Theorem.

$$se = \sqrt{\frac{p(1-p)}{n}}$$

Proof. It is sufficient to prove

$$\mathbb{V}\left(\hat{p}_n\right) = \frac{p(1-p)}{n}$$

We already know that

$$\mathbb{E}_\theta\left(\hat{p}_n\right) = \sum_{i=1}^{n} \frac{1}{n} \mathbb{E}\left(X_i\right) = p$$

On the other hand, because $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Bernoulli}(p)$,

$$\begin{aligned}
\mathbb{E}_\theta\left(\hat{p}_n^2\right) &= \frac{1}{n^2} \sum_{i,j=1}^{n} \mathbb{E}_\theta\left(X_i X_j\right) \\
&= \frac{1}{n^2}\left(\sum_{i=1}^{n} \mathbb{E}_\theta\left(X_i^2\right) + 2\sum_{i \neq j} \mathbb{E}_\theta\left(X_i\right) \mathbb{E}_\theta\left(X_j\right)\right) \\
&= \frac{1}{n^2}\left(\sum_{i=1}^{n} p + 2\binom{n}{2} p^2\right) = p^2 + \frac{p(1-p)}{n}
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{V}_\theta\left(\hat{p}_n\right) &= \mathbb{E}_\theta(\hat{p}_n^2) - (\mathbb{E}_\theta(\hat{p}_n))^2 \\
&= p^2 + \frac{p(1-p)}{n} - p^2 = \frac{p(1-p)}{n}
\end{aligned}$$

The theorem is proved. □