

Isomorphism of memBers 1

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1 Introduction 5

Definition 1.1 (memBer). Take $\forall x$ such that (there exists $\exists S$ such that $x \in S$). 6

Then x is said a memBer. 7

■ 8

This article:(1) defines that (x,y) as memBers are isomorphic,(2) proves that 9

if (x,y) as memBers are isomorphic then (y,x) as memBers are isomorphic too, 10

(3) proves that all homeomorphic topological spaces are isomorphic as mem- 11

Bers,(4) defines that a memBer $S1$ is a minor of a memBer $S2$. 12

I expect that readers will realize that the newly defined isomorphisms are 13

somehow more fundamental than ,e.g., homeomorphisms of topological spaces. 14

Because "homeomorphisms" logically resolve to "isomorphisms of memBers" 15

whereas the inverse of it does not hold. 16

2 Deep member 17

Definition 2.1 (Deep member of memBer). This definition uses a style of 18
recursion. 19

Take $\forall(x, y)$ such that *1 holds. Then it is said as *2 and written as *3. 20

1 $x = y$ else (there exists $\exists z$ such that $x \in z \in^{deep} y$). 21

2 x is a deep member of y 22

3 $x \in^{deep} y$ 23

■ 24

Definition 2.2 (Space of memBer). Take $\forall(x, y)$ such that *1 holds. Then it	25
is said as *2.	26
1 $y = \{d \mid d \in^{deep} x \text{ then } d \text{ is a point } \}$.	27
2 y is the space of x .	28
	29
Definition 2.3 (Proper deep member of memBer). Take $\forall x$ such that *1 holds.	30
Then it is said as *2.	31
1 There exists $\exists z$ such that $x \in^{deep} z \in y$.	32
2 x is a proper deep member of y .	33
	34
Definition 2.4 (Paradoxical memBer). Take $\forall x$ such that (*1 else *2) holds.	35
Then it is said as *2.	36
1 " $\{ d \mid d \in^{deep} x \}$ " is not definable.	37
2 x is a proper deep member of x .	38
3 x is paradoxical.	39
	40
Definition 2.5 (Non-paradoxical memBer). In the rest, informally speaking,	41
" $\forall x$ " must be interpreted as " $\forall x$ which is not paradoxical".	42
Formally, in the rest, the domain of discourse does not include any paradoxical	43
memBer.	44

3 Notations 45

Definition 3.1 (Symbols). $*1 \equiv *2$ and $*3 \equiv *4$.	46
1 A then B .	47
2 $A \text{ then} \wedge B$.	48
3 A else B .	49
4 $A \text{ else} \vee B$.	50

As a remark, *1 says that $((A \wedge B)$ and (the meaning of B may depend on that A holds)).

As a remark, *3 says that $((A \vee B)$ and (the meaning of B may depend on that A fails)).

Definition 3.2 (Propositional function). Following *1 is true because *1 calls *2 passing integer 123 as the value of x . In other words, *2 is a propositional function.

1 Let $x:=123$ then *2 holds.

2 $x \in Z$.

Definition 3.3 (Colon). A colon(:) may be used in introducing a new variable. Some examples follow.

1 $x:=1$.

2 $\forall x : \in S$.

Definition 3.4 (Restriction of binary relation). Take $\forall(L, X, Y, X1)$ such that *1 holds. Then define *2.

1 L is a binary relation on $X * Y$ then $\wedge X1 \subset X$.

2 $L[X1] := \{ (x, y) \in L \mid x \in X1 \}$.

