Recall:

Definition

A square matrix A is a *diagonalizable* if A is of the form

$$A = PDP^{-1}$$

where D is a diagonal matrix and P is an invertible matrix.

Diagonalization Theorem

1) An $n \times n$ matrix A is a diagonalizable if and only if it has n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

2) In such case $A = PDP^{-1}$ where :

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \qquad \begin{array}{l} \lambda_1 = \text{eigenvalue corresponding to } \mathbf{v}_1 \\ \lambda_2 = \text{eigenvalue corresponding to } \mathbf{v}_2 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \lambda_n = \text{eigenvalue corresponding to } \mathbf{v}_n \end{array}$$

Example.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Note. Not every matrix is diagonalizable.

Example.

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

Recall:

Definition

If

$$\mathbf{u} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

are vectors in $\mathbb{R}^n$ then the *dot product* of $\mathbf{u}$ and $\mathbf{v}$ is the number

$$\mathbf{u} \cdot \mathbf{v} = a_1 b_1 + \ldots + a_n b_n$$

Definition

If $\mathbf{u} \in \mathbb{R}^n$ then the *length* (or the *norm*) of $\mathbf{u}$ is the number

$$||\mathbf{u}|| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$$

Definition

Vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are *orthogonal* if $\mathbf{u} \cdot \mathbf{v} = 0$.

Definition

A square matrix $Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{bmatrix}$ is an *orthogonal matrix* if $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is an orthonormal set of vectors, i.e.:

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Example.

$$A = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}$$

Theorem

If Q is an orthogonal matrix then Q is invertible and $Q^{-1} = Q^T$.

Definition

A square matrix A is *symmetric* if $A^T = A$.

Spectral Theorem

If A is an $n \times n$ symmetric matrix then A has n orthonormal eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$.

As a consequence the matrix A is diagonalizable:

$$A = QDQ^{-1} = QDQ^T$$

where

$$Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_n \end{bmatrix}$$

is an orthogonal matrix with the orthonormal eigenvectors as columns and

$$D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

is a diagonal matrix, where λ_i is an eigenvalue of A corresponding to the eigenvector $\mathbf{u}_i$.