Definition

A matrix A is *totally unimodular* if every square matrix obtained by removing some rows and columns of A has determinant 0, 1, or -1.

Proposition

Consider a linear program of the equality form: maximize

$$z = c_1x_1 + \ldots + c_nx_n$$

subject to constraints:

$$a_{i1}x_1 + \ldots + a_{in}x_n = b_i$$

for $i = 1, \ldots, m$ and $x_j \geq 0$ for $j = 1, \ldots, n$.

If the the coefficient matrix

$$A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix}$$

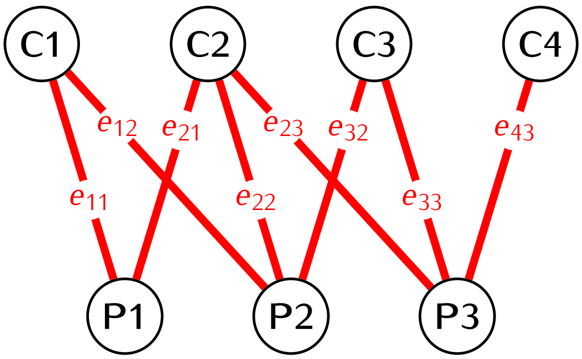
is totally unimodular and $b_i \in \mathbb{Z}$ for $i = 1, \ldots, m$ then values of $x_1, \ldots, x_n$ for every basic feasible solution of the linear program are integers. In particular, if the linear program has a maximum, then there is a solution with integer values which gives this maximum.

Proof of Proposition.

Proposition

For any bipartite graph $G = (V_1 \cup V_2, E)$ the incidence matrix of G is totally unimodular.

Example.



$$
\begin{array}{c}
\\
\mathrm{C1}\\ \mathrm{C2}\\ \mathrm{C3}\\ \mathrm{C4}\\ \mathrm{P1}\\ \mathrm{P1}\\ \mathrm{P3}
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} e_{11} & e_{12} & e_{21} & e_{22} & e_{23} & e_{32} & e_{33} & e_{43} \end{array}\\
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1
\end{bmatrix}
\end{array}
$$

Proposition

If A is a totally unimodular matrix and B is a matrix obtained by appending to A by a column that that has only one non-zero entry equal to 1, then B is totally unimodular.

Corollary

If the linear program for an assignment problem is feasible, then the simplex method always gives a solution that consists of integers.