Training Scale-Invariant Neural Networks on the Sphere Can Happen in Three Regimes

Maxim Kodryan*, Ekaterina Lobacheva*, Maksim Nakhodnov*, Dmitry Vetrov

Introduction

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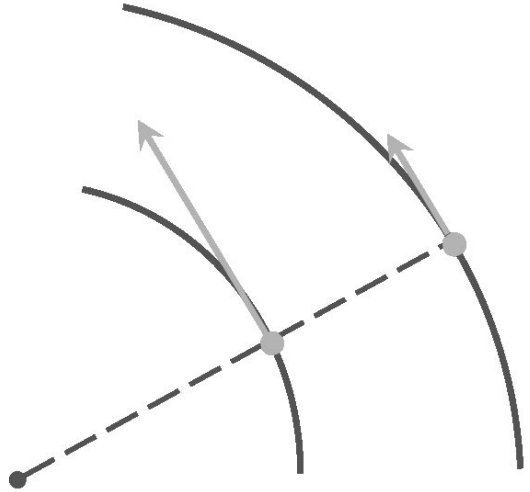
SGD optimization with a **fixed** LR ⇒ **varying** *effective learning rate* (ELR):

$\mathrm{ELR} = \mathrm{LR}/||\theta||^2$ — learning rate on the *unit sphere*.



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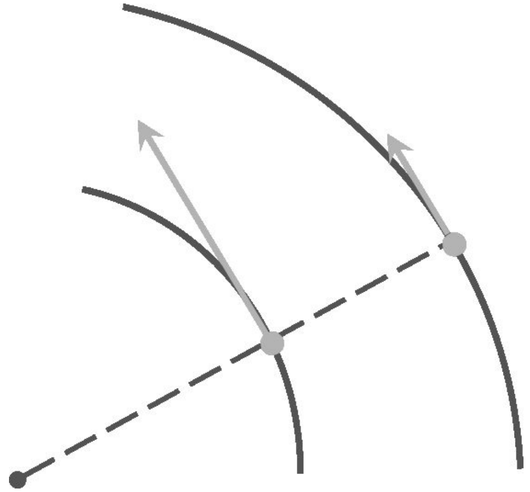
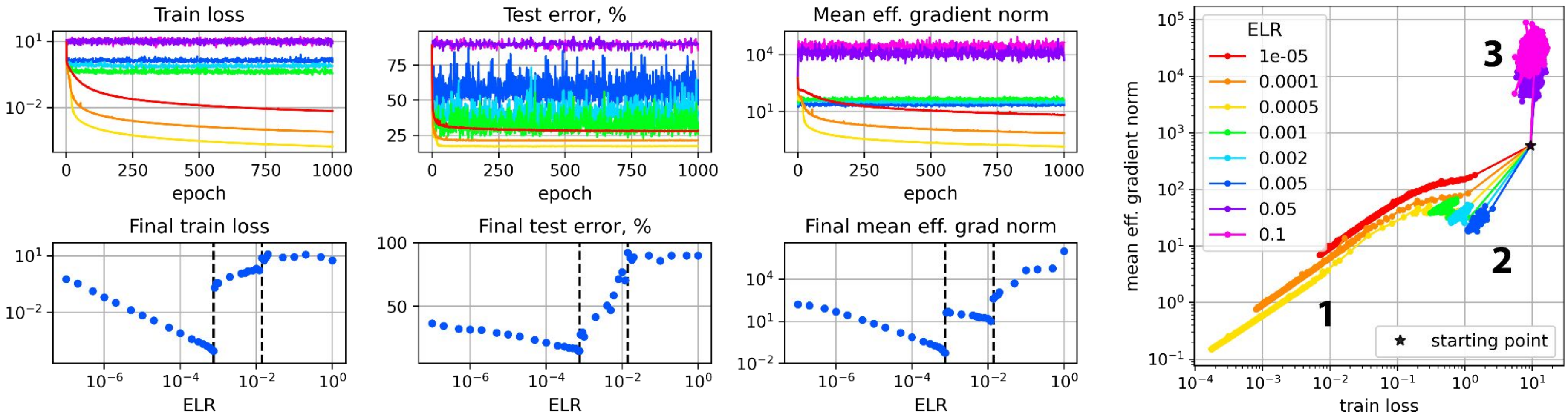


Let's optimize SI models directly on the sphere!
(with a <u>fixed</u> effective learning rate)

