Q,long=R))
> MN <- list(format="MWnets",info=info,ways=ways,nodes=nodes,links=V,data=list())
> write(toJSON(MN),"RobinsBank.json")
```

Data from the paper

Label	Relation	Label	Relation
Advice − seeking	0 1 1 1 0 0 0 0 0 0 0	*Close*	0 1 1 1 1 0 0 0 0 0 0
A	1 0 0 0 0 0 0 0 0 0 0	*friendship*	0 0 0 0 0 0 0 0 0 0 0
	0 1 0 0 0 0 0 0 0 0 0	**F**	0 0 0 1 0 1 0 0 0 0 0
	1 0 0 0 0 0 0 0 0 0 0		0 0 1 0 0 1 0 0 0 0 1
	0 0 0 0 0 0 0 0 0 0 0		0 0 0 0 0 0 0 0 0 0 0
	0 0 1 1 1 0 0 0 0 0 0		0 0 1 1 0 0 0 0 0 0 1
	1 1 1 1 1 1 0 0 0 0 0		0 0 1 1 0 1 0 0 1 0 0
	1 1 1 1 0 1 1 0 1 0 0		0 0 0 1 0 0 0 0 1 0 0
	1 1 1 0 0 0 0 0 0 0 0		0 0 0 0 0 0 0 1 0 0 0
	1 0 0 0 0 0 0 0 0 0 0		0 0 0 1 0 0 0 0 0 0 0
	0 0 1 1 0 1 1 0 0 0 0		0 0 0 0 0 0 0 0 0 0 0
Satisfying	0 0 0 0 0 0 0 0 0 0 0	*Confiding*	0 1 1 1 0 0 0 0 0 0 0
interaction	1 0 1 1 1 1 1 1 1 1 1	**C**	1 0 0 0 0 0 0 0 0 0 0
S	1 1 0 0 0 0 0 0 0 0 0		0 0 0 0 0 0 0 0 0 0 0
	1 1 1 0 1 1 1 1 1 1 1		1 0 0 0 0 0 0 0 0 0 0
	1 1 1 1 0 1 1 1 1 1 1		1 1 1 1 0 1 1 0 0 0 0
	0 0 1 1 0 0 0 0 0 0 0		0 0 1 1 0 0 0 0 0 0 0
	1 0 1 1 1 1 0 1 0 0 0		0 0 1 1 1 0 0 0 0 0 0
	0 0 1 1 1 0 1 0 1 0 1		1 0 0 1 0 0 0 0 1 0 0
	0 0 0 0 0 0 0 0 0 0 0		1 0 0 0 0 0 0 1 0 0 0
	0 0 0 1 0 0 0 0 0 0 0		1 0 0 1 0 0 0 0 0 0 0
	0 0 0 1 0 1 1 0 0 1 0		0 0 0 1 0 1 1 0 0 1 0

RobinsBank.json structure

```
> str(MN)
List of 6
 $ format: chr "MWnets"
 $ info  :List of 7
  ..$ network: chr "Robins bank"
  ..$ title  : chr "Robins: organizational structure in a bank branch"
  ..$ by     : chr "Garry Robins"
  ..$ ref    : chr "Pattison, etal: Statistical Evaluation of Algebraic Constraints for Social N
  ..$ href   : chr "https://www.sciencedirect.com/science/article/pii/S0022249699912610"
  ..$ creator: chr "V. Batagelj"
  ..$ date   : chr "Wed Mar 29 03:29:28 2023"
 $ ways  :List of 3
  ..$ P1: chr "first person"
  ..$ P2: chr "second person"
  ..$ R : chr "relation"
 $ nodes :List of 3
  ..$ P1:'data.frame':  11 obs. of  2 variables:
  .. ..$ ID  : chr [1:11] "BM" "DM" "A1" "A2" ...
  .. ..$ long: chr [1:11] "branch manager" "deputy manager" "service adviser 1" "service adviser
  ..$ P2:'data.frame':  11 obs. of  1 variable:
  .. ..$ ID: chr [1:11] "BM" "DM" "A1" "A2" ...
  ..$ R :'data.frame':  4 obs. of  2 variables:
  .. ..$ ID  : chr [1:4] "advice" "friend" "satisfy" "confide"
  .. ..$ long: chr [1:4] "Advice seeking" "Close friendship" "Satisfying interaction" "Confiding
 $ links :'data.frame': 128 obs. of  4 variables:
  ..$ P1: int [1:128] 1 1 1 2 3 4 6 6 6 7 ...
  ..$ P2: int [1:128] 2 3 4 1 2 1 3 4 5 1 ...
  ..$ R : int [1:128] 1 1 1 1 1 1 1 1 1 1 ...
  ..$ w : num [1:128] 1 1 1 1 1 1 1 1 1 1 ...
 $ data  : list()
>
```

Converting a data frame into a multiway network

Olympics

`DF2MWN(DF,vars,w=weights,network=name,title=tit)` – builds the basic multiway network structure from a given data frame *DF* for selected variables *vars* and weights *weights*.

An example is the construction of the Olympics network. An interesting data set *120 years of Olympic history: athletes and results* in the form of a data frame is available at Kaggle.
From it I built the multiway network of summer medals.

