

Bayesian Thinking: Fundamentals, Regression and Multilevel Modeling

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Webinar 1-2: Normal Models for Continuous Data

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- 2 Conjugate prior and posterior inferences for μ
- 3 Inferences for μ and σ
- 4 Wrap-up and additional material

Section 1

Example: Expenditures in the CE and normal distribution

The Consumer Expenditure Surveys data (CE)

- Conducted by the U.S. Census Bureau for the BLS.
- Contains data on expenditures, income, and tax statistics about consumer units (CU) across the country.
- Provides information on the buying habits of U.S. consumers.

The Consumer Expenditure Surveys data (CE)

- Conducted by the U.S. Census Bureau for the BLS.
- Contains data on expenditures, income, and tax statistics about consumer units (CU) across the country.
- Provides information on the buying habits of U.S. consumers.
- We work with PUMD micro-level data, with the continuous variable **TOTEXPPQ**: CU total expenditures last quarter.
- We work with Q1 2017 sample: $n = 6,208$.

The Total Expenditure variable

```
library(readr)
CEsample <- read_csv("CEsample.csv")

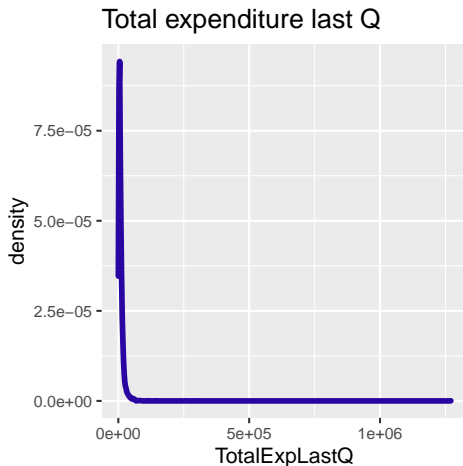
summary(CEsample$TotalExpLastQ)
```

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
30	3522	6417	9513	11450	1270598

```
sd(CEsample$TotalExpLastQ)
```

```
[1] 19341.25
```

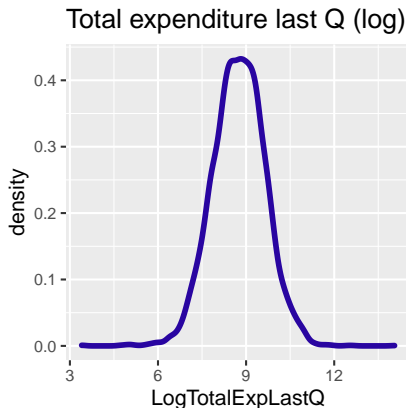
The Total Expenditure variable cont'd



- Very skewed to the right.
- Take log and transform it to the log scale.

Log transformation of the Total Expenditure variable

```
CEsample$LogTotalExpLastQ <- log(CEsample$TotalExpLastQ)
```

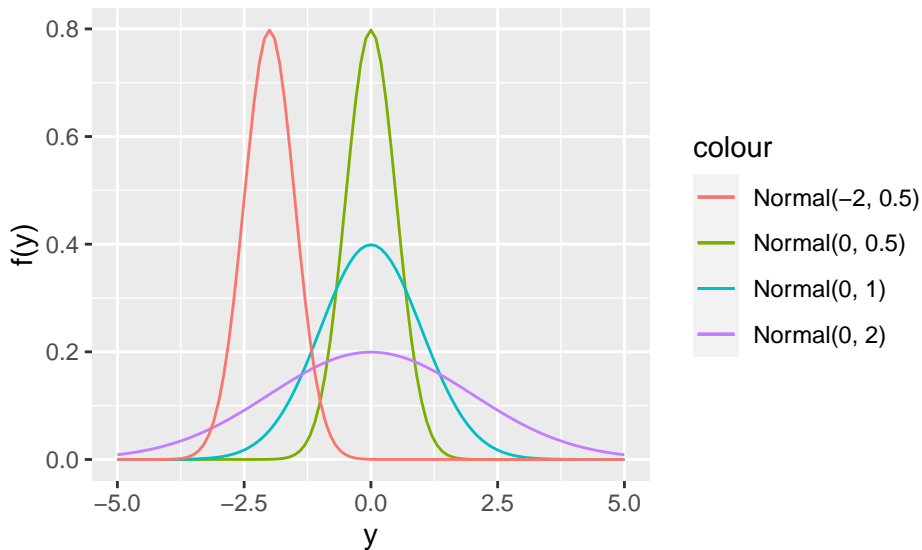


The normal distribution

- The normal distribution is a symmetric, bell-shaped distribution.
- It has two parameters: mean μ and standard deviation σ .
- The probability density function (pdf) of $\text{Normal}(\mu, \sigma)$ is:

