Data Analysis in R

Hypothesis Testing

Ken Stiller

14th December 2022

Syllabus: Data Analysis in R

1. Introduction
2. Causality & Basics of Statistics
3. Sampling & Measurement
4. Prediction
5. Multivariate Regression
6. Probability & Uncertainty
7. **Hypothesis Testing**
8. Assumptions & Limits of OLS
9. Interactions & Non-Linear Effects

Plan for Today

- Accuracy
  - Standard Errors
  - Confidence Intervals
- Hypothesis Testing
  - What is a hypothesis?
  - p-values
  - Type I and Type II errors
  - One and two-sided tests

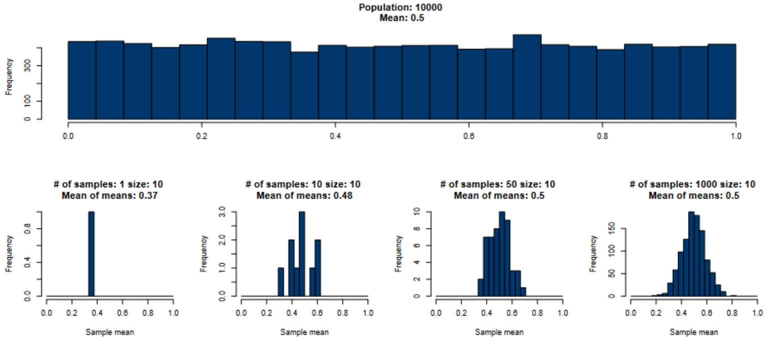
Table of Contents

Recap: Statistical Inference

- What we want to know: parameter $\theta \rightsquigarrow$ unobservable
- What you do observe: data
- We use data to compute an estimate of the parameter ($\hat{\theta}$)

- **But how good is $\hat{\theta}$ an estimate of θ?**
- Ideally, we want to know the estimation error $= \hat{\theta} - \theta$
- The problem remains unchanged: θ is unknown

Recap: Central Limit Theorem



Standard Error

- Standard Error $(\overline{X}) = \frac{s}{\sqrt{n}}$
- The standard error is an estimate of how far any sample mean 'typically' deviates from the population mean

Confidence Intervals

- We know the standard error and are aware of the Central Limit Theorem
- Thus, we can calculate how 'likely' it is that a specific range around sample mean contains the population mean
- This is called a confidence interval

Confidence Intervals II

- An m-percent confidence interval establishes a boundary around the sample mean in which the true mean will lie m out of 100 times under repeated sampling
- Common values for m are 95 and 99 (sometimes 90)
- m is specified by choosing a significance level $\alpha : m = (1 - \alpha) * 100$
- Common significance levels are therefore 0.05 and 0.01 (sometimes 0.1)

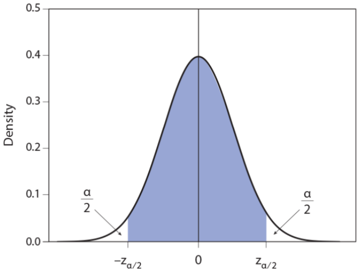
Confidence Intervals: Defining Boundaries

- In order to provide an interval estimate of the population mean μ we need to identify a lower bound (LB) and an upper bound (UB) such that $\mathrm{P}(LB \leq \mu \leq UB)$
- Recall last week on probability: We can use our knowledge of the normal distribution to find this boundary
- After z-transformation of any normal distribution

$$z = \frac{x_i - \overline{x}}{s_x}$$

  - Probability between -1 and 1 is 0.68
  - Probability between -1.96 and 1.96 is 0.95
  - Probability between -3 and 3 is 0.997
- $z_{\alpha/2}$ is the value associated with $(1 - \alpha) * 100\%$ coverage in the standard normal distribution

Example: Critical Values of Normal Distribution



- The lower and upper critical values, $-z_{\alpha/2}$ and $z_{\alpha/2}$, shown on horizontal axis
- Area under the density curve between these critical values (in blue) equals $1 - \alpha$

