Isomorphism between general objects

with simple applications in topological geometry

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1 Introduction

Prerequisites are only some first chapters of graduate level texts of general topology and graph theory; (1),(2). This article introduces a new fundamental language of mathematics. Using it, you can divide preexisting mathematics into tow classes; one which can be translated into the new language and the other one which cannot.

Take $\forall x$. Then x is said a general objects if it can be equivalently expressed as a nested graph. Blue texts indicate that the notions will be defined later.

This article defines when two given general objects, say (x, y), are said isomorphic, written $x \cong y$. For example, take $\forall(x, y)$ as numbers, then: $x \cong y \equiv x = y$. Contrary two points in the sense of elementary geometry will be unconditionally said isomorphic.

Defining the new notion, I have met one problem that any object can be regarded as a point in mathematics. It yields problems because I can not define that all two points are unconditionally said isomorphic. To simply solve this problem of titled points, let us bring in a new notion "identity".

Definition 1.1 (Identity)**.** Take $\forall x$, then $\exists p$ written p=ID(x). p is said the **identity** of x. Also write, x=ID$^{-1}(p)$.

x may be said a full version of p.

Take $\forall(x, y)$, then $x = y \equiv$ ID(x)=ID(y). It happens to be that x=ID(x) if x is a point in the sense of elementary geometry.

ID(x) may literally **represent** x, and x also may literally represent ID(x).

Notice that we are not interested in distinguishing ID(x) from x outside of expressions of isomorphism $\cong$. Meanwhile two identities will be unconditionally said isomorphic.

Definition 1.2 (Identity, point and vertex)**.**
It is true in the rest of this article:
p is a point $\equiv$ p is a vertex $\equiv$ p is an identity.

Hence, you cannot regard a general object x as a point in general. Instead you can regard ID(x) as a point. With this definition, all homeomorphic topological spaces will be unconditionally said isomorphic because their points will be unconditionally said isomorphic.

Though the definition of isomorphisms between general objects will still work with general objects regarded as points.

A topological space X is a set of points defined the topology T. (X, T) also may be said a topological space.

Warning: **Inside expressions of isomorphism**, no convention implicitly relates X to T; X is just a set of points, no topology is implicitly accompanied.

Let us continue to enumerate more examples of isomorphism. It will be said: $(X, T, p, 321)_1 \cong (X, T, p, 321)_2$ if $(X, T)_1$ and $(X, T)_2$ are homeomorphic by some homeomorphism $\exists f$ and $f(p_1) = p_2$. $(x, 321)_1 \ncong (x, 123)_2$ even if $x_1 \cong x_2$ because different numbers are not isomorphic. $\{x, y\} \ncong (x, y)$ because $(x, y) := \{\{x\}, \{x, y\}\}$.

Needless to say, $\cong$ can express more complex examples like $(X, T, F, k, p)_1 \cong (X, T, F, k, p)_2$ where $(X, T)_{\forall i}$ is a topological space, $F_{\forall i}$ is a set of some ambient isotopies on X_i*[0,1], $k_{\forall i}$ is an embedding into X_i and $p_{\forall i}$ is a point in X_i.

Moreover, $\cong$ can express that two logical expressions containing general objects are isomorphic because all logical expressions can be expressed as nested graphs. Isomorphisms of general objects will be defined in words of elementary graph theory.

2 Isomorphism between general objects

Definition 2.1 (Deep member)**.** Take $\forall(c, x, y)$ such that: c is a chain of set membership. x is the maximum member of c. y is a member of c. Then y is said a deep member of x and you write as $y \in^{deep} x$. ■

For example:

$$m \in ... \in y \in .... \in x$$

For example:

$$\{y1, y2\} \in^{deep} \{y1, y2\}$$

$$y \in^{deep} \{1, \{2, y\}\}$$

Definition 2.2 ($\overset{\text{ID}}{Deep}$)**.** Take $\forall X$.

$\overset{\text{ID}}{Deep}(X) := \{p \mid p \in^{deep} X \overset{\text{and}}{\wedge} p \text{ is an identity } \}$.