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y - \mu)^2}{2\sigma^2}\right), -\infty < y < \infty.$$

The normal distribution cont'd



i.i.d. normals

- Suppose there are a sequence of n responses: Y_1, Y_2, \dots, Y_n .
- Further suppose each response **independently and identically** follows a normal distribution:

$$Y_i \stackrel{i.i.d.}{\sim} \text{Normal}(\mu, \sigma).$$

- Then the joint probability density function (joint pdf) of y_1, \dots, y_n is:

$$f(y_1, \dots, y_n) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(y_i - \mu)^2}{2\sigma^2}\right), -\infty < y_i < \infty.$$

Recap from beta-binomial

- Bayesian inference procedure:
 - The prior distribution: $p \sim \text{Beta}(\alpha, \beta)$
 - The sampling density: $Y \sim \text{Binomial}(N, p)$
 - The posterior distribution: $p \mid Y \sim \text{Beta}(\alpha + Y, \beta + N - Y)$

Recap from beta-binomial

- Bayesian inference procedure:
 - The prior distribution: $p \sim \text{Beta}(\alpha, \beta)$
 - The sampling density: $Y \sim \text{Binomial}(N, p)$
 - The posterior distribution: $p \mid Y \sim \text{Beta}(\alpha + Y, \beta + N - Y)$
- What to do for a normal model $Y_i \stackrel{i.i.d.}{\sim} \text{Normal}(\mu, \sigma)$?
 - Data model/sampling density is chosen: normal.
 - What to do with two parameters μ and σ ?
 - How to specify priors?

Section 2

Conjugate prior and posterior inferences for μ

Overview

- The data model/sampling density for N observations:

$$Y_i \stackrel{i.i.d.}{\sim} \text{Normal}(\mu, \sigma).$$

- There are two parameters μ and σ in the normal model.
- Need a joint prior distribution (if both μ and σ are unknown):

$$g(\mu, \sigma).$$

- Bayes' rule will help us derive a joint posterior:

$$g(\mu, \sigma \mid Y_1, \dots, Y_n) \propto g(\mu, \sigma) f(Y_1, \dots, Y_N \mid \mu, \sigma)$$

If only mean μ is unknown

- Special case: μ is unknown, σ is known.
- There is only one parameter μ in $Y_i \stackrel{i.i.d.}{\sim} \text{Normal}(\mu, \sigma)$.
- The Bayesian inference procedure simplifies to:
 - The data model for N observations with σ known:

$$Y_i \stackrel{i.i.d.}{\sim} \text{Normal}(\mu, \sigma).$$

- Need a prior distribution for μ :

$$g(\mu \mid \sigma).$$

- Bayes' rule will help us derive a posterior for μ :

$$g(\mu \mid Y_1, \dots, Y_N, \sigma) \propto g(\mu \mid \sigma) f(Y_1, \dots, Y_N \mid \mu, \sigma).$$

Normal conjugate prior

- For this special case, normal prior for μ is a conjugate prior:
 - The prior distribution:

$$\mu \mid \sigma \sim \text{Normal}(\mu_0, \sigma_0).$$

- The sampling density:

$$Y_1, \dots, Y_N \mid \mu, \sigma \stackrel{i.i.d.}{\sim} \text{Normal}(\mu, \sigma).$$

Normal conjugate prior

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 - The prior distribution:

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- The sampling density:

$$Y_1, \dots, Y_N \mid \mu, \sigma \stackrel{i.i.d.}{\sim} \text{Normal}(\mu, \sigma).$$

- The posterior distribution:

$$\mu \mid Y_1, \dots, Y_N, \phi \sim \text{Normal} \left(\frac{\phi_0 \mu_0 + N \phi \bar{Y}}{\phi_0 + N \phi}, \sqrt{\frac{1}{\phi_0 + N \phi}} \right),$$

where $\phi = \frac{1}{\sigma^2}$ (and $\phi_0 = \frac{1}{\sigma_0^2}$), the precision. Since σ (and σ_0) is known, ϕ (and ϕ_0) is known too.

