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# CHARCTERISTIC POLYNOMIAL AND EIGENVALUES OF ANTIADJACENCY MATRICES OF DIRECTED UNICYCLIC FLOWER VASE GRAPH

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# ABSTRACT

This research discussed characteristic polynomial and eigenvalues of antiadjacency matrix of directed unicyclic flower vase graph. The entries in antiadjacency matrix of a directed graph can be represented as the presence or absence of directed arc from one vertex to the others. If $A$ is the adjacency matrix of a graph $G$ then the antiadjacency matrix $B$ of graph $G$ is $B = J - A$, where $J$ is the square matrix with all entries equal to one. The general form of characteristic polynomial coefficients of the antiadjacency matrix of directed unicyclic flower vase graph can be obtained by calculating the sum of determinants of the antiadjacency matrices of all induced cyclic and acyclic subgraphs, while the eigenvalues were obtained by using polynomial factorization and Horner's method. In this paper, we give the characteristic polynomial coefficients and eigenvalues of antiadjacency matrix of directed unicyclic flower vase graph. The characteristic polynomial can be considered as a function that depends on the number of vertex.

**Keywords**: Flower vase graph; Unicylic graph; Antiadjacency matrix; Characteristic polynomial; Eigenvalues

# INTRODUCTION

Wildan (2015)

Characteristic polynomials of the adjacency matrix and antiadjacency matrix of the directed cyclic graph.

Putra (2017)

Polynomial characteristics of the antiadjacency matrix from directed cycle.

# BASIC DEFINITIONS

- Directed graph $\vec{G}$ can be defined as an ordered pair of two sets $V$ and $E$, where $V$ is a non empty finite set and $E$ is the set of ordered pairs of members of $V$. The set $V$ is called the set of vertices and the set $E$ is called the set of directed arcs. If $a = (u, v)$ is the arc of a directed graph $\vec{G}$, then $u$ is said to be adjacent to $v$ and $v$ is adjacent from $u$ (Chartrand, Lesniak & Zhang, 2016).

- The adjacency matrix of a graph $\vec{G}$ is the matrix $A = [a_{ij}]$ with size $n \times n$ with $a_{ij} = 1$ if there is a directed arc from vertex $v_i$ to vertex $v_j$ and $a_{ij} = 0$ for the others. Meanwhile, the antiadjacency matrix of $\vec{G}$ is $B = J - A$ where $J$ is a matrix of size $n \times n$ with all entries of $1$ (Bapat, 2010).

- Flower vase graph $C_nS_m$ is constructed from star graph and cycle graph connected by arcs from the center vertex of the star graph $S_m$ with one vertex on cycle graph $C_n$ (Ahmad, 2012). A directed flower vase graph is a graph that can be cyclic by giving direction for each arc on its cycle graph as can be seen in Figure 1.

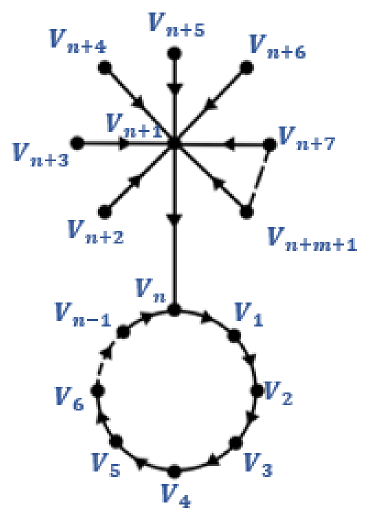



Figure 1. Directed unicyclic flower vase graph $\overrightarrow{C_nS_m}$

# KNOWN RESULTS

**Theorem 1**. Suppose $\vec{G}$ is a directed acyclic graph with $V(\vec{G}) = (v_1, v_2, v_3, \ldots, v_{n-1}, v_n\}$ and $B$ is the antiadjacency matrix of $\vec{G}$, then $det(B) = 1$, if $\vec{G}$ has a Hamiltonian path and $det(B)$ = 0, if $\vec{G}$ does not have a Hamiltonian path (Bapat, 2010).

