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Application of Auxiliary ODE to BVPE model

First, we begin with the BVPE model, which can be reduced to

$$\frac{1}{\kappa}\frac{\delta u}{\delta t} = \frac{\delta \sigma}{\delta z}, \quad 0 < z < h,\ 0 < t < t_f \tag{1}$$

Where $\sigma = \sigma_{zz}$ is modeled well by

$$\sigma(z,t) = H_A \int_0^t G(t-s)\frac{d\epsilon}{ds}(z,s)ds = H_A G * \frac{d\epsilon}{dt} \tag{2}$$

where

$$G(t) = 1 + c\int_{\tau_1}^{\tau_2} g(t,\tau)dF(\tau) = G_\infty + G_d \mathbb{E}_F[g] \tag{3}$$

In other words, $G_\infty \equiv 1, G_d \equiv c$ with

$$g(t,\tau) = \frac{e^{-t/\tau}}{\tau}, \quad \tau = \text{"relaxation time"} \tag{4}$$

Note that

$$\hat{g} = \frac{1}{1+i\omega\tau} \tag{5}$$

Then, $\sigma(z,t)$ can be expressed as

$$\sigma(z,t) = H_A G_\infty\, \epsilon|_0^t + H_A P \tag{6}$$

where $P := G_d \mathbb{E}_F[g] * d\epsilon/dt$, giving

$$\hat{P} = G_d \mathbb{E}_F[\hat{g}] \cdot \frac{d\hat{\epsilon}}{dt} = \underbrace{\int_{\tau_1}^{\tau_2} \frac{G_d}{1+i\omega\tau} dF(\tau)}_{\text{“}\hat{G}_r(\omega)\text{”}} \cdot \frac{d\hat{\epsilon}}{dt} \tag{7}$$

“Can show that” $P = \mathbb{E}_F[\mathcal{P}]$ where $\mathcal{P}$ satisfies an (auxiliary) ODE:

$$\tau\dot{\mathcal{P}} + \mathcal{P} = G_d \frac{d\epsilon}{dt}, \text{ with } \tau \sim F(\tau) = U[m-r, m+r] = r\xi + m,\ \xi \sim U[-1,1]. \tag{8}$$

Which is similar in nature to the stochastic polarization (Banks and Gibson) defined as

$$\mathcal{P}(t,z) = \int_{\tau_a}^{\tau_b} P(t,z;\tau)dF(\tau) \tag{9}$$

Note the similarity to Equation 7. Furthermore, using theDebye model the stochastic ordinary differential equation (SODE) given by

$$\tau\frac{\delta \mathcal{P}}{\delta t} + \mathcal{P} = \tilde{\epsilon}_d E \tag{10}$$

Which is, forcing term aside, the same as Equation 8. We can thus use a similar process to that described by Bela and Hortsch to apply polynomial chaos to Equation 8. Applying this process to the ODE given in Equation 8 (using the same notation) we first rewrite $\mathcal{P}$ as an expansion of orthogonal polynomials

$$\mathcal{P}(\xi) = \sum_{i=0}^{\infty} \alpha_i(t)\phi_i(\xi) \tag{11}$$

Where $\phi_i(\xi)$ is an orthogonal basis with the property

$$\int \phi_i\phi_j dW = \delta_{ij} \tag{12}$$

where δ_{ij} represents the Kronecker delta function.
In order for this to be helpful, need solution for α. First, truncate the expansion to a (somewhat arbitrary, higher term means better approximation) value Q, giving

$$P_Q(\xi) = \sum_{i=0}^{Q} \alpha_i(t)\phi_i(\xi) \tag{13}$$

Then we make the assumption that the ODE is satisfied by this approximation, e.g.

$$\tau \dot{P}_Q + P_Q = \epsilon E \tag{14}$$

Which can thus be rewritten as

$$\tau\left(\sum_{i=0}^{Q} \dot{\alpha}_i(t)\phi_i(\xi)\right) + \left(\sum_{i=0}^{Q} \alpha_i(t)\phi_i(\xi)\right) = \epsilon E \tag{15}$$

Note that the only element in P that possesses a derivative with respect to time t is α_i. Furthermore, using the definition of τ found in Equation 7 as a uniform distribution with expected value $\mu = m$ and deviation $\sigma = r$, can rewrite our truncated ODE as

$$(r\xi + m)\left(\sum_{i=0}^{Q} \dot{\alpha}_i(t)\phi_i(\xi)\right) + \left(\sum_{i=0}^{Q} \alpha_i(t)\phi_i(\xi)\right) = \epsilon E \tag{16}$$

