

The background features several intersecting lines in yellow, orange, green, and blue. A large, light teal triangle is positioned on the left side, with its vertices near the top and bottom left corners. The text is centered over these elements.

The Chow-Lam Form

joint work with Bernd Sturmfels

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Slides: <https://lizziepratt.com/notes>

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The Chow Form

Definition (Chow form)

Let $X \subset \mathbb{P}^{n-1}$ be a d -dimensional projective variety. The *Chow locus* of X is

$$\mathcal{C}_X = \{L \in \mathrm{Gr}(n - d - 1, n) : X \cap L \neq \emptyset\}.$$

The *Chow form* C_X is the defining equation of \mathcal{C}_X .

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Example (Hypersurface)

The Chow form of a hypersurface $V(F)$ is F .

Example (Linear space)

The Chow locus of a linear space is a Schubert divisor.

Coordinate Systems

A linear space L can be represented multiple ways.

- ▶ **Primal** : as the kernel of an $(n - k) \times n$ matrix
- ▶ **Dual** : as the rowspan of a $k \times n$ matrix

The primal and dual Plücker coordinates are the maximal minors of these matrices. They are related by complementing indices and multiplying by a sign.

Example (Coordinates on $\text{Gr}(3, 5)$)

p_{12}	p_{13}	p_{14}	p_{15}	p_{23}	p_{24}	p_{25}	p_{34}	p_{35}	p_{45}
q_{345}	$-q_{245}$	q_{235}	$-q_{234}$	q_{145}	$-q_{135}$	q_{134}	q_{125}	$-q_{124}$	q_{123}

The twisted cubic

The Chow locus of the twisted cubic X is the set of $[L] \in \text{Gr}(2, 4)$ such that the line L meets X in \mathbb{P}^3 . The Chow form is given by the determinant of the *Bézout matrix* :

$$C_X = \det \begin{bmatrix} p_{12} & p_{13} & p_{14} \\ p_{13} & p_{14} + p_{23} & p_{24} \\ p_{14} & p_{24} & p_{34} \end{bmatrix}.$$

Its expansion is

$$-p_{14}^3 - p_{14}^2 p_{23} + 2p_{13} p_{14} p_{24} - p_{12} p_{24}^2 - p_{13}^2 p_{34} + p_{12} p_{14} p_{34} + p_{12} p_{23} p_{34}$$

For more on computing Chow forms of curves using resultants, see [David Eisenbud and Frank-Olaf Schreyer: *Resultants and Chow Forms via Exterior Syzygies*]

Intersection

Proposition (Intersection Formula)

Suppose $\dim X = d$, and let L and M be linear subspaces of \mathbb{P}^{n-1} such that $\operatorname{codim}(L \cap M) = \operatorname{codim}(L) + \operatorname{codim}(M) = d + 1$. Then

$$C_{X \cap L}(M) = C_X(L \cap M).$$

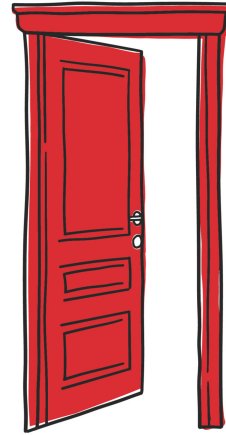
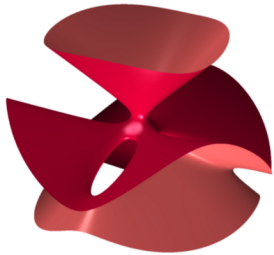
Example (Coordinates of intersection)

Fix $n = 6, d = 2$. Let $\operatorname{codim} L = 1$ and $\operatorname{codim} M = 2$ with **primal Plücker coordinates** l_i, m_{jk} . Then

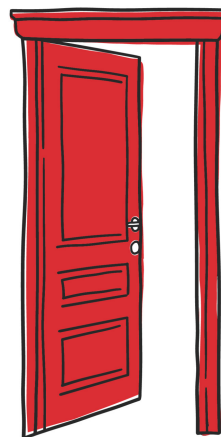
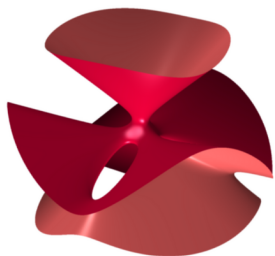
$$L \cap M = \begin{bmatrix} l_1 & l_2 & l_3 & l_4 & l_5 & l_6 \\ u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{bmatrix}.$$

So $p_{ijk}(L \cap M) = l_i m_{jk} - l_j m_{ik} + l_k m_{ij}$.

