Proof of that
Homeomorphisms are isomorphisms of memBers

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1 Notations

Definition 1.1 (Restriction of binary relation)**.**
Take $\forall(L, X, Y)$ as a binary relation L and sets (X, Y).
$L[X] := \{(x, y) \in L \mid x \in X\}$.
$L[, Y] := \{(x, y) \in L \mid y \in Y\}$.

2 Properties of equivalence relation

Proposition 1 (Reflexive,symmetry,transitive properties)**.**
The relation by isomorphisms of memBers has properties of reflexive, symmetry and transitive. ■

Proof.

- *1 has been proved in graph theory.
- It is trivial that (*2 $_{and}\wedge$ ..... $_{and}\wedge$ *5) holds.
- Hence this proposition holds.

1 The relation by graph isomorphisms has properties of reflexive, symmetry and transitive.

2 Take $\forall f_1, f_2, f_3$ as graph isomorphisms such that
$domain(f_2) = image(f_1)$ $_{and}\wedge$
f_3 is the identity function on domain(f_3).

3 relate-constant-memBer(f_3) $_{and}\wedge$

4 relate-constant-memBer(f_1) $\equiv$ relate-constant-memBer(f_1^{-1}) $_{and}\wedge$

5 (relate-constant-memBer(f_1) $_{and}\wedge$ relate-constant-memBer(f_2)) $\equiv$
relate-constant-memBer($f_2 \circ f_1$)

□

3 Homeomorphic topological spaces as isomorphic memBers

Definition 3.1.
Take $\forall(m1, m2, c)$ such that (
c is a chain of set membership $_{and}\wedge$
$m1$ is the [1]minimum member of c
$m2$ is the [2]maximum member of c.
).
Then define (*1 $_{and}\wedge$ ..... $_{and}\wedge$ *5).

1 $m1$ is said a deep member of $m2$.

Hence all memBer is a deep member of itself.

2 $|c| - 1$ is said a power of $(m1, m2)$.

3 It is written as $m1 \in^{|c|-1} m2$.

4 Let p be the maximum power of $(m1, m2)$.
Then $depth(m1, m2) := p$.

5 Let $S := \{d \mid \text{there exists } \exists m \text{ such that } d = depth(m, m2)\}$.
Then $depth(m2)$:="the maixmum member of S".

■

[1]No member of c is a member of $m1$.
[2]No member of c has $m2$ as a member.

Definition 3.2 (Space of memBer)**.**
Take $\forall m$.
Then define that
$Deep(m) := \{d \mid d \text{ is a deep member of } m\ \}$.
$Space(m) := \{p \in Deep(m) \mid p \text{ is a point }\}$.

Proposition 2 (Isomorphism of vertices)**.**
Take $\forall(m1, m2, f, v1)$ such that (
$(m1, m2)$ are isomorphic by $f\ \ _{and}\wedge\ v1 \in Deep(m1)$
).
Then $v1, f(v1)$ are isomorphic by $f[Deep(v1)]$.

Proof.

- Let $v2 := f(v1)$.
- As **C1**, claim that $Deep(v2) \subset image(f[Deep(v1)])$.
- Assume that the claim fails.
- There exists $\exists w2 :\in Deep(v2)$
  as a minimum counterexample to *C1 compared by $depth(w2, v2)$.
- It is trivial that $w2 \neq v2$.
- There exists $\exists x2 :\in Deep(v2)$ such that $w2 \in x2$.
- Hence x2 is not a counterexample to *C1
  because $depth(w2, v2) < depth(x2, v2)$.
- Hence There exists $\exists x1 :\in Deep(v1)$ such that $f(x1) = x2$.
- Hence There exists $\exists w1 :\in x1$ such that
  $(f(w1) = w2\ \ _{and}\wedge\ w1 \in Deep(v1))$. A contradiction.
- Hence The assumption on ($\neg$ *C1) is false.
- As **C2**, claim that ($Deep(v1) \subset image(\ f^{-1}[Deep(v2)]\ ))$.
- Though it is trivial that the same logic for the proof of *C1 proves *C2.
- Hence $Deep(v2) = image(f[Deep(v1)])$.
- Hence $f[Deep(v1)]$ is a graph isomorphism
  from*to $Deep(v1) * Deep(v2)$.

- And it is trivial that relate-constant-memBer(f) $\Rightarrow$ relate-constant-memBer($f[Deep(v1)]$).

$\square$

Proposition 3 (Isomorphism of Spaces)**.**
Take $\forall(m1, m2, f)$ such that $(m1, m2)$ are isomorphic by f.
Then $f[Space(m1)]$ is a bijection from*to $Space(m1) * Space(m2)$. $\blacksquare$

Proof.

