Isomorphism between general objects

Generalization of category theory

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1 Introduction

Prerequisites are only some first chapters of graduate level texts on general topology and graph theory; (1),(2). As you know, two objects are regarded as equivalent if they are isomorphic. This article introduces: how to find two general objects are isomorphic or not; given two non-isomorphic general objects, how to non-trivially abstract them so that they can be regarded as isomorphic.

Before we go ahead, let us glance at some trivial examples.

If $\{p_i\}_{i\in\{1,2,3\}}$ is a set of 3 objects pairwise isomorphic then $(p_1, p_2) \cong (p_1, p_3)$. Though $(p_1, p_2) \ncong (\mathrm{p}_1, p_1)$.

In general, two homeomorphic topological spaces are not isomorphic. For example, two Euclidean spaces (R^1_a, R^1_b) of the dimension 1 are homeomorphic but are not necessarily isomorphic, e.g., $(X, T)_i := R^1_i$ and $X_b \subset 2^{X_a}$.

$$(R^1_a \cong_h R^1_b) \wedge (R^1_a \ncong R^1_b)$$

It is somehow easy to non-trivially abstract (R^1_a, R^1_b) so that (R^1_a, R^1_b) can be regarded as isomorphic. Though the same is not always trivial for pairs of general objects. Especially when the pair are not topological spaces.

A topological space X is a set of points defined the topology T. (X, T) also may be said a topological space.

Warning: **Inside expressions of isomorphism**, no convention implicitly relates X to T; X is just a set of points, no topology is implicitly accompanied.

2 Isomorphism

Definition 2.1 (Deep member)**.** Take $\forall(c, x, y, n)$ such that: c is a chain of set membership $\overset{\text{and}}{\wedge}$ $|c| = n$. x is the maximum member of c. y is the minimum member of c. Then c is said a deep chain of x; y is said a $(n-1)$th deep member of x and you write as $y \in^{deep} x$ or $y \in^{n-1} x$; (x, y) are also written as (max(c),min(c)) respectively.

For example:

$$m \in ... \in y \in .... \in x$$

For example:

$$\{y1, y2\} \in^{deep} \{y1, y2\}$$
$$y \in^{deep} \{1, \{2, y\}\}$$

Definition 2.2 (Deep graph and tree)**.** Take $\forall(x, V, E_v, E_h)$ such that: V is the set of all deep chains of x; E_v is a set of directed edges on V such that $E_v := \{(c_1 > c_2) \in V^2 \mid c_1 \supset c_2) \wedge min(c_1) \in min(c_2)\ \}$; E_h is a set of edges on V such that $E_h := \{\{c_1, c_2\} \subset V \mid min(c_1) = min(c_2)\ \}$.

$G := (V, E_v, E_h)$ is said the deep graph of x; (V, E_v) is said the deep tree of x.

All vertex of a deep tree is said an end node if it is not a sub chain of some vertex of the deep tree.

(E_v, E_h) are described as (vertical,horizontal) respectively.

Definition 2.3 (Isomorphism)**.** Take $\forall(x, y)$. Then $x \cong y \equiv$ (*1 $\overset{\text{and}}{\wedge}$ ... $\overset{\text{and}}{\wedge}$ *3).

1. Deep graphs of (x, y) are graph isomorphic by a graph isomorphism f.

2. $\exists F$ as a bijection on a set of indentities.

3. Take all end vertex v of the deep tree of x, then (*3a $\overset{\text{or}}{\vee}$ *3b).

3a. $F(\min(v)) = F(\min(f(v)))$.

3b. $v \notin$ domain(F) $\overset{\text{and}}{\wedge}$ min(v) =min$(f(v))$.

3 Abstraction

Axiom 3.1. Take $\forall(x, y)$, then x has its identity written ID(x) such that (*1 $\overset{\text{and}}{\wedge}$ ... $\overset{\text{and}}{\wedge}$ *3).

