Theorem 3.1

In a group G there is only one identity element.

Proof. Assume that $e, e' \in G$ are two identity elements:

$$ae = ea = a = ae' = e'a$$

for all $a \in G$. Then

$$e = ee' = e'$$

$\square$

Theorem 3.2 (Cancellation Property)

If G is a group $a, b, c \in G$ and either $ac = bc$ or $ca = cb$ then $a = b$.

Proof. Assume that $ac = bc$. Then we have

$$a = acc^{-1} = (ac)c^{-1} = (bc)c^{-1} = bcc^{-1} = b$$

$\square$

Theorem 3.3

If G is a group then every element of G has only one inverse element.

Proof. Let $a \in G$ and let $b, b' \in G$ be two inverses of a:

$$ab = e = ab'$$

Then, by the cancellation property $b = b'$. $\square$

Theorem 3.4

If G is a group and $a, b \in G$ then $(ab)^{-1} = b^{-1}a^{-1}$.

Proof.

$$(ab)(b^{-1}a^{-1}) = abb^{-1}a^{-1} = e$$

Similarly, $(b^{-1}a^{-1})(ab) = e$. □

Exercise 3.5. Write multiplication tables of all possible groups with 1, 2 and 3 elements.

Exercise 3.6. Write multiplication tables of all possible groups with 4 elements.