Isomorphism of memBers

Shigeo Hattori
bayship.org@gmail.com

October 24, 2019
First version: September, 2019

1 Introduction

Definition 1.1 (memBer)**.** Take $\forall x$ such that (there exists $\exists S$ such that $x \in S$). Then x is said a memBer.

This article:(1) defines a binary relation that (x,y) as memBers are isomorphic,(2) proves that the binary relation is an equvalence relation, (3) proves that all homeomorphic topological spaces are isomorphic as memBers,(4) defines that a memBer $S1$ is a minor of a memBer $S2$.
I expect that readers will realize that the newly defined isomorphisms are somehow more fundamental than ,e.g., homeomorphisms of topological spaces. Because "homeomorphisms" logically resolve to "isomorphisms of memBers" whereas the inverse of it does not hold.

2 Notation

Definition 2.1. Consider "A and B". It is almost equivalent to "$B \wedge A$". But some times they are different. Because the meaning of B may depend on A.

"$A \ {}_{and}\!\wedge\ B$" $\equiv$ "A and B where the meaning of B may depend on A".
"$A \ {}_{or}\!\vee\ B$" $\equiv$ "A or B where the meaning of B may depend on $\neg A$".
"$\forall x :\in S$" $\equiv$ "for all x such that $x \in S$".
"$\forall x$ as an integer" $\equiv$ "for all x such that x is an integer".

3 Deep member

Definition 3.1 (Deep member of memBer)**.** This definition uses a style of recursion.
Take $\forall(x, y)$ such that *1 holds. Then define *2 $_{and}\wedge$ *3.

1 If $x \neq y$ then (there exists $\exists z$ such that $x \in z \in^{\geq 0} y$).

2 $x \in^{\geq 0} y$.

3 x is a deep member of y.

Definition 3.2 (Space of memBer)**.** Take $\forall(x, y)$ such that *1 holds. Then define *2.

1 $y = \{d \mid d \in^{\geq 0} x \ \ _{and}\wedge\ d$ is a point $\}$.

2 y is the space of x.

4 Notations

Definition 4.1 (Restriction of binary relation)**.** Take $\forall(L, X, Y, X1, Y1)$ such that *1 holds. Then define (*2 $_{and}\wedge$ *3 $_{and}\wedge$ *4).

1 L is a binary relation on $X * Y \ \ _{and}\wedge\ X1 \subset X \ \ _{and}\wedge\ Y1 \subset Y$.

2 $L[X1]$:={ $(x, y) \in L \mid x \in X1$ }.

3 $L[, Y1]$:={ $(x, y) \in L \mid y \in Y1$ }.

4 $L[X1, Y1]$:=$L[X1,] \cap L[, Y1]$.

5 Isomorphic memBers

Definition 5.1 (Isomorphic memBers)**.** Take all $\forall x$. Then (x, x) are said isomorphic.

Definition 5.2 (Isomorphic memBers by binary relation)**.** This definition uses a style of recursion.

Take $\forall(x, y, F)$ such that *A holds. Then define (*B1 $_{and}\wedge$ *B2).

A (F is a binary relation $_{and}\wedge$ *0) holds.

- **0** If there exists $\exists v :\in \{x, y\}$ such that $space(v) = \varnothing$ Then $x = y$ Else *1.
- **1** If there exists $\exists v :\in \{x, y\}$ such that v is a point Then ((x, y) are points $_{and}\wedge$ $(x, y) \in F$) Else (*2 $_{and}\wedge$ *3).
- **2** $F[space(x), space(y)]$ is a [1]bijection from*to space(x)*space(y).
- **3** There exists $\exists f$ such that (*4 $_{and}\wedge$ *5 $_{and}\wedge$ *6).
- **4** f is a bijection from*to $x * y$.
- **5** Take $\forall(m1, m2) \in f$.
- **6** *A holds for $(m1, m2, F)$ in place of (x, y, F))).