■

Definition 2.3 (Nested graph)**.** All nested graph (V, E, U) is a directed graph (V, E) with the following extra definitions.

Take $\forall G$ as a nested graph. If no vertex v of G represents a nested graph, i.e., $\text{ID}^{-1}(v)$ is not a nested graph, then the nest number of G is defined to be 0. Otherwise the nested number of G is defined to be $m + 1$ where m denotes the maximum nested number over all nested graphs its vertices represent. And it is exclusively defined that the nested number of G is decidable and finite.

U is the set to collect $\forall u$ such that: $\exists v :\in V \overset{\text{and}}{\wedge} u{=}(v, \text{ID}^{-1}(v))$.

■

Warning: Consider U of the definition. U is important in the following respect. Take $\forall(e, t, p, x)$ such that: e is an expression of isomorphism. t is a (left | right) term of e. p is an identity $\overset{\text{and}}{\wedge} p \in^{deep} t \overset{\text{and}}{\wedge} p{=}\text{ID}(x) \overset{\text{and}}{\wedge} x \notin^{deep} t$. Then t does not account x in terms of e.

A supplement follows. Consider the relation between objects by representation in terms of the notion of identity. And consider the relation between objects by deep set membership. **The former is weaker than the latter**. In other words, all representation never implies any implicit deep set membership.

Definition 2.4 (Isomorphism between vertices of nested graphs)**.** Take $\forall(F, p_1, p_2)$ such that: F is a bijection between sets of identities. (p_1, p_2) are vertices of nested graphs. Recall that $p_{\forall i}$ represents $\text{ID}^{-1}(\text{p}_i)$.

Let $S_F := \overset{\text{ID}}{Deep}(\text{ domain}(F) \cup \text{image}(F) \text{ })$.

Define that: $*1 \equiv (*2 \overset{\text{or}}{\vee} *3)$

1. $p_1 \cong_F p_2$.

2. $*\mathrm{a} \overset{\text{and}}{\wedge} (\mathrm{a1} \overset{\text{or}}{\vee} \ldots \overset{\text{or}}{\vee} *\mathrm{a3})$.

a. take $\forall i \in \{1,2\} \overset{\text{and}}{\wedge} p_i$ is not a nested graph.

a1. $F(p_1) = p_2$

a2. $p_1 = p_2 \overset{\text{and}}{\wedge} \varnothing = \overset{\text{ID}}{Deep}(\{p_1, p_2\}) \cap S_F$.

a3. take $\forall i \in \{1,2\} \overset{\text{and}}{\wedge} \varnothing \neq \overset{\text{ID}}{Deep}(p_i) \cap S_F$.

3. $*\mathrm{b1} \overset{\text{and}}{\wedge} *\mathrm{b2}$.

b1. take $\forall i \in \{1,2\} \overset{\text{and}}{\wedge} p_i$ is a nested graph.

b2. $p_1 \cong^F p_2$.

$p_1 \cong^F p_2$, will be defined later.

Proposition 1. Take $\forall m :\in \mathbb{N}$. Let $\mathbb{G}_m$ denote the set of all nested graphs having nested numbers at most m. Then $*1 \to *2$.

1. For $\mathbb{G}_m$: Proposition 2 holds true.

2. For $\mathbb{G}_m$: $p1 \cong_F p2 \quad \equiv \quad p2 \cong_{F^{-1}} p1$.

Proof. Let Def be an alias for Definition 2.4. Take $\forall(F, p1, p2)$ as a counterexample. Hence $(*2 \overset{\text{or}}{\vee} *3)$ of Def holds for $(F, p1, p2)$ in place of $(F, p1, p2)$.

p1. Assume $*2$ of Def holds for $(F, p1, p2)$.