4 Isomorphic memBers 72

Definition 4.1 (Isomorphic memBers). This definition uses a style of recursion. 73

74

Take $\forall(x, y, F)$ such that $*A$ holds. Then define $*B$. 75

A $*0 \text{ }_{else} \vee *1 \text{ }_{else} \vee (*2 \text{ }_{then} \wedge *3)$. 76

77

0 $space(x) = \emptyset \text{ }_{then} \wedge x = y$. 78

1 (x, y) are points $\text{ }_{then} \wedge (x, y) \in F$ 79

2 $F[space(x)]$ is an ¹injection from $*to \text{ } space(x) * space(y)$. 80

3 There exists $\exists f$ such that $(*4 \text{ }_{then} \wedge *5 \text{ }_{then} \wedge *6)$. 81

4 f is a bijection from $*to \text{ } x * y \text{ }_{then} \wedge f \neq \emptyset$. 82

5 Take $\forall(m1, m2) \in f$. 83

6 $*A$ holds for $(m1, m2, F)$ in place of (x, y, F)). 84

B (x, y) are said isomorphic by F as an isomorphism. ■ 85

¹Some alternative definitions are to weaken $*2$ by replacing "injection" with "function" or "binary relation". Though such weakend condition should not be titled as "isomorphic".

5 Minors of memBers 86

Definition 5.1 (Minors). Take $\forall(x, y)$ such that $*A$ holds. Then it is said as 87
 $*B$. 88

A $*1$ *then* \wedge $*2$. 89

1 Take $\forall d$. Then $d \in^{deep} x \Rightarrow d \in^{deep} y$. 90

2 Take $\forall(d1, d2, d3)$ such that (take $\forall d \in \{d1, d2, d3\}$, then $d \in^{deep} x$). 91

Then $(*3 \leftarrow *4)$. 92

3 $((x, d1, d3), (x, d2, d3))$ are isomorphic. 93

4 $((y, d1, d3), (y, d2, d3))$ are isomorphic. 94

B $*5$ *then* \wedge $*6$. 95

5 x is a minor of y . 96

6 $x \leq^{minor} y$. 97

■ 98

6 Deep graph 99

Definition 6.1 (Deep graph). Take $\forall(x, V, E, P, P_{ord})$ such that $*A$ holds. 100
 Then define $*B$. 101

A $*1$ *then* \wedge $*2$ *then* \wedge $*3$ *then* \wedge $*4$ 103

1 $V = \{d \mid d \in^{deep} x\}$. 105

2 $E = \{(d1, d2) \in V * V \mid d2 \in d1\}$. 106

3 $P = \{(v, |p|) \mid v \in V \text{ then } \wedge p \text{ is a shortest path from } x \text{ to } v \text{ on } (V, E)\}$. 107
108

4 P_{ord} is P with the order on $P * P$ as $*5$. 109

5 $((v1, n1) < (v2, n2)) \equiv n1 < n2$. 110

B $*6$ *then* \wedge $*7$ *then* \wedge $*8$ 111

6 (V, E) is said the deep graph of x . 113

- 7 Take $\forall(v, n) : \in P$. Then n is said the depth of v on x . 114
- 8 There exists $\exists(v, n) : \in P_{ord}$ such that " (v, n) is a maximum member of P_{ord} ". Then n is said "the maximum depth on x ". 115
116
- 117

7 Propositions 118

Definition 7.1. In this section, *Def refers to the definition titled as "Isomorphic memBers". 119
120

And *1 \equiv *2, without any explicit proof because it is trivial by *Def. 121

And *3 holds, without any explicit proof because it is trivial by *Def. 122

1 (x_i, y_i) are isomorphic by F_i . 123

2 *Def.A holds for (x_i, y_i, F_i) in place of (x, y, F) . 124

3 Take $\forall(x, y, F)$ such that (*4 $_{else} \vee$ *5). Then *6 holds. 125

4 $(space(x) = \emptyset \text{ then } \wedge x = y)$. 126

5 $((x, y)$ are points $\text{ then } \wedge (x, y) \in F)$. 127

6 (x, y) are isomorphic by F . ■ 128

Proposition 1 (Empty space). Take $\forall(x, y, F)$ such that *A1. Then $x = y$. 129

