Isomorphism of memBers

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1 Introduction

Definition 1.1 (memBer)**.** Take $\forall x$ such that (there exists $\exists S$ such that $x \in S$). Then x is said a memBer.

This article:(1) defines a binary relation that (x,y) as memBers are isomorphic,(2) proves that the binary relation is an equvalence relation, (3) proves that all homeomorphic topological spaces are isomorphic as memBers,(4) defines that a memBer $S1$ is a minor of a memBer $S2$.
I expect that readers will realize that the newly defined isomorphisms are somehow more fundamental than ,e.g., homeomorphisms of topological spaces. Because "homeomorphisms" logically resolve to "isomorphisms of memBers" whereas the inverse of it does not hold.

2 Notation

Definition 2.1.
"A $_{then}\wedge$ B" $\equiv$ "A holds then B holds".
"A $_{else}\vee$ B" $\equiv$ "if A fails then B holds".
"$\forall x :\in S$" $\equiv$ "for all x such that $x \in S$".
"$\forall x$ as an integer" $\equiv$ "for all x such that x is an integer".

Definition 2.2. Few exceptional conventions in this article are that:
Take $\forall(x, I, X)$ as a family X, the index set I and the function x, then x is surjective.
In other words, $X = \{x_i \mid i \in I\}$.

And x is said a family's function.

3 Deep member

Definition 3.1 (Deep member of memBer)**.** This definition uses a style of recursion.
Take $\forall(x, y)$ such that *1 holds. Then define *2 $_{then}\wedge$ *3.

1 $x = y$ else (there exists $\exists z$ such that $x \in z \in^{\geq 0} y$).

2 x is a deep member of y

3 $x \in^{\geq 0} y$

Definition 3.2 (Power of membership)**.** This definition uses a style of recursion.
Take $\forall(x, y, z, n)$ such [1]that $n \in N$. [2]Then define *1 $_{then}\wedge$ *2.

1 $x \in^0 x$.

2 If $x \in y \in^n z$ Then $x \in^{n+1} z$.

Definition 3.3 (Sum of powers of membership)**.** Take $\forall m \geq 1$. Then let $I := [1, m+1] \subset N$. Take $\forall x$ as a family's function on I. Then define *1.

1 If $x_1 \in^{p_1} .... \in^{p_m} x_{m+1}$ Then $(\in^{p1+...+p_m}, x_1, ..., x_{m+1})$.

2 For example,
If $x_1 \in^{n_1} x_2$ Then $(\in^{n_1}, x_1, x_2)$.

3 For example,
If $x_1 \in^{n_1} x_2 \in^{n_2} x_3$ Then $(\in^{n_1+n_2}, x_1, x_2, x_3)$.

Definition 3.4 (Space of memBer)**.** Take $\forall(x, y)$ such that *1 holds. Then define *2.

1 $y = \{d \mid d \in^{\geq 0} x$ then d is a point $\}$.

2 y is the space of x.

[1] N denotes the set of all natural numbers.
[2] It may happen to be that $x \in^1 y$ and $x \in^2 y$.

4 Notations

Definition 4.1 (Restriction of binary relation)**.** Take $\forall(L, X, Y, X1, Y1)$ such that *1 holds. Then define (*2 $_{then}\wedge$ *3 $_{then}\wedge$ *4).

1 L is a binary relation on $X * Y \ {}_{then}\wedge\ X1 \subset X \ {}_{then}\wedge\ Y1 \subset Y$.

2 $L[X1]$:={ $(x, y) \in L \mid x \in X1$ }.

3 $L[, Y1]$:={ $(x, y) \in L \mid y \in Y1$ }.

4 $L[X1, Y1]$:={ $(x, y) \in L \mid x \in X1 \ {}_{then}\wedge\ y \in Y1$ }.

5 Isomorphic memBers

Definition 5.1 (Isomorphic memBers)**.** Take all $\forall x$. Then (x, x) are said isomorphic.