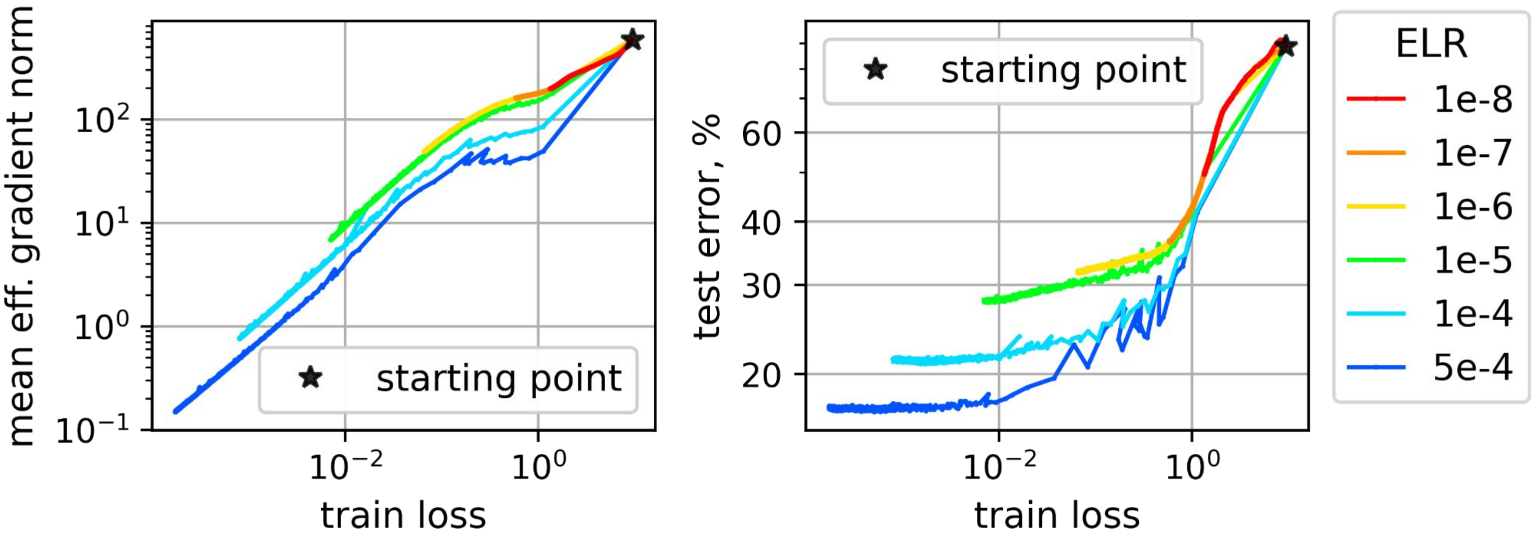
Three regimes of optimization on the sphere

1 Convergence **2** Chaotic equilibrium **3** Divergence



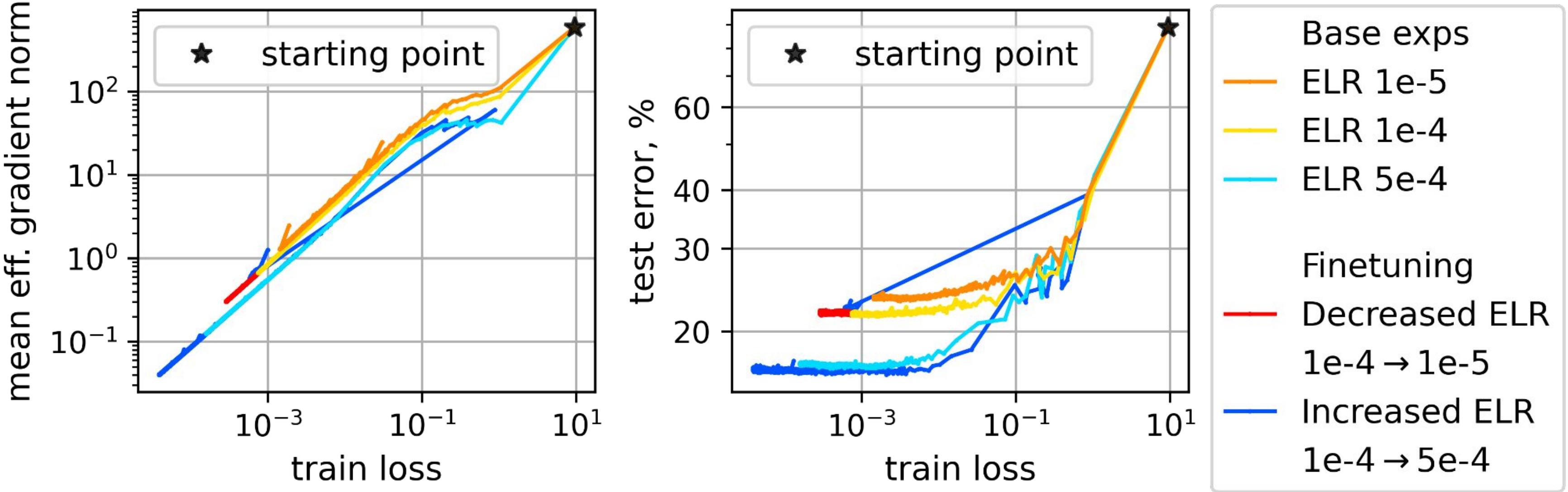
Regime 1: convergence

- Small ELR ⇒ ***convergence***
- Different optima depending on the ELR: **higher** ELR = **better** final solution (in terms of sharpness/generalization)