- Assume it is false.

- $image(f[Space(m1)]) \neq Space(m2)$.

- $image(f[Space(m1)]) \not\subset Space(m2)$ $\ _{or}\vee$ $image(f[Space(m1)]) \not\supset Space(m2)$.

- Then there exists $\exists(m1, m2, f, p1, p2)$ as a counterexample such that (*A0 $\ _{and}\wedge$ (*A1 $\ _{or}\vee$ *A2)) holds.
  A0 $(p1, p2) :\in Space(m1) * Space(m2))$.
  A1 $f(p1) \notin Space(m2)$.
  A2 $p2 \notin image(f[Space(m1)])$.

- Assume *A1 holds.

- Then $f(p1)$ is either a constant-memBer (or a non-constant-memBer as a set).

- Though $f(p1)$ can not be a constant-memBer by that relate-constant-memBer(f).

- Hence $f(p1)$ is a non-constant-memBer as a set.

- Though it contradicts to that f is a graph isomorphism because $f(p1)$ has edge to some its member.

- Hence the assumption of *A1 is false $\ _{and}\wedge$ *A2 holds.

- There exists $\exists c1 :\notin Space(m1)$ such that $f(c1) = p2$.

- Hence $f^{-1}(p2) = c1$

- Though this condition has been denied in the disproof of *A1.

- Hence the assumption of *A2 is false $\ _{and}\wedge$ the main assumption is false.

□

Proposition 4 (Pair of member's isomorphisms)**.**
Take $\forall(I := \{1,2,3,4\},\ \ \{m_i\}_{i\in I},\ \ f_{1,2},\ \ f_{3,4})$
such that (*1 $\ _{and}\wedge$ ..... $\ _{and}\wedge$ *4) holds.
Then (*5 $\ _{and}\wedge$ *6) holds.

1 $(m1, m2)$ are isomorphic by $f_{1,2}$.

2 $(m3, m4)$ are isomorphic by $f_{3,4}$.

3 Let $f := f_{1,2} \cup f_{3,4}\ \ _{and}\wedge$ let $f_s := f[Space(f)]$.

4 Then f_s is a bijection.

5 f is a function.

6 f is a bijection.

7 relate-constant-memBer(f).

■

*Proof of *5.*

- Let $(V, E)_{i:\in\{1,2,3,4\}}$ be the deep graph of m_i.
- Assume it is false.
- Then there exists $\exists((m1, m3), (m2, m4))$ as a minimum counterexample by $depth((m1, m3))$ such that f is not a function.
- Let us make sure that f is a union of a set of bijections.
- There exists $\exists v :\in V_1 \cap V_3$ such that $|f[\{v\}]| \geq 1\ \ _{and}\wedge\ v \notin \{m1, m3\}$.
- By the way, this proposition accepts the following $args_v$
  in place of $(m1, m2, m3, m4, f_{1,2}, f_{3,4})$.
- $args_v :=$ (
  v,
  $f_{1,2}(v)$,
  v,
  $f_{3,4}(v)$,
  $f_{1,2}[Deep(v)]$,
  $f_{3,4}[Deep(v)]$
).

- In the rest, this $args_v$ is proved to be a counterexample smaller than a minimal counterexample.

- As the first step, the such-that clause of this proposition holds for $args_v$ as follows.

- Equivalently (*1 $_{and}\wedge$ ..... $_{and}\wedge$ *4) holds for $args_v$ as follows.

- Assume *1 fails for $args_v$.

- Hence $(v, f_{1,2}(v))$ is not isomorphic by $f_{1,2}[Deep(v)]$.

- Though it contradicts to the proposition titled as "Isomorphism of vertices".

- Hence the last assumption is false.

- Hence *1 holds for $args_v$.

- Hence *2 holds for $args_v$ because (for $args_v$, *1 and *2 are logically equivalent).

- Assume *4 fails for $args_v$.

- Let $f_v := f_{1,2}[Deep(v)] \cup f_{3,4}[Deep(v)]$ $_{and}\wedge$
  let $f_{v,s} := f_{1,2}[Deep(v)][Space(f_v)] \cup f_{3,4}[Deep(v)][Space(f_v)]$.

- Then $f_{v,s}$ is not a bijection.

- Though it is false because $f_{v,s} \subset f_s$. Hence *4 holds for $args_v$.

- Hence (*1 $_{and}\wedge$ ..... $_{and}\wedge$ *4) holds for $args_v$.

- Moreover *5 fails for $args_v$ as follows.

- Assume *5 holds for $args_v$.