**Theorem 2.** Suppose B is the antiadjacency matrix of a directed cyclic graph $\overrightarrow{(C_n)}$, then $det(B(\overrightarrow{C_n})) = n - 1$ (Putra, 2017).

**Theorem 3.** Let $P\left(B(\vec{G})\right) = \lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \cdots + b_{n-1}\lambda + b_n$ be the characteristic polynomial of the antiadjacency matrix $B(\vec{G})$ of a directed graph, let $\left|B(\langle U\rangle_{acyclic})_i{}^{(j_1)}\right|$ is the determinant of the antiadjacency matrix of an acyclic induced subgraph with $i$ vertices and $j_1 = 1,2, \ldots, w_1$ where $w_1$ is the number of acyclic induced subgraphs $\langle U\rangle_{acyclic}$ with $i$ vertices of the cyclic directed graph $\vec{G}$ and let $\left|B(\langle U\rangle_{cyclic})_i{}^{(j_2)}\right|$ is the determinant of the antiadjacency matrix of a cyclic induced graph with $i$ vertices and $j_2 = 1,2, \ldots, w_2$ where $w_2$ is the number of cyclic induced graphs $\langle U\rangle_{cyclic}$ with $i$ vertices of a cyclic directed graph $\vec{G}$, then

$$b_i = (-1)^i\left(\sum_{j_1=1}^{w_1}\left|B\left(\langle U\rangle_{acyclic}\right)_i{}^{(j_1)}\right| + \sum_{j_2=1}^{w_2}\left|B\left(\langle U\rangle_{cyclic}\right)_i{}^{(j_2)}\right|\right)$$

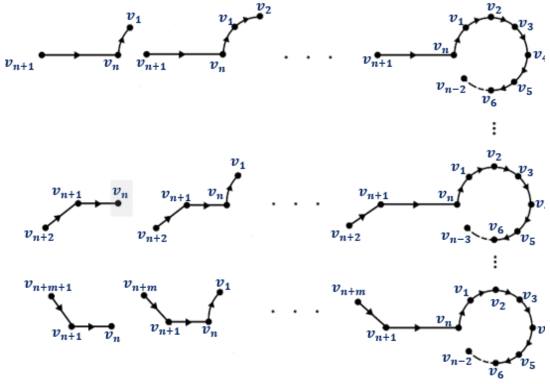
with $i = 1,2, \ldots, n$ (Wildan, 2015).

**Theorem 4**. If $P\left(B(\vec{G})\right) = \lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \cdots + b_{n-1}\lambda + b_n$ is the characteristic polynomial of the antiadjacency matrix $B(\vec{G})$ of a directed graph with $n$ vertices and $m$ arcs, then $b_1 = -n$ and $b_2 = m$ (Wildan, 2015).

