Isomorphism between general objects

Generalization of category theory

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1 Introduction

Prerequisites are only some first chapters of graduate level texts of general topology and graph theory; (1),(2). This article introduces a new fundamental language of mathematics which can be regarded as a generalization of category theory. As you know, two objects are regarded as equivalent if they are isomorphic.

For example, if $\{p_i\}_{i\in\{1,2,3\}}$ is a set of 3 objects pairwise isomorphic then (p_1, p_2) and (p_1, p_3) are isomorphic. Though (p_1, p_2) and (p_1, p_1) are not isomorphic

Analogously, assume that given two non-isomorphic objects, you may be able to abstracted the two by some rule to result isomorphic outputs. Then the original objects are regarded as equivalent for observations which accept the rule of abstraction. A new notion named "the identity of an object" will be used to abstract objects. We will define that: For all two objects, their identities are isomorphic. In addition, their identities are identical if and only if they are identical.

For example, $\{p_1, p_2\}$ and $\{\{p_1\}, \{p_3\}\}$ are not isomorphic whereas abstracting their members by their identities results isomorphic objects, namely $\{d_1, d_2\}$ and $\{D_1, D_3\}$ where d_i denotes the identity of p_i and D_i denotes the identity of $\{p_i\}$.

Take $\forall x$. Then x is said a general objects if it can be equivalently expressed as a nested graph. Blue texts indicate that the notions will be defined later.

This article defines when two given general objects, say (x, y), are said isomorphic, written $x \cong y$. For example, take $\forall(x, y)$ as numbers, then it will be defined that: $x \cong y \equiv x = y$. Contrary two points in the sense of [1]elementary geometry will be unconditionally said isomorphic.

Defining the main notion, I have met one problem that any complex object can be regarded as a point in mathematics. It yields problems because I can not define that all two points are unconditionally said isomorphic. To simply solve this problem, let us bring in a new notion "identity".

Definition 1.1 (Identity)**.** Take $\forall x$, define that: $\exists p$ written p=ID(x). p is said the **identity** of x. Also write, x=ID$^{-1}(p)$.

To distinguish x from ID(x), call x as a full point. All full point is not said a point unless x=ID(x).

Take $\forall(x, y)$, define that: $x = y \equiv$ ID(x)=ID(y). It happens to be that x=ID(x) if x is a point in the sense of elementary geometry.

Two identities will be unconditionally said isomorphic.

Definition 1.2 (Identity, point and vertex)**.**
Define that:

p is a point $\equiv$ p is a vertex $\equiv$ p is an identity.

w is a full point $\equiv$ w is a full vertex $\equiv$ w is a full identity.

By default you can regard ID(x) as a [2]titled point. If your texts are very explicit, then you can regard a full point as a titled point. With **the default definition**, all homeomorphic topological spaces will be unconditionally said isomorphic because their titled points are points and points will be unconditionally said isomorphic.

Definition 1.3 (Titled point)**.**
By default:

p is a titled point $\rightarrow$ p is a point.

A topological space X is a set of titled points defined the topology T. (X, T) also may be said a topological space.

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[1]Or geometry in ancient times.

[2]It may sound very bad in English. An alternative may be "a said point".

Warning: **Inside expressions of isomorphism**, no convention implicitly relates X to T; X is just a set of titled points, no topology is implicitly accompanied.

Let us continue to enumerate more examples of isomorphism. It will be said: $(X, T, p, 321)_1 \cong (X, T, p, 321)_2$ if $(X, T)_1$ and $(X, T)_2$ are homeomorphic by some homeomorphism $\exists f$ and $f(p_1) = p_2$. $(x, 321)_1 \ncong (x, 123)_2$ even if $x_1 \cong x_2$ because different numbers are not isomorphic. $\{x, y\} \ncong (x, y)$ because $(x, y) := \{\{x\}, \{x, y\}\}$.