- Then f_v is a function.

- Though
  $v \in Deep(v)$ $_{and}\wedge$
  $f_v[\{v\}] = f[\{v\}]$ $_{and}\wedge$
  $|f_v[\{v\}]| = |f[\{v\}]| \geq 1$.

- Hence *5 fails for $args_v$.

- $args_v$ is a counterexample.

- And the size as a counterexample of $args_v$ equals to $depth((v,v))$.
- Though $depth((v,v)) < depth((m1,m3))$ [3]because $depth((v,v)) = depth(v) + 2 < depth(m1) + 2 \leq depth(m1,m3)$.
- Hence arg_v is a counterexample smaller than a minimum counterexample.
- Hence the main assumption is false.

$\square$

*Proof of *6.*

- Consider the proposition *P$_S$ titled as "Reflexive,symmetry,transitive properties".
- Consider the proposition *P$_I$ titled as "Isomorphism of spaces".
- Then ((*P$_S$ $_{and}\wedge$ *P$_I$) $_{and}\wedge$ (*1 $_{and}\wedge$ ..... $_{and}\wedge$ *4)) implies (*S1 $_{and}\wedge$ ..... $_{and}\wedge$ *S4).

S1 $(m2, m1)$ are isomorphic by $f^{-1}_{1,2}$ as an isomorphism.

S2 $(m4, m3)$ are isomorphic by $f^{-1}_{3,4}$ as an isomorphism.

S3 Let $f_{-1} := f^{-1}_{1,2} \cup f^{-1}_{3,4}$ $_{and}\wedge$ let $f_{s,-1} := f_{-1}[Space(f_{-1})]$.

S4 Then $f_{s,-1}$ is a bijection.

- For *S4, it holds because (it is trivial that ($f_{-1} = f^{-1}$ $_{and}\wedge$ $f_{s,-1} = f^{-1}_s$).
- Moreover *5 implies that f_{-1} is a function.
- Hence f^{-1} is a function.
- Hence *5 implies that f is an injection.
- By the way,f is surjective because f is not defined the codomain.
- Hence f is a bijection.

$\square$

*Proof of *7.*

[3] $(x,y) := \{\{x\},\{x,y\}\}$

- Assume it is false.
- There exists $\exists(x, y) :\in f$ such that
  (either x or y is a constant-memBer) $_{and}\wedge$ $(x \neq y)$.
- Though $f = f_{1,2} \cup f_{3,4}$.
- Hence $(x, y) \in f_{1,2}$ $_{or}\vee$ $(x, y) \in f_{3,4}$.
- There exists $\exists g :\in \{f_{1,2}, f_{3,4}\}$ such that
  $\neg$(relate-constant-memBer(g)).
- It contradicts to (*1 $_{and}\wedge$ *2).
- The assumption is false.

$\square$

Definition 3.3 (Constant space)**.**
A constant space D is most likely a function to be used to state conditions on variables.
For example, let D be a function and let $x, y, z :\in Z * Z * Z$ such that $x = D(z)$ and $y = D(z)$.
Then $x = y$.
In this case, D is used to make sure that variables hold equal values.
Be careful that all constant space is just a usual variable but a global constant.

Proposition 5 (Isomorphism by member's isomorphisms)**.**
Let *P$_P$ denote the proposition titled as "Pair of member's isomorphisms".
Take $\forall(S1, S2, f, F)$ as sets $(S1, S2)$ such that (*A1 $_{and}\wedge$ ..... $_{and}\wedge$ *A7).
Then (*10 $_{and}\wedge$ ..... $_{and}\wedge$ 12) holds.

A1 $\mid$ $Deep(\{S1, S2\})$ $\mid \leq$ continuum.

A2 f is a bijection from*to $S1 * S2$.

A3 There exists $\exists D$ as a function and as a constant space.

A4 Take $\forall((m1, m2), (m3, m4)) :\in f^2$.

A5 There exists $\exists f_{1,2}, f_{3,4}$
such that $f_{1,2} = D((m1, m2))$ $_{and}\wedge$ $f_{3,4} = D((m3, m4))$.

A5 Let $args$:= (
$m1, m2, m3, m4,$
$f_{1,2},$
$f_{3,4}$
).
Then $*\mathrm{P}_P$ accepts $args$
in place of $(m1, m2, m3, m4, f_{1,2}, f_{3,4})$.

A6 $*\mathrm{P}_P.(*1 \ {}_{and}\wedge \ ..... \ {}_{and}\wedge \ *4)$ holds for $args$.

A7 Let $D_{1,2} := \{D((m1, m2)) \mid (m1, m2) \in f\}$.
Then F = union $D_{1,2}$.

