The Chow-Lam Form

joint work with Bernd Sturmfels

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Slides: https://lizziepratt.com/notes

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The Chow Form

Definition (Chow form)

Let $X \subset \mathbb{P}^{n-1}$ be a d-dimensional projective variety. The *Chow locus* of X is

$$\mathcal{C}_X = \{L \in \mathsf{Gr}(n-d-1, n) : X \cap L \neq \varnothing\}.$$

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Example (Hypersurface)

The Chow form of a hypersurface $V(F)$ is F.

Example (Linear space)

The Chow locus of a linear space is a Schubert divisor.

Coordinate Systems

A linear space L can represented multiple ways.

- Primal : as the kernel of an $(n-k) \times n$ matrix
- Dual : as the rowspan of a $k \times n$ matrix

The primal and dual Plücker coordinates are the maximal minors of these matrices. They are related by complementing indices and multiplying by a sign.

Example (Coordinates on Gr$(3,5)$)

$$\begin{array}{cccccccccc} p_{12} & p_{13} & p_{14} & p_{15} & p_{23} & p_{24} & p_{25} & p_{34} & p_{35} & p_{45} \\ q_{345} & -q_{245} & q_{235} & -q_{234} & q_{145} & -q_{135} & q_{134} & q_{125} & -q_{124} & q_{123} \end{array}$$

The twisted cubic

The Chow locus of the twisted cubic X is the set of $[L] \in \mathrm{Gr}(2,4)$ such that the line L meets X in $\mathbb{P}^3$. The Chow form is given by the determinant of the *Bézout matrix* :

$$C_X = \det \begin{bmatrix} p_{12} & p_{13} & p_{14} \\ p_{13} & p_{14} + p_{23} & p_{24} \\ p_{14} & p_{24} & p_{34} \end{bmatrix}.$$

Its expansion is

$$-p_{14}^3 - p_{14}^2 p_{23} + 2p_{13}p_{14}p_{24} - p_{12}p_{24}^2 - p_{13}^2 p_{34} + p_{12}p_{14}p_{34} + p_{12}p_{23}p_{34}$$

For more on computing Chow forms of curves using resultants, see [David Eisenbud and Frank-Olaf Schreyer: *Resultants and Chow Forms via Exterior Syzygies*]

Intersection

Proposition (Intersection Formula)

Suppose $\dim X = d$, *and let* L *and* M *be linear subspaces of* $\mathbb{P}^{n-1}$ *such that* $\operatorname{codim}(L \cap M) = \operatorname{codim}(L) + \operatorname{codim}(M) = d + 1$. *Then*

$$\mathrm{C}_{X \cap L}(M) \;=\; \mathrm{C}_X(L \cap M).$$

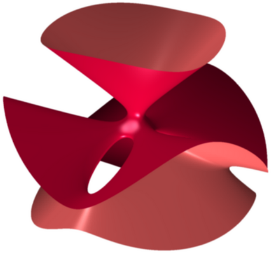
Example (Coordinates of intersection)

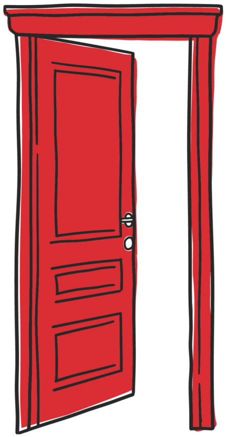
Fix $n = 6, d = 2$. Let codim $L = 1$ and codim $M = 2$ with primal Plücker coordinates l_i, m_{jk}. Then

$$L \cap M = \begin{bmatrix} l_1 & l_2 & l_3 & l_4 & l_5 & l_6 \\ u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{bmatrix}.$$

So $p_{ijk}(L \cap M) = l_i m_{jk} - l_j m_{ik} + l_k m_{ij}$.

A variety walks through the door...





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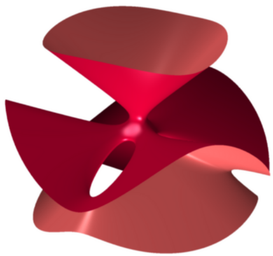

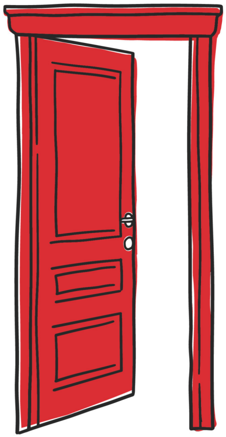

Proposition (Degree)

The degree of C_X equals the degree of X.

Proof.