Confidence Intervals: Overview

- CI: boundaries in which μ will lie m-times out of a 100
- $(1-\alpha) * 100\%$ confidence intervals:

$$CI_\alpha = [\overline{X} - z_{\alpha/2} * SE, \overline{X} + z_{\alpha/2} * SE]$$

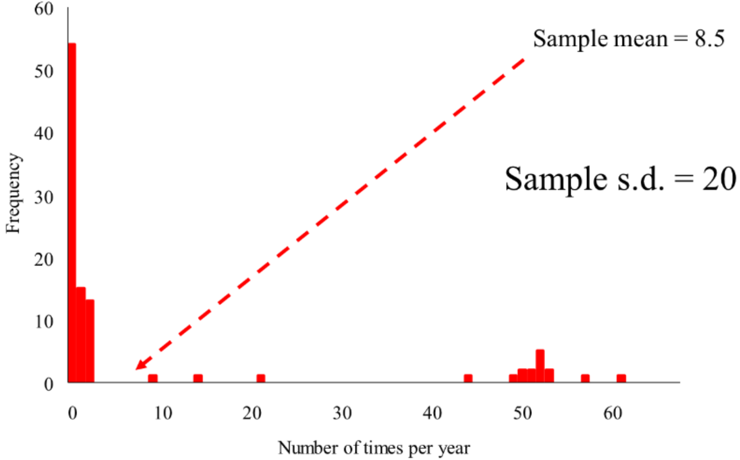
  where $z_{\alpha/2}$ is the critical value and α reflects our chosen significance level

- $P(Z > z_{\alpha/2}) = \alpha/2$ and $Z \sim \mathcal{N}(0,1)$
  1. $\alpha = 0.01$ gives $z_{\alpha/2} = 2.58$
  2. $\alpha = 0.05$ gives $z_{\alpha/2} = 1.96$
  3. $\alpha = 0.10$ gives $z_{\alpha/2} = 1.64$

Example: Confidence Intervals & Standard Error for Sample Mean

- How often do German people attend some form of religious worship?
- Take a random sample of 100 people from the German population and record how many times they attended a form of religious worship last year
  - Distribution is extremely skewed
  - Some went a lot, most went infrequently or not at all
- From that sample we get a sample mean and a sample standard deviation

Example: Confidence Intervals & Standard Error for Sample Mean II



Example: Confidence Intervals & Standard Error for Sample Mean III

- From the *sd* and *mean* of the sample we can calculate the standard error for the sample mean:

$$\frac{s}{\sqrt{n}} = \frac{20}{\sqrt{100}}$$

  where $s =$ sample standard deviation
- From this we can calculate any confidence interval

$$\text{CI}_\alpha = \left[\bar{X} - z_{\alpha/2} \times \text{ standard error}, \bar{X} + z_{\alpha/2} \times \text{ standard error } \right]$$

- Usually, we are interested in a 95% CI:

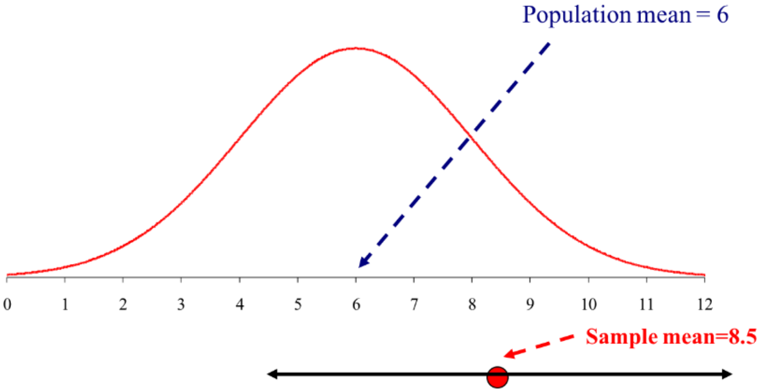
$$[8.5 - 1.96 \times 2, 8.5 + 1.96 \times 2] = [4.58, 12.42]$$