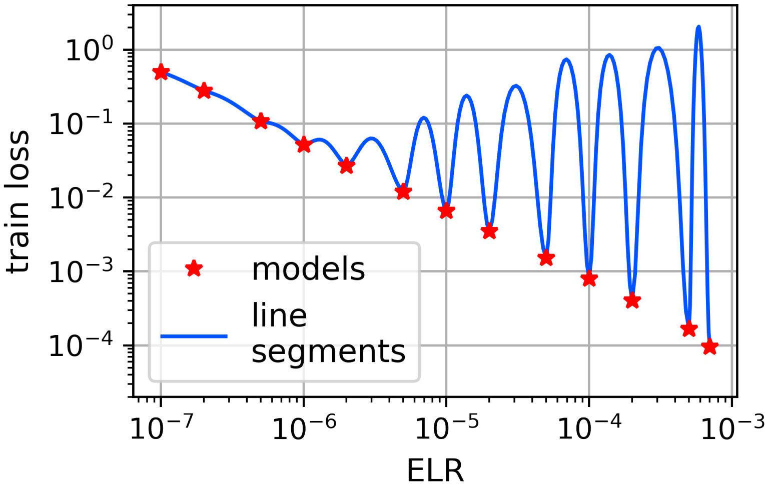
Regime 1: convergence

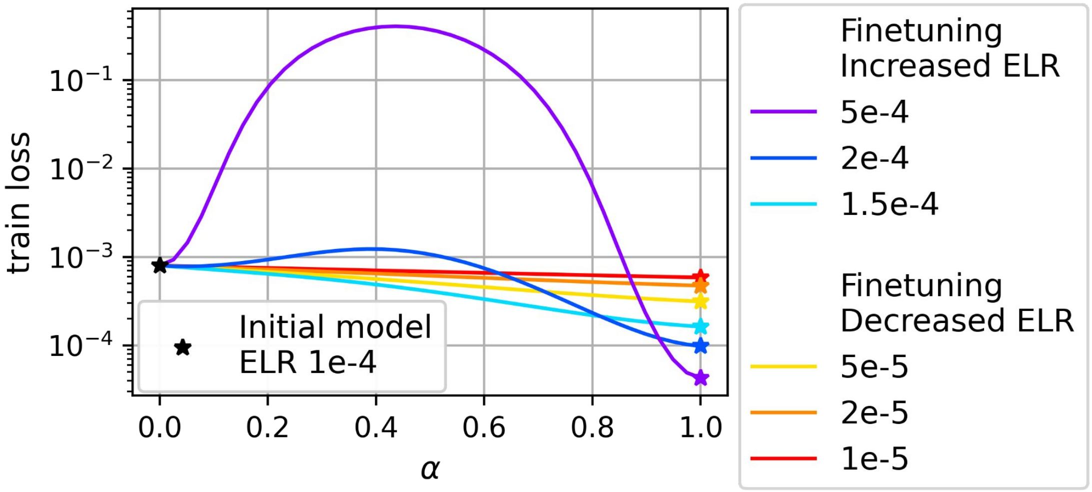
- Fine-tuning with **lower** ELR ⇒ the **same** trajectory
- Fine-tuning with **higher** ELR ⇒ jumping out into a **new wider** basin



Regime 1: convergence

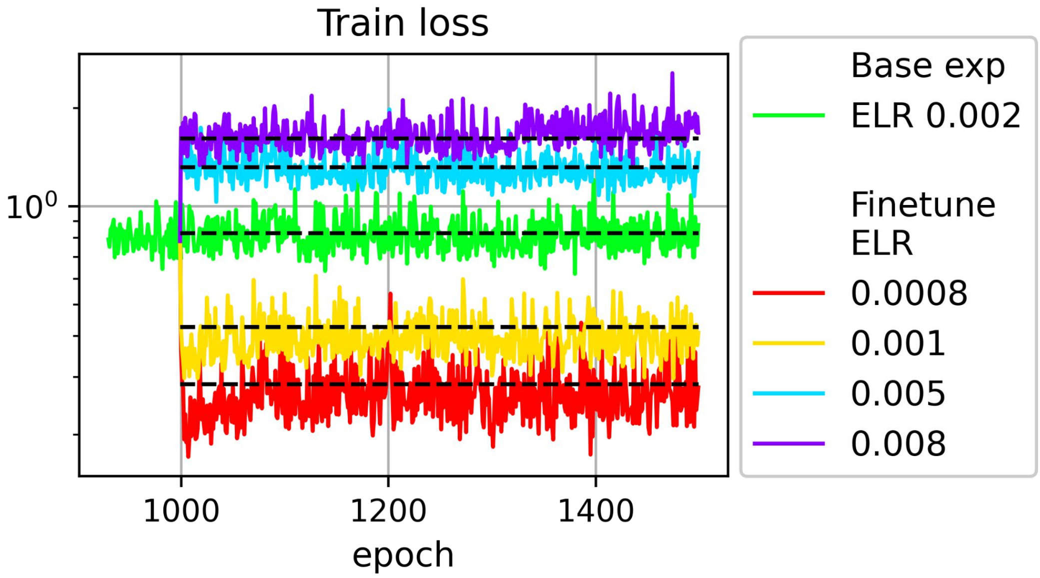
- Optima from **different** ELRs are **not** linearly connected (LC)
- Pre-trained and fine-tuned (FT) weights **are** LC for **low** FT ELR and **not** for **high** FT ELR