```
> MT <- DF2MWN(OS,c("Name","Games","Team","NOC","Year","Sport","Event","Sex","Medal"),
+   w=c("Age","Height","Weight"),network="Olympic16SO",title="Summer Olympic medals till 2016")
```

For details see Summer Olympic medals till 2016.

3D layout

To get some insight into the structure of the network and obtained cores we can inspect their 3D layouts in X3D format. They are created using the MWnets function `mwnX3D`.

`mwnX3D(MN,u,v,z,w,lu="ID",lv="ID",lz="ID", pu=NULL,pv=NULL,pz=NULL,shape="Box",col=c(1,0,0), bg=c(0.8,0.8,0.8),maxsize=1,file="MWnets.x3d")` – Exports a 3D layout description of the 3-way (sub)network of network MN on ways u, v, z with weight w in X3D format to the selected file. The parameters lu, lv, lz are the names of node properties containing node labels. The vectors pu, pv, pz contain the permutations of nodes in the corresponding ways. The parameter shape can be either "Box" or "Sphere". The parameter col specifies the color of each link, and bg the background color. The parameter maxsize controls the maximum size of the shape.

Inspecting a 3D layout

The 3D layouts can be interactively inspected using special viewers such as **view3dscene** [20]. They can be also included in HTML web pages and inspected (rotate, move, zoom in/out) using a Javascript-based library X3DOM [19]. Inspecting a 3D layout the user can also dynamically get information about the selected link (shape) by clicking on it.

The current implementation of the function `mwnX3D` provides only the basic functionalities – for a given simple/simplified multiway network displaying three ways by given orderings, coloring each shape, and the volume of the shape representing a link is proportional to its weight. It will be improved.

In producing the layout it is very important to provide "clever" orderings of the ways. For ordinal ways we usually stick to natural ordering; for nominal (categorical) ways the ordering is often determined using clustering or by sorting some important quantity defined on a selected way.

Creating and inspecting an X3Dom layout

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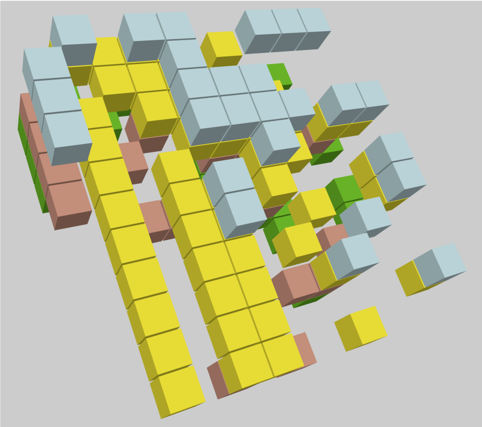
http://vladowiki.fmf.uni-lj.si/doku.php?id=vlado:work:2m:mwn:x3d:inline

Clicked link description: link 104 : branch manager , service adviser 2 , confide , 1



3D layout

Projection to a selected way

Because of the reordering option, we can assume that we selected the way $\mathcal{V}_1$. A projection to a selected way is a generalization of the projection of two-mode networks. The projection creates an ordinary weighted network $(\mathcal{V}_1, \mathcal{A}, p)$, $\mathcal{A} \subseteq \mathcal{V}_1 \times \mathcal{V}_1$ and $p : \mathcal{A} \to \mathbb{R}$. Let $u, t \in \mathcal{V}_1$ then

$$p(u,t) = \sum_{(v_2,\ldots,v_k) \in \mathcal{V}_2 \times \cdots \times \mathcal{V}_k} w(u, v_2, \ldots, v_k) \cdot w(t, v_2, \ldots, v_k)$$

This network can be analyzed using traditional methods for the analysis of weighted networks. Sometimes it is more appropriate to apply projection(s) to a normalized version of the original multi-way network.

The operation can be generalized to the product of compatible multiway networks.

Salton index has the following properties

From the projection p we can get the corresponding measure of similarity – *Salton index* $S(u,t)$ [2]

$$S(u,t) = \frac{p(u,t)}{\sqrt{p(u,u) \cdot p(t,t)}}$$

that can be used for clustering the set $\mathcal{V}_1$.

The Salton index has the following properties

1. $S(u,t) \in [-1,1]$
2. $S(u,t) = S(t,u)$
3. $S(u,u) = 1$
4. $w : L \to \mathbb{R}_0^+ \Rightarrow S(u,t) \in [0,1]$
5. $S(\alpha u, \beta t) = S(u,t), \quad \alpha, \beta > 0$
6. $S(\alpha u, u) = 1, \quad \alpha > 0$

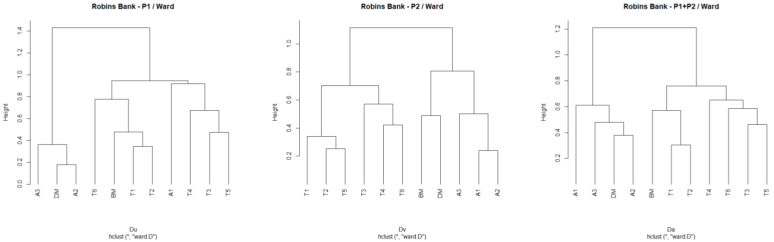
Robins Bank analysis

```
> source("https://raw.githubusercontent.com/bavla/ibm3m/master/
    multiway/MWnets.R")
> Cou <- projection(MN,"P1","w")
> Sau <- salton(Cou); Du <- as.dist(1-Sau); Du[is.na(Du)] <- 1
> tu <- hclust(Du,method="ward")
> plot(tu,hang=-1,cex=1,main="Robins Bank - P1 / Ward")

> Cov <- projection(MN,"P2","w")
> Sav <- salton(Cov); Dv <- as.dist(1-Sav); Dv[is.na(Dv)] <- 1
> tv <- hclust(Dv,method="ward")
> plot(tv,hang=-1,cex=1,main="Robins Bank - P2 / Ward")

