Bayesian Thinking: Fundamentals, Regression and Multilevel Modeling

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Webinar 2-1: Regression Models for Continuous Data

1. Introduction: adding a continuous predictor variable
2. A simple linear regression for the CE sample
3. A multiple linear regression for the CE sample
4. Wrap-up and additional material

Section 1

Introduction: adding a continuous predictor variable

Review: the normal model

- When you have continuous outcomes, you can use a normal model:

$$Y_i \mid \mu, \sigma \overset{i.i.d.}{\sim} \text{Normal}(\mu, \sigma), \quad i = 1, \cdots, n. \tag{1}$$

- Suppose now you have another continuous variable available, x_i. And you want to use the information in x_i to learn about Y_i.
  1. Y_i is the log of expenditure of CU's
  2. x_i is the log of total income of CU's
- Is the model in Equation (1) flexible to include x_i?

An observation specific mean

- We can adjust the model in Equation (1) to Equation (2), where the common mean μ is replaced by an observation specific mean μ_i:

$$Y_i \mid \mu_i, \sigma \overset{ind}{\sim} \mathrm{Normal}(\mu_i, \sigma), \;\; i = 1, \cdots, n. \tag{2}$$

- How to link μ_i and x_i?

Linear relationship between the mean and the predictor

- One basic approach: use a linear relationship:

$$\mu_i = \beta_0 + \beta_1 x_i, \quad i = 1, \cdots, n. \tag{3}$$

- x_i's are known constants.
- β_0 (intercept) and β_1 (slope) are unknown parameters.
- Bayesian approach:
  1. assign a prior distribution to $(\beta_0, \beta_1, \sigma)$
  2. perform inference
  3. summarize posterior distribution of these parameters

The simple linear regression model

- To put everything together, a linear regression model:

$$Y_i \mid x_i, \beta_0, \beta_1, \sigma \overset{ind}{\sim} \text{Normal}(\beta_0 + \beta_1 x_i, \sigma), \quad i = 1, \cdots, n. \tag{4}$$

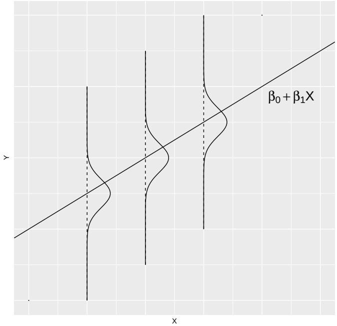
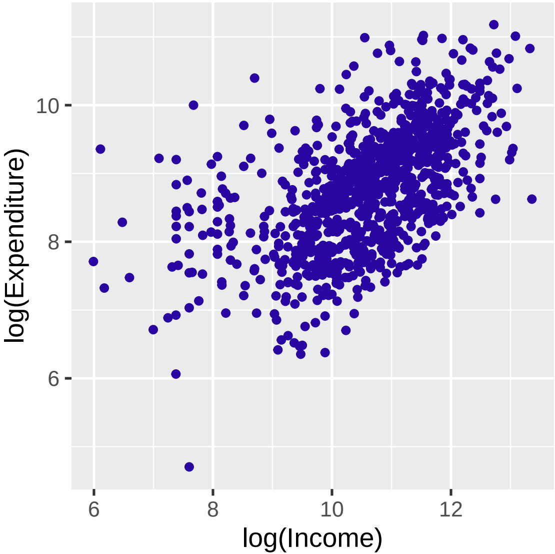


Figure 1: Display of linear regression model. The line represents the unknown regression line $\beta_0 + \beta_1 x$ and the normal curves represent the distribution of the response Y about the line.

The simple linear regression model cont'd



Section 2

A simple linear regression for the CE sample

The CE sample

The CE sample comes from the 2017 Q1 CE PUMD: 4 variables, 994 observations.