B1 (x, y) are said isomorphic by F as an isomorphism.

B2 Take $\forall(x, y, F)$ such that (x, y) are isomorphic by F. Then (x, y) are said isomorphic. ■

[1]To weaken the definition, replace "bijection" with "function" or with "binary relation".

6 Minors of memBers

Definition 6.1 (Minors)**.** Take $\forall(x, y)$ such that *A holds. Then define *B.

A *1 $_{and}\wedge$ *2.

- **1** Take $\forall d$. Then $d \in^{\geq 0} x \Rightarrow$ d$\in^{\geq 0} y$.
- **2** Take $\forall(d1, d2, d3)$ such that (take $\forall d \in \{d1, d2, d3\}$, then $d \in^{\geq 0} x$). Then (*3 $\Leftarrow$ *4).
- **3** (($x, d1, d3$),($x, d2, d3$)) are isomorphic.
- **4** (($y, d1, d3$),($y, d2, d3$)) are isomorphic.

B x is a minor of y.

7 Notations

Definition 7.1 (Set indexed by itself)**.**
Take $\forall x$ as a set. Then define (*1 $_{and}\wedge$ ..... $_{and}\wedge$ *6).

1 By default, members of x are indexed by the identity function on x, as $\{x_i\}_{i \in x}$.
For example, x is a set and $x_i \in x$.

2 x_{gen} denotes the general member of x to imply general conditions on all member of x.

3 Take the set of all minimum members of x.
If it's not empty Then x_{min} denotes the general member of them.

4 Take the set of all maximum members of x.
If it's not empty Then x_{max} denotes the general member of them.

5 x_{i+1} denotes the immediate follower of x_i if it uniquely exists.

6 x_{i-1} denotes the member immediately followed by x_i if it uniquely exists.

Definition 7.2 (Another form of set definition)**.**
Consider,for example, $S := \{x \in N \mid 0 < x < 3\}$.
That is, $S = \{1, 2\}$.
And consider,for example, $S1 := \{... \mid x \in N$ $_{and}\wedge$ $0 < x < 3$ $_{and}\wedge$ $y = x+1\}$.
That is, $S1 = \{\{x = 1, y = 2\}, \{x = 2, y = 3\}\}$.
Generally, given $S := \{... \mid L$ as a logical expression),
you find all free variables in L,
then S is the set of all possible substitutions into the free variables of L.

8 Depth of memBer

Definition 8.1 (Powers of set membership)**.**
Take $\forall(c, x, y)$ such that *1 $_{and}\wedge$ *2. Then define *3 $_{and}\wedge$ *4.

1 c is a chain of set [2]membership.

2 $|c| \geq 1 \ _{and}\wedge \ c_{min} = x \ _{and}\wedge \ c_{max} = y$.

3 $power(x, y) :\ni |c| - 1$.

4 $x \in^{|c|-1} y$.

For example: let y:={1,{1}}.
Then $1 \in^1 y \ _{and}\wedge \ 1 \in^2 y$.

Definition 8.2 (Depth of deep membership)**.**
Take $\forall(x, y)$ such that *1.
Then define (*2 $_{and}\wedge$ *3).

1 $S := \{... \mid power(x, y) \ni p\} \ _{and}\wedge \ S \neq \varnothing$.

2 Let S be ordered by p.

3 $depth(x, y) := p$ for S_{max}.

For example: let y:={1,{1}}.
Then $depth(1, y) = 2$.

Definition 8.3 (Sum of depths of deep membership)**.**
Take $\forall c$ such that *1. Then define *2 $_{and}\wedge$ *3.

1 c is a chain of deep [3]membership.

2 Let $S1 := \{... \mid$
$c_i < c_j \ _{and}\wedge$
take $\forall h$ such that $(c_i \leq c_h \leq c_j)$, then $h \in \{i, j\}$
$\}$

3 $depth(c) := \sum depth(c_i, c_j)$ over $S1$.