C10 $F[Space(F)]$ is bijective.

C11 F is a function.

C12 F is bijective.

C13 relate-constant-memBer(F).

C14 $(S1, S2)$ are isomorphic by $F \cup \{S1, S2\}$.

■

*Proof of *C10.*

- First of all, it is trivial that
  $domain(F[Space(F)]) = Space(S1) \ {}_{and}\wedge$
  $image(F[Space(F)]) = Space(S2)$.
- Assume it is false.
- There exists $\exists(p1, p2) :\in Space(S1) * Space(S2)$ such that
  $|F(p1)| \geq 1 \ {}_{or}\vee \ |F^{-1}(p2)| \geq 1$.
- Though it implies that the antecedent of this proposition have failed.
- Namely, there exists $\exists((m1, m2), (m3, m4))$
  which has been taken as $\forall((m1, m2), (m3, m4))$ in *A4
  such that, of *A6, $*\mathrm{P}_P.(*4)$ have failed for $((m1, m2), (m3, m4))$.
- Hence the assumption is false.

□

*Proof of (*C11 $_{and}\wedge$ *C12 $_{and}\wedge$ *C13).*

- First of all, consider the proposition titled as "Pair of member's isomorphisms".
- The proposition implies that the antecedent of this proposition implies that *A6 can be modified as the following *A6 typed in red.
- That is, the original "*4" has been replaced with "*7".
- **A6** *P_P.(*1 $_{and}\wedge$ ..... $_{and}\wedge$ *7) holds for $args$.
- Call this modified antecedent as the modified antecedent.
- By the way, assume (*C11 $_{and}\wedge$ *C12 $_{and}\wedge$ *C13) is false.
- (*B1 $_{or}\vee$ *B2) holds.
- **B1** There exists $\exists(x1, x2) :\in S1 * S2$ such that
  $|F(x1)| \geq 1$ $_{or}\vee$ $|F^{-1}(x2)| \geq 1$.
- **B2** There exists $\exists f_{1,2} :\in D_{1,2}$ such that
  $\neg$ relate-constant-memBer($f_{1,2}$).
- Though it implies that the modified antecedent have failed.
- Namely, there exists $\exists((m1, m2), (m3, m4))$
  which has been taken as $\forall((m1, m2), (m3, m4))$ in *A4
  such that, of *A6,
  P_P.(*5 $_{and}\wedge$ *6 $_{and}\wedge$ *7) have failed for $((m1, m2), (m3, m4))$.
- Hence the assumption is false.

□

*Proof of *C14.*

- Assume it is false.
- Let $F_+ := F \cup \{S1, S2\}$, Then (*B1 $_{or}\vee$ *B2) holds.
- As **B1**, $(S1, S2)$ are not graph-isomorphic by F_+.
- As **B2**, $\neg$ relate-constant-memBer(F_+).
- Assume *B2 holds.
- Hence $\neg$ relate-constant-memBer($\{S1, S2\}$).

- Hence there exists$\exists(T1, T2) :\in \{(S1, S2), (S2, S1)\}$ such that
  $T1$ is a constant-memBer $_{and}\wedge$ $T2$ is not a constant-memBer.

- There exists $\exists(c_1, p_2) :\in F$ such that
  (c_1 is a constant-memBer $_{and}\wedge$ p_2) is not a point.
  By this contradiction, the assumption on *B2 is false.

- Hence *B1 holds.

- There exists $\exists(v1, v2) :\in S1 * S2$ such that
  $F(v1) \notin S2$ $_{or}\vee$ $F^{-1}(v2) \notin S1$.

- Though there exists $\exists f_{1,2} :\in D_{1,2}$ such that (
  $(v1, F(v1)) \in f_{1,2}$ $_{and}\wedge$
  $f_{1,2}$ is a bijection from*to Deep($v1$)*Deep($F(v1)$)
).

- Moreover $F \supset f_{1,2}$.

- Hence the assumption on *B1 is false.

- The main assumption is false.

□

Definition 3.4 (Variations of Indexed set)**.**
As you know, for example, $\{x_i\}_{i\in\{1,2\}} := \{x_1, x_2\}$, in mathematics.
In this article,
analogously, $(x_i)_{i\in\{1,2\}} := (x_1, x_2)$.
As an alternative simplified form, $(x)_{i\in\{1,2\}} := (x_1, x_2)$.
As one of many variations, $(\{x\})_{i\in\{1,2\}} := (\{x_1\}, \{x_2\})$.
As a comment, the order on the composed sequence should respect the most natural order on the index set. ■

Proposition 6 (Isomorphisms by spaces)**.**
Take $\forall(S)_{i:\in\{1,2\}}, \forall(f, g)$ such that (
$(S)_{i:\in\{1,2\}}$ are isomorphic by f and also by g $_{and}\wedge$
f[Space(f)]=g[Space(g)]
).
Then $f = g$.