1. $(x = y \overset{\text{or}}{\vee}$ x=ID(y) $\overset{\text{or}}{\vee}$ ID(x)=y. $) \equiv$ ID(x) =ID(y).

2. ID(x)=ID(ID(x)).

3. The mathematics on ID(x) and the mathematics on ID(y) are equivalent.

We call y as the ID holder of ID(x) if $\exists n \overset{\text{and}}{\wedge} y$=ID$^{-n}(x) \neq x$.

If no such n exists then we call ID(x) as the ID holder of ID(x).

$\blacksquare$

Take $\forall((X_1, T_1), (X_2, T_2))$ as homeomorphic topological spaces where $\forall T_i$ is a topology. In the rest we prefer that $((X_1, T_1), (X_2, T_2))$ are also isomorphic. In other words we prefer all points of $((X_1, T_1), (X_2, T_2))$ to be identities.

In the rest, we transform all topological space into the point abstraction.

Definition 3.1 (Point abstraction)**.** Take $\forall(x,G)_{i\in\{1,2\}}$ such that (*0 $\overset{\text{and}}{\wedge}$ ... $\overset{\text{and}}{\wedge}$ *11).

0. $G_{i\in\{1,2\}}$ denotes the deep graph of x_i.

1. $\neg(\exists y$ such that *1a).

1a. $\{y,\mathrm{ID}(y)\} \subset \{d \mid d \in^{deep} x_2\ \}\ \overset{\text{and}}{\wedge}\ \mathrm{ID}(y) \neq y$.

2. $\exists(f,G_3)$.

3. G_3 is a sub graph of G_2.

4. f is a graph isomorphism from G_1 to G_3.

5. let $G_{i\in\{1,2,3\}}$ be decomposed as $(V,E^v,E^h)_i := G_i$.

6. take $\forall v \in V_1$, then f preserves the length of v as the deep chain.

7. take $\forall v_2$ such that: $v_2 \in V_2 - V_3\ \overset{\text{or}}{\vee}\ v_2$ is an end vertex of V_3.

8. then $\exists v_3 \in V_3$.

9. v_3 is a sub chain of $v_2\ \overset{\text{and}}{\wedge}\ v_3$ is an end vertex of V_3.

10. $\min(f^{-1}(v_3))$=ID($\min(v_3)$).

11. All point of V_2 is an instance of v_2.

x_1 is said the **point abstraction** of x_2.

4 Applications in geometrical topology

4.1 Natural automorphism

Definition 4.1 (Natural automorphism)**.**
Let I:=[0,1], i.e., I is a unit interval.

As you know I is more than a topological space. It is defined a metric table, and decided which end point is said as 0.

Let (Y,T_Y) denote the topological space of I where Y is the set of points and T_Y is the topology on Y. We use I as a bijective index set for Y.

Take $\forall X$ such that: X is a topological space defined the topology T_X. Let (P_{XY}, T_{XY}) denote the product space for $X * Y$. Take $\forall p :\in P_{XY}$, Write p as $[x, y]$. That is, (x, y) =$\text{ID}^{-1}(p)$.

Take $\forall F$ as an injection from X*Y to P_{XY} such that (*0 $\overset{\text{and}}{\wedge}$ ... $\overset{\text{and}}{\wedge}$ *2). Hence F takes a pair of points as the input. Then F outputs a point which is the identity of a pair of points.

0. Take $\forall(x_1, y) :\in X$*Y.

1. $\exists x_2 :\in X \overset{\text{and}}{\wedge} F(x_1, y) = [x_2, y]$.

2. $F(\forall x, 0) := [x, 0]$.

Let F_0 be a solution of F with (X, I) fixed such that: $F_0(\forall x, \forall y) := [x, y]$.

Take $\forall F_i$ as a solution of F with (X, I) fixed such that: *1.

1. $(F_i, T_X, T_Y, T_{XY}) \cong (F_0, T_X, T_Y, T_{XY})$.