[2]For example, $c_1 \in c_2$.
[3]For example, $c_1 \in^n c_2$

Definition 8.4 (Depth of memBber)**.**
Take $\forall x$. Then define $*1 \ {}_{and}\wedge \ *2 \ {}_{and}\wedge \ *3$.

1 Let $S := \{... \mid depth(y,x) = d\}$.

2 Let S be ordered by d.

3 $depth(x) := d$ for S_{max}.

For example: let x:={1,{1}}.
Then $depth(x) = 2$.

Proposition 1 (Depths of chains)**.**
Take $\forall(x,y,z)$ such that $x \in^{\geq 0} y \in^{\geq 0} z$. Then $depth(x,z) \geq depth(x,y,z)$.

Proof.

- Assume it is false.
- There exists $\exists(x,y,z)$ such that it is a counterexample.
- Hence $depth(x,z) < depth(x,y,z) = depth(x,y) + depth(y,z)$.
- Meanwhile $depth(x,z)$ equals to (length-1) of the longest membership chain from,to (x,z).
- And $depth(x,y,z)$ equals to (sum of (length-1)) of the longest membership chains from,to (x,y) and from,to (y,z).
- Hence $depth(x,y,z)$ equals to (length-1) of the longest membership chain from,via,to (x,y,z).
- Hence $depth(x,z) \geq depth(x,y,z)$.
- The assumption is false.

□

Proposition 2 (Relative depths of member and set)**.**
Take $\forall(x,y,z)$ such that $x \in y \in^{\geq 0} z$. Then $depth(x,z) > depth(x,y)$.

Proof.

- Assume it is false.
- There exists $\exists(x, y, z)$ such that it is a counterexample.
- Hence $\mathit{depth}(x, z) \leq \mathit{depth}(x, y)$.
- Hence $\mathit{depth}(x, z) \geq \mathit{depth}(x, y, z) > \mathit{depth}(x, y) \geq \mathit{depth}(x, z)$.
- The assumption is false.

$\square$

Proposition 3 (Depths of member and set)**.**
Take $\forall(x, y)$ such that $x \in y$. Then $\mathit{depth}(x) < \mathit{depth}(y)$.

Proof.

- Assume it is false.
- There exists $\exists(x, y)$ such that it is a counterexample.
- Hence $\mathit{depth}(x) \geq \mathit{depth}(y)$.
- There exists $\exists v :\in^{\geq 0} x$ such that ($\mathit{depth}(x) = \mathit{depth}(v, x) \geq \mathit{depth}(v, x, y) \leq \mathit{depth}(y)$).
- Though $\mathit{depth}(v, x) + 1 = \mathit{depth}(v, x) + \mathit{depth}(x, y) = \mathit{depth}(v, x, y)$.
- The assumption is false.

$\square$

9 Isomorphic memBers as equivalence relation

Definition 9.1. In this section, *Def refers to the definition titled as "Isomorphic memBers by binary relation".
And *1 $\equiv$ *2, without any explicit proof because it is trivial by *Def.
And *3 holds, without any explicit proof because it is trivial by *Def.

1 (x_i, y_i) are isomorphic by F_i.

2 *Def.A holds for (x_i, y_i, F_i) in place of (x, y, F).

3 Take $\forall(x, y, F)$ such that (*4 ${}_{and}\wedge$ (*5 ${}_{or}\vee$ *6)). Then *7 holds.

4 F is a binary relation.

5 ($space(x) = \varnothing$ ${}_{and}\wedge$ $x = y$).

6 ((x, y) are points ${}_{and}\wedge$ $(x, y) \in F$).

7 (x, y) are isomorphic by F.

Proposition 4 (Restriction)**.** Take $\forall(x, y, F1, F2)$ such that (*A1 ${}_{and}\wedge$ *A2 ${}_{and}\wedge$ *A3) holds. Then *B holds.

A1 $(F1, F2)$ are binary relations.

