We begin by stating the nondimensionalized form of the PDE model

$$\frac{\partial \bar{v}}{\partial \bar{t}} = \left(G_\infty + \frac{G_d}{\tau}\right) + \frac{\partial^2 \bar{v}}{\partial \bar{z}^2} - \frac{t_c}{\tau}\frac{\partial^2 \bar{P}}{\partial \bar{z}^2} \tag{1}$$

With the coupled auxiliary ODE

$$\tau \frac{\partial \bar{P}}{\partial \bar{t}} = G_d \bar{v} - \frac{1}{\tau}\bar{P} \tag{2}$$

Using the following change of variables

$$u = h\bar{u},\ z = h\bar{z},\ \sigma = H_a\bar{\sigma},\ t = t_c\bar{t},\ P = \rho\bar{P}$$

And setting $t_c = h^2/(\kappa H_A)$ and $\rho = h$. We then define $\bar{P}$ as the expected value of $\mathcal{P}$, i.e.

$$\bar{P} = \mathbb{E}_F\left[\bar{P}(\bar{z},\bar{t})\right] := G_d \cdot \bar{g}(\bar{t};\tau) * \bar{v} \tag{3}$$

Where $\bar{\mathcal{P}}$ satisfies the auxiliary ODE

$$\tau\frac{\partial \bar{\mathcal{P}}}{\partial \bar{t}} + t_c\bar{\mathcal{P}} = G_d\bar{v},\ \text{with } \tau \sim \mathrm{F} \tag{4}$$

Applying generalized Polynomial Chaos, we can say $\tau = r\xi + m$, with $\xi \in [-1,1]$. Now, assuming non-dimensionalization (i.e. ceasing bar notation)

$$(rM + mI)\frac{\partial \vec{\alpha}}{\partial t} + t_c\vec{\alpha} = G_d v(z,t)\hat{e}_1,\ \text{with } \mathcal{P}(\mathrm{t},\mathrm{z},\xi) = \sum_{\mathrm{j}=0}^{\mathrm{p}} \mathrm{a_j}(\mathrm{z},\mathrm{t})\phi_{\mathrm{j}}(\mathrm{xi}) \tag{5}$$

Where $\mathbb{E}_F\left[\mathcal{P}\right] = \alpha_0(t,z)$. Introducing matrix notation of the form such that an $n \times m$ matrix A is denoted as $[A_{ij}]_{m\times n}$, we can write α in the form

$$\alpha_j(t_n, z_i) = [\alpha_{ij}]_{m\times p} = \begin{bmatrix} \alpha_0(z_0) & \alpha_1(z_0) & \cdots & \alpha_p(z_0) \\ \alpha_0(z_1) & \ddots & & \\ \vdots & & & \\ \alpha_0(z_m) & \cdots & & \alpha_p(z_m) \end{bmatrix} \tag{6}$$

Using our new notation method and matrix form, equation (5) can be written in the form

$$(r\,[M_{ij}]_{p\times p} + m\,[I_{ij}]_{p\times p})\frac{\partial}{\partial t}[\alpha_{ij}]^T_{m\times p} + t_c\,[\alpha_{ij}]^T_{m\times p} = G_d\left(\vec{v}_{1\times m+2}\cdot \hat{e}_1\right) \tag{7}$$

Discretizing with respect to time, this becomes

$$(r\,[M_{ij}]_{p\times p} + m\,[I_{ij}]_{p\times p})\frac{\left([\alpha_{ij}]^{n+1}_{m\times p}\right)^T - \left([\alpha_{ij}]^{n}_{m\times p}\right)^T}{\Delta t} + t_c\left([\alpha_{ij}]^{n+\theta}_{m\times p}\right)^T = G_d\hat{e}_1\left(\vec{v}^{\,n+\theta}_{1\times m+2}\right)^T \tag{8}$$