A1 *Def.A holds for $(x, y, F) \text{ then } \wedge space(X) = \emptyset$. 130

■ 131

Proof. 132

- Assume it is false. 133
- There exists $\exists(x, y, F)$ such that (x, y, F) is a minimum counterexample by the maximum depth on x . 134
135
- Let us follow *Def.A for (x, y, F) . 136
- At *0, it fails because $x \neq y$. 137
- At *1, it fails because $space(x) = \emptyset$. 138
- Hence (*2 $\text{ then } \wedge$ *3) holds. 139

- Consider f of *3 together with that $x \neq y$. 140
- Hence there exists $\exists(m1, m2) \in f$ such that 141
 $m1 \neq m2$ then \wedge *Def.A holds for $(m1, m2, F)$. 142
- Additionally $space(m1) \subset space(x) = \emptyset$. 143
- Moreover $m1 < x1$ compared by the order by their maximum depths on 144
them respectively. 145
- Hence $(m1, m2, F)$ is a counterexample smaller than a minimum coun- 146
terexample. 147
- The assumption is false. 148

□ 149

Proposition 2 (Members' isomorphisms as consequent). Take $\forall(x, y, F)$ such 150
that (*A1 then \wedge *A2). Then (*B1 then \wedge *B2) holds. 151

A1 *Def.A holds for (x, y, F) in place of $(x1, x2, F)$. 152

A2 F is an injection. 153

B1 Take $\forall m1 : \in^{deep} x$. Then there exists $\exists m2 : \in^{deep} y$ such that *Def.A holds 154
for $(m1, m2, F)$ in place of $(x1, x2, F)$. 155

B2 Take $\forall m2 : \in^{deep} y$. Then there exists $\exists m1 : \in^{deep} x$ such that *Def.A holds 156
for $(m1, m2, F)$ in place of $(x1, x2, F)$. ■ 157

*Proof of *B1.* 158

- Assume it is false. 159
- Then there exists $\exists(x, y, F, m1)$ such that $(x, y, F, m1)$ is a minimum coun- 160
terexample by the depth of $m1$ on x . 161
- It is trivial that $(x \neq m1)$. 162
- There exists $\exists x1$ such that $(m1 \in x1$ then \wedge $x1$ in place of $m1$ is not a 163
counterexample). 164
- Hence *B1 should hold for $x1$ in place of $m1$. 165
- Hence there exists $2 : \in^{deep} y$ such that *Def.A holds for $(x1, y2, F)$. 166
- Let us follow *Def.A for $(x1, y2, F)$. 167

- Assume $*0$ holds. 168
- Then $space(x1) = \emptyset \text{ then } \wedge x1 = y2$. 169
- Hence $space(m1) = \emptyset \text{ then } \wedge m1 = m1 \text{ then } \wedge m1 \in^{deep} y$. 170
- Hence $*B1$ holds for $m1$ in place of $m1$. 171
- Hence $(x, y, F, m1)$ is not a counterexample. 172
- Hence the last assumption is false. 173
- Assume $*1$ holds. 174
- Hence $(x1 \text{ is a point}) \text{ then } \wedge (m1 \in x1)$. 175
- Hence the last assumption is false. 176
- Hence $(*2 \text{ then } \wedge *3)$ must hold. 177
- Hence, $(*4 \text{ then } \wedge *5 \text{ then } \wedge *6)$ holds. 178
- Hence $*B1$ holds for $m1$ in place of $m1$. 179
- Hence $m1$ is not a counterexample. 180
- The first assumption is false. 181

□ 182

*Proof of $*B2$.* 183

- Assume it is false. 184
- Then there exists $\exists(x, y, F, m2)$ such that $(x, y, F, m2)$ is a minimum counterexample by the depth of $m2$ on y . 185
186
- It is trivial that $(y \neq m2)$. 187
- There exists $\exists y2$ such that $(m2 \in y2 \text{ then } \wedge y2 \text{ in place of } m2 \text{ is not a counterexample})$. 188
189
- Hence $*B2$ should hold for $y2$ in place of $m2$. 190
- Hence there exists $1 : \in^{deep} x$ such that $*Def.A$ holds for $(x1, y2, F)$. 191
- Let us follow $*Def.A$ for $(x1, y2, F)$. 192
- Assume $*0$ holds. 193