A2 $F1[space(x)] = F2[space(x)]$.

A3 *Def.A holds for $(x, y, F1)$.

B *Def.A holds for $(x, y, F2)$.

Proof.

- Assume it is false.
- There exists $\exists(x, y, F1, F2)$ such that it is a minimum counterexample by $depth(x)$.
- Let us follow *Def.A for $(x, y, F1)$.
- Assume the antecedent of *0 holds.
- Hence $space(x) = \varnothing$ ${}_{and}\wedge$ $x = y$.
- Then *0 holds for $(x, y, F2)$.
- The last assumption is false.
- Assume the antecedent of *1 holds.
- Hence (x, y) are points ${}_{and}\wedge$ $(x, y) \in F1$.
- Then *1 holds for $(x, y, F2)$.
- The last assumption is false.
- Then (*2 ${}_{and}\wedge$ *3) holds.

- Hence F1[space(x),space(y)] is a bijection from*to space(x)*space(y).
- Hence *2 holds for $(x, y, F2)$.
- Hence *3 fails for $(x, y, F2)$.
- Hence there exists $\exists(m1, m2) \in f$ such that
- *Def.A holds for $(m1, m2, F1)$ ${}_{and}\wedge$ *Def.A fails for $(m1, m2, F2)$.
- By the way F1[space(m1)]=F2[space(m1)].
- Hence $(m1, m2, F1, F2)$ is a counterexample smaller than $(x, y, F1, F2)$.
- The first assumption is false.

□

Proposition 5 (Members' isomorphisms as consequent)**.** Take $\forall(x, y, F)$ such that (*A1 ${}_{and}\wedge$ *A2). Then (*B1 ${}_{and}\wedge$ *B2) holds.

A1 *Def.A holds for (x, y, F) in place of (x, y, F).

A2 F is an injection.

B1 Take $\forall m1 :\in^{\geq 0} x$. Then there exists $\exists m2 :\in^{\geq 0} y$ such that *Def.A holds for $(m1, m2, F)$ in place of (x, y, F).

B2 Take $\forall m2 :\in^{\geq 0} y$. Then there exists $\exists m1 :\in^{\geq 0} x$ such that *Def.A holds for $(m1, m2, F)$ in place of (x, y, F).

*Proof of *B1.*

- Assume it is false.
- Then there exists $\exists(x, y, F, m1)$ such that it is a minimum counterexample by $depth(m1, x)$.
- It is trivial that $(x \neq m1)$.
- Consider the proposition titled as "Depth of deep member".
- There exists $\exists x1$ such that $(m1 \in x1$ ${}_{and}\wedge$ $(x, y, F, x1)$ is not a counterexample).
- Hence *B1 holds for $x1$ in place of $m1$.
- Hence there exists $2 :\in^{\geq 0} y$ such that *Def.A holds for $(x1, y2, F)$.

- Let us follow *Def.A for $(x1, y2, F)$.
- Assume the antecedent of *0 holds.
- Then $space(x1) = \varnothing \ _{and}\wedge\ x1 = y2$.
- Hence $space(m1) = \varnothing \ _{and}\wedge\ m1 = m1 \ _{and}\wedge\ m1 \in^{\geq 0} y$.
- Hence *B1 holds for $m1$ in place of $m1$.
- Hence $(x, y, F, m1)$ is not a counterexample.
- Hence the last assumption is false.
- Assume the antecedent of *1 holds.
- Hence ($x1$ is a point) $_{and}\wedge$ $(m1 \in x1)$.
- Hence the last assumption is false.
- Hence (*2 $_{and}\wedge$ *3) must hold.
- Hence, (*4 $_{and}\wedge$ *5 $_{and}\wedge$ *6) holds.
- Hence *B1 holds for $(x, y, F, m1)$ in place of $(x, y, F, m1)$.
- Hence $(x, y, F, m1)$ is not a counterexample.
- The first assumption is false.

