Bayesian Thinking: Fundamentals, Regression and Multilevel Modeling

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Outline

1. **Example: Expenditures in the CE and normal distribution**
2. **Conjugate prior and posterior inferences for μ**
3. **Conjugate prior and posterior inferences for σ**
4. **Recap**

Section 1

Example: Expenditures in the CE and normal distribution

The Consumer Expenditure Surveys Data (CE)

- Conducted by the U.S. Census Bureau for the BLS.
- Contains data on expenditures, income, and tax statistics about consumer units (CU) across the country.
- Provides information on the buying habits of U.S. consumers.

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- We work with PUMD micro-level data, with the continuous variable TOTEXPPQ: CU total expenditures last quarter.
- We work with Q1 2017 sample: $n = 6,208$.

The Total Expenditure variable

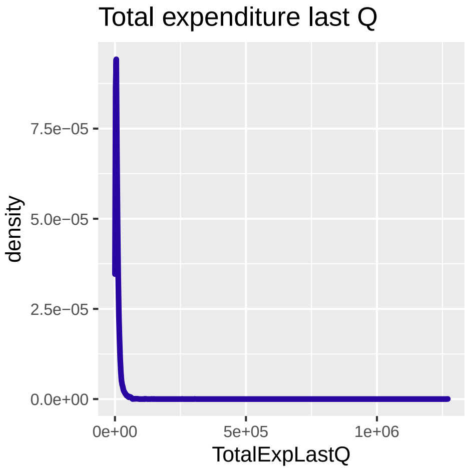
```
library(readr)
CEsample <- read_csv("CEsample.csv")

summary(CEsample$TotalExpLastQ)
```

```
##     Min.  1st Qu.   Median     Mean  3rd Qu.     Max.
##       30     3522     6417     9513    11450  1270598
```

```
sd(CEsample$TotalExpLastQ)
```

```
## [1] 19341.25
```

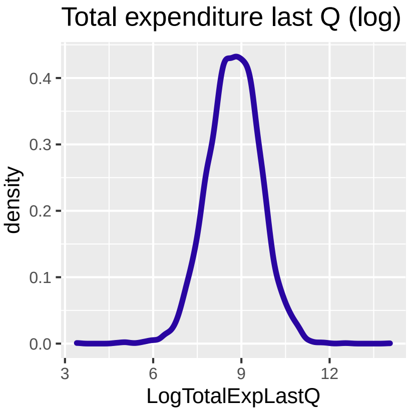
The Total Expenditure variable cont'd



- Very skewed to the right.
- Take log and transform it to the log scale.

Log transformation of the Total Expenditure variable

```
CEsample$LogTotalExpLastQ <- log(CEsample$TotalExpLastQ)
```

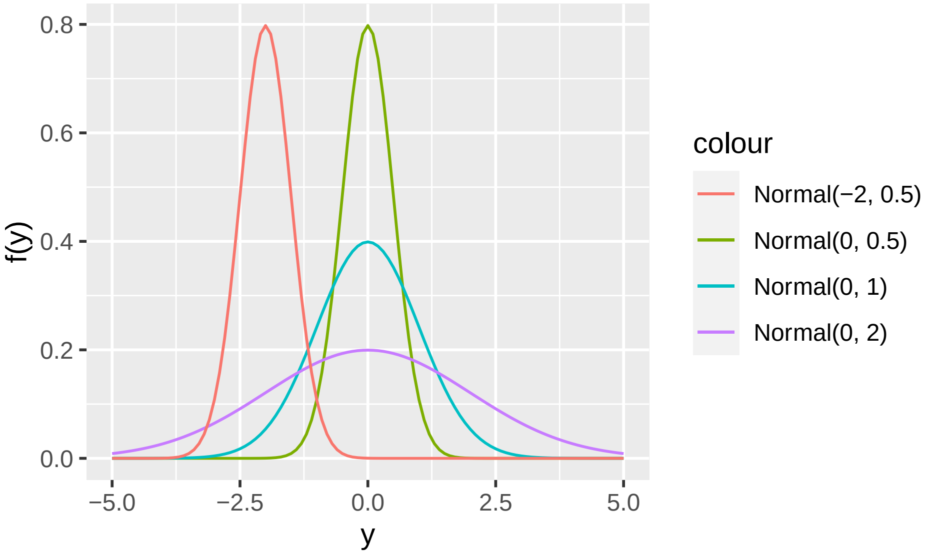


The normal distribution

- The normal distribution is a symmetric, bell-shaped distribution.
- It has two parameters: mean μ and standard deviation σ.
- The probability density function (pdf) of $\mathrm{Normal}(\mu, \sigma)$ is:

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(y-\mu)^2}{2\sigma^2}\right), -\infty < y < \infty.$$

The normal distribution cont'd



i.i.d. normals

- Suppose there are a sequence of n responses: $Y_1, Y_2, \cdots, Y_n$.
- Further suppose each response independently and identically follows a normal distribution:

$$Y_i \overset{i.i.d.}{\sim} \mathrm{Normal}(\mu, \sigma).$$

- Then the joint probability density function (joint pdf) of $y_1, \cdots, y_n$ is:

$$f(y_1, \cdots, y_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(y_i - \mu)^2}{2\sigma^2}\right), -\infty < y_i < \infty. \tag{1}$$

Recap from beta-binomial

- Bayesian inference procedure:
  - The prior distribution: $p \sim \mathrm{Beta}(\alpha, \beta)$
  - The sampling density: $Y \sim \mathrm{Binomial}(N, p)$
  - The posterior distribution: $p \mid Y \sim \mathrm{Beta}(a + Y, b + N - Y)$

Recap from beta-binomial

- Bayesian inference procedure:
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  - The sampling density: $Y \sim \mathrm{Binomial}(N, p)$
  - The posterior distribution: $p \mid Y \sim \mathrm{Beta}(a + Y, b + N - Y)$
- What to do for a normal model $Y_i \overset{i.i.d.}{\sim} \mathrm{Normal}(\mu, \sigma)$?
  - Data model/sampling density is chosen: normal.
  - What to do with two parameters μ and σ?
  - How to specify priors?

Section 2

Conjugate prior and posterior inferences for μ

Overview

- The data model/sampling density for N observations:

$$Y_i \overset{i.i.d.}{\sim} \mathrm{Normal}(\mu, \sigma).$$

- There are two parameters μ and σ in the normal model.
- Need a joint prior distribution (if both μ and σ are unknown):

$$g(\mu, \sigma). \tag{2}$$

- Bayes' rule will help us derive a joint posterior:

$$g(\mu, \sigma \mid y_1, \cdots, y_n) \propto g(\mu, \sigma) f(Y_1, \cdots, Y_N \mid \mu, \sigma) \tag{3}$$

If only mean μ is unknown

- Special case: μ is unknown, σ is known.
- There is only one parameter μ in $Y_i \overset{i.i.d.}{\sim} \mathrm{Normal}(\mu, \sigma)$.
- The Bayesian inference procedure simplifies to:
  - The data model for N observations with σ known:

$$Y_i \overset{i.i.d.}{\sim} \mathrm{Normal}(\mu, \sigma).$$

  - Need a prior distribution for μ:

$$g(\mu \mid \sigma). \tag{4}$$

  - Bayes' rule will help us derive a posterior for μ:

$$g(\mu \mid Y_1, \cdots, Y_N, \sigma) \propto g(\mu \mid \sigma) f(Y_1, \cdots, Y_N \mid \mu, \sigma). \tag{5}$$

Normal conjugate prior

- For this special case, normal prior for μ is a conjugate prior:
  - The prior distribution:

$$\mu \mid \sigma \sim \mathrm{Normal}(\mu_0, \sigma_0). \tag{6}$$

  - The sampling density:

$$Y_1, \cdots, Y_N \mid \mu, \sigma \overset{i.i.d.}{\sim} \mathrm{Normal}(\mu, \sigma). \tag{7}$$

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  - The sampling density:

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  - The posterior distribution:

$$\mu \mid Y_1, \cdots, Y_N, \phi \sim \mathrm{Normal}\left(\frac{\phi_0\mu_0 + N\phi\bar{Y}}{\phi_0 + N\phi}, \sqrt{\frac{1}{\phi_0 + N\phi}}\right), \tag{8}$$

  where $\phi = \frac{1}{\sigma^2}$ (and $\phi_0 = \frac{1}{\sigma_0^2}$), the precision. Since σ (and σ_0) is known, ϕ (and ϕ_0) is known too.