A variety walks through the door...



A variety walks through the door...



Proposition (Degree)

The degree of C_X equals the degree of X .

Proof.

The degree of C_X equals the degree of $C_{X \cap L}$. So, cut X with hyperplanes to get a finite number of points.



Projection

Theorem (Projection)

Suppose $\dim X = d$, and let L be a linear subspace of \mathbb{P}^{n-1} . Let Z be a $r \times n$ matrix, where $r \leq d - 2$. That is,

$$Z : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{r-1}.$$

Then $C_{Z(X)}(L) = C_X(Z^{-1}(L))$.

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Claim (Twistor Coordinates)

Let L be given in primal coordinates, as an $r \times (r - d - 1)$ matrix. The primal Plucker coordinate $p_I(Z^{-1}(L))$ is the maximal minor of the $r \times (n + r - d - 1)$ matrix

$$[Z_1 \dots Z_n \mid L]$$

*which includes I columns of Z and all columns of L . This is called the ***I*th twistor coordinate** of Z and L .*

The twisted cubic revisited

Recall the Chow form:

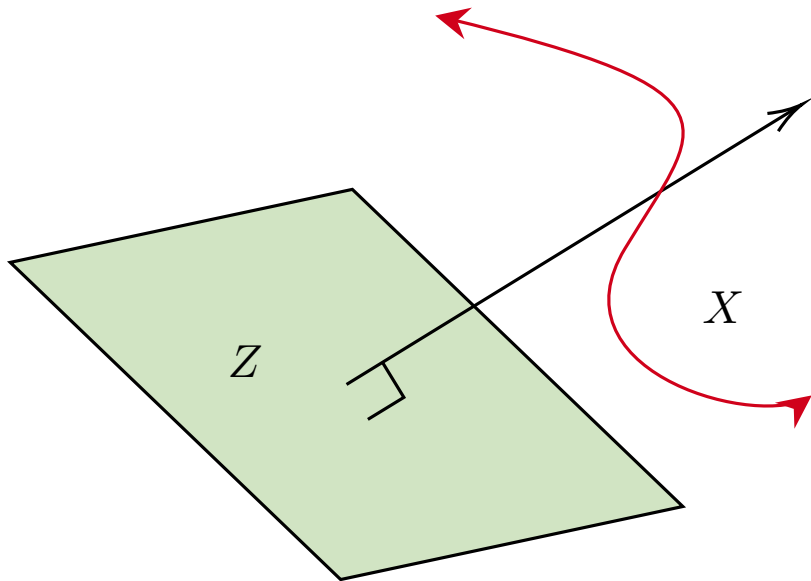
$$-p_{14}^3 - p_{14}^2 p_{23} + 2p_{13}p_{14}p_{24} - p_{12}p_{24}^2 - p_{13}^2 p_{34} + p_{12}p_{14}p_{34} + p_{12}p_{23}p_{34}$$

We make the substitution where the p_{ij} index 3×3 minors of

$$[Z|L] = \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} & l_1 \\ z_{21} & z_{22} & z_{23} & z_{24} & l_2 \\ z_{31} & z_{32} & z_{33} & z_{34} & l_3 \end{bmatrix}$$

involving l . Now $C_{Z(X)}(L) = C_X(Z^{-1}(L))$ is a function of $[l_1 : l_2 : l_3]$ on \mathbb{P}^3 and the matrix entries of Z . Choosing a particular Z gives the equation of $\overline{Z(X)}$ in \mathbb{P}^2 .

The twisted cubic revisited



The Chow form $C_{Z(X)}(L)$ computes the **universal projection variety** :

$$\mathcal{Y}_X := \overline{\{(Z, Z(p)) : p \in X\}} \subset \text{Mat}(r, n) \times \mathbb{P}^{r-1}.$$

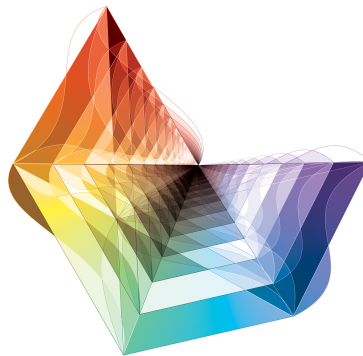
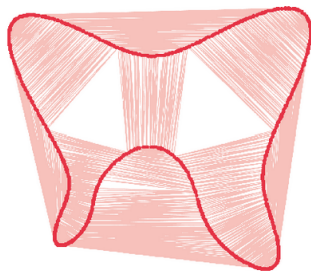
[Thomas Lam: *Totally nonnegative Grassmannian and Grassmann polytopes*, 2015]