$\square$

*Proof of *B2.*

- Assume it is false.
- Then there exists $\exists(x, y, F, m2)$ such that $(x, y, F, m2)$ is a minimum counterexample by $depth(m2, y)$.
- It is trivial that $(y \neq m2)$.
- There exists $\exists y2$ such that ($m2 \in y2 \ _{and}\wedge$ $(x, y, F, y2)$ is not a counterexample).
- Hence *B2 should hold for $y2$ in place of $m2$.
- Hence there exists $1 :\in^{\geq 0} x$ such that *Def.A holds for $(x1, y2, F)$.
- Let us follow *Def.A for $(x1, y2, F)$.

- Assume the antecedent of *0 holds.
- Then $space(x1) = \varnothing \ _{and}\wedge \ x1 = y2$.
- Hence $space(m2) = \varnothing \ _{and}\wedge \ m2 = m2 \ _{and}\wedge \ m2 \in^{\geq 0} x$.
- Hence *B2 holds for $m2$ in place of $m2$.
- Hence $(x, y, F, m2)$ is not a counterexample.
- Hence the last assumption is false.
- Assume the antecedent of *1 holds.
- Hence ($y2$ is a point) $_{and}\wedge$ $(m2 \in y2)$.
- Hence the last assumption is false.
- Hence (*2 $_{and}\wedge$ *3) must hold.
- Hence, (*4 $_{and}\wedge$ *5 $_{and}\wedge$ *6) holds.
- Hence *B2 holds for $(x, y, F, m2)$ in place of $(x, y, F, m2)$.
- Hence $(x, y, F, m2)$ is not a counterexample.
- The first assumption is false.

□

Proposition 6 (Symmetric property)**.** Take $\forall B$ such that B is a binary relation. Then let B^{-1} denote $\{(b2, b1) \mid (b1, b2) \in B\}$.
Take $\forall(x, y, F)$. Then *A1 implies *A2.

A1 Def.A holds for (x, y, F).

A2 Def.A holds for (y, x, F^{-1}).

■

Proof.

- Assume it is false.
- There exists $\exists(x, y, F)$ such that it is a minimum counterexample by $depth(x)$.
- Let us follow *Def.A for (x, y, F) in terms of *A1.

- Assume the antecedent of *0 holds for (x, y, F) in terms of *A1.
- Hence $space(x) = \varnothing \ {}_{and}\wedge \ x = y$.
- Hence *0 holds for (x, y, F) in terms of *A2.
- Hence the last assumption is false.
- Assume the antecedent of *1 holds for (x, y, F) in terms of *A1.
- Hence (x, y) are points ${}_{and}\wedge$ $(x, y) \in F$.
- Hence (y, x) are points ${}_{and}\wedge$ $(y, x) \in F^{-1}$.
- Hence *1 holds for (x, y, F) in terms of *A2.
- Hence the last assumption is false.
- Hence (*2 ${}_{and}\wedge$ *3) must hold for (x, y, F) in terms of *A1.
- Hence $F[space(x), space(y)]$ is a bijection from*to $space(x) * space(y)$.
- Hence $F^{-1}[space(y), space(x)]$ is a bijection from*to $space(y) * space(x)$.
- Hence *2 holds for (x, y, F) in terms of *A2.
- Hence *3 must fail for (x, y, F) in terms of *A2.
- At same time, *3 hold for (x, y, F) in terms of *A1.
- Hence there exists $\exists(m1, m2) \in f$ such that (
  Def.A holds for $(m1, m2, F)$ ${}_{and}\wedge$
  Def.A fails for $(m2, m1, F^{-1})$.
  item).
- Hence $(m1, m2, F)$ is a counterexample.
- Consider the proposition titled as "Depth of memBer".
- Moreover $depth(m1) < depth(x)$.
- It contradicts to the title of (x, y, F) as a minimum counterexample.
- Hence the first assumption is false.

