**Example 1.** A farmer plans to plant two types of crops on a 100 acre farm: $C_1$ and $C_2$. It costs

- \$100 and 4 hour of labor to grow 1 acre of $C_1$
- \$300 and 1 hour of labor to grow 1 acre of $C_1$. 2

Each acre of $C_1$ will bring \$200 profit, and each acre of $C_2$ will bring \$100 profit. The farmer can spend up to \$27,000 on the production costs and up to 280 hours of labor. How many acres of each crop should be planted to maximize the profit?

<u>Mathematical formulation</u> (linear program)

decision variables

<u>Unknown:</u> $x_1$ = the number of acres of $C_1$

$x_2$ = the number of acres of $C_2$

<u>We want to maximize the profit function:</u>

the objective function

$$z = 200x_1 + 100x_2$$

<u>Restrictions:</u>

constraints

$$x_1 + x_2 \leq 100$$
$$100x_1 + 300x_2 \leq 27{,}000$$
$$4x_1 + 1 \cdot x_2 \leq 280$$

nonnegativity constraints

$$x_1 \geq 0$$
$$x_2 \geq 0$$

**Example 2.** A company manufacturing widgets has 2 factories and 3 warehouses. The cost of sending 1000 widgets from each factory to each warehouse is as follows:

| | $W_1$ | $W_2$ | $W_3$ |
|---|---|---|---|
| $F_1$ | 5 | 5 | 3 |
| $F_2$ | 6 | 4 | 1 |

The factory $\mathbf{F_1}$ can produce up to 10,000 widgets per week and $\mathbf{F_2}$ up to 7,000 widgets per week. The warehouses must receive exactly 8,000 widgets per week for $\mathbf{W_1}$, 5,000 widgets per week for $\mathbf{W_2}$, and 2,000 widgets per week for $\mathbf{W_3}$.

How many widgets should be shipped each week from each factory to each warehouse to minimize the shipping costs?

Linear program

Decision variables:

$x_{ij}$ = the number of widgets (in thousands) shipped from $F_i$ to $W_j$ $(i = 1,2 \ , j = 1,2,3)$

The objective function to minimize:

$$z = 5x_{11} + 5x_{12} + 3x_{13} + 6x_{21} + 4x_{22} + 1 \cdot x_{23}$$

Constraints:

$$x_{11} + x_{12} + x_{13} \leq 10$$
$$x_{21} + x_{22} + x_{23} \leq 7$$
$$x_{11} + x_{21} = 8$$
$$x_{12} + x_{22} = 5$$
$$x_{13} + x_{23} = 2$$
$$x_{ij} \geq 0$$

## The general form of a linear program

For the decision variables $x_1, \ldots, x_n$ find the minimum (or the maximum) of the objective function

$$z = c_1 x_1 + \ldots + c_n x_n$$

Subject to the constraints:

$$a_{i1}x_1 + \ldots + a_{in}x_n \begin{matrix} \leq \\ = \\ \geq \end{matrix} b_i$$

for $i = 1, \ldots, m$, and possibly $x_j \geq 0$ for $j = 1, \ldots, n$.

**Sidenote:** The growth of linear functions

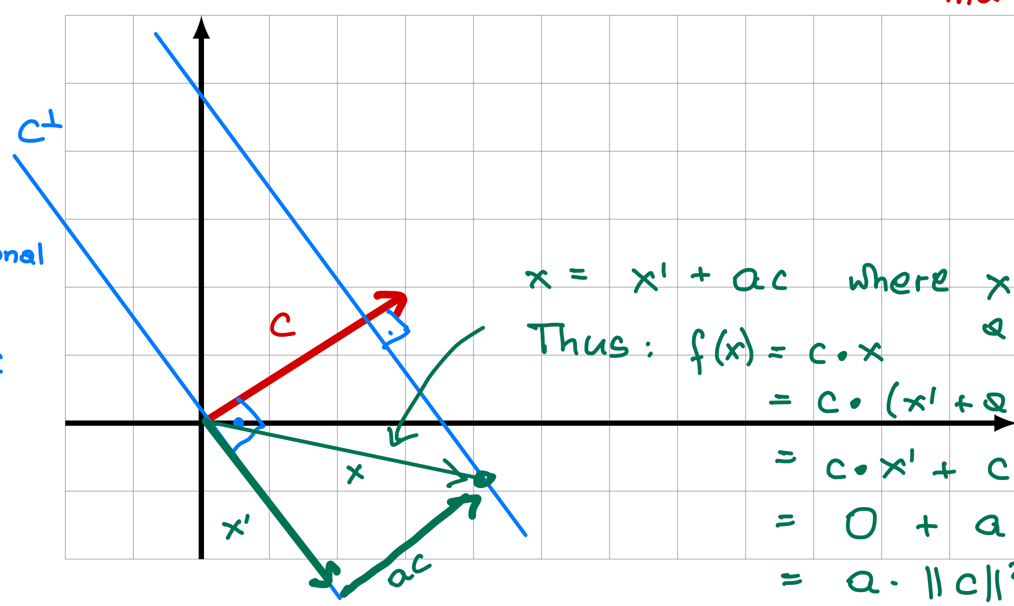
**Example:** $f(x_1, x_2) = 3x_1 + 2x_2$ $\qquad f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

