A polynomial chaos-based method for the continuous spectrum biphasic poroviscoelastic model of articular cartilage

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Abstract

We present a numerical method for solving the linear biphasic poroviscoelastic model of articular cartilage, specifically by replacing the convolution representing history effects with an auxiliary ODE to which generalized Polynomial Chaos (gPC) is applied to account for a continuous spectrum of relaxation times. The resulting system of ODEs is solved using a finite difference method of the same order of accuracy as the method for the PDE. The results of this solution are then compared to existing literature to determine overall accuracy and the previous, Gauss-Legendre solution, is recovered as a special case of gPC.

This is a URISC funded undergraduate research project with an emphasis on reformulation of the model and implementation of gPC, and can be considered as an extension of the work done in [1], [2] as well as an application of new methods in electromagnetics. Note that this project is still in progress.

Application

The ultimate purpose of this project is to improve the simplicity, efficiency, and utility of the BPVE model of articular cartilage. Upon doing so, this will make further studies into the behaviors of cartilage better informed, and add value to not only mathematical literature but medical as well, furthering joint research.

Primary Problem

We desire a solution u, to the linear biphasic viscoporoelastic (BPVE) model of articular cartilage

$$\frac{1}{\kappa}\frac{\partial u}{\partial t} = \frac{\partial \sigma}{\partial z},\ 0 < z < h,\ 0 < t < t_f.$$

Where the stress, σ, is defined to be

$$\sigma(z,t) = H_A \int_{-\infty}^{t} G(t-s)\frac{\partial \epsilon}{\partial s} ds.$$

The quantity $\epsilon(z,t) = \frac{\partial u}{\partial z}$ is the strain. Substituting the definition of $\epsilon(z,t)$ into the original model and rewriting in terms of the velocity $v = \frac{\partial u}{\partial t}$, then

$$v = \kappa H_A \int_{-\infty}^{t} G(t-s)\frac{\partial^2 v}{\partial z^2}(z,s) ds.$$

where typically $G(t) = G_\infty + G_d\, g(t;\tau)$ and

$$g(t;\tau) = \frac{e^{-t/\tau}}{\tau}.$$

This method was implemented by [1, 2], however we wish to employ an auxiliary ordinary differential equation (ODE) approach. To that end we define $Q(z,t) := G_d\, g(t;\tau) * v_{zz}$. Taking a time derivative on both sides gives

$$\frac{\partial v}{\partial t} = \kappa H_A G_\infty \frac{\partial^2 v}{\partial z^2} + \kappa H_A \dot{Q}.$$

We further introduce a new variable P such that $\frac{\partial^2 P}{\partial z^2} = Q$ and

$$P(z,t) := G_d\, g(t;\tau) * v(z,t).$$

Note that P satisfies the auxiliary ODE

$$\tau \dot{P} + P = G_d v.$$

After substituting $\dot{Q} = \frac{\partial^2 \dot{P}}{\partial z^2}$, the original PDE becomes

$$\frac{\partial v}{\partial t} = \kappa H_A \left(G_\infty + \frac{G_d}{\tau}\right)\frac{\partial v}{\partial z} - \frac{\kappa H_A}{\tau}\frac{\partial^2 P}{\partial z^2}$$

Development of New Method

The above assumes a single relaxation time parameter τ. In the presence of a continuous spectrum of relaxation times, one may redefine the kernel in the convolution integral to be

$$G(t) = G_\infty + G_d \int_{\tau_1}^{\tau_2} \frac{e^{-t/\tau}}{\tau} dF(\tau)$$

where $F(\tau)$ is the probability density of relaxation times on the interval $[\tau_1, \tau_2]$. In previous efforts, this kernel was approximated via Gaussian quadrature in order to produce $G^Q(t)$ with arbitrary order of accuracy. Here we propose to use the approach defined in [3], where we explicitly model the relaxation as a random variable. Then polynomial chaos is applied to the resulting random solution.

