Isomorphism between general objects

with a simple application in topological geometry

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1 Introduction

Prerequisites are only some first chapters of graduate level texts of general topology and graph theory; (1),(2). Though, in almost all sub areas of mathematics, there are expected valuable applications of the defined new notion.

This article defines when two given general objects are said isomorphic. For example, two numbers $\forall(x, y)$ are isomorphic if and only if $x = y$. Contrary **two points in the sense of elementary geometry are unconditionally said isomorphic**.

Meanwhile two topological spaces are said isomorphic if the two are homeomorphic and their points are points of elementary geometry.

So what to do if their points are not points of elementary geometry? To keep the texts simple, let us define for the rest that **all points of topology must be points of elementary geometry**. And "$\forall x$ is a point" is meant **in the sense of elementary geometry**.

Example.1:

- $T1 := \{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$.
- $T2 := \{t2 \mid \exists t1 :\in T1 \overset{\text{and}}{\wedge} t2 = \{ID(x) \mid x \in t1\ \}\ \}$.

Example.1 shows how to convert a topology $T1$ of non-points to a topology $T2$ of points. Namely **you convert all non-point $\forall x$ to a point** using a function

ID(x) which returns the identity of x. That is, it is defined that **"point" is an alias of "identity"**.

Let us continue to enumerate more examples of isomorphism. $(X1, p1, 321) \cong (X2, p2, 321)$ if $X1 \cong X2$ by some homeomorphism $\exists f$ and $f(p1) = p2$. $(x1, 321) \ncong (x2, 123)$ even if $x1 \cong x2$. $\{x, y\} \ncong (x, y)$ if $x \neq y$ because $(x, y) := \{\{x\}, \{x, y\}\}$.

Needless to say, $\cong$ can express more complex examples like $(X, F, k, p)_1 \cong (X, F, k, p)_2$ where $X_{\forall i}$ is a topological space, $F_{\forall i}$ is a set of some ambient isotopies on Space(X_i)*[0,1], $k_{\forall i}$ is an embedding into X_i and $p_{\forall i}$ is a point of X.

Moreover, $\cong$ can express that two logical expressions containing general objects are isomorphic because all logical expressions can be expressed as graphs.

2 Isomorphism between general objects

A topological space X is a pair as (Space(X), T) where Space(X) denotes the set of all points of X and T denotes the topology.

Definition 2.1 (Deep member)**.** Take $\forall(c, x, y)$ such that: c is a chain of set membership. x is the maximum member of c. y is a member of c. Then y is said a deep member of x and you write as $y \in^{deep} x$.

For example:

$$m \in ... \in y \in .... \in x$$

For example:

$$\{y1, y2\} \in^{deep} \{y1, y2\}$$
$$y \in^{deep} \{1, \{2, y\}\}$$

Definition 2.2 (Space)**.** Space($\forall X$) := $\{p \mid p \in^{deep} X \overset{\text{and}}{\wedge} p \text{ is a point } \}$.

Definition 2.3 (Nested graph)**.** A nested graph is a graph of which some vertices may be nested graphs. The graph isomorphism of nested graphs is defined to regard all vertices as points even if some vertex is a nested graph.

For example, two nest graphs, $(V, E)_{i \in \{1,2\}}$ as $V_1 := \{1\}$,$V_2 := \{(V_1, E_1)\}$, are graph isomorphic if $E_{\forall i}$ is empty.

Take $\forall G$ as a nested graph. If no vertex of G is a nested graph. Then the nest number of G is defined to be 0. Otherwise the nested number of G is defined to be $m+1$ where m denotes the maximum nested number among all its vertices which are nested graphs. And G is defined to have the finite nested number.

Definition 2.4 (Isomorphism between vertices of nested graphs)**.** Take $\forall(F, p_1, p_2)$ such that: F is a bijection between sets of points. (p_1, p_2) are vertices. Let S_F :=Space(domain(F) $\cup$ image(F)).

Define that: $*1 \equiv (*2 \overset{\text{or}}{\vee} ... \overset{\text{or}}{\vee} *5)$.