- In 95 out of a 100 times will the true mean lie between 4.58 and 12.42

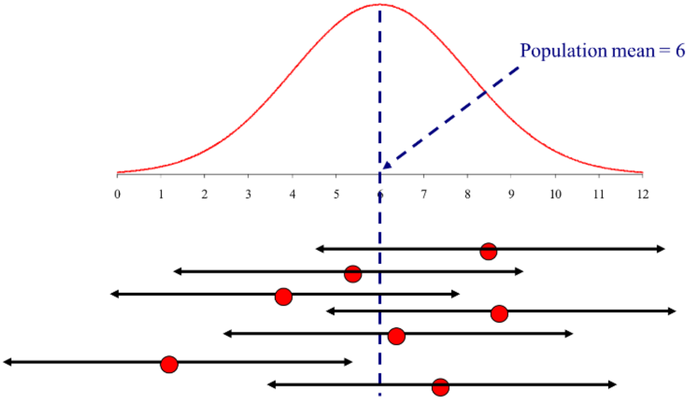
Example: Confidence Intervals & Standard Error for Sample Mean IV

- Suppose we know that the population mean for religious worship attendance in Germany is actually 6 times per year
  - Our particular sample is off by 2.5
  - The mean of all the possible sample means is equal to the population mean so the centre of the sampling distribution is 6
- In 95 out of a 100 times will the true mean lie between 4.58 and 12.42

Example: Confidence Intervals & Standard Error for Sample Mean V



Example: Confidence Intervals & Standard Error for Sample Mean VI



Example: Confidence Intervals & Standard Error for Sample Mean VII

- Of the 7 samples, all the confidence intervals around the sample mean enclosed the actual true population mean apart from one
- If we repeated this lots of times, we would expect 95% of the confidence intervals to enclose the actual population mean
  - 95% because that's the level we set
  - If we had set 99%, the confidence intervals would be larger

Statistical Hypothesis Testing: Overview

1. Construct a null hypothesis (H_0) and its alternative (H_1)
2. Pick a test statistic T
3. Figure out the sampling distribution of T under H_0 (reference distribution)
   - For hypothesis tests regarding the mean, if sample size large, use the normal distribution
   - For other test statistics, you need to use other distributions
4. Is the observed value of T likely to occur under H_0?
   - **Yes** - Retain H_0
   - **No** - Reject H_0

What is a Hypothesis?

- Hypotheses = testable statements about the world
- Hypotheses = falsifiable
  - We test hypotheses by attempting to see if they could be false, rather than 'proving' them to be true
- Hypotheses come from:
  - Theory
  - Past empirical work
  - Common sense (?)
  - Anecdotal observations

Null and Alternative Hypotheses

- We need to choose between two conflicting statements:

  1. The null hypothesis (H_0) is directly tested
     - This is a statement that the parameter we are interested in has a value similar to no effect (i.e., usually 0 for coefficients)
     - e.g. regarding ideology, old people are the same as young people

  2. Alternative (H_1) contradicts the null hypothesis
     - This is a statement that the parameter falls into a different set of values than those predicted by (H_0)
     - e.g. regarding ideology, old people are more right-wing than young people

- **Note that we actually 'test' the null hypothesis!**

Hypothesis Testing: Test Statistic

- In any statistical hypothesis test, a test statistic is computed from the data in order to test the null hypothesis.

$$T = \frac{\text{sample estimate} - \text{parameter value } \textit{under } H_0}{\text{standard error}}$$

- The larger T, the more the data contradict the null hypothesis
- For a given estimate, T becomes larger as the standard error decrease

Statistical Hypothesis Testing: Overview II

- Hypotheses - H_0: $\mu = \mu_0$ and $H_1 : \mu \neq \mu_0$
- Test statistic:

$$\text{z-score} \;=\; \frac{\bar{X} - \mu_0}{\text{standard error}} = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$

- Under the null, by the central limit theorem

$$\text{z-score} \;\overset{\text{approx.}}{\sim}\; \mathcal{N}(0, 1)$$

- Is Z_{obs} unusual under the null?
  - Reject the null when $|Z_{obs}| > z_{\alpha/2}$
  - Retain the null when $|Z_{obs}| \leq z_{\alpha/2}$