Let A denote the set of all solutions of F_i with (X, I) fixed. Take $\forall F_i :\in A, \forall g$ such that g is a function on X as $g(\forall x)$:=$\text{ID}^{-1} \circ F_i(x, 1)$. Then g is said a **natural automorphism** on X. $\blacksquare$

Definition 4.2 (Natural-automorphic)**.**
Take $\forall X$ such that: X is a topological space. Take $\forall(s1, s2)$. Then $(s1, s2)$ are said X**-natural-automorphic** if: $\exists F$ as a super set of some natural automorphism on $X \overset{\text{and}}{\wedge} s1 \cong^F s2$.

4.2 Ideal set of sub spaces

Definition 4.3 (Ideal set of sub spaces)**.**
Take $\forall(X, S)$ such that: X is a topological space. S is a set of sub spaces of X. S is said **ideal** if: (*1 $\overset{\text{and}}{\wedge}$ ... $\overset{\text{and}}{\wedge}$ *7).

1. Let S_P be the set to collect $\forall(s, p)$ such that $s \in S \overset{\text{and}}{\wedge} p \in s$.

2. $\exists B$ as an open basis to generate X.

Regard B as a subset of the power set of X.

3. Let $S_B := \{S_b \mid \exists b \in B \overset{\text{and}}{\wedge} S_b = \{(s,p) \in S_P \mid s \subset b\ \}\ \}$.

4. Let $S_P := \{\mathrm{ID}((s,p)) \mid (s,p) \in S_P\ \}$.

5. Let $S_B := \{S_b1 \mid \exists S_b2 \in S_B \overset{\text{and}}{\wedge} S_b1 = \{\mathrm{ID}((s,p)) \mid (s,p) \in S_b2\ \}\ \}$.

6. S_B is an open basis on S_P.

7. Members of S_P are pairwise S_P-natural-automorphic.

$\blacksquare$

Conjecture 4.1 (Ideal set of sub spaces and ambient isotopies)**.** Take $\forall(X,T,S,F,A)$ such that: S is an ideal set of sub spaces of (X,T) where T is the topology. F is the set to collect: $\forall f$: X*[0,1] $\rightarrow X$ such that f is an ambient isotopy. A is the set to collect $\forall(g,S_1,S_2)$ such that: g is a natural automorphism on $X \overset{\text{and}}{\wedge} (S1,S2)$ are subsets of $S \overset{\text{and}}{\wedge} (S_1,T) \cong^g (S_2,T)$.

Then (*1 $\overset{\text{and}}{\wedge}$ ... $\overset{\text{and}}{\wedge}$ *4) holds.

1. take $\forall(g,S_1,S_2) :\in A$.

2. $\exists f :\in F$

3. take $\forall t :\in (0,1] \overset{\text{and}}{\wedge}$ let $f_t(\forall x :\in X){:=}f(x,t)$.

4. $(f_t,S,S) \in A \overset{\text{and}}{\wedge}$ if $t=1$ then $f_t = g$.

$\blacksquare$

Definition 4.4 (Prime topological space)**.** Take $\forall X$ as a topological space. Then X is said prime if *1.

1. $\exists S$ as a set of sub spaces of $X \overset{\text{and}}{\wedge} S$ is ideal $\overset{\text{and}}{\wedge} S$ is an open basis to generate X.

$\blacksquare$

Conjecture 4.2 (Ideal set of sub spaces)**.** Take $\forall(X,S)$ such that: X is a prime topological space. S is a set of sub spaces of X. Then S is ideal if (*1 $\overset{\text{and}}{\wedge}$ *2).

1. Members of $\{S\}$*X are pairwise X-natural-automorphic.

2. Let $S_p := \{(s,p) \mid s \in S \overset{\text{and}}{\wedge} p \in s\ \}$.

Members of $\{S\}$*S_p are pairwise X-natural-automorphic.

