Probab. Theory Relat. Fields (2009) 144:247–279
DOI 10.1007/s00440-008-0147-9

Self-interacting diffusions: a simulated annealing version

Olivier Raimond

Received: 30 October 2006 / Revised: 6 February 2008 / Published online: 26 March 2008

Abstract We study asymptotic properties of processes X, living in a Riemannian compact manifold M, solution of the stochastic differential equation (SDE)

$$dX_t = dW_t(X_t) - \beta(t)\nabla V\mu_t(X_t)dt$$

with W a Brownian vector field, $\beta(t) = a\log(t+1)$, $\mu_t = \frac{1}{t}\int_0^t \delta_{X_s} ds$ and $V\mu_t(x) = \frac{1}{t}\int_0^t V(x, X_s)ds$, V being a smooth function. We show that the asymptotic behavior of μ_t can be described by a non-autonomous differential equation. This class of processes extends simulated annealing processes for which $V(x, y)$ is only a function of x. In particular we study the case $M = \mathbb{S}^n$, the n-dimensional sphere, and $V(x, y) = -\cos(d(x, y))$, where $d(x, y)$ is the distance on $\mathbb{S}^n$, which corresponds to a process attracted by its past trajectory. In this case, it is proved that μ_t converges almost surely towards a Dirac measure.

Mathematics Subject Classification (2000) 60K35

1 Introduction

Let M be a smooth compact Riemannian manifold without boundary. We denote by n the dimension of M. Let $\mathcal{M}(M)$ be the Banach space of Borel bounded measures on M, equipped with the norm

$$|\mu| = \sup\{|\mu f| : \ f \in C(M), \ \|f\|_\infty = 1\},$$

O. Raimond (✉)
Département de Mathématiques, Université Paris-Sud, Bâtiment 425, 91405 Orsay cedex, France
e-mail: olivier.raimond@math.u-psud.fr

where μf denotes $\int_M f(x)\mu(dx)$. Let $C^r(M)$, $r \geq 1$, be the space of r times continuously differentiable real functions on M. We also let $C(M)$ be the space of continuous functions on M. Let $V : M \times M \to \mathbb{R}$ be a smooth function. The function V is called the interacting potential. For all measures μ in $\mathcal{M}(M)$, we set

$$V\mu(x) = \int V(x, y)\mu(dy)$$

(without loss of generality, in the following, we will choose V such that $\|V\|_\infty \leq 1$). Let Δ be the Laplacian on M. For all measures μ in $\mathcal{M}(M)$, we define the operator A_μ acting on $C^2(M)$ by

$$A_\mu f = \Delta f - \langle \nabla(V\mu), \nabla f \rangle.$$

Let $\beta : \mathbb{R}^+ \to \mathbb{R}$ be a C^1-function. We now consider the process X_t solution of the SDE

$$dX_t = \sum_i e_i(X_t) \circ dB_t^i - \beta(t)\nabla(V\mu_t)(X_t)dt, \tag{1}$$

where

$$\mu_t = \frac{1}{t}\int_0^t \delta_{X_s} ds, \tag{2}$$

B^i are independent Wiener processes and e_i are vector fields such that $\sum_i e_i(e_i f) = 2\Delta f$. This process is such that

$$f(X_t) - f(X_0) - \int_0^t A_{\beta(s)\mu_s} f(X_s)ds$$

is a martingale for all $f \in C^2(M)$. The process X could also have been defined as the unique solution (in law) of this martingale problem.

The case $\beta(t) = 1$ has been studied in [2–4]. In this case, the asymptotics of μ_t can be described by a dynamical system on $\mathcal{M}(M)$. This dynamical system is generated by the ordinary differential equation (ODE)

$$\frac{d\mu}{dt} = F(\mu),$$

the vector field F being defined by

$$F(\mu) = -\mu + \Pi(\mu),$$

where

$$\Pi(\mu)(dx) = \frac{e^{-V\mu(x)}}{Z_\mu}\lambda(dx)$$

with λ, the normalized Riemann measure on M, and

$$Z_\mu = \int_M e^{-V\mu(x)}\lambda(dx).$$

When β is not constant, we can expect to describe the asymptotics of μ_t by a non-autonomous dynamical system on $\mathcal{M}(M)$, generated by the non-autonomous ordinary differential equation

$$\frac{d\mu}{dt} = F_t(\mu), \tag{3}$$

where F_t, the vector field at time t, is defined by

$$F_t(\mu) = -\mu + \Pi_t(\mu),$$

with

$$\Pi_t(\mu) = \Pi(\beta(e^t)\mu).$$

We intend to describe the asymptotic behavior of μ_t for β of the form $\beta(t) = a\log(t+1)$ for a sufficiently small constant a. The case $\beta(t) = t$ has been studied by different authors (see [5,6,10]). In [5,6], the process lives in $\mathbb{R}$ (and in $\mathbb{R}^d$ in [10]). The result proved in these works is that for a class of functions V of the form $V(x, y) = U(x - y)$, with U a convex function having a unique minimum at 0, the process converges a.s. The technics employed in these papers do not involve any approximation by an ODE.

When $\beta(t) = a\log(t+1)$, one cannot hope for the almost sure convergence of the process X. In Sect. 2, we show a theorem that describes how close is the trajectory $(\mu_{e^{t+s}})_{s\geq 0}$ to the solution of the non autonomous ODE (3) started at time t at position μ_{e^t}. In Sect. 3, the example $M = \mathbb{S}^n$ and $V(x, y) = -\cos(d(x, y))$ (where d is the geodesic distance on $\mathbb{S}^n$) is investigated. The fact that μ_t converges towards a Dirac measure is proved. The proofs of these results use the techniques of [2] and involves an approximation by an ODE.

When $V(x, y)$ is only a function of x (i.e. when there is no interaction with the past trajectory), the processes defined by the SDE (1) are simulated annealing processes (see [7,9]). In this case, the process X_t converges in law towards the uniform measure on $\mathrm{Argmin}(V)$, the set of points x such that $V(x) = \inf\{V(x);\ x \in M\}$.

Note that when $\beta(t)$ converges towards ∞ as t goes to ∞, then $\Pi_t(\mu)$ converges towards the uniform measure on $\mathrm{Argmin}(V\mu)$. Thus, for the class of processes we study in this paper, one can guess that μ_t converges towards a measure μ that is the uniform measure on $\mathrm{Argmin}(V\mu)$. One would also like to replace the non-autonomous ODE (3) by the ODE (if it could be properly defined)

$$\frac{d\mu}{dt} = -\mu + \lambda_\mu, \tag{4}$$

where λ_μ is the uniform probability measure on Argmin$(V\mu)$. For the example developed in Sect. 3, when $\beta(t) = a$ for a sufficiently large constant a, μ_t converges towards a random measure μ satisfying $\mu = \Pi(a\mu)$, that is not the uniform probability measure on M. When a goes to ∞, $\Pi(a\mu)$ converges towards a Dirac measure. This explains the result of Sect. 3, where $\beta(t)$ is a function converging towards ∞. Unfortunately, with the approximation technics used here, we are not in position to study all the functions $\beta(t)$ that converges towards ∞.

In the whole paper K (respectively C) denotes a constant (respectively a random constant) that may change from line to line, and that depends only on V, on β and on M.

In the following, the function β will satisfy the hypothesis

Hypothesis 1.1 *There exists a positive a and t_0 such that for $t \geq t_0$,*

$$|\beta(t)| \leq a \log(t)$$

and there exists $\gamma \in]0, 1]$ such that

$$|\beta'(t)| = O(t^{-\gamma}).$$

This hypothesis is satisfied for the functions $\beta(t) = a \log(t + 1)$, $\beta(t) = a \log(\log(t + 2))$ and $\beta(t) = a$, but is not satisfied for $\beta(t) = t$.

2 Approximation by a non-autonomous ODE

For s and t positive, $f \in C(M)$, set

$$\epsilon_t(s)f = \int_{e^t}^{e^{t+s}} \frac{f(X_u) - \Pi(\beta(u)\mu_u)f}{u} du.$$

Then

$$\mu_{e^{t+s}} f - \mu_{e^t} f = \int_0^s \left(-\mu_{e^{t+u}} f + \Pi(\beta(e^{t+u})\mu_{e^{t+u}}) f\right) du + \epsilon_t(s)f.$$

In this section we prove the following

Theorem 2.1 *Assume that β satisfies Hypothesis 1.1. There exists a constant $a_c > 0$ such that if a and γ satisfy $a < a_c\gamma$, then there are constants $K > 0$ and $\alpha \in]0, 1]$ such that for all $f \in C(M)$*

$$\mathsf{E}\left[\sup_{s\geq 0}(\epsilon_t(s)f)^2|\mathcal{F}_{e^t}\right] \leq K \exp(-\alpha t) \|f\|_\infty^2,$$

which implies that almost surely,

$$\limsup_{t\to\infty} \frac{1}{t} \log\left(\sup_{s\geq 0} |\epsilon_t(s) f| \right) \leq -\frac{\alpha}{2}.$$

When $\gamma = 1$ and $\beta(t) = o(\log(t))$, the theorem is satisfied for all $\alpha < 1$, and when $\beta(t) = a$ it is satisfied with $\alpha = 1$.