Variable	Description
log(Expenditure)	Continuous; CU's total expenditures in last quarter (log)
log(Income)	Continuous; the amount of CU income before taxes in past 12 months (log)
Rural	Binary; the urban/rural status of CU: 0 = Urban, 1 = Rural
Race	Categorical; the race category of the reference person: 1 = White, 2 = Black, 3 = Native American, 4 = Asian, 5 = Pacific Islander, 6 = Multi-race

An SLR for the CE sample

- For now, we focus on a simple linear regression:

$$
\begin{aligned}
Y_i \mid \mu_i, \sigma &\overset{ind}{\sim} \mathrm{Normal}(\mu_i, \sigma), \qquad (5) \\
\mu_i &= \beta_0 + \beta_1 x_i. \qquad (6)
\end{aligned}
$$

Variable	Description
log(Expenditure)	Continuous; CU's total expenditures in last quarter (log)
log(Income)	Continuous; the amount of CU income before taxes in past 12 months (log)

A weakly informative prior

- Assume know little about $(\beta_0, \beta_1, \sigma)$.
- Assuming independence: $g(\beta_0, \beta_1, \sigma) = g(\beta_0)g(\beta_1)g(\sigma)$.
- For example:

$$\begin{aligned} \beta_0 &\sim \mathrm{Normal}(0, 10), \\ \beta_1 &\sim \mathrm{Normal}(0, 10), \\ \sigma &\sim \mathrm{Cauchy}(0, 1). \end{aligned}$$

Fitting the model

- Use the `brm()` function with `family = gaussian`.

```
library(brms)
SLR_fit <- brm(data = CEData, family = gaussian,
               log_TotalExp ~ 1 + log_TotalIncome,
               prior = c(prior(normal(0, 10), class = Intercept),
                         prior(normal(0, 10), class = b),
                         prior(cauchy(0, 1), class = sigma)),
               iter = 10000, warmup = 8000, chains = 2, seed = 123)
```

Saving posterior draws

- Save post as a matrix of simulated posterior draws

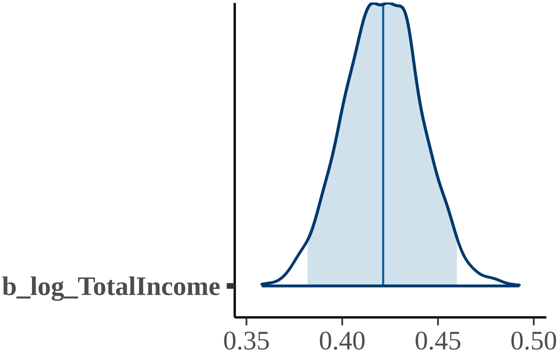
```
post <- posterior_samples(SLR_fit)
head(post)
```

```
##   b_Intercept b_log_TotalIncome     sigma      lp__
## 1    4.297944         0.4211047 0.7394589 -1096.985
## 2    4.479542         0.4061439 0.7350549 -1096.494
## 3    4.565761         0.3979486 0.7380264 -1097.019
## 4    4.223149         0.4324778 0.7562594 -1098.440
## 5    4.507156         0.4045579 0.7162845 -1096.480
## 6    4.247718         0.4290111 0.7079412 -1096.564
```

Posterior plots

- Function `mcmc_areas()` displays a density estimate of the simulated posterior draws with a specified credible interval.

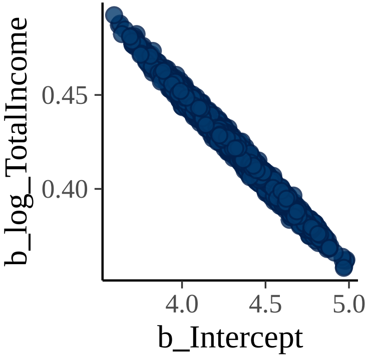
```
library(bayesplot)
mcmc_areas(post, pars = "b_log_TotalIncome", prob = 0.95)
```