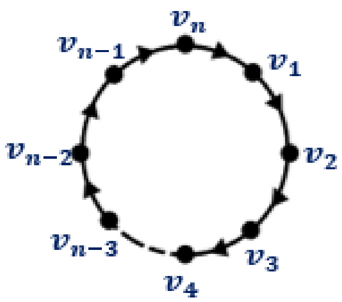
# RESULT AND DISCUSSION 1

## Induced Subgraph

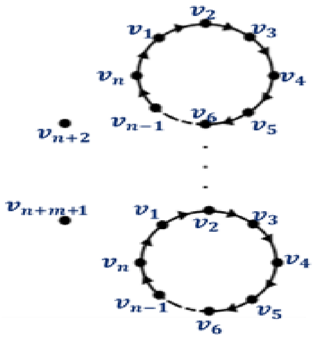
Type 1



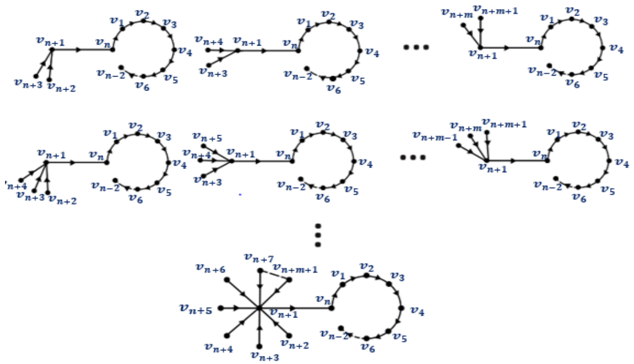
Type 2



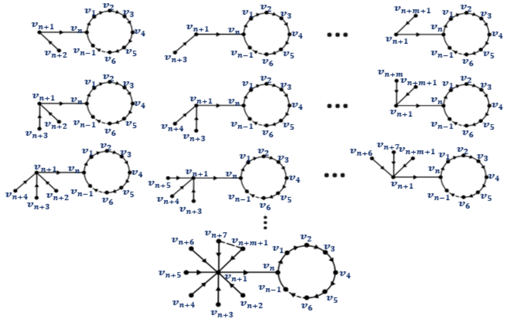

Type 3



Type 4



Type 5



**Lemma 1**. Suppose $\vec{G}$ is a directed unicylic flower vase graph, then the number of acyclic induced subgraphs that have Hamiltonian path in subgraph type 1 is $m + n + 1$ path for $1 \leq i \leq n-1$ vertrices, $m + 1$ path for $n$ vertices, $m$ paths for $n + 1$ vertices.

**Lemma 2**. Suppose $B(T1)$ is the antiadjacency matrix of the subgraph type 1, then $det\ (B(T1)) = 1$.

**Lemma 3**. Suppose $\vec{G}$ is a directed unicylic flower vase graph, then the number of cyclic induced subgraphs that have Hamiltonian path in subgraph type 2 is only one.

**Lemma 4**. Suppose $B(T2)$ is the antiadjacency matrix of the subgraph type 2, then $det(B(T2)) = n-1$.

**Lemma.5**. Suppose $\vec{G}$ is a directed unicylic flower vase graph, then the number of cyclic induced subgraphs that have Hamiltonian path in subgraph type 3 is $m$.

**Lemma 6**. Suppose $B(T2)$ is the antiadjacency matrix of the subgraph type 3, then $det(B(T3)) = -1$.

# RESULT AND DISCUSSION 2

**Theorem 5.** If $P\left(B(\overrightarrow{C_nS_m})\right) = \lambda^{m+n+1} + b_1\lambda^{m+n} + b_2\lambda^{m+n-1} + \ldots + b_{n-1}\lambda^m + b_n\lambda^{m+1} + \cdots + b_{n+m}\lambda + b_{n+m+1}$ is the characteristic polynomials of a directed unicyclic flower vase graph, then:
$b_i = (-1)^i(m+n+1)$, for $1 \leq i < n$,
$b_i = (-1)^i(m+n)$, for $i = n$,
$b_i = 0$, for $n+1 \leq i \leq n+m+1$.

Proof :
Case $1 \leq i < n$.
Case $b_1$.
Since $P\left(B(\overrightarrow{C_nS_m})\right) = \lambda^{m+n+1} + b_1\lambda^{m+n} + b_2\lambda^{m+n-1} + \ldots + b_{n-1}\lambda^m + b_n\lambda^{m+1} + \cdots + b_{n+m}\lambda + b_{n+m+1}$ is the characteristic polynomials of the directed unicyclic flower vase graph with $m+n+1$ vertrices, then based on **Theorem 4**, we have $b_1 = m+n+1$.
Case $b_2$.
Since $P\left(B(\overrightarrow{C_nS_m})\right) = \lambda^{m+n+1} + b_1\lambda^{m+n} + b_2\lambda^{m+n-1} + \ldots + b_{n-1}\lambda^m + b_n\lambda^{m+1} + \cdots + b_{n+m}\lambda + b_{n+m+1}$ is the characteristic polynomials of the directed unicyclic flower vase graph with $m+n+1$ arcs, then based on **Theorem 4**, we have $b_2 = m+n+1$.
Case $3 \leq i < n$
Acylic case.
Based on **Lemma 1**, there are $m + n + 1$ induced subgraph of type 1 which have a Hamiltonian path. Then, based on **Lemma 2**, we have $\det(B) = 1$ and based on **Theorem 3**, we get $\sum_{j_1}^{w_1}\left|B(\langle U\rangle_{acyclic})_i^{(j_1)}\right| = (m+n+1)\cdot 1 = m+n+1$.
Cyclic case.
There is no cyclic subgraph for $b_i$, $3 \leq i < n$, then $\sum_{j_1}^{w_1}\left|B(\langle U\rangle_{acyclic})_i^{(j_2)}\right| = 0$.
So we can conclude
$b_i = (-1)^i(m+n+1)$, for $3 \leq i < n$. (3.1)

