Simple Population Models (Discrete-Time)

- **<u>Discrete-time</u>:** time is measured in discontinuous steps (days, years, generations)
    a. **<u>Projection Interval</u>:** the size of each step
    b. Most appropriate for life cycles, seasonal/annual plants/crops, bacteria grown on a petri dish

- **<u>Difference Equation/Recursion Equation</u>:** general form for a single species discrete-time model

$$N_{t+1} = f(N_t) = N_t F(N_t)$$

$f(N_t)$; net growth rate, current pop $\qquad$ $f(N_t)$; per-capita growth on avg. total pop.

- Dimensionless "rates" "ratios" "net production" "net per-capita production"

- **<u>Net Growth Rate</u>:** the percentage increase or decrease of a population based on births and deaths within that timeframe, **Births - Deaths**

$$f(N) \geq 0 \text{ with } f(0) = 0\text{; for a population to grow}$$

- **<u>Net Per Capita Growth Rate</u>:** the average rate at which a population increases per individual, essentially showing the overall change in population size *per individual* over a given period of time, **(Births - Deaths) / Total Population**

$$F(N) \geq 0$$

1. **<u>Linear Population Models w/ Exponential Growth</u>**

- Simple Linear Model with Births and Deaths

$$N_{t+1} = N_t + \Delta N_t = N_t + bN_t - dN_t$$
$$N_{t+1} = N_t + \Delta N_t$$
$$= \frac{N_t}{present\ pop.} + \frac{bN_t}{total\ births} - \frac{dN_t}{total\ deaths}$$

$$N_t(1 + b - d)$$

- **Net Per-Capita Growth Rate** in the exponential growth model, density-independent:

$$F(N) = 1 + b - d = \lambda$$

- b is the average number of surviving offspring per individual over a lifetime
- d is the probability that a given individual dies during the year
- λ is the net per capita growth rate/ratio

- Dimensionless units
- Biologically realistic ranges for these parameters → $b \geq 0, \lambda \geq 1, 0 \leq d < 1$

- Population growth is density-independent as $F(N)$ does not depend on population size. Gives a linear difference equation:

$$N_{t+1} = \lambda N_t$$

- The solution to this model with **initial condition** $N(0) = N_0$:

$$N_t = \lambda^t N_0$$

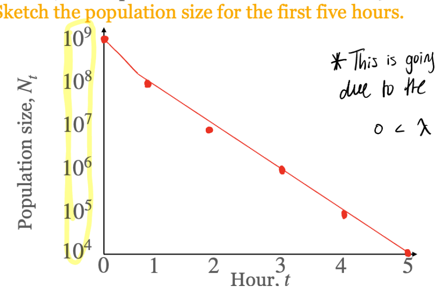
Growth	Decay
$\lambda \to \infty$ as $t \to \infty$	$\lambda \to 0$ as $t \to \infty$
$\lambda > 1$	$0 \leq \lambda < 1$

A bacterial colony initially contains a billion cells. An antibiotic is administered, which prevents cell division and kills 90% of cells each hour. Sketch the population size for the first five hours.

$N_0 = 10^9$
$\left.\begin{matrix} b = 0 \\ d = 0.9 \end{matrix}\right\} \begin{matrix} 1 + 0 - 0.9 \\ = 0.1 = \lambda \end{matrix}$

$N_t = 0.1^5(10^9)$
$= 10^4$

t	N_t
0	10^9
1	10^8
2	10^7
3	10^6
4	10^5
5	10^4



* This is going to be linear due to the log scale

$0 < \lambda < 1$

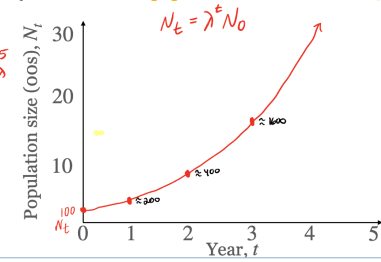
A population of rabbits initially contains 100 individuals. Each rabbit produces 1.5 surviving offspring (on average) and has a 50% chance of dying each year. Sketch the population size for the next 5 years.