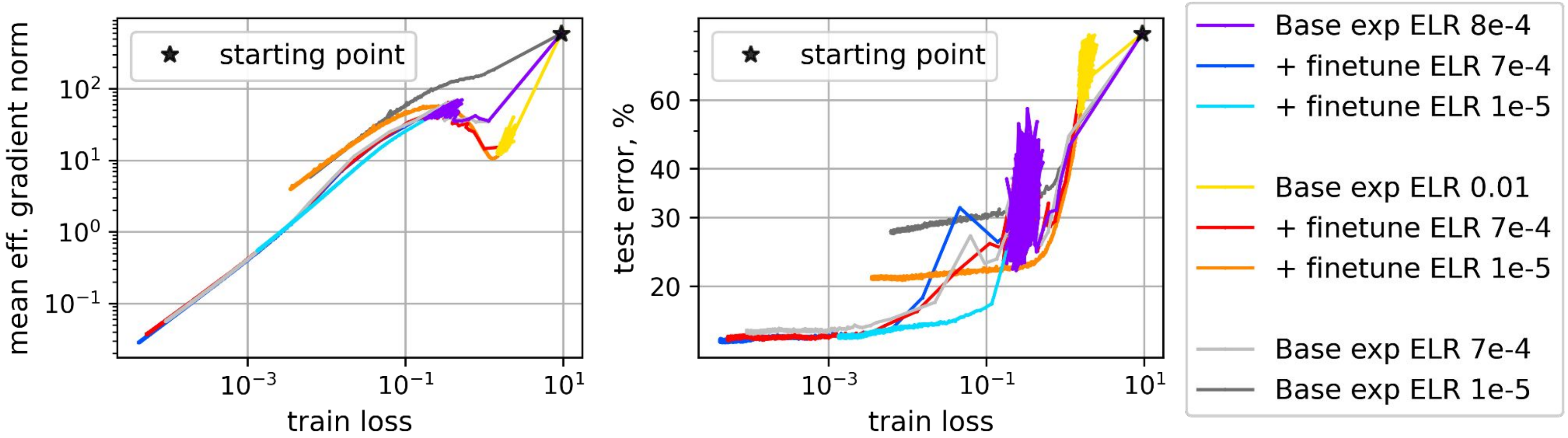
Regime 2: chaotic equilibrium

- Medium ELR ⇒ ***chaotic equilibrium***: loss noisily stabilizes at some *level*
- Changing the **ELR** ⇒ changing the **level**
- Optimization is “hopping along the walls” of the loss landscape



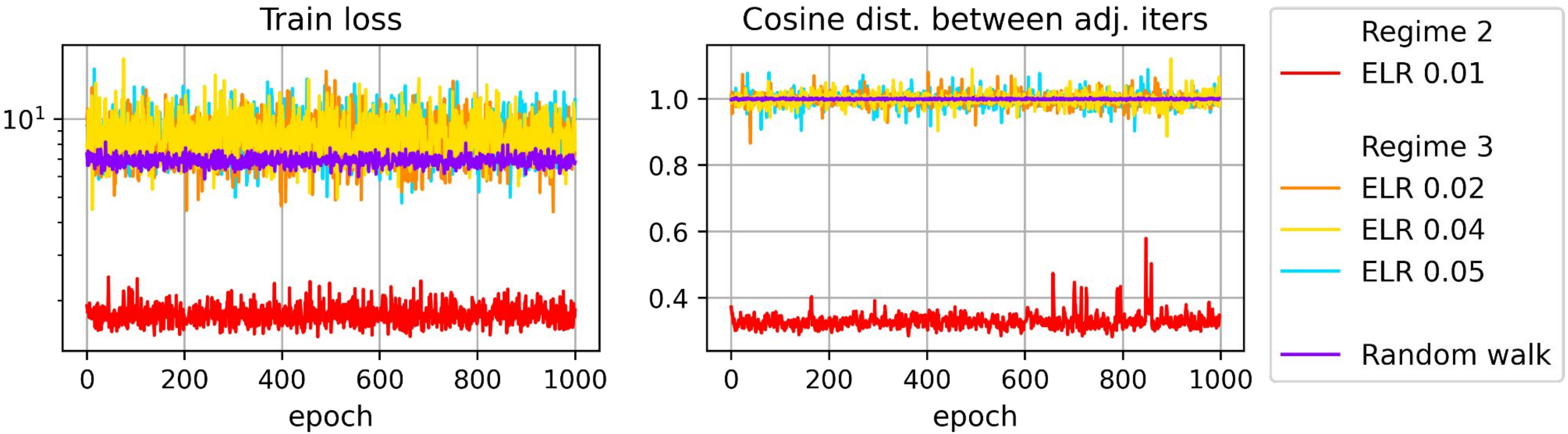
Regime 2: chaotic equilibrium

- Fine tuning with regime-1 ELRs from **low** regime-2 ELRs ⇒ convergence to the **widest** optimum
- Fine tuning with regime-1 ELRs from **high** regime-2 ELRs ⇒ convergence to **various** optima



Regime 3: divergence

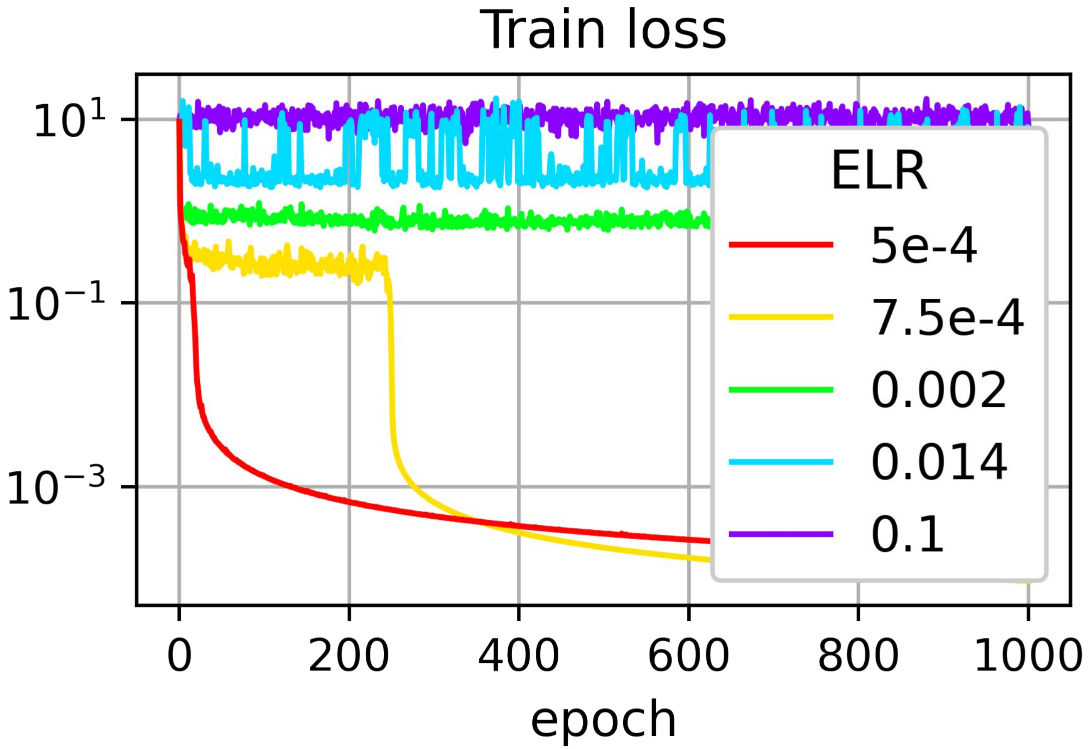
- High ELR ⇒ ***divergence***: near random guess behavior (close to random walk)