# RESULT AND DISCUSSION 2

Case $i = n$.
Acylic case.
Proof is similar with case $3 \leq i < n$, then we get $\sum_{j_1}^{w_1} \left| B\left(\langle U \rangle_{acyclic}\right)_n^{(j_1)} \right| = (m+1) \cdot 1 = m+1$.
Cylic case.
Based on **Lemma 3**, there are 1 induced cyclic subgraph of type 2 which have a Hamiltonian path. Because in each directed cyclic subgraph there is $\overrightarrow{C_n}$, then based on **Lemma 4**, we have $\det(B) = n-1$ and based on **Theorem 3**, we get $\sum_{j_2}^{w_2} \left| B\left(\langle U \rangle_{cyclic}\right)_n^{(j_2)} \right| = 1.(n-1) = n-1$.
So we can conclude
$b_i = (-1)^i[(m+1)+(n-1)] = (-1)^i(m+n)$. (3.2)
Case $i > n$.
Case $i = n+1$.
Acylic case.
Proof is similar with case $3 \leq i < n$, then we get $\sum_{j_1}^{w_1} \left| B\left(\langle U \rangle_{acyclic}\right)_i^{(j_1)} \right| = m \cdot 1 = m$.
Cylic case.
Based on **Lemma 5**, there are $m$ induced cyclic subgraph of type 3 which have a Hamiltonian path. Based on **Lemma 6**, we have $\det(B) = -1$. And based on **Theorem 3**, we get $\sum_{j_2}^{w_2} \left| B\left(\langle U \rangle_{cyclic}\right)_i^{(j_2)} \right| = m.(-1) = -m$.

So we can conclude
$b_i = (-1)^i(m-m) = 0$. (3.3)
Case $n+2 \leq i \leq n+m+1$.
Directed subgraph type 4 and type 5 doesn't have Hamiltonian path. So based on **Theorem 1** and **Theorem 3** we can conclude
$b_i = (-1)^i(0+0) = 0$. (3.4)
From equations (3.1) and (3.2), based on **Theorem 3**, we get
$b_i = (-1)^i(0+0) = 0$, for $n+1 \leq i \leq n+m+1$. (3.5)
From equations (3.1), (3.2) and (3.5), we can conclude that the characteristic polynomials of the directed unicyclic flower vase graph $\overrightarrow{C_nS_m}$ is $P\left(B\left(\overrightarrow{C_nS_m}\right)\right) = \lambda^{m+n+1} + b_1\lambda^{m+n} + b_2\lambda^{m+n-1} + \ldots + b_{n-1}\lambda^m + b_n\lambda^{m+1} + \cdots + b_{n+m}\lambda + b_{n+m+1}$, with:
$b_i = (-1)^i(m+n+1)$, for $1 \leq i < n$,
$b_i = (-1)^i(m+n)$, for $i = n$,
$b_i = 0$, for $n+1 \leq i \leq n+m+1$. ■