Needless to say, $\cong$ can express more complex examples like $(X, T, F, k, p)_1 \cong (X, T, F, k, p)_2$ where $(X, T)_{\forall i}$ is a topological space, $F_{\forall i}$ is a set of some ambient isotopies on X_i*[0,1], $k_{\forall i}$ is an embedding into X_i and $p_{\forall i}$ is a titled point in X_i.

Moreover, $\cong$ can express that two logical expressions containing general objects are isomorphic because all logical expressions can be expressed as nested graphs. Isomorphisms of general objects will be defined in words of elementary graph theory.

2 Isomorphism between general objects

Definition 2.1 (Deep member)**.** Take $\forall(c, x, y)$ such that: c is a chain of set membership. x is the maximum member of c. y is a member of c. Then y is said a deep member of x and you write as $y \in^{deep} x$.

For example:

$$m \in ... \in y \in .... \in x$$

For example:

$$\{y1, y2\} \in^{deep} \{y1, y2\}$$

$$y \in^{deep} \{1, \{2, y\}\}$$

Definition 2.2 ($\overset{\text{ID}}{Deep}$)**.** Take $\forall X$.

$\overset{\text{ID}}{Deep}(X) := \{p \mid p \in^{deep} X \overset{\text{and}}{\wedge} p \text{ is an identity } \}$.

Definition 2.3 (Nested graph)**.** All nested graph (V, E) is a directed graph (V, E) of which vertices are defined to be **full vertices**.

Take $\forall G$ as a nested graph. If no titled vertex v of G a nested graph, then the nest number of G is defined to be 0. Otherwise the nest number of G is defined to be $m+1$ where m denotes the maximum nest number over all nested graphs as its full vertices. And it is exclusively defined that the nest number of G is decidable and finite.

Definition 2.4 (Isomorphism between full vertices of nested graphs)**.** Take $\forall(F, p_1, p_2)$ such that: F is a bijection between sets of identities. (p_1, p_2) are full vertices of nested graphs.

Let $S_F := \overset{\text{ID}}{Deep}(\text{ domain}(F) \cup \text{image}(F) \text{ })$.

Define that: $*1 \equiv (*2 \overset{\text{or}}{\vee} *3)$

1. $p_1 \cong_F p_2$.

2. $*\text{a} \overset{\text{and}}{\wedge} (\text{a1} \overset{\text{or}}{\vee} \dots \overset{\text{or}}{\vee} *\text{a3})$.

a. take $\forall i \in \{1, 2\} \overset{\text{and}}{\wedge} p_i$ is not a nested graph.

a1. $F(p_1) = p_2$

a2. $p_1 = p_2 \overset{\text{and}}{\wedge} \varnothing = \overset{\text{ID}}{Deep}(\{p_1, p_2\}) \cap S_F$.

a3. take $\forall i \in \{1, 2\} \overset{\text{and}}{\wedge} \varnothing \neq \overset{\text{ID}}{Deep}(p_i) \cap S_F$.

3. $*\text{b1} \overset{\text{and}}{\wedge} *\text{b2}$.

b1. take $\forall i \in \{1, 2\} \overset{\text{and}}{\wedge} p_i$ is a nested graph.

b2. $p_1 \cong^F p_2$.

$p_1 \cong^F p_2$, will be defined later.

Proposition 1. Take $\forall m :\in \mathbb{N}$. Let $\mathbb{G}_m$ denote the set of all nested graphs having nest numbers at most m. Then $*1 \to *2$.

1. For $\mathbb{G}_m$: Proposition 2 holds true.

2. For $\mathbb{G}_m$: $p1 \cong_F p2 \quad \equiv \quad p2 \cong_{F^{-1}} p1$.

Proof. Let Def be an alias for Definition 2.4. Take $\forall(F, p1, p2)$ as a counterexample. Hence $(*2 \overset{\text{or}}{\vee} *3)$ of Def holds for $(F, p1, p2)$ in place of $(F, p1, p2)$.

p1. Assume $*2$ of Def holds for $(F, p1, p2)$.