Definition 5.2 (Isomorphic memBers by binary relation)**.** This definition uses a style of recursion.

Take $\forall(x, y, F)$ such that *A holds. Then define (*B1 ${}_{then}\wedge$ *B2).

A (F is a binary relation ${}_{then}\wedge$ *0) holds.

- **0** If there exists $\exists v :\in \{x, y\}$ such that $space(v) = \varnothing$ Then $x = y$ Else *1.
- **1** If there exists $\exists v :\in \{x, y\}$ such that v is a point Then ((x, y) are points ${}_{then}\wedge$ $(x, y) \in F$) Else (*2 ${}_{then}\wedge$ *3).
- **2** $F[space(x), space(y)]$ is a [3]bijection from*to space(x)*space(y).
- **3** There exists $\exists f$ such that (*4 ${}_{then}\wedge$ *5 ${}_{then}\wedge$ *6).
- **4** f is a bijection from*to $x * y$.
- **5** Take $\forall(m1, m2) \in f$.
- **6** *A holds for $(m1, m2, F)$ in place of (x, y, F))).

B1 (x, y) are said isomorphic by F as an isomorphism.

B2 Take $\forall(x, y, F)$ such that (x, y) are isomorphic by F. Then (x, y) are said isomorphic.

[3]To weaken the definition, replace "bijection" with "function" or with "binary relation".

6 Minors of memBers

Definition 6.1 (Minors)**.** Take $\forall(x, y)$ such that *A holds. Then it is said as *B.

A $*1 \ {}_{then}\wedge \ *2$.

- **1** Take $\forall d$. Then $d \in^{\geq 0} x \Rightarrow$ $\mathrm{d}\in^{\geq 0} y$.
- **2** Take $\forall(d1, d2, d3)$ such that (take $\forall d \in \{d1, d2, d3\}$, then $d \in^{\geq 0} x$). Then $(*3 \Leftarrow *4)$.
- **3** $((x, d1, d3)$,$(x, d2, d3))$ are isomorphic.
- **4** $((y, d1, d3)$,$(y, d2, d3))$ are isomorphic.

B $*5 \ {}_{then}\wedge \ *6$.

- **5** x is a minor of y.
- **6** $x \leq^{minor} y$.

7 Notations

Notations defined bellow are used to express typical patterns of logic.
They are used to make the texts of proofs shorter or simpler if the original texts are consists of long or complex repetitions of typical logic.

Definition 7.1 (Symbol for omitting)**.**
For example:

Let $S := \{x \in N \mid ... \}$.
Then $S = N$.

Definition 7.2 (Set definition with symbol for omitting)**.** Bellow, P denotes a logical expression.
For example:

Let $P := (x \in N \ {}_{then}\wedge \ x < 2)$.
Let $S := \{... \mid P \}$.

Then S is the set of all valid substitutions into free variables of P.

Namely $S = \{(x = 0), (x = 1)\}$.

For example:

Let $Q := P \ _{then}\wedge \ (\forall y :\in N, (\exists z :\in N$ such that $y = z - 1))$.
Let $S := \{ ... \mid Q \}$.
Then $S = \{(x = 0), (x = 1)\}$.

As a remark on Q, only x is a free variable.

Definition 7.3 (Pipes)**.**
Pipe to express a function:

Let $S := \{x * 2 \ _{fun}|| \ x \in N \mid ... \ \}$.

Then S is the set of all even members from N.

Pipe to define a subset:

$S1, S2 := \{...| \ x \in N \ \ _{then}\wedge \ x^2 = y \ \} \ ||_{sub}$ (S1,where y is smallest).

Then $S1 = \{...| \ x \in N \ \ _{then}\wedge \ x^2 = y \ \}$.
And $S2$ is the subset of $S1$ exactly including all minimum members in terms of y.