Following [2], the mapping

$$(t, \mu) \mapsto -\mu + \Pi_t(\mu)$$

is a locally Lipschitz function mapping $\mathbb{R}^+ \times \mathcal{P}(M)$ into $\mathcal{P}(M)$, with $\mathcal{P}(M)$ equipped with the variation norm. This implies the existence of a family $(\phi_{s,t})_{s\leq t}$ such that for all $s > 0$ and $\mu \in \mathcal{P}(M)$, $\phi_{s,s+\cdot}(\mu)$ is the solution of the non autonomous differential equation

$$\phi_{s,t}(\mu) = \mu + \int_s^t (-\phi_{s,u}(\mu) + \Pi_u(\phi_{s,u}(\mu)))du.$$

This family is also continuous with respect to the weak topology and satisfies, for all $s < t < u$,

$$\phi_{s,u} = \phi_{t,u} \circ \phi_{s,t}.$$

Theorem 2.1 shows how close is $(\mu_{e^{t+s}})_{s\geq 0}$ to be a solution to this non autonomous differential equation. More can be proved if in addition to Hypothesis 1.1, we assume $\beta(t) = o(\log(t))$ and $V(x, y) = \sum_{i=1}^n f_i(y)g_i(x)$ for some smooth functions f_i and g_i. Let d_w be a distance on $\mathcal{P}(M)$ compatible with the weak topology. For convenience we choose f_i such that $\|f_i\|_\infty \leq 1$ and take for d_w the distance defined by

$$d_w(\mu, \nu) = \sum_{i=1}^{\infty} 2^{-i} |\mu f_i - \nu f_i|,$$

where $(f_i)_{i>n}$ are chosen such that f_i is continuous, $\|f_i\|_\infty \leq 1$ and $\{f_i\}_{i\geq 1}$ is dense in $\{f \in C^0(M) : \|f\|_\infty \leq 1\}$. Then we have

Theorem 2.2 *For all* $T > 0$,

$$\lim_{t\to\infty} \sup_{s\in[0,T]} d_w(\mu_{e^{t+s}}, \phi_{t,t+s}(\mu_{e^t})) = 0.$$

Proof There exists a constant K such that for all $t > 0$, all $f \in C^0(M)$, and all μ and $\nu \in \mathcal{P}(M)$,

$$\begin{aligned}|\Pi_t(\mu)f - \Pi_t(\nu)f| &\leq K\beta(e^t)\|V\mu - V\nu\|_\infty \times \|f\|_\infty\\ &\leq K\beta(e^t) \sup_{1\leq i\leq n} |\mu f_i - \nu f_i| \times \|f\|_\infty\\ &\leq K\beta(e^t) d_w(\mu,\nu) \times \|f\|_\infty.\end{aligned}$$

Fix $T > 0$ and take $\epsilon < \frac{\alpha}{2T}$. There exists t_0 such that for all $t > t_0$, $\beta(e^t) \leq \epsilon \log(t)$. This implies that

$$d_w(-\mu + \Pi_t(\mu), -\nu + \Pi_t(\nu)) \leq K(1 + \beta(e^t))d_w(\mu,\nu).$$

A standart application of Gronwall's lemma implies the existence of a constant K_T depending only on T such that for all $t > t_0$,

$$\sup_{s\in[0,T]} d_w(\mu_{e^{t+s}}, \phi_{t,t+s}) \leq K_T(1 + te^{\epsilon T t}) \times \left(\sum_{i=1}^{\infty} 2^{-i} \sup_{s\in[0,T]} |\epsilon_t(s)f_i|\right)$$

which converges towards 0 as $t \to \infty$ because of Theorem 2.1. $\square$

2.1 Semigroup estimates

Let μ be a positive Borel finite measure on M. We define P_t^μ as the Feller semigroup with generator A_μ and invariant probability measure $\Pi(\mu)$. For $f \in L^2(\lambda)$ (which coincides with $L^2(\Pi(\mu))$ for all μ), we set

$$K_\mu f = f - \Pi(\mu)f \tag{5}$$

and

$$\mathsf{Q}_\mu f = -\int_0^\infty \mathsf{P}_t^\mu K_\mu f\, dt. \tag{6}$$

Then Q_μ is the "inverse" of $-A_\mu$ and we have

$$A_\mu \mathsf{Q}_\mu f = -K_\mu f.$$

The next crucial lemma bounds $\|\mathsf{Q}_\mu f\|_\infty$ and $\|\nabla \mathsf{Q}_\mu f\|_\infty$ in terms of the L^∞ norm of f.

Lemma 2.3 *There exist positive constants K and κ such that*

$$\|\mathsf{Q}_\mu f\|_\infty \leq K e^{\kappa\|V\mu\|_\infty}\|f\|_\infty \tag{7}$$

for all $f \in L^\infty$. *Furthermore, if* f *is* C^∞, $\mathsf{Q}_\mu f \in C^1$ *and*

$$\|\nabla \mathsf{Q}_\mu f\|_\infty \le K(1+\|Hess(V\mu)\|_\infty)^{1/2} e^{\kappa\|V\mu\|_\infty}\|f\|_\infty. \tag{8}$$

Proof We first prove inequality (7). In order to do this, We need to show the following ultra-contractive inequality:

Lemma 2.4 *There exist constants* $\kappa_1 > 0$ *and* $K > 0$ *such that for all* $f \in L^\infty$, $t \in]0,1]$,

$$\|\mathsf{P}^\mu_t f\|_\infty \le K e^{\kappa_1\|V\mu\|_\infty} t^{-n/2}\|f\|_{2,\mu}, \tag{9}$$

where $\|\cdot\|_{2,\mu}$ *denotes the norm in* $L^2(\Pi(\mu))$.

Proof The Dirichlet form $\mathcal{E}_\mu$ associated with P^μ_t is defined by

$$\mathcal{E}_\mu(f,f) = \int \|\nabla f\|^2 d\Pi(\mu),$$

with $f \in C^1(M)$. Note that (see [1]) A_μ satisfies the following log-Sobolev inequality

$$\Pi(\mu)(f^2\log(f^2)) - \Pi(\mu)(f^2)\log(\Pi(\mu)(f^2)) \le \Pi(\mu)(f^2)\Phi_\mu\left(\frac{\mathcal{E}_\mu(f,f)}{\Pi(\mu)(f^2)}\right) \tag{10}$$

where Φ_μ is defined by

$$\Phi_\mu(x) = \frac{n}{2}\log(c_1(\mu)+c_2(\mu)x).$$

The constants $c_1(\mu)$ and $c_2(\mu)$ can be taken of the form $c_1 e^{2\|V\mu\|_\infty}$ and $c_2 e^{2\|V\mu\|_\infty}$. In [1] is proved the following: for all $s > 0$, if we set

$$t_\mu(s) = \frac{1}{2}\int_1^\infty \Phi'_\mu(sx)\frac{dx}{\sqrt{x(x-1)}}$$

$$m_\mu(s) = \frac{1}{2}\int_1^\infty \frac{\Psi_\mu(sx)}{x}\frac{dx}{\sqrt{x(x-1)}}$$

where $\Psi_\mu(x) = \Phi_\mu(x) - x\Phi'_\mu(x)$, then

$$\|P^\mu_{t_\mu(s)}\|_{1,\infty} \le e^{m_\mu(s)}, \tag{11}$$

where

$$\|P^\mu_{t_\mu(s)}\|_{1,\infty} = \sup\left\{\frac{\|P^\mu_{t_\mu(s)}f\|_\infty}{\|f\|_{1,\mu}};\quad f \in L^1(\mu)\right\}$$

with $\|\cdot\|_{1,\mu}$ denoting the norm in $L^1(\Pi(\mu))$.

Since

$$\Phi_\mu = n\|V\mu\|_\infty + \Phi_0,$$

we have $t_\mu(s) = t_0(s)$ and $m_\mu(s) = \kappa_1\|V\mu\|_\infty + m_0(s)$, with

$$\kappa_1 = \frac{n}{2}\int_1^\infty \frac{dx}{x\sqrt{x(x-1)}}.$$

This implies that

$$\|P^\mu_{t_0(s)}\|_{1,\infty} \leq e^{\kappa_1\|V\mu\|_\infty} e^{m_0(s)}.$$

As $s \to \infty$ (see [1]), $st_0(s)$ converges towards $n/2$ and

$$m_0(s) = \frac{n}{2}\log(s) + K + o(1).$$

This implies that there exists a constant K (that does not depend on μ) such that for all $t \in]0, 1]$

$$\|P^\mu_t\|_{1,\infty} \leq Ke^{\kappa_1\|V\mu\|_\infty} t^{-n/2}.$$

This inequality implies the lemma. □

In addition to this ultra-contractive inequality, we also have the spectral gap inequality: there exists a positive constant b such that for all $f \in L^\infty$,

$$\lambda f^2 - (\lambda f)^2 \leq b\lambda(\|\nabla f\|^2). \tag{12}$$

Using the fact that for all $\mu \in \mathcal{M}(M)$,

$$\Pi(\mu)f^2 - (\Pi(\mu)f)^2 = \frac{1}{2}\int_{M^2}(f(x) - f(y))^2\Pi(\mu)(dx)\Pi(\mu)(dy),$$

inequality (12) implies the following spectral gap inequality: for all $\mu \in \mathcal{M}(M)$, $f \in L^\infty$,

$$\Pi(\mu)f^2 - (\Pi(\mu)f)^2 \leq be^{4\|V\mu\|_\infty}\Pi(\mu)(\|\nabla f\|^2). \tag{13}$$

With these inequalities in hand, we can now prove (7). The ultra-contractive inequality, with the semigroup property, implies that

$$\begin{aligned}\|\mathsf{Q}_\mu f\|_\infty &\le \int_0^\infty \|\mathsf{P}_t^\mu(K_\mu f)\|_\infty dt \\ &\le 2\|f\|_\infty + \int_0^\infty \|\mathsf{P}_1^\mu(\mathsf{P}_t^\mu(K_\mu f))\|_\infty dt \\ &\le 2\|f\|_\infty + Ke^{\kappa_1\|V\mu\|_\infty}\int_0^\infty \|\mathsf{P}_t^\mu(K_\mu f))\|_{2,\mu} dt. \end{aligned} \tag{14}$$

The spectral gap inequality then implies that

$$\begin{aligned}\|\mathsf{P}_t^\mu(K_\mu f))\|_{2,\mu} &\le \exp\left\{-\frac{t}{be^{4\|V_\mu\|_\infty}}\right\}\|K_\mu f\|_{2,\mu} \\ &\le 2\exp\left\{-\frac{t}{be^{4\|V_\mu\|_\infty}}\right\}\|f\|_\infty. \end{aligned} \tag{15}$$

Combining (14) and (15) shows that

$$\|\mathsf{Q}_\mu f\|_\infty \le \|f\|_\infty(2 + Ke^{\kappa_1\|V\mu\|_\infty}be^{4\|V\mu\|_\infty}).$$

Inequality (7) is proved.

