2 Species Competition/Lotka-Volterra: with logistic growth for both species and competition terms that reduce their growth rates due to the presence of the other species.
u: population size of species 1
v: population size of species 2
a: competition coefficient (effect of species 2 on species 1)
b: competition coefficient (effect of species 1 on species 2)
s: relative growth rate of species 2 (can be used to rescale time)
Nullcline:

a > 1	Species 2 hurts species 1 more than species 1 hurts itself
b > 1	Species 1 hurts species 2 more than species 2 hurts itself

Interspecific competition is stronger than intraspecific competition for both species. They are each more negatively affected by the other than by their own population. Unstable coexistence: outcome depends on initial conditions. If species 1 starts higher → wins, if species 2 starts higher → win. Nullclines intersect, saddle.

a < 1	Species 2 has less effect on species 1 than species 1 does on itself
b > 1	Species 1 has more effect on species 2 than species 2 does on itself

Species 1 is resilient/Species 2 is sensitive to competition. Species 1 wins regardless of initial conditions. Species 1 will exclude species 2. Nullcline for species 1 will lie above nullcline for species 2. Stable equilibrium where species 2 is extinct.

T = arbitrary α_{ii} = intraspecific
$\alpha_{ii} = \frac{1}{K}$ α_{ij} = interspecific

$\frac{dN_1}{dt} = r_1 N_1(1-\alpha_{11}N_1 - \alpha_{12}N_2) \Rightarrow \boxed{\frac{1}{r_1}\frac{dN_1}{dT}} = N_1(1-\alpha_{11}N_1-\alpha_{12}N_2)$
$\frac{dN_2}{dT} = r_2 N_2(1-\alpha_{21}N_1 - \alpha_{22}N_2) \Rightarrow \boxed{\frac{1}{r_1}\frac{dN_2}{dT}} = \frac{r_2}{r_1}N_2(1-\alpha_{21}N_1-\alpha_{22}N_2)$
$t = r_1 T$, $T = \frac{1}{r_1}t$, $s = \frac{r_2}{r_1}$
① Rescale time
r = rate
Chain Rule:
$\frac{dN_1}{dt} = \frac{dN_1}{dT}\cdot\frac{dT}{dt} = \frac{1}{r_1}\frac{dN_1}{dt}$ | $\frac{dN_2}{dt} = \frac{1}{r_1}\frac{dN_2}{dT} = s\cancel{\frac{1}{r_1}}N_2(1-\alpha_{21}N_1-\alpha_{22}N_2)$
6 → 5 parameters

$\frac{du}{dt} = u(1-u-av)$
$\frac{dv}{dt} = sv(1-bu-v)$
6 → 3 parameters
② Pop size
$\alpha_{ii} = \frac{1}{K_i}$, $u = \alpha_{11}N_1$, $v = \alpha_{22}N_2$ $\alpha = \frac{\alpha_{12}}{\alpha_{22}}$
$N_1 = \frac{u}{\alpha_{11}}$, $N_2 = \frac{v}{\alpha_{22}}$
$\frac{dN_1}{dt} = \frac{d}{dT}\left(\frac{u}{\alpha_{11}}\right) = \underbrace{\frac{1}{\alpha_{11}}}_{\text{species 1}}\frac{du}{dt} = \frac{u}{\alpha_{11}}\left(1-u-\frac{\alpha_{12}}{\alpha_{22}}v\right)$
$V = \alpha_{22}N_2$, $N_2 = \frac{v}{\alpha_{22}}$ | $s = \frac{r_2}{r_1}$, $N_1 = \frac{u}{\alpha_{11}}$ $\boxed{\frac{du}{dt} = u(1-u-\alpha v)}$
$b = \frac{\alpha_{21}}{\alpha_{11}}$
what is N_2?
$\frac{dN_2}{dt} = \frac{d}{dt}\left(\frac{v}{\alpha_{22}}\right) = \frac{1}{\alpha_{22}}\frac{dv}{dt} = \frac{sv}{\alpha_{22}}\left(1-\frac{\alpha_{21}}{\alpha_{11}}u - v\right)$
$\frac{dv}{dt} = sv\left(1-\frac{\alpha_{21}}{\alpha_{11}}u-v\right)$
$\frac{dv}{dt} = sv(1-bu-v)$

