

**Example 1.** A farmer plans to plant two types of crops on a 100 acre farm:  $C_1$  and  $C_2$ . It costs

- \$100 and 4 hour of labor to grow 1 acre of  $C_1$
- \$300 and 1 hour of labor to grow 1 acre of  $C_2$ .

Each acre of  $C_1$  will bring \$200 profit, and each acre of  $C_2$  will bring \$100 profit. The farmer can spend up to \$27,000 on the production costs and up to 280 hours of labor. How many acres of each crop should be planted to maximize the profit?

### Mathematical formulation (linear program)

decision variables [ Unknown:  $x_1$  = the number of acres of  $C_1$   
 $x_2$  = the number of acres of  $C_2$

the objective function [ We want to maximize the profit function:  

$$z = 200x_1 + 100x_2$$

constraints [ Restrictions:  

$$x_1 + x_2 \leq 100$$

$$100x_1 + 300x_2 \leq 27,000$$

$$4x_1 + 1x_2 \leq 280$$

nonnegativity constraints [  $x_1 \geq 0$   
 $x_2 \geq 0$

**Example 2.** A company manufacturing widgets has 2 factories and 3 warehouses. The cost of sending 1000 widgets from each factory to each warehouse is as follows:

	$W_1$	$W_2$	$W_3$
$F_1$	5	5	3
$F_2$	6	4	1

The factory  $F_1$  can produce up to 10,000 widgets per week and  $F_2$  up to 7,000 widgets per week. The warehouses must receive exactly 8,000 widgets per week for  $W_1$ , 5,000 widgets per week for  $W_2$ , and 2,000 widgets per week for  $W_3$ .

How many widgets should be shipped each week from each factory to each warehouse to minimize the shipping costs?

### Linear program

Decision variables :

$x_{ij}$  = the number of widgets (in thousands)  
shipped from  $F_i$  to  $W_j$  ( $i=1,2$  ,  $j=1,2,3$ )

The objective function to minimize :

$$z = 5x_{11} + 5x_{12} + 3x_{13} + 6x_{21} + 4x_{22} + 1 \cdot x_{23}$$

Constraints :

$$x_{11} + x_{12} + x_{13} \leq 10$$

$$x_{21} + x_{22} + x_{23} \leq 7$$

$$x_{11} + x_{21} = 8$$

$$x_{12} + x_{22} = 5$$

$$x_{13} + x_{23} = 2$$

$$x_{ij} \geq 0$$

## The general form of a linear program

For the decision variables  $x_1, \dots, x_n$  find the minimum (or the maximum) of the objective function

$$Z = c_1 x_1 + \dots + c_n x_n$$

Subject to the constraints:

$$a_{i1}x_1 + \dots + a_{in}x_n \begin{matrix} \leq \\ \geq \end{matrix} b_i$$

for  $i = 1, \dots, m$ , and possibly  $x_j \geq 0$  for  $j = 1, \dots, n$ .

Sidenote: The growth of linear functions

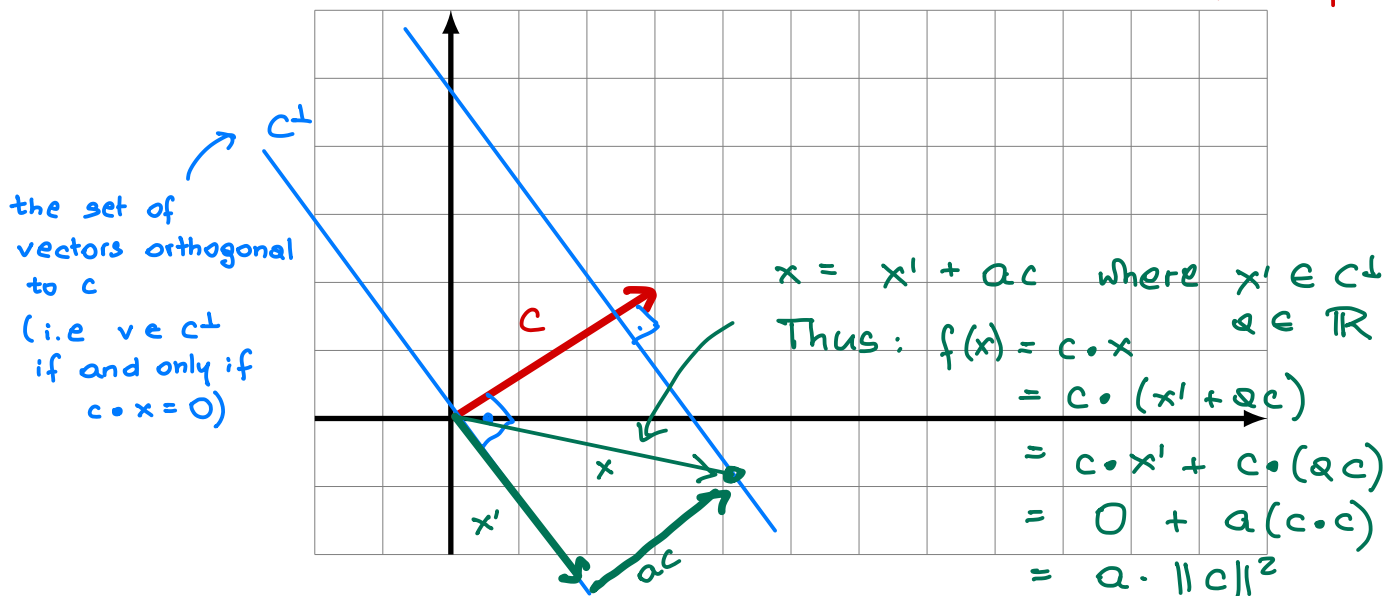
Example:  $f(x_1, x_2) = 3x_1 + 2x_2$

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}$$

Denote:  $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, c = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

