

Example. Maximize

$$z = 3x_1 + x_2$$

subject to:

$$-x_1 + x_2 \leq 1$$

$$x_1 \leq 3$$

$$2x_1 + x_2 \leq 7$$

$$x_1, x_2 \geq 0$$

Convert to the equality form:

$$\text{Constraints: } -x_1 + x_2 + s_1 = 1$$

$$x_1 + s_2 = 3$$

$$2x_1 + x_2 + s_3 = 7$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

augmented
matrix

$$\begin{array}{ccccc|c} \text{free variables} & & \text{basic variables} & & & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ x_1 & x_2 & s_1 & s_2 & s_3 & \\ \hline -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 3 \\ 2 & 1 & 0 & 0 & 1 & 7 \end{array}$$

We want to maximize

$$z = 3x_1 + x_2 + \underbrace{0s_1 + 0s_2 + 0s_3}_{\substack{\uparrow \text{coefficients of} \\ \text{are all 0.}}}$$

basic variables

Note: By setting the free variables to 0 we get a basic feasible solution:

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ s_1 = 1 \\ s_2 = 3 \\ s_3 = 7 \end{cases}$$

this gives : $z = 0$

Goal: Look for other basic feasible solutions that make the value of z larger.

Simplex tableau

	x_1	x_2	s_1	s_2	s_3		
constraints	-1	1	1	0	0	1	s_1
	1	0	0	1	0	3	s_2
	2	1	0	0	1	7	s_3
the objective function	3	1	0	0	0	z	

The current basic feasible solution:

$$\begin{array}{l}
 \text{free} \\
 \text{basic}
 \end{array}
 \left\{ \begin{array}{l}
 x_1 = 0 \\
 x_2 = 0 \\
 s_1 = 1 \\
 s_2 = 3 \\
 s_3 = 7
 \end{array} \right. \rightarrow z = 0$$

The pivot step:

- 1) We can increase the value of z by increasing either x_1 or x_2 . Let's try to increase x_2 , keep $x_1 = 0$.
- 2) Change of x_2 will affect the values of the basic variables s_1, s_2, s_3 . These variables must stay ≥ 0 , which restricts possible values of x_2 .

Since by assumption $x_1 = 0$ we have:

$$\begin{aligned}
 x_2 + s_1 &= 1 \Rightarrow s_1 = 1 - x_2 \geq 0 \quad \underline{\text{so:}} \quad x_2 \leq 1 \\
 0 \cdot x_2 + s_2 &= 3 \Rightarrow s_2 = 3 \geq 0 \quad - \text{no restriction on } x_2 \\
 x_2 + s_3 &= 7 \Rightarrow s_3 = 7 - x_2 \geq 0 \quad \underline{\text{so:}} \quad x_2 \leq 7
 \end{aligned}$$

All these conditions are satisfied if $x_2 \leq 1$.

For the biggest increase of z we set $x_2 = 1$ which makes $s_1 = 0$. Since we want free variables to have value 0, and basic variables to have non-zero values, we want to make x_2 into a basic variable and make s_1 free.

The diagram illustrates the pivot operation in the simplex method. It shows two tableaux separated by a red arrow indicating the transformation.

Initial Tableau (Left):

	x_1	x_2	s_1	s_2	s_3	RHS
-1	1	1	0	0	1	
1	0	0	1	0	3	
2	1	0	0	1	7	
3	1	0	0	0	z	

Red annotations on the initial tableau: A red arrow labeled (-1) points from the x_1 column of the first row to the x_1 column of the second row. Another red arrow labeled (-1) points from the x_1 column of the third row to the x_1 column of the second row.

Resulting Tableau (Right):

	x_1	x_2	s_1	s_2	s_3	RHS
-1	1	1	0	0	1	x_2
1	0	0	1	0	3	s_2
3	0	-1	0	1	6	s_3
4	0	-1	0	0	$z-1$	

Red annotations on the resulting tableau: A red bracket labeled "free" is placed above the x_1 and s_1 columns. A red arrow points from the x_1 column of the first row to the x_1 column of the second row. Another red arrow points from the s_1 column of the first row to the s_1 column of the second row.

The new basic feasible solution

free $\left[\begin{cases} x_1 = 0 \\ s_1 = 0 \end{cases} \right.$ $\rightarrow z - 1 = 0$
basic $\left[\begin{cases} x_2 = 1 \\ s_2 = 3 \\ s_3 = 6 \end{cases} \right.$ (or $z = 1$)

The pivot step:

The free variables are x_1 and s_1 .

Increasing s_1 will decrease z ,

but increasing x_1 will increase z .

Thus we will increase x , while keeping $x_2, s_2, s_3 \geq 0$.

(Note: we keep $s_1 = 0$)

$$-x_1 + x_2 = 1 \Rightarrow x_2 = 1 + x_1 \geq 0 \quad \text{- no restriction on } x_1$$

$$x_1 + s_2 = 3 \Rightarrow s_2 = 3 - x_1 \geq 0 \Rightarrow x_1 \leq 3$$

$$3x_1 + s_3 = 6 \Rightarrow s_3 = 6 - 3x_1 \geq 0 \quad \underline{\text{so}} \quad \begin{array}{l} 3x_1 \leq 6 \\ x_1 \leq 2 \end{array}$$

The biggest value of x_i satisfying all conditions is $x_i = 2$

Then $s_3 = 0$, so s_3 becomes free and x_1 basic.

x_1	x_2	s_1	s_2	s_3	
-1	1	1	0	0	1
1	0	0	1	0	3
3	0	-1	0	1	6
4	0	-1	0	0	$z - 1$

x_1	x_2	s_1	s_2	s_3	
0	1	$\frac{2}{3}$	0	$\frac{1}{3}$	3
0	0	$\frac{1}{3}$	1	$-\frac{1}{3}$	1
1	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	2
0	0	$\frac{1}{3}$	0	$-\frac{4}{3}$	$z - 9$

Geometric interpretation of the simplex method

Recall: Maximize

$$z = 3x_1 + x_2$$

subject to:

$$-x_1 + x_2 \leq 1$$

$$x_1 \leq 3$$

$$2x_1 + x_2 \leq 7$$

$$x_1, x_2 \geq 0$$