We now prove the second inequality (8). We will use the Γ_2 criterion of Bakry and Emery (see [1] or [8]). Let $R(\mu)$ be a lower bound of Ric + Hess$(V\mu)$, where Ric is the Ricci tensor on M and Hess$(V\mu)$ is the Hessian of $V\mu$. It is well-known that the semigroup P_t^μ satisfies a curvature-dimension inequality $CD(R(\mu), \infty)$. This curvature-dimension inequality implies that (see Eq. (2.3) in [8]) for all $t > 0$,

$$\|\nabla P_t^\mu f\|_\infty^2 \le \frac{-R(\mu)}{1 - e^{2R(\mu)t}}\|f\|_\infty^2. \tag{16}$$

We set $K(\mu) = -R(\mu) > 0$. Using the fact that $\frac{s}{1-e^{-s}} \le 1 + s$ for all positive s, we get

$$\frac{-R(\mu)}{1 - e^{2R(\mu)t}} \le \frac{1 + 2tK(\mu)}{2t}.$$

Thus, for $t \in [0, 2]$

$$\|\nabla P_t^\mu f\|_\infty \le \frac{D(\mu)}{\sqrt{t}}\|f\|_\infty. \tag{17}$$

with

$$D(\mu) = \sqrt{\frac{1 + 4K(\mu)}{2}}. \tag{18}$$

Using this inequality with the semigroup property, we obtain that

$$\begin{aligned}\|\nabla Q_\mu f\|_\infty &\le \int_0^\infty \|\nabla \mathsf{P}_t^\mu (K_\mu f)\|_\infty dt \\ &\le 2D(\mu)\|f\|_\infty \int_0^2 \frac{dt}{\sqrt{t}} + \int_0^\infty \|\nabla \mathsf{P}_1^\mu (\mathsf{P}_{t+1}^\mu K_\mu f)\|_\infty dt \\ &\le 4\sqrt{2}D(\mu)\|f\|_\infty + D(\mu)\int_0^\infty \|\mathsf{P}_{t+1}^\mu K_\mu f\|_\infty dt.\end{aligned}$$

The conclusion follows by the same arguments as in (14) and the fact that there exists a constant K such that $D(\mu) \le K(1 + \|\mathrm{Hess}(V\mu)\|_\infty)^{1/2}$. □

Notice that the assumption $\|V\|_\infty \le 1$ implies the existence of a constant K such that

$$\|\mathsf{Q}_\mu f\|_\infty \le K e^{\kappa|\mu|}\|f\|_\infty \tag{19}$$

and that

$$\|\nabla \mathsf{Q}_\mu f\|_\infty \le K(1 + |\mu|)^{1/2} e^{\kappa|\mu|}\|f\|_\infty. \tag{20}$$

We also have

Lemma 2.5 *There exists a constant $K > 0$ such that for all μ and ν in $\mathcal{M}(M)$ and all $f \in C(M)$,*

$$\begin{aligned}\|\nabla \mathsf{Q}_\mu f - \nabla \mathsf{Q}_\nu f\|_\infty \le\ & K(1 + \|\mathrm{Hess}(V\mu)\|_\infty)^{1/2} e^{\kappa\|V\mu\|_\infty} \\ &\times (1 + \|\mathrm{Hess}(V\nu)\|_\infty)^{1/2} e^{\kappa\|V\nu\|_\infty} \\ &\times \|\nabla V\mu - \nabla V\nu\|_\infty \times \|f\|_\infty\end{aligned}$$

Proof Set $u = \mathsf{Q}_\nu f$. Then, by definition of Q_ν,

$$f = -A_\nu u + \Pi(\nu) f.$$

Since

$$A_\mu u - A_\nu u = -\langle \nabla V\mu - \nabla V\nu, \nabla u\rangle$$

we have

$$\begin{aligned}\mathsf{Q}_\mu f &= \mathsf{Q}_\mu(-A_\nu u + \Pi(\nu) f) \\ &= -\mathsf{Q}_\mu A_\nu u \\ &= -\mathsf{Q}_\mu(A_\mu u + \langle \nabla V\mu - \nabla V\nu, \nabla u\rangle) \\ &= u - \Pi(\mu) u - \mathsf{Q}_\mu(\langle \nabla V\mu - \nabla V\nu, \nabla u\rangle).\end{aligned}$$

So that we get

$$\mathsf{Q}_\mu f - \mathsf{Q}_\nu f = -\Pi(\mu)\mathsf{Q}_\nu f - \mathsf{Q}_\mu(\langle \nabla V\mu - \nabla V\nu, \nabla \mathsf{Q}_\nu f\rangle)$$

and

$$\nabla \mathsf{Q}_\mu f - \nabla \mathsf{Q}_\nu f = -\nabla \mathsf{Q}_\mu(\langle \nabla V\mu - \nabla V\nu, \nabla \mathsf{Q}_\nu f\rangle)$$

The lemma then follows from Lemma 2.3. □

2.2 Proof of the theorem

For $f \in C(M)$, $t > 0$ and $x \in M$, we set $F_t(x) = \frac{1}{t}\mathsf{Q}_{\beta(t)\mu_t} f(x)$. Then we have

$$\begin{aligned}\epsilon_t(s) f &= \int_{e^t}^{e^{t+s}} (f(X_u) - \Pi(\beta(u)\mu_u) f)\frac{du}{u}\\ &= \int_{e^t}^{e^{t+s}} K_{\beta(u)\mu_u} f(X_u)\frac{du}{u}\\ &= -\int_{e^t}^{e^{t+s}} A_{\beta(u)\mu_u} F_u(X_u) du.\end{aligned}$$

For $t > 0$, we set $A_t = A_{\beta(t)\mu_t}$, $\mathsf{Q}_t = \mathsf{Q}_{\beta(t)\mu_t}$ and $K_t = K_{\beta(t)\mu_t}$. Itô's formula then implies that

$$\epsilon_t(s) f = \epsilon_t^1(s) f + \epsilon_t^2(s) f + \epsilon_t^3(s) f$$

with

$$\begin{aligned}\epsilon_t^1(s) f &= F_{e^t}(X_{e^t}) - F_{e^{t+s}}(X_{e^{t+s}})\\ \epsilon_t^2(s) f &= \int_{e^t}^{e^{t+s}} \partial_u F_u(X_u) du\\ \epsilon_t^3(s) f &= \sum_i \int_{e^t}^{e^{t+s}} e_i(F_u)(X_u) dB_u^i\end{aligned}$$

In order to upper-estimate ϵ^1, ϵ^2 and ϵ^3, in addition to the estimates for $\|\mathsf{Q}_t f\|_\infty$ and for $\|\nabla \mathsf{Q}_t f\|_\infty$, we also need an upper-estimate for $\|\frac{d}{dt}\mathsf{Q}_t f\|_\infty$.

2.2.1 An upper-estimate for $\|\frac{d}{dt}\mathsf{Q}_t f\|_\infty$

Given two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ we let $\mathcal{L}(\mathcal{X},\mathcal{Y})$ denote the Banach space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$, equipped with the operator norm. For $f \in \mathcal{X}$ we let $\mathcal{L}_f(\mathcal{X},\mathcal{Y})$ denote the closed subset of $\mathcal{L}(\mathcal{X},\mathcal{Y})$ consisting of operators A such that $Af = 0$. Set $E = \mathcal{L}_1(\mathcal{D}^2, L^2)$, $F = \mathcal{L}_1(L^2, \mathcal{D}^2)$ and $G = \mathcal{L}_1(\mathcal{D}^2, \mathcal{D}^2)$, where $\mathcal{D}^2$ is the L^2-domain of Δ (i.e. $\mathcal{D}^2$ is the completion in L^2 of the C^∞ functions for the norm $\|f\|_2 + \|\Delta f\|_{L^2}$). For $\mu \in \mathcal{M}(M)$, the $L^2(\mu)$-domain of A_μ coincides with $\mathcal{D}^2$. Then, for a measure μ in $\mathcal{M}(M)$, equivalent to λ, $A_\mu \in E$, $\mathsf{Q}_\mu \in F$ and $K_\mu \in G$.

Lemma 2.6 *Let g_t be the function defined by*

$$g_t(x) = \frac{\beta(t)}{t} V(x, X_t) + t\frac{d}{dt}\left(\frac{\beta(t)}{t}\right) V\mu_t(x).$$

Then

(i) $t \mapsto A_t$ *is* C_1 *and for all* $f \in C^2(M)$,

$$\frac{d}{dt} A_t f = -\langle \nabla g_t, \nabla f\rangle.$$

(ii) $t \mapsto K_t$ *is* C_1 *and, for all* $f \in \mathcal{D}^2$,

$$\left(\frac{d}{dt} K_t\right) f = c(t, f)\mathbf{1}$$

with $\mathbf{1}(x) = 1$ *and*

$$c(t, f) = \Pi_t(f g_t) - (\Pi_t f)(\Pi_t g_t).$$

Proof Since

$$\frac{d}{dt}(\beta(t)\mu_t) = \frac{\beta(t)}{t}\delta_{X_t} + t\frac{d}{dt}\left(\frac{\beta(t)}{t}\right)\mu_t,$$

we have

$$g_t = \frac{d}{dt} V(\beta(t)\mu_t).$$

Let $f \in C^2(M)$. Then

$$\begin{aligned}\frac{d}{dt} A_t f &= -\frac{d}{dt}\langle \nabla V(\beta(t)\mu_t), \nabla f\rangle\\ &= -\langle \nabla g_t, \nabla f\rangle.\end{aligned}$$

This proves (i).

Since, for μ and ν in $\mathcal{M}(M)$,

$$\frac{d\Pi(\mu)}{d\mu}\cdot\nu = -\left(V\nu(x) - \Pi(\mu)(V\nu)\right)\Pi(\mu)(dx),$$

we get that

$$\begin{aligned}\frac{d}{dt}K_t = -\frac{d}{dt}\Pi_t &= -\frac{d\Pi}{d\mu}(\beta(t)\mu_t)\cdot\left(\frac{d}{dt}(\beta(t)\mu_t)\right)\\ &= (g_t(x) - \Pi_t g_t)\Pi_t(dx).\end{aligned}$$

This proves (ii). □

Lemma 2.7 *$t \mapsto \mathsf{Q}_t$ is a C^1 map from $\mathbb{R}^+$ into F, with vector derivative*

$$\frac{d}{dt}\mathsf{Q}_t = \left(\frac{d}{dt}K_t + \mathsf{Q}_t\frac{d}{dt}A_t\right)\mathsf{Q}_t. \tag{21}$$

Proof Let $L : \mathbb{R}^+ \times F \to G$, $(t, \mathsf{Q}) \mapsto \mathsf{Q}A_t + K_t$. The map L is C^1 and (t, Q_t) satisfies the implicit equation $L(t, \mathsf{Q}_t) = 0$. Set $L_1 = \frac{\partial L}{\partial Q}(t, \mathsf{Q}_t)$. Then $L_1 \in \mathcal{L}(F, G)$ is the operator defined by $L_1(B) = BA_t$.

