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A SIMPLE EXPRESSION FOR THE SHAPLEY VALUE IN A SPECIAL CASE*

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We present a simple and easily calculated expression for the Shapley value whenever the characteristic function is a "cost" function with the property that the cost of any subset of players is equal to the cost of the "largest" player in that subset. It turns out that a simple rule previously proposed for calculating airport landing charges generates precisely the Shapley value for an appropriately defined game.

In the twenty years since its proposal, the Shapley value [4] has received surprisingly few applications. Perhaps one reason is the difficulty of computing it for large games. Owen [3 p. 72] has defined a multilinear extension of a game which "allows us to obtain some reasonable approxmiations (of the Shapley value) with comparative ease". It would also seem worthwhile to explore special cases in which simple and exact expressions of the Shapley value are available.

The present note concerns one such case. It was prompted by some work of Baker [1] and Thompson [5] on setting airport landing charges for different types of aircraft. They suggest that the cost of providing an airport runway essentially depends upon the largest type of aircraft to land there. They propose the following simple rule for allocating these costs.

> Divide the cost of catering for the smallest type of aircraft equally among the number of landings of all aircraft.
>
> Divide the incremental cost of catering for the second smallest type of aircraft (above the cost of the smallest type) equally among the number of landings of all but the smallest type of aircraft. Continue thus until finally the incremental cost of the largest type of aircraft (above the cost of the second largest type) is divided equally among the number of landings made by the largest aircraft type.

An implication of the present note is that the resulting set of landing charges is precisely the Shapley value for an appropriately defined game. As a matter of interest, details of this application are given in the accompanying Table 1.

The general result is that a simple and easily calculated expression may be obtained for the Shapley value whenever the characteristic function is a "cost" function with the property that the cost of any subset is equal to the cost of the "largest" player in that subset. (Such a cost function is subadditive but its negative is clearly superadditive. The Shapley value of a game based on the negative of the cost function is equal to minus the Shapley value of a game based on the cost function itself.)

Terminology

Let N_i denote the set of players type i, for $i = 1, \cdots, m$, where $n_i > 0$ is the number of players type i. Let $N = \bigcup_{i=1}^{m} N_i$ and $n = \sum_{i=1}^{m} n_i$. Let c_i be the "cost" associated with a player type i. Without loss of generality these types may be ordered so that

$$0 = c_0 < c_1 < c_2 < \cdots < c_m .$$

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TABLE 1

Aircraft Landings, Runway Costs and Landing Charges at Birmingham Airport, 1968–69

Aircraft Type	Subscript	Number of Aircraft Landings	Annual Capital Cost	Shapley Value Capital Charge	Main-tenance Cost Per Landing	Total Shapley Value Landing Fee	Present Landing Fee
	i	n_i	c_i	Φ_i	m_i	$\Phi_i + m_i$	f_i
Fokker Friendship 27	1	42	£65,899	£4.86	£5.23	£10.09	£5.80
Viscount 800	2	9,555	76,725	5.66	6.09	11.75	11.40
Hawker Siddeley Trident	3	288	95,200	10.30	7.55	17.85	21.70
Britannia 100	4	303	97,200	10.85	7.71	18.56	29.80
Caravelle VI R	5	151	97,436	10.92	7.73	18.65	20.30
BAC 111 (500)	6	1,315	98,142	11.13	7.79	18.92	16.70
Vanguard 953	7	505	102,496	13.40	8.13	21.53	26.40
Comet 4B	8	1,128	104,849	15.07	8.32	23.39	29.40
Britannia 300	9	151	113,322	44.80	8.99	53.79	34.70
Convair Corronado	10	112	115,440	60.61	9.16	69.77	48.30
Boeing 707	11	22	117,676	162.24	9.34	171.58	66.70

Define a game on N by the subadditive characteristic function

$$C(\emptyset) = 0, \qquad C(S) = \max\{c_i\}, \tag{1}$$

where the maximisation is taken over all i such that $N_i \cap S \neq \emptyset$. Finally define

$$R_k = \textstyle\bigcup_{i=k}^m N_i \quad \text{and} \quad r_k = \sum_{i=k}^m n_i \qquad \text{for } k = 1, \cdots, m. \tag{2}$$

Note that $R_1 = N$ and $r_1 = n$. Recall that the Shapley value for the game (N, C) is defined by

$$\phi_j(C) = \textstyle\sum_{S\subseteq N; S\ni j} \frac{(n-s)!(s-1)!}{n!}[C(S) - C(S - \{j\})] \quad \text{for } j \in N, \tag{3}$$

where s is the number of players in subset S.

PROPOSITION. *If the characteristic function is defined by* (1) *then the expression for the Shapley value may be simplified to*

$$\phi_j(C) = \textstyle\sum_{k=1}^i (c_k - c_{k-1})/r_k \quad \text{for} \quad j \in N_i, i = 1, \cdots, m. \tag{4}$$

REMARKS. (a) If we set $\phi_j(C) = \Phi_i$ for $j \in N_i$, then

$$\Phi_i = \Phi_{i-1} + (c_i - c_{i-1})/r_i. \tag{5}$$

(b) If $n_i = 1$ for all i, then

$$\Phi_i = \Phi_{i-1} + (c_i - c_{i-1})/(n - i + 1). \tag{6}$$

PROOF OF PROPOSITION. By assumption, the Shapley value for the sum of two games is equal to the sum of the Shapley values of the two component games taken separately. Here the characteristic function C can be represented as the sum of m games, each of which is symmetric and hence has an easily-derived Shapley value.

Formally, for each $k = 1, \cdots, m$, define the characteristic function C_k on N by

$$\begin{aligned} C_k(S) &= 0 && \text{if } S \cap R_k = \emptyset, \\ &= c_k - c_{k-1} && \text{if } S \cap R_k \neq \emptyset. \end{aligned} \tag{7}$$

This is clearly a symmetric game for r_k players, with $n - r_k$ dummies, giving a value

$$(8) \qquad \begin{aligned} \phi_j(C_k) &= 0 && \text{if } j \notin R_k, \\ &= (c_k - c_{k-1})/r_k && \text{if } j \in R_k. \end{aligned}$$

Now, for each $S \subseteq N$,

$$(9) \qquad C(S) = \textstyle\sum_{k=1}^{n} C_k(S)$$

and so

$$(10) \qquad \phi_j(C) = \textstyle\sum_{k=1}^{n} \phi_j(c_k).$$

If $j \in N_i$, the summands on the right-hand side of the last expression will vanish for $k > i$ and will equal $(c_k - c_{k-1})/r_k$ for $k \leqq i$. Thus

$$(11) \qquad \phi_j(C) = \textstyle\sum_{k=1}^{i} (c_k - c_{k-1})/r_k \quad \text{for } j \in N_i,\ i = 1, \cdots, m.$$

Q.E.D.

Example

At Birmingham airport in the year 1968–69 there were 13,572 landings of 11 different aircraft types. Runway maintenance costs were incurred on a per-landing basis whereas capital costs depended upon the largest aircraft type to land there. Assume a landing fee for aircraft type i comprises a maintenance charge (m_i) plus a capital charge (Φ_i) where the latter is computed as the Shapley value of the appropriate game. Table 1 sets out the calculation and compares the resulting fee with the present landing fee (f_i). The two fee structures are not dissimilar and recoup the same total cost, but the Shapley value structure is higher for the smallest and largest aircraft. The data is taken from Thompson [5], who used precisely the rule given by (5). More general game-theoretic approaches to this runway pricing problem are discussed in Littlechild and Thompson [2].

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