It is clear that each term of $(*\mathrm{a} \overset{\text{and}}{\wedge} (*\mathrm{a1} \overset{\text{or}}{\vee} \ldots \overset{\text{or}}{\vee} *\mathrm{a3}))$ is logically neutral between $(F, p1, p2)$ and $(F^{-1}, p2, p1)$. Hence each holds for $(F^{-1}, p2, p1)$ in place of $(F, p1, p2)$. A contradiction.

p2. Assume $*3$ of Def holds for $(F, p1, p2)$. It is trivial that $*\mathrm{b1}$ holds for $(F^{-1}, p2, p1)$ in place of $(F, p1, p2)$. And $*1$ of this proposition implies that Proposition 2 holds for $(F, p1, p2)$ in place of $(F, G1, G2)$. Hence $*\mathrm{b2}$ of Def holds for $(F^{-1}, p2, p1)$ too in place of $(F, p1, p2)$. A contradiction.

$\square$

Definition 2.5 (Isomorphism between nested graphs)**.** Take $\forall F$ as a bijection between sets of identities. Take $\forall \{G_i\}_{i \in \{1,2\}}$ as a pair of nested graphs. Decompose G_i as $\exists (V, E)_i$.

Then F is said an isomorphism from G_1 to G_2 if (*0 $\overset{\text{and}}{\wedge}$ *1). Define that: (*0 $\overset{\text{and}}{\wedge}$ *1) $\equiv$ *2. And define that: *2 $\rightarrow$ *3.

0. $\exists f$ as a graph isomorphism from G_1 to G_2.

1. Take $\forall v :\in V_1$, then $v \cong_F f(v)$.

2. $G_1 \cong^F G_2$.

3. $G_1 \cong G_2$.

$\blacksquare$

Proposition 2. Take $\forall m :\in \mathbb{N}$. Let $\mathbb{G}_m$ denote the set of all nested graphs having nested numbers at most m.

For $\mathbb{G}_m$:

$$G_1 \cong^F G_2 \quad \equiv \quad G_2 \cong^{F^{-1}} G_1$$

$\blacksquare$

Proof. Let Def be an alias for Definition 2.5.

Take $\forall (m, G_1, G_2, F)$ as a minimum counterexample by m. Though at least F^{-1} is a bijection between sets of identities. Hence the antecedent of Def holds for (G_2, G_1, F^{-1}) in place of (G_1, G_2, F) except (*0 $\overset{\text{and}}{\wedge}$ *1) of Def. By the way, f^{-1} is a graph isomorphism from G_2 to G_1.

Hence *1 of Def fails for $(G_2, G_1, F^{-1}, f^{-1})$ in place of (G_1, G_2, F, f).

q1. Hence: $\exists v :\in V_2 \overset{\text{and}}{\wedge} \neg(\ v \cong_{F^{-1}} f^{-1}(v)\ )$.

q2. Though: $(\ G_1 \cong^F G_2\ ) \rightarrow (\ f^{-1}(v) \cong_F f \circ f^{-1}(v)\ )$.

Let Prop be an alias for Proposition 1 together with its proof.

The right term of *q2 implies (*p1 $\overset{\text{or}}{\vee}$ *p2) of Prop.

If *p1 of Prop then (*q1 $\overset{\text{and}}{\wedge}$ *q2) is a contradiction.

Hence *p2 of Prop.

Hence Prop fails for $\mathbb{G}_n$ for $\exists n :< m$ because this proposition holds for $\mathbb{G}_n$. $\square$

Proposition 3. Take $\forall F, \forall (X,T)_{i \in \{1,2\}}$ such that: $(X,T)_{\forall i}$ is a topological space. (X_1, X_2) are homeomorhic by F as a homeomorphism from X_1 to X_2.

Then $(X,T)_1 \cong^F (X,T)_2$.

Proof. Let (DefV | DefG) be aliases for Definition (2.4 | 2.5) respectively. You can equivalently express $(X,T)_i$ as a nested graph G_i as follows. I show that the antecedent of DefG holds for (F, G_1, G_2) in place of (F, G_1, G_2).

As a prerequisite, define a function toG($\forall S$) to return a nested graph (V,E) as follows. As a supplement, S is a deep member of some $(X,T)_{\exists i}$.