Let $L_2 \in \mathcal{L}(G, F)$ be defined by $L_2(C) = -C\mathsf{Q}_t$. Since for all $C \in F$ and $B \in G$, $C1 = B1 = 0$, the fact that $A_t\mathsf{Q}_t f = -K_t f$ shows that L_2 is the inverse of L_1. Therefore by application of the implicit function theorem in Banach spaces, the map $t \mapsto \mathsf{Q}_t$ is C^1 and its derivative is given by

$$\begin{aligned}\frac{d}{dt}\mathsf{Q}_t &= -\left(\frac{\partial L}{\partial \mathsf{Q}}\right)^{-1}(t, \mathsf{Q}_t)\left(\frac{d}{dt}K_t + \mathsf{Q}_t\frac{d}{dt}A_t\right)\\ &= \left(\frac{d}{dt}K_t + \mathsf{Q}_t\frac{d}{dt}A_t\right)\mathsf{Q}_t.\end{aligned}$$

□

Lemma 2.8 *There exists a constant K such that for every $t \geq t_0$ and $f \in \mathcal{D}^2$*

$$\left\|\frac{d}{dt}\mathsf{Q}_t f\right\|_\infty \leq K\frac{(\log(t))^{3/2}}{t^{\gamma-2a\kappa}} \times \|f\|_\infty.$$

Proof This is a straightforward consequence of the two previous lemmas and the estimate for $\|\mathsf{Q}_t f\|_\infty$:

$$\left\|\frac{d}{dt}\mathsf{Q}_t f\right\|_\infty \leq \left\|\left(\frac{d}{dt}K_t\right)\mathsf{Q}_t f\right\|_\infty + \left\|\mathsf{Q}_t\left(\frac{d}{dt}A_t\right)\mathsf{Q}_t f\right\|_\infty.$$

On one hand,

$$\begin{aligned}\left\|\left(\frac{d}{dt}K_t\right)\mathsf{Q}_t f\right\|_\infty &\le c(t,\mathsf{Q}_t f)\\ &\le 2\|g_t\|_\infty \times \|\mathsf{Q}_t f\|_\infty\\ &\le K\times\left(t\left|\frac{d}{dt}\left(\frac{\beta(t)}{t}\right)\right| + \frac{|\beta(t)|}{t}\right)\times e^{\kappa|\beta(t)|}\times\|f\|_\infty.\end{aligned}$$

On the other hand,

$$\begin{aligned}\left\|\mathsf{Q}_t\left(\frac{d}{dt}A_t\right)\mathsf{Q}_t f\right\|_\infty &\le Ke^{\kappa|\beta(t)|}\left\|\left(\frac{d}{dt}A_t\right)\mathsf{Q}_t f\right\|_\infty\\ &\le Ke^{\kappa|\beta(t)|}\times\|\nabla g_t\|_\infty\times\|\nabla\mathsf{Q}_t f\|_\infty\\ &\le K\times\left(t\left|\frac{d}{dt}\left(\frac{\beta(t)}{t}\right)\right|+\frac{|\beta(t)|}{t}\right)\times e^{\kappa|\beta(t)|}\\ &\quad\times(1+|\beta(t)|)^{1/2}\times e^{\kappa|\beta(t)|}\times\|f\|_\infty.\end{aligned}$$

We conclude using the fact that β satisfies Hypothesis 1.1. □

2.2.2 Estimates of ϵ^1, ϵ^2 and ϵ^3

Take $a_c = \gamma(2\kappa)^{-1}$ and $\alpha \in]0, \gamma - 2a_c\kappa[$.

Lemma 2.9 *For all $a \in [0, a_c[$, there exists a constant K such that for all positive t,*

(i) $\sup_{s\ge 0}|\epsilon_t^1(s)f + \epsilon_t^2(s)f| \le Ke^{-\alpha t}\|f\|_\infty$.

(ii) $\mathsf{E}[\sup_{s\ge 0}(\epsilon_t^3(s)f)^2|\mathcal{F}_{e^t}] \le Ke^{-\alpha t}\|f\|_\infty^2$.

Proof One has for all s and t,

$$\begin{aligned}|\epsilon_t^1(s)f| &\le K(e^{-t}e^{\kappa|\beta(e^t)|} + e^{-(t+s)}e^{\kappa|\beta(e^{t+s})|})\|f\|_\infty\\ &\le Ke^{-(1-a\kappa)t}\|f\|_\infty.\end{aligned}$$

Since

$$\begin{aligned}|\epsilon_t^2(s)f| &\le \int_{e^t}^{e^{t+s}}\left(\left\|\frac{1}{u^2}\mathsf{Q}_u f\right\|_\infty + \left\|\frac{1}{u}\frac{d}{du}\mathsf{Q}_u f\right\|_\infty\right)du\\ &\le K\|f\|_\infty\int_{e^t}^{e^{t+s}}\left(u^{-(2-a\kappa)} + (\log(u))^{3/2}u^{-(1+\gamma-2a\kappa)}\right)du\\ &\le Ke^{-\alpha t}\|f\|_\infty.\end{aligned}$$

This proves (i). Since $\epsilon_t^3(\cdot)f$ is a martingale whose quadratic variation satisfies

$$\begin{aligned}\frac{d}{ds}\langle \epsilon_t^3(\cdot)f\rangle_s &= e^{t+s}\|\nabla F_{e^{t+s}}(X_{e^{t+s}})\|^2\\ &\leq e^{t+s}\times e^{-2(t+s)}\|\nabla \mathsf{Q}_{e^{t+s}}f\|_\infty^2\\ &\leq Ke^{-(1-2a\kappa)(t+s)}\|f\|_\infty^2.\end{aligned}$$

Applying Doob's inequality, one gets (ii). □

This lemma implies that for all $f\in C(M)$,

$$\mathsf{E}\left[\sup_{s\geq 0}(\epsilon_t(s)f)^2|\mathcal{F}_{e^t}\right]\leq K\exp(-\alpha t)\|f\|_\infty^2.$$

To finish the proof of the theorem, we prove that this inequality implies that almost surely,

$$\limsup_{t\to\infty}\frac{1}{t}\log\left(\sup_{s\geq 0}|\epsilon_t(s)f|\right)\leq -\frac{\alpha}{2}.$$

Let $\delta<\alpha$. Then,

$$\mathsf{P}\left(\sup_{s\geq 0}|\epsilon_n(s)f|>e^{-\frac{\delta}{2}n}\right)\leq Ke^{-(\alpha-\delta)n}\|f\|_\infty^2.$$

This implies that a.s.,

$$\limsup_{n\to\infty}\frac{1}{n}\log\left(\sup_{s\geq 0}|\epsilon_n(s)f|\right)\leq -\frac{\delta}{2}. \tag{22}$$

Now, remark that for all $t\in[n,n+1[$,

$$\sup_{s\geq 0}|\epsilon_t(s)f|\leq 2\sup_{s\geq 0}|\epsilon_n(s)f|.$$

Using this estimate and (22), we prove that for all $\delta<\alpha$, a.s.

$$\limsup_{n\to\infty}\frac{1}{n}\log\left(\sup_{s\geq 0}|\epsilon_t(s)f|\right)\leq -\frac{\delta}{2}. \tag{23}$$

This implies the last claim of the theorem. □

3 Study of one example

Let $M = \mathbb{S}^n$ be the unit sphere in $\mathbb{R}^{n+1}$, with $n \geq 1$. The geodesic distance on $\mathbb{S}^n$ is denoted by d. To define the process X, we take for the interacting potential the function V defined by

$$V(x, y) = -\cos(d(x, y)) = -\langle x, y\rangle, \tag{24}$$

where $\langle \cdot, \cdot\rangle$ is the euclidean scalar product in $\mathbb{R}^{n+1}$. We choose a function β satisfying Hypothesis 1.1, with $\gamma = 1$ and a such that $a < a_c$, with a_c like in theorem 2.1. In addition to this assumption, we will assume that

$$\lim_{t\to\infty} \beta(t) = \infty.$$

In this section, we prove the following theorem

Theorem 3.1 *There exists a random variable X_∞ in $\mathbb{S}^n$ such that almost surely, $\mu_t = \frac{1}{t}\int_0^t \delta_{X_s}\, ds$ converges weakly towards δ_{X_∞}.*

In the proof of this theorem given here, for simplicity, we assume that β is nondecreasing.

Note that we do not have precise information on the distribution of X_∞. It depends on the initial condition. The only case one can say something is in the case where X_0 is uniformly distributed. Then the law of X_∞ is rotationally invariant, which implies that X_∞ is uniformly distributed.

3.1 Definition of Z_t and of z_t

For a finite measure μ on $\mathbb{S}^n$, we define

$$\begin{aligned}
m(\mu) &= \int_{\mathbb{S}^n} x\mu(dx),\\
r(\mu) &= \|m(\mu)\|,\\
u(\mu) &= m(\mu)/r(\mu) \quad \text{if} \quad r(\mu) > 0 \quad \text{and}\\
u(\mu) &= 0 \quad \text{if} \quad r(\mu) = 0.
\end{aligned}$$

Then,

$$V\mu(x) = -r(\mu)\cos(d(x, u(\mu))) = -r(\mu)\langle x, u(\mu)\rangle. \tag{25}$$

Set ν_t to be the measure defined by

$$\nu_t f = \mu_t f + \frac{1}{t}\mathsf{Q}_{\beta(t)\mu_t} f(X_t)$$

for all $f \in C(M)$. Thus, as $t \to \infty$, $|\nu_t - \mu_t| \to 0$. Set $z_t = m(\mu_{e^t})$, $r_t = r(\mu_{e^t})$ and $u_t = u(\mu_{e^t})$. Set $Z_t = m(\nu_{e^t})$, $R_t = r(\nu_{e^t})$ and $U_t = u(\nu_{e^t})$. Then, as $t \to \infty$, $\|Z_t - z_t\| \to 0$.

Let us remark that for all $\mu \in \mathcal{M}(M)$,

$$m(\Pi(\mu)) = \Lambda(r(\mu))u(\mu), \tag{26}$$

with $\Lambda = Z'/Z$, the function Z being defined by

$$Z(\rho) = \int_{\mathbb{S}^n} e^{\rho\cos(d(x,e))}\lambda(dx), \tag{27}$$

where $e \in \mathbb{S}^n$ (the definition of Z of course does not depend on the choice of e). We also have that

$$Z(\rho) = K_n \int_0^{\pi/2} \cosh(\rho\cos\theta)(\sin\theta)^{n-1}d\theta \tag{28}$$

for some constant K_n.

3.2 Study of Λ

Let us briefly study the function Λ: It is an increasing concave function, $\Lambda(0) = 0$, $\Lambda'(0) > 0$ and as $\rho \to \infty$,

$$\Lambda(\rho) = 1 - K/\rho + o(1/\rho), \tag{29}$$

where K is a positive constant. The claim (29) holds since [using the change of variable $v = \rho(1-\cos\theta)$]

$$\begin{aligned}
1 - \Lambda(\rho) &= \frac{\int_0^{\pi/2} e^{-\rho}[\cosh(\rho\cos\theta) - \cos\theta\sinh(\rho\cos\theta)](\sin\theta)^{n-1}d\theta}{\int_0^{\pi/2} e^{-\rho}\cosh(\rho\cos\theta)(\sin\theta)^{n-1}d\theta} \\
&= \frac{\int_0^{\rho}[e^{-v}+e^{-2\rho}e^{v}-(1-v/\rho)(e^{-v}-e^{-2\rho}e^{v})](v/\rho)^{\frac{n-2}{2}}(2-v/\rho)^{\frac{n-2}{2}}dv}{\int_0^{\rho}(e^{-v}+e^{-2\rho}e^{v})(v/\rho)^{\frac{n-2}{2}}(2-v/\rho)^{\frac{n-2}{2}}dv} \\
&= \frac{\int_0^{\rho}((v/\rho)e^{-v}+(2-v/\rho)e^{-2\rho}e^{v})(v/\rho)^{\frac{n-2}{2}}(2-v/\rho)^{\frac{n-2}{2}}dv}{\int_0^{\rho}(e^{-v}+e^{-2\rho}e^{v})(v/\rho)^{\frac{n-2}{2}}(2-v/\rho)^{\frac{n-2}{2}}dv} \\
&\sim \frac{1}{\rho}\frac{\int_0^{\infty}(ve^{-v})v^{\frac{n-2}{2}}dv}{\int_0^{\infty}(e^{-v})v^{\frac{n-2}{2}}dv}
\end{aligned}$$

using Lebesgue dominated convergence theorem.