A new context: Grassmannians

Let Z be a $r \times n$ matrix. This gives a map

$$Z : \operatorname{Gr}(k, n) \dashrightarrow \operatorname{Gr}(k, r).$$

Can we compute “universal” equations of projected varieties which are codimension 1 in the target? Examples: canonical curves, positroid varieties, torus orbits...



The Chow-Lam form

Definition (Grassmannian)

For a linear space P in \mathbb{P}^{n-1} , let $\text{Gr}(k, P)$ denote k -planes in \mathbb{P}^{n-1} contained in P .

Definition (Chow-Lam form)

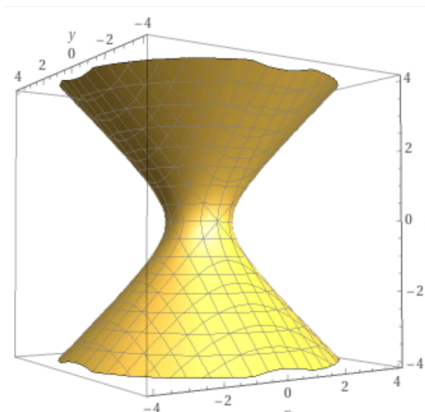
Let \mathcal{V} be a variety of dimension $k(r - k) - 1$ for some $r \leq n$. We define the *Chow-Lam locus* to be

$$\mathcal{CL}_{\mathcal{V}} := \{P \in \text{Gr}(k + n - r, n) : \text{Gr}(k, P) \cap \mathcal{V} = \emptyset\}.$$

When $\mathcal{CL}_{\mathcal{V}}$ has codimension 1, its defining equation is the *Chow form* $CL_{\mathcal{V}}$.

Example: Ruled surface

Let $\mathcal{V} \subset \text{Gr}(2, 4)$ be a curve in the Grassmannian, and let S be the surface swept out by the lines parameterized by \mathcal{V} .



Then $\mathcal{CL}_{\mathcal{V}}$ is a surface in $\text{Gr}(3, 4) = (\mathbb{P}^3)^{\vee}$. It parametrizes planes P in \mathbb{P}^3 which contain a line Q from the curve \mathcal{V} , ie planes tangent to \mathcal{V} . Thus $\mathcal{CL}_{\mathcal{V}}$ is the projective dual of S .

Example: Schubert varieties

Consider the following Schubert varieties in $\text{Gr}(2, 5)$ of dimension $3 = 2(4 - 2) - 1$:

► $\Sigma_{24} = V(q_{12}, q_{13}, q_{14}, q_{15}, q_{23})$

► $\Sigma_{15} = V(q_{12}, q_{13}, q_{14}, q_{23}, q_{24}, q_{34})$

(A dense subset of) these can be parameterized by rowspans of 2×5 matrices of the forms $\begin{bmatrix} 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \end{bmatrix}$ and

$\begin{bmatrix} 1 & \star & \star & \star & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, respectively. It turns out that

► $\mathcal{CL}_{\Sigma_{24}} = V(p_{45})$

► $\mathcal{CL}_{\Sigma_{15}} = V(p_{15}, p_{25}, p_{35}, p_{45})$

Thus we get an example of a Chow-Lam locus with codimension higher than 1.

Example: Schubert varieties

Hueristic on projection. Recall

$$\blacktriangleright \Sigma_{24}^{\circ} = \left\{ \text{rowspan} \begin{bmatrix} 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \end{bmatrix} \right\}$$

$$\blacktriangleright \Sigma_{15}^{\circ} = \left\{ \text{rowspan} \begin{bmatrix} 1 & \star & \star & \star & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right\}$$

Fix a 4×5 matrix Z Then points in $Z(\Sigma_{24})$ and $Z(\Sigma_{15})$ look like

$$\blacktriangleright \text{rowspan} \begin{bmatrix} Z_2 + \star Z_3 + \star Z_5 \\ Z_4 + \star Z_5 \end{bmatrix}$$

$$\blacktriangleright \text{rowspan} \begin{bmatrix} Z_1 + \star Z_2 + \star Z_3 + \star Z_4 \\ Z_5 \end{bmatrix}$$

Geometrically, these are “lines which intersect the line $\text{span}(Z_4, Z_5)$ in \mathbb{P}^3 ” and “lines which pass through the point Z_5 in \mathbb{P}^3 ”. As a condition on a line $\text{rowspan}(L)$, their equations should be given by $\det[Z_{45}|L] = 0$ and $\det[Z_{i5}|L] = 0$ for $i = 1, 2, 3, 4$ respectively.