Example on log(Total Expenditure)

- Prior for μ is $\mu \sim \text{Normal}(5, 1)$, i.e. $\mu_0 = 5, \phi_0 = 1$
- Our log(Total Expenditure): $N = 6208, \bar{Y} = 8.75$
- Assume $\phi = 1.25$, i.e. $\sigma = \sqrt{1/1.25}$
- Use these quantities to obtain posterior for μ :

$$\mu \mid Y_1, \dots, Y_N, \phi \sim \text{Normal} \left(\frac{\phi_0 \mu_0 + N \phi \bar{Y}}{\phi_0 + N \phi}, \sqrt{\frac{1}{\phi_0 + N \phi}} \right).$$

Posterior for μ

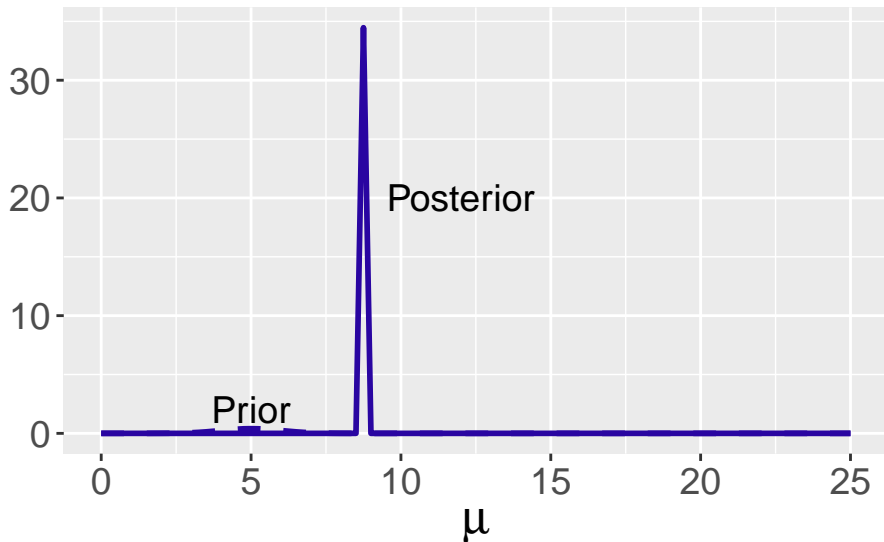
```
mu_0 <- 5
sigma_0 <- 1
phi_0 <- 1/sigma_0^2
ybar <- mean(CESample$LogTotalExpLastQ)
phi <- 1.25
n <- dim(CESample)[1]
mu_n <- (phi_0*mu_0+n*ybar*phi)/(phi_0+n*phi)
sd_n <- sqrt(1/(phi_0+n*phi))

mu_n
```

```
[1] 8.747753
```

```
sd_n
```

```
[1] 0.01135118
```

Posterior for μ cont'd

Bayesian inferences: hypothesis testing

- Suppose someone thinks the $\log(\text{Total Expenditure})$ of CUs in the U.S. on average is at least \$8.5 (i.e. \$4914), is this statement reasonable?
- Exact solution:

```
1 - pnorm(8.5, mean = mu_n, sd = sd_n)
```

```
[1] 1
```

- Monte Carlo simulation solution:

```
set.seed(123)
S <- 1000
mu_post <- rnorm(S, mean = mu_n, sd = sd_n)
sum(mu_post >= 8.5) / S
```

```
[1] 1
```

Bayesian inferences: credible interval

- Bayesian credible interval: an interval contains the unknown parameter with a certain probability.
- What is a 95% credible interval for μ ?
- Exact solution:

```
qnorm(c(0.025, 0.975), mean = mu_n, sd = sd_n)
```

```
[1] 8.725506 8.770001
```

- Monte Carlo simulation solution:

```
quantile(mu_post, c(0.025, 0.975))
```

```
      2.5%      97.5%  
8.725714 8.770886
```

Bayesian inferences: prediction

- Suppose we are interested in predicting $\log(\text{Total Expenditure})$ of another CU.
- The posterior predictive distribution is

$$f(Y^* \mid Y_1, \dots, Y_N) = \int f(Y^* \mid \mu, \sigma) g(\mu \mid Y_1, \dots, Y_N, \sigma) d\mu. \quad (1)$$

- The integration step in Equation (1) can be approximated through simulation.
 - Step 1: Sample a value of μ from its posterior distribution

$$\mu \mid Y_1, \dots, Y_N, \phi \sim \text{Normal} \left(\frac{\phi_0 \mu_0 + N \phi \bar{Y}}{\phi_0 + N \phi}, \sqrt{\frac{1}{\phi_0 + N \phi}} \right).$$

