

# Isomorphism of memBers 1

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## 1 Notation 5

**Definition 1.1** (Pair). Take  $\forall\{x, y\}$  such that  $x \neq y$ . Then  $(x, y) \neq (y, x)$ . 6

■ 7

## 2 Introduction 8

**Definition 2.1** (memBer). Take  $\forall x$  such that (there exists  $\exists S$  such that  $x \in S$ ). 9

Then  $x$  is said a memBer. 10

■ 11

This article:(1) defines a binary relation that (x,y) as memBers are isomor- 12  
phic,(2) proves that the binary relation is an equivalence relation, (3) proves 13  
that all homeomorphic topological spaces are isomorphic as memBers,(4) de- 14  
fines that a memBer  $S1$  is a minor of a memBer  $S2$ . 15

I expect that readers will realize that the newly defined isomorphisms are 16  
somehow more fundamental than ,e.g., homeomorphisms of topological spaces. 17  
Because "homeomorphisms" logically resolve to "isomorphisms of memBers" 18  
whereas the inverse of it does not hold. 19

## 3 Deep member 20

**Definition 3.1** (Deep member of memBer). This definition uses a style of 21  
recursion. 22

Take  $\forall(x, y)$  such that \*1 holds. Then it is said as \*2 and written as \*3. 23

1  $x = y$  else (there exists  $\exists z$  such that  $x \in z \in^{\geq 0} y$ ). 24

<b>2</b>	$x$ is a deep member of $y$	25
<b>3</b>	$x \in^{\geq 0} y$	26
		27
<b>Definition 3.2</b>	(Space of memBer). Take $\forall(x, y)$ such that *1 holds. Then it is said as *2.	28 29
<b>1</b>	$y = \{d \mid d \in^{\geq 0} x \text{ then } d \text{ is a point } \}$ .	30
<b>2</b>	$y$ is the space of $x$ .	31
		32
<b>Definition 3.3</b>	(Proper deep member of memBer). Take $\forall x$ such that *1 holds. Then it is said as *2 and written as *3.	33 34
<b>1</b>	There exists $\exists z$ such that $x \in^{\geq 0} z \in y$ .	35
<b>2</b>	$x$ is a proper deep member of $y$ .	36
<b>3</b>	$x \in^{\neq 0} y$ .	37
		38
<b>Definition 3.4</b>	(Paradoxical memBer). Take $\forall x$ such that (*1 else *2) holds. Then it is said as *2.	39 40
<b>1</b>	" $\{ d \mid d \in^{\geq 0} x \}$ " is not definable.	41
<b>2</b>	$x$ is a proper deep member of $x$ .	42
<b>3</b>	$x$ is paradoxical.	43 44
<b>Definition 3.5</b>	(Non-paradoxical memBer). In the rest, informally speaking, " $\forall x$ " must be interpreted as " $\forall x$ which is not paradoxical". Formally, in the rest, the domain of discourse does not include any paradoxical memBer.	45 46 47 48
<b>4</b>	<b>Notations</b>	49
<b>Definition 4.1</b>	(Symbols). *1 $\equiv$ *2 and *3 $\equiv$ *4.	50
<b>1</b>	$A$ then $B$ .	51

<b>2</b>	$A \text{ then } \wedge B.$	52
<b>3</b>	$A \text{ else } B.$	53
<b>4</b>	$A \text{ else } \vee B.$	54
	As a remark, *1 says that $((A \wedge B)$ and (the meaning of $B$ may depend on that $A$ holds)).	55 56
	As a remark, *3 says that $((A \vee B)$ and (the meaning of $B$ may depend on that $A$ fails)).	57 58
	<b>Definition 4.2</b> (Propositional function). Following *1 is true because *1 calls *2 passing integer 123 as the value of $x$ . In other words, *2 is a propositional function.	59 60 61
		62
<b>1</b>	Let $x:=123$ then *2 holds.	63
<b>2</b>	$x \in Z.$	64
		65
	<b>Definition 4.3</b> (Colon). A colon(:) may be used in introducing a new variable. Some examples follow.	66 67
		68
<b>1</b>	$x:=1.$	69
<b>2</b>	$\forall x : \in S.$	70
		71
	<b>Definition 4.4</b> (Restriction of binary relation). Take $\forall(L, X, Y, X1)$ such that *1 holds. Then define *2.	72 73
<b>1</b>	$L$ is a binary relation on $X * Y \text{ then } \wedge X1 \subset X.$	74
<b>2</b>	$L[X1] := \{ (x, y) \in L \mid x \in X1 \}.$	75