Example:

$S1, S2, S3 := \{...| \ x \in N \ \ _{then}\wedge \ x^2 = y \ \} \ ||_{sub}$
($S1$,where y is smallest) $||_{fun}$
($S2$,return $x + y$).

The last line says that(take $S2$, then return $x + y$ for each member).
In other words, $S3 = \{x + y \ _{fun}|| \ P(x, y) \in S2 \mid ... \ \}$.
Namely $S3 = \{(x = 0, y = 0)\}$.

8 Depth of deep member

Definition 8.1 (Depth of deep member)**.** Recall the defined type of expressions titled as "Sum of powers of membership".
Take $\forall m \geq 1$. Take $\forall x$ as a family's function on $[1, m + 1] :\subset N$. Then define

as follows.

Let $S1, S2, S3 := \{...|\ (\in^{n_1+...+n_m}, x_1, ..., x_{m+1})\ \}\ ||_{sub}$
($S1$,where $n1 + ... + n_m$ is largest) $||_{fun}$
($S2$,return $n1 + ... + n_m$).
If $S1 \neq \varnothing$ Then
take $\forall n :\in S3$,
let $\#depth(x_1, ..., x_{m+1}) := n$.

As a remark, all n_i are free variables whereas all x_i are not.

Definition 8.2 (Depth of memBer)**.** Take $\forall x_2$.

Let $S1, S2, S3 := \{...|\ (\in^{n1}, x_1, x_2)\ \}\ ||_{sub}$
($S1$,where $n1$ is largest) $||_{fun}$
($S2$,return $n1$).
Take $\forall n :\in S3$.
Let $\#depth(x_2) := n$.

As a remark, all n_i and x_1 are free variables whereas x_2 is not.

Proposition 1 (Depth of deep member)**.** Take $\forall(x, y, z)$ such that $z \in y \in^{\geq 0} x$. Then $\#depth(z, x) > \#depth(y, x)$.

Proof.

- Assume it is false.
- There exists $\exists(x, y, z)$ such that it is a counterexample.
- Hence $\#depth(z, x) \leq \#depth(y, x)$.
- Hence $\#depth(z, x) \geq \#depth(z, y, x) > \#depth(y, x) \geq \#depth(z, x)$.
- The assumption is false.

$\square$

Proposition 2 (Depth of memBer)**.** Take $\forall(x, y)$ such that $y \in x$. Then $\#depth(y) < \#depth(x)$.

Proof.

- Assume it is false.
- There exists $\exists(x, y)$ such that it is a counterexample.
- Hence $\#depth(y) \geq \#depth(x)$.
- There exists $\exists v :\in y$ such that ($\#depth(y) = \#depth(v, y) \geq \#depth(v, y, x) \leq \#depth(x)$).
- Though $\#depth(v, y) + 1 = \#depth(v, y, x)$.
- The assumption is false.

□

9 Isomorphic memBers as equivalence relation

Definition 9.1. In this section, *Def refers to the definition titled as "Isomorphic memBers by binary relation".
And *1 $\equiv$ *2, without any explicit proof because it is trivial by *Def.
And *3 holds, without any explicit proof because it is trivial by *Def.

1 (x_i, y_i) are isomorphic by F_i.

2 *Def.A holds for (x_i, y_i, F_i) in place of (x, y, F).

3 Take $\forall(x, y, F)$ such that (*4 ${}_{then}\wedge$ (*5 ${}_{else}\vee$ *6)). Then *7 holds.

4 F is a binary relation.

5 $(space(x) = \varnothing \ {}_{then}\wedge \ x = y)$.

6 $((x, y)$ are points ${}_{then}\wedge$ $(x, y) \in F)$.

7 (x, y) are isomorphic by F.

Proposition 3 (Restriction)**.** Take $\forall(x, y, F1, F2)$ such that (*A1 ${}_{then}\wedge$ *A2 ${}_{then}\wedge$ *A3) holds. Then *B holds.

A1 $(F1, F2)$ are binary relations.