$N_0 = 100$
$\left.\begin{matrix} b = 1.5 \\ d = 0.5 \end{matrix}\right\} \begin{matrix} 1 + 1.5 - 0.5 \\ = 2 = \lambda \end{matrix}$

$N_t = 2^6(100)$
$= 6400$

t	N_t
0	100
1	200
2	400
3	1600
4	3200
5	6400

$N_t = \lambda^t N_0$



- Problems with exponential growth model → Population approaches infinity as time approaches infinity, does not account for density dependent factors
- Can be improved by:

$$\lambda = 1 + b - d$$
$$\lambda = 1 + b(N) - d(N) \rightarrow \text{resource competition}$$

2. Nonlinear Population Model

- Density-dependent net per-capita growth rate
- Density-independent births and deaths, and density-dependent deaths

Assume: $r = b - d_1$ and $K = \frac{r}{d_2}$, $d_2 = \frac{r}{K}$

$$N_{t+1} = N_t + bN_t - d_1N_t - d_2N_tN_t$$
$$N_{t+1} = N_t + N_t(b - d_1 - d_2N_t)$$
$$= N_t + N_t(r - \frac{r}{K}N_t)$$
$$= N_t + rN_t(1 - \frac{N_t}{K})$$

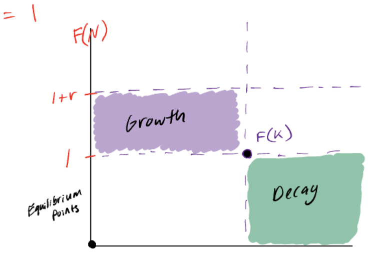
- **Net Per-Capita Growth Rate**

$$F(N) = 1 + r(1 - \frac{N_t}{K})$$

- *Intrinsic Growth Rate*: the maximum rate at which a population can grow when there are no environmental limits, $r = b - d_1$
- *Carrying Capacity*: $K = \frac{r}{d_2}$

$1+r(1-N_t/K)$

$F(0) = 1+r$
$F(K) = 1$



- *Steady States*: occur when $N_t + 1 = N_t = N^*$ for some value N^* (finding the equilibria). Satisfy $N^* = f(N^*)$.

- The steady states of the logistic growth model:

Growth rate and decay rate are the same

$f(N) = N^* + rN^*\left(1 - \frac{N^*}{K}\right) = N^*$

$f(N^*) = N^*$

$N^* = 1$

$f(N) = N^* + rN^*\left(1 - \frac{N^*}{K}\right) = N^*$

$f(N) = \cancel{1} + r\left(1 - \frac{N^*}{K}\right) = \cancel{1}$

$f(N) = r\left(1 - \frac{N^*}{K}\right) = 0$

$r - \frac{rN^*}{K} = 0$

$\cancel{r} = \frac{\cancel{r}N^*}{K}$

Plug in 0: $N^* = 0$ (trivial steady state)

$1 = \frac{N^*}{K}$ Plug in 1:

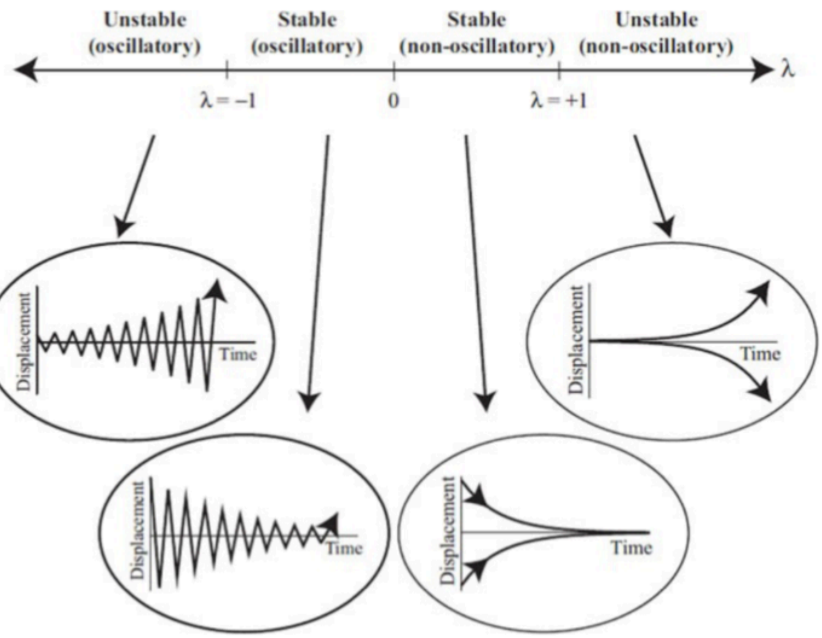
$K = N^*$ (non trivial steady state)

- *Stability*: if a steady state is stable or unstable
- *Linear Stability Analysis*: determining what happens close to a steady state, use Taylor expansion to linearize point to approximate value, does the population return to the steady state or move away?