## 5 Isomorphic memBers 76

**Definition 5.1** (Isomorphic memBers). This definition uses a style of recursion. 77

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Take  $\forall(x, y, F)$  such that  $*A$  holds. Then define  $*B$ . 79

**A**  $*0 \text{ }_{else} \vee *1 \text{ }_{else} \vee (*2 \text{ }_{then} \wedge *3)$ . 80

81

**0**  $space(x) = \emptyset \text{ }_{then} \wedge x = y$ . 82

**1**  $(x, y)$  are points  $\text{ }_{then} \wedge (x, y) \in F$  83

**2**  $F[space(x)]$  is an <sup>1</sup>injection from  $*to \text{ } space(x) * space(y)$ . 84

**3** There exists  $\exists f$  such that  $(*4 \text{ }_{then} \wedge *5 \text{ }_{then} \wedge *6)$ . 85

**4**  $f$  is a bijection from  $*to \text{ } x * y \text{ }_{then} \wedge f \neq \emptyset$ . 86

**5** Take  $\forall(m1, m2) \in f$ . 87

**6**  $*A$  holds for  $(m1, m2, F)$  in place of  $(x, y, F)$ ). 88

**B**  $(x, y)$  are said isomorphic by  $F$  as an isomorphism. 89

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<sup>1</sup>Some alternative definitions are to weaken  $*2$  by replacing "injection" with "function" or "binary relation". Though such weakend condition should not be titled as "isomorphic".

## 6 Minors of memBers 90

**Definition 6.1** (Minors). Take  $\forall(x, y)$  such that  $*A$  holds. Then it is said as 91  
 $*B$ . 92

**A**  $*1 \text{ then} \wedge *2$ . 93

**1** Take  $\forall d$ . Then  $d \in^{\geq 0} x \Rightarrow d \in^{\geq 0} y$ . 94

**2** Take  $\forall(d1, d2, d3)$  such that (take  $\forall d \in \{d1, d2, d3\}$ , then  $d \in^{\geq 0} x$ ). 95

Then  $(*3 \Leftarrow *4)$ . 96

**3**  $((x, d1, d3), (x, d2, d3))$  are isomorphic. 97

**4**  $((y, d1, d3), (y, d2, d3))$  are isomorphic. 98

**B**  $*5 \text{ then} \wedge *6$ . 99

**5**  $x$  is a minor of  $y$ . 100

**6**  $x \leq^{minor} y$ . 101

■ 102

## 7 Deep graph 103

**Definition 7.1** (Deep graph). Take  $\forall(x, V, E)$  such that  $*A1 \text{ then} \wedge *A2$  holds. 104  
Then define  $*B$ . 105

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**A1**  $V = \{d \mid d \in^{\geq 0} x\}$ . 108

**A2**  $E = \{(d1, d2) \in V * V \mid d2 \in d1\}$ . 109

**B**  $(V, E)$  is said the deep graph of  $x$ . 110

■ 111

**Definition 7.2** (Depth of deep member). 112

It is defined as  $*1 \text{ then} \wedge *2$ . 113

It is defined as  $*5 \text{ then} \wedge *6$ . 114

It is defined as  $*7 \text{ then} \wedge *8$ . 115

**1** Take  $\forall(x, y, z)$  such that  $z \in^{\geq 0} y \in^{\geq 0} x$ . 116

**2** Let  $down(x) := \{p \mid *3 \text{ then} \wedge *4\}$ . 117

- 3  $p$  is a path on the deep graph of  $x$ . 118
- 4  $p$  is a <sup>2</sup>directed path from  $x$ . 119
- 5 Let  $down(y, x) := \{p \mid p \in down(x) \text{ then } \wedge p \text{ has } y \text{ as the last vertex}\}$ . 120
- 6 Let  $down(z, y, x) := \{p \mid p \in down(z, x) \text{ then } \wedge p \text{ has } y \text{ as a vertex}\}$ . 121
- 7 Take  $\forall(x, D, n)$  such that 122  
 $n$  is the maximum member of a subset  $D$  of  $down(x)$ . 123
- 8 Let  $depth(D) := n$ . 124

■ 125

**Notation rule 7.1.** Unitl redefined, let  $\#$  be a <sup>3</sup>short for  $depth(\ )$ . 126

**Proposition 1** (Depths on memBer). Take  $\forall(x, y, z)$  such that  $z \in y \in^{\geq 0} x$ . 127  
Then  $\#down(z, x) > \#down(y, x)$ . 128