Transitions between regimes

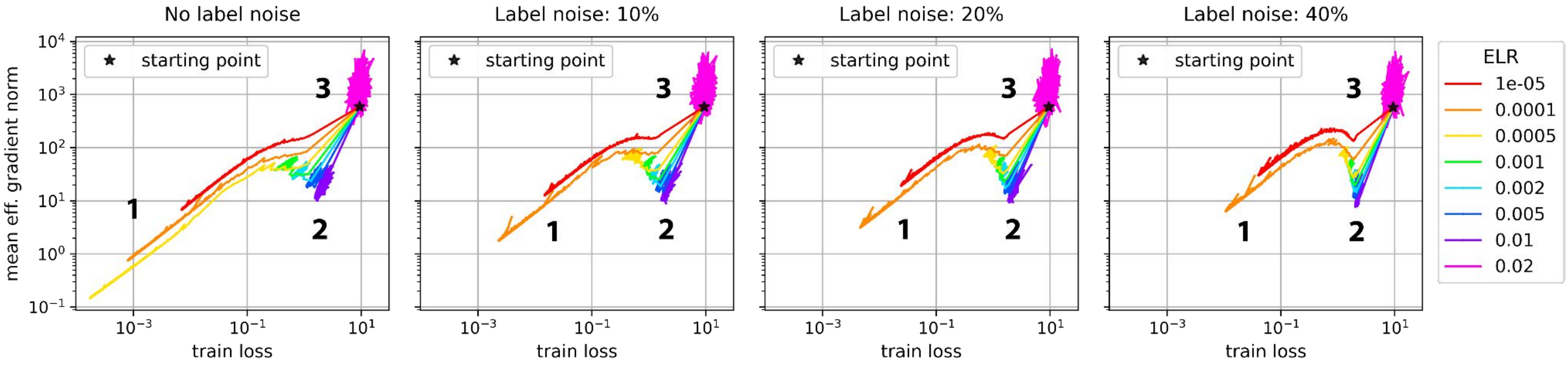
Typical regime ELRs:
- Regime 1
- Regime 2
- Regime 3