$V := \{v \mid \exists d \in S \overset{\text{and}}{\wedge} \text{ if } d \text{ has some member then } v{=}\text{toG}(d) \text{ else } v = d\ \} \cup \{S\}$.

$E := \{(d1, d2) \in V^2 \mid d1 \in d2\ \}$.

Let $(V,E)_i{:=}\text{toG}(X_i)$.

Then $(V,E)_1 \cong^F (V,E)_2$.

As a proof, a graph isomorphism f can be defined as:

$f := F \cup \{(X_1,\ X_2)\}$.

It is trivial that f is a graph isomorphism from $(V,E)_1$ to $(V,E)_2$.

Consider (*a1, *a3) of DefV. Take $\forall v :\in V_1$, then $v \cong_F f(v)$.

Let $T_{12} := \{(t1, t2) :\in T_1 * T_2 \mid t2{=}\text{image}(F[t1])\ \}$.

Take $\forall (t1, t2) :\in T_{12}$.

Then $\text{toG}(t_1) \cong^F \text{toG}(t_2)$.

It is clear that an analogous proof exists.

Let $(V,E)_i{:=}\text{toG}(T_i)$.

Then $(V,E)_1 \cong^F (V,E)_2$.

As a proof, a graph isomorphism f can be defined as:

Let $T_{g12} := \{(\ \text{toG}(t1),\ \text{toG}(t2)) \mid (t1, t2) \in T_{12}\ \}$

$f := \{(T_1,\ T_2)\} \cup T_{g12}$.

It is trivial that f is a graph isomorphism from $(V,E)_1$ to $(V,E)_2$.

Consider (*a3, *b2) of DefV. Take $\forall v$, then $v \cong_F f(v)$.

Let $(V,E)_i{:=}\text{toG}((X,T)_i)$.

Then $(V,E)_1 \cong^F (V,E)_2$.

As a proof, a graph isomorphism f can be defined as:

$f := \{(\ (X,T)_1,\ (X,T)_2\ )\} \cup$

$\{($ toG(X_1), toG(X_2) $)\} \cup$
$\{($ toG(T_1), toG(T_2) $)\}$.
It is trivial that f is a graph isomorphism from $(V, E)_1$ to $(V, E)_2$.
Consider (*a3, *b2) of DefIsoV. Take $\forall v$, then $v \cong_F f(v)$.

$\square$

3 Applications in geometrical topology

3.1 Natural automorphism

Definition 3.1 (Natural automorphism)**.**
Let I:=[0,1], i.e., I is a unit interval.

As you know I is more than a topological space. It is defined a metric table, and decided which end point is 0.

Let (Y, T_Y) denote the topological space correspond to I where Y is the set of points and T_Y is the topology on Y. We use I as a bijective index set for Y.

Take $\forall X$ such that: X is a topological space defined the topology T_X. Let (P_{XY}, T_{XY}) denote the product space for $X * Y$. Recall that all point $\forall x :\in P_{XY}$ is an identity, i.e., $\mathrm{ID}^{-1}(x) \in X * Y$. We let $\mathrm{ID}^{-1}(x)$ represent x.

As you know, the topology of P_{XY} is said a product topology.

Take $\forall F$ as an injection from X*Y to P_{XY} such that (*0 $\overset{\text{and}}{\wedge}$ ... $\overset{\text{and}}{\wedge}$ *2). Hence F takes a pair of points as the input. Then F outputs a point which is the identity of a pair of points.

0. Take $\forall(x_1, y) :\in X$*Y.

1. $\exists x_2 :\in X \overset{\text{and}}{\wedge} F(x_1, y) = (x_2, y)$.

2. $F(\forall x, 0) := (x, 0)$.

Let F_0 be a solution of F with (X, I) fixed such that: $F_0(\forall x, \forall y) := (x, y)$.

Take $\forall F_i$ as a solution of F with (X, I) fixed such that: *1.