■ 129

*Proof.* 130

- Assume it is false. 131
- There exists  $\exists(x, y, z)$  such that it is a counterexample. 132
- Hence  $\#down(z, x) \leq \#down(y, x)$ . 133
- Hence  $\#down(z, x) \geq \#down(z, y, x) > \#down(y, x) \geq \#down(z, x)$ . 134
- The assumption is false. 135

□ 136

**Proposition 2** (Maximum depth on memBer). Take  $\forall(x, y)$  such that  $y \in x$ . 137  
Then  $\#down(y) < \#down(x)$ . ■ 138

*Proof.* 139

- Assume it is false. 140
- There exists  $\exists(x, y)$  such that it is a counterexample. 141
- Hence  $\#down(y) \geq \#down(x)$ . 142

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<sup>2</sup>The edges are all oriented in the same direction.

<sup>3</sup>For example,  $\#D = depth(D)$ .

- There exists  $\exists v : \in y$  such that ( 143  
 $\#down(y) = \#down(v, y) \geq \#down(v, y, x) \leq \#down(x)$  144  
 $)$ . 145
  - Though  $\#down(v, y) + 1 = \#down(v, y, x)$ . 146
  - The assumption is false. 147
- 148

## 8 Propositions 149

**Definition 8.1.** In this section, \*Def refers to the definition titled as "Isomorphic memBers". 150

And \*1  $\equiv$  \*2, without any explicit proof because it is trivial by \*Def. 152

And \*3 holds, without any explicit proof because it is trivial by \*Def. 153

**1**  $(x_i, y_i)$  are isomorphic by  $F_i$ . 154

**2** \*Def.A holds for  $(x_i, y_i, F_i)$  in place of  $(x, y, F)$ . 155

**3** Take  $\forall(x, y, F)$  such that  $(*4 \text{ else } \vee *5)$ . Then \*6 holds. 156

**4**  $(space(x) = \emptyset \text{ then } \wedge x = y)$ . 157

**5**  $((x, y) \text{ are points then } \wedge (x, y) \in F)$ . 158

**6**  $(x, y)$  are isomorphic by  $F$ . ■ 159

**Proposition 3** (Restriction). Take  $\forall(x, y, F1, F2)$  such that  $(*A1 \text{ then } \wedge *A2)$  160  
holds. Then \*B holds. 161

**A1** Def.A holds for  $(x, y, F1)$ . 162

**A2**  $F1[space(x)] = F2[space(x)]$ . 163

**B** Def.A holds for  $(x, y, F2)$ . 164

■ 165

*Proof.* 166

- Assume it is false. 167
- There exists  $\exists(x, y, F1, F2)$  such that it is a minimum counterexample by 168  
 $\#down(x)$ . 169

- Let us follow \*Def.A for  $(x, y, F1)$ . 170
- Assume \*0 holds. 171
- Hence  $space(x) = \emptyset$  then  $\wedge x = y$ . 172
- Then \*0 holds for  $(x, y, F2)$ . 173
- The last assumption is false. 174
- Assume \*1 holds. 175
- Hence  $(x, y)$  are points then  $\wedge (x, y) \in F1$ . 176
- Then \*1 holds for  $(x, y, F2)$ . 177
- The last assumption is false. 178
- Then (\*2 then  $\wedge$  \*3) holds. 179
- Hence \*2 holds for  $(x, y, F2)$ . 180
- Hence \*3 fails for  $(x, y, F2)$ . 181
- Hence there exists  $\exists(m1, m2) \in f$  such that 182
- \*Def.A holds for  $(m1, m2, F1)$  then  $\wedge$  \*Def.A holds for  $(m1, m2, F2)$ . 183
- Hence  $(m1, m2, F1, F2)$  is a counterexample smaller than  $(x, y, F1, F2)$ . 184
- The first assumption is false. 185

□ 186

**Proposition 4** (Empty space). Take  $\forall(x, y, F)$  such that \*A1. Then  $x = y$ . 187

**A1** \*Def.A holds for  $(x, y, F)$  then  $\wedge space(x) = \emptyset$ . 188

■ 189

*Proof.* 190

- Assume it is false. 191
- There exists  $\exists(x, y, F)$  such that  $(x, y, F)$  is a minimum counterexample 192  
by  $\#down(x)$ . 193
- Let us follow \*Def.A for  $(x, y, F)$ . 194