A2 $F1[space(x)] = F2[space(x)]$.

A3 Def.A holds for $(x, y, F1)$.

B Def.A holds for $(x, y, F2)$.

■

Proof.

- Assume it is false.
- There exists $\exists(x, y, F1, F2)$ such that it is a minimum counterexample by $\#depth(x)$.
- Let us follow *Def.A for $(x, y, F1)$.
- Assume the antecedent of *0 holds.
- Hence $space(x) = \varnothing \ {}_{then}\wedge \ x = y$.
- Then *0 holds for $(x, y, F2)$.
- The last assumption is false.
- Assume the antecedent of *1 holds.
- Hence (x, y) are points ${}_{then}\wedge$ $(x, y) \in F1$.
- Then *1 holds for $(x, y, F2)$.
- The last assumption is false.
- Then (*2 ${}_{then}\wedge$ *3) holds.
- Hence *2 holds for $(x, y, F2)$.
- Hence *3 fails for $(x, y, F2)$.
- Hence there exists $\exists(m1, m2) \in f$ such that
- *Def.A holds for $(m1, m2, F1)$ ${}_{then}\wedge$ *Def.A fails for $(m1, m2, F2)$.
- Hence $(m1, m2, F1, F2)$ is a counterexample smaller than $(x, y, F1, F2)$.
- The first assumption is false.

□

Proposition 4 (Members' isomorphisms as consequent)**.** Take $\forall(x, y, F)$ such that (*A1 ${}_{then}\wedge$ *A2). Then (*B1 ${}_{then}\wedge$ *B2) holds.

A1 *Def.A holds for (x, y, F) in place of (x, y, F).

A2 F is an injection.

B1 Take $\forall m1 :\in^{\geq 0} x$. Then there exists $\exists m2 :\in^{\geq 0} y$ such that *Def.A holds for $(m1, m2, F)$ in place of (x, y, F).

B2 Take $\forall m2 :\in^{\geq 0} y$. Then there exists $\exists m1 :\in^{\geq 0} x$ such that *Def.A holds for $(m1, m2, F)$ in place of (x, y, F).

*Proof of *B1.*

- Assume it is false.
- Then there exists $\exists(x, y, F, m1)$ such that it is a minimum counterexample by $\#depth(m1, x)$.
- It is trivial that $(x \neq m1)$.
- Consider the proposition titled as "Depth of deep member".
- There exists $\exists x1$ such that $(m1 \in x1 \ {}_{then}\wedge\ (x, y, F, x1)$ is not a counterexample).
- Hence *B1 holds for $x1$ in place of $m1$.
- Hence there exists $2 :\in^{\geq 0} y$ such that *Def.A holds for $(x1, y2, F)$.
- Let us follow *Def.A for $(x1, y2, F)$.
- Assume the antecedent of *0 holds.
- Then $space(x1) = \varnothing \ {}_{then}\wedge\ x1 = y2$.
- Hence $space(m1) = \varnothing \ {}_{then}\wedge\ m1 = m1 \ {}_{then}\wedge\ m1 \in^{\geq 0} y$.
- Hence *B1 holds for $m1$ in place of $m1$.
- Hence $(x, y, F, m1)$ is not a counterexample.
- Hence the last assumption is false.
- Assume the antecedent of *1 holds.
- Hence ($x1$ is a point) ${}_{then}\wedge$ $(m1 \in x1)$.
- Hence the last assumption is false.
- Hence (*2 ${}_{then}\wedge$ *3) must hold.

- Hence, $(*4 \ {}_{then}\!\wedge\ *5 \ {}_{then}\!\wedge\ *6)$ holds.
- Hence *B1 holds for $(x, y, F, m1)$ in place of $(x, y, F, m1)$.
- Hence $(x, y, F, m1)$ is not a counterexample.
- The first assumption is false.