> clrs <- c("green4","tomato3","gold1","cadetblue3")
> pie(rep(1,4), col=clrs)
> CC <- col2rgb(clrs)/255
> Col <- cbind(CC[1,MN$links$R],CC[2,MN$links$R],CC[3,MN$links$R])
> MN$nodes$P2$long <- MN$nodes$P1$long
> I <- inv(tu$order); J <- inv(tv$order); K <- c(3,4,1,2)
> mwnX3D(MN,"P1","P2","R","w",maxsize=0.95,pu=I,pv=I,pz=K,
+   lu="long",lv="long",col=Col,file="RbankClu1.x3d")
> mwnX3D(MN,"P1","P2","R","w",maxsize=0.95,pu=J,pv=J,pz=K,
+   lu="long",lv="long",col=Col,file="RbankClu2.x3d")
> mwnX3D(MN,"P1","P2","R","w",maxsize=0.95,pu=I,pv=J,pz=K,
+   lu="long",lv="long",col=Col,file="RbankClu3.x3d")
```

wiki

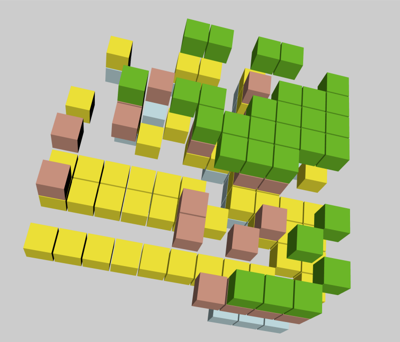
Clusterings

```
> Da <- (Du+Dv)/2; ta <- hclust(Da,method="ward")
> plot(ta,hang=-1,cex=1,main="Robins Bank - P1+P2 / Ward")
> L <- inv(ta$order)
> mwnX3D(MN,"P1","P2","R","w",maxsize=0.95,pu=L,pv=L,pz=K,
+   lu="long",lv="long",col=Col,file="RbankClu4.x3d")
```

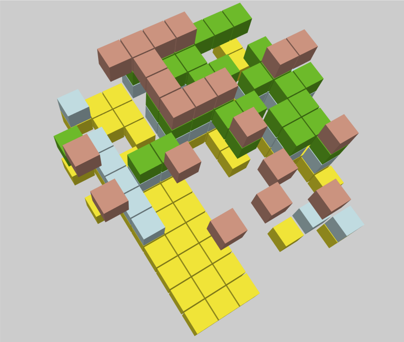


Robins Bank 3D layouts

Clicked link description: link 34 : branch manager , service adviser 3 , friend , 1



Clicked link description: link 33 : branch manager , service adviser 2 , friend , 1



Recoding (binning)

Let $\mathcal{B} = \{B_1, B_2, \ldots, B_s\}$ be a partition of $\mathbb{R}$. The *recoding* transformation transforms the weight function w into a new weight w' determined for a link $e \in L$ as

$$w'(e) = i \Leftrightarrow w(e) \in P_i$$

It could be used also for transforming weights to ways.

Code 0 corresponds to the case $e = (v_1, v_2, \ldots, v_k) \notin L$ which is usually equivalent to $w(v_1, v_2, \ldots, v_k) = 0$ in the array representation. If 0 is also a legal weight value we have to introduce another zero, $\square$, that indicates the absence of a link.

A special case is a *binarization* for which $B_0 = \{0\}$ and $B_1 = \mathbb{R} \setminus B_0$.

Recoding is often used to get more readable matrix visualizations of a given network. wiki

```
Mc <- recodecol2bins(MN,"w","code",bins=c(1,5,10,20,Inf))
Mr <- recodeway2part(MN,"prov","IDreg","regs","region")
Mre <- flatten(Mr,"w",c("regs","univ","prog","year"))
```

European airports and flight companies dataset

The *European airports and flight companies* dataset was collected by A. Cardillo and his collaborators for the paper

> *A. Cardillo, J. Gómez-Gardeñes, M. Zanin, M. Romance, D. Papo, F. del Pozo, S. Boccaletti: Emergence of Network Features from Multiplexity, Scientific Reports 3, 1344 (2013).*

It is an example of a multiway network. It contains 450 airports making two ways `airA` (departure airports) and `airB` (arrival airports), the third way is `line` (37 flight companies). There are 7176 links $(\mathit{from}, \mathit{to}, \mathit{line}) \in \mathcal{L}$ telling that there is a flight provided by the company *line* from the airport *from* to the airport *to*. Each link has the weight $w = 1$.

The multiway network `AirEu2013` represented in JSON format and an R package MWnets for analysis of multiway networks are available at GitHub/Bavla.

Dataset