The degree of C_X equals the degree of $C_{X\cap L}$. So, cut X with generic hyperplanes to get a finite number of points. $\square$

Projection

Theorem (Projection)

Suppose $\dim X = d$, *and let* L *be a linear subspace of* $\mathbb{P}^{n-1}$. *Let* Z *be a* $r \times n$ *matrix, where* $r \leq d - 2$. *That is,*

$$Z : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{r-1}.$$

Then $C_{Z(X)}(L) = C_X(Z^{-1}(L))$.

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Claim (Twistor Coordinates)

Let L *be given in primal coordinates, as an* $r \times (r-d-1)$ *matrix. The primal Plucker coordinate* $p_I(Z^{-1}(L))$ *is the maximal minor of the* $r \times (n+r-d-1)$ *matrix*

$$[Z_1 \ldots Z_n \mid L]$$

which includes I *columns of* Z *and all columns of* L. *This is called the* *Ith twistor coordinate* *of* Z *and* L.

The twisted cubic revisited

Recall the Chow form:

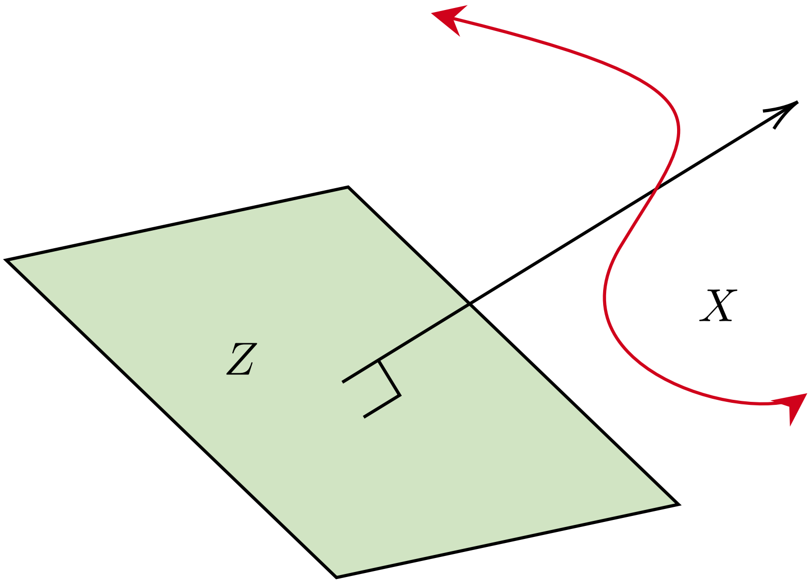
$$-p_{14}^3 - p_{14}^2 p_{23} + 2p_{13}p_{14}p_{24} - p_{12}p_{24}^2 - p_{13}^2 p_{34} + p_{12}p_{14}p_{34} + p_{12}p_{23}p_{34}$$

We make the substitution where the p_{ij} index 3×3 minors of

$$[Z|L] = \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} & l_1 \\ z_{21} & z_{22} & z_{23} & z_{24} & l_2 \\ z_{31} & z_{32} & z_{33} & z_{34} & l_3 \end{bmatrix}$$

involving l. Now $C_{Z(X)}(L) = C_X(Z^{-1}(L))$ is a function of $[l_1 : l_2 : l_3]$ on $\mathbb{P}^3$ and the matrix entries of Z. Choosing a particular Z gives the equation of $\overline{Z(X)}$ in $\mathbb{P}^2$.

The twisted cubic revisited



The Chow form $C_{Z(X)}(L)$ computes the universal projection variety :

$$\mathcal{Y}_X := \overline{\{(Z, Z(p)) : p \in X\} \subset \mathsf{Mat}(r,n) \times \mathbb{P}^{r-1}}.$$

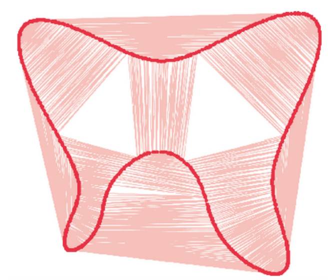
[Thomas Lam: *Totally nonnegative Grassmannian and Grassmann polytopes*, 2015]

A new context: Grassmannians

Let Z be a $r \times n$ matrix. This gives a map

$$Z : \mathrm{Gr}(k, n) \dashrightarrow \mathrm{Gr}(k, r).$$

Can we compute "universal" equations of projected varieties which are codimension 1 in the target? Examples: canonical curves, positroid varieties, torus orbits...



The Chow-Lam form

Definition (Grassmannian)

For a linear space P in $\mathbb{P}^{n-1}$, let $\mathrm{Gr}(k, P)$ denote k-planes in $\mathbb{P}^{n-1}$ contained in P.