Transition ELRs:
- Regime 1/2
- Regime 2/3

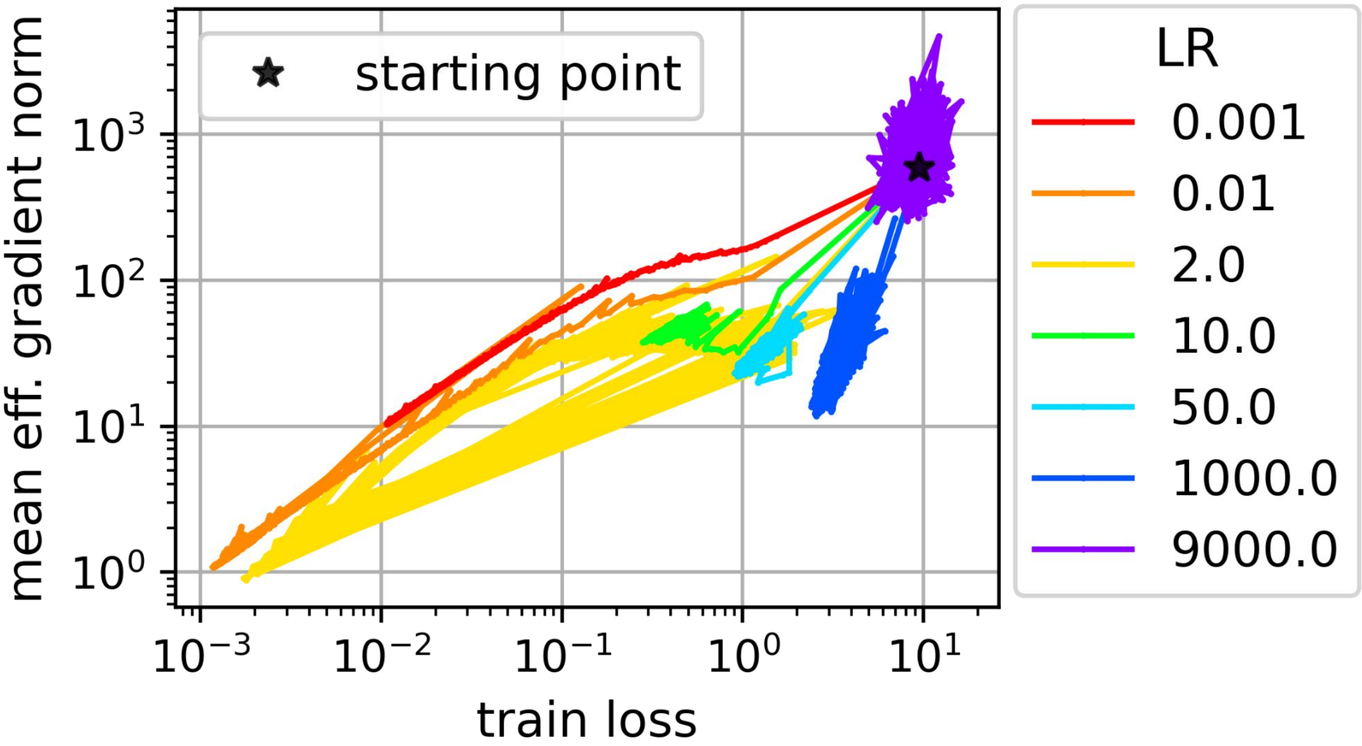


Transition between regimes 1/2 and DD

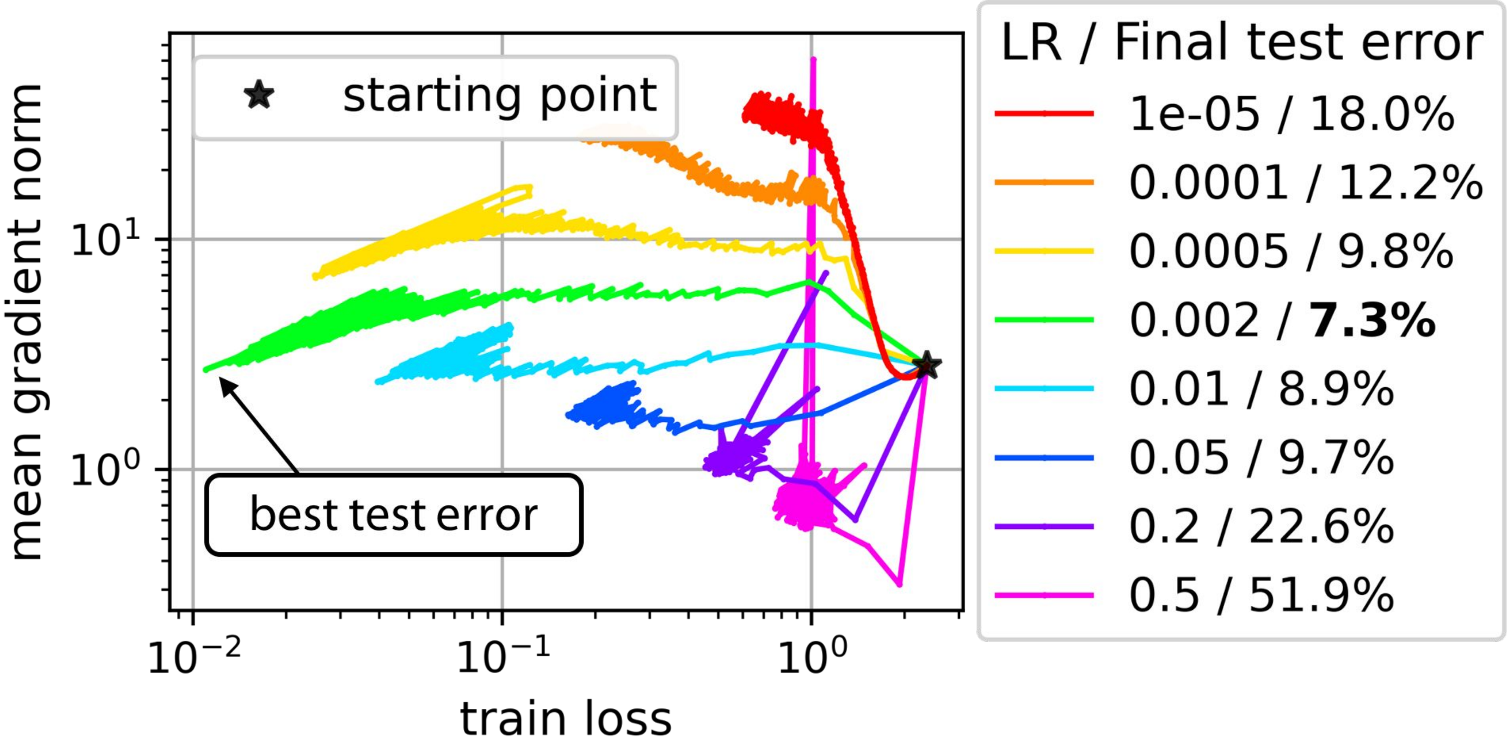
Double Descent peak ⇔ high-sharpness zone ⇔ transition between R1 and R2