```
List of 6
 $ format: chr "MWnets"
 $ info  :List of 7
  ..$ network: chr "AirEu2013"
  ..$ title  : chr "Air Transportation Multiplex"
  ..$ ref    : chr "Cardillo A. et al. Emergence of network features from multiplexity, Scientif
  ..$ href   : chr "http://complex.unizar.es/~atnmultiplex/"
 $ ways  :List of 3
  ..$ airA: chr "first airport"
  ..$ airB: chr "second airport"
  ..$ line: chr "airline"
 $ nodes :List of 3
  ..$ airA:'data.frame':       450 obs. of  7 variables:
  .. ..$ ID     : chr [1:450] "LCLK" "EDDF" "EDDK" "EGNX" ...
  .. ..$ lon    : num [1:450] 33.63 8.57 7.14 -1.33 -3.41 ...
  .. ..$ lat    : num [1:450] 34.9 50 50.9 52.8 50.7 ...
  .. ..$ iata   : chr [1:450] "LCA" "FRA" "CGN" "EMA" ...
  .. ..$ long   : chr [1:450] "Larnaca International Airport" "Frankfurt Airport" "Cologne Bonn
  .. ..$ region : chr [1:450] "Larnaka" "Hessen" "Nordrhein-Westfalen" "England" ...
  .. ..$ country: chr [1:450] "CY" "DE" "DE" "GB" ...
  ..$ airB:'data.frame':       450 obs. of  2 variables:
  .. ..$ ID  : chr [1:450] "LCLK" "EDDF" "EDDK" "EGNX" ...
  .. ..$ long: chr [1:450] "Larnaca International Airport" "Frankfurt Airport" "Cologne Bonn Air
  ..$ line:'data.frame':       37 obs. of  1 variable:
  .. ..$ ID: chr [1:37] "Lufthansa" "Ryanair" "Easyjet" "British A" ...
 $ links :'data.frame': 7176 obs. of  4 variables:
  ..$ airA: int [1:7176] 1 1 2 2 2 2 2 2 2 2 ...
  ..$ airB: int [1:7176] 2 38 1 7 8 10 14 15 17 18 ...
  ..$ line: int [1:7176] 1 1 1 1 1 1 1 1 1 1 ...
  ..$ w   : int [1:7176] 1 1 1 1 1 1 1 1 1 1 ...
 $ data  :List of 1
  ..$ Eu:'data.frame':  40 obs. of  6 variables:
  .. ..$ alpha2   : chr [1:40] "AT" "BA" "BE" "BG" ...
  .. ..$ alpha3   : chr [1:40] "AUT" "BIH" "BEL" "BGR" ...
  .. ..$ Ccode    : int [1:40] 40 70 56 100 756 196 203 276 208 233 ...
  .. ..$ long     : chr [1:40] "Austria" "Bosnia and Herzegovina" "Belgium" "Bulgaria" ...
  .. ..$ region   : chr [1:40] "Europe" "Europe" "Europe" "Europe" ...
  .. ..$ subregion: chr [1:40] "Western Europe" "Southern Europe" "Western Europe" "Eastern Euro
```

Derived networks

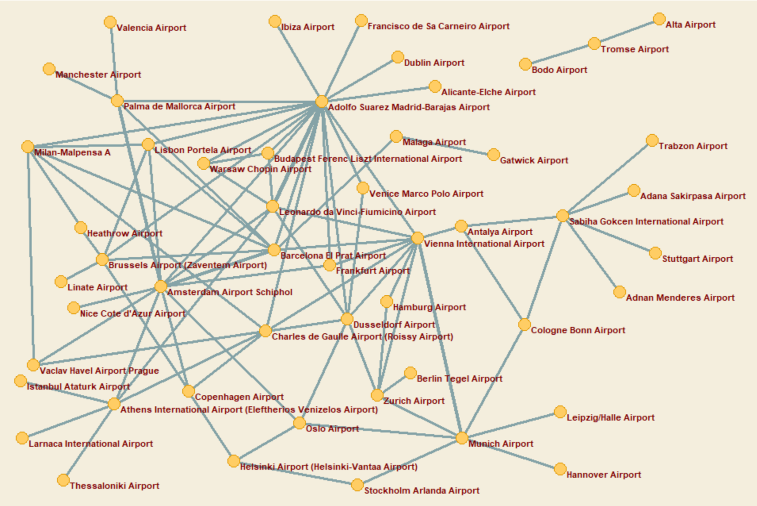
Using transformations of multiway networks we can create ordinary networks and analyze them using network analysis tools.

```
> wdir <- "C:/Users/vlado/DL/data/multi/cores/air"; setwd(wdir)
> library(jsonlite)
> source("https://raw.githubusercontent.com/bavla/Rnet/master/R/Pajek.R")
> source("https://raw.githubusercontent.com/bavla/ibm3m/master/multiway/MWnets.R")
> MN <- fromJSON("https://raw.githubusercontent.com/bavla/ibm3m/master/data/AirEu2013Ext.json")
> str(MN)
> Net <- flatten(MN,"w",c("airA","airB"))
> head(Net$links)
> mwn2net(Net,"airA","airB",ID1="long",ID2="long",w="w",twomode=FALSE,Net="lines.net")
```

We obtained a network of flights between airports with the weight $w(a, b)$ counting the number of companies flying from the airport a to the airport b.

We analyze it in Pajek: draw, transform to undirected (max), info weights, link-cut at level 3, draw; skeletons, cores, islands.

Link-cut at level 3

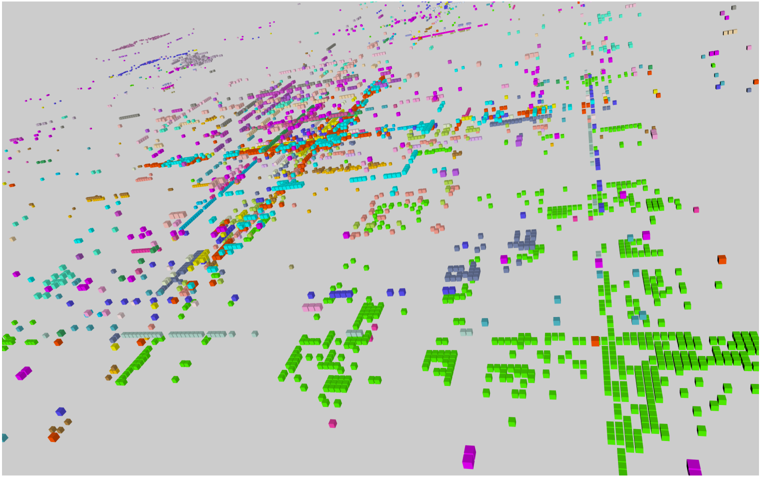


Clustering