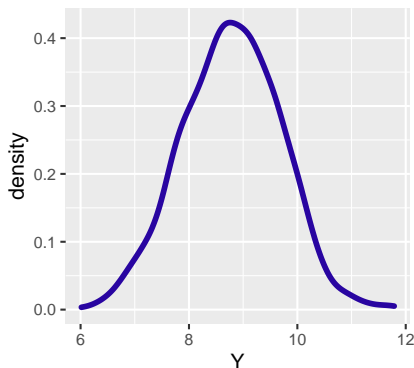
- Step 2: Sample a new observation Y^* from the sampling model

$$Y^* \sim \text{Normal}(\mu, \sigma)$$

Bayesian inferences: prediction cont'd

```
set.seed(123)
S <- 1000
mu_post <- rnorm(S, mean = mu_n, sd = sd_n)
y_pred <- rnorm(S, mean = mu_post, sd = sqrt(1 / phi))
```

Density plot of predictions



Using Stan

- Write a Stan script defining the Bayesian model.

```
data {  
  int<lower=0> N;  // number of observations  
  real y[N];      // vector of continuous observations  
}  
parameters {  
  real mu; // mean parameter  
}  
model {  
  mu ~ normal(5, 1); // prior  
  for (i in 1:N) {  
    y[i] ~ normal(mu, sqrt(1 / 1.25)); // observation model  
  }  
}
```

Section 3

Inferences for μ and σ

Overview

- The data model/sampling density for N observations:

$$Y_i \stackrel{i.i.d.}{\sim} \text{Normal}(\mu, \sigma).$$

- There are two parameters μ and σ in the normal model.
- Need a joint prior distribution (if both μ and σ are unknown):

$$g(\mu, \sigma).$$

- Bayes' rule will help us derive a joint posterior:

$$g(\mu, \sigma \mid Y_1, \dots, Y_n) \propto g(\mu, \sigma) f(Y_1, \dots, Y_N \mid \mu, \sigma)$$

Priors for μ and for σ

- Suppose we keep the normal prior for μ :

$$\mu \sim \text{Normal}(\mu_0, \sigma_0).$$

- Now let's also specify a prior for $\sigma > 0$:

$$\sigma \sim \text{Cauchy}(\gamma_1, \gamma_2).$$

- We know that from Bayes' rule, we obtain our joint posterior:

$$g(\mu, \sigma \mid Y_1, \dots, Y_n) \propto g(\mu)g(\sigma)f(Y_1, \dots, Y_n \mid \mu, \sigma).$$

Markov chain Monte Carlo methods

- Our goal is to estimate the joint posterior:

$$g(\mu, \sigma \mid Y_1, \dots, Y_n).$$

- As an approximation, we can iterate by sampling:
 - Sample μ at iteration i :

$$\mu^{(i)} \sim g(\mu \mid Y_1, \dots, Y_N, \sigma^{(i-1)}).$$

- Sample σ at iteration i :

$$\sigma^{(i)} \sim g(\sigma \mid Y_1, \dots, Y_N, \mu^{(i)}).$$

- After convergence, $\{\mu^{(1)}, \dots, \mu^{(S)}\}$ and $\{\sigma^{(1)}, \dots, \sigma^{(S)}\}$ serve as approximations to the posterior distribution.

Using Stan

- Write a Stan script defining the Bayesian model.

```
data {
  int<lower=0> N;  // number of observations
  real y[N];      // vector of continuous observations
}
parameters {
  real mu; // mean parameter
  real<lower=0> sigma; // sd parameter
}
model {
  mu ~ normal(5, 1); // prior for mu
  sigma ~ cauchy(0, 1); // prior for sigma
  for (i in 1:N) {
    y[i] ~ normal(mu, sigma); // observation model
  }
}
```

Run Stan using the rstan package

- Enter data by a list

```
n <- dim(CESample)[1]
my_data <- list(N = n, y = CESample$LogTotalExpLastQ)
```

- Inputs are Stan model file and the data list.

```
library(rstan)
fit_normal <- stan(file = "normal_2unknowns.stan",
                  data = my_data,
                  refresh = 0)
```

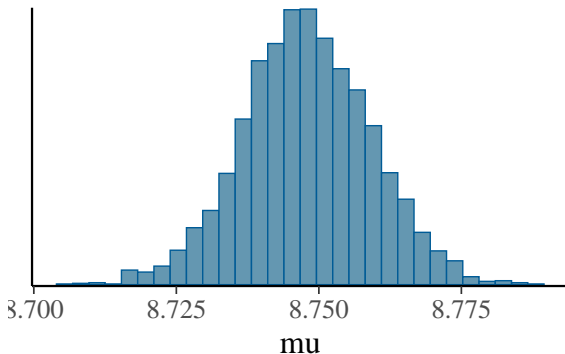

Extract the posterior draws

```
draws <- as.data.frame(fit_normal)
head(draws)
```

	mu	sigma	lp__
1	8.731845	0.8921777	-2419.087
2	8.742834	0.8996197	-2418.376
3	8.750744	0.8951207	-2418.105
4	8.737180	0.8900023	-2418.647
5	8.753385	0.9037658	-2418.871
6	8.749248	0.8940300	-2418.074

