Recall: The general form of a linear program

For the objective variables $x_1, \ldots, x_n$ find the minimum (or the maximum) of the objective function

$$z = c_1 x_1 + \ldots + c_n x_n$$

subject to the constraints:

$$a_{i1} x_1 + \ldots + a_{in} x_n \begin{array}{c} \leq \\ = \\ \geq \end{array} b_i$$

for $i = 1, \ldots, m$, and possibly $x_j \geq 0$ for $j = 1, \ldots, n$.

The *equality* (or *standard*) form of a linear program:

- we are looking for the maximum;
- all constraints are of the form

$$a_{i1} x_1 + \ldots + a_{in} x_n = b_i$$

- we require that $x_j \geq 0$ for $j = 1, \ldots, n$.

Fact

Every linear program can be converted to the equality form.

- finding minimum of $z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

  $\parallel$

  finding maximum of $z = -(c_1x_1 + c_2x_2 + \dots + c_nx_n)$
  $= -c_1x_1 - c_2x_2 - \dots - c_nx_n$

- if we have a constraint of the form

  $$a_{i1}x_1 + \dots + a_{in}x_n \leqslant b_i$$

  then we can replace it by

  $$a_{i1}x_1 + \dots + a_{in}x_n + s_i = b_i$$

  where $s_i \geqslant 0$ is a new <u>slack variable</u>.

- a constraint of the form

  $$a_{i1}x_1 + \dots + a_{in}x_n \geqslant b_i$$

  is equivalent to

  $$-a_{i1}x_1 - \dots - a_{in}x_n \leqslant -b_i$$

  so we can use a slack variable again:

  $$-a_{i1}x_1 - \dots - a_{in}x_n + s_i = -b_i$$

  where $s_i \geqslant 0$.

- if x_j is an urestricted variable (i.e. x_j can be any real number) then we can replace it by $x_j = x_j^+ - x_j^-$ where x_j^+, x_j^- are new variables such that $x_j^+, x_j^- \geqslant 0$.

Example. Convert the following linear program to the equality form.

Minimize the function

$$z = 6x_1 - 10x_2$$

subject to the constraints:

$$\begin{aligned} 5x_1 + 7x_2 &\leq 8 \\ 4x_1 + 2x_2 &\geq 10 \\ x_1 &\geq 0 \\ x_2 &\in \mathbb{R} \end{aligned}$$

Solution:

Maximize

$z = -6x_1 + 10x_2$

Constraints:

$$\begin{aligned} 5x_1 + 7x_2 + s_1 &= 8 \\ -4x_1 - 2x_2 + s_2 &= -10 \\ x_1 &\geqslant 0 \\ s_1 &\geqslant 0 \\ s_2 &\geqslant 0 \\ x_2 &\in \mathbb{R} \end{aligned}$$

$\xrightarrow{\quad}$ $x_2 = x_2^+ - x_2^-$

Maximize:

$z = -6x_1 + 10x_2^+ - 10x_2^- + 0s_1 + 0s_2$

$$\underbrace{\begin{aligned} 5x_1 + 7x_2^+ - 7x_2^- + s_1 &= 8 \\ -4x_1 - 2x_2^+ + 2x_2^- + s_2 &= -10 \\ x_1 &\geqslant 0 \\ x_2^+ &\geqslant 0 \\ x_2^- &\geqslant 0 \\ s_1 &\geqslant 0 \\ s_2 &\geqslant 0 \end{aligned}}_{\text{the equality form}}$$

The *inequality* form of a linear program:

- we are looking for the maximum;
- all constraints are of the form

$$a_{i1}x_1 + \ldots + a_{in}x_n \leq b_i$$

- we require that $x_j \geq 0$ for $j = 1, \ldots, n$.

Fact

Every linear program can be converted to the inequality form.

- if we have a constraint of the form

$$a_{i1}x_1 + \ldots + a_{in}x_n = b_i$$

then we can replace it by two constraints

$$a_{i1}x_1 + \ldots + a_{in}x_n \leq b_i$$
$$a_{i1}x_1 + \ldots + a_{in}x_n \geq b_i$$

Linear programs with Python:

`scipy.optimize.linprog` (looks for the minimum of the objective function).