Definition (Chow-Lam form)

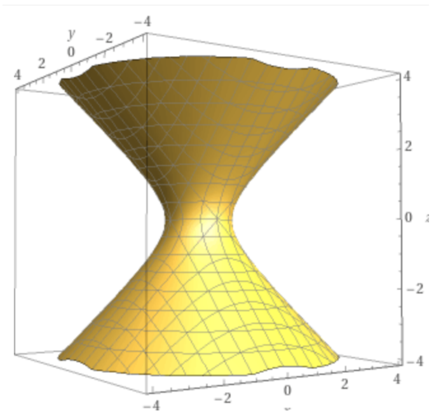
Let $\mathcal{V}$ be a variety of dimension $k(r-k)-1$ for some $r \leq n$. We define the *Chow-Lam locus* to be

$$\mathcal{CL}_{\mathcal{V}} := \{P \in \mathrm{Gr}(k+n-r, n) : \mathrm{Gr}(k, P) \cap \mathcal{V} = \varnothing\}.$$

When $\mathcal{CL}_{\mathcal{V}}$ has codimension 1, its defining equation is the *Chow form* $CL_{\mathcal{V}}$.

Example: Ruled surface

Let $\mathcal{V} \subset \text{Gr}(2,4)$ be a curve in the Grassmannian, and let S be the surface swept out by the lines parameterized by $\mathcal{V}$.



Then $\mathcal{CL}_{\mathcal{V}}$ is a surface in $\mathrm{Gr}(3,4) = (\mathbb{P}^3)^\vee$. It parametrizes planes P in $\mathbb{P}^3$ which contain a line Q from the curve $\mathcal{V}$, ie planes tangent to $\mathcal{V}$. Thus $\mathcal{CL}_{\mathcal{V}}$ is the projective dual of S.

Example: Schubert varieties

Consider the following Schubert varieties in Gr$(2,5)$ of dimension $3 = 2(4-2)-1$:

- $\Sigma_{24} = V(q_{12}, q_{13}, q_{14}, q_{15}, q_{23})$
- $\Sigma_{15} = V(q_{12}, q_{13}, q_{14}, q_{23}, q_{24}, q_{34})$

(A dense subset of) these can be parameterized by rowspans of 2×5 matrices of the forms $\begin{bmatrix} 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \end{bmatrix}$ and $\begin{bmatrix} 1 & \star & \star & \star & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, respectively. It turns out that

- $\mathcal{CL}_{\Sigma_{24}} = V(p_{45})$
- $\mathcal{CL}_{\Sigma_{15}} = V(p_{15}, p_{25}, p_{35}, p_{45})$

Thus we get an example of a Chow-Lam locus with codimension higher than 1.

Example: Schubert varieties

Hueristic on projection. Recall

- $\Sigma_{24}^\circ = \left\{ \text{rowspan} \begin{bmatrix} 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \end{bmatrix} \right\}$
- $\Sigma_{15}^\circ = \left\{ \text{rowspan} \begin{bmatrix} 1 & \star & \star & \star & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right\}$

Fix a 4×5 matrix Z Then points in $Z(\Sigma_{24})$ and $Z(\Sigma_{15})$ look like

- rowspan $\begin{bmatrix} Z_2 + \star Z_3 + \star Z_5 \\ Z_4 + \star Z_5 \end{bmatrix}$
- rowspan $\begin{bmatrix} Z_1 + \star Z_2 + \star Z_3 + \star Z_4 \\ Z_5 \end{bmatrix}$

Geometrically, these are "lines which intersect the line span(Z_4, Z_5) in $\mathbb{P}^3$" and "lines which pass through the point Z_5 in $\mathbb{P}^3$". As a condition on a line rowspan(L), their equations should be given by $\det[Z_{45}|L] = 0$ and $\det[Z_{i5}|L] = 0$ for $i = 1, 2, 3, 4$ respectively.

Properties of the Chow-Lam form

Proposition (P-Sturmfels,Projection Formula)

The Chow-Lam form $\mathrm{CL}_{Z(\mathcal{V})}$ *is obtained from the Chow-Lam form* $\mathrm{CL}_{\mathcal{V}}$ *by replacing the primal Plücker coordinates with twistor coordinates:*

$$p_{i_1 i_2 \cdots i_{r-k}} \;=\; [\,Z \,|\, L\,]_{i_1 i_2 \cdots i_{r-k}} \quad \text{for } 1 \leq i_1 < i_2 < \cdots < i_{r-k} \leq n.$$

This expresses $\mathrm{CL}_{Z(\mathcal{V})}$ *in dual Plücker coordinates on* $\mathrm{Gr}(k, r)$*, given by the* $r \times k$ *matrix* L*.*

Example: Positroid varieties

Fix $k{=}2, n{=}9, r{=}7$ and the positroid variety

$$\mathcal{V} \ = \ V(q_{12}, q_{13}, q_{23}, q_{45}, q_{67}, q_{89}) \ \subset \ \mathrm{Gr}(2,9).$$

This variety consists of rowspans of 2×9 matrices $L = [L_1 ... L_9]$ with rk(L_{123}) = rk(L_{45}) = rk(L_{67}) = rk$(L_{89}) = 1$. Then $\mathcal{CL}_{\mathcal{V}}$ is in Gr$(4,9)$.

