Report for Rare-Events Simulation

1

1

Abstract

This report presents an experiment on estimating rare events under a multimodal distribution. The experimental result is accurate and the relative error is small. However, in low-dimensional examples, the result is not sufficient to illustrate the advantages of the Normalizing Flows approach. I will discuss this issue and give directions for improvement.

Keywords: Rare-Event Simulation; Normalizing Flows; Importance Sampling

Expriment

This experiment is trying to estimate the probability of the following probability under a rare event, and its density function is:

$$p(\boldsymbol{x}) = \pi_1 \mathcal{N}(\boldsymbol{x}|\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1) + \pi_2 \mathcal{N}(\boldsymbol{x}|\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2)$$

where $\pi_1 = \pi_2 = 0.5$, $\boldsymbol{\mu}_1 = (0, -3)^T$, $\boldsymbol{\mu}_2 = (0, 3)^T$ and $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The density is illustrates in Figure 1. The rare event is defined as: $\boldsymbol{x} \in (-\infty, +\infty) \times (3, +\infty)$.

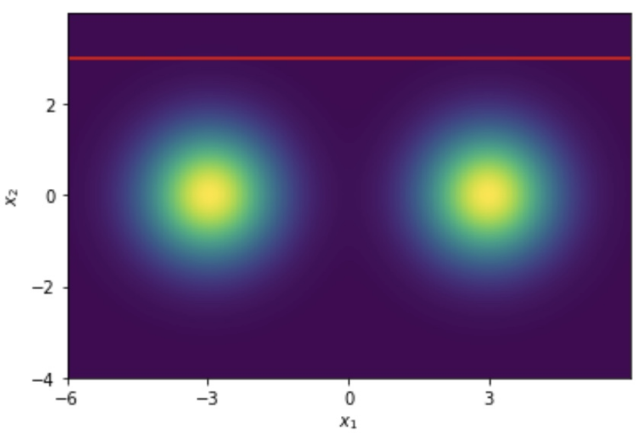

Figure 1 This is the density of the mixture Gaussian with two components, whose means are (-3, 0) and (3, 0) respectively, and the covariant matrix is the Identity matrix, and the mixture weights are both 0.5. The area above the red line defines the rare events.

I use Normalizing Flows to fit the rare events, and get a fitted distribution, which density is shown in Figure 2. Then, I apply Importance Sampling by using this fitted distribution as proxy distribution to estimate the probability of the rare events. I run 100 times and get an average is 0.0013459, which is very close to the actual probability 0.0013499. The relative error is 0.00295, and the standard deviation is 0.00008425.

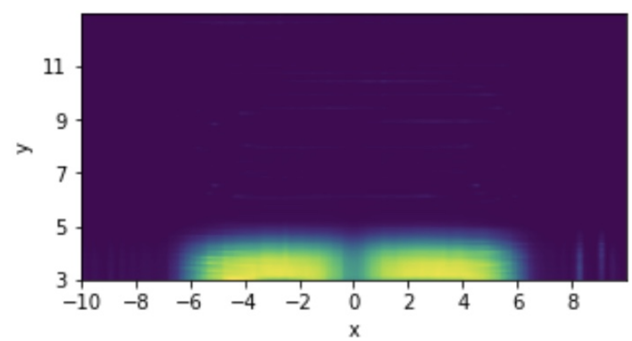

Figure 2 The fitted density of rare event by using Normalizing flows. This is used as the proxy distribution in Importance Sampling.

Discussion

This experiments proved that the method of Normalizing flows has the potential to simulate rare events in multi-modal distribution, at least in low-dimensional context. However, when I use a tailored uniform distribution as the proxy distribution, the result is also very accurate.

For example, I used a tailored uniform distribution as shown in Figure 3 to estimate the rare event. I run 100 times and get an average is 0.0013445. The relative error is 0.00397, and the standard deviation is 0.0000784878. The relative error and standard deviation are only slightly bigger than the method of Normalizing flow in this low-dimensional context.

This is because there will be many sampled points falling in the area of rare events if the samples are generated enough. However, when the dimension is high, enough sampling will not be possible. For example, let's define an event in every dimension as half of the Uniform. And the area of the event is 0.5^N of the original Uniform distribution. When N is high, there

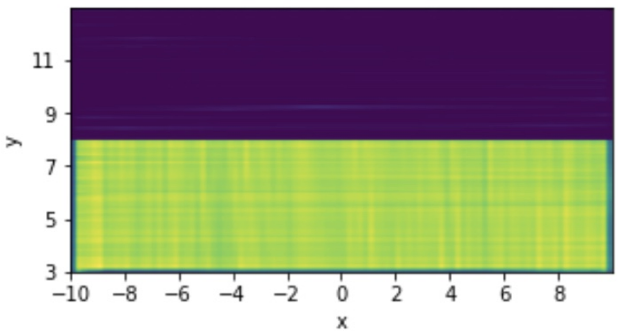

Figure 3 The density of tailered 2D Uniform distibution defined in $(-10, 10) \times (3, 8)$. Using this distribution to estimate rare event can also get an accuracy results in this low dimensional context.

will never be enough samples following in this area.

So, high-dimensional examples are necessary to demonstrate whether the method of Normalizing Flows actually works. Next, I will work on the high-dimensional examples as described in the handbook.

References