Posterior plots cont'd

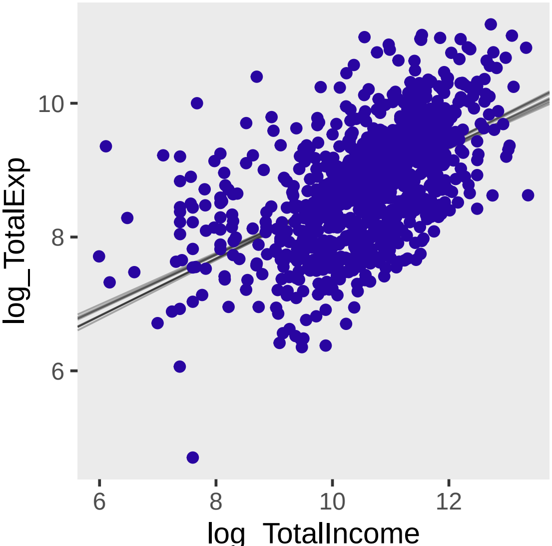
- Function `mcmc_scatter()` creates a simple scatterplot of two parameters.

```
mcmc_scatter(post, pars = c("b_Intercept", "b_log_TotalIncome"))
```



Plotting posterior inference against the data

- Plot the first 10 (β_0, β_1) fits to the data



Predictions

- Use the `predict()` function to make predictions of observed CUs.

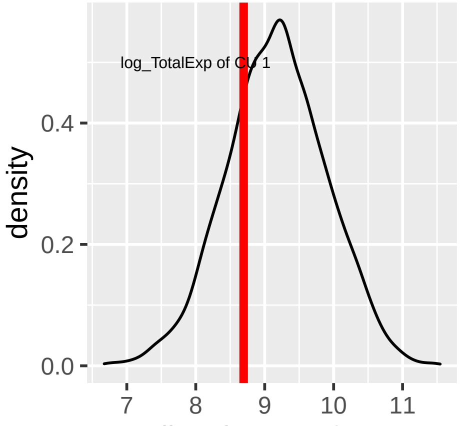
```
pred_logExp_obs <- predict(SLR_fit, newdata = CEData)
head(pred_logExp_obs)
```

```
##      Estimate Est.Error     Q2.5     Q97.5
## [1,] 9.157685 0.7294317 7.692801 10.590172
## [2,] 8.563797 0.7271387 7.097595  9.998737
## [3,] 9.079879 0.7274548 7.626594 10.478905
## [4,] 9.370100 0.7295246 7.961262 10.742722
## [5,] 9.282554 0.7280344 7.863468 10.708152
## [6,] 8.699488 0.7211794 7.295259 10.110860
```

Predictions cont'd

- If we focus on one CU, i.e.g CU 1; set `summary = FALSE` to obtain predicted values.

```
pred_logExp_obs_1 <- predict(SLR_fit, newdata = CEData[1, ],
                             summary = FALSE)
```



Predictions cont'd

- Now suppose we get to know a new CU with log_TotalIncome = 10, and we want to predict its log_TotalExp

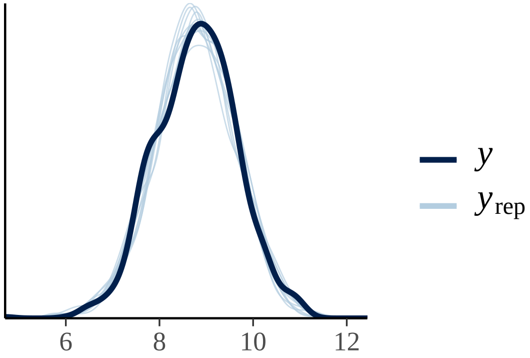
```
newdata <- data.frame(log_TotalIncome = c(10))
pred_logExp_new <- predict(SLR_fit, newdata = newdata)
pred_logExp_new
```

```
##      Estimate Est.Error     Q2.5    Q97.5
## [1,] 8.544662  0.723126 7.157743 9.974127
```

Model checking

- Function `pp_check()` performs posterior predictive checks
  - plot density estimates for 10 replicated samples from the posterior predictive distribution and overlay the observed log income distribution

```
pp_check(SLR_fit)
```



Section 3

A multiple linear regression for the CE sample

Adding a binary predictor

Variable	Description
log(Expenditure)	Continuous; CU's total expenditures in last quarter (log)
Rural	Binary; the urban/rural status of CU: $0 =$ Urban, $1 =$ Rural

- Consider Rural as a binary categorical variable to classify two groups:
  - The urban group
  - The rural group
- Such classification puts an emphasis on the difference of the expected outcomes between the two groups.