We have:  $f(x) = c \cdot x = c^T x$

↓ the dot product  
↑ matrix product



Upshot: The values of the function  $f$  increase as we move in the direction of the vector  $c$ , and decrease as we move in the opposite direction.

## Back to Example 1

Maximize:

$$f(x_1, x_2) = 200x_1 + 100x_2$$

subject to the constraints:

$$x_1 + x_2 \leq 100$$

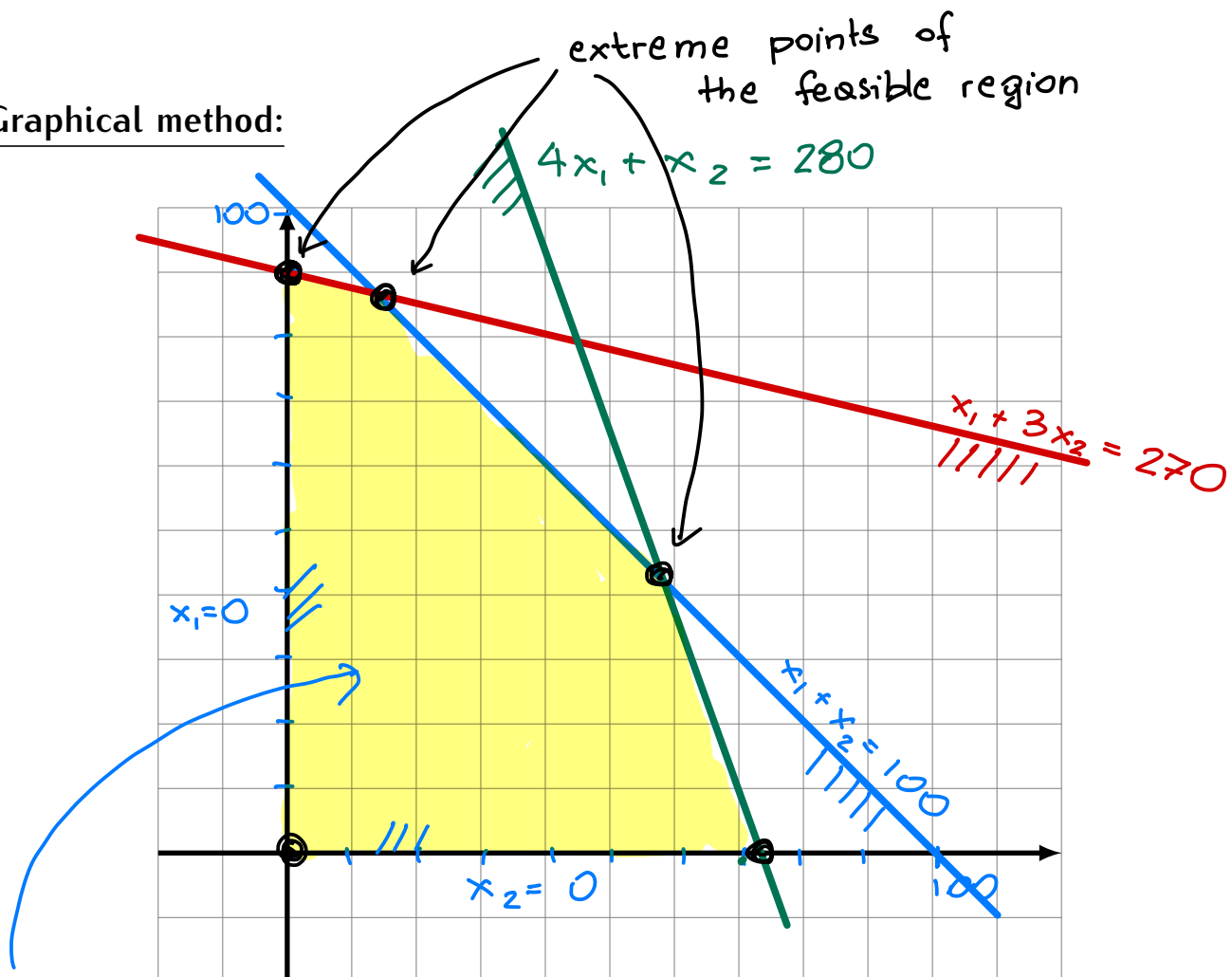
$$100x_1 + 300x_2 \leq 27000 \quad x_1 + 3x_2 \leq 270$$

$$4x_1 + x_2 \leq 280$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

Graphical method:



the feasible region: consists of points  $(x_1, x_2)$  that satisfy all constraints. Such points are called feasible solutions

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