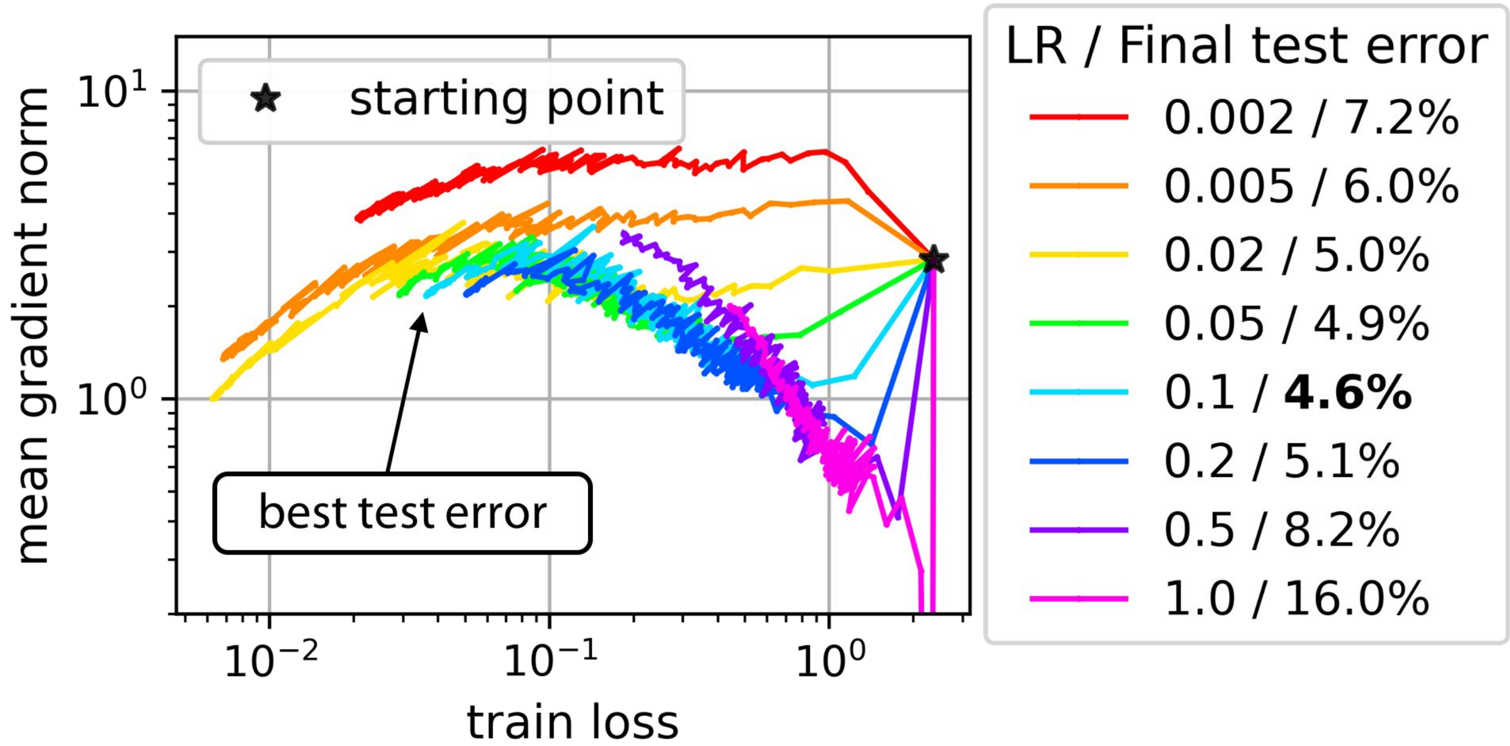
Regimes when training SI model in the whole space



Regimes in standard training: fixed LR



Regimes in standard training: cosine LR schedule



Theoretical analysis

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Define:

ELR for θ_i as $\tilde{\eta}_i \equiv \eta/||\theta_i||^2$

Eff. grad. norm for θ_i as $\tilde{g}_i \equiv ||\nabla_{\theta_i} F|| \cdot ||\theta_i||$

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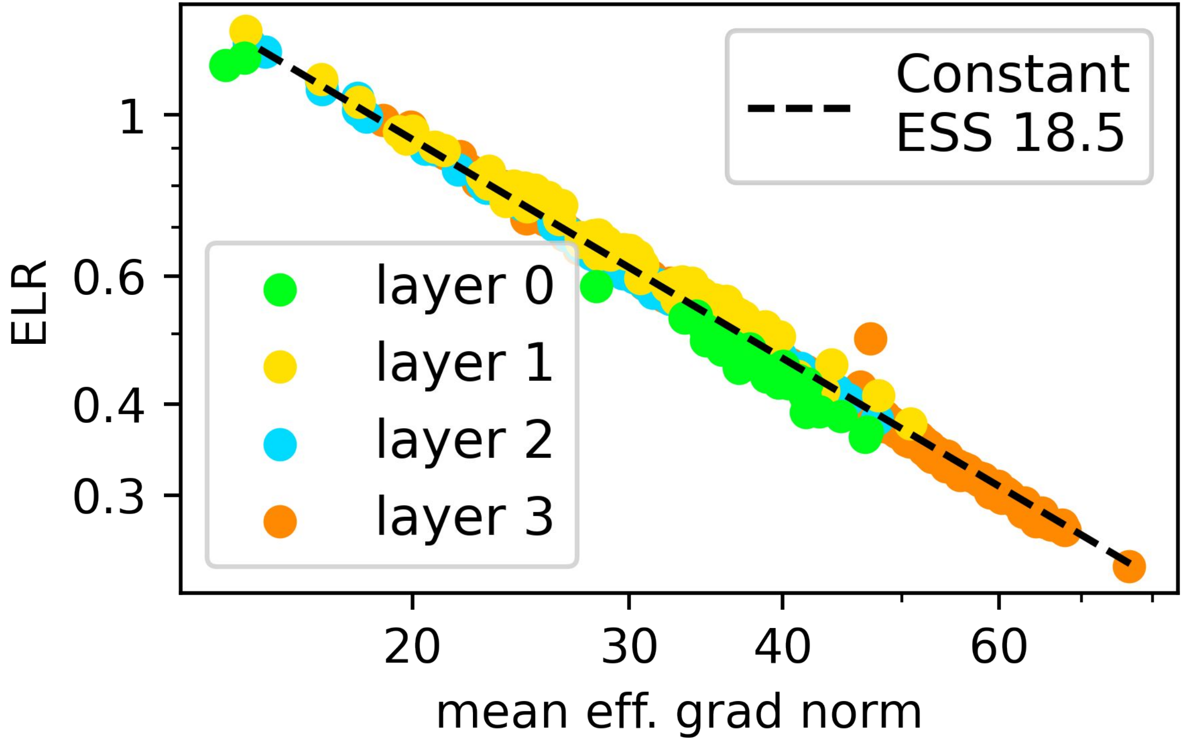
ELRs dynamics: $\tilde{\eta}_i^{(t+1)} \leftarrow \tilde{\eta}_i^{(t)} \dfrac{1 + \left(\tilde{g}^{(t)} \tilde{\eta}\right)^2}{1 + \left(\tilde{g}_i^{(t)} \tilde{\eta}_i^{(t)}\right)^2}$