With only one binary predictor

- For simplicity, consider a simplified regression model with a single predictor: the binary indicator for rural area x_i.

$$\mu_i = \beta_0 + \beta_1 x_i = \begin{cases} \beta_0, & \text{the urban group;} \\ \beta_0 + \beta_1, & \text{the rural group.} \end{cases} \tag{7}$$

- The expected outcome μ_i for CUs in the urban group: β_0.
- The expected outcome μ_i for CUs in the rural group: $\beta_0 + \beta_1$.
- β_1 represents the change in the expected outcome μ_i from the urban group to the rural group.

The multiple linear regression model

$$
\begin{aligned}
Y_i \mid \mu_i, \sigma &\overset{ind}{\sim} \mathrm{Normal}(\mu_i, \sigma), && (8)\\
\mu_i &= \beta_0 + \beta_1 x_{i,logIncome} + \beta_2 x_{i,Rural}. && (9)
\end{aligned}
$$

A weakly informative prior

- Assume know little about $(\beta_0, \beta_1, \beta_2, \sigma)$.

$$
\begin{aligned}
\beta_0 &\sim \text{Normal}(0, 10), \\
\beta_1 &\sim \text{Normal}(0, 10), \\
\beta_2 &\sim \text{Normal}(0, 10), \\
\sigma &\sim \text{Cauchy}(0, 1).
\end{aligned}
$$

Fitting the model

- Use the `brm()` function with `family = gaussian`.
- Use `as.factor()` for binary / categorical predictors.

```
MLR_fit <- brm(data = CEData, family = gaussian,
               log_TotalExp ~ 1 + log_TotalIncome + as.factor(Rural),
               prior = c(prior(normal(0, 10), class = Intercept),
                         prior(normal(0, 10), class = b),
                         prior(cauchy(0, 1), class = sigma)),
               iter = 10000, warmup = 8000, chains = 2, seed = 123)
```

Saving posterior draws

- Save post as a matrix of simulated posterior draws

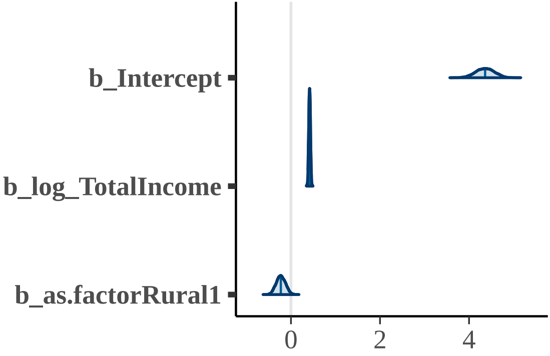
```
post_MLR <- posterior_samples(MLR_fit)
head(post_MLR)
```

```
##   b_Intercept b_log_TotalIncome b_as.factorRural1     sigma      lp__
## 1    4.676683         0.3890965       -0.19268408 0.7407412 -1098.599
## 2    4.554845         0.4013413       -0.19753931 0.7191823 -1097.365
## 3    4.629246         0.3971874       -0.29455991 0.6810329 -1102.570
## 4    4.275641         0.4220288       -0.16950690 0.7505153 -1100.285
## 5    4.517729         0.4016574       -0.07030555 0.7302973 -1098.663
## 6    4.296507         0.4251669       -0.21423636 0.7115447 -1097.055
```

Posterior plots

- Function `mcmc_areas()` displays a density estimate of the simulated posterior draws with a specified credible interval.

```
mcmc_areas(post_MLR,
           pars = c("b_Intercept", "b_log_TotalIncome", "b_as.factorRural1"
           prob = 0.95)
```