Polynomial Chaos

We use polynomial chaos expansions for the function $\mathcal{P}$, defined analogously to P as $\mathcal{P}(z,t) := G_d g(t;\tau) * v$ and satisfying the ODE

$$\tau \dot{\mathcal{P}} + \mathcal{P} = G_d v \text{ with } \tau \sim \mathrm{F}. \tag{1}$$

Applying gPC, we can rewrite as $\tau = r\xi + m$, where $\xi \in [-1, 1]$. and approximate the solution

$$\mathcal{P}(t,z,\xi) \approx \sum_{j=0}^{p} \alpha_j(t,z)\varphi_j(\xi) \tag{2}$$

where the expansion coefficients satisfy the following tridiagonal system of ODEs

$$A\dot{\vec{\alpha}} + \vec{\alpha} = \vec{f}. \tag{3}$$

The PDE depends on the quantity $\mathbb{E}_F[\mathcal{P}] = \alpha_0(t,z)$. The φ_j's above are (for example) Jacobi polynomials in ξ, which are orthogonal on $[-1,1]$ with respect to a weight function proportional to the distribution F, e.g.,

$$W(\xi) = (1-\xi)^a(1+\xi)^b.$$

System of Equations

Equation 3 represents a system of equations expressed in matrix form, where

$$A := (rM + mI),$$

$$\vec{f} := \begin{bmatrix} G_d v(t,z)\hat{e}_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$M := \begin{bmatrix} b_0 & a_1 & 0 & \dots & 0 \\ c_0 & b_1 & \ddots & & \vdots \\ 0 & c_1 & \ddots & a_{Q-1} & 0 \\ \vdots & & \ddots & b_{p-1} & a_p \\ 0 & \dots & 0 & c_{p-1} & b_p \end{bmatrix}.$$

and a_i, b_i, c_i are the recursion coefficients. Note that the deterministic value $\vec{f}$ forces the system and is dependent on $v(t,z)$, which is itself dependent on the expected value $P = \mathbb{E}[\mathcal{P}]$, which can be approximated by α_0.

Alternative to Numerical Integration

- Direct approach to approximating solutions to the ODEs for $\mathcal{P}$ is to compute inner products, $\langle \varphi_i, \varphi_j \rangle_W$, with numerical integration (quadrature)
- Numerical integration needed at each time t and can be expensive
- Instead, we use an efficient method for approximating α_j that avoids computing the integrals directly in favor of a finite difference approach.

Iterative method for determining α_j

Using matrix notation:

$$\alpha_j(t_n, z_i) = [\bar{\bar{\alpha}}_{ij}^n] = \begin{bmatrix} \alpha_0(z_0) & \alpha_1^n(z_0) & \cdots & \alpha_p^n(z_0) \\ \alpha_0(z_1) & \cdots & & \\ \vdots & & & \\ \alpha_0(z_M) & & & \end{bmatrix} = \bar{\bar{\alpha}}^n$$

Using finite difference approximation, 3 can be written as

$$(rM + mI)\frac{(\bar{\bar{\alpha}}^{n+1})^T - (\bar{\bar{\alpha}}^n)^T}{\Delta t} + (\bar{\bar{\alpha}}^{n+\theta})^T = G_d \hat{e}_1 (\vec{v}^{n+\theta})^T \tag{4}$$

Let

$$B_+ := rM + mI + \Delta t \theta I$$
$$B_- := rM + mI - \Delta t(1-\theta) I$$

Then this simply becomes

$$B_+(\bar{\bar{\alpha}}^{n+1})^T = B_-(\bar{\bar{\alpha}}^n)^T + G_d \hat{e}_1 (\vec{v}^{n+\theta})^T \Delta t \tag{5}$$

Solving for $\bar{\bar{\alpha}}^{n+1}$ this becomes

$$\bar{\bar{\alpha}}^{n+1} = \bar{\bar{\alpha}}^n B_-^T B_+^T + G_d \vec{v}^{n+\theta} \hat{e}_1^T B_+^{-T} \Delta t \tag{6}$$

However, we are ultimately interested in P, not α, thus we extract the first column from the α matrix.

Current, Future Work

- Finish MATLAB implementation of BPVE model, gPC application numerical method.
- Compare results of numerical method to those utilized in [1], [2].
- Determine error bounds in order to find the truncation level of the orthogonal polynomials used in gPC in order to reach and surpass the accuracy of existing methods.
- Beginning with the uniform distribution, apply various probability densities in order to determine the best fit for the relaxation time and numerical practicality.

References

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[2] [2004] M. A. Haider and R. C. Schugart, A numerical method for the continuous spectrum biphasic poroviscoelastic model of articular cartilage, *Journal of Biomechanics*, Vol. 39, No. 1, pp. 177-183.

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