It is clear that each term of $(*\mathrm{a} \overset{\text{and}}{\wedge} (*\mathrm{a1} \overset{\text{or}}{\vee} \ldots \overset{\text{or}}{\vee} *\mathrm{a3}))$ is logically neutral between $(F, p1, p2)$ and $(F^{-1}, p2, p1)$. Hence each holds for $(F^{-1}, p2, p1)$ in place of $(F, p1, p2)$. A contradiction.

p2. Assume $*3$ of Def holds for $(F, p1, p2)$. It is trivial that $*\mathrm{b1}$ holds for $(F^{-1}, p2, p1)$ in place of $(F, p1, p2)$. And $*1$ of this proposition implies that Proposition 2 holds for $(F, p1, p2)$ in place of $(F, G1, G2)$. Hence $*\mathrm{b2}$ of Def holds for $(F^{-1}, p2, p1)$ too in place of $(F, p1, p2)$. A contradiction.

□

Definition 2.5 (Isomorphism between nested graphs)**.** Take $\forall F$ as a bijection between sets of identities. Take $\forall \{G_i\}_{i \in \{1,2\}}$ as a pair of nested graphs. Decompose G_i as $\exists (V, E)_i$.

Then F is said an isomorphism from G_1 to G_2 if $(*0 \overset{\text{and}}{\wedge} *1)$. Define that: $(*0 \overset{\text{and}}{\wedge} *1) \equiv *2$. And define that: $*2 \rightarrow *3$.

0. $\exists f$ as a graph isomorphism from G_1 to G_2.

1. Take $\forall v :\in V_1$, then $v \cong_F f(v)$.

2. $G_1 \cong^F G_2$.

3. $G_1 \cong G_2$.

Proposition 2. Take $\forall m :\in \mathbb{N}$. Let $\mathbb{G}_m$ denote the set of all nested graphs having nest numbers at most m.

For $\mathbb{G}_m$:

$$G_1 \cong^F G_2 \quad \equiv \quad G_2 \cong^{F^{-1}} G_1$$

Proof. Let Def be an alias for Definition 2.5.

Take $\forall(m, G_1, G_2, F)$ as a minimum counterexample by m. Though at least F^{-1} is a bijection between sets of identities. Hence the antecedent of Def holds for (G_2, G_1, F^{-1}) in place of (G_1, G_2, F) except (*0 $\overset{\text{and}}{\wedge}$ *1) of Def. By the way, f^{-1} is a graph isomorphism from G_2 to G_1.

Hence *1 of Def fails for $(G_2, G_1, F^{-1}, f^{-1})$ in place of (G_1, G_2, F, f).

q1. Hence: $\exists v :\in V_2 \overset{\text{and}}{\wedge} \neg(\ v \cong_{F^{-1}} f^{-1}(v)\ )$.

q2. Though: $(\ G_1 \cong^F G_2\ ) \rightarrow (\ f^{-1}(v) \cong_F f \circ f^{-1}(v)\ )$.

Let Prop be an alias for Proposition 1 together with its proof.

The right term of *q2 implies (*p1 $\overset{\text{or}}{\vee}$ *p2) of Prop.

If *p1 of Prop then (*q1 $\overset{\text{and}}{\wedge}$ *q2) is a contradiction.

Hence *p2 of Prop.

Hence Prop fails for $\mathbb{G}_n$ for $\exists n :< m$ because this proposition holds for $\mathbb{G}_n$. □

Proposition 3. Take $\forall F, \forall (X, T)_{i \in \{1,2\}}$ such that: $(X, T)_{\forall i}$ is a topological space. (X_1, X_2) are homeomorhic by F as a homeomorphism from X_1 to X_2.

Then $(X, T)_1 \cong^F (X, T)_2$.

Proof. Let (DefV | DefG) be aliases for Definition (2.4 | 2.5) respectively. You can equivalently express $(X, T)_i$ as a nested graph G_i as follows. I show that the antecedent of DefG holds for (F, G_1, G_2) in place of (F, G_1, G_2).