1. $p_1 \cong_F p_2$.

2. $F(p_1) = p_2$

3. $p_1 = p_2 \overset{\text{and}}{\wedge} \varnothing = \text{Space}(\{p_1, p_2\}) \cap S_F$.

4. take $\forall i \in \{1, 2\} \overset{\text{and}}{\wedge} \varnothing \neq \text{Space}(p_i) \cap S_F$.

5. If $p_{\forall i}$ is a nested graph then $p_1 \cong^F p_2$.

$p_1 \cong^F p_2$, is defined later.

Proposition 1. Take $\forall m :\in \mathbb{N}$. Let $\mathbb{G}_m$ denote the set of all nested graphs having nested numbers at most m. Then $*1 \to *2$.

1. For $\mathbb{G}_m$: the proposition 2 holds true.

2. For $\mathbb{G}_m$: $p1 \cong_F p2 \quad \equiv \quad p2 \cong_{F^{-1}} p1$.

Proof. Take $\forall(p1, p2, F)$ as a counterexample. Hence $(*2 \overset{\text{or}}{\vee} ... \overset{\text{or}}{\vee} *5)$ of the definition 2.4 holds for $(p1, p2, F)$ in place of $(p1, p2, F)$.

p1. Assume $(*2 \overset{\text{or}}{\vee} ... \overset{\text{or}}{\vee} *4)$ of the definition holds for $(p1, p2, F)$.

It is clear that each logical expression of the disjunctions are logically symmetric between $(p1, p2, F)$ and $(p2, p1, F^{-1})$. Hence each holds for $(p2, p1, F^{-1})$ in place of $(p1, p2, F)$. A contradiction.

p2. Assume $*5$ of the definition holds for $(p1, p2, F)$. By $*1$ of this proposition, the proposition 2 holds for $(p1, p2, F)$ in place of $(G1, G2, F)$ then $*5$ of the definition holds for $(p2, p1, F^{-1})$ too in place of $(p1, p2, F)$. A contradiction.

$\square$

Definition 2.5 (Isomorphism between nested graphs)**.** Take $\forall F$ as a bijection between sets of points. Take $\forall \{G_i\}_{i \in \{1,2\}}$ as a pair of nested graphs. Decompose G_i as $\exists (V, E)_i$.

Then F is said an isomorphism from G_1 to G_2 if $(*0 \overset{\text{and}}{\wedge} *1)$. Define that: $(*0 \overset{\text{and}}{\wedge} *1) \equiv *2$. And define that: $*2 \to *3$.

0. $\exists f$ as a graph isomorphism from G_1 to G_2.

1. Take $\forall v :\in V_1$, then $v \cong_F f(v)$.

2. $(G, C)_1 \cong^F (G, C)_2$.

3. $(G, C)_1 \cong (G, C)_2$.

■

Proposition 2. $G_1 \cong^F G_2. \quad \equiv \quad G_2 \cong^{F^{-1}} G_1.$

Proof. Let DefIsoGraph be an alias for the definition 2.5.

Take $\forall (G_1, G_2, F)$ as a minimum counterexample compared by the maximum nest number for G_1. Though at least F^{-1} is a bijection between sets of points. Hence the antecedent of DefIsoGraph holds for (G_2, G_1, F^{-1}) in place of (G_1, G_2, F) except $(*0 \overset{\text{and}}{\wedge} *1)$ of DefIsoGraph.

Hence $*1$ of DefIsoGraph fails for $(G_2, G_1, F^{-1}, f^{-1})$ in place of (G_1, G_2, F, f).

q1. Hence: $\exists v :\in V_2 \overset{\text{and}}{\wedge} \neg(\ v \cong_{F^{-1}} f^{-1}(v)\ )$.

q2. Though: $(\ G_1 \cong^F G_2\ ) \to (\ f^{-1}(v) \cong_F f \circ f^{-1}(v)\ )$.

Let $\mathrm{PropCong}_F$ be an alias for the proof of the proposition 1.

