Isomorphism of memBers

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1 Introduction

Definition 1.1 (memBer)**.** Take $\forall x$ such that (there exists $\exists S$ such that $x \in S$). Then x is said a memBer.

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This article:(1) defines that (x,y) as memBers are isomorphic,(2) proves that if (x,y) as memBers are isomorphic then (y,x) as memBers are isomorphic too, (3) proves that all homeomorphic topological spaces are isomorphic as memBers,(4) defines that a memBer $S1$ is a minor of a memBer $S2$.
I expect that readers will realize that the newly defined isomorphisms are somehow more fundamental than ,e.g., homeomorphisms of topological spaces. Because "homeomorphisms" logically resolve to "isomorphisms of memBers" whereas the inverse of it does not hold.

2 Deep member

Definition 2.1 (Deep member of memBer)**.** This definition uses a style of recursion.
Take $\forall(x, y)$ such that *1 holds. Then it is said as *2 and written as *3.

1 $x = y$ else (there exists $\exists z$ such that $x \in z \in^{deep} y$).

2 x is a deep member of y

3 $x \in^{deep} y$

■

Definition 2.2 (Space of memBer)**.** Take $\forall(x, y)$ such that *1 holds. Then it is said as *2.

1 $y = \{d \mid d \in^{deep} x$ then d is a point $\}$.

2 y is the space of x.

Definition 2.3 (Proper deep member of memBer)**.** Take $\forall x$ such that *1 holds. Then it is said as *2.

1 There exists $\exists z$ such that $x \in^{deep} z \in y$.

2 x is a proper deep member of y.

Definition 2.4 (Paradoxical memBer)**.** Take $\forall x$ such that (*1 else *2) holds. Then it is said as *2.

1 "$\{\ d \mid d \in^{deep} x\ \}$" is not definable.

2 x is a proper deep member of x.

3 x is paradoxical.

Definition 2.5 (Non-paradoxical memBer)**.** In the rest, informally speaking, "$\forall x$" must be interpreted as "$\forall x$ which is not paradoxical".
Formally, in the rest, the domain of discourse does not include any paradoxical memBer.

3 Notations

Definition 3.1 (Symbols)**.** *1 $\equiv$ *2 and *3 $\equiv$ *4.

1 A then B.

2 $A \ {}_{then}\wedge \ B$.

3 A else B.

4 $A \ {}_{else}\vee \ B$.

As a remark, *1 says that $((A \wedge B)$ and (the meaning of B may depend on that A holds)).
As a remark, *3 says that $((A \vee B)$ and (the meaning of B may depend on that A fails)).

Definition 3.2 (Propositional function)**.** Following *1 is true because *1 calls *2 passing integer 123 as the value of x. In other words, *2 is a propositional function.

1 Let x:=123 then *2 holds.

2 $x \in Z$.

Definition 3.3 (Colon)**.** A colon(:) may be used in introducing a new variable. Some examples follow.

1 x:=1.

2 $\forall x :\in S$.

Definition 3.4 (Restriction of binary relation)**.** Take $\forall(L, X, Y, X1)$ such that *1 holds. Then define *2.

1 L is a binary relation on $X * Y \ {}_{then}\wedge\ X1 \subset X$.

2 $L[X1]$:=$\{\ (x, y) \in L \mid x \in X1\ \}$.

4 Isomorphic memBers

Definition 4.1 (Isomorphic memBers)**.** This definition uses a style of recursion.

Take $\forall(x, y, F)$ such that *A holds. Then define *B.

A $*0 \ {}_{else}\vee \ *1 \ {}_{else}\vee \ (*2 \ {}_{then}\wedge \ *3)$.

0 $space(x) = \varnothing \ {}_{then}\wedge \ x = y$.

1 (x, y) are points ${}_{then}\wedge \ (x, y) \in F$

2 $F[space(x)]$ is an [1]injection from*to $space(x)$*$space(y)$.

3 There exists $\exists f$ such that $(*4 \ {}_{then}\wedge \ *5 \ {}_{then}\wedge \ *6)$.

4 f is a bijection from*to $x * y \ {}_{then}\wedge \ f \neq \varnothing$.

5 Take $\forall(m1, m2) \in f$.

6 *A holds for $(m1, m2, F)$ in place of $(x, y, F)))$.