Predictions

- Use the `predict()` function to make predictions of observed CUs.

```
pred_logExp_obs <- predict(MLR_fit, newdata = CEData)
head(pred_logExp_obs)
```

```
##      Estimate Est.Error     Q2.5    Q97.5
## [1,] 9.171882 0.7397145 7.729109 10.62510
## [2,] 8.577200 0.7310542 7.122587 10.03840
## [3,] 9.093761 0.7239746 7.669618 10.50746
## [4,] 9.346656 0.7282312 7.887735 10.76552
## [5,] 9.310681 0.7239532 7.891724 10.73004
## [6,] 8.734337 0.7170328 7.324649 10.17302
```

Predictions cont'd

- Now suppose we get to know two new CU with log_TotalExp = 10, one is rural and the other is urban, and we want to predict its log_TotalIncome.
- Can also use the `posterior_predict()` function.

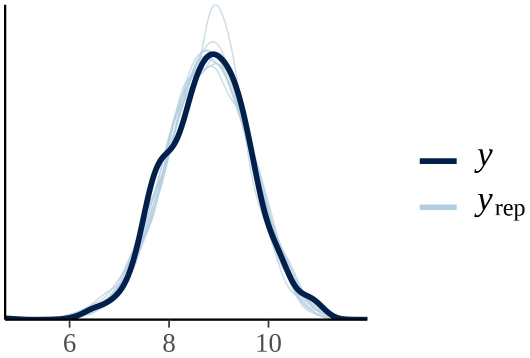
```
newdata <- data.frame(log_TotalIncome = c(10, 10), Rural = c(1, 0))
pred_logExp_new <- posterior_predict(MLR_fit, newdata = newdata)
apply(pred_logExp_new, 2, summary)
```

```
##               [,1]      [,2]
## Min.      5.374280  6.115223
## 1st Qu.   7.832950  8.054784
## Median    8.346794  8.555401
## Mean      8.327357  8.543794
## 3rd Qu.   8.817202  9.041177
## Max.     10.527223 11.170398
```

Model checking

- Function `pp_check()` plots density estimates for 10 replicated samples from the posterior predictive distribution and overlay the observed log income distribution.

```
pp_check(MLR_fit)
```



Section 4

Wrap-up and additional material

Wrap-up

- Bayesian linear regression:
  - Linear relationship between the expected outcome and the predictor(s)
  - Continuous predictors, binary predictors
  - Using the `brms` package; prior choices
- Bayesian inferences
  - Bayesian hypothesis testing and credible interval
  - Bayesian prediction
  - Posterior predictive checks

Additional material: adding a categorical predictor

Variable	Description
log(Expenditure)	Continuous; CU's total expenditures in last quarter (log)
Race	Categorical; the race category of the reference person: $1 =$ White, $2 =$ Black, $3 =$ Native American, $4 =$ Asian, $5 =$ Pacific Islander, $6 =$ Multi-race

- It is common to consider it as a categorical variable to classify multiple groups:
  - How many groups? What are the groups?
- Such classification puts an emphasis on the difference of the expected outcomes between one group to the reference group.

With only one categorical predictor

- For simplicity, consider a simplified regression model with a single predictor: the race category of the reference person x_i.

$$
\begin{aligned}
\mu_i &= \beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \beta_3 x_{i,3} + \beta_4 x_{i,4} + \beta_5 x_{i,5} \\
&= \begin{cases} \beta_0, & \text{White;} \\ \beta_0 + \beta_1, & \text{Black;} \\ \beta_0 + \beta_2, & \text{Native American;} \\ \beta_0 + \beta_3, & \text{Asian;} \\ \beta_0 + \beta_4, & \text{Pacific Islander;} \\ \beta_0 + \beta_5, & \text{Multi-race.} \end{cases}
\end{aligned} \tag{10}
$$

- What is the expected outcome μ_i for CUs in the White group?
- What is the expected outcome μ_i for CUs in the Asian group?
- What does β_5 represent?