```
> setwd("C:/test/cores")
> library(jsonlite)
> library(Polychrome)
> source("https://raw.githubusercontent.com/bavla/ibm3m/master/multiway/MWnets.R")
> MN <- fromJSON("https://raw.githubusercontent.com/bavla/ibm3m/master/data/AirEu2013Ext.json")
> str(MN)
> MN$nodes$line$short <- substr(MN$nodes$line$ID,1,4)
> Cou <- projection(MN,"airA","w")
> Sau <- salton(Cou); Du <- as.dist(1-Sau); Du[is.na(Du)] <- 1
> tu <- hclust(Du,method="ward")
> plot(tu,hang=-1,cex=0.2,main="EU airports - airports / Ward")
> Coz <- projection(MN,"line","w")
> Saz <- salton(Coz); Dz <- as.dist(1-Saz); Dz[is.na(Dz)] <- 1
> tz <- hclust(Dz,method="ward")
> plot(tz,hang=-1,cex=0.8,main="EU airports - companies / Ward")
> I <- inv(tu$order); K <- inv(tz$order)
> CC <- col2rgb(createPalette(37,c("#ff0000","#00ff00","#0000ff")))/255
> Col <- cbind(CC[1,MN$links$line],CC[2,MN$links$line],CC[3,MN$links$line])
> mwnX3D(MN,"airA","airB","line","w",maxsize=0.85,pu=I,pv=I,pz=K,lu="long",lv="long",
+   col=Col,file="EUair.x3d")
> row.names(Cou) <- MN$nodes$airA$long
> svg("airports.svg",width=18,height=8)
> plot(tu,hang=-1,cex=0.2,main="EU airports - airports / Ward")
> dev.off()
```

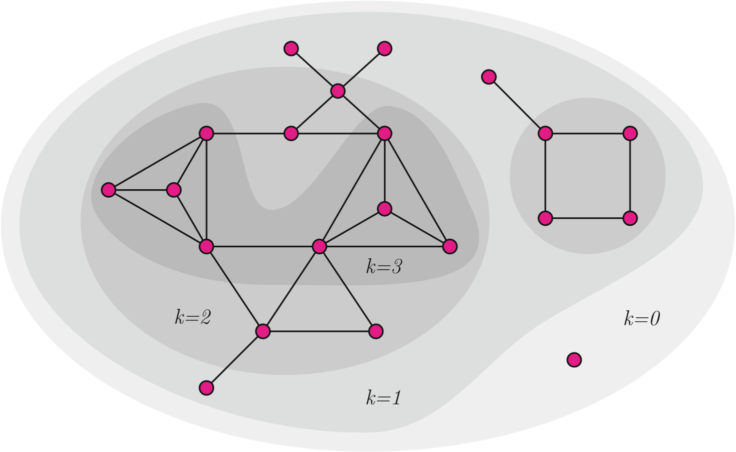
Airports dendrogram, 3D layout.

Screenshot of a part of the 3D layout of AirEu2013



The green links in the bottom right corner are provided by Ryan air – it is mostly linking airports that are not linked by other companies.

Cores in networks



Extended to other node properties, two-mode networks, and temporal networks [5, 7].

Node properties

A list of subways $\mathcal{C} = (C_1, C_2, \ldots, C_r)$, $C_i \subseteq \mathcal{V}_{h(i)}$ we will call a *selection*. Note that the selection $\mathcal{C}$ defines a function $h : 1..r \to 1..k$.
For a link $e \in \mathcal{L}$ we introduce abbreviations

$$h(e) \equiv (e(h(1)), e(h(2)), \ldots, e(h(r)))$$

for a *selected part* of the link e; and

$$h(e) \in \mathcal{C} \equiv \forall i \in 1..r : e(h(i)) \in C_i$$

for a statement that all nodes of the selected part of the link e belong to selected subways.
For a node $u \in \mathcal{V}_i$ and a selection $\mathcal{C}$, we call a *star* in the node u for the selection $\mathcal{C}$, the set

$$S(u, \mathcal{C}) = \{e \in \mathcal{L} : e(i) = u \wedge h(e) \in \mathcal{C}\}.$$

. . . Node properties

For a node $u \in \mathcal{V}_i$ we call the set of *neigbors* of the node u in the way $\mathcal{V}_j$ for the selection $\mathcal{C}$, the set of nodes $N(u, \mathcal{C}, \mathcal{V}_j) = \mathcal{V}_j(S(u, \mathcal{C}))$.

Using stars and sets of neighbors we can define some node property functions on multiway networks. For a node $u \in \mathcal{V}_i$ and a selection $\mathcal{C}$

degree: $p_d(u, \mathcal{C}) = \mathrm{card}(S(u, \mathcal{C}))$

weighted degree for the weight w: $p_S(u, \mathcal{C}; w) = \sum_{e \in S(u,\mathcal{C})} w(e)$

diversity of the way $\mathcal{V}_j$: $p_\delta(u, \mathcal{C}, \mathcal{V}_j) = \mathrm{card}(N(u, \mathcal{C}, \mathcal{V}_j))$

contribution of a (measured) node property/attribute a to the way $\mathcal{V}_j$: $p_c(u, \mathcal{C}, \mathcal{V}_j; a) = \sum_{v \in N(u,\mathcal{C},\mathcal{V}_j)} a(v)$

We say that the node property function $p(v, \mathcal{C})$:

- is *local* iff: $p(v, \mathcal{C}) = p(v, N(v, \mathcal{C}))$ for all $v \in \mathcal{V}$.
- is *monotonic* iff: $\mathcal{C}_1 \subset \mathcal{C}_2 \Rightarrow \forall v \in \mathcal{V} : p(v, \mathcal{C}_1) \leq p(v, \mathcal{C}_2)$.

Node properties in MWnets

Currently in `MWnets` the node property function(s) can be selected among

- p_d: `pDeg(MN,u,cip,C)`
- p_S: `pWsum(MN,u,cip,C,weight=))`
- p_M: `pWmax(MN,u,cip,C,weight=))`
- p_δ: `pDiv(MN,u,cip,C,way=)`
- p_c: `pAttr(MN,u,cip,C,way=,attr=,FUN=sum)`

The user can implement and use in MWcore his/her own node property functions.

Important nodes

Using node properties we can identify important nodes (with respect to the selected property). In network analysis, such properties are degree, weighted degree, closeness, betweenness, etc. For multiway networks, similar concepts are still to be developed.