The right term of $*$q2 implies $(*\text{p1} \overset{\text{or}}{\vee} *\text{p2})$ of $\mathrm{PropCong}_F$.

If $*$p1 of $\mathrm{PropCong}_F$ then $(*\text{q1} \overset{\text{and}}{\wedge} *\text{q2})$ is a contradiction.

Hence $*$p2 of $\mathrm{PropCong}_F$.

By the way, consider the right term of $*$q2; $(f^{-1}(v), v, F)$ is smaller than a minimum counterexample. Hence $(*\text{q1} \overset{\text{and}}{\wedge} *\text{q2})$ contradicts to $\mathrm{PropCong}_F$. $\square$

Proposition 3. Take $\forall (X_1, X_2, F)$ such that: (X_1, X_2) are topological spaces homeomorhic by F as a homeomorphism from $\mathrm{Space}(X_1)$ to $\mathrm{Space}(X_2)$. Then $X1 \cong^F X2$.

Proof. You can express X_i as an graph G_i as follows so that G_i is equivalent to the definition of X_i. Meanwhile let MainDef be an alias for the definition 2.5. I show that the antecedent of MainDef holds for $(F, G_{i\in\{1,2\}})$ in place of $(F, G_{i\in\{1,2\}})$.

In the rest, indices may be omitted if the meanings are clear.

Decompose X as $X := (\text{Space}(X), \exists T)$.

Define toG($\forall S$) as a function to return a graph (V, E) as follows.

$V := \{v \mid \exists d \in S \overset{\text{and}}{\wedge}$ if d has some member then v=toG(d) else $v = d$ $\} \cup \{S\}$.

$E := \{(d1, d2) \in V^2 \mid d1 \in d2\ \}$.

Then toG(Space(X_1) $\cong^F$ toG(Space(X_2).

As a proof, a graph isomorphism can be defined as:

$f := F \cup \{(\text{Space}(X_1), \text{Space}(X_2))\}$.

It is trivial that f is a graph isomorphism between them.

By (*2 $\overset{\text{or}}{\vee}$ *4) of the definition 2.4, $\forall v \cong_F f(v)$.

Let $T_{12} := \{(t1, t2) :\in T_1 * T_2 \mid F$ takes $t1$ to $t2$ $\}$.

Take $\forall(t1, t2) :\in T_{12}$. Then toG($t_1$) $\cong^F$ toG(t_2).

It is clear that an analogous proof exists.

Moreover toG(T_1) $\cong^F$ toG(T_2).

As a proof, a graph isomorphism can be defined as:

Let $T_{12} := \{($ toG($t1$), toG($t2$)) $\mid (t1, t2) \in T_{12}\ \}$

$f := \{(T_1,\ T_2)\} \cup T_{12}$.

It is trivial that f is a graph isomorphism between them.

By (*4 $\overset{\text{or}}{\vee}$ *5) of the definition 2.4, $\forall v \cong_F f(v)$.

Finally toG(X_1) $\cong^F$ toG(X_2).

As a proof, a graph isomorphism can be defined as:

$f := \{(X_1,\ X_2)\} \cup$

$\{($ toG(Space(X_1)), toG(Space(X_2)) $)\} \cup$

$\{($ toG(T_1), toG(T_2) $)\}$.

It is trivial that f is a graph isomorphism between them.

By (*4 $\overset{\text{or}}{\vee}$ *5) of the definition 2.4, $\forall v \cong_F f(v)$.

□

3 Applications in geometrical topology

Take $\forall(X, S, F)$ such that: X is a topological space on a set of points called Space(X). S is a set of sub spaces of X. F is the set to collect: $\forall f$: Space(X)*[0,1] → Space(X) as an ambient isotopy. That is, F is the set of all such ambient isotopies.

For ideal cases, S can be regarded as a topological space Y. To find such Y, we define a set P of paths in S as follows.