As a prerequisite, define a function toG($\forall S$) to return a nested graph (V, E) as follows. As a supplement, S is a deep member of some $(X, T)_{\exists i}$.

$V := \{v \mid \exists d \in S \overset{\text{and}}{\wedge}$ if d has some member then v=toG(d) else $v = d\ \} \cup \{S\}$.

$E := \{(d1, d2) \in V^2 \mid d1 \in d2\ \}$.

Let $(V, E)_i$:=toG(X_i).

Then $(V, E)_1 \cong^F (V, E)_2$.

As a proof, a graph isomorphism f can be defined as:

$f := F \cup \{(X_1,\ X_2)\}$.

It is trivial that f is a graph isomorphism from $(V, E)_1$ to $(V, E)_2$.

Consider (*a1, *a3) of DefV. Take $\forall v :\in V_1$, then $v \cong_F f(v)$.

Let $T_{12} := \{(t1, t2) :\in T_1 * T_2 \mid t2$=image($F[t1]$) $\}$.

Take $\forall (t1, t2) :\in T_{12}$.

Then toG(t_1) $\cong^F$ toG(t_2).

It is clear that an analogous proof exists.

Let $(V, E)_i$:=toG(T_i).

Then $(V, E)_1 \cong^F (V, E)_2$.

As a proof, a graph isomorphism f can be defined as:

Let $T_{g12} := \{($ toG($t1$), toG($t2$)) $\mid (t1, t2) \in T_{12}$ }

$f := \{(T_1, T_2)\} \cup T_{g12}$.

It is trivial that f is a graph isomorphism from $(V, E)_1$ to $(V, E)_2$.

Consider (*a3, *b2) of DefV. Take $\forall v$, then $v \cong_F f(v)$.

Let $(V, E)_i$:=toG($(X, T)_i$).

Then $(V, E)_1 \cong^F (V, E)_2$.

As a proof, a graph isomorphism f can be defined as:

$f := \{(\ (X, T)_1,\ (X, T)_2\ )\} \cup$

$\{($ toG(X_1), toG(X_2))} $\cup$

$\{($ toG(T_1), toG(T_2))}.

It is trivial that f is a graph isomorphism from $(V, E)_1$ to $(V, E)_2$.

Consider (*a3, *b2) of DefIsoV. Take $\forall v$, then $v \cong_F f(v)$.

□

3 Applications in geometrical topology

3.1 Natural automorphism

For the notation of the index set, $x_{i \in S}$, it may be written as [i] if the indexed set is known. For example, consider an indexing $f(x, y) = x^2 + 1$. Then [2,3]=7 if we know the indexed set is image(f).

Definition 3.1 (Natural automorphism)**.**
Let I:=[0,1], i.e., I is a unit interval.

As you know I is more than a topological space. It is defined a metric table, and decided which end point is 0.

Let (Y, T_Y) denote the topological space correspond to I where Y is the set of points and T_Y is the topology on Y. We use I as a bijective index set for Y.

Take $\forall X$ such that: X is a topological space defined the topology T_X. Let (P_{XY}, T_{XY}) denote the product space for $X * Y$. In the standard topology, all point p of P_{XY} is some full point $\exists(x, y) \in X * Y$. Define that: take $\forall p :\in P_{XY}$ $\overset{\text{and}}{\wedge}$ p=ID(x, y). Write p as $[x, y]$.

As you know, the topology of P_{XY} is said a product topology.

Take $\forall F$ as an injection from X*Y to P_{XY} such that (*0 $\overset{\text{and}}{\wedge}$ ... $\overset{\text{and}}{\wedge}$ *2). Hence F takes a pair of points as the input. Then F outputs a point which is the identity of a pair of points.

0. Take $\forall(x_1, y) :\in X{*}Y$.

1. $\exists x_2 :\in X \overset{\text{and}}{\wedge} F(x_1, y) = [x_2, y]$.

2. $F(\forall x, 0) := [x, 0]$.

Let F_0 be a solution of F with (X, I) fixed such that: $F_0(\forall x, \forall y) := [x, y]$.