```
> wdir <- "C:/Users/vlado/DL/data/multi/cores/air"
> setwd(wdir)
> library(jsonlite)
> source("https://raw.githubusercontent.com/bavla/ibm3m/master/multiway/MWnets.R")
> MN <- fromJSON("https://raw.githubusercontent.com/bavla/ibm3m/master/data/AirEu2013Ext.json")
> str(MN)
> cw <- c("airA","airB","line")
> # complete selection
> C <- lapply(cw, \(x) 1:nrow(MN$nodes[[x]])); names(C) <- cw
> for(line in 1:37) cat(line,MN$nodes$line$ID[line],pWsum(MN,line,c(3,1,2),C,"w"),"\n")
> Dep <- rep(0,450)
> for(a in 1:450) Dep[a] <- pWsum(MN,a,c(1,2,3),C,"w")
> I <- order(Dep,decreasing=TRUE)
> top20=Dep[I[1:20]]
> names(top20) <- MN$nodes$airA$long[I[1:20]]
> top20
> Lin <- rep(0,37)
> for(line in 1:37) Lin[line] <- pWsum(MN,line,c(3,1,2),C,"w")
> I <- order(Lin,decreasing=TRUE)
> top10 <- Lin[I[1:10]]
> names(top10) <- MN$nodes$line$ID[I[1:10]]
> top10
```

. . . Important nodes


```
> top20
            Amsterdam Airport Schiphol
                                   156
    Adolfo Suarez Madrid-Barajas Airport
                                   152
                        Munich Airport
                                   139
           Barcelona El Prat Airport
                                   137
                     Frankfurt Airport
                                   127
                       Gatwick Airport
                                   124
        Vienna International Airport
                                   122
   Leonardo da Vinci-Fiumicino Airport
                                   119
Charles de Gaulle Airport (Roissy Airport)
                                   118
               London Stansted Airport
                                   118
                    Dusseldorf Airport
                                   115
                        Dublin Airport
                                   110
           Palma de Mallorca Airport
                                   106
   Brussels Airport (Zaventem Airport)
                                   104
```

```
> top10
   Ryanair
      1202
   Easyjet
       614
 Lufthansa
       488
Air Berlin
       368
   Netjets
       360
 Turkish A
       236
       SAS
       220
     Flybe
       198
  Alitalia
       186
  Wizz Air
       184
```

Cores in multiway networks

Let $\mathcal{C} = (C_1, C_2, \ldots, C_k) \subseteq \mathcal{V}$ be a selection, $\mathbf{p} = (p_i)$ be a list of monotonic node property functions over the selected ways, and $\mathbf{t} = (t_i)$ a list of the corresponding thresholds.

The multiway subnetwork $Core(\mathbf{p}, \mathbf{t}) = (\mathcal{C}, \mathcal{L}(\mathcal{C}))$ induced by the selection $\mathcal{C}$ in a multiway network $\mathcal{N}$ is a *generalized multiway core* for node property functions $\mathbf{p}$ at levels $\mathbf{t}$ if and only if it holds that for all $v \in \mathcal{C}_i : p_i(v, \mathcal{C}) \geq t_i$ and $\mathcal{C}$ is the maximal such selection.

Then for monotonic node property functions $\mathbf{p}$ the generalized core can be determined by the following algorithm

```
function CoreMW(N, p, t)
    C ← (V_h(1), V_h(2), ..., V_h(r))
    while ∃u ∈ C_i : p_i(u, C) < t_i do C_i ← C_i \ {u}
    return C
end function
```

$$\begin{array}{l}
\textbf{function } \textsc{CoreMW}(\mathcal{N}, \mathbf{p}, \mathbf{t}) \\
\quad \mathcal{C} \leftarrow (\mathcal{V}_{h(1)}, \mathcal{V}_{h(2)}, \ldots, \mathcal{V}_{h(r)}) \\
\quad \textbf{while } \exists u \in C_i : p_i(u, \mathcal{C}) < t_i \textbf{ do } C_i \leftarrow C_i \setminus \{u\} \\
\quad \textbf{return } \mathcal{C} \\
\textbf{end function}
\end{array}$$

Cores in multiway networks

Multiway cores are partially nested

$$\mathbf{t}' < \mathbf{t} \Rightarrow \mathit{Core}(\mathbf{p}, \mathbf{t}) \subseteq \mathit{Core}(\mathbf{p}, \mathbf{t}')$$

In an elaboration of this algorithm, we have many options. Again the result does not depend on the node elimination order. Here we present a simple algorithm that is eliminating nodes way-wise.

The core conditions are given in a list P. Each entry of the list is a triple (p, t, args) where p is a node property function, t is the corresponding level, and args is a list of ways on which the function p is defined. It is assumed that the node u belongs to the way $\mathit{args}[1]$.

In an elaboration of this algorithm, we have many options. Again the result does not depend on the node elimination order. Here we present a simple algorithm that is eliminating nodes way-wise.

It is implemented as the function `MWcore(MN,P)` where `MN` is a multiway network and `P` is a list of core conditions. The user can implement and use in MWcore his/her own node property functions.

Cores algorithm

function $\textsc{MWcore}(\mathcal{N}, h, P)$
  $\mathcal{C} \leftarrow (\mathcal{V}_{h(1)}, \mathcal{V}_{h(2)}, \ldots, \mathcal{V}_{h(r)})$
  repeat
    $exit \leftarrow true$
    for $(p, t, args) \in P$ **do**
      $R \leftarrow \emptyset$; $i \leftarrow args[1]$
      for $u \in C_i$ **do**
        if $p(\mathcal{N}, u, \mathcal{C}) < t$ **then**
          $R \leftarrow R \cup \{u\}$; $exit \leftarrow false$
        end if
      end for
      $C_i \leftarrow C_i \setminus R$
    end for
  until exit
  return $\mathcal{C}$
end function

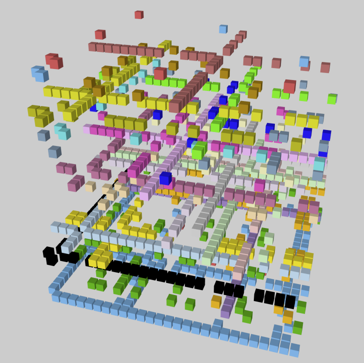
Cores

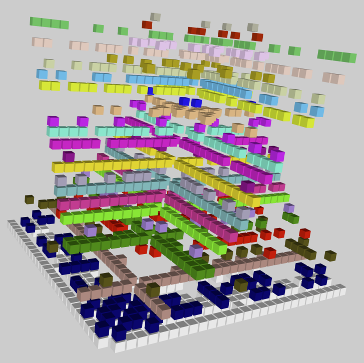
```
> source("https://raw.githubusercontent.com/bavla/Rnet/master/R/Pajek.R")
> source("https://raw.githubusercontent.com/bavla/ibm3m/master/multiway/MWnets.R")
> cw <- c("airA","airB")
> ci <- unname(sapply(cw,\(x) which(names(MN$ways)==x)))
> pDIV <- function(MN,v,cip,C,...) pDiv(MN,v,cip,C,way="line")
> P <- list(
+   p1 = list(p = pDIV, t = 10, cwp = c("airA","airB"), cip = NULL),
+   p2 = list(p = pDIV, t = 10, cwp = c("airB","airA"), cip = NULL),
+   cways = list(cw=cw,ci=ci)  )
> for(i in 1:(length(P)-1)) P[[i]]$cip <- unname(sapply(P[[i]]$cwp,\(x) which(cw==x)))
> str(P)
> P[[1]]$t <- 13; P[[2]]$t <- 13
> CC <- MWcore(MN,P)
> listCore(MN,CC,P)
> Ap <- MN$nodes$airA$long[CC[[1]]]