3.3 An SDE satisfied by Z_t

For all positive s and t,

$$z_{t+s} - z_t = \int_0^s \left(-r_{t+u} + \Lambda(\beta(e^{t+u})r_{t+u})\right) u_{t+u} du + \int_{\mathbb{S}^n} x\, \epsilon_t(s)(dx).$$

This implies that

$$\begin{aligned} Z_{t+s} - Z_t &= z_{t+s} - z_t - \int x\, \epsilon_t^1(s)(dx) \\ &= \int_0^s \left(-r_{t+u} + \Lambda(\beta(e^{t+u})r_{t+u})\right) u_{t+u} du + \int_{\mathbb{S}^n} x\, (\epsilon_t^2(s) + \epsilon_t^3(s))(dx). \end{aligned}$$

For all positive t, set

$$\begin{aligned} H_t &= (\partial_u F_u)_{|u=e^t}(X_{e^t}) \\ &\quad + \left(-r_t + \Lambda(\beta(e^t)r_t)\right) u_t - \left(-R_t + \Lambda(\beta(e^t))R_t)\right) U_t, \end{aligned}$$

where $F_t(x) = \frac{1}{t}\mathsf{Q}_{\beta(t)\mu_t} f(x)$, with $f : \mathbb{S}^n \to \mathbb{R}^{n+1}$ defined by $f(x) = x$.

Lemma 3.2 *There exists a positive constant K such that for all $t \geq \log(t_0)$,*

$$|H_t| \leq K e^{-(1+\alpha)t/2}.$$

Proof We have

$$\begin{aligned} |\partial_u F_u|_{|u=e^t}(X_{e^t}) &\leq e^{-2t}\|\mathsf{Q}_{e^t} f\|_\infty + e^{-t}\| \left(\frac{d}{du}\mathsf{Q}_u\right)_{|u=e^t} f\|_\infty \\ &\leq K(e^{-2t}e^{a\kappa t} + t^{3/2}e^{-t}e^{-(\gamma-2a\kappa)t}) \\ &\leq K e^{-(1+\alpha)t}. \end{aligned}$$

We also have that

$$\begin{aligned} r_t u_t - R_t U_t &= m(\mu_{e^t}) - m(\nu_{e^t}) \\ &= e^{-t}\mathsf{Q}_{e^t} f(X_{e^t}) \end{aligned}$$

with $f : \mathbb{S}^n \to \mathbb{R}^{n+1}$ defined by $f(x) = x$. This implies that

$$|r_t u_t - R_t U_t| \leq K e^{-t} e^{a\kappa t} \leq K e^{-\frac{(1+\alpha)}{2}t}.$$

Finally, using the fact that $\Pi : \mathcal{M}(M) \to \mathcal{M}(M)$ is Lipschitzian (see Lemma 3.1 in [2]),

$$\begin{aligned}\left\| \Lambda(\beta(e^t)R_t)U_t - \Lambda(\beta(e^t)r_t)u_t \right\| &= \left\| m(\Pi(\beta(e^t)\mu_{e^t})) - m(\Pi(\beta(e^t)\nu_{e^t})) \right\| \\ &\leq (n+1) \left\| \Pi(\beta(e^t)\mu_{e^t}) - \Pi(\beta(e^t)\nu_{e^t}) \right\| \\ &\leq K\beta(e^t)|\mu_{e^t} - \nu_{e^t}| \\ &\leq Kte^{-t}e^{a\kappa t} \\ &\leq Ke^{-\frac{(1+\alpha)}{2}t}.\end{aligned}$$

The lemma follows from these estimates. □

We thus have

$$dZ_t = \left(-R_t + \Lambda(\beta(e^t)R_t)\right) U_t dt + H_t dt + dM_t, \tag{30}$$

where M is the martingale defined by $M_0 = 0$ and for all positive s and t,

$$M_{t+s} - M_t = \int x\epsilon_t^3(s)(dx).$$

For $1 \leq j \leq n+1$,

$$M_{t+s}^j - M_t^j = \sum_i \int_{e^t}^{e^{t+s}} e_i(F_u^j)(X_u)dB_u^i$$

with

$$F_t^j(x) = \frac{1}{t}\mathsf{Q}_{\beta(e^t)\mu_t} f^j(x)$$

where $f^j(x) = x^j$. The martingale brackets of M are given by

$$\langle M^i, M^j\rangle_t = \int_0^t e^{-s}\langle \nabla Q_{\beta(e^s)\mu_{e^s}} f^i, \nabla Q_{\beta(e^s)\mu_{e^s}} f^j\rangle(X_{e^s})ds,$$

with $f^i(x) = x_i$, $1 \leq i \leq n+1$.

3.4 The bilinear form Φ

For all $x \in M$ and $\mu \in \mathcal{M}(M)$, we define the positive symmetric matrix $\Phi_\mu(x)$ by

$$\Phi_\mu^{i,j}(x) = \langle \nabla Q_\mu f^i, \nabla Q_\mu f^j\rangle(x), \quad 1 \leq i, j \leq n+1. \tag{31}$$

The associated bilinear form is defined by

$$\Phi_\mu(u,v)(x) = \sum_{i,j} \Phi_\mu^{i,j}(x) u_i v_j,$$

with u and v in $\mathbb{R}^{n+1}$. For all positive t, the positive symmetric matrix $\Phi_{\beta(e^t)\mu_{e^t}}(X_{e^t})$ will be denoted by Φ_t.

For $\mu \in \mathcal{M}(M)$, we set $\Gamma_\mu(x) = \mathrm{Tr}(\Phi_\mu(x))$ and $\Gamma_t = \mathrm{Tr}(\Phi_t)$, respectively the trace of $\Phi_\mu(x)$ and of Φ_t.

Equation (30) implies that

$$\begin{aligned} dR_t^2 &= 2R_t\left(-R_t + \Lambda(\beta(e^t)R_t)\right)dt \\ &\quad +2R_t\langle U_t, H_t\rangle dt + e^{-t}\Gamma_t dt \\ &\quad +2R_t\langle U_t, dM_t\rangle. \end{aligned} \tag{32}$$

The martingale bracket of the martingale $N_t = \int_0^t \langle U_s, dM_s\rangle$ is given by

$$\langle N\rangle_t = \int_0^t e^{-s}\Phi_s(U_s, U_s)ds. \tag{33}$$

Since $\Phi_s(U_s, U_s) \le K \sup_j \|\nabla \mathsf{Q}_s f^j\|_\infty^2$, we have the following estimate

$$\Phi_s(U_s, U_s) \le K(1 + \mathrm{sr}_s)e^{2a\kappa \mathrm{sr}_s}. \tag{34}$$

With this estimate and using the fact that $\alpha < 1 - 2a\kappa$, one can prove

Lemma 3.3 *There exists a positive constant K such that for all $t \ge \log(t_0)$,*

$$\frac{d}{dt}\langle N\rangle_t \le Ke^{-\alpha t}.$$

Proof This is straightforward since $r_t \le 1$. □

The following lemma will be used in the following section.

Lemma 3.4 *There exists a constant K such that for all $\mu \in \mathcal{M}(M)$,*

$$\sup_{i,j} \|\Phi_\mu^{i,j} - \Phi_\lambda^{i,j}\|_\infty \le K(1 + r(\mu))r(\mu)e^{2\kappa r(\mu)}$$

Proof We have

$$\begin{aligned} \Phi_\mu^{i,j}(x) - \Phi_\lambda^{i,j}(x) &= \langle \nabla \mathsf{Q}_\mu f^i, \nabla \mathsf{Q}_\mu f^j\rangle(x) - \langle \nabla \mathsf{Q}_\lambda f^i, \nabla \mathsf{Q}_\lambda f^j\rangle(x) \\ &= \langle \nabla \mathsf{Q}_\mu f^i - \nabla \mathsf{Q}_\lambda f^i, \nabla \mathsf{Q}_\mu f^j\rangle(x) \\ &\quad + \langle \nabla \mathsf{Q}_\lambda f^i, \nabla \mathsf{Q}_\mu f^j - \nabla \mathsf{Q}_\lambda f^j\rangle(x) \end{aligned}$$

Thus

$$\|\Phi_\mu^{i,j} - \Phi_\lambda^{i,j}\|_\infty \leq \|\nabla \mathsf{Q}_\mu f^i - \nabla \mathsf{Q}_\lambda f^i\|_\infty \times \|\nabla \mathsf{Q}_\mu f^j\|_\infty \\ + \|\nabla \mathsf{Q}_\mu f^j - \nabla \mathsf{Q}_\lambda f^j\|_\infty \times \|\nabla \mathsf{Q}_\lambda f^i\|_\infty.$$

The lemma then follows from Lemmas 2.3 and 2.5. □

3.5 Study of the process R

In this section, we prove the following

Theorem 3.5 *The process R_t converges towards* 1, *a.s.*

This implies in particular that a.s., all limiting values μ of $\{\mu_t\}$ satisfy $r(\mu) = 1$. By strict convexity of the sphere, this implies that all limiting value of $\{\mu_t\}$ is a Dirac measure.

3.5.1 Non convergence towards 0

We first prove the following lemma (note that to prove this lemma, we don't use the fact that $\lim_{t\to\infty} \beta(t) = \infty$, the only additional assumption needed is that $\liminf_{t\to\infty} \beta(t) > 1/\Lambda'(0)$)

Lemma 3.6 *Almost surely*, $\liminf R_t > 0$.

In order to prove this lemma, we will need to have a lower estimate for Γ_t when $\beta(e^t)\mu_{e^t}$ is close to λ. This will be obtained using the fact that $\Gamma_\lambda(x) = nc(n)^2$, with $c(n) = \frac{1}{n-1}$: indeed, for all $i \in \{1, \dots, n+1\}$, $\mathsf{Q}_\lambda f^i = -\frac{1}{n-1} f^i$, thus

$$\begin{aligned}\Phi_\lambda^{i,j}(x) &= \langle \nabla \mathsf{Q}_\lambda f^i, \nabla \mathsf{Q}_\lambda f^j \rangle(x) \\ &= c(n)^2 \langle \nabla f^i, \nabla f^j \rangle(x) \\ &= c(n)^2 (\delta_{i,j} - x_i x_j)\end{aligned}$$

which gives $\Gamma_\lambda(x) = \sum_i \Phi_\lambda^{i,i}(x) = nc(n)^2$.