• At *0, it fails because $x \neq y$ .	195
• At *1, it fails because $space(x) = \emptyset$ .	196
• Hence $(*2 \text{ then } \wedge *3)$ holds.	197
• Consider $f$ of *3 together with that $x \neq y$ .	198
• Hence there exists $\exists(m1, m2) \in f$ such that	199
$m1 \neq m2 \text{ then } \wedge *Def.A$ holds for $(m1, m2, F)$ .	200
• Additionally $space(m1) \subset space(x) = \emptyset$ .	201
• Moreover $\#down(m1) < \#down(x)$ .	202
• Hence $(m1, m2, F)$ is a counterexample smaller than a minimum coun-	203
terexample.	204
• The assumption is false.	205
	□ 206
<b>Proposition 5</b> (Members' isomorphisms as consequent). Take $\forall(x, y, F)$ such	207
that $(*A1 \text{ then } \wedge *A2)$ . Then $(*B1 \text{ then } \wedge *B2)$ holds.	208
<b>A1</b> *Def.A holds for $(x, y, F)$ in place of $(x1, x2, F)$ .	209
<b>A2</b> $F$ is an injection.	210
<b>B1</b> Take $\forall m1 : \in^{\geq 0} x$ . Then there exists $\exists m2 : \in^{\geq 0} y$ such that *Def.A holds	211
for $(m1, m2, F)$ in place of $(x1, x2, F)$ .	212
<b>B2</b> Take $\forall m2 : \in^{\geq 0} y$ . Then there exists $\exists m1 : \in^{\geq 0} x$ such that *Def.A holds	213
for $(m1, m2, F)$ in place of $(x1, x2, F)$ . ■	214
<i>Proof of *B1.</i>	215
• Assume it is false.	216
• Then there exists $\exists(x, y, F, m1)$ such that $(x, y, F, m1)$ is a minimum coun-	217
terexample by $\#down(m1, x)$ .	218
• It is trivial that $(x \neq m1)$ .	219
• Consider the proposition titled as "Depth of deep member".	220
• There exists $\exists x1$ such that $(m1 \in x1 \text{ then } \wedge x1$ in place of $m1$ is not a	221
counterexample).	222

- Hence  $*B1$  should hold for  $x1$  in place of  $m1$ . 223
- Hence there exists  $2 : \in^{\geq 0} y$  such that  $*Def.A$  holds for  $(x1, y2, F)$ . 224
- Let us follow  $*Def.A$  for  $(x1, y2, F)$ . 225
- Assume  $*0$  holds. 226
- Then  $space(x1) = \emptyset \text{ then } \wedge x1 = y2$ . 227
- Hence  $space(m1) = \emptyset \text{ then } \wedge m1 = m1 \text{ then } \wedge m1 \in^{\geq 0} y$ . 228
- Hence  $*B1$  holds for  $m1$  in place of  $m1$ . 229
- Hence  $(x, y, F, m1)$  is not a counterexample. 230
- Hence the last assumption is false. 231
- Assume  $*1$  holds. 232
- Hence  $(x1 \text{ is a point}) \text{ then } \wedge (m1 \in x1)$ . 233
- Hence the last assumption is false. 234
- Hence  $(*2 \text{ then } \wedge *3)$  must hold. 235
- Hence,  $(*4 \text{ then } \wedge *5 \text{ then } \wedge *6)$  holds. 236
- Hence  $*B1$  holds for  $m1$  in place of  $m1$ . 237
- Hence  $m1$  is not a counterexample. 238
- The first assumption is false. 239

□ 240

*Proof of  $*B2$ .* 241

- Assume it is false. 242
- Then there exists  $\exists(x, y, F, m2)$  such that  $(x, y, F, m2)$  is a minimum coun- 243  
terexample by  $\#down(m2, y)$ . 244
- It is trivial that  $(y \neq m2)$ . 245
- There exists  $\exists y2$  such that  $(m2 \in y2 \text{ then } \wedge y2 \text{ in place of } m2 \text{ is not a}$  246  
counterexample). 247
- Hence  $*B2$  should hold for  $y2$  in place of  $m2$ . 248

- Hence there exists  $1 : \in^{\geq 0} x$  such that  $*\text{Def.A}$  holds for  $(x1, y2, F)$ . 249
- Let us follow  $*\text{Def.A}$  for  $(x1, y2, F)$ . 250
- Assume  $*0$  holds. 251
- Then  $\text{space}(x1) = \emptyset \text{ then } \wedge x1 = y2$ . 252
- Hence  $\text{space}(m2) = \emptyset \text{ then } \wedge m2 = m2 \text{ then } \wedge m2 \in^{\geq 0} x$ . 253
- Hence  $*B2$  holds for  $m2$  in place of  $m2$ . 254
- Hence  $(x, y, F, m2)$  is not a counterexample. 255
- Hence the last assumption is false. 256
- Assume  $*1$  holds. 257
- Hence  $(y2 \text{ is a point}) \text{ then } \wedge (m2 \in y2)$ . 258
- Hence the last assumption is false. 259
- Hence  $(*2 \text{ then } \wedge *3)$  must hold. 260
- Hence,  $(*4 \text{ then } \wedge *5 \text{ then } \wedge *6)$  holds. 261
- Hence  $*B2$  holds for  $m2$  in place of  $m2$ . 262
- Hence  $m2$  is not a counterexample. 263
- The first assumption is false. 264

