SLDG integration following Qiu et al. 2011

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Consider a partition of the domain $I = [\,\alpha, \beta\,]$ into subintervals $I_i = [\,x_{i-1}, x_{i-1}\,], \quad i = 1, \ldots, N$ such that

$$\alpha = x_0 < x_1 < \cdots < x_N = \beta.$$

We assume that $x^* = \check{X}^n(x)$ is the foot of the characteristic reaching a fixed node $x \in I_j$ at time $\tau = \Delta t$, so that

$$\check{X}^n(x) \coloneqq X^n(0; x) = x - \int_0^{\Delta t} a_n(X^n(\tau; x))\,\mathrm{d}\tau.$$

Define the trace interval $\mathcal{I}_n(x)$ for a fixed node $x \in I_j$ by (see Figure 1)

$$\mathcal{I}_n(x) \coloneqq \left[x, \check{X}^n(x)\right] \cup \left[\check{X}^n(x), x\right].$$

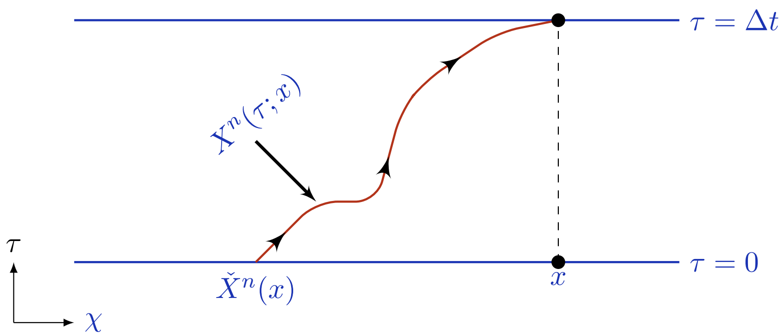

Figure 1: Illustration of the characteristic curve $X^n(\tau; x)$, characteristic foot $\check{X}^n(x)$ and trace interval $\mathcal{I}_n(x) = \left[\check{X}^n(x), x\right]$.

We want to evaluate the following integral:

$$\begin{aligned} L_h^{\Delta t}(u_h^n(x)) &= \int_0^{\Delta t} a(x,\tau) S_h(\tau) u_h^n(x)\,\mathrm{d}\tau = \int_{\check{X}^n(x)}^{x} u_h^n(\xi)\,\mathrm{d}\xi \\ &= \operatorname{sign}\,\big(x - \check{X}^n(x)\big) \sum_{k=1}^{M} \int_{\mathcal{I}_n(x)\cap I_k} u_h^n(\xi)\,\mathrm{d}\xi \end{aligned} \tag{1}$$

Let

$$L := |\,I\,| = x_N - x_0 = \beta - \alpha.$$

Assume periodic boundary conditions. Then we have two important assumptions to make.

Assumption L. $x^* \le x$.

Case 1. $x_0 \le x^* \le x_N$:
If $x^* \in I_i$, then $i \le j$.

$$\begin{aligned} L_h^{\Delta t} u_h^n(x) &= \int_{x^*}^{x} u_h^n(\xi)\,\mathrm{d}\xi \\ &= \int_{x^*}^{x_i} u_h^n(\xi)\,\mathrm{d}\xi + \int_{x_i}^{x_{j-1}} u_h^n(\xi)\,\mathrm{d}\xi + \int_{x_{j-1}}^{x} u_h^n(\xi)\,\mathrm{d}\xi \qquad (i<j) \end{aligned}$$

Case 2. $x^* < x_0$:

$$\begin{aligned} L_h^{\Delta t} u_h^n(x) &= \int_{x^*}^{x} u_h^n(\xi)\,\mathrm{d}\xi \\ &= \int_{x_0}^{x} u_h^n(\xi)\,\mathrm{d}\xi + \int_{y^*}^{x_N} u_h^n(\xi)\,\mathrm{d}\xi + k\int_{x_0}^{x_N} u_h^n(\xi)\,\mathrm{d}\xi \\ &= \int_{y^*}^{x_N} u_h^n(\xi)\,\mathrm{d}\xi + (k+1)\int_{x_0}^{x_N} u_h^n(\xi)\,\mathrm{d}\xi, \end{aligned}$$

where

$$\begin{aligned} & y^* = x_N - (\,(x_0 - x^*) \bmod L\,) \in [\,x_0, x_N\,]\,, \\ & k = (x_0 - x^*)\mathrm{div}L \in \mathbb{N} \end{aligned}$$

If $y^* \in I_i$ and $i \le j$, then Case 1 can be used to evaluate the integral over $[\,y^*, x\,]$, otherwise we use Case 1 of Assumption R. Remark also that if $i = j$, Case 1 becomes $L_h^{\Delta t} u_h^n(x) = \int_{x^*}^{x} u_h^n(\xi)\,\mathrm{d}\xi$.

Assumption R. $x^* \geq x$.

Case 1. $x_0 \leq x^* \leq x_N$:
If $x^* \in I_i$, then $i \geq j$.

$$\begin{aligned} L_h^{\Delta t} u_h^n(x) &= -\int_x^{x^*} u_h^n(\xi)\,\mathrm{d}\xi \\ &= -\left[\int_x^{x_j} u_h^n(\xi)\,\mathrm{d}\xi + \int_{x_j}^{x_{i-1}} u_h^n(\xi)\,\mathrm{d}\xi + \int_{x_{i-1}}^{x^*} u_h^n(\xi)\,\mathrm{d}\xi\right] \quad (i > j) \end{aligned}$$

In this case it suffices to reverse the roles of $x^* \longleftrightarrow x$, apply Case 1 of Assumption L and tag a minus sign to the result.

Case 2. $x^* > x_N$:

$$\begin{aligned} L_h^{\Delta t} u_h^n(x) &= -\int_x^{x^*} u_h^n(\xi)\,\mathrm{d}\xi \\ &= -\left[\int_x^{x_N} u_h^n(\xi)\,\mathrm{d}\xi + \int_{x_0}^{y^*} u_h^n(\xi)\,\mathrm{d}\xi + k\int_{x_0}^{x_N} u_h^n(\xi)\,\mathrm{d}\xi\right] \\ &= \int_{y^*}^{x_N} u_h^n(\xi)\,\mathrm{d}\xi - (k+1)\int_{x_0}^{x_N} u_h^n(\xi)\,\mathrm{d}\xi, \end{aligned}$$

where

$$\begin{aligned} &y^* = x_0 + \left((x^* - x_N) \bmod L\right), \\ &k = (x^* - x_N) \operatorname{div} L. \end{aligned}$$