B (x, y) are said isomorphic by F as an isomorphism. ■

[1]Some alternative definitions are to weaken *2 by replacing "injection" with "function" or "binary relation". Though such weakend condition should not be titled as "isomorphic".

5 Minors of memBers

Definition 5.1 (Minors)**.** Take $\forall(x, y)$ such that *A holds. Then it is said as *B.

A *1 ${}_{then}\wedge$ *2.

- **1** Take $\forall d$. Then $d \in^{deep} x \Rightarrow$ d$\in^{deep} y$.
- **2** Take $\forall(d1, d2, d3)$ such that (take $\forall d \in \{d1, d2, d3\}$, then $d \in^{deep} x$). Then (*3 $\Leftarrow$ *4).
- **3** $((x, d1, d3)$,$(x, d2, d3))$ are isomorphic.
- **4** $((y, d1, d3)$,$(y, d2, d3))$ are isomorphic.

B *5 ${}_{then}\wedge$ *6.

- **5** x is a minor of y.
- **6** $x \leq^{minor} y$.

6 Deep graph

Definition 6.1 (Deep graph)**.** Take $\forall(x, V, E, P, P_{ord})$ such that *A holds. Then define *B.

A *1 ${}_{then}\wedge$ *2 ${}_{then}\wedge$ *3 ${}_{then}\wedge$ *4

- **1** $V = \{d \mid d \in^{deep} x\}$.
- **2** $E = \{(d1, d2) \in V * V \mid d2 \in d1\}$.
- **3** $P = \{(v, |p|) \mid v \in V \;\; {}_{then}\wedge$ p is a shortest path from x to v on (V, E) $\}$.
- **4** P_{ord} is P with the order on $P * P$ as *5.
- **5** $((v1, n1) < (v2, n2)) \equiv n1 < n2$.

B *6 ${}_{then}\wedge$ *7 ${}_{then}\wedge$ *8

- **6** (V, E) is said the deep graph of x.

7 Take $\forall(v, n) :\in P$. Then n is said the depth of v on x.

8 There exists $\exists(v, n) :\in P_{ord}$ such that "(v, n) is a maximum member of P_{ord}". Then n is said "the maximum depth on x".

$\blacksquare$

7 Propositions

Definition 7.1. In this section, *Def refers to the definition titled as "Isomorphic memBers".
And *1 $\equiv$ *2, without any explicit proof because it is trivial by *Def.
And *3 holds, without any explicit proof because it is trivial by *Def.

1 (x_i, y_i) are isomorphic by F_i.

2 *Def.A holds for (x_i, y_i, F_i) in place of (x, y, F).

3 Take $\forall(x, y, F)$ such that (*4 ${}_{else}\vee$ *5). Then *6 holds.

4 $(space(x) = \varnothing \ {}_{then}\wedge \ x = y)$.

5 $((x, y)$ are points ${}_{then}\wedge$ $(x, y) \in F)$.

6 (x, y) are isomorphic by F. $\blacksquare$

Proposition 1 (Empty space)**.** Take $\forall(x, y, F)$ such that *A1. Then $x = y$.

A1 *Def.A holds for (x, y, F) ${}_{then}\wedge$ $space(X) = \varnothing$.

$\blacksquare$

Proof.

- Assume it is false.
- There exists $\exists(x, y, F)$ such that (x, y, F) is a minimum counterexample by the maximum depth on x.
- Let us follow *Def.A for (x, y, F).
- At *0, it fails because $x \neq y$.
- At *1, it fails because $space(x) = \varnothing$.
- Hence (*2 ${}_{then}\wedge$ *3) holds.

- Consider f of *3 together with that $x \neq y$.
- Hence there exists $\exists(m1, m2) \in f$ such that $m1 \neq m2 \ {}_{then}\wedge$ *Def.A holds for $(m1, m2, F)$.
- Additionally $space(m1) \subset space(x) = \varnothing$.
- Moreover $m1 < x1$ compared by the order by their maximum depths on them respectively.
- Hence $(m1, m2, F)$ is a counterexample smaller than a minimum counterexample.
- The assumption is false.

$\square$

Proposition 2 (Members' isomorphisms as consequent)**.** Take $\forall(x, y, F)$ such that (*A1 ${}_{then}\wedge$ *A2). Then (*B1 ${}_{then}\wedge$ *B2) holds.