Normal conjugate prior cont'd

- The posterior distribution:

$$\mu \mid Y_1, \cdots, Y_N, \phi \sim \text{Normal}\left(\frac{\phi_0\mu_0 + N\phi\bar{Y}}{\phi_0 + N\phi}, \sqrt{\frac{1}{\phi_0 + N\phi}}\right). \quad (9)$$

- We can then use the `rnorm()` R function to sample posterior draws of μ from Equation (9). Known quantities: $\phi_0, \mu_0, N, \bar{Y}, \phi$

Example on log(Total Expenditure)

- Prior for μ is $\mu \sim \mathrm{Normal}(5, 1)$, i.e. $\mu_0 = 5, \phi_0 = 1$
- Our log(Total Expenditure): $N = 6208$, $\bar{Y} = 8.75$
- Assume $\phi = 1.25$, i.e. $\sigma = \sqrt{1/1.25}$
- Use these quantities to obtain posterior for μ:

$$\mu \mid Y_1, \cdots, Y_N, \phi \sim \mathrm{Normal}\left(\frac{\phi_0\mu_0 + N\phi\bar{Y}}{\phi_0 + N\phi}, \sqrt{\frac{1}{\phi_0 + N\phi}}\right). \tag{10}$$

Posterior for μ

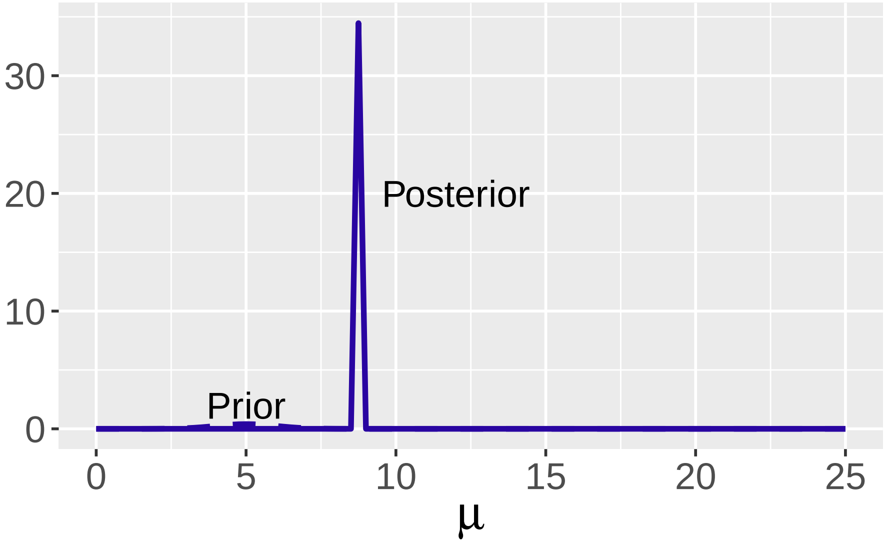
```
mu_0 <- 5
sigma_0 <- 1
phi_0 <- 1/sigma_0^2
ybar <- mean(CEsample$LogTotalExpLastQ)
phi <- 1.25
n <- dim(CEsample)[1]
mu_n <- (phi_0*mu_0+n*ybar*phi)/(phi_0+n*phi)
sd_n <- sqrt(1/(phi_0+n*phi))

mu_n
```

```
## [1] 8.747753
```

```
sd_n
```

```
## [1] 0.01135118
```

Posterior for μ cont'd



Bayesian inferences: hypothesis testing

```
mu_0 <- 5
set.seed(123)
S <- 1000
mu_post <- rnorm(S, mean = mu_n, sd = sd_n)
```