Properties of the Chow-Lam form

Proposition (Projection Formula, P-Sturmfels)

The Chow-Lam form $\text{CL}_{Z(\mathcal{V})}$ is obtained from the Chow-Lam form $\text{CL}_{\mathcal{V}}$ by replacing the primal Plücker coordinates with twistor coordinates:

$$p_{i_1 i_2 \dots i_{r-k}} = [Z \mid L]_{i_1 i_2 \dots i_{r-k}} \quad \text{for } 1 \leq i_1 < i_2 < \dots < i_{r-k} \leq n.$$

This expresses $\text{CL}_{Z(\mathcal{V})}$ in dual Plücker coordinates on $\text{Gr}(k, r)$, given by the $r \times k$ matrix L .

Example: Positroid varieties

Fix $k=2, n=9, r=7$ and the positroid variety

$$\mathcal{V} = V(q_{12}, q_{13}, q_{23}, q_{45}, q_{67}, q_{89}) \subset \text{Gr}(2, 9).$$

This variety consists of rowspaces of 2×9 matrices $L = [L_1 \dots L_9]$ with $\text{rk}(L_{123}) = \text{rk}(L_{45}) = \text{rk}(L_{67}) = \text{rk}(L_{89}) = 1$. Then $\mathcal{CL}_{\mathcal{V}}$ is in $\text{Gr}(4, 9)$.

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Let $X = [X_1 \dots X_9]$ be a 4×9 matrix. Note that $L \subset X$ if and only if there exists a 2×4 matrix T with $L = TX$. Then TX is in \mathcal{V} iff

$$\text{rk}(TX_{123}) = \text{rk}(TX_{45}) = \text{rk}(TX_{67}) = \text{rk}(TX_{89}) = 1.$$

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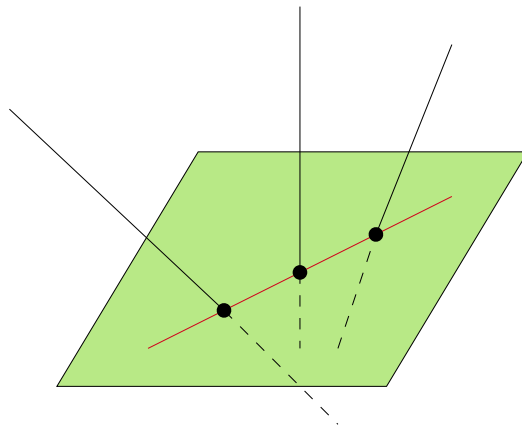
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Geometrically: The line in \mathbb{P}^3 given by T is contained in the plane X_{123} and intersects the lines X_{45} , X_{67} , and X_{89} .

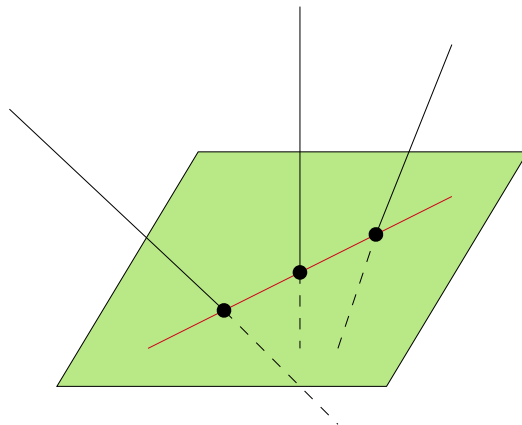
Example: Positroid varieties

For which X does there exist a line T in \mathbb{P}^3 contained in the plane X_{123} and intersecting the lines X_{45} , X_{67} , and X_{89} ?