# RESULT AND DISCUSSION 3

**Theorem 6.** If $P\left(B\left(\overrightarrow{C_nS_m}\right)\right) = \lambda^{m+n+1} - (m+n+1)\lambda^{m+n} + (m+n+1)\lambda^{m+n-1} - (m+n+1)\lambda^{m+n-2} + \ldots + (-1)^n(m+n)\lambda^{m+1}$ is the characteristic polynomials of the directed unicyclic flower vase graph, then :
for $n = 2k$, where $k = 2,3,4, \ldots$

$\lambda_l = 0;\ l = 1,2,\ldots,m+1;\ \lambda_{m+2} = m+n;\ \lambda_{m+3} = 1;\ \lambda_l = e^{i\frac{2j\pi}{n}};\ l = m+4, m+5, \ldots, m+n+1; j = 1,2,3,\ldots,n-2; j \neq \frac{n}{2}$ and $j \neq 0$.

for $n = 2k+1$, where $k = 1,2,3, \ldots$

$\lambda_l = 0; l = 1,2,\ldots,m+1;\ \lambda_{m+2} = m+n;\ \lambda_l = e^{i\pi\left(\frac{2j+1}{n}\right)}; l = m+3, m+4, \ldots, m+n+1;\ j = 0,1,2,\ldots,n-1; j \neq \frac{n-1}{2}$.

**Proof:**

For case $n = 2k$, $k = 2,3,4, \ldots$, from **Theorem 5**, it is known that the characteristic equation of the antiandjacency matrix of directed unicylic flower vase graph is as follows.

$$P\left(B\left(\overrightarrow{C_nS_m}\right)\right) = \lambda^{m+n+1} - (m+n+1)\lambda^{m+n} + (m+n+1)\lambda^{m+n-1} - (m+n+1)\lambda^{m+n-2} + \ldots + (-1)^n(m+n)\lambda^{m+1}$$

$$\lambda^{m+1}(\lambda^{2k} - (m+2k+1)\lambda^{2k-1} + (m+2k+1)\lambda^{2k-2} - (m+2k+1)\lambda^{2k-3} + \cdots + (m+2k)) = 0 \quad (3.6)$$

By factorizing polynomial characteristic we have

$\lambda^{m+1} = 0$, $\lambda - (m+2k) = 0$, $\lambda - 1 = 0$, or $\frac{\lambda^{2k}-1}{\lambda^2-1} = 0$.

For $\lambda^{m+1} = 0$, we get $\lambda_1 = \lambda_2 = \cdots = \lambda_{m+1} = 0$.

For $\lambda - (m+2k) = 0$, we get $\lambda_{m+2} = (m+2k) = m+n$.

For $\lambda - 1 = 0$, we get $\lambda_{m+3} = 1$.

For $\frac{\lambda^{2k}-1}{\lambda^2-1} = 0$, we get $\lambda^{2k} = 1$, with $\lambda \neq 1$ and $\lambda \neq -1$. (3.9)

By using complex number equations from equations (3.9) we get

$\lambda_{m+4} = \lambda_{m+5} = \cdots = \lambda_{m+2k+1} = e^{i\frac{j\pi}{k}}$, for $j \neq \frac{n}{2}$ and $j \neq 0$.

So for n = 2k, k = 2,3,4, ..., the eigenvalues of the antiadjacency matrix of directed unicyclic flower vase graph are:

$\lambda_l = 0;\ l = 1,2,\ldots,m+1;\ \lambda_{m+2} = m+n;\ \lambda_{m+3} = 1;\ \lambda_l = e^{i\frac{2j\pi}{n}}; l = m+4, m+5, \ldots, m+n+1; j = 1,2,\ldots,n-2; j \neq \frac{n}{2}$ and $j \neq 0$.

For case $n = 2k+1$, $k = 1,2,3, \ldots$, the proof is similar with case $n = 2k$, $k = 2,3,4, \ldots$ . Then we obtain.

$\lambda^{m+1} = 0$, $\lambda - (m+2k+1) = 0$, or $\frac{-\lambda^{2k+1}-1}{-\lambda-1} = 0$.

For $\lambda^{m+1} = 0$, we get $\lambda_1 = \lambda_2 = \cdots = \lambda_{m+1} = 0$.