For all positive t, set

$$S_t = \inf\{s > t; \quad R_s > 1/s\}.$$

We will prove that for large t, S_t is finite with a great probability, which roughly means that if R_s goes to 0, it cannot go as fast as $1/s$. In order to prove this, we will first prove that

$$S_t^0 = \inf\{s > t; \quad R_s > (\beta(e^s))^{-1/4} e^{-s/2}\}$$

is finite with a great probability.

Lemma 3.7 *For all $\beta \le a$, $\Lambda(\beta) \ge \beta\Lambda'(a)$.*

Proof For $\beta \le a$ and $r \in [0,1]$, set $\varphi_\beta(r) = \Lambda(\beta r)$. The function φ_β satisfies $\varphi_\beta(0) = 0$ and $\varphi'_\beta(r) = \beta\Lambda'(\beta r)$. Since Λ is concave, for $r \le 1$ and $\beta \le a$, $\varphi'_\beta(r) \ge \beta\Lambda'(a)$. This implies the lemma. □

We use this lemma to prove

Lemma 3.8 *There exist positive constants K and T such that*

$$\mathsf{P}(S_t^0 = \infty | \mathcal{F}_{e^t}) \le K\beta(e^t)^{-1/2}. \tag{35}$$

Proof For $s \in]t, S_t^0[$, when $R_t < (\beta(e^t))^{-1/4}e^{-t/2}$ we have that

$$\begin{aligned} -R_s + \Lambda(\beta(e^s)R_s) &\ge R_s(-1 + \beta(e^s)\Lambda'(\beta(e^s)R_s) \\ &\ge R_s(-1 + \beta(e^s)\Lambda'(\beta(e^s)s^{-1/4}e^{-s/2})). \end{aligned}$$

Since

$$\begin{aligned} \liminf_{s\to\infty} \beta(e^s)\Lambda'(\beta(e^s)^{1-1/4}e^{-s/2}) &= \liminf_{s\to\infty} \beta(e^s)\Lambda'(0) \\ &= \infty, \end{aligned}$$

there exists T such that for all $t \ge T$ and $s \in]t, S_t^0[$,

$$-R_s + \Lambda(\beta(e^s)R_s) \ge 0.$$

We now fix $t > T$. For $s \in [t, S_t^0]$, when $R_t < (\beta(e^t))^{-1/4}e^{-t/2}$ we have [using (32)] that

$$\begin{aligned} R_s^2 \ge R_t^2 &+ \int_t^s e^{-u}\Gamma_\lambda(X_{e^u})du \\ &+ 2\int_t^s R_u\langle U_u, H_u\rangle du + \int_t^s e^{-u}(\Gamma_u - \Gamma_\lambda(X_{e^u}))du \\ &+ 2\int_t^s R_u dN_u \end{aligned}$$

For $u \in]t, S_t^0[$, we have that

$$|R_u\langle U_u, H_u\rangle| \le K(\beta(e^u))^{-1/4}e^{-(1+\alpha/2)u}$$

and that

$$\begin{aligned} r_u &\le R_u + \|m(\mu_{e^u} - \nu_{e^u})\| \\ &\le R_u + Ke^{-(1-a\kappa)u}. \end{aligned}$$

This implies that

$$\begin{aligned} ur_u &\le uR_u + Ke^{-\left(\frac{1+\alpha}{2}\right)u} \\ &\le Ku(\beta(e^u))^{-1/4}e^{-u/2} \end{aligned}$$

Lemma 3.4 implies that

$$\begin{aligned} e^{-u}|\Gamma_u - \Gamma_\lambda(X_{e^u})| &\le K(1+ur_u)ur_u e^{2a\kappa ur_u}e^{-u} \\ &\le Ku(\beta(e^u))^{-1/4}e^{-3u/2} \\ &\le K(\beta(e^u))^{-1/4}e^{-(1+\alpha/2)u}. \end{aligned}$$

Thus (since $\Gamma_\lambda(x) = nc(n)^2$ for all x)

$$\begin{aligned} \mathsf{E}[R^2_{S^0_t}|\mathcal{F}_{e^t}] &\ge (nc(n)^2 - K(\beta(e^t))^{-1/4}e^{-\frac{\alpha}{2}t})e^{-t}\mathsf{P}(S^0_t = \infty|\mathcal{F}_{e^t}) \\ &\ge \frac{nc(n)^2}{2}e^{-t}\mathsf{P}(S^0_t = \infty|\mathcal{F}_{e^t}) \end{aligned}$$

for sufficiently large t. Since

$$\mathsf{E}[R^2_{S^0_t}|\mathcal{F}_{e^t}] \le (\beta(e^t))^{-1/2}e^{-t}\mathsf{P}(S^0_t < \infty|\mathcal{F}_{e^t}),$$

we get

$$\mathsf{P}(S^0_t = \infty|\mathcal{F}_{e^t}) \le \frac{1}{1 + \frac{nc(n)^2}{2}(\beta(e^t))^{1/2}}.$$

This proves the lemma. □

We then prove that

Lemma 3.9

$$\mathsf{P}(S_t = \infty|\mathcal{F}_{e^t}) \le K(\beta(e^t))^{-1/2}. \tag{36}$$

Proof Let $S'_t = \inf\{s \ge S^0_t;\ R_s \in \{e^{-s}, 1/s\}\}$. For all $s \in [S^0_t, S'_t]$, by Itô's formula we have

$$\begin{aligned} R_s - R_{S^0_t} = &\int_{S^0_t}^{s} (-R_u + \Lambda(\beta(e^u)R_u))du + \int_{S^0_t}^{s} \langle U_u, H_u\rangle du \\ &+ \int_{S^0_t}^{s} \frac{e^{-u}}{2R_u}(\Gamma_u - \Phi_u(U_u, U_u))du \\ &+ N_s - N_{S^0_t}. \end{aligned} \tag{37}$$

There exists $c > 0$ such that for t sufficiently large and $u \in [S_t^0, S_t']$, the following properties hold (using the fact that $\beta(e^u)R_u \le a$):

- $-R_u + \Lambda(\beta(e^u)R_u) \ge (-1 + \beta(e^u)\Lambda'(a))R_u \ge c\beta(e^u)R_u$;
- $\langle U_u, H_u\rangle \ge -Ke^{-(1+\alpha)u/2}$.

We now lower estimate $\frac{e^{-u}}{2R_u}(\Gamma_u - \Phi_u(U_u, U_u))$. For all $u \in \mathbb{S}^n$,

$$\begin{aligned}\Gamma_\lambda(x) - \Phi_\lambda(x)(u, u) &= c(n)^2(n - 1 + \langle u, x\rangle^2)\\ &\ge 0.\end{aligned}$$

We also have (for $u \in [S_t^0, S_t']$)

$$\begin{aligned}\Gamma_u - \Phi_u(U_u, U_u) \ge\ & \Gamma_\lambda(X_{e^u}) - \Phi_\lambda(X_{e^u})(U_u, U_u)\\ & -|\Gamma_u - \Gamma_\lambda(X_{e^u})|\\ & -|\Phi_u(U_u, U_u) - \Phi_\lambda(X_{e^u})(U_u, U_u)|\\ \ge\ & -K(1 + ur_u)ur_u e^{2a\kappa ur_u}\end{aligned}\tag{38}$$

which is greater that $-Kur_u$, for some positive constant K. Since $r_u \le R_u + Ke^{-u}e^{a\kappa ur_u}$, $ur_u \le 2$ for t large enough and $u \in]S_t^0, S_t'[$. Thus, $r_u \le R_u + Ke^{-u}$ and

$$\begin{aligned}\frac{e^{-u}}{2R_u}(\Gamma_u - \Phi_u(U_u, U_u)) &\ge -Kue^{-u}(1 + e^{-u}/R_u)\\ &\ge -Kue^{-u}\\ &\ge -Ke^{-(1+\alpha)u/2}.\end{aligned}$$

With the above estimates, relation (37) implies that for t large enough

$$\begin{aligned}R_s - R_{S_t^0} \ge\ & c\int_{S_t^0}^{s} \beta(e^u)R_u du\\ & - \int_{S_t^0}^{s} Ke^{-(1+\alpha)u/2}du\\ & + N_s - N_{S_t^0}.\end{aligned}$$

This implies that for $s \in [S_t^0, S_t']$ and t large enough,

$$\exp\left(-\int_{S_t^0}^{s} c\beta(e^u)du\right) R_s \ge R_{S_t^0}$$

$$
\begin{aligned}
&+\int_{S_t^0}^{s} \exp\left(-\int_{S_t^0}^{u} c\beta(e^v)dv\right)(dN_u - Ke^{-(1+\alpha)u/2}du)\\
&\geq K(\beta(e^{S_t^0}))^{-1/4}e^{-S_t^0/2}\\
&+\int_{S_t^0}^{s} \exp\left(-\int_{S_t^0}^{u} c\beta(e^v)dv\right)dN_u.
\end{aligned}
$$

On the event $\{R_{S_t'} = e^{-S_t'}\} \cup \{S_t' = \infty\}$,

$$
\lim_{s\uparrow S_t'} \exp\left(-\int_{S_t^0}^{s} c\beta(e^u)du\right)R_s = \exp\left(-\int_{S_t^0}^{S_t'} c\beta(e^u)du\right)e^{-S_t'}
$$

which is dominated by $e^{-S_t^0}$. Thus there exists a positive constant K such that for t large enough, on the event $\{R_{S_t'} = e^{-S_t'}\} \cup \{S_t' = \infty\} = \{R_{S_t'} \in \{0, e^{-S_t'}\}\}$ there exists $s > S_t^0$ such that

$$
\int_{S_t^0}^{s} \exp\left(-\int_{S_t^0}^{u} c\beta(e^v)dv\right)dN_u \leq -K(\beta(e^{S_t^0}))^{-1/4}e^{-S_t^0/2}.
$$

This implies that on the event $\{S_t^0 < \infty\}$, (to simplify $\int_{S_t^0}^{u} c\beta(e^v)dv$ will be denoted by $c(u)$)

$$
\begin{aligned}
\mathsf{P}(R_{S_t'} \in \{0, e^{-S_t'}\}|\mathcal{F}_{e^{S_t^0}}) &\leq \mathsf{P}\left(\sup_{s\in[S_t^0, S_t']}\left|\int_{S_t^0}^{s} e^{-c(u)}dN_u\right| > K(\beta(e^{S_t^0}))^{-1/4}e^{-S_t^0/2}\middle|\mathcal{F}_{e^{S_t^0}}\right)\\
&\leq K(\beta(e^{S_t^0}))^{1/2}e^{S_t^0}\mathsf{E}\left(\int_{S_t^0}^{S_t'} e^{-2c(u)}e^{-u}e^{2a\kappa u R_u}du\middle|\mathcal{F}_{e^{S_t^0}}\right)\\
&\leq K(\beta(e^{S_t^0}))^{1/2}e^{S_t^0}\int_{S_t^0}^{\infty} e^{-2c(u)}e^{-u}du.
\end{aligned}
$$