Bayesian inferences: credible interval

Bayesian inferences: prediction

Using Stan

Section 3

Conjugate prior and posterior inferences for σ

If only standard deviation σ is unknown

- Special case: μ is known, σ is unknown.
- There is only one parameter μ in $Y_i \overset{i.i.d.}{\sim} \mathrm{Normal}(\mu, \sigma)$.
- The Bayesian inference procedure simplifies to:
  - The data model/sampling density for n observations with μ known:

$$Y_i \overset{i.i.d.}{\sim} \mathrm{Normal}(\mu, \sigma).$$

  - The likelihood is in terms of unknown parameter σ:

$$f(y_1, \cdots, y_n) = L(\sigma). \tag{11}$$

  - Need a prior distribution for σ:

$$\pi(\sigma \mid \mu). \tag{12}$$

  - Bayes' rule will help us derive a posterior for σ:

$$\pi(\sigma \mid y_1, \cdots, y_n, \mu). \tag{13}$$

If only standard deviation σ is unknown: Gamma conjugate prior for $1/\sigma^2$

- For this special case, Gamma prior for $1/\sigma^2$ is a conjugate prior:
  - The prior distribution:

$$1/\sigma^2 \mid \mu \sim \text{Gamma}(\alpha, \beta). \tag{14}$$

  - The sampling density:

$$y_1, \cdots, y_n \mid \mu, \sigma \overset{i.i.d.}{\sim} \text{Normal}(\mu, \sigma). \tag{15}$$

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  - The posterior distribution:

$$1/\sigma^2 \mid y_1, \cdots, y_n, \mu \sim \text{Gamma}\left(\alpha + \frac{n}{2}, \beta + \frac{1}{2}\sum_{i=1}^{n}(y_i - \mu)^2\right) \tag{16}$$

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- We can then use `rgamma()` R function to sample posterior draws of σ from Equation (16). Known quantities: $\alpha, n, \beta, \{y_i\}, \mu$

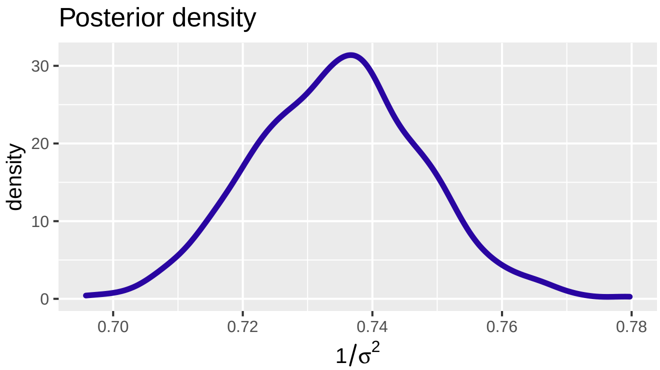
Simulate posterior draws of σ

```
alpha <- 1
beta <- 1
mu <- 8
n <- dim(CEsample)[1]
alpha_n <- alpha+n/2
beta_n <- beta+1/2*sum((CEsample$LogTotalExpLastQ-mu)^2)

set.seed(123)
S <- 1000
invsigma2_post <- rgamma(S, shape=alpha_n, rate=beta_n)
df <- as.data.frame(invsigma2_post)
```

Simulate posterior draws of σ cont'd

```
ggplot(data = df, aes(invsigma2_post)) +
  geom_density(color = crcblue, size = 1) +
  labs(title = "Posterior density") +
  xlab(expression(1/sigma^2)) +
  theme_grey(base_size = 8, base_family = "")
```



Section 4

Recap

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- Bayesian inference procedure:
  - Step 1: express an opinion about the location of mean μ and standard deviation σ (or precision ϕ) before sampling (prior).
  - Step 2: take the sample (data/likelihood).
  - Step 3: use Bayes' rule to sharpen and update the previous opinion about μ and σ (or precision ϕ) given the information from the sample (posterior).

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- For Normal data/likelihood, Normal distributions are conjugate priors for μ, and Gamma distributions are conjugate priors for ϕ.

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- Bayesian inference
  - Bayesian hypothesis testing
  - Bayesian credible interval
  - Bayesian prediction
  - Posterior predictive checking

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- Bayesian inference
  - Bayesian hypothesis testing
  - Bayesian credible interval
  - Bayesian prediction
  - Posterior predictive checking
- What if we want to use a different prior for μ? What if both μ and σ are unknown?