Take $\forall F_i$ as a solution of F with (X, I) fixed such that: *1.

1. $(F_i, T_X, T_Y, T_{XY}) \cong (F_0, T_X, T_Y, T_{XY})$.

Let A denote the set of all solutions of F_i with (X, I) fixed. Take $\forall F_i :\in A, \forall g$ such that g is a function on X as $g(\forall x) := \mathrm{ID}^{-1} \circ F_i(x, 1)$. Then g is said a **natural automorphism** on X. $\blacksquare$

Definition 3.2 (Natural-automorphic)**.**
Take $\forall X$ such that: X is a topological space. Take $\forall(s1, s2)$. Then $(s1, s2)$ are said **X-natural-automorphic** if: $\exists F$ as a super set of some natural automorphism on $X \overset{\text{and}}{\wedge} s1 \cong^F s2$.

3.2 Ideal set of sub spaces

Definition 3.3 (Ideal set of sub spaces)**.**
Take $\forall(X, S)$ such that: X is a topological space. S is a set of sub spaces of X. S is said **ideal** if: (*1 $\overset{\text{and}}{\wedge}$ ... $\overset{\text{and}}{\wedge}$ *7).

1. Let S_P be the set to collect $\forall(s, p)$ such that $s \in S \overset{\text{and}}{\wedge} p \in s$.

2. $\exists B$ as an open basis to generate X.

Regard B as a subset of the power set of X.

3. Let $S_B := \{S_b \mid \exists b \in B \overset{\text{and}}{\wedge} S_b = \{(s, p) \in S_P \mid s \subset b\ \}\ \}$.

4. Let $S_P := \{\mathrm{ID}((s, p)) \mid (s, p) \in S_P\ \}$.

5. Let $S_B := \{S_b1 \mid \exists S_b2 \in S_B \overset{\text{and}}{\wedge} S_b1 = \{\mathrm{ID}((s,p)) \mid (s,p) \in S_b2\ \}\ \}$.

6. S_B is an open basis on S_P.

7. Members of S_P are pairwise S_P-natural-automorphic.

$\blacksquare$

Conjecture 3.1 (Ideal set of sub spaces and ambient isotopies)**.** Take $\forall(X,T,S,F,A)$ such that: S is an ideal set of sub spaces of (X,T) where T is the topology. F is the set to collect: $\forall f$: X*[0,1] $\to X$ such that f is an ambient isotopy. A is the set to collect $\forall(g,S_1,S_2)$ such that: g is a natural automorphism on $X \overset{\text{and}}{\wedge} (S1,S2)$ are subsets of $S \overset{\text{and}}{\wedge} (S_1,T) \cong^g (S_2,T)$.

Then (*1 $\overset{\text{and}}{\wedge}$ ... $\overset{\text{and}}{\wedge}$ *4) holds.

1. take $\forall(g,S_1,S_2) :\in A$.

2. $\exists f :\in F$

3. take $\forall t :\in (0,1] \overset{\text{and}}{\wedge}$ let $f_t(\forall x :\in X){:=}f(x,t)$.

4. $(f_t,S,S) \in A \overset{\text{and}}{\wedge}$ if $t=1$ then $f_t = g$.

$\blacksquare$

Definition 3.4 (Prime topological space)**.** Take $\forall X$ as a topological space. Then X is said prime if *1.

1. $\exists S$ as a set of sub spaces of $X \overset{\text{and}}{\wedge} S$ is ideal $\overset{\text{and}}{\wedge} S$ is an open basis to generate X.

$\blacksquare$

Conjecture 3.2 (Ideal set of sub spaces)**.** Take $\forall(X,S)$ such that: X is a prime topological space. S is a set of sub spaces of X. Then S is ideal if (*1 $\overset{\text{and}}{\wedge}$ *2).

1. Members of $\{S\}$*X are pairwise X-natural-automorphic.

2. Let $S_p := \{(s,p) \mid s \in S \overset{\text{and}}{\wedge} p \in s\ \}$.

   Members of $\{S\}$*S_p are pairwise X-natural-automorphic.