1. $(F_i, T_X, T_Y, T_{XY}) \cong (F_0, T_X, T_Y, T_{XY})$.

Let A denote the set of all solutions of F_i with (X, I) fixed. Take $\forall F_i :\in A, \forall g$ such that g is a function on X as $g(\forall x)$:=$\mathrm{ID}^{-1} \circ F_i(x, 1)$. Then g is said a **natural automorphism** on X. ■

Definition 3.2 (Natural-automorphic)**.**
Take $\forall X$ such that: X is a topological space. Take $\forall(s1, s2)$. Then $(s1, s2)$ are said X**-natural-automorphic** if: $\exists F$ as a super set of some natural automorphism on $X \overset{\text{and}}{\wedge} s1 \cong^F s2$.

3.2 Ideal set of sub sets

Definition 3.3 (Ideal set of sub spaces)**.**
Take $\forall(X, S)$ such that: X is a topological space. S is a set of sub spaces of X. S is said **ideal** if: (*1 $\overset{\text{and}}{\wedge}$ ... $\overset{\text{and}}{\wedge}$ *7).

1. Let S_P be the set to collect $\forall(s, p)$ such that $s \in S \overset{\text{and}}{\wedge} p \in s$.
2. $\exists B$ as an open basis to generate X.

   Regard B as a subset of the power set of Space(X).
3. Let $S_B := \{S_b \mid \exists b \in B \overset{\text{and}}{\wedge} S_b = \{(s, p) \in S_P \mid s \subset b\ \}\ \}$.
4. Let $S_P := \{\mathrm{ID}((s, p)) \mid (s, p) \in S_P\ \}$.
5. Let $S_B := \{S_b 1 \mid \exists S_b 2 \in S_B \overset{\text{and}}{\wedge} S_b 1 = \{\mathrm{ID}((s, p)) \mid (s, p) \in S_b 2\ \}\ \}$.
6. S_B is an open basis on S_P.
7. Members of S_P are pairwise S_P-natural-automorphic.

■

Conjecture 3.1 (Ideal set of sub spaces and ambient isotopies)**.**
Take $\forall(X, T, S, F, A)$ such that: S is an ideal set of sub spaces of (X, T) where T is the topology. F is the set to collect: $\forall f$: X*[0,1] $\to X$ such that f is an ambient isotopy. A is the set to collect $\forall(f, S_1, S_2)$ such that: f is a natural automorphism on $X \overset{\text{and}}{\wedge} (S1, S2)$ are subsets of $S \overset{\text{and}}{\wedge} (S_1, T) \cong^f (S_2, T)$.

Then (*1 $\overset{\text{and}}{\wedge}$ ... $\overset{\text{and}}{\wedge}$ *3) holds.

1. take $\forall(g, S_1, S_2) :\in A$.
2. $\exists f :\in F$ such that (let $f_1(\forall x :\in X{:=}f(x, 1)$, then $(S_1, T) \cong^{f_1} (S_2, T)$).
3. take $\forall t :\in (0, 1] \overset{\text{and}}{\wedge}$ (let $f_t(\forall x :\in X{:=}f(x, t)$, then $(S, T) \cong^{f_t} (S, T)$).

■

Definition 3.4 (Prime topological space)**.** Take $\forall X$ as a topological space. Then X is said prime if *1.

1. $\exists S$ as a set of sub spaces of $X \overset{\text{and}}{\wedge} S$ is ideal $\overset{\text{and}}{\wedge} S$ is an open basis to generate X.

■

Conjecture 3.2 (Ideal set of sub spaces)**.** Take $\forall(X, S)$ such that: X is a prime topological space. S is a set of sub spaces of X. Then S is ideal if $(*1 \overset{\text{and}}{\wedge} *2)$.

1. Members of $\{S\}^*X$ are pairwise X-natural-automorphic.

2. Let $S_p := \{(s, p) \mid s \in S \overset{\text{and}}{\wedge} p \in s\}$.

Members of $\{S\}^*S_p$ are pairwise X-natural-automorphic.

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