- *Perturbation*: abiotic/biotic disturbance (disease, invasive species), affecting steady state, want to know if it grows or decays over time, dynamics given by:

$n_{t+1} = N_{t+1} - N^*$ $n_t = \lambda^t n_0$

- Summarizing the behavior of n_t and the stability of the steady state:



	Behavior	Stability
$\lambda > 1$	monotonic growth "growing towards something"	unstable
$0 < \lambda < 1$	monotonic decay	stable
$-1 < \lambda < 0$	oscillatory decay	Stable
$\lambda < -1$	oscillatory growth	unstable

- An increase in r results in higher oscillations above and below the carrying capacity.

- **Stability of the steady states $N^* = 0$ *and* $N^* = K$, assuming $r > 0$:**

$N_{t+1} = f(N_t)$

$$N_t + 1 = N_t(1 + r(1 - \frac{N_t}{K})) = f(N_t)$$

$$f(N_t) = N_t + rN_t - \frac{rN_t}{K}$$

$$f'(N_t) = 1 + r - \frac{2rN_t}{K}$$

steady states

$f'(0) = 1 + r$ trivial steady state

$f'(K) = 1 - r$ nontrivial steady state

$\lambda = f'(0)$ $1 + r > 1$ monotonic unstable

$\lambda = f'(K)$
$= 1 - r$

if $0 < r < 1$ monotonic stable

if $1 < r < 2$ oscillatory stable

if $r > 2$ oscillatory unstable

- *Asymptotic Behavior:* when data approaches/levels off as t −> ∞
- *Cobwebbing*: geometric technique to analyze the stability and dynamics of steady states of simple discrete-time models of the form $N_{t+1} = f(N_t)$. Shows how the population changes from one time state to another using the net growth rate, $f(N)$, to plot:

$$\overset{x\quad y}{(N_0, N_0)} \longrightarrow \underbrace{(N_0, N_1)}_{\text{updating}} \longrightarrow (N_1, N_1) \longrightarrow \underbrace{(N_1, N_2)}_{\text{updating}}$$

- N_0 : initial population
- pop. changes from one time point to the next

- **Cobwebbing Method**:

1. Set up axes with N_t on the horizontal axis and N_{t+1} on the vertical axis
2. Sketch the diagonal line $N_{t+1} = N_t$
3. Sketch the net growth function, $f(N_t)$
4. Choose an initial condition, N_0, and plot (N_0, N_0)
5. Since $N_t + 1 = f(N_t)$ move vertically (up or down) to the curve $f(N_t)$ and plot (N_t, N_{t+1})
6. Move horizontally to the diagonal line $N_{t+1} = N_t$ and plot (N_{t+1}, N_{t+1})
7. Repeat steps 5-6 until the long-time (**asymptotic**) behavior becomes clear

$f(N) = N\left(1 + r\left(1 - \frac{N}{K}\right)\right)$

1. $f(0)$, what is the initial condition: typically $f(0) = 0$
2. $f(N)$, what happens to the equation / pop. as N gets large
   $f(N) = -\infty$, approach $-\infty$

<u>Logistic Model</u>: use this info to sketch cobweb

* is growth going to be positive / negative?

$$f'(N) = 1 + r - \frac{2rN_t}{K}$$

$$f'(0) = 1 + r > 1 \text{ if } r > 0$$

$$f'(K) = 1 - r \begin{cases} 0 < f'(K) < 1 & \text{if } r < 1 \\ -1 < f'(K) < 0 & \text{if } 1 < r < 2 \\ f'(K) < -1 & \text{if } r > 2 \end{cases}$$

First 2 steady states

$r = 1/2$

$f'(K) = 1/2$