$\square$

*Proof of *B2.*

- Assume it is false.
- Then there exists $\exists(x, y, F, m2)$ such that $(x, y, F, m2)$ is a minimum counterexample by $\#depth(m2, y)$.
- It is trivial that $(y \neq m2)$.
- There exists $\exists y2$ such that $(m2 \in y2 \ {}_{then}\!\wedge\ (x, y, F, y2)$ is not a counterexample).
- Hence *B2 should hold for $y2$ in place of $m2$.
- Hence there exists $1 :\in^{\geq 0} x$ such that *Def.A holds for $(x1, y2, F)$.
- Let us follow *Def.A for $(x1, y2, F)$.
- Assume the antecedent of *0 holds.
- Then $space(x1) = \varnothing \ {}_{then}\!\wedge\ x1 = y2$.
- Hence $space(m2) = \varnothing \ {}_{then}\!\wedge\ m2 = m2 \ {}_{then}\!\wedge\ m2 \in^{\geq 0} x$.
- Hence *B2 holds for $m2$ in place of $m2$.
- Hence $(x, y, F, m2)$ is not a counterexample.
- Hence the last assumption is false.
- Assume the antecedent of *1 holds.
- Hence ($y2$ is a point) ${}_{then}\!\wedge$ $(m2 \in y2)$.
- Hence the last assumption is false.
- Hence $(*2 \ {}_{then}\!\wedge\ *3)$ must hold.
- Hence, $(*4 \ {}_{then}\!\wedge\ *5 \ {}_{then}\!\wedge\ *6)$ holds.

- Hence *B2 holds for $(x, y, F, m2)$ in place of $(x, y, F, m2)$.
- Hence $(x, y, F, m2)$ is not a counterexample.
- The first assumption is false.

$\square$

Proposition 5 (Symmetric property)**.** Take $\forall B$ such that B is a binary relation. Then let B^{-1} denote $\{(b2, b1) \mid (b1, b2) \in B\}$.
Take $\forall(x, y, F)$. Then *A1 implies *A2.

A1 Def.A holds for (x, y, F).

A2 Def.A holds for (y, x, F^{-1}).

$\blacksquare$

Proof.

- Assume it is false.
- There exists $\exists(x, y, F)$ such that it is a minimum counterexample by $\#depth(x)$.
- Let us follow *Def.A for (x, y, F) in terms of *A1.
- Assume the antecedent of *0 holds for (x, y, F) in terms of *A1.
- Hence $space(x) = \varnothing \ {}_{then}\wedge\ x = y$.
- Hence *0 holds for (x, y, F) in terms of *A2.
- Hence the last assumption is false.
- Assume the antecedent of *1 holds for (x, y, F) in terms of *A1.
- Hence (x, y) are points ${}_{then}\wedge$ $(x, y) \in F$.
- Hence (y, x) are points ${}_{then}\wedge$ $(y, x) \in F^{-1}$.
- Hence *1 holds for (x, y, F) in terms of *A2.
- Hence the last assumption is false.
- Hence (*2 ${}_{then}\wedge$ *3) must hold for (x, y, F) in terms of *A1.
- Hence $F[space(x), space(y)]$ is a bijection from*to $space(x) * space(y)$.

- Hence $F^{-1}[space(y), space(x)]$ is a bijection from*to $space(y) * space(x)$.
- Hence *2 holds for (x, y, F) in terms of *A2.
- Hence *3 must fail for (x, y, F) in terms of *A2.
- At same time, *3 hold for (x, y, F) in terms of *A1.
- Hence there exists $\exists(m1, m2) \in f$ such that (
- *Def.A holds for $(m1, m2, F) \;\; _{then}\wedge$
- *Def.A fails for $(m2, m1, F^{-1})$.
-).
- Hence $(m1, m2, F)$ is a counterexample.
- Consider the proposition titled as "Depth of memBer".
- Moreover $\#depth(m1) < \#depth(x)$.
- It contradicts to the title of (x, y, F) as a minimum counterexample.
- Hence the first assumption is false.