$\square$

Proposition 7 (Reflexive property)**.** Take $\forall(x, F)$ such that *A holds. Then *B holds.

A F is the identity function on $space(x)$.

B Def.A holds for (x, x, F).

Proof.

- Assume it is false.
- There exists $\exists(x, F)$ such that it is a minimum counterexample by $depth(x)$.
- Let us follow *Def.A for (x, x, F).
- Assume the antecedent of *0 holds.
- Then *0 holds.
- The last assumption is false.
- Assume the antecedent of *1 holds.
- Then *1 holds.
- The last assumption is false.
- It is trivial that *2 holds. Hence *3 must fail.
- Let $f1$ be the identity function on x.
- Then *3 must fail for $f1$ in place of f.
- Though *4 holds.
- Hence (*5 $_{and}\wedge$ *6) must fail.
- Hence there exists $\exists(m1, m1) :\in f1$ such that *Def.A fails for $(m1, m1, F)$.
- Meanwhile *Def.A holds for $(m1, m1, F[space(m1)])$ because otherwise $(m1, F[space(m1)])$ would be a counterexample smaller than a minimum counterexample,(x, F).
- Though consider the proposition titled as ”Restriction”.
- Then *Def.A holds for $(m1, m1, F)$.
- The first assumption is false.

$\square$

Proposition 8 (Transitive property)**.** Take $\forall(B1, B2)$ such that $(B1, B2)$ are binary relations. Then let $B2 \circ B1$ denote $\{(b1, b3) \mid \exists b2$ such that $(b1, b2) \in B1 \ {}_{and}\wedge\ (b2, b3) \in B2\ \}$.
Take $\forall(x, y, z, F1, F2)$ such that (*A1 $_{and}\wedge$ *A2) holds. Then *B holds.

A1 Def.A holds for $(x, y, F1)$.

A2 Def.A holds for $(y, z, F2)$.

B Def.A holds for $(x, z, F2 \circ F1)$.

Proof.

- Assume it is false.
- There exists $\exists(x, y, z, F1, F2)$ such that it is a minimum counterexample by $depth(x)$.
- Let us follow *Def.A for $(x, y, F1)$ and for $(y, z, F2)$.
- Assume the antecedent of *0 holds for $(x, y, F1)$.
- Hence $space(x) = \varnothing \ {}_{and}\wedge\ x = y$.
- Hence the antecedent of *0 holds for $(y, z, F2)$.
- Hence $x = y = z$.
- Hence *0 holds for $(x, z, F2 \circ F1)$.
- The last assumption is false.
- Assume the antecedent of *0 holds for $(y, z, F2)$.
- Hence $space(y) = \varnothing \ {}_{and}\wedge\ y = z$.
- Hence the antecedent of *0 holds for $(x, y, F1)$.
- The last assumption is false.
- Assume the antecedent of *1 holds $(x, y, F1)$.
- Hence (x, y) are points $_{and}\wedge$ $(x, y) \in F1)$.
- Hence *1 also hold for $(y, z, F2)$ because otherwise *G.A cannot hold for $(y, z, F2)$.