Example: Exam Scores

- Suppose there's a standardised exam with marks ranging from 0-100
- Suppose further we know test scores are normally distributed with mean $\mu = 88$ and standard deviation $\sigma = 5$
- Now, in five tests cohorts receive test scores of $\overline{X} = 95$
  $H_0 : \mu = 88$ $H_1 : \mu \neq 88$

Example: Exam Scores II

- We know that the standard deviation of test in the population is 5. The sample size is 5 so we calculate the standard error as:

$$SE = \frac{5}{\sqrt{5}}$$

- Assuming H_0 was true, we know that the sampling distribution is

$$\mathcal{N}(88, (\frac{5}{\sqrt{5}})^2)$$

- Based on sampling distribution, how many standard deviations away is the observed mean from the hypothesized mean?

$$z = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{95 - 88}{\frac{5}{\sqrt{5}}} = 5.82$$

- What is the probability of observing a z-value of more than 5.82 or below -5.82?

Example: Exam Scores III

The probability is smaller than 0.00001 What do we make of this?

Example: Exam Scores III

The probability is smaller than 0.00001 What do we make of this?

p-Value

- Ok, so what's a p-value then?
- A p-value indicates the probability, under H_0, of observing a value of the test statistic at least as extreme as its observed value
- A smaller p-value presents stronger evidence against H_0
- The level of the test: Pr(rejection—H_0) $= \alpha$
- A p-value less than α conventionally indicates statistical significance
  - Conventional values of α: 0.05 & 0.01

p-Value II

- A p-value is an *arbitrary* that our test must meet
  - we might want to be 99% confident that we are correctly rejecting the null hypothesis
  - or we might make the judgement that p-values of e.g. 0.05 and below are **probably** good evidence the null hypothesis can be rejected

- Keep in mind: the p-value is **not** the probability that H_0 (H_1) is true (false)

Type I and Type II Errors

- Concern false rejection if the null is true (*type I error*)
- Two types of errors:

	Reject H_0	Retain H_0
H_0 is true	Type I error	Correct
H_0 is false	Correct	Type II error

- Type I error occurs when we reject H_0 even though it is true
  - Happens 5% of the time if we choose $\alpha = 0.05$
- Type II error occurs when we do not reject H_0 even though it is false
  - If $\alpha = 0.05$, sometimes a real difference won't be detected

Type I and Type II Errors

- There's a trade-off between the two types of error
  - What probability do you want to minimize? False positive or false negative?

One- or Two-Sided (Tailed) Tests

- In the example above, we were interested in the difference to the true value
  - one-sided alternative hypothesis: $H_1 : \mu > \mu_0$ or $\mu < \mu_0$
  - one-sided p-value= $\Pr(Z > Z_{obs})$ or $\Pr(Z < Z_{obs})$
- Convention is to use two-tailed tests
  - making it even more difficult to find results just due to chance
  - normally don't have very strong prior information about the difference

Differences Between 2 Samples

- The example above was a one-sample test
- There are also two-sample tests
  - often, we wish to compare two samples
  - e.g . examine H_0 that means of two populations are equal

- Consider the following example
  - Suppose we're interested in whether religiosity differs between men and women
  - We have 2 samples, 45 men and 55 women
  - Men: mean attendance of 6.5 days a year, standard deviation of 15
  - Women: mean attendance of 11 days a year, standard deviation of 15

Differences Between 2 Samples II

- Null hypothesis = no difference between male and female mean attendance of religious worship
- This time, the difference between sample means is our statistic
- Significance test on this statistic to discover whether samples likely to represent real differences between the populations of men and women
- Work out z-score as before:

$$z = \frac{\text{Estimate of parameter} \ - \ \text{null hypothesis value}}{\text{Standard error of estimate}},$$

$$z = \frac{\left(\bar{X}_{\text{women}} - \bar{X}_{\text{men}}\right) - 0}{\text{SE}\left(\bar{X}_{\text{women}} - \bar{X}_{\text{men}}\right)} = \frac{\bar{X}_{\text{women}} - \bar{X}_{\text{men}}}{\sqrt{\frac{s^2_{\text{women}}}{n_{\text{women}}} + \frac{s^2_{\text{men}}}{n_{\text{men}}}}},$$