$\blacksquare$

5 Abstract conjectures

5.1 Main abstract conjecture

Conjecture 5.1 (Abstract conjecture of ideal set and metric)**.**
Take $\forall(M, X, S1, f)$ such that *A.

Consider (*B $\to$ *C). It is independent from the topological class of members of $S1$ if f is **enough general** for topological classes of members of solutions of $S1$ with (M, X) fixed.

The claim converges to true if generality approaches to the perfect.

A. *1 $\overset{\text{and}}{\wedge}$ ... $\overset{\text{and}}{\wedge}$ *3.

1. M is a metric table to define X as a topological space $\overset{\text{and}}{\wedge}$ X is prime.
2. $S1$ is an ideal set of sub spaces of X.
3. f is a function on $S1$.

B. *1 $\overset{\text{and}}{\wedge}$ ... $\overset{\text{and}}{\wedge}$ *3.

1. Take $\forall k1 :\in S1$
2. Let $S2 := \{k2 \in S1 \mid f(k2) = f(k1)\ \}$.
3. $S2$ is unique for (M, X).

   Unique?: For example, take $\forall x :\in \overset{\text{ID}}{Deep}(X)$. If $S2$ is the set to collect $\forall k :\in S1$ such that $x \in \overset{\text{ID}}{Deep}(k)$ then $S2$ is not unique for (M, X) **in general** because x is not unique for (M, X) in general. Instead $S2$ is unique for (M, X, x).

C. $S2$ is ideal.

5.2 Application on knots

Let Conj be an alias for Conjecture 5.1. Let Def be an alias for the following Definition 5.1. The antecedent of Conj apparently holds for (M, X, K, K_f, f) of Def in place of $(M, X, S1, S2, f)$. And f is apparently enough general as required in Conj.

Definition 5.1 (A set of knots)**.** Take $\forall(M, X, K, K_f)$ such that:
M is a metric table to define X as a Euclidean space of 3-dimension.
Take $\forall k_0$ as a knot and a subspace of X.
K is the set to collect $\forall k$ such that: (k, k_0) are X-natural-automorphic.

$K_f := \{k \in K \mid f(k) = f(k_0)\ \}$.

Definition of f:

- $j1(\forall k :\in K) := \{j \mid$
  j is an orthogonal [1]projection of k onto some infinite plane $\}$.
- $j2(\forall k :\in K) := \{j \in j1(k) \mid$
  $\neg\ (\exists p \overset{\text{and}}{\wedge} p \in \text{image}(j) \overset{\text{and}}{\wedge} \mid j^{-1}(p) \mid > 2)\ \}$.
- $j3(\forall k :\in K) := \{n \mid$
  $\exists j \overset{\text{and}}{\wedge} j \in j2(k) \overset{\text{and}}{\wedge} n$ is the number of [2]double points on $j\ \}$.
- $f(\forall k :\in K) := \{m \mid$
  m is the maximal member from $j3(k)\ \}$.

6 Notation

- take $\forall x \equiv$ for $\forall x \equiv \forall x$.

  In other words, "take" means nothing.
- $\forall x$ as a set $\equiv \forall x$ such that x is a set.
- Assume y is dependent on z then:

  $\forall x$ as a solution of y with z fixed $\equiv \forall x$ as a solution of y.
- $\{x \mid p(x)\ \} \equiv$ the set to collect $\forall x$ such that $p(x)$.

In definitions, I rarely write "if and only if". In stead I write "if" even if I know that "if and only if" can replace the "if".

References

[1] Glen E. Bredon,Topology and Geometry, Springer, ISBN 978-1-4419-3103-0

[2] Reinhard Diestel, Graph Theory, Springer-Verlag, ISBN 0-387-98976-5

[1]Hence, j is a function from k to an infinite plane.

[2]Double point?: That is, the inverse image of a double point has exactly 2 distinct points of k; no matter the double point represents a crossing or a tangent point.