Multi-way networks

Implementation

References

Analysis of multiway networks

Vladimir Batagelj

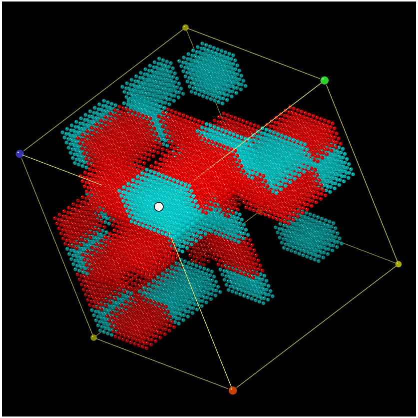
IMFM Ljubljana

1327. sredin seminar

Ljubljana, 9. november 2022

Outline

1 Multi-way networks
2 Implementation
3 References



Vladimir Batagelj: vladimir.batagelj@fmf.uni-lj.si
Current version of slides (November 6, 2022 at 16 : 33): slides PDF
https://github.com/bavla/NormNet/blob/main/docs/

Multi-way networks
Implementation
References

IFCS 2022

Viszards 2009

Indirect block modeling of 3-mode data

Multi-way networks

A multi-way network $N = (V, L, w)$ is based on nodes from k pairwise disjoint sets (ways or dimensions)

$$V = \{V_1, V_2, \ldots, V_k\}, \quad V_i \cap V_j = \emptyset \ \text{ for } \ i \neq j$$

The set of links $L \subseteq V_1 \times V_2 \times \cdots \times V_k$. The weight $w : L \to \mathbb{R}$. It can be represented by a k-dimensional array W

$$W[v_1, v_2, \ldots, v_k] = w(v_1, v_2, \ldots, v_k) \ \text{ for } \ (v_1, v_2, \ldots, v_k) \in L$$

otherwise $W[v_1, v_2, \ldots, v_k] = 0$.

Multiway analysis

Transformations

In data analysis of multi-way networks, some transformations could prove to be useful:

- reordering of ways
- joining the ways
- flattening of a way
- projection to a selected way
- normalization
- recoding (binarization)

Reordering of ways

Trivial.

Joining the ways

Selected ways V_i and V_j, $i < j$, are replaced by a new joint way

$$V_{ij} = \{(u : v) : u \in V_i \wedge v \in V_j \wedge \exists(\ldots, u, \ldots, v, \ldots) \in L\}$$

*** add a detailed description of the transformed network.
This transformation reduces the number of ways for 1.
"Commutativity".

Flattening of a way

A selected way V_i is removed from the network.

$$V' = \{V_1, V_2, \ldots, V_{i-1}, V_{i+1}, \ldots, V_k\}$$

$$w'(v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k) = \sum_{v \in V_i} w(v_1, v_2, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_k)$$

This transformation reduces the number of ways for 1.
"Commutativity".

Normalization

TBA

Recoding

Let $P = \{P_1, P_2, \ldots, P_p\}$ be a partition of $\mathbb{R}$. The recoding transformation transforms the weight function w into a new weight w' determined for a link $(v_1, v_2, \ldots, v_k) \in L$ as

$$w'(v_1, v_2, \ldots, v_k) = i \Leftrightarrow w(v_1, v_2, \ldots, v_k) \in P_i$$

Code 0 corresponds to the case $(v_1, v_2, \ldots, v_k) \notin L$ which is usually equivalent to $w(v_1, v_2, \ldots, v_k) = 0$ in the array representation. If 0 is also a legal weight value we have to introduce another zero, $\square$, that indicates the absence of a link.
A special case is a binarization for which $P_0 = \{0\}$ and $P_1 = \mathbb{R} \setminus P_0$.
Recoding is often used to get more readable matrix visualizations of a given network.

Projection to a selected way

Because of the reordering option, we can assume that we selected the way V_1. A projection to a selected way is a generalization of the projection of two-mode networks. The projection creates an ordinary weighted network (V_1, A, p), $A \subseteq V_1 \times V_1$ and $p : A \to \mathbb{R}$. Let $u, t \in V_1$ then

$$p(u, t) = \sum_{(v_2,\ldots,v_k)\in V_2\times\cdots\times V_k} w(u, v_2, \ldots, v_k) \cdot w(t, v_2, \ldots, v_k)$$

This network can be analyzed using traditional methods for the analysis of weighted networks. Sometimes it is more appropriate to apply projection(s) to a normalized version of the original multi-way network.

From the projection p we can get the corresponding measure of similarity – Salton index $S(u, t)$

$$S(u, t) = \frac{p(u, t)}{\sqrt{p(u, u) \cdot p(t, t)}}$$

that can be used for clustering the set V_1.

Salton index has the following properties

The Salton index has the following properties

1. $S(u, t) \in [-1, 1]$
2. $S(u, t) = S(t, u)$
3. $S(u, u) = 1$
4. $w : L \to \mathbb{R}_0^+ \Rightarrow S(u, t) \in [0, 1]$
5. $S(\alpha u, \beta t) = S(u, t), \quad \alpha, \beta > 0$
6. $S(\alpha u, u) = 1, \quad \alpha > 0$

Representation

A natural representation of a weighted multi-way network is by a multi-dimensional array. In real-life networks many (most of) array entries have the value 0. In such cases, this representation is computationally inefficient – takes more space and requires unnecessary computations with 0s.

Representation

An alternative representation follows the formal definition

```
{
title = "TITLE";

nodes1 = ["v11", ...,"v1n1"];
...
nodesk = ["vk1", ...,"vknk"];
nodes = [ nodes1, ..., nodesk ];
links = [
...
[link: [v1,v2,...,vk], w: W ];
...
]
}
```

Acknowledgments

This work is supported in part by the Slovenian Research Agency (research program P1-0294 and research projects J1-9187 and J5-2557), and prepared within the framework of the HSE University Basic Research Program.

References I

- Batagelj, V, Cerinšek, M.: On bibliographic networks. Scientometrics 96 (2013), pages 845–864. Springer
- Batagelj, V.: On fractional approach to the analysis of linked networks. Scientometrics 123 (2020), pages621–633. Springer
- Batagelj, V, Ferligoj, A, Doreian, P: Indirect Blockmodeling of 3-Way Networks. In: Brito, P, Cucumel, G, Bertrand, P, de Carvalho, F (eds) Selected Contributions in Data Analysis and Classification pp 151–159 Springer, 2007; preprint.
- Borgatti, SP, Everett, MG: Regular blockmodels of multiway, multimode matrices. Social Networks 14(1992)1-2, 91–120.
- Everett, MG, Borgatti, SP: Chapter 9, Partitioning Multimode Networks. In *** Patrick Doreian,Vladimir Batagelj,Anuška Ferligoj (Eds): Wiley, 2019.