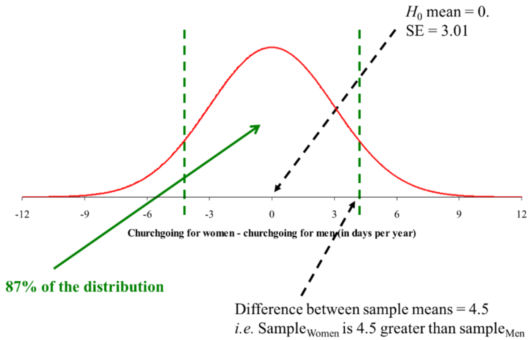
$$z = \frac{11 - 6.5}{\sqrt{\frac{15^2}{55} + \frac{15^2}{45}}} = \frac{4.5}{\sqrt{4.09 + 5}}$$

$$z = \mathbf{1.5}$$

Differences Between 2 Samples III

- Standard error of estimator = 3.01
- z-score = 1.5
- No priors → we test the possibility that the statistics for men and women are the same
    - i.e. how likely are we to get an individual estimate of the difference between the sample means that is either 1.5 SE greater than the null hypothesis (i.e. zero) or 1.5 SE less than the null hypothesis?

Two-Tailed Test



- The p-value for a 2-sided test is 0.134
- This value is higher than our 5% cut off value, so we reject H_1 that men and women differ in their church attendance

Significance Tests and CIs

- Note that our significance test looks similar to the CIs
- We could use a CI around the difference between the two sample means to 'test' the hypothesis that they are the same
- A 95% CI would just be $1.96 * SE$
  - We've just worked out that the $SE \approx 3$

Significance Tests and CIs II

- 95% confidence interval:

$$(\mu_{\text{women}} - \mu_{\text{men}}) = \left(\bar{X}_{\text{women}} - \bar{X}_{\text{men}}\right) \pm 1.96 * SE$$
$$(\mu_{\text{women}} - \mu_{\text{men}}) = 4.5 \pm 1.96 * 3$$
$$(\mu_{\text{women}} - \mu_{\text{men}}) = 4.5 \pm 5.88$$

- Note that the 95% CI encloses zero (which was our null hypothesis, that women are the same as men)

- CIs and significance tests are doing the same job, just presenting the information in a slightly different way

Binary Variables and Proportions

- We have been working with continuous variables and means
- This works for binary variables too, where the mean is just the proportion:
  - Population mean = μ = population proportion = π
  - Population standard deviation = $\sigma = \sqrt{\pi(1-\pi)}$
  - Sample proportion = P
  - Standard deviation $(P) = \frac{\sqrt{\pi(1-\pi)}}{\sqrt{n}}$
  - Standard error $(P) = \frac{\sqrt{P(1-P)}}{\sqrt{n}} = \sqrt{\frac{P(1-P)}{n}}$

Example: Is Boris Johnson the best PM?

- In a survey of whether people think Johnson is the best PM with a sample size of 1675 people, 29% think he's the best PM (i.e. mean = .29)
- From this we can work out the standard error:
  - Sample proportion $= P = .29$
  - Standard error $(P) = \sqrt{\frac{P(1-P)}{n}} = \sqrt{\frac{.29*(.71)}{1675}} = .011087$
- You can then calculate CIs and test hypotheses based on this

Take Aways

- Knowing the shape of the sampling distribution, we can work out:
  - ranges around a sample mean that will enclose the population mean $X\%$ of the time
  - the probability that a hypothesis about the population mean is true, given a particular sample mean
  - the probability that population means for different groups are different, given two sample means
  - all of the above for proportions
- Note that this allows us to make a **probabilistic** statement. Not more, not less.
- In expectation a (non-negligible) share will be false positives!

The way ahead

- Uncertainty:
  - More on uncertainty n relation to the linear regression model
- Regression Diagnostics:
  - What if OLS assumptions are violated?
  - When do we care?
  - What can we do?
  - What do we actually do?