For $\lambda - (m+2k+1) = 0$, we get $\lambda_{m+2} = (m+2k+1) = m+n$

For $\frac{-\lambda^{2k+1}-1}{-\lambda-1} = 0$, we get $\lambda^{2k+1} = -1$ , with $\lambda \neq -1$. (3.13)

By using complex number equations from equations (3.13) we get

$\lambda_{m+3} = \lambda_{m+4} = \cdots = \lambda_{m+n+1} = e^{i\pi\left(\frac{2j+1}{n}\right)}$, $j \neq \frac{n-1}{2}$

So, for $n = 2k+1$, $k = 1,2,3, \ldots$, the eigenvalues of the antiadjacency matrix of directed unicyclic flower vase graph are:

$\lambda_l = 0;\ l = 1,2,\ldots,m+1;$

$\lambda_{m+2} = m+n;\ \lambda_l = e^{i\pi\left(\frac{2j+1}{n}\right)}; l = m+3, m+4, \ldots, m+n+1; j = 0,1,2,\ldots,n-1;\ j \neq \frac{n-1}{2}$. ■

# CONCLUSION

Suppose $B(\overrightarrow{C_nS_m})$ is the antiadjacency matrix of directed unicyclic flower vase graph and $P\left(B(\overrightarrow{C_nS_m})\right) = \lambda^{m+n+1} + b_1\lambda^{m+n} + b_2\lambda^{m+n-1} + \dots + b_{n-1}\lambda^{m} + b_n\lambda^{m+1} \dots + b_{n+m}\,\lambda + b_{n+m+1}$ is the characteristic polynomial of antiadjacency matrix of directed unicylic flower pot graph $\overrightarrow{C_nS_m}$

Table 1. Characteristic Polynomial Coefficients of Matrix $B(\overrightarrow{C_nS_m})$.

| $i$ | Coefficient Value |
|---|---|
| $i = 1,2,\dots,n-1$ | $b_i = (-1)^i(m+n+1)$ |
| $i = n$ | $b_i = (-1)^i(m+n)$ |
| $i = n+1, n+2, \dots, m+n+1$ | $b_i = 0$ |

Table 2. Eigen Values of Matrix $B(\overrightarrow{C_nS_m})$.

| $n$ | Real Eigen Values | Complex Eigen Values |
|---|---|---|
| Even | $\lambda_l = 0$, $l = 1,2,\dots,m+1$ | $\lambda_l = e^{i\frac{2j\pi}{n}}$, $j \neq \frac{n}{2}$ $dan\ j \neq 0$, $l = m+4, m+5, \dots, m+n+1$. |
| | $\lambda_{m+2} = m+n$ | |
| | $\lambda_{m+3} = 1$ | |
| Odd | $\lambda_l = 0$, $l = 1,2,\dots,m+1$ | $\lambda_l = e^{i\pi\left(\frac{2j+1}{n}\right)}$, $j \neq \frac{n-1}{2}$ , |
| | $\lambda_{m+2} = m+n$ | |

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# REFERENCES

1. Chartrand, G Lesniak L & Zhang P 2016 *Graph and Digraphs* 6th ed. (London: Chapman & Hall)
2. Bapat R B 2010 *Graphs and Matrices* (London: Springer London)
3. Ahmad M 2012 *Pelabelan Graceful dan Pelabelan ρ pada Graf Pot Bunga dan Graf Pohon Palem* (Depok: Tesis Universitas Indonesia)
4. Wildan 2015 *Polinomial Karakteristik dari Matriks Antiketetanggaan dan Ketetanggaan dari Graf Berarah yang Diberi Orientasi* (Depok: Tesis Universitas Indonesia)
5. Diwyacitta D, Putra A P, Sugeng K A and Utama S 2017 *The Determinant of An Antiadjacency Matrix of a Directed Cycle Graph with Chords* AIP Conference Proceedings 1862, 030127 (2017); doi:10.1063/1.4991231