- Hence (y, z) are points $_{and}\wedge$ $(y, z) \in F2)$.
- Hence (x, z) are points $_{and}\wedge$ $(x, z) \in F2 \circ F1$.
- Hence *1 holds for $(x, z, F2 \circ F1)$.
- The last assumption is false.
- Assume the antecedent of *1 holds $(y, z, F2)$.
- Hence (y, z) are points $_{and}\wedge$ $(y, z) \in F2)$.
- Hence the antecedent of *1 also hold for $(x, y, F1)$ because otherwise *G.A cannot hold for $(x, y, F1)$.
- The last assumption is false.
- Hence (*2 $_{and}\wedge$ *3) holds for $(x, y, F1)$ and for $(y, z, F2)$.
- Hence $F1[space(x), space(Y)]$ is a bijection from*to $space(x) * space(y)$.
- And $F2[space(y), space(z)]$ is a bijection from*to $space(y) * space(z)$.
- Hence $(F2 \circ F1)[space(x), space(z)]$ is a bijection from*to $space(x) * space(z)$.
- Hence *2 holds for $(x, z, F2 \circ F1)$.
- Hence *3 fails for $(x, z, F2 \circ F1)$.
- By the way,there exists $(f1, f2)$ such that (
  3 holds for $(x, y, F1, f1)$ in place of (x, y, F, f) $_{and}\wedge$
  3 holds for $(y, z, F2, f2)$ in place of (x, y, F, f)
).
- Then *3 fails for $(x, z, F2 \circ F1, f2 \circ f1)$ in place of (x, y, F, f).
- Hence, there exists $\exists(m1, m2, m3)$ such that (
  $(m1, m2) \in f1$ $_{and}\wedge$
  $(m2, m3) \in f2$ $_{and}\wedge$
  (the antecedent of this proposition accepts
  $(m1, m2, m3, F1, F2)$ as $(x, y, z, F1, F2)$
) $_{and}\wedge$
  $(m1, m2, m3, F1, F2)$ is a counterexample
).
- Though $(m1, m2, m3, F1, F2)$ is smaller than a minimum counterexample.

- The first assumption is false.

$\square$

10 Homeomorphism as Isomorphism

Proposition 9 (Members' isomorphisms as antecedent)**.** Take $\forall(x, y, F, f)$ such that (*A1 $_{and}\wedge$ *A2 $_{and}\wedge$ *A3). Then *B holds.

A1 F is an injection.

A2 f is a bijection from*to x*y.

A3 Take $\forall(m1, m2) :\in f$. Then *Def.A holds for $(m1, m2, F)$.

B *Def.A holds for (x, y, F) in place of (x, y, F).

Proof.

- Assume B fails.
- Hence there exists $\exists(x, y, F)$ such that *Def.A fails for (x, y, F).
- Let us follow *Def.A for (x, y, F).
- (the antecedent of *0 fails $_{and}\wedge$ the antecedent of *1 fails $_{and}\wedge$ (*2 fails $_{or}\vee$ *3 fails)).
- Hence $(space(x) \neq \varnothing \neq space(y))$ $_{and}\wedge$ both of (x, y) are not points.
- Assume *2 fails.
- Hence $F[space(x), space(y)]$ is not a bijection from*to $space(x) * space(y)$.
- Consider *A1 which says F is an injection.
- Hence there exists $\exists(p_x, p_y) :\in space(x) * space(y)$ such that $p_x \notin domain(F)$ $_{or}\vee$ $p_y \notin image(F)$.
- Consider *A2,*A3 and the proposition titled as "Members' isomorphisms as the consequent".
- There exists $\exists y2 \in^{\geq 0} y$ such that *Def.A holds for $(p_x, y2, F)$.
- There exists $\exists x1 \in^{\geq 0} x$ such that *Def.A holds for $(x1, p_y, F)$.

- Meanwhile, for each of the 2 lines just above, (*Def.A holds only by the if-then condition of *1) because (each of (p_x, p_y) is a point).
- Hence $p_x \in domain(F) \ _{and}\wedge \ p_y \in image(F)$.
- Hence the last assumption is false.
- Hence *3 must fail.
- Hence *3 fails for f in place of f.
- Though by (*A2 $_{and}\wedge$ *A3), (*4 $_{and}\wedge$ *5 $_{and}\wedge$ *6) holds.
- Hence the first assumption is false.

□

Definition 10.1 (Pair)**.** Take $\forall\{x, y\}$. [4]Then $(x, y) := \{\{x\}, \{x, y\}\}$.

■

Proposition 10 (Topological space)**.** Take $\forall((X1, T1), (X2, T2))$ such that *A holds. Then *B1 $\Rightarrow$ *B2.