- Then $space(x1) = \emptyset \text{ then } \wedge x1 = y2$. 194
- Hence $space(m2) = \emptyset \text{ then } \wedge m2 = m2 \text{ then } \wedge m2 \in^{deep} x$. 195
- Hence *B2 holds for $m2$ in place of $m2$. 196
- Hence $(x, y, F, m2)$ is not a counterexample. 197
- Hence the last assumption is false. 198
- Assume *1 holds. 199
- Hence $(y2 \text{ is a point}) \text{ then } \wedge (m2 \in y2)$. 200
- Hence the last assumption is false. 201
- Hence $(*2 \text{ then } \wedge *3)$ must hold. 202
- Hence, $(*4 \text{ then } \wedge *5 \text{ then } \wedge *6)$ holds. 203
- Hence *B2 holds for $m2$ in place of $m2$. 204
- Hence $m2$ is not a counterexample. 205
- The first assumption is false. 206

□ 207

Proposition 3 (Surjectivity). Take $\forall(x, y, F)$ such that *A1. Then $F[space(x)]$ 208
is a surjection from *to $space(x) * space(y)$. 209

A1 *Def.A holds for (x, y, F) by $(*2 \text{ then } \wedge *3)$. 210

■ 211

Proof. 212

- Assume it is false. 213
- There exists $\exists(x, y, F)$ such that (x, y, F) is a counterexample. 214
- (there exists $\exists p_x : \in space(x)$ such that $p_x \notin domain(F)$) 215
 $\text{else } \vee$ 216
 (there exists $\exists p_y : \in space(y)$ such that $p_y \notin image(F)$). 217
- Though this logical disjunction contradicts to the proposition titled as 218
 "MemBers' isomorphisms as consequent" as follows. 219

- A contradiction for p_x : 220
- There exists $y2 : \in^{deep} y$ such that $*Def.A$ holds for $(p_x, y2, F)$. 221
- Let us follow $*Def.A$ for $(p_x, y2, F)$. Then $(*0 \text{ else } \vee (*2 \text{ then } \wedge *3))$ fails. 222
- Hence $*1$ holds. Hence $(p_x, y2) \in F$. Hence $p_x \in domain(F)$. A contra- 223
diction. 224
- A contradiction for p_y : 225
- There exists $x1 : \in^{deep} x$ such that $*Def.A$ holds for $(x1, p_y, F)$. 226
- Let us follow $*Def.A$ for $(x1, p_y, F)$. Then $(*0 \text{ else } \vee (*2 \text{ then } \wedge *3))$ fails. 227
- Hence $*1$ holds. Hence $(x1, p_y) \in F$. Hence $p_y \in image(F)$. A contra- 228
diction. 229
- Finally, the assumption is false. 230

□ 231

Proposition 4 (Symmetric). Take $\forall B$ such that B is a binary relation. Then 232
let B^{-1} denote $\{(b2, b1) \mid (b1, b2) \in B\}$. 233
Take $\forall(x, y, F)$. Then $*A1$ implies $*A2$. 234

A1 $Def.A$ holds for (x, y, F) . 235

A2 $Def.A$ holds for (y, x, F^{-1}) . 236

■ 237

Proof. 238

- Assume it is false. 239
- There exists $\exists(x, y, F)$ such that it is a minimum counterexample by the 240
maximum depth on x . 241
- Let us follow $*Def.A$ for (x, y, F) in terms of $*A1$. 242
- Assume $*0$ holds for (x, y, F) in terms of $*A1$. 243
- Hence $space(x) = \emptyset \text{ then } \wedge x = y$. 244
- Hence $space(y) = \emptyset \text{ then } \wedge y = x$. 245
- Hence $*0$ holds for (x, y, F) in terms of $*A2$. 246