Regime 2: ESS alignment

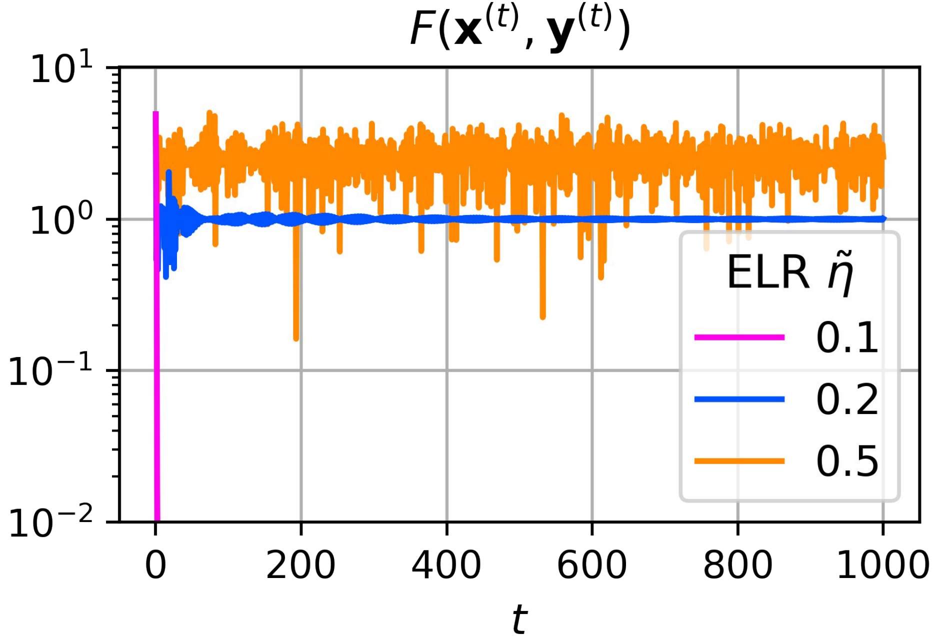


Toy example

$$F(x_1, y_1, \ldots, x_n, y_n) = \sum_{i=1}^{n} \alpha_i \frac{x_i^2}{x_i^2 + y_i^2}, \; \alpha_i > 0$$

$$\tilde{\eta} < 1/\sum_{i=1}^{n} \alpha_i \;\Rightarrow\; \text{Regime 1}$$

$$\tilde{\eta} > 1/\sum_{i=1}^{n} \alpha_i \;\Rightarrow\; \text{Regime 2/3}$$



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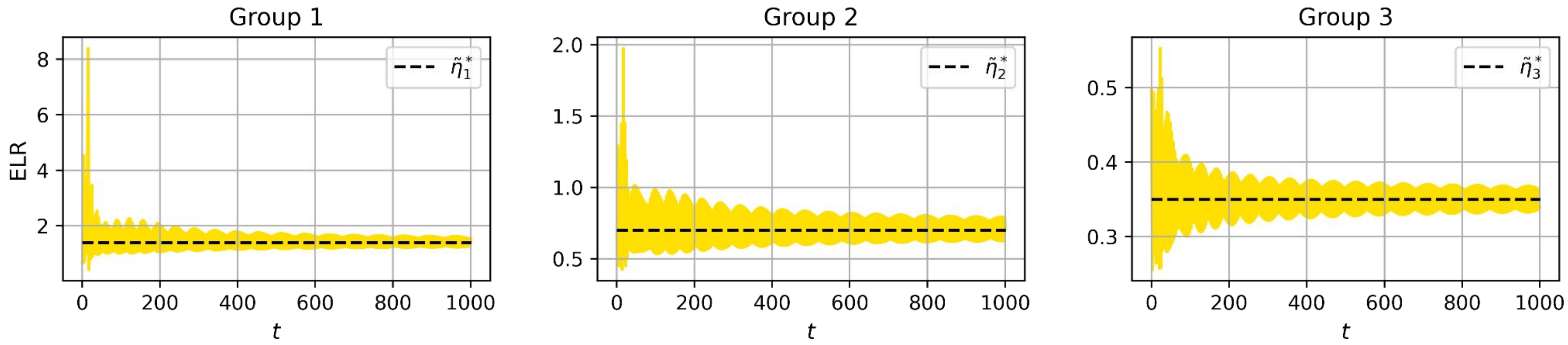
$$\text{Regime 2/3 with equilibrium ELRs } \tilde{\eta}_i^* \equiv \frac{\tilde{\eta} \sum_{j=1}^{n} \alpha_j}{\alpha_i}$$



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- **Other results:** LC of regime-2 checkpoints, dependence on model width/data complexity, additional experiments, etc.