A1 *Def.A holds for (x, y, F) in place of $(x1, x2, F)$.

A2 F is an injection.

B1 Take $\forall m1 :\in^{deep} x$. Then there exists $\exists m2 :\in^{deep} y$ such that *Def.A holds for $(m1, m2, F)$ in place of $(x1, x2, F)$.

B2 Take $\forall m2 :\in^{deep} y$. Then there exists $\exists m1 :\in^{deep} x$ such that *Def.A holds for $(m1, m2, F)$ in place of $(x1, x2, F)$.

*Proof of *B1.*

- Assume it is false.
- Then there exists $\exists(x, y, F, m1)$ such that $(x, y, F, m1)$ is a minimum counterexample by the depth of $m1$ on x.
- It is trivial that $(x \neq m1)$.
- There exists $\exists x1$ such that ($m1 \in x1 \ {}_{then}\wedge$ $x1$ in place of $m1$ is not a counterexample).
- Hence *B1 should hold for $x1$ in place of $m1$.
- Hence there exists $2 :\in^{deep} y$ such that *Def.A holds for $(x1, y2, F)$.
- Let us follow *Def.A for $(x1, y2, F)$.

- Assume *0 holds.
- Then $space(x1) = \varnothing \ {}_{then}\wedge \ x1 = y2$.
- Hence $space(m1) = \varnothing \ {}_{then}\wedge \ m1 = m1 \ {}_{then}\wedge \ m1 \in^{deep} y$.
- Hence *B1 holds for $m1$ in place of $m1$.
- Hence $(x, y, F, m1)$ is not a counterexample.
- Hence the last assumption is false.
- Assume *1 holds.
- Hence ($x1$ is a point) ${}_{then}\wedge$ $(m1 \in x1)$.
- Hence the last assumption is false.
- Hence (*2 ${}_{then}\wedge$ *3) must hold.
- Hence, (*4 ${}_{then}\wedge$ *5 ${}_{then}\wedge$ *6) holds.
- Hence *B1 holds for $m1$ in place of $m1$.
- Hence $m1$ is not a counterexample.
- The first assumption is false.

□

*Proof of *B2.*

- Assume it is false.
- Then there exists $\exists(x, y, F, m2)$ such that $(x, y, F, m2)$ is a minimum counterexample by the depth of $m2$ on y.
- It is trivial that $(y \neq m2)$.
- There exists $\exists y2$ such that ($m2 \in y2 \ {}_{then}\wedge \ y2$ in place of $m2$ is not a counterexample).
- Hence *B2 should hold for $y2$ in place of $m2$.
- Hence there exists $1 :\in^{deep} x$ such that *Def.A holds for $(x1, y2, F)$.
- Let us follow *Def.A for $(x1, y2, F)$.
- Assume *0 holds.

- Then $space(x1) = \varnothing \ _{then}\wedge \ x1 = y2$.
- Hence $space(m2) = \varnothing \ _{then}\wedge \ m2 = m2 \ _{then}\wedge \ m2 \in^{deep} x$.
- Hence *B2 holds for $m2$ in place of $m2$.
- Hence $(x, y, F, m2)$ is not a counterexample.
- Hence the last assumption is false.
- Assume *1 holds.
- Hence ($y2$ is a point) $_{then}\wedge$ ($m2 \in y2$).
- Hence the last assumption is false.
- Hence (*2 $_{then}\wedge$ *3) must hold.
- Hence, (*4 $_{then}\wedge$ *5 $_{then}\wedge$ *6) holds.
- Hence *B2 holds for $m2$ in place of $m2$.
- Hence $m2$ is not a counterexample.
- The first assumption is false.

□

Proposition 3 (Surjectivity)**.** Take $\forall(x, y, F)$ such that *A1. Then $F[space(x)]$ is a surjection from*to $space(x) * space(y)$.

A1 *Def.A holds for (x, y, F) by (*2 $_{then}\wedge$ *3).

■

Proof.