P collects $\forall p : [0,1] \to S$ such that: $\exists s :\in S$, $\exists f :\in F \overset{\text{and}}{\wedge}$ take $\forall t :\in [0,1] \overset{\text{and}}{\wedge} \exists s_t :\in S \overset{\text{and}}{\wedge} s_{t=0} = s \overset{\text{and}}{\wedge} p(t) = s_t \overset{\text{and}}{\wedge}$ image(f[Space(s)*$\{t\}$])=Space(s_t). Denote P as Path(S). Recall the function ID($\forall x$) which returns the identity of x. Using ID(), give some change on Path(). Make sure that the input of Path() are sets of sub spaces, e.g., S. Namely, new Path(S) := $\{q$ | *1 $\overset{\text{and}}{\wedge}$ ... $\overset{\text{and}}{\wedge}$ *3 $\}$. In the rest, Path() refers to this new Path().

1. q: [0,1] → image(toP[S]).

2. $\exists p \in$ old Path(S).

3. Take $\forall t :\in [0,1]$. Then $q(t) = (\mathrm{ID} \circ p)(t)$.

Let us check if Path(S) is enough large. To do it, we use the trivial set T_X of sub spaces of X; that is T_X=$\{\{x\}$ | $x \in$ Space(X) $\}$.

Take $\forall S$ as an instance of S such that | Path(S) | $\geq$ | Path(T_X) |. Take $\forall Y$ such that: (Path(S) $\cong$ Path(T_Y)) where T_Y denotes the trivial set of sub spaces of Y. Then we can expect that S is possibly equivalent to Y.

Let us check if Y is enough symmetric. If members of Space(Y) are pairwise Y-ambient-isomorphic then S is said **ideal**.

Definition 3.1 (Ambient-isomorphic)**.**

Take $\forall(X, S, K)$ such that: X is a topological space $\overset{\text{and}}{\wedge}$ S is a set of sub spaces of X. Take $\forall(s1, s2, s3) :\in S^3$. If $\exists s4 :\in S$ such that $(K, s4, s1) \cong (K, s4, s2)$ then write: $(s1 \cong^S_K s2)$. If $s1 \cong^S_K s2 \overset{\text{and}}{\wedge} s2 \cong^S_K s3$ then write: $s1 \cong^S_K s3$. Members of S are pairwise said K**-ambient-isomorphic** if: Take $\forall(s1, s2) :\in S^2$, then $s1 \cong^S_K s2$. ■

Take $\forall(X, S, F, A)$ such that: S is an ideal set of sub spaces of X. F is the set to collect: $\forall f$: Space(X)*[0,1] → Space(X). A is the set of all automorphisms on S. Speaking informally, there is no room to make S more symmetric. So it is natural to expect that the following paragraph is true.

There exists $\exists F_A :\subset F$ and the following (*1 $\overset{\text{and}}{\wedge}$ *2) holds for F_A. Moreover (*1 $\overset{\text{and}}{\wedge}$ *3) also holds for F_A.

1. Take $\forall f :\in F_A$.

2. $\exists g :\in A$ $\overset{\text{and}}{\wedge}$ take $\forall (s_0, s_1) :\in g$ $\overset{\text{and}}{\wedge}$

image(f[Space(s_0) $* \{1\}$] =Space(s_1).

3. Take $\forall t :\in (0,1]$, $\forall f_t :\in F$ such that $f_t(\forall x, \forall r) = f(x, t * r)$.

Then $f_t \in F_A$.

Let us shift the subject to how to find an ideal set S of sub spaces of a topological space X. It is natural to expect that if (X, S) are enough or absolutely symmetric then S is ideal.

I conjecture that: (*1 $\overset{\text{and}}{\wedge}$ ... $\overset{\text{and}}{\wedge}$ *3) $\rightarrow$ (S is ideal). **Conjecture 1.1.**

Define that: *1 $\rightarrow$ (X is a **prime topological space**).

1. The trivial set of sub spaces of X is ideal.

2. Members of Space(X) are pairwise (X, S)-ambient-isomorphic.

3. Let $S_p := \{(s, p) \mid s \in S$ $\overset{\text{and}}{\wedge}$ $p \in$ Space(s) $\}$.

Then Members of S_p are pairwise (X, S)-ambient-isomorphic.

References

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