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Let $X = [X_1 ... X_9]$ be a 4×9 matrix. Note that $L \subset X$ if and only if there exists a 2×4 matrix T with $L = TX$. Then TX is in $\mathcal{V}$ iff

$$\mathrm{rk}(TX_{123}) = \mathrm{rk}(TX_{45}) = \mathrm{rk}(TX_{67}) = \mathrm{rk}(TX_{78}) = 1.$$

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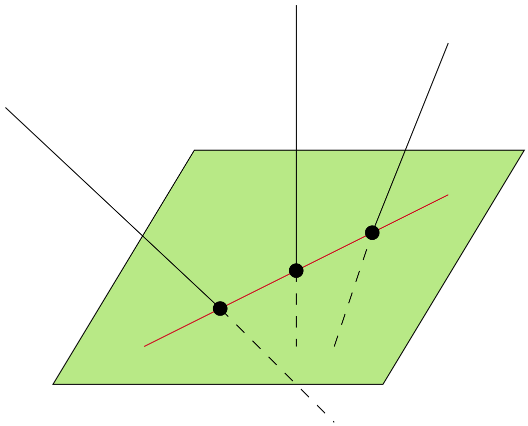
Geometrically: The line in $\mathbb{P}^3$ given by T is contained in the plane X_{123} and intersects the lines X_{45}, X_{67}, and X_{89}.

Example: Positroid varieties

For which X does there exists a line T in $\mathbb{P}^3$ contained in the plane X_{123} and intersecting the lines X_{45}, X_{67}, and X_{89}?



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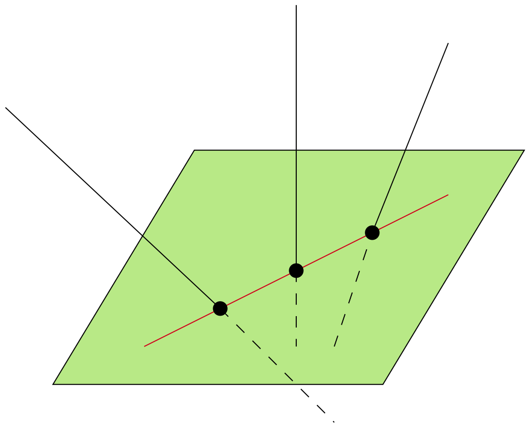


Answer : When the points $X_{123} \cap X_{45}, X_{123} \cap X_{67}, X_{123} \cap X_{89}$ are collinear! In Plücker coordinates, we have

$$X_{123} \cap X_{ij} = q_{23ij} X_1 - q_{13ij} X_2 + q_{12ij} X_3.$$

The Chow-Lam form is

$$\mathrm{CL}_{\mathcal{V}} = \det \begin{bmatrix} q_{2345} & -q_{1345} & q_{1245} \\ q_{2367} & -q_{1367} & q_{1267} \\ q_{2389} & -q_{1389} & q_{1289} \end{bmatrix}.$$

Example: Positroid varieties

Fix $k = 2, n = 10, r = 8$ and consider the positroid variety

$$\mathcal{V} = V(q_{12}, q_{34}, q_{56}, q_{78}, q_{90}) \subset \mathsf{Gr}(2, 10).$$

By similar reasoning, the Chow-Lam form is given by matrices $X = \begin{bmatrix} X_1 & \dots & X_9 & X_0 \end{bmatrix}$ where the fives lines $\overline{x_1x_2}$, $\overline{x_3x_4}$, $\overline{x_5x_6}$, $\overline{x_7x_8}$, $\overline{x_9x_0}$ are transversal. This codimension 1 condition is given by the Chow-Lam form

$$\mathrm{CL}_{\mathcal{V}} = \det \begin{bmatrix} 0 & q_{1234} & q_{1256} & q_{1278} & q_{1290} \\ q_{1234} & 0 & q_{3456} & q_{3478} & q_{3490} \\ q_{1256} & q_{3456} & 0 & q_{5678} & q_{5690} \\ q_{1278} & q_{3478} & q_{5678} & 0 & q_{7890} \\ q_{1290} & q_{3490} & q_{5690} & q_{7890} & 0 \end{bmatrix}.$$

Hurwitz forms and higher Chow forms

Higher Chow forms characterize linear spaces that intersect $\mathcal{V}$ non-transversally. These are also known as coisotropic hypersurfaces.

