Linear diffusion models

Grigory Bartosh

7 October 2022

Simple ODEs and SDEs

- Drift Ordinary Differential Equation (ODE):

$$dz_t = \bar{r}_t dt, \quad q(z_t|z_0) = \delta\left(z_0 + \int_0^t \bar{r}_s ds - z_t\right) \tag{1}$$

- Random walk 1:

$$dz_t = dw_t, \quad q(z_t|z_0) = \mathcal{N}\left(z_t; z_0, t\right) \tag{2}$$

- Random walk 2:

$$dz_t = \bar{g}_t dw_t, \quad q(z_t|z_0) = \mathcal{N}\left(z_t; z_0, \int_0^t \bar{g}_s^2 ds\right) \tag{3}$$

- Itô drift-diffusion processes:

$$dz_t = \bar{r}_t dt + \bar{g}_t dw_t, \quad q(z_t|z_0) = \mathcal{N}\left(z_t; z_0 + \int_0^t \bar{r}_s ds, \int_0^t \bar{g}_s^2 ds\right) \tag{4}$$

Itô's lemma

- Let z_t satisfies the SDE:

$$dz_t = \bar{r}_t dt + \bar{g}_t dw_t \tag{5}$$

- If $F_t(z_t) \in C^2$:

$$dF_t(z_t) = \frac{\partial F_t(z_t)}{\partial t} dt + \frac{\partial F_t(z_t)}{\partial z_t} dz_t + \frac{1}{2} \frac{\partial^2 F_t(z_t)}{\partial z_t^2} dz_t^2 + \dots \tag{6}$$

$$= \frac{\partial F_t(z_t)}{\partial t} dt + \frac{\partial F_t(z_t)}{\partial z_t} (\bar{r}_t dt + \bar{g}_t dw_t) + \frac{1}{2} \frac{\partial^2 F_t(z_t)}{\partial z_t^2} \left(r_t^2 dt^2 + 2\bar{r}_t \bar{g}_t dt dw_t + \bar{g}_t^2 dw_t^2 \right) + \dots \tag{7}$$

$$= \left[\frac{\partial F_t(z_t)}{\partial t} + \bar{r}_t \frac{\partial F_t(z_t)}{\partial z_t} + \frac{\bar{g}_t^2}{2} \frac{\partial^2 F_t(z_t)}{\partial z_t^2} \right] dt + \bar{g}_t \frac{\partial F_t(z_t)}{\partial z_t} dw_t \tag{8}$$

Linear 1D SDE: I

- What we'd like to solve:

$$dx_t = r_t x_t dt + g_t dw_t, \quad q(x_t|x_0) = ??? \tag{9}$$

- **Idea**: Let's find function $F_t(z_t) : z_t \mapsto x_t$
- SDE for z_t:

$$dz_t = \bar{g}_t dw_t \tag{10}$$

- Function $F_t(z_t)$:

$$x_t = F_t(z_t) = \alpha_t z_t \tag{11}$$

- New SDE:

$$\begin{aligned} dx_t &= dF_t(z_t) &(12)\\ &= \left[\frac{\partial F_t(z_t)}{\partial t} + \bar{r}_t \frac{\partial F_t(z_t)}{\partial z_t} + \frac{\bar{g}_t^2}{2}\frac{\partial^2 F_t(z_t)}{\partial z_t^2}\right] dt + \bar{g}_t \frac{\partial F_t(z_t)}{\partial z_t} dw_t &(13)\\ &= [\dot{\alpha}_t z_t]\, dt + \bar{g}_t \alpha_t dw_t &(14) \end{aligned}$$

