Numerical solutions implemented in determining values of v, $\dot{Q}$

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May 7th, 2012

In order to determine v, defined as

$$v = \frac{\partial u}{\partial t} \tag{1}$$

for use with an advection/diffusion model, we need to (1) characterize $\dot{\mathcal{Q}}$, defined as

$$\mathcal{Q} = \frac{\partial \mathcal{P}}{\partial z} \tag{2}$$

and then (2) use this characterization to solve for v. It was determined that a numerical solution should be used, specifically using finite difference equations.

1 Finding $\dot{\mathcal{Q}}$

$\dot{\mathcal{Q}}$ can be determined using the forward difference approximation for first derivatives:

$$\dot{Q}_{n+\frac{1}{2}} = \frac{Q_{n+1} - Q_n}{\Delta t} \tag{3}$$

Where $\mathcal{Q}$ is determined also using a numerical method, specifically

$$\mathcal{Q}_{n+1} = \mathcal{Q}_n + \Delta t \frac{G_d}{\tau} A v \tag{4}$$

where the update step is dependent upon v, and can be reformulated and written as

$$\mathcal{Q}_{n+1} = \mathcal{Q}_n + \frac{\Delta t \cdot G_d \cdot v_{zz}}{\tau} - \frac{\Delta t \mathcal{Q}_n}{\tau} \tag{5}$$

2 Finding v_{zz}

Due to the fact that v and $\mathcal{Q}$ are co-dependent (i.e. their values depend upon each other) they are thus solved simultaneously. The second order, center, finite difference approximation of v_{zz} (read: $\frac{\partial^2 v}{\partial z^2}$) is simplified using the fact that v represents a vector of length M (the number of nodes approximated at). Using a diffusion coefficient γ, where

$$\gamma = \frac{\Delta t}{(\Delta z)^2} \tag{6}$$

and a tridiagonal matrix A in order to take the center finite difference for the values in vector v, specifically of the form

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix} \tag{7}$$

Then it follows simply that

$$\vec{v}_{zz} = A\vec{v} \tag{8}$$

3 Finding v

After a solution for v_{zz} is determined, it can be used in determining the value of v (as this is done numerically, this is *iteratively* determined), specifically as

$$v_n = v_{n-1} + \kappa H_A G_\infty \cdot v_{zz} + \Delta t \cdot \kappa H_A \dot{\mathcal{Q}} \tag{9}$$

4 Ignore past here

v is also solved using finite difference, however this is done using a second order central difference approximation, defined as

$$f''(x) = \frac{f(x+h) - 2 \cdot f(x) + f(x-h)}{h^2} \tag{10}$$

Specifically of the form

$$v_n(j) = v_{n-1}(j) + D\gamma \left(v_{n-1}(j+1) - 2 \cdot v_{n-1}(j) + v_{n-1}(j-1)\right) + \Delta t \kappa \dot{\mathcal{Q}}(j) \tag{11}$$

Although it may appear that we are missing the h^2 term, due to the fact that $h = 1$, this is simply $h^2 = 1$, which is not necessary to include.