□ 265

**Proposition 6** (Surjectivity). Take  $\forall(x, y, F)$  such that  $*A1$ . Then  $F[\text{space}(x)]$  266  
is a surjection from  $*$  to  $\text{space}(x) * \text{space}(y)$ . 267

**A1**  $*\text{Def.A}$  holds for  $(x, y, F)$  by  $(*2 \text{ then } \wedge *3)$ . 268

■ 269

*Proof.* 270

- Assume it is false. 271
- There exists  $\exists(x, y, F)$  such that  $(x, y, F)$  is a counterexample. 272
- (there exists  $\exists p_x : \in \text{space}(x)$  such that  $p_x \notin \text{domain}(F)$  ) 273  
 $\text{else } \vee$  274  
 (there exists  $\exists p_y : \in \text{space}(y)$  such that  $p_y \notin \text{image}(F)$  ). 275

- Though this logical disjunction contradicts to the proposition titled as "MemBers' isomorphisms as consequent" as follows.
 

276  
277
- A contradiction for  $p_x$ :
 

278
- There exists  $y2 : \in^{\geq 0} y$  such that \*Def.A holds for  $(p_x, y2, F)$ .
 

279
- Let us follow \*Def.A for  $(p_x, y2, F)$ . Then  $(*0 \text{ else } \vee (*2 \text{ then } \wedge *3))$  fails.
 

280
- Hence \*1 holds. Hence  $(p_x, y2) \in F$ . Hence  $p_x \in \text{domain}(F)$ . A contradiction.
 

281  
282
- A contradiction for  $p_y$ :
 

283
- There exists  $x1 : \in^{\geq 0} x$  such that \*Def.A holds for  $(x1, p_y, F)$ .
 

284
- Let us follow \*Def.A for  $(x1, p_y, F)$ . Then  $(*0 \text{ else } \vee (*2 \text{ then } \wedge *3))$  fails.
 

285
- Hence \*1 holds. Hence  $(x1, p_y) \in F$ . Hence  $p_y \in \text{image}(F)$ . A contradiction.
 

286  
287
- Finally, the assumption is false.
 

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□ 289

**Proposition 7** (Symmetric property). Take  $\forall B$  such that  $B$  is a binary relation. Then let  $B^{-1}$  denote  $\{(b2, b1) \mid (b1, b2) \in B\}$ . Take  $\forall(x, y, F)$ . Then \*A1 implies \*A2.

**A1** Def.A holds for  $(x, y, F)$ . 293

**A2** Def.A holds for  $(y, x, F^{-1})$ . 294

■ 295

*Proof.* 296

- Assume it is false.
 

297
- There exists  $\exists(x, y, F)$  such that it is a minimum counterexample by  $\#down(x)$ .
 

298  
299
- Let us follow \*Def.A for  $(x, y, F)$  in terms of \*A1.
 

300
- Assume \*0 holds for  $(x, y, F)$  in terms of \*A1.
 

301
- Hence  $\text{space}(x) = \emptyset \text{ then } \wedge x = y$ .
 

302

- Hence  $space(y) = \emptyset$  then  $\wedge y = x$ . 303
- Hence \*0 holds for  $(x, y, F)$  in terms of \*A2. 304
- Hence the last assumption is false. 305
- Assume \*1 holds for  $(x, y, F)$  in terms of \*A1. 306
- Hence  $(x, y)$  are points then  $\wedge (x, y) \in F$ . 307
- Hence  $(y, x)$  are points then  $\wedge (y, x) \in F^{-1}$ . 308
- Hence \*1 holds for  $(x, y, F)$  in terms of \*A2. 309
- Hence the last assumption is false. 310
- Hence (\*2 then  $\wedge$  \*3) must hold for  $(x, y, F)$  in terms of \*A1. 311
- Consider the proposition titled as "Surjectivity". 312
- Then  $F[space(x)]$  is a bijection from \*to  $space(x) * space(y)$ . 313
- Hence  $F^{-1}[space(y)]$  is a bijection from \*to  $space(y) * space(x)$ . 314
- Hence \*2 holds for  $(x, y, F)$  in terms of \*A3. 315
- Hence \*3 must fail for  $(x, y, F)$  in terms of \*A3. 316
- At same time, \*3 hold for  $(x, y, F)$  in terms of \*A1. 317
- Hence there exists  $\exists(m1, m2) \in f$  such that ( 318
- \*Def.A holds for  $(m1, m2, F)$  then  $\wedge$  319
- \*Def.A does not hold for  $(m2, m1, F^{-1})$ . 320
- ). 321
- Hence  $(m1, m2, F)$  is a counterexample. 322
- Moreover  $\#down(m1) < \#down(x)$ . 323
- It contradicts to the title of  $(x, y, F)$  as a minimum counterexample. 324
- Hence the first assumption is false. 325