Equilibria: set both derivatives to zero
Sp1 $\frac{du}{dt} = u(1-u-av)$, $\frac{du}{dt} = 0$
$\boxed{u=0}$ or $1-u-av=0 \Rightarrow \boxed{u=1-av}$
Sp2 $\frac{dv}{dt} = sv(1-bu-v)$, $\frac{dv}{dt} = 0$
$\boxed{v=0}$ or $1-bu-v=0 \Rightarrow \boxed{v=1-bu}$

Equilibrium Points:
(0,0) trivial, both extinct
(1,0) species 1 wins
(0,1) species 2 wins
Semi-trivial
$\left(\frac{1-a}{1-ab}, \frac{1-b}{1-ab}\right)$ for $ab \neq 1$ coexistence equilibrium

Nullclines Sp 1: $\left(\frac{du}{dt}=0\right)$
$\boxed{u=0,\ v=\frac{1}{a}(1-u)}$
plug into $\frac{dv}{dt}$: $sv(1-bu-v) = sv(1-v)$
if $0<b<1$ ↑
if $b>1$ ↓
When $v = \frac{1}{a}(1-u)$
$\frac{dv}{dt} > 0$ if $u < \frac{1-a}{1-ab}$ Sp 2 grows
$\frac{dv}{dt} < 0$ if $u > \frac{1-a}{1-ab}$ Sp 2 shrinks

Nullcline Sp 2: $\left(\frac{dv}{dt}=0\right)$
$\boxed{v=0,\ v=1-bu}$
$u(1-u)$ $\begin{cases} >0 \text{ if } 0<u<1 \\ <0 \text{ if } u>1 \end{cases}$
When on sp2 nullcline, $v=1-bu$
$\frac{du}{dt} > 0$ if $u < \frac{1-a}{1-ab}$ Sp1 grows
$\frac{du}{dt} < 0$ if $u > \frac{1-a}{1-ab}$ Sp1 shrinks

Directional Field:
$\frac{du}{dt} = 0$ ↑↓ w/ respect to v
$\frac{dv}{dt} = 0$ ← → w/ respect to u

Sign of $\frac{du}{dt}, \frac{dv}{dt}$: same for v
if $\frac{du}{dt} > 0$, u is increasing
if $\frac{du}{dt} < 0$, u is decreasing

Graphing Phase Plane:
$\frac{du}{dt} = 0$ $\begin{cases} u=0 \\ v=\frac{1}{a}(1-u) \\ u=0 \Rightarrow v=\frac{1}{a} \\ u=1 \Rightarrow v=0 \end{cases}$ decline, plot
$\frac{dv}{dt} =$ $\begin{cases} v=0 \\ v=1-bu \\ u=0 \Rightarrow v=1 \\ v=0 \Rightarrow u=1/b \end{cases}$

At $u=0$, when $v<1$ Sp 2 grows
$v>1$ Sp 2 shrinks
$v=1$ $\frac{dv}{dt}=0$

At $v=0$, when $u<1$ Sp1 grows
$u>1$ Sp1 shrinks
$u=1$ $\frac{du}{dt}=0$

a	b	Outcome	Phase Plane	Equilibrium/Stability	Direction Field
< 1	< 1	Stable coexistence			Point to coexistence eq.
< 1	> 1	Sp1 wins	Sp1 nullcline: shallower, starts higher (since a < 1) Sp2 nullcline: steeper, drops sharply (since b > 1)	Nullclines intersect but not meaningful. Stable equilibrium on u-axis (v=0). Coexistence eq. not valid	Point right and down. Trajectories lead to Sp1 dominance.
> 1	< 1	Sp2 wins	Sp1 nullcline: steep, starts lower (since a > 1) Sp2 nullcline: shallower, declines slowly (since b < 1)	Nullclines intersect but not valid. Stable equilibrium on the v-axis (u=0). Coexistence eq. not valid	Point left and up. Trajectories lead to Sp2 dominance
> 1	> 1	Unstable coexistence (Bi-stabilityt)	Sp1 nullcline: steep Sp2 nullcline: steep	Nullclines intersect in relevant region. Initial conditions determine winner. Saddle point-unstable equilibrium.	Arrows point away from intersections. Trajectories move toward either axis depending on starting point.