Since $\int_s^\infty \exp\left(-2c\int_s^u \beta(e^v)dv\right)e^{-u}du = O(e^{-s}/\beta(e^s))$, on the event $\{S_t^0 < \infty\}$,

$$
\mathsf{P}(R_{S_t'} \in \{0, e^{-S_t'}\}|\mathcal{F}_{e^{S_t^0}}) \leq K(\beta(e^{S_t^0}))^{-1/2}.
$$

Since $\mathsf{P}(S_t = \infty | \mathcal{F}_{e^{S_t^0}}) \le \mathsf{P}(R_{S_t'} \in \{0, e^{-S_t'}\} | \mathcal{F}_{e^{S_t^0}})$,

$$\begin{aligned} \mathsf{P}(S_t = \infty | \mathcal{F}_{e^t}) &= \mathsf{E}[\mathsf{P}(S_t = \infty | \mathcal{F}_{e^{S_t^0}}) 1_{\{S_t^0 < \infty\}} | \mathcal{F}_{e^t}] \\ &\quad + \ \mathsf{P}(S_t^0 = \infty | \mathcal{F}_{e^t}) \\ &\le K(\beta(e^t))^{-1/2} \end{aligned}$$

This proves the lemma. □

Proof of Lemma 3.6 Let $H = \{\liminf R_t > 0\}$. Fix $\beta_0 > 1/\Lambda'(0)$ and let T be such that for all $t \ge T$, $\beta(e^t) \ge \beta_0$. For $t \ge 1$, set

$$T_t = \inf\{s \ge S_t; \quad R_s = e^{-(1-a\kappa)s}\}.$$

On the event $\{S_t < \infty\}$, for all $s \in [S_t, T_t]$,

$$\begin{aligned} R_s - R_{S_t} = &\int_{S_t}^{s} (-R_u + \Lambda(\beta(e^u) R_u)) du + \int_{S_t}^{s} \langle U_u, H_u \rangle du \\ &+ \int_{S_t}^{s} \frac{e^{-u}}{2R_u} (\Gamma_u - \Phi_u(U_u, U_u)) du \\ &+ N_s - N_{S_t}. \end{aligned} \tag{39}$$

Fix r_1 and r_2 such that $0 < r_1 < r_2 < 1$ and such that for all $r < r_2$ and $\beta \ge \beta_0$, $-r + \Lambda(\beta r) > 0$.[1] Let φ be a C^2 increasing concave function such that $\varphi(r) = r$ if $r < r_1$ and $\varphi'(r) = 0$ if $r > r_2$. Then it is clear that $\liminf_{t\to\infty} R_t \ge \liminf_{t\to\infty} \varphi(R_t)$. On the event $\{S_t < \infty\}$, for all $s \in [S_t, T_t]$, by Itô's formula,

$$\varphi(R_s) - \varphi(R_{S_t}) = \int_{S_t}^{s} \varphi'(R_u)(-R_u + \Lambda(\beta(e^u) R_u)) du \tag{40}$$

$$+ \int_{S_t}^{s} \varphi'(R_u) \langle U_u, H_u \rangle du \tag{41}$$

$$+ \int_{S_t}^{s} \varphi'(R_u) \frac{e^{-u}}{2R_u} (\Gamma_u - \Phi_u(U_u, U_u)) du \tag{42}$$

$$+ \frac{1}{2} \int_{S_t}^{s} \varphi''(R_u) d\langle N \rangle_u \tag{43}$$

[1] Such r_2 exists since $-r + \Lambda(\beta r) \ge -r + \Lambda(\beta_0 r)$ and $\partial_r(-r + \Lambda(\beta_0 r))_{|r=0} = -1 + \beta_0 \Lambda'(0) > 0$.

$$+\int_{S_t}^{s} \varphi'(R_u)dN_u \tag{44}$$

It is clear that (40) is nonnegative, because of our choice of r_2. The term (41) is greater than

$$-K\int_{S_t}^{\infty} e^{-\frac{(1+\alpha)}{2}u}du = -Ke^{-\frac{(1+\alpha)}{2}S_t}.$$

Using Lemma 3.4, we get that

$$\begin{aligned}\Gamma_u - \Phi_u(U_u, U_u) &\geq -K(1+ur_u)ur_u e^{2a\kappa ur_u}\\ &\geq -Ku^2 r_u e^{2a\kappa u}.\end{aligned}$$

There exists a constant K such that for t large enough and $u \in [S_t, T_t]$, $r_u \leq KR_u$. This implies that (since $R_u \geq e^{-u}e^{a\kappa u}$)

$$\begin{aligned}\frac{e^{-u}}{2R_u}(\Gamma_u - \Phi_u(U_u, U_u)) &\geq -Ku^2 e^{-(1-2a\kappa)u}\\ &\geq -Ke^{-\alpha u}.\end{aligned}$$

Thus (42) is greater than $-Ke^{-\alpha S_t}$. The term (43) is greater than

$$-K\int_{S_t}^{\infty} e^{-\alpha u}du = -Ke^{-\alpha S_t}$$

where K is a constant also depending on φ.

This implies that for all $s \in [S_t, T_t]$,

$$\varphi(R_s) \geq 1/S_t - Ke^{-\alpha S_t} + \inf_{s\in[S_t,T_t]} \int_{S_t}^{s} \varphi'(R_u)dN_u.$$

Thus

$$\inf_{s\in[S_t,T_t]} \varphi(R_s) \geq \frac{1}{2S_t} + \inf_{s\in[S_t,T_t]} \int_{S_t}^{s} \varphi'(R_u)dN_u.$$

for t large enough. Since (using Doob–Meyer inequality)

$$\mathsf{P}\left(\inf_{s \in [S_t, T_t]} \int_{S_t}^{s} \varphi'(R_u) dN_u \le -\frac{1}{4S_t} \middle| \mathcal{F}_{e^{S_t}} \right) \le K S_t^2 \int_{S_t}^{\infty} (\varphi'(R_u))^2 d\langle N \rangle_u$$
$$\le K e^{-\alpha S_t}$$

we obtain that

$$\mathsf{P}\left(\inf_{s \in [S_t, T_t]} \varphi(R_s) \ge \frac{1}{4S_t} \middle| \mathcal{F}_{e^{S_t}} \right) \ge 1 - K e^{-\alpha S_t}.$$

Since

$$\left\{ \inf_{s \in [S_t, T_t]} \varphi(R_s) \ge \frac{1}{4S_t} \right\} \cap \{S_t < \infty\} \subset H,$$

we have for t large enough that

$$\begin{aligned}
\mathsf{P}(H) &\ge \mathsf{P}(H \cap \{S_t < \infty\}) \\
&\ge \mathsf{E}\left[\mathsf{P}\left(\left\{ \inf_{s \in [S_t, T_t]} \varphi(R_s) \ge \frac{1}{4S_t} \right\} \middle| \mathcal{F}_{e^{S_t}} \right) 1_{\{S_t < \infty\}} \right] \\
&\ge \mathsf{E}[(1 - K e^{-\alpha S_t}) 1_{\{S_t < \infty\}}] \\
&\ge (1 - K e^{-\alpha t}) \mathsf{P}(S_t < \infty) \\
&\ge (1 - K e^{-\alpha t})(1 - K(\beta(e^t))^{-1/2}),
\end{aligned}$$

which converges towards 1 as $t \to \infty$. This proves that $\mathsf{P}(H) = 1$. □

3.5.2 Convergence towards 1

Fix $\epsilon > 0$. Since $\mathsf{P}(\liminf_{s\to\infty} R_s > 0) = 1$, there exist $r > 0$ and $t > 0$ such that $\mathsf{P}(\Omega_{r,t}) \ge 1 - \epsilon$, where $\Omega_{r,t} = \{\inf_{s>t} R_s > r\}$. On $\Omega_{r,t}$, $\Lambda(\beta(e^s) R_s) \ge \Lambda(\beta(e^s) r)$. We fix t such that for all $s > t$,

$$\Lambda(\beta(e^s) r) \ge 1 - k/\beta(e^s)$$

for some constant $k > 0$. For all $s \le \inf\{u \ge t;\ R_u \le r\}$,

$$(1 - R_s) - (1 - R_t) = \int_t^s K_u du - (N_s - N_t),$$

with K_u being dominated by $-(1 - R_u) + \epsilon(u)$, with

$$\epsilon(u) = k/\beta(e^u) + K e^{-(1+\alpha)u}.$$

This implies that for $s \le \inf\{u \ge t;\ R_u \le r\}$

$$e^s(1-R_s) - e^t(1-R_t) \le \int_t^s e^u(\epsilon(u)du - dN_u).$$

Thus, for $s \le \inf\{u \ge t;\ R_u \le r\}$

$$(1-R_s) \le e^{-(s-t)}(1-R_t) + e^{-s}\int_t^s \epsilon(u)e^u du - e^{-s}\int_t^s e^u dN_u. \tag{45}$$

We first remark that since $\lim_{u\to\infty}\epsilon(u) = 0$, $e^{-s}\int_t^s \epsilon(u)e^u du$ converges towards 0 as $s \to \infty$.