- Assume it is false.
- There exists $\exists(x, y, F)$ such that (x, y, F) is a counterexample.
- Then (there exists $\exists p_x :\in space(x)$ such that $p_x \notin domain(F)$) $_{else}\vee$ (there exists $\exists p_y :\in space(y)$ such that $p_y \notin image(F)$).
- Though this logical disjunction contradicts to the proposition titled as ”MemBers’ isomorphisms as consequent” as follows.
- A contradiction for p_x:

- There exists $y2 :\in^{deep} y$ such that *Def.A holds for $(p_x, y2, F)$.
- Let us follow *Def.A for $(p_x, y2, F)$. Then (*0 $_{else}\vee$ (*2 $_{then}\wedge$ *3)) fails.
- Hence *1 holds. Hence $(p_x, y2) \in F)$. Hence $p_x \in domain(F)$. A contradiction.
- A contradiction for p_y:
- There exists $x1 :\in^{deep} x$ such that *Def.A holds for $(x1, p_y, F)$.
- Let us follow *Def.A for $(x1, p_y, F)$. Then (*0 $_{else}\vee$ (*2 $_{then}\wedge$ *3)) fails.
- Hence *1 holds. Hence $(x1, p_x) \in F)$. Hence $p_y \in image(F)$. A contradiction.
- Finally, the assumption is false.

□

Proposition 4 (Symmetric)**.** Take $\forall B$ such that B is a binary relation. Then let B^{-1} denote $\{(b2, b1) \mid (b1, b2) \in B\}$.
Take $\forall(x, y, F)$. Then *A1 implies *A2.

A1 Def.A holds for (x, y, F).

A2 Def.A holds for (y, x, F^{-1}).

■

Proof.

- Assume it is false.
- There exists $\exists(x, y, F)$ such that it is a minimum counterexample by the maximum depth on x.
- Let us follow *Def.A for (x, y, F) in terms of *A1.
- Assume *0 holds for (x, y, F) in terms of *A1.
- Hence $space(x) = \varnothing$ $_{then}\wedge$ $x = y$.
- Hence $space(y) = \varnothing$ $_{then}\wedge$ $y = x$.
- Hence *0 holds for (x, y, F) in terms of *A2.
- Hence the last assumption is false.

- Assume *1 holds for (x, y, F) in terms of *A1.
- Hence (x, y) are points ${}_{then}\wedge$ $(x, y) \in F$.
- Hence (y, x) are points ${}_{then}\wedge$ $(y, x) \in F^{-1}$.
- Hence *1 holds for (x, y, F) in terms of *A2.
- Hence the last assumption is false.
- Hence (*2 ${}_{then}\wedge$ *3) must hold for (x, y, F) in terms of *A1.
- Consider the proposition titled as "Surjectivity".
- Then $F[space(x))$ is a bijection from*to $space(x) * space(y)$.
- Hence $F^{-1}[space(y))$ is a bijection from*to space(y)*space(x).
- Hence *2 holds for (x, y, F) in terms of *A3.
- Hence *3 must fail for (x, y, F) in terms of *A3.
- At same time, *3 hold for (x, y, F) in terms of *A1.
- Hence there exists $\exists(m1, m2) \in f$ such that (
- *Def.A holds for $(m1, m2, F)$ ${}_{then}\wedge$
- *Def.A does not hold for $(m2, m1, F^{-1})$.
-).
- Hence $(m1, m2, F)$ is a counterexample.
- Moreover $(m1, m2, F) < (x, y, F)$ compared by the order by the maximum depths on $m1$ and x respectively.
- It contradicts to the title of (x, y, F) as a minimum counterexample.
- Hence the first assumption is false.

$\square$

Proposition 5 (Members' isomorphisms as antecedent)**.** Take $\forall(x, y, F, f)$ such that (*A1 ${}_{then}\wedge$ *A2 ${}_{then}\wedge$ *A3). Then *B holds.

A1 F is an injection.

A2 f is a bijection from*to x*y ${}_{then}\wedge$ $f \neq \varnothing$.

A3 Take $\forall(m1, m2) :\in f$. Then *Def.A holds for $(m1, m2, F)$.

B *Def.A holds for (x, y, F) in place of (x, y, F).

Proof.