□ 326

**Proposition 8** (Reflexive property). Take  $\forall(x, F)$  such that \*A holds. Then 327  
 \*B holds. 328

<b>A</b>	$F$ is the identity function on $space(x)$ .	329
<b>B</b>	Def.A holds for $(x, x, F)$ .	330
		■ 331
	<i>Proof.</i>	332
•	Assume it is false.	333
•	There exists $\exists(x, F)$ such that it is a minimum counterexample by $\#down(x)$ .	334
•	Let us follow *Def.A for $(x, x, F)$ .	335
	At *0, assume $space(x) = \emptyset$ .	336
	Then *0 holds.	337
	The last assumption is false.	338
	At *1, assume $(x, x)$ are points.	339
	Then *1 holds.	340
	The last assumption is false.	341
	It is trivial that *2 holds.	342
	Let $f1$ be the identity function on $x$ .	343
	Then *3 must fail for $f1$ in place of $f$ .	344
	Though *4 holds.	345
	Hence $(*5 \text{ then } \wedge *6)$ must fail.	346
	Hence there exists $\exists(m1, m1) : \in f1$ such that *Def.A fails for $(m1, m1, F)$ .	347
	Meanwhile *Def.A holds for $(m1, m1, F[space(m1)])$ because otherwise	348
	$(m1, F[space(m1)])$ would be a counterexample smaller than a minimum	349
	counterexample, $(x, F)$ .	350
	Though consider the proposition titled as "Restriction".	351
	Then *Def.A holds for $(m1, m1, F)$ .	352
	The first assumption is false.	353
		□ 354
<b>Proposition 9</b>	(Transitive property). Take $\forall(B1, B2)$ such that $(B1, B2)$ are	355
	binary relations. Then let $B2 \circ B1$ denote $\{(b1, b3) \mid \exists b2 \text{ such that } (b1, b2) \in B1$	356
	$\text{ then } \wedge (b1, b2) \in B1 \}$ .	357
	Take $\forall(x, y, z, F1, F2)$ such that $(*A1 \text{ then } \wedge *A2)$ holds. Then *B holds.	358
<b>A1</b>	Def.A holds for $(x, y, F1)$ .	359
<b>A2</b>	Def.A holds for $(y, z, F2)$ .	360
<b>B</b>	Def.A holds for $(x, z, F2 \circ F1)$ .	361

*Proof.* 363

- Assume it is false. 364
- There exists  $\exists(x, y, z, F1, F2)$  such that it is a minimum counterexample 365  
by  $\#down(x)$ . 366
- Let us follow \*Def.A for  $(x, y, F1)$  and for  $(y, z, F2)$ . 367
  - Assume \*0 holds for  $(x, y, F1)$ . 368
  - Hence  $space(x) = \emptyset$  then  $x = y$ . 369
  - Consider the proposition titled as "Empty space". 370
  - Hence  $x = y = z$ . 371
  - Hence \*0 holds for  $(x, z, F2 \circ F1)$ . 372
  - The last assumption is false for  $(x, y, F1)$ . 373
  - Assume \*0 holds for  $(y, z, F2)$ . 374
  - Hence  $space(y) = \emptyset$  then  $y = z$ . 375
  - Consider the proposition titled as "Empty space". 376
  - Hence  $x = y = z$ . 377
  - Hence \*0 holds for  $(x, z, F2 \circ F1)$ . 378
  - The last assumption is false for  $(y, z, F2)$ . 379
  - Assume \*1 holds  $(x, y, F1)$ . 380
  - Hence  $(x, y)$  are points then  $(x, y) \in F1$ . 381
  - Hence \*1 also hold for  $(y, z, F2)$  because otherwise \*G.A cannot hold for 382  
 $(y, z, F2)$ . 383
  - Hence  $(x, z)$  are points then  $(x, z) \in F2 \circ F1$ . 384
  - Hence \*1 holds for  $(x, z, F2 \circ F1)$ . 385
  - The last assumption is false  $(x, y, F1)$ . 386
  - Assume \*1 holds  $(y, z, F2)$ . 387
  - Hence  $(y, z)$  are points then  $(y, z) \in F2$ . 388
  - Hence \*1 also hold for  $(x, y, F1)$  because otherwise \*G.A cannot hold for 389  
 $(x, y, F1)$ . 390
  - Hence  $(x, z)$  are points then  $(x, z) \in F2 \circ F1$ . 391
  - Hence \*1 holds for  $(x, z, F2 \circ F1)$ . 392
  - The last assumption is false for  $(y, z, F2)$ . 393
  - Hence (\*2 then  $\wedge$  \*3) holds for  $(x, y, F1)$  and for  $(y, z, F2)$ . 394
  - Consider the proposition titled as "Surjectivity". 395
  - Then  $F1[space(x)]$  is a bijection from \*to  $space(x) * space(y)$ . 396
  - And  $F2[space(y)]$  is a bijection from \*to  $space(y) * space(z)$ . 397