Definition (Hurwitz locus)

The *Hurwitz locus* of a projective variety X of dimension d consists of linear spaces of codimension d which intersect X non-transversally.

Definition (Hurwitz-Lam locus)

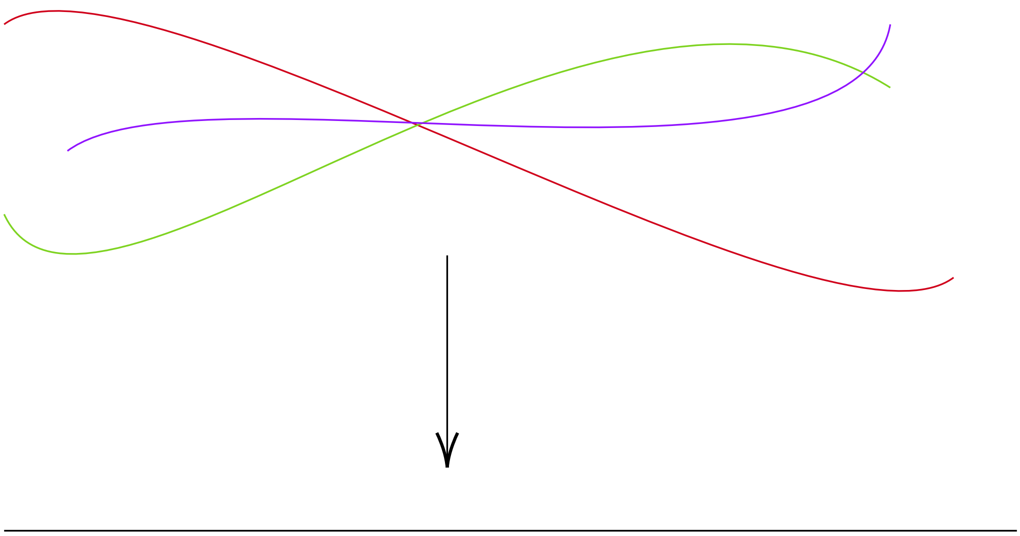
Let $\mathcal{V}$ have dimension $k(r-k)$ for some r. The *Hurwitz-Lam locus* of $\mathcal{V}$ is

$$\mathcal{HL}_{\mathcal{V}} \; = \; \left\{ P \in \mathrm{Gr}(k+n-r,n) \; : \; \mathcal{V} \cap \mathsf{Gr}(k,P) \text{ is not transverse} \right\}.$$

Hurwitz-Lam forms

Theorem (P-Sturmfels, Computing Branch Loci)

The branch locus of Z : Gr$(k, n) \to$ Gr(k, r) in $\mathcal{V}$ is a hypersurface in $\mathrm{Gr}(k, r)$, and its equation is obtained from the Hurwitz-Lam form by replacing the primal Plücker coordinates with twistor coordinates.



Example: Positroid variety

Proposition (P-Sturmfels)

The Hurwitz-Lam form of the positroid variety $\mathcal{V} = V(q_{12}, q_{34}, q_{56}, q_{78}) \subset \mathrm{Gr}(2,8)$ *equals*

$$\begin{array}{rcl} \mathrm{HL}_{\mathcal{V}} & = & p_{1235}^2 p_{4678}^2 + p_{1236}^2 p_{4578}^2 + p_{1245}^2 p_{3678}^2 + p_{1246}^2 p_{3578}^2 \end{array}$$
$$+4p_{1235}p_{1246}p_{3578}p_{4678} - 2\,(p_{1234}p_{1256}p_{3478}p_{5678} + p_{1234}p_{1256}p_{3578}p_{4678}$$
$$+p_{1235}p_{1236}p_{4578}p_{4678} + p_{1235}p_{1245}p_{3678}p_{4678} + p_{1235}p_{1246}p_{3478}p_{5678}$$
$$+p_{1236}p_{1246}p_{3578}p_{4578} + p_{1245}p_{1246}p_{3578}p_{3678}).$$

Geometrically: The amplituhedron $\mathcal{A}_{8,6,2} \subset \mathrm{Gr}(2,6)$ has a degree four and codimension two boundary given by this ramification locus, which is NOT itself the projection of a positroid variety.

Open Questions (free! please take!)

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Thank you for listening!