Theorem. *Suppose* $X_1, \ldots, X_n \stackrel{i.i.d.}{\sim} F$. *Then*

$$\mathbb{E}_\theta\left(\hat{F}_n(x)\right) = F(x)$$
$$\mathbb{V}_\theta\left(\hat{F}_n(x)\right) = \frac{F(x)(1-F(x))}{n}$$
$$MSE = \frac{F(x)(1-F(x))}{n} \to 0$$

Proof. Recall the definition of $\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{X_i \le x\}}$. Observe that

$$\mathbb{E}_\theta\left(\hat{F}_n(x)\right) = \frac{1}{n}\sum_{i=1}^n \mathbb{E}_\theta\left(\mathbf{1}_{\{X_i \le x\}}\right) = \frac{1}{n}\sum_{i=1}^n \mathbb{P}_\theta(X_i \le x) = F(x)$$

where the last equality holds due to the i.i.d. assumption on $X_i \sim F$. Furthermore,

$$\begin{aligned}
\mathbb{E}_\theta\left(\frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{X_i \le x\}}\right)^2 &= \frac{1}{n^2}\sum_{i,j}\mathbb{E}_\theta\left(\mathbf{1}_{\{X_i \le x\}}\mathbf{1}_{\{X_j \le x\}}\right) \\
&= \frac{1}{n^2}\left\{\sum_{i=1}^n \mathbb{E}_\theta\left(\mathbf{1}_{\{X_i \le x\}}\right) + 2\sum_{i \ne j}\mathbb{E}_\theta\left(\mathbf{1}_{\{X_i \le x\}}\right)\mathbb{E}_\theta\left(\mathbf{1}_{\{X_j \le x\}}\right)\right\} \\
&= \frac{1}{n^2}\left\{n\mathbb{P}_\theta(X_i \le x) + 2\binom{n}{2}\mathbb{P}_\theta(X_i \le x)\mathbb{P}_\theta(X_j \le x)\right\} \\
&= \frac{F(x) + (n-1)F^2(x)}{n} = F^2(x) + \frac{F(x)(1-F(x))}{n}
\end{aligned}$$

Hence

$$\mathbb{V}_\theta\left(\hat{F}_n(x)\right) = \mathbb{E}_\theta\left(\frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{X_i \le x\}}\right)^2 - \left(\mathbb{E}_\theta\left(\hat{F}_n(x)\right)\right)^2 = \frac{F(x)(1-F(x))}{n}$$

Since $\mathbb{E}\left(\hat{F}_n(x)\right) = F(x)$, $\hat{F}_n(x)$ is unbiased hence MSE $= \mathbb{V}_\theta\left(\hat{F}_n(x)\right)$. The theorem is proved. $\square$