□

Proposition 6 (Reflexive property)**.** Take $\forall(x, F)$ such that *A holds. Then *B holds.

A F is the identity function on $space(x)$.

B Def.A holds for (x, x, F).

■

Proof.

- Assume it is false.
- There exists $\exists(x, F)$ such that it is a minimum counterexample by $\#depth(x)$.
- Let us follow *Def.A for (x, x, F).
- Assume the antecedent of *0 holds.
- Then *0 holds.

- The last assumption is false.
- Assume the antecedent of *1 holds.
- Then *1 holds.
- The last assumption is false.
- It is trivial that *2 holds. Hence *3 must fail.
- Let $f1$ be the identity function on x.
- Then *3 must fail for $f1$ in place of f.
- Though *4 holds.
- Hence (*5 $\ {}_{then}\wedge$ *6) must fail.
- Hence there exists $\exists(m1, m1) :\in f1$ such that *Def.A fails for $(m1, m1, F)$.
- Meanwhile *Def.A holds for $(m1, m1, F[space(m1)])$ because otherwise $(m1, F[space(m1)])$ would be a counterexample smaller than a minimum counterexample,(x, F).
- Though consider the proposition titled as "Restriction".
- Then *Def.A holds for $(m1, m1, F)$.
- The first assumption is false.

$\square$

Proposition 7 (Transitive property)**.** Take $\forall(B1, B2)$ such that $(B1, B2)$ are binary relations. Then let $B2 \circ B1$ denote $\{(b1, b3) \mid \exists b2$ such that $(b1, b2) \in B1$ $\ {}_{then}\wedge\ (b2, b3) \in B2\ \}$.
Take $\forall(x, y, z, F1, F2)$ such that (*A1 $\ {}_{then}\wedge$ *A2) holds. Then *B holds.

A1 Def.A holds for $(x, y, F1)$.

A2 Def.A holds for $(y, z, F2)$.

B Def.A holds for $(x, z, F2 \circ F1)$.

$\blacksquare$

Proof.

- Assume it is false.

- There exists $\exists(x, y, z, F1, F2)$ such that it is a minimum counterexample by $\#depth(x)$.
- Let us follow *Def.A for $(x, y, F1)$ and for $(y, z, F2)$.
- Assume the antecedent of *0 holds for $(x, y, F1)$.
- Hence $\mathit{space}(x) = \varnothing \;_{then}\wedge\; x = y$.
- Hence the antecedent of *0 holds for $(y, z, F2)$.
- Hence $x = y = z$.
- Hence *0 holds for $(x, z, F2 \circ F1)$.
- The last assumption is false.
- Assume the antecedent of *0 holds for $(y, z, F2)$.
- Hence $\mathit{space}(y) = \varnothing \;_{then}\wedge\; y = z$.
- Hence the antecedent of *0 holds for $(x, y, F1)$.
- The last assumption is false.
- Assume the antecedent of *1 holds $(x, y, F1)$.
- Hence (x, y) are points $\;_{then}\wedge\; (x, y) \in F1)$.
- Hence *1 also hold for $(y, z, F2)$ because otherwise *G.A cannot hold for $(y, z, F2)$.
- Hence (y, z) are points $\;_{then}\wedge\; (y, z) \in F2)$.
- Hence (x, z) are points $\;_{then}\wedge\; (x, z) \in F2 \circ F1$.
- Hence *1 holds for $(x, z, F2 \circ F1)$.
- The last assumption is false.
- Assume the antecedent of *1 holds $(y, z, F2)$.
- Hence (y, z) are points $\;_{then}\wedge\; (y, z) \in F2)$.
- Hence the antecedent of *1 also hold for $(x, y, F1)$ because otherwise *G.A cannot hold for $(x, y, F1)$.
- The last assumption is false.