A Take $i \in \{1, 2\}$. Then (X_i, T_i) is a topological space.

B1 $((X1, T1), (X2, T2))$ are homeomorphic.

B2 There exists $\exists F$ such that $((X1, T1), (X2, T2))$ are isomorphic by F.

■

Proof.

B1 implies *C.

C There exists $\exists(G, g)$ such that (*C1 $_{and}\wedge$ ..... $_{and}\wedge$ *C4).

C1 G is a bijection from $X1$ to $X2$.

C2 G is a homeormorphism for *B1.

C3 g is a bijection from $T1$ to $T2$.

[4]By Kazimierz Kuratowski.

C4 Take $\forall(t1, t2) :\in g$. Then (G takes $t1$ to $t2$).

Consider the previous proposition titled as Members' isomorphisms as antecedent and refer it as *P.
Then *P accepts arguments as (*D1 $_{and}\wedge$ ..... $_{and}\wedge$ *D6).

D1 *P accepts $(X1, X2, G, G)$ in place of (x, y, F, f).

D2 *P accepts $(\{X1\}, \{X2\}, G, \{(X1, X2)\})$ in place of (x, y, F, f).

D3 Take $\forall(t1, t2) :\in g$. Then *P accepts $(t1, t2, G, G)$ in place of (x, y, F, f).

D4 *P accepts $(T1, T2, G, g)$ in place of (x, y, F, f).

D5 *P accepts (
$\{X1, T1\}$,
$\{X2, T2\}$,
G,
$\{(X1, X2), (T1, T2)\}$
) in place of (x, y, F, f).

D6 *P accepts (
$\{\{X1\}, \{X1, T1\}\}$,
$\{\{X2\}, \{X2, T2\}\}$,
G,
$\{(\{X1\}, \{X2\}), (\{X1, T1\}, \{X2, T2\})\}$
) in place of (x, y, F, f).

Hence *P implies (*E1 $_{and}\wedge$ ..... $_{and}\wedge$ *E6).
Finally, *E6 implies this proposition.

E1 $(X1, X2)$ are isomorphic by G.

E2 $(\{X1\}, \{X2\})$ are isomorphic by G.

E3 Take $\forall(t1, t2) :\in g$. Then $(t1, t2)$ are isomorphic by G.

E4 $(T1, T2)$ are isomorphic by G.

E5 $\{X1, T1\}, \{X2, T2\})$ are isomorphic by G.

E6 $(\{\{X1\}, \{X1, T1\}\}, \{\{X2\}, \{X2, T2\}\})$ are isomorphic by G.

□

11 Restriction of memBer by space

Definition 11.1. This definition uses a style of recursion.
Take $\forall(S, X, NULL)$ such that (*A1 $_{and}\wedge$ *A2 $_{and}\wedge$ *A3) [5]holds. Then define *B.

A1 X is a [6]space.

A2 $NULL$ is not a set.

A3 $NULL \notin^{\geq 0} (S, X)$.

B

- **1** If $\mathit{space}(S) \subset X$ Then $S[X] := S$ Else *2.
- **2** If S is not a set Then $S[X] := NULL$ Else *3.
- **3** $S[X] := \{s[X] \mid s \in S \ _{and}\wedge \ s[X] \neq NULL\}$.

■

12 Deep space

Definition 12.1. Take $\forall(S1, S2)$ such that (*1 $_{and}\wedge$ *2 $_{and}\wedge$ *3). Then define *4.

1 $S2 \subset \{m \mid m \in^{\geq 0} S1\}$.

2 Take $\forall(p, C)$ such that
($p :\in \mathit{space}(S1)$ $_{and}\wedge$ C is a chain from $S1$ down to p by set membership).

3 Then $C \cap S2 \neq \varnothing$

4 $S2$ is a deep space of $S1$.

---

[5]*A3 says that $\neg(NULL \in^{\geq 0} S)$.
[6]That is, x is a set of points.

References