Linear 1D SDE: II

- Let's find α_t:

$$\dot{\alpha}_t z_t = r_t x_t \tag{15}$$

$$\dot{\alpha}_t \cancel{z_t} = r_t \alpha_t \cancel{z_t} \tag{16}$$

$$\frac{\dot{\alpha}_t}{\alpha_t} = r_t \tag{17}$$

$$\frac{\partial \log \alpha_t}{\partial t} = r_t \tag{18}$$

$$\alpha_t = e^{\int_0^t r_s ds} \tag{19}$$

- Let's find $\bar{g}_t$:

$$\bar{g}_t \alpha_t = g_t \tag{20}$$

$$\bar{g}_t = g_t \alpha_t^{-1} \tag{21}$$

Linear 1D SDE: III

- Linear SDE:

$$dx_t = r_t x_t dt + g_t dw_t \tag{22}$$

- Function $F_t(z_t)$:

$$x_t = F_t(z_t) = \alpha_t z_t, \quad \alpha_t = e^{\int_0^t r_s ds} \tag{23}$$

- SDE for z_t:

$$dz_t = \bar{g}_t dw_t, \quad q(z_t|z_0) = \mathcal{N}\left(z_t; z_0, \int_0^t \bar{g}_s^2 ds\right), \quad \bar{g}_t = g_t \alpha_t^{-1} \tag{24}$$

- Let's find $q(z_t|z_0)$:

$$q(x_t|x_0) = F_t\left(q(z_t|z_0)\right) \tag{25}$$

$$= \mathcal{N}\left(x_t; \alpha_t x_0, \alpha_t^2 \int_0^t \alpha_s^{-2} g_s^2 ds\right) \tag{26}$$

Designing a linear 1D SDE

- Linear SDE:

$$q(x_t|x_0) = \mathcal{N}\left(x_t; \alpha_t x_0, \sigma_t^2\right), \quad dx_t = r_t x_t dt + g_t dw_t, \quad r_t =?, \ g_t =? \tag{27}$$

- Let's find r_t:

$$e^{\int_0^t r_s ds} = \alpha_t, \quad r_t = \frac{\partial \log \alpha_t}{\partial t} \tag{28}$$

- Let's find g_t:

$$\alpha_t^2 \int_0^t \alpha_s^{-2} g_s^2 ds = \sigma_t^2 \tag{29}$$

$$\int_0^t \alpha_s^{-2} g_s^2 ds = \alpha_t^{-2} \sigma_t^2 \tag{30}$$

$$g_t^2 = \alpha_s^2 \frac{\partial}{\partial t}\left(\alpha_t^{-2}\sigma_t^2\right) = \alpha_s^2 \left(\alpha_t^{-2}\frac{\partial \sigma_t^2}{\partial t} + \frac{\partial \alpha_t^{-2}}{\partial t}\sigma_t^2\right) \tag{31}$$

$$= \cancel{\alpha_s^2}\left(\cancel{\alpha_t^{-2}}\frac{\partial \sigma_t^2}{\partial t} - 2\cancel{\alpha_t^{-2}} r_t \sigma_t^2\right) = \frac{\partial \sigma_t^2}{\partial t} - 2 r_t \sigma_t^2 \tag{32}$$

Multidimensional linear SDE

- Linear SDE:

$$dx_t = R_t x_t dt + G_t dw_t, \quad q(x_t|x_0) = \mathcal{N}(x_t; A_t x_0, \Sigma_t) \tag{33}$$

- R_t and G_t:

$$R_t = \dot{A}_t A_t^{-1}, \quad G_t G_t^T = \dot{\Sigma}_t - R_t \Sigma_t - \Sigma_t R_t^T \tag{34}$$

- R_t and G_t:

$$A_t = e^{\int_0^t R_s ds}, \quad \Sigma_t = A_t \int_0^t A_s G_s G_s^T A_s^T ds A_t^T \tag{35}$$

Density estimation

- Linear SDE:

$$dx_t = R_t x_t dt + G_t dw_t \tag{36}$$

- ODE (shares marginals):

$$dx_t = \left[R_t x_t - \frac{1}{2} G_t G_t^T \nabla_{x_t} \log q(x_t) \right] dt \tag{37}$$

- Can use NeuralODE approach for density estimation:

$$dx_t = f_t(x_t) dt \tag{38}$$

$$d \log q_t(x_t) = -Tr \left(\frac{\partial f_t(x_t)}{\partial x_t} \right) dt \tag{39}$$