Hence  $(F2 \circ F1)[space(x)]$  is a bijection from  $*$ to  $space(x) * space(z)$ . 398  
Hence  $*2$  holds for  $(x, z, F2 \circ F1)$ . 399  
Hence  $*3$  fails for  $(x, z, F2 \circ F1)$ . 400  
There exists  $(f1, f2)$  such that ( 401  
3 holds for  $(x, y, F1, f1)$  in place of  $(x, y, F, f)$  *then*  $\wedge$  402  
3 holds for  $(y, z, F2, f2)$  in place of  $(x, y, F, f)$  403  
). 404  
Then  $*3$  fails for  $(x, z, F2 \circ F1, f2 \circ f1)$  in place of  $(x, y, F, f)$ . 405  
There exists  $\exists(m1, m2, m3) \in f2 \circ f1$  such that ( 406  
the antecedent of this proposition accepts  $(m1, m2, m3, F1, F2)$  as  $(x, y, z, F1, F2)$   
*then*  $\wedge$  408  
 $(m1, m2, m3, F1, F2)$  is a counterexample 409  
). 410  
Though  $(m1, m2, m3, F1, F2)$  is smaller than a minimum counterexample. 411  
The first assumption is false. 412

□ 413

**Proposition 10** (Members' isomorphisms as antecedent). Take  $\forall(x, y, F, f)$  414  
such that  $(*A1 \text{ *then* } \wedge *A2 \text{ *then* } \wedge *A3)$ . Then  $*B$  holds. 415

**A1**  $F$  is an injection. 416

**A2**  $f$  is a bijection from  $*$ to  $x*y$  *then*  $\wedge f \neq \emptyset$ . 417

**A3** Take  $\forall(m1, m2) : \in f$ . Then  $*Def.A$  holds for  $(m1, m2, F)$ . 418

**B**  $*Def.A$  holds for  $(x, y, F)$  in place of  $(x, y, F)$ . ■ 419

*Proof.* 420

- Assume B fails. 421
- Hence there exists  $\exists(x, y, F)$  such that  $*Def.A$  fails for  $(x, y, F)$ . 422
- Let us follow  $*Def.A$  for  $(x, y, F)$ . 423
- $(*0 \text{ fails } \text{*then* } \wedge *1 \text{ fails } \text{*then* } \wedge (*2 \text{ fails } \text{*else* } \vee *3 \text{ fails})$  ). 424
- Assume  $*2$  fails. 425
- Hence  $F[space(x)]$  is not an injection from  $*$ to  $space(x) * space(y)$ . 426
- Consider the proposition titled as "MemBers' isomorphisms as conse- 427  
quent". 428



- Then  $(space(x) = \emptyset = space(y)) \text{ else } \vee (space(x) \neq \emptyset \neq space(y))$ . 429
- Meanwhile  $F$  is an injection by \*A1. 430
- Hence  $(space(x) \neq \emptyset \neq space(y))$  because otherwise \*2 holds. 431
- Hence there exists  $\exists(p_x, p_y) : \in space(x) * space(y)$  such that 432  
 $p_x \notin domain(F) \text{ else } \vee p_y \notin image(F)$ . 433
- Consider \*A2,\*A3 and the proposition titled as "Members' isomorphisms 434  
as the consequent". 435
- There exists  $\exists y2 \in^{\geq 0} y$  such that \*Def.A holds for  $(p_x, y2, F)$ . 436
- There exists  $\exists x1 \in^{\geq 0} x$  such that \*Def.A holds for  $(x1, p_y, F)$ . 437
- Meanwhile, for each of the 2 lines just above, 438  
(\*Def.A holds only at \*1) because 439  
(each of  $(p_x, p_y)$  is a point). 440
- Hence  $p_x \in domain(F) \text{ then } \wedge p_y \in image(F)$ . 441
- Hence the last assumption is false. 442
- Hence \*3 must fail. 443
- Hence \*3 fails for  $f$  in place of  $f$ . 444
- Though by  $(*A2 \text{ then } \wedge *A3), (*4 \text{ then } \wedge *5 \text{ then } \wedge *6)$  holds. 445
- Hence the first assumption is false. 446