Lemma 3.10 *For all $t > 0$, the process $e^{-s}\int_t^s e^u dN_u$ converges a.s. towards 0 as $s \to \infty$.*

Proof We fix $t > 0$. Set $Y_s = e^{-s}\int_t^s e^u dN_u$. Then

$$\mathsf{E}[Y_s^2] \le e^{-2s}\int_t^s e^{2u}e^{-u}e^{2\alpha\kappa u}du \le \frac{e^{-\alpha s}}{2-\alpha}.$$

Let $s_0 \ge t$, and $s > s_0$, then

$$Y_s = e^{-(s-s_0)}Y_{s_0} + e^{-s}\int_{s_0}^s e^u dN_u.$$

This implies that for all $\epsilon > 0$

$$\begin{aligned}
\mathsf{P}\left(\sup_{s\in[s_0,s_0+1]}|Y_s - Y_{s_0}| \ge \epsilon\right) &\le \mathsf{P}(|Y_{s_0}| \ge \epsilon/2) \\
&\quad + \mathsf{P}\left(e^{-s_0}\sup_{s\in[s_0,s_0+1]}\left|\int_{s_0}^s e^u dN_u\right| > \epsilon/2\right) \\
&\le \frac{K}{\epsilon^2}\left(e^{-\alpha s_0} + e^{-2s_0}\int_{s_0}^{s_0+1} e^{2u}e^{-u}e^{2\alpha\kappa u}du\right) \\
&\le \frac{K}{\epsilon^2}\left(e^{-\alpha s_0} + e^{-2s_0}e^{(1+2\alpha\kappa)(s_0+1)}\right) \\
&\le \frac{K}{\epsilon^2}e^{-\alpha s_0}.
\end{aligned}$$

Borel-Cantelli lemma implies that for all $\epsilon > 0$, a.s., there exists N such that for all $n \geq N$, $\sup_{s \in [n,n+1]} |Y_s - Y_n| < \epsilon$ and $|Y_n| < \epsilon$. This implies that for all positive ϵ, a.s. $\limsup_{s\to\infty} |Y_s| \leq 2\epsilon$. This implies the lemma. □

Proof of Theorem 3.5 Using this lemma, Eq. (45) and the fact that on $\Omega_{r,t}$, $\inf\{u \geq t;\ R_u \leq r\} = \infty$, we prove that a.s. on $\Omega_{r,t}$, R_s converges to 1. Let $\Omega_\infty = \cup_{r>0} \cup_{t>0} \Omega_{r,t}$, $\mathsf{P}(\Omega_\infty) = 1$. We then have that a.s. on Ω_∞, R_s converges to 1. This proves Theorem 3.5. □

3.6 Study of the process U_t

We prove here the following

Lemma 3.11 *The process $U_t = Z_t/R_t$ converges a.s.*

Proof We recall that

$$dZ_t = (-R_t + \Lambda(\beta(e^t)R_t))U_t dt + H_t dt + dM_t$$

and that

$$\begin{aligned} dR_t &= (-R_t + \Lambda(\beta(e^t)R_t))dt + \langle U_t, H_t\rangle dt \\ &\quad + \frac{e^{-t}}{2R_t}(\Gamma_t - \Phi_t(U_t, U_t))dt + \langle U_t, dM_t\rangle. \end{aligned}$$

By Itô's formula,

$$\begin{aligned} dU_t &= \frac{dZ_t}{R_t} - U_t\frac{dR_t}{R_t} - \frac{1}{R_t^2}d\langle Z, R\rangle_t + U_t\frac{d\langle R\rangle_t}{R_t^2} \\ &= \frac{1}{R_t}(dM_t - \langle dM_t, U_t\rangle U_t) \\ &\quad + \frac{1}{R_t}(H_t - \langle H_t, U_t\rangle U_t)dt \\ &\quad + A_t\, dt \end{aligned}$$

with

$$A_t = -\frac{e^{-t}}{2R_t^2}(\Gamma_t - 3\Phi_t(U_t, U_t))U_t - \frac{e^{-t}}{R_t^2}\Phi_t U_t.$$

Note that Γ_t, $\Phi_t(U_t, U_t)$ and $\|\Phi_t U_t\|$ are dominated by

$$\begin{aligned} K \sup_i \|\nabla Q_{\beta(e^t)\mu_{e^t}} f^i\|_\infty^2 &\leq K(1+t)e^{2\alpha\kappa t} \\ &\leq K e^{(1-\alpha)t}. \end{aligned}$$

This implies that

$$\|A_t\| \leq \frac{K}{R_t^2} e^{-\alpha t}. \tag{46}$$

We also have that

$$\begin{aligned}\left\| \frac{1}{R_t}(H_t - \langle H_t, U_t\rangle U_t) \right\| &\leq \frac{K}{R_t} e^{-\left(\frac{1+\alpha}{2}\right)t} \\ &\leq \frac{K}{R_t} e^{-\alpha t}. \end{aligned} \tag{47}$$

Since R_t converges a.s. towards 1,

$$\sup_{s \geq 0} \left| \int_t^s A_u du + \int_t^s \frac{1}{R_u}(H_u - \langle H_u, U_u\rangle U_u) du \right|$$

which is dominated by

$$K \int_t^\infty e^{-\alpha u} \left(\frac{1}{R_u^2} + \frac{1}{R_u} \right) du,$$

converges towards 0 as $t \to \infty$.

Lemma 3.12 *Set* $\tilde{M}_t = M_t - \int_0^t \langle dM_u, U_u\rangle U_u$. *Almost surely,*

$$\lim_{t \to \infty} \sup_{s \geq t} \left\| \int_t^s \frac{d\tilde{M}_u}{R_u} \right\| = 0.$$

Proof Fix $r \in]0, 1[$ and $T > 0$. Set $\Omega_{r,T} = \{\inf_{s>T} R_s > r\}$. Then

$$\lim_{r \downarrow 0} \lim_{T \uparrow \infty} \mathsf{P}(\Omega_{r,T}) = 1.$$

For $t \geq T$, set $\tau_t = \inf\{s \geq t;\ R_s \leq r\}$ and

$$V_t = \sup_{s \geq t} \left\| \int_t^s \frac{d\tilde{M}_u}{R_u} \right\|.$$

Then, on $\Omega_{r,T}$, for all $t \geq T$, $\tau_t = \infty$ and

$$V_t = \sup_{s \in [t, \tau_t]} \left\| \int_t^s \frac{d\tilde{M}_u}{R_u} \right\|.$$

We then have that

$$\begin{aligned}
\mathsf{E}[V_t 1_{\Omega_{r,T}}] &\leq \mathsf{E}\left[\sup_{s\in[t,\tau_t]}\left\|\int_t^s \frac{d\tilde{M}_u}{R_u}\right\|^2\right]^{1/2} \mathsf{P}(\Omega_{r,T})^{1/2}\\
&\leq 2\mathsf{E}\left[\int_t^{\tau_t} \frac{\sum_i d\langle\tilde{M}^i\rangle_u}{R_u^2}\right]^{1/2} \mathsf{P}(\Omega_{r,T})^{1/2}\\
&\leq 2\mathsf{E}\left[\int_t^{\tau_t} \frac{e^{-u}}{R_u^2}(\Gamma_u - \Phi_u(U_u, U_u))du\right]^{1/2} \mathsf{P}(\Omega_{r,T})^{1/2}\\
&\leq K\mathsf{E}\left[\int_t^{\tau_t} e^{-\alpha u}du\right]^{1/2} \mathsf{P}(\Omega_{r,T})^{1/2}\\
&\leq Ke^{-\frac{\alpha}{2}t}\mathsf{P}(\Omega_{r,T})^{1/2}.
\end{aligned}$$

This implies that for all positive ϵ, by Markov inequality,

$$\mathsf{P}(V_t \geq \epsilon|\Omega_{r,T}) \leq \frac{Ke^{-\frac{\alpha}{2}t}}{\epsilon\mathsf{P}(\Omega_{r,T})^{1/2}}.$$

We also have that

$$\begin{aligned}
\mathsf{E}\left[\sup_{s\in[t,t+1[}|V_s - V_t|1_{\Omega_{r,T}}\right] &\leq \mathsf{E}\left[\sup_{s\in[t,t+1[}\left|\int_t^{s\wedge\tau_t} \frac{d\tilde{M}_u}{R_u}\right|1_{\Omega_{r,T}}\right]\\
&\leq \mathsf{E}\left[\sup_{s\in[t,t+1[}\left|\int_t^{s\wedge\tau_t} \frac{d\tilde{M}_u}{R_u}\right|^2\right]^{1/2} \mathsf{P}(\Omega_{r,T})^{1/2}\\
&\leq Ke^{-\frac{\alpha}{2}t}\mathsf{P}(\Omega_{r,T})^{1/2}.
\end{aligned}$$

This implies that for all positive ϵ,

$$\mathsf{P}\left(\sup_{s\in[t,t+1[}|V_s - V_t| \geq \epsilon\middle|\Omega_{r,T}\right) \leq \frac{Ke^{-\frac{\alpha}{2}t}}{\epsilon\mathsf{P}(\Omega_{r,T})^{1/2}}.$$

Using Borel-Cantelli lemma, we prove that for all positive ϵ, a.s. on $\Omega_{r,T}$, there exists N such that for all $n \geq N$, $V_n \leq \epsilon$ and $\sup_{s\in[n,n+1[}|V_s - V_n| \leq \epsilon$. This implies that for all positive ϵ, a.s. on $\Omega_{r,T}$, $\limsup_{t\to\infty} V_t \leq \epsilon$. This implies that a.s. on $\Omega_{r,T}$, $\lim_{t\to\infty} V_t = 0$. We conclude since $\lim_{r\downarrow 0}\lim_{T\uparrow\infty}\mathsf{P}(\Omega_{r,T}) = 1$. □

Estimates (46) and (47), with lemma 3.12 implies that, a.s.,

$$\lim_{t\to\infty} \sup_{s\geq t} \|U_s - U_t\| = 0.$$

This proves the lemma. □

Since the a.s. convergence of z_t towards a random variable z_∞ such that $\|z_\infty\| = 1$ implies that μ_t converges a.s. towards the Dirac measure δ_{z_∞}. Theorem 3.5 is proved.

References

1. Bakry, D.: L'hypercontractivité et son utilisation en théorie des semigroupes. Ecole d'Eté de Probabilités de St-Flour. Lecture Notes in Mathematics **1581**, 1–114, Springer, Heidelberg (1994)
2. Benaïm, M., Ledoux, M., Raimond, O.: Self-interacting diffusions. Probab. Theory Relat. Fields 122(1), 1–41 (2002)
3. Benaïm, M., Raimond, O.: Self-interacting diffusions II: convergence in law. Ann. Inst. H. Poincaré Probab. Statist. **39**(6), 1043–1055 (2003)
4. Benaïm, M., Raimond, O.: Self-interacting diffusions III: the symmetric case. Ann. Probab. **33**(5), 1717-1759 (2005)
5. Cranston, M., Le Jan, Y.: Self-attracting diffusions: two case studies. Math. Ann. **303**, 87–93 (1995)
6. Herrmann, S., Roynette, B.: Boundedness and convergence of some self-attracting diffusions. Math. Ann. **325**(1), 81–96 (2003)
7. Holley, R.A., Kusuoka, S., Stroock, D.W.: Asymptotics of the spectral gap with applications to the theory of simulated annealing. J. Funct. Anal. **83**(2), 333–347 (1989)
8. Ledoux, M.: The geometry of Markov diffusion generators. Ann. Fac. Sci. Toulouse (to appear)
9. Miclo, L.: Recuit simulé sans potentiel sur une variété riemannienne compacte. Stochastics Stochastics Rep. **41**(1–2), 23–56 (1992)
10. Raimond, O.: Self attracting diffusions: case of the constant interaction. Probab. Theor. Relat. Fields **107**, 177–196 (1996)