- Hence the last assumption is false. 247
- Assume $*1$ holds for (x, y, F) in terms of $*A1$. 248
- Hence (x, y) are points $\text{then} \wedge (x, y) \in F$. 249
- Hence (y, x) are points $\text{then} \wedge (y, x) \in F^{-1}$. 250
- Hence $*1$ holds for (x, y, F) in terms of $*A2$. 251
- Hence the last assumption is false. 252
- Hence $(*2 \text{ then} \wedge *3)$ must hold for (x, y, F) in terms of $*A1$. 253
- Consider the proposition titled as "Surjectivity". 254
- Then $F[\text{space}(x)]$ is a bijection from $*to \text{space}(x) * \text{space}(y)$. 255
- Hence $F^{-1}[\text{space}(y)]$ is a bijection from $*to \text{space}(y) * \text{space}(x)$. 256
- Hence $*2$ holds for (x, y, F) in terms of $*A3$. 257
- Hence $*3$ must fail for (x, y, F) in terms of $*A3$. 258
- At same time, $*3$ hold for (x, y, F) in terms of $*A1$. 259
- Hence there exists $\exists(m1, m2) \in f$ such that (260
- $*\text{Def.A}$ holds for $(m1, m2, F) \text{ then} \wedge$ 261
- $*\text{Def.A}$ does not hold for $(m2, m1, F^{-1})$. 262
-). 263
- Hence $(m1, m2, F)$ is a counterexample. 264
- Moreover $(m1, m2, F) < (x, y, F)$ compared by the order by the maximum 265
depths on $m1$ and x respectively. 266
- It contradicts to the title of (x, y, F) as a minimum counterexample. 267
- Hence the first assumption is false. 268

□ 269

Proposition 5 (Members' isomorphisms as antecedent). Take $\forall(x, y, F, f)$ such 270
that $(*A1 \text{ then} \wedge *A2 \text{ then} \wedge *A3)$. Then $*B$ holds. 271

A1 F is an injection. 272

A2	f is a bijection from x^* to x^*y then $\wedge f \neq \emptyset$.	273
A3	Take $\forall(m1, m2) : \in f$. Then $^*\text{Def.A}$ holds for $(m1, m2, F)$.	274
B	$^*\text{Def.A}$ holds for (x, y, F) in place of (x, y, F) .	275
	<i>Proof.</i>	276
	• Assume B fails.	277
	• Hence there exists $\exists(x, y, F)$ such that $^*\text{Def.A}$ fails for (x, y, F) .	278
	• Let us follow $^*\text{Def.A}$ for (x, y, F) .	279
	• (*0 fails then $\wedge ^*1$ fails then $\wedge (^*2$ fails else $\vee ^*3$ fails)).	280
	• Assume *2 fails.	281
	• Hence $F[\text{space}(x)]$ is not an injection from x^* to $\text{space}(x) * \text{space}(y)$.	282
	• Consider the proposition titled as "MemBers' isomorphisms as consequent".	283 284
	• Then $(\text{space}(x) = \emptyset = \text{space}(y))$ else $\vee (\text{space}(x) \neq \emptyset \neq \text{space}(y))$.	285
	• Meanwhile F is an injection by *A1 .	286
	• Hence $(\text{space}(x) \neq \emptyset \neq \text{space}(y))$ because otherwise *2 holds.	287
	• Hence there exists $\exists(p_x, p_y) : \in \text{space}(x) * \text{space}(y)$ such that	288
	$p_x \notin \text{domain}(F)$ else $\vee p_y \notin \text{image}(F)$.	289
	• Consider $^*A2, ^*A3$ and the proposition titled as "Members' isomorphisms as the consequent".	290 291
	• There exists $\exists y2 \in^{deep} y$ such that $^*\text{Def.A}$ holds for $(p_x, y2, F)$.	292
	• There exists $\exists x1 \in^{deep} x$ such that $^*\text{Def.A}$ holds for $(x1, p_y, F)$.	293
	• Meanwhile, for each of the 2 lines just above,	294
	($^*\text{Def.A}$ holds only at *1) because	295
	(each of (p_x, p_y) is a point).	296
	• Hence $p_x \in \text{domain}(F)$ then $\wedge p_y \in \text{image}(F)$.	297
	• Hence the last assumption is false.	298
	• Hence *3 must fail.	299