□ 447

**Proposition 11** (Topological space). Take  $\forall((X1, T1), (X2, T2))$  such that \*A 448  
holds. Then \*B1  $\Rightarrow$  \*B2. 449

**A** Take  $i \in \{1, 2\}$ . Then  $(X_i, T_i)$  is a topological space. 450

**B1**  $((X1, T1), (X2, T2))$  are homeomorphic. 451

**B2** There exists  $\exists F$  such that  $((X1, T1), (X2, T2))$  are isomorphic by  $F$ . 452

■ 453

<i>Proof.</i>	454
First of all, let me define how to express a pair as a set.	455
Take $\forall(p1, p2)$ . Then $(p1, p2) = \{PAIR, p1, \{p1, p2\}\}$ .	456
In the expression, "PAIR" is a constant keyword.	457
By the way, *B1 implies *C.	458
	459
<b>C</b> There exists $\exists(G, g)$ such that $(*C1 \text{ then} \wedge *C2 \text{ then} \wedge *C3 \text{ then} \wedge *C4)$ .	460
	461
<b>C1</b> $G$ is a bijection from $X1$ to $X2$ .	462
<b>C2</b> $G$ is a homeomorphism for *B1.	463
<b>C3</b> $g$ is a bijection from $T1$ to $T2$ .	464
<b>C4</b> Take $forall(t1, t2) : \in g$ . Then $(G \text{ takes } t1 \text{ to } t2)$ .	465
	466
Consider the previous proposition titled as Members' isomorphisms as antecedent	467
and refer it as *P.	468
P accepts arguments as follows.	469
	470
<b>D1</b> *P accepts $(X1, X2, G, G)$ in place of $(x, y, F, f)$ .	471
<b>D2</b> Take $\forall(t1, t2) : \in g$ . Then *P accepts $(t1, t2, G, G)$ in place of $(x, y, F, f)$ .	472
<b>D3</b> *P accepts $(T1, T2, G, g)$ in place of $(x, y, F, f)$ .	473
<b>D4</b> *P accepts (	474
$\{X1, T1\}$ ,	475
$\{X2, T2\}$ ,	476
$G$ ,	477
$\{(X1, X2), (T1, T2)\}$	478
) in place of $(x, y, F, f)$ .	479
<b>D5</b> *P accepts (	480
$\{PAIR, X1, \{X1, T1\}\}$ ,	481
$\{PAIR, X2, \{X2, T2\}\}$ ,	482
$G$ ,	483
$\{(PAIR, PAIR), (X1, X2), (\{X1, T1\}, \{X2, T2\})\}$	484
) in place of $(x, y, F, f)$ .	485

486

Hence \*P implies (\*E1 *then*  $\wedge$  \*E2 *then*  $\wedge$  \*E3 *then*  $\wedge$  \*E4 *then*  $\wedge$  \*E5). 487  
 Finally, \*E5 implies this proposition. 488

489

**E1** ( $X1, X2$ ) are isomorphic by  $G$ . 490

**E2** Take  $\forall(t1, t2) : \in g$ . Then  $(t1, t2)$  are isomorphic by  $G$ . 491

**E3** ( $T1, T2$ ) are isomorphic by  $G$ . 492

**E4**  $\{X1, T1\}, \{X2, T2\}$  are isomorphic by  $G$ . 493

**E5**  $(\{PAIR, X1, \{X1, T1\}\}, \{PAIR, X2, \{X2, T2\}\})$  are isomorphic by  $G$ . 494

495

□ 496

## 9 Appendix 497

### 9.1 Paradox of the set of all sets 498

This article somehow leads to paradoxes which resemble to "the set of all sets". 499  
For example, the usage of the constant keyword PAIR. Though I expect it does 500  
not matter for almost all of readers. Because it is not specific to this article but 501  
the entire mathematics and logic. ■ 502

## References 503