> w1 <- CC$airA; w2 <- CC$airB
> Score <- extract(MN,c("airA","airB"),c("w1","w2"))
> act <- as.integer(names(table(Score$links$line)))
> Rcore <- extract(Score,"line","act")
> str(Rcore)
> c27 <- glasbey.colors(27); CC <- col2rgb(c27)/255
> Col <- cbind(CC[1,Rcore$links$line],CC[2,Rcore$links$line],CC[3,Rcore$links$line])
> ts <- c(1,20,10,26,27,19,24,9,7,8,21,13,18,4,2,15,11,23,6,22,5,16,3,12,14,17,25,28)
> t <- inv(ts)
> qs <- c(1,5,10,19,3,7,12,4,16,13,21,8,20,6,11,18,17,2,23,9,15,24,25,14,22,27,26)
> qq <- inv(qs)
> mwnX3D(Rcore,"airA","airB","line","w",pu=t,pv=t,pz=qq,lu="long",lv="long",maxsize=0.85,
+   col=Col,file="EuAirCore13.x3d")

> percents(MN,Rcore,"airA","airB","line","w")
```

Core 3D layout, arXiv

3D layout of EUair diversity core at level 13

initial (left) and reordered by crosses (right)

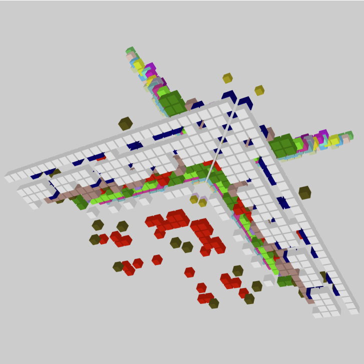


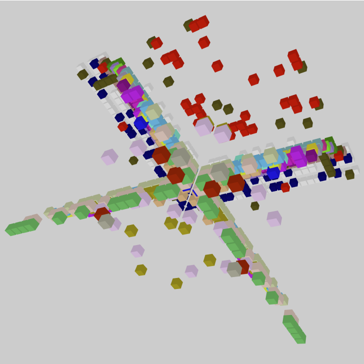


(1) Frankfurt Ap, (2) Berlin Tegel Ap, (3) Munich Ap, (4) Dusseldorf Ap, (5) Hamburg Ap, (6) Zurich Ap, (7) Geneva Ap, (8) Milan-Malpensa Ap, (9) Copenhagen Ap, (10) Stockholm Arlanda Ap, (11) Heathrow Ap, (12) Warsaw Chopin Ap, (13) Vienna IAp, (14) Amsterdam Ap Schiphol, (15) Charles de Gaulle Ap (Roissy Ap), (16) Adolfo Suarez Madrid-Barajas Ap, (17) Barcelona El Prat Ap, (18) Malaga Ap, (19) Vaclav Havel Ap Prague, (20) Leonardo da Vinci-Fiumicino Ap, (21) Brussels Ap (Zaventem Ap), (22) Budapest Ferenc Liszt IAp, (23) Athens IAp (Eleftherios Venizelos Ap), (24) Ben Gurion Ap, (25) Henri Coanda IAp, (26) Venice Marco Polo Ap, (27) Sofia Ap, (28) Nice Cote d'Azur Ap.

3D layout of EUair diversity core at level 13

two views exposing companies based on multiple airports





(1) Lufthansa, (2) Air Berlin, (3) Swiss IAL, (4) Netjets, (5) Easyjet, (6) SAS, (7) Norwegian AS, (8) British A, (9) LOT Polish A, (10) Austrian A, (11) Niki, (12) KLM, (13) Transavia H, (14) Air France, (15) Iberia, (16) Air Nostrum, (17) Vueling A, (18) Ryanair, (19) Czech A, (20) Alitalia, (21) Brussels A, (22) European AT, (23) Malev HA, (24) Wizz Air, (25) Aegean A (26) Olympic Air, (27) TNT Airways.

The companies that were not providing any line between core airports are Turkish A, Flybe, TAP Portugal, Finnair, Air Lingus, Germanwings, Pegasus A, SunExpress, Air Baltic, and Wideroe.

Some observations

- Lufthansa and Air Berlin are serving lines from German airports Frankfurt, Berlin, Munich, Dusseldorf, Hamburg, and Milan in Italy
- Swiss IAL and Netjets are serving lines from Swiss airports Zurich and Geneva
- SAS and Norwegian AS are serving lines from Copenhagen and Stockholm
- Spanish airports Madrid, Barcelona, and Malaga are served by companies Iberia, Air Nostrum, Vueling, and Ryanair
- companies serving lines from their base airport: (British A, Heathrow), (LOT, Warsaw), (Austrian A, Niki; Vienna), (KLM, Transavia; Amsterdam), (Air France, CDG), (Czech A, Prague), (Alitalia, Fiumicino), (Brussels A, European AT; Brussels), (Malev, Budapest), (Aegean, Olympic Air; Athens); the only "irregular" link is by Iberia between Barcelona and Budapest

. . . Some observations

- Companies with dispersed services inside the core are Easyjet, Netjets, Wizz Air, European AT, and TNT Airways
  - Easyjet is serving lines from Milan and also from CDG, Amsterdam, Fiumicino, Madrid
  - Netjets is serving lines also from Barcelona, Venice, Vienna, and Nice
  - Wizz Air is serving lines from Budapest and Fiumicino
  - European AT has also some lines from Heathrow, Milan, Venice, Barcelona, and Madrid
  - TNT Airways is linking airports CDG and Athens, and Sofia and Henri Coanda

Olympics