• Hence *3 fails for f in place of f .	300
• Though by (*A2 $\text{ then } \wedge$ *A3), (*4 $\text{ then } \wedge$ *5 $\text{ then } \wedge$ *6) holds.	301
• Hence the first assumption is false.	302
□	303
Proposition 6 (Topological space). Take $\forall((X1, T1), (X2, T2))$ such that *A	304
holds. Then *B1 \Rightarrow *B2.	305
A Take $i \in \{1, 2\}$. Then (X_i, T_i) is a topological space.	306
B1 $((X1, T1), (X2, T2))$ are homeomorphic.	307
B2 There exists $\exists F$ such that $((X1, T1), (X2, T2))$ are isomorphic by F .	308
■	309
<i>Proof.</i>	310
First of all, let me define how to express a pair as a set.	311
Take $\forall(p1, p2)$. Then $(p1, p2) = \{Pair, p1, \{p1, p2\}\}$.	312
In the expression, "Pair" is a constant keyword.	313
By the way, *B1 implies *C.	314
	315
C There exists $\exists(G, g)$ such that (*C1 $\text{ then } \wedge$ *C2 $\text{ then } \wedge$ *C3 $\text{ then } \wedge$ *C4).	316
	317
C1 G is a bijection from $X1$ to $X2$.	318
C2 G is a homeomorphism for *B1.	319
C3 g is a bijection from $T1$ to $T2$.	320
C4 Take $\text{forall}(t1, t2) : \in g$. Then $(G \text{ takes } t1 \text{ to } t2)$.	321
	322
Consider the previous proposition titled as Members' isomorphisms as antecedent	323
and refer it as *P.	324
P accepts arguments as follows.	325
	326
D1 *P accepts $(X1, X2, G, G)$ in place of (x, y, F, f) .	327

D2	Take $\forall(t1, t2) :\in g$. Then *P accepts $(t1, t2, G, G)$ in place of (x, y, F, f) .	328
D3	*P accepts $(T1, T2, G, g)$ in place of (x, y, F, f) .	329
D4	*P accepts (<td>330</td>	330
	$\{X1, T1\}$,	331
	$\{X2, T2\}$,	332
	G ,	333
	$\{(X1, X2), (T1, T2)\}$	334
) in place of (x, y, F, f) .	335
D5	*P accepts (<td>336</td>	336
	$\{Pair, X1, \{X1, T1\}\}$,	337
	$\{Pair, X2, \{X2, T2\}\}$,	338
	G ,	339
	$\{(Pair, Pair), (X1, X2), (\{X1, T1\}, \{X2, T2\})\}$	340
) in place of (x, y, F, f) .	341
		342
	Hence *P implies $(*E1 \text{ then} \wedge *E2 \text{ then} \wedge *E3 \text{ then} \wedge *E4 \text{ then} \wedge *E5)$.	343
	Finally, *E5 implies this proposition.	344
		345
E1	$(X1, X2)$ are isomorphic by G .	346
E2	Take $\forall(t1, t2) :\in g$. Then $(t1, t2)$ are isomorphic by G .	347
E3	$(T1, T2)$ are isomorphic by G .	348
E4	$\{X1, T1\}, \{X2, T2\})$ are isomorphic by G .	349
E5	$(\{Pair, X1, \{X1, T1\}\}, \{Pair, X2, \{X2, T2\}\})$ are isomorphic by G .	350
		351
	□	352

References

353