What is aij? aij is the competition coefficient representing the effect of species j on species i; tells us how much species j reduces the growth of species i, measured in units of species
aii: intraspecific competition (effect of species i on itself) ajj: interspecific competition (effect of species j on species i)
aii = 1/k; the intraspecific competition strength is inversely proportional to the carrying capacity
What does aij represent for i = j? determines how quickly growth slows as the species approaches it's own carrying capacity →
What does aij represent for i =/= j? determines how many individuals of species j are equivalent to one individual of species i in terms of resource use or competitive pressure
What are sensible bounds for aij? aij >/ 0 (negative values do not imply competition), typical vals 0 </ aij </ 1
If aij = 0, no interspecific competition (neutralism), If aij = 1, ind of species j compete exactly as much as species i, For aij > 1, ind of species j have stronger competitive effect on species i. High aij values can lead to unstable coexistence or competitive exclusion.
The value of aij reflects the degree of niche overlap between species i and j. **Niche overlap:** how similarly two species use the same types of resources, if species use resources exactly the same way they compete strongly (high aij), if species specialize in different parts of resource spectrum competition low (small aij). More overlap = high aij, less overlap = low aij. Ex: Two bird species both eat seeds, but one prefers small seeds and the other prefers medium seeds. Their **resource utilization curves** (normal distributions) will only partially overlap. Moderate αij, because they compete for some but not all of the same seeds
Non-dimensionalization: simplifying math model by removing/combining units/parameters without changing how the system behaves, rescles time and pop size. Context: making time unitless and eliminating parameter r1, introduces S=r2/r1, a dimensionless ratio of growth rates (and not two separate rates), each population is scaled by carrying capacity. a=α11/α12: the effect of species 2 on species 1, relative to species 1's self-limitation
b=α21/α22: the effect of species 1 on species 2, relative to species 2's self-limitation
They represent interspecific competition strength compared to intraspecific competition.
If α11>α12, species 1 is more limited by its own crowding than by competition with species 2 → coexistence is more likely. If α12>α11, species 2 hurts species 1 more than species 1 hurts itself → species 1 may decline or go extinct.

a < 1, b < 1	Species limit themselves more than each other	Stable coexistence
a > 1, b > 1	One species harms the other too much	Exclusion/unstable

Resource (R)	$\frac{dR}{dt} = f(R) - g(C,R)$	Renewal minus consumptio
Consumer (C)	$\frac{dC}{dt} = \epsilon g(C,R) - h(C)$	Growth from consuming R (efficiency ϵ) minus death
ϵ	$0 < \epsilon < 1$	Conversion efficiency
$f(R)$	e.g. $rR(1-R/K)$	Resource renewal function (logistic, linear, or constant
$g(C,R)$	Functional response	Rate at which consumers consume resources
$h(C)$	Consumer death rate	Can be constant or density-dependent

Predator-Prey Model
Trivial Equilibrium (R*, C*) = (0,0) Non-trivial equilibrium (d/Eb, a/b)

$$J(R,C) = \begin{bmatrix} a-bC & -bR \\ \epsilon bC & \epsilon bR - d \end{bmatrix}$$

Type I	$g(C,R) = aCR$	Linear	Constant attack rate, no handling time
Type II	$g(C,R) = \frac{aCR}{1+ahR}$	Saturating	Includes handling time; rate plateaus
Type III	$g(C,R) = \frac{aCR^k}{1+ahR^k}$	Sigmoidal / Generalized Saturating	Slow start at low prey densities; prey switching

No consumers $(C=0)$	$\frac{dR}{dt} = f(R)$	Resource grows on its own logistic)
No resources $(R=0)$	$\frac{dC}{dt} = -h(C)$	Consumer declines via deat rate
Constant per capita death	$h(C) = dC$	Linear mortality
Density-dependent death	$h(C) = dC^2$	Mortality increases with crowding