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Answer : When the points $X_{123} \cap X_{45}$, $X_{123} \cap X_{67}$, $X_{123} \cap X_{89}$ are collinear! In Plücker coordinates, we have

$$X_{123} \cap X_{ij} = q_{23ij}X_1 - q_{13ij}X_2 + q_{12ij}X_3.$$

The Chow-Lam form is

$$\text{CL}_{\mathcal{V}} = \det \begin{bmatrix} q_{2345} & -q_{1345} & q_{1245} \\ q_{2367} & -q_{1367} & q_{1267} \\ q_{2389} & -q_{1389} & q_{1289} \end{bmatrix}.$$

Example: Positroid varieties

Fix $k = 2, n = 10, r = 8$ and consider the positroid variety

$$\mathcal{V} = V(q_{12}, q_{34}, q_{56}, q_{78}, q_{90}) \subset \text{Gr}(2, 10).$$

By similar reasoning, the Chow-Lam form is given by matrices $X = [X_1 \ \dots \ X_9 \ X_0]$ where the five lines $\overline{x_1x_2}, \overline{x_3x_4}, \overline{x_5x_6}, \overline{x_7x_8}, \overline{x_9x_0}$ are transversal. This codimension 1 condition is given by the Chow-Lam form

$$\text{CL}_{\mathcal{V}} = \det \begin{bmatrix} 0 & q_{1234} & q_{1256} & q_{1278} & q_{1290} \\ q_{1234} & 0 & q_{3456} & q_{3478} & q_{3490} \\ q_{1256} & q_{3456} & 0 & q_{5678} & q_{5690} \\ q_{1278} & q_{3478} & q_{5678} & 0 & q_{7890} \\ q_{1290} & q_{3490} & q_{5690} & q_{7890} & 0 \end{bmatrix}.$$

Hurwitz forms and higher Chow forms

Higher Chow forms characterize linear spaces that intersect \mathcal{V} non-transversally. These are also known as **coisotropic hypersurfaces**.

Definition (Hurwitz locus)

The **Hurwitz locus** of a projective variety X of dimension d consists of linear spaces of codimension d which intersect X non-transversally.

Definition (Hurwitz-Lam locus)

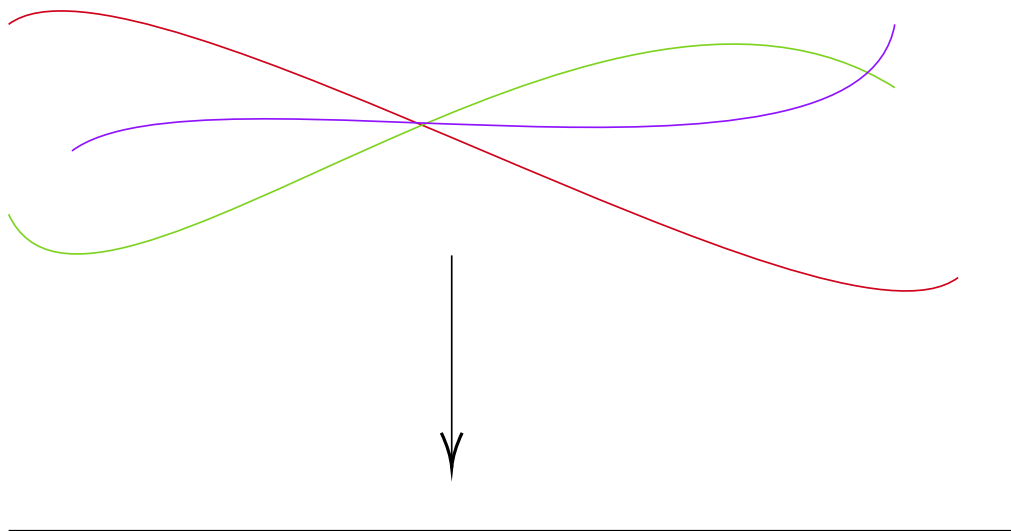
Let \mathcal{V} have dimension $k(r - k)$ for some r . The **Hurwitz-Lam locus** of \mathcal{V} is

$$\mathcal{HL}_{\mathcal{V}} = \{ P \in \mathrm{Gr}(k + n - r, n) : \mathcal{V} \cap \mathrm{Gr}(k, P) \text{ is not transverse} \}.$$

Hurwitz-Lam forms

Theorem (P-Sturmfels, Computing Branch Loci)

The branch locus of $Z : \text{Gr}(k, n) \rightarrow \text{Gr}(k, r)$ in \mathcal{V} is a hypersurface in $\text{Gr}(k, r)$, and its equation is obtained from the Hurwitz-Lam form by replacing the primal Plücker coordinates with twistor coordinates.



Open Questions (free! please take!)

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Thank you for listening!