From here, find the Galerkin Projection in order to convert this continuous operator problem into a discrete problem by multiplying both sides of Equation 16 by ϕ_j and integrating

$$\int (r\xi + m)\left(\sum_{i=0}^{Q} \dot{\alpha}_i(t)\phi_i(\xi)\right)\phi_j W(\xi)d\xi + \int \left(\sum_{i=0}^{Q} \alpha_i(t)\phi_i(\xi)\right)\phi_j W(\xi)d\xi = \int \epsilon E\phi_j W(\xi)d\xi \tag{17}$$

Using the definition for the Kronecker delta function in Equation 12, this can be rewritten as

$$\int (r\xi + m)\left(\sum_{i=0}^{Q} \dot{\alpha}_i(t)\phi_i(\xi)\right)\phi_j W(\xi)d\xi + \sum_{i=0}^{Q} \alpha_i \delta ij = \int \epsilon E\delta_{0j} \tag{18}$$

Although this is more compact, it is still not completely simplified. In order to simplify further, we can use the fact that α is not dependent on ξ, giving

$$\sum_{i=0}^{Q} \dot{\alpha}_i(t) \int (r\xi + m)\left(\phi_i(\xi)\right)\phi_j W(\xi) d\xi + \sum_{i=0}^{Q} \alpha_i \delta ij = \int \epsilon E \delta_{0j} \tag{19}$$

Then multiplying out r and m to separate terms

$$r\sum_{i=0}^{Q} \dot{\alpha}_i(t) \int \xi\phi_i(\xi)\phi_j(\xi)W(\xi)d\xi + m\sum_{i=0}^{Q} \dot{\alpha}_i(t) \int \phi_i(\xi)\phi_j(\xi)W(\xi)d\xi + \sum_{i=0}^{Q} \alpha_i\delta_{ij} = \int \epsilon E\delta_{0j} \tag{20}$$

Giving a final form of

$$r\sum_{i=0}^{Q} \dot{\alpha}_i(t) \int \xi\phi_i(\xi)\phi_j(\xi)W(\xi)d\xi + m\sum_{i=0}^{Q} \dot{\alpha}_i(t)\delta_{ij} + \sum_{i=0}^{Q} \alpha_i\delta_{ij} = \epsilon E \tag{21}$$

With the integral dependent on the value of j, a system of equations is formed

j-value	Resulting Equation
0	$r(\gamma_0) + m\dot{\alpha}_0 + \alpha_0 = \epsilon E$
1	$r(\gamma_1) + m\dot{\alpha}_1 + \alpha_1 = 0$
$\vdots$	$\vdots$
j	$\epsilon E\delta_{0j}$
	$r\Gamma\dot{\vec{\alpha}} + mI\dot{\vec{\alpha}} + I\vec{\alpha} = \vec{f}$

Where γ_j is representative of the term

$$\sum_{i=0}^{Q} \dot{\alpha}_i(t) \int \xi\phi_i(\xi)\phi_j(\xi)W(\xi)d\xi \tag{22}$$

And Γ is the matrix composed of the individual γ_j values. In order to reach a more meaningful definition of Γ, we can use the fact that all orthogonal polynomials have a recurrence relationship of the form

$$\xi\phi_n(\xi) = a_n\phi_{n+1}(\xi) + b_n\phi_n(\xi) + c_n\phi_{n-1}(\xi) \tag{23}$$

Which allows reformulation of γ_j as

$$\sum_{i=0}^{Q} \dot{\alpha}_i \int (a_n\phi_{n+1}(\xi) + b_n\phi_n(\xi) + c_n\phi_{n-1}(\xi))dW \tag{24}$$

Integrating

$$\sum_{i=0}^{Q} \dot{\alpha}_i\left(a_i\delta_{i+1,j} + b_i\delta i, j + c_i\delta_{i-1,j}\right) \tag{25}$$

Finally giving

$$\Gamma = \begin{bmatrix} b_0 & c_1 & 0 & \cdots & 0 \\ a_0 & b_1 & c_2 & & \ddots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & a_{Q-2} & b_{Q-1} & c_Q \\ 0 & \cdots & 0 & a_{Q-1} & b_Q \end{bmatrix} \tag{26}$$

Where a_i, b_i, and c_i are the recursion coefficients, and $\vec{f}$ from earlier forces the system and has a deterministic value

$$\vec{f} = \begin{pmatrix} \tilde{\epsilon}E \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{27}$$