Proof.

- Assume it is false.

- There exists $\exists v_1 :\in Deep(S1)$ as a minimum counterexample compared by depth(v_1) such that
$f(v_1) \neq g(v_1)$.

- It is trivial that depth(v_1)> 0.

- Hence v_1 is a set.

- $f[v_1] = g[v_1]$ because (
take $\forall w_1 :\in v_1$,
then (depth(w_1) <depth(v_1) $_{and}\wedge$ w_1 is not a counterexample)
).

- Hence $f(v_1) = image(f[v_1]) = image(g[v_1]) = g(v_1)$.

- The assumption is false.

□

Definition 3.5 (Isomorphism by spaces)**.**
Take $\forall(S)_{i:\in\{1,2\}}, \forall(f, F)$ such that
$(S)_{i:\in\{1,2\}}$ are isomorphic by F $_{and}\wedge$ $Space(F) \subset f \subset F$.
Then $(S)_{i:\in\{1,2\}}$ are also said isomorphic by f. ■

Proposition 7 (Homeomorphism as isomorphism)**.**
As you know, the set theory defines that
$(x, y) := \{\{x\}, \{x, y\}\}$.
Take $\forall((X, T))_{i:\in\{1,2\}}, \forall H$ such that (
$((X, T))_{i:\in\{1,2\}}$ is a pair of topological spaces $_{and}\wedge$
H is a bijection from*to $X_1 * X_2$ $_{and}\wedge$
$((X, T))_{i:\in\{1,2\}}$ are homeomorphic by H
).
Then (*1 $_{and}\wedge$ ..... $_{and}\wedge$ *5) holds.

1. $(X)_{i:\in\{1,2\}}$ are isomorphic by H.

2. Take $\forall(t_1, t_2) :\in T1 * T2$ such that $t_2 = image(H[t_1])$.
Then $(t)_{i:\in\{1,2\}}$ are isomorphic by $H[t_1]$.

3. $(T)_{i:\in\{1,2\}}$ are isomorphic by H.

4. $(\{X\})_{i:\in\{1,2\}}$ are isomorphic by H.

5. $(\{X, T\})_{i:\in\{1,2\}}$ are isomorphic by H.

6. $(\{\{X\},\{X,T\}\})_{i:\in\{1,2\}}$ are isomorphic by H.

■

*Proof of *1.*

- Consider the proposition titled as "Isomorphism by member's isomorphisms".
- $(X)_{i:\in\{1,2\}}$ are isomorphic by $H \cup \{(X1, X2)\}$.

□

*Proof of *2.*

- Consider the proposition titled as "Isomorphism by member's isomorphisms".
- $(t)_{i:\in\{1,2\}}$ are isomorphic by $H[t_1] \cup \{(t1, t2)\}$.

□

*Proof of *3.*

- Consider the proposition titled as "Isomorphism by member's isomorphisms".
- Consider *2.
- Let $t_{1,2} := \{(t_1, t_2) \in T1 * T2 \mid t_2 = image(H[t_1])\}$.
- $(T)_{i:\in\{1,2\}}$ are isomorphic by $H \cup t_{1,2} \cup \{(T1, T2)\}$.

□

*Proof of *4.*

- Consider the proposition titled as "Isomorphism by member's isomorphisms".
- Consider *1.
- $(\{X\})_{i:\in\{1,2\}}$ are isomorphic by $H \cup \{(X1, X2), (\{X1\}, \{X2\})\}$.

□

*Proof of *5.*

- Consider the proposition titled as "Isomorphism by member's isomorphisms".
- Consider *1 and *3.
- $(\{X, T\})_{i:\in\{1,2\}}$ are isomorphic
  by $H \cup \{(X1, X2), (T1, T2), (\{X1, T1\}, \{X2, T2\})\}$.

$\square$

*Proof of *6.*

- Consider the proposition titled as "Isomorphism by member's isomorphisms".
- Consider *4 and *5.
- $(\{\{X\}, \{X, T\}\})_{i:\in\{1,2\}}$ are isomorphic
- by $H \cup \{$
  $(X1, X2)$,
  $(T1, T2)$,
  $(\{X1, T1\}, \{X2, T2\})$,
  $(\{\{X1\}, \{X1, T1\}\}, \{\{X2\}, \{X2, T2\}\})$
  $\}$.

$\square$