- Hence (*2 $_{then}\wedge$ *3) holds for $(x, y, F1)$ and for $(y, z, F2)$.
- Hence $F1[space(x), space(Y)]$ is a bijection from*to $space(x) * space(y)$.
- And $F2[space(y), space(z)]$ is a bijection from*to $space(y) * space(z)$.
- Hence $(F2 \circ F1)[space(x), space(z)]$ is a bijection from*to $space(x) * space(z)$.
- Hence *2 holds for $(x, z, F2 \circ F1)$.
- Hence *3 fails for $(x, z, F2 \circ F1)$.
- By the way,there exists $(f1, f2)$ such that (
  3 holds for $(x, y, F1, f1)$ in place of (x, y, F, f) $_{then}\wedge$
  3 holds for $(y, z, F2, f2)$ in place of (x, y, F, f)
).
- Then *3 fails for $(x, z, F2 \circ F1, f2 \circ f1)$ in place of (x, y, F, f).
- Hence, there exists $\exists(m1, m2, m3)$ such that (
  $(m1, m2) \in f1$ $_{then}\wedge$
  $(m2, m3) \in f2$ $_{then}\wedge$
  (the antecedent of this proposition accepts
  $(m1, m2, m3, F1, F2)$ as $(x, y, z, F1, F2)$
) $_{then}\wedge$
  $(m1, m2, m3, F1, F2)$ is a counterexample
).
- Though $(m1, m2, m3, F1, F2)$ is smaller than a minimum counterexample.
- The first assumption is false.

$\square$

10 Homeomorphism as Isomorphism

Proposition 8 (Members' isomorphisms as antecedent)**.** Take $\forall(x, y, F, f)$ such that (*A1 $_{then}\wedge$ *A2 $_{then}\wedge$ *A3). Then *B holds.

A1 F is an injection.

A2 f is a bijection from*to x*y.

A3 Take $\forall(m1, m2) :\in f$. Then *Def.A holds for $(m1, m2, F)$.

B *Def.A holds for (x, y, F) in place of (x, y, F).

Proof.

- Assume B fails.
- Hence there exists $\exists(x, y, F)$ such that *Def.A fails for (x, y, F).
- Let us follow *Def.A for (x, y, F).
- (the antecedent of *0 fails $_{then}\wedge$ the antecedent of *1 fails $_{then}\wedge$ (*2 fails $_{else}\vee$ *3 fails)).
- Hence $(\mathit{space}(x) \neq \varnothing \neq \mathit{space}(y))$ $_{then}\wedge$ both of (x, y) are not points.
- Assume *2 fails.
- Hence $F[\mathit{space}(x), \mathit{space}(y)]$ is not a bijection from*to $\mathit{space}(x) * \mathit{space}(y)$.
- Consider *A1 which says F is an injection.
- Hence there exists $\exists(p_x, p_y) :\in \mathit{space}(x) * \mathit{space}(y)$ such that $p_x \notin \mathit{domain}(F)$ $_{else}\vee$ $p_y \notin \mathit{image}(F)$.
- Consider *A2,*A3 and the proposition titled as "Members' isomorphisms as the consequent".
- There exists $\exists y2 \in^{\geq 0} y$ such that *Def.A holds for $(p_x, y2, F)$.
- There exists $\exists x1 \in^{\geq 0} x$ such that *Def.A holds for $(x1, p_y, F)$.
- Meanwhile, for each of the 2 lines just above,
(*Def.A holds only by the if-then condition of *1) because
(each of (p_x, p_y) is a point).
- Hence $p_x \in \mathit{domain}(F)$ $_{then}\wedge$ $p_y \in \mathit{image}(F)$.
- Hence the last assumption is false.
- Hence *3 must fail.
- Hence *3 fails for f in place of f.
- Though by (*A2 $_{then}\wedge$ *A3), (*4 $_{then}\wedge$ *5 $_{then}\wedge$ *6) holds.
- Hence the first assumption is false.