- Assume B fails.
- Hence there exists $\exists(x, y, F)$ such that *Def.A fails for (x, y, F).
- Let us follow *Def.A for (x, y, F).
- (*0 fails ${}_{then}\wedge$ *1 fails ${}_{then}\wedge$ (*2 fails ${}_{else}\vee$ *3 fails)).
- Assume *2 fails.
- Hence $F[space(x)]$ is not an injection from*to $space(x) * space(y)$.
- Consider the proposition titled as "MemBers' isomorphisms as consequent".
- Then $(space(x) = \varnothing = space(y))$ ${}_{else}\vee$ $(space(x) \neq \varnothing \neq space(y))$.
- Meanwhile F is an injection by *A1.
- Hence $(space(x) \neq \varnothing \neq space(y))$ because otherwise *2 holds.
- Hence there exists $\exists(p_x, p_y) :\in space(x) * space(y)$ such that
  $p_x \notin domain(F)$ ${}_{else}\vee$ $p_y \notin image(F)$.
- Consider *A2,*A3 and the proposition titled as "Members' isomorphisms as the consequent".
- There exists $\exists y2 \in^{deep} y$ such that *Def.A holds for $(p_x, y2, F)$.
- There exists $\exists x1 \in^{deep} x$ such that *Def.A holds for $(x1, p_y, F)$.
- Meanwhile, for each of the 2 lines just above,
  (*Def.A holds only at *1) because
  (each of (p_x, p_y) is a point).
- Hence $p_x \in domain(F)$ ${}_{then}\wedge$ $p_y \in image(F)$.
- Hence the last assumption is false.
- Hence *3 must fail.
- Hence *3 fails for f in place of f.

- Though by (*A2 $_{then}\wedge$ *A3), (*4 $_{then}\wedge$ *5 $_{then}\wedge$ *6) holds.
- Hence the first assumption is false.

$\square$

Proposition 6 (Topological space)**.** Take $\forall((X1, T1), (X2, T2))$ such that *A holds. Then *B1 $\Rightarrow$ *B2.

A Take $i \in \{1, 2\}$. Then (X_i, T_i) is a topological space.

B1 $((X1, T1), (X2, T2))$ are homeomorphic.

B2 There exists $\exists F$ such that $((X1, T1), (X2, T2))$ are isomorphic by F.

Proof.
First of all, let me define how to express a pair as a set.
Take $\forall(p1, p2)$. Then $(p1, p2)$=$\{Pair, p1, \{p1, p2\}\}$.
In the expression, "Pair" is a constant keyword.
By the way, *B1 implies *C.

C There exists $\exists(G, g)$ such that (*C1 $_{then}\wedge$ *C2 $_{then}\wedge$ *C3 $_{then}\wedge$ *C4).

C1 G is a bijection from $X1$ to $X2$.

C2 G is a homeormorphism for *B1.

C3 g is a bijection from $T1$ to $T2$.

C4 Take $forall(t1, t2) :\in g$. Then (G takes $t1$ to $t2$).

Consider the previous proposition titled as Members' isomorphisms as antecedent and refer it as *P.
P accepts arguments as follows.

D1 *P accepts $(X1, X2, G, G)$ in place of (x, y, F, f).

D2 Take $\forall(t1, t2) :\in g$. Then *P accepts $(t1, t2, G, G)$ in place of (x, y, F, f).

D3 *P accepts $(T1, T2, G, g)$ in place of (x, y, F, f).

D4 *P accepts (
$\{X1, T1\}$,
$\{X2, T2\}$,
G,
$\{(X1, X2), (T1, T2)\}$
) in place of (x, y, F, f).

D5 *P accepts (
$\{Pair, X1, \{X1, T1\}\}$,
$\{Pair, X2, \{X2, T2\}\}$,
G,
$\{(Pair, Pair), (X1, X2), (\{X1, T1\}, \{X2, T2\})\}$
) in place of (x, y, F, f).

Hence *P implies (*E1 $_{then}\wedge$ *E2 $_{then}\wedge$ *E3 $_{then}\wedge$ *E4 $_{then}\wedge$ *E5).
Finally, *E5 implies this proposition.

E1 $(X1, X2)$ are isomorphic by G.

E2 Take $\forall(t1, t2) :\in g$. Then $(t1, t2)$ are isomorphic by G.

E3 $(T1, T2)$ are isomorphic by G.

E4 $\{X1, T1\}, \{X2, T2\})$ are isomorphic by G.

E5 $(\{Pair, X1, \{X1, T1\}\}, \{Pair, X2, \{X2, T2\}\})$ are isomorphic by G.

$\square$

References

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