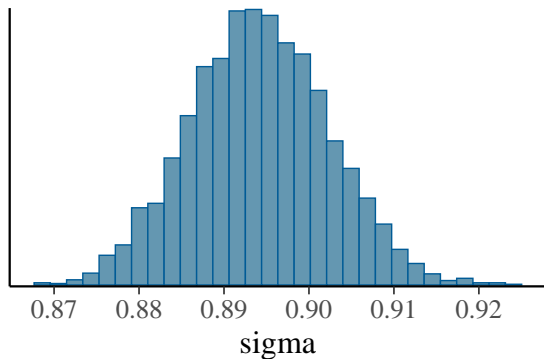
Histogram of the simulated draws of μ

```
library(bayesplot)
mcmc_hist(draws, pars = 'mu')
```



Histogram of the simulated draws of Σ

```
mcmc_hist(draws, pars = 'sigma')
```



Posterior summaries

```
summary(fit_normal)
```

```
$summary
```

	mean	se_mean	sd	2.5%	25%
mu	8.7478510	0.0001936590	0.01131484	8.7251974	8.7402609
sigma	0.8941723	0.0001364165	0.00816012	0.8783017	0.8885314
lp__	-2419.0794746	0.0241412019	1.03262094	-2421.8243926	-2419.4754635
	50%	75%	97.5%	n_eff	Rhat
mu	8.7475162	8.755525	8.7703460	3413.673	1.0002326
sigma	0.8940308	0.899628	0.9103544	3578.155	0.9990457
lp__	-2418.7566891	-2418.357007	-2418.0890224	1829.633	1.0004411

```
$c_summary
```

```
, , chains = chain:1
```

parameter	stats				
	mean	sd	2.5%	25%	50%
mu	8.7480701	0.011431011	8.7257254	8.7401911	8.74805
sigma	0.8941528	0.007983593	0.8787577	0.8885802	0.89414
lp__	-2419.0671886	0.984825323	-2421.5421235	-2419.4612982	-2418.77608

Bayesian inferences: hypothesis testing and credible interval

- Since exact posterior distribution is not available, our inferential methods are mainly Monte Carlo simulation.
- Hypothesis testing: μ at least \$8.5?

```
sum(draws$mu > 8.5) / dim(draws)[1]
```

```
[1] 1
```

- Credible interval: a 95% credible interval for σ ?

```
quantile(draws$sigma, c(0.025, 0.975))
```

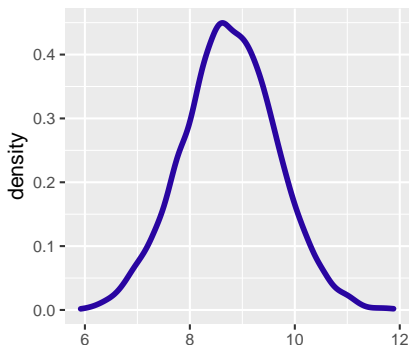
```
      2.5%      97.5%  
0.8783017 0.9103544
```

Bayesian inferences: prediction

$$Y^* \sim \text{Normal}(\mu, \sigma)$$

```
set.seed(123)
S <- dim(draws)[1]
y_pred2 <- rnorm(S, draws$mu, draws$sigma)
```

Density plot of predictions



Section 4

Wrap-up and additional material

Wrap-up

- Bayesian inference procedure:
 - Step 1: express an opinion about the location of the parameters before sampling (prior).
 - Step 2: take the sample (data/likelihood).
 - Step 3: use Bayes' rule to sharpen and update the previous opinion about the parameters given the information from the sample (posterior).
- Bayesian inferences
 - Bayesian hypothesis testing
 - Bayesian credible interval
 - Bayesian prediction

Additional material: posterior predictive checks

- A way to check model fitting
- Sample S copies of predictions of the same sample size as the original data

```
set.seed(123)
S <- dim(draws)[1]
sim_ytilde <- function(j){
  rnorm(n, draws$mu, draws$sigma)
}
ytilde <- t(sapply(1:S, sim_ytilde))
```

Posterior predictive checks cont'd

- Use some statistics to check, e.g. the average

```
pred_ybar_sim <- apply(ytilde, 1, mean)
```