$\square$

Definition 10.1 (Pair)**.** Take $\forall\{x, y\}$. [4]Then $(x, y) := \{\{x\}, \{x, y\}\}$.

Proposition 9 (Topological space)**.** Take $\forall((X1, T1), (X2, T2))$ such that *A holds. Then *B1 $\Rightarrow$ *B2.

A Take $i \in \{1, 2\}$. Then (X_i, T_i) is a topological space.

B1 $((X1, T1), (X2, T2))$ are homeomorphic.

B2 There exists $\exists F$ such that $((X1, T1), (X2, T2))$ are isomorphic by F.

Proof.

B1 implies *C.

C There exists $\exists(G, g)$ such that (*C1 ${}_{then}\wedge$ *C2 ${}_{then}\wedge$ *C3 ${}_{then}\wedge$ *C4).

C1 G is a bijection from $X1$ to $X2$.

C2 G is a homeormorphism for *B1.

C3 g is a bijection from $T1$ to $T2$.

C4 Take $\forall(t1, t2) :\in g$. Then (G takes $t1$ to $t2$).

Consider the previous proposition titled as Members' isomorphisms as antecedent and refer it as *P.
Then *P accepts arguments as (*D1 to *D6) [5]combined by "${}_{then}\wedge$".

D1 *P accepts $(X1, X2, G, G)$ in place of (x, y, F, f).

D2 *P accepts $(\{X1\}, \{X2\}, G, \{(X1, X2)\})$ in place of (x, y, F, f).

D3 Take $\forall(t1, t2) :\in g$. Then *P accepts $(t1, t2, G, G)$ in place of (x, y, F, f).

D4 *P accepts $(T1, T2, G, g)$ in place of (x, y, F, f).

[4]By Kazimierz Kuratowski.

[5]*D1 ${}_{then}\wedge$ .... ${}_{then}\wedge$ *D5.

D5 *P accepts (
$\{X1, T1\}$,
$\{X2, T2\}$,
G,
$\{(X1, X2), (T1, T2)\}$
) in place of (x, y, F, f).

D6 *P accepts (
$\{\{X1\}, \{X1, T1\}\}$,
$\{\{X2\}, \{X2, T2\}\}$,
G,
$\{(\{X1\}, \{X2\}), (\{X1, T1\}, \{X2, T2\})\}$
) in place of (x, y, F, f).

Hence *P implies (*E1 to *E6) combined by ${}_{then}\wedge$.
Finally, *E6 implies this proposition.

E1 $(X1, X2)$ are isomorphic by G.

E2 $(\{X1\}, \{X2\})$ are isomorphic by G.

E3 Take $\forall(t1, t2) :\in g$. Then $(t1, t2)$ are isomorphic by G.

E4 $(T1, T2)$ are isomorphic by G.

E5 $\{X1, T1\}, \{X2, T2\})$ are isomorphic by G.

E6 $(\{\{X1\}, \{X1, T1\}\}, \{\{X2\}, \{X2, T2\}\})$ are isomorphic by G.

□

11 Restriction of memBer by space

Definition 11.1. This definition uses a style of recursion.
Take $\forall(S, X, N_{null})$ such that (*A1 ${}_{then}\wedge$ *A2 ${}_{then}\wedge$ *A3) [6]holds. Then define *B.

[6]*A2 says that $\neg(N_{null} \in^{\geq 0} S)$.

A1 X is a [7]space.

A2 N_{null} is not a set.

A3 $N_{null} \notin^{\geq 0} (S, X)$.

B

- **1** If $space(S) \subset X$ Then $S[X] := S$ Else *2.
- **2** If S is not a set Then $S[X] := N_{null}$ Else *3.
- **3** $S[X] := \{s[X] \mid s \in S \ {}_{then}\wedge\ s[X] \neq N_{null}\}$.

$\blacksquare$

[7] That is, x is a set of points.

References

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