$\blacksquare$

4 Abstract conjectures

4.1 Main abstract conjecture

Conjecture 4.1 (Abstract conjecture of ideal set and metric)**.**
Take $\forall(M, X, S1, f)$ such that *A.

Consider (*B → *C). It is independent from the topological class of members of $S1$ if f is **enough general** for topological classes of members of solutions of $S1$ with (M, X) fixed.

The claim converges to true if generality approaches to the perfect.

A. $*1 \overset{\text{and}}{\wedge} \ldots \overset{\text{and}}{\wedge} *3$.

1. M is a metric table to define X as a topological space $\overset{\text{and}}{\wedge}$ X is prime.
2. $S1$ is an ideal set of sub spaces of X.
3. f is a function on $S1$.

B. $*1 \overset{\text{and}}{\wedge} \ldots \overset{\text{and}}{\wedge} *3$.

1. Take $\forall k1 :\in S1$
2. Let $S2 := \{k2 \in S1 \mid f(k2) = f(k1)\ \}$.
3. $S2$ is unique for (M, X).

   Unique?: For example, take $\forall x :\in \overset{\text{ID}}{Deep}(X)$. If $S2$ is the set to collect $\forall k :\in S1$ such that $x \in \overset{\text{ID}}{Deep}(k)$ then $S2$ is not unique for (M, X) **in general** because x is not unique for (M, X) in general. Instead $S2$ is unique for (M, X, x).

C. $S2$ is ideal.

■

4.2 Application on knots

Let Conj be an alias for Conjecture 4.1. Let Def be an alias for the following Definition 4.1. The antecedent of Conj apparently holds for (M, X, K, K_f, f) of Def in place of $(M, X, S1, S2, f)$. And f is apparently enough general as required in Conj.

Definition 4.1 (A set of knots)**.** Take $\forall(M, X, K, K_f)$ such that:
M is a metric table to define X as a Euclidean space of 3-dimension.
Take $\forall k_0$ as a knot and a subspace of X.
K is the set to collect $\forall k$ such that: (k, k_0) are X-natural-automorphic.

$K_f := \{k \in K \mid f(k) = f(k_0)\ \}$.

Definition of f:

- $j1(\forall k :\in K) := \{j \mid$
  j is an orthogonal [3]projection of k onto some infinite plane $\}$.
- $j2(\forall k :\in K) := \{j \in j1(k) \mid$
  $\neg\ (\exists p \ \overset{\text{and}}{\wedge}\ p \in \text{image}(j)\ \overset{\text{and}}{\wedge}\ \mid j^{-1}(p) \mid > 2)\ \}$.
- $j3(\forall k :\in K) := \{n \mid$
  $\exists j \ \overset{\text{and}}{\wedge}\ j \in j2(k)\ \overset{\text{and}}{\wedge}\ n$ is the number of [4]double points on j $\}$.
- $f(\forall k :\in K) := \{m \mid$
  m is the maximal member from $j3(k)$ $\}$.

5 Notation

- take $\forall x \equiv$ for $\forall x \equiv \forall x$.

  In other words, "take" means nothing.
- $\forall x$ as a set $\equiv \forall x$ such that x is a set.
- Assume y is dependent on z then:

  $\forall x$ as a solution of y with z fixed $\equiv \forall x$ as a solution of y.
- $\{x \mid p(x)\ \} \equiv$ the set to collect $\forall x$ such that $p(x)$.

In definitions, I rarely write "if and only if". In stead I write "if" even if I know that "if and only if" can replace the "if".

References

[1] Glen E. Bredon,Topology and Geometry, Springer, ISBN 978-1-4419-3103-0

[2] Reinhard Diestel, Graph Theory, Springer-Verlag, ISBN 0-387-98976-5

[3]Hence, j is a function from k to an infinite plane.

[4]Double point?: That is, the inverse image of a double point has exactly 2 distinct points of k; no matter the double point represents a crossing or a tangent point.