The data set *120 years of Olympic history – athletes and results* with basic bio data on athletes and medal results from Athens 1896 to Rio 2016 is available at Kaggle [16]. The original data describe 134732 participants of the Summer and Winter Olympics. We transformed it into the multiway network `Olympics16S` about the medalists of the Summer Olympics till 2016 (available at GitHub/Bavla [1]). It contains data about 34088 medals. Omitting the athlete Name info and simplifying the network we obtain a multiway network with 10429 links on five ways (Games (29), NOC (147), Sport (52), Sex (2), Medal (3)), the weight w counting the medals, and additional weights Age, Height, and Weight.

We decided to search for a core with a large number of medals on the ways Games, NOC, and Sport.

Olympics core

After examing the distributions of values of the node property function p_S on all three ways we decided to use the core conditions list P that contains three triples
$(p = pWsum, t = 100, args = (Games, NOC, Sport))$,
$(p = pWsum, t = 100, args = (NOC, Games, Sport))$, and
$(p = pWsum, t = 100, args = (Sport, Games, NOC))$.

The obtained core was too large. We increased the thresholds to (500, 300, 350). The corresponding p_S-core $\mathcal{C} = (C_{\mathrm{Games}}, C_{\mathrm{NOC}}, C_{\mathrm{Sport}})$ is of order (577, 308, 390) – inside the core, each Olympics accounts for at least 577 medals, each country (NOC) for at least 308 medals, and each sport discipline for at least 390 medals.

. . . Olympics core

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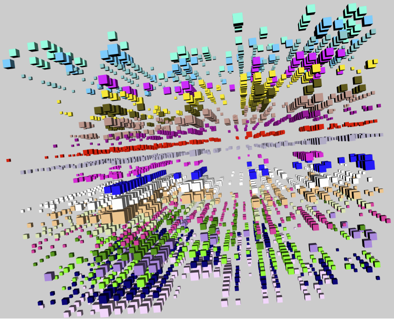
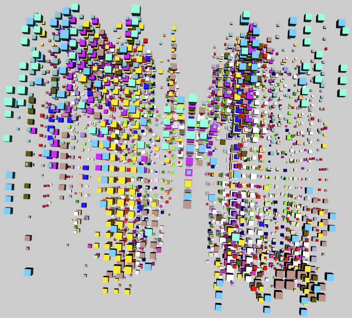
C_{Games} (25): 1988 Seoul (1405), 1984 Los Angeles (1354), 2008 Beijing (1352), 2012 London (1338), 2000 Sydney (1324), 2016 Rio de Janeiro (1320), 2004 Athens (1305), 1980 Moscow (1270), 1996 Atlanta (1251), 1976 Montreal (1220), 1992 Barcelona (1112), 1972 Munich (1080), 1920 Antwerp (1054), 1964 Tokyo (952), 1968 Mexico City (949), 1912 Stockholm (853), 1960 Rome (826), 1952 Helsinki (797), 1956 Melbourne + Stockholm (795), 1936 Berlin (754), 1948 London (731), 1924 Paris (681), 1908 London (656), 1928 Amsterdam (610), 1932 Los Angeles (577).

C_{NOC} (29): USA (4153), URS (2027), GBR (1594), GER (1573), ITA (1325), FRA (1184), AUS (1165), HUN (1052), SWE (1019), NED (854), GDR (843), RUS (776), CHN (662), JPN (661), ROU (632), CAN (613), NOR (546), POL (520), DEN (519), FRG (500), FIN (446), BRA (446), ESP (416), YUG (379), SUI (352), KOR (347), BEL (337), BUL (317), TCH (308).

C_{Sport} (21): Athletics (2911), Swimming (2815), Rowing (2535), Gymnastics (2043), Fencing (1584), Football (1216), Cycling (1129), Hockey (1126), Canoeing (1010), Shooting (995), Sailing (995), Wrestling (941), Handball (924), Basketball (914), Equestrianism (883), Water Polo (876), Volleyball (839), Boxing (596), Judo (425), Weightlifting (419), Diving (390).

Screenshot of a 3D layout of Olympics weighted degree core at levels (577, 308,390)





It turns out that the core contains $25566/34088 = 75\%$ of all medals, $3607/10429 = 34.6\%$ of all links, and occupies 6.9% of the total space. Note also that the set $\mathcal{C}_{\mathrm{NOC}}$ contains the Soviet Union and Russia; and Germany, West Germany, and East Germany.

The links belonging to the same sport discipline are of the same color. The Olympics are ordered chronologically. The order of NOCs (countries) and sport disciplines was determined using clustering.

Additional datasets

Vlado wiki: Multiway networks

Conclusions

1. Work in progress
   1. more testing,
   2. improving some code (efficiency, programming solutions – objects?, dplyr?),
   3. unification,
   4. robustness.
2. Apply the methods to different data sets (test, experience, ideas)
3. Normalizations
4. Additional data sets. New problems.
5. Julia and/or Python version
6. Additional Javascript support for X3DOM visualization (slicing, ...)

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