Denote: $\quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad c = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

the dot product

We have : $\quad f(x) = c \bullet x = c^T x$

matrix product




the set of vectors orthogonal to $c$ (i.e $v \in c^\perp$ if and only if $c \bullet x = 0$)

$x = x' + ac$ where $x' \in c^\perp$, $a \in \mathbb{R}$

Thus: $f(x) = c \bullet x$
$= c \bullet (x' + ac)$
$= c \bullet x' + c \bullet (ac)$
$= 0 + a(c \bullet c)$
$= a \cdot \|c\|^2$

Upshot: The values of the function $f$ increase as we move in the direction of the vector $c$, and decrease as we move in the opposite direction.

## Back to Example 1

Maximize:

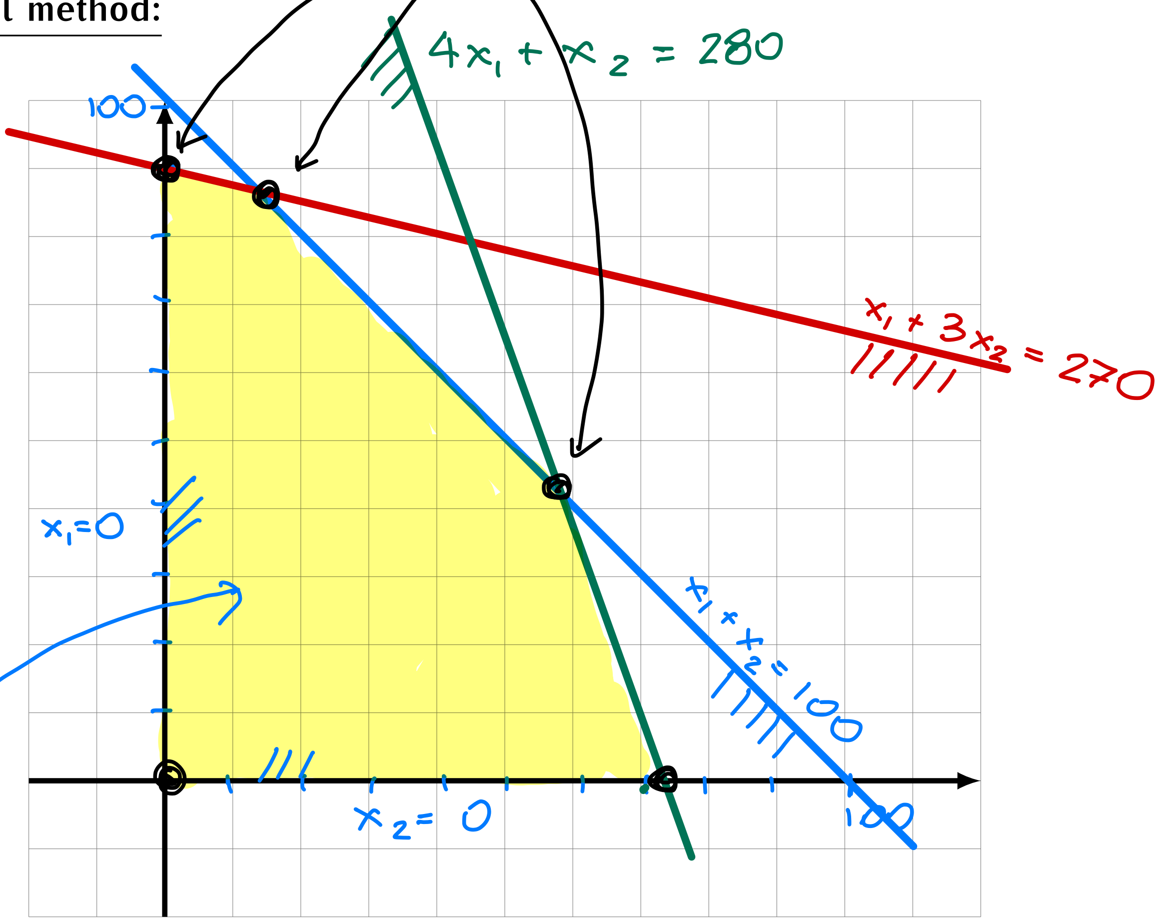
$$f(x_1, x_2) = 200x_1 + 100x_2$$

subject to the constraints:

$$\begin{aligned} x_1 + x_2 &\le 100 \\ 100x_1 + 300x_2 &\le 27000 \qquad x_1 + 3x_2 \le 270 \\ 4x_1 + x_2 &\le 280 \\ x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}$$

extreme points of the feasible region

## Graphical method:

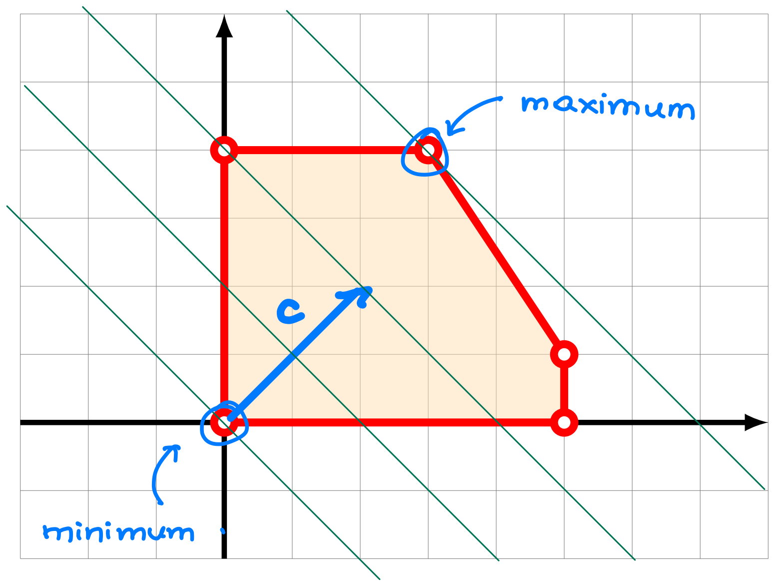



the feasible region : consists of points $(x_1, x_2)$ that satisfy all constraints. Such points are called feasible solutions

**Fact**

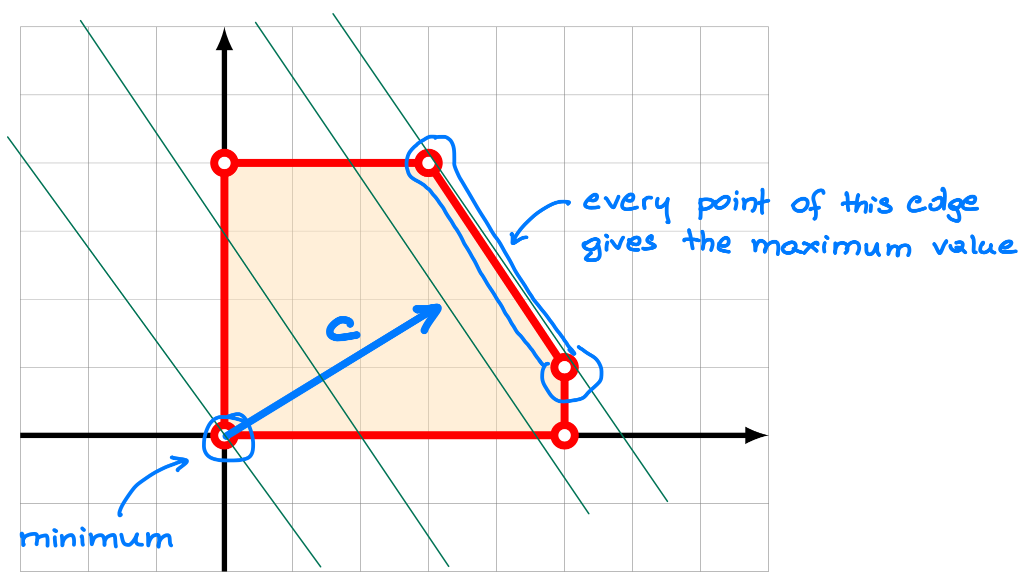
The maximum and the minimum of the objective function, if it exists, always occurs in one of extreme points of the feasible region.

$f(x) = c \cdot x$




**Note.** It may happen that the objective function assumes the minimum or the maximum in infinitely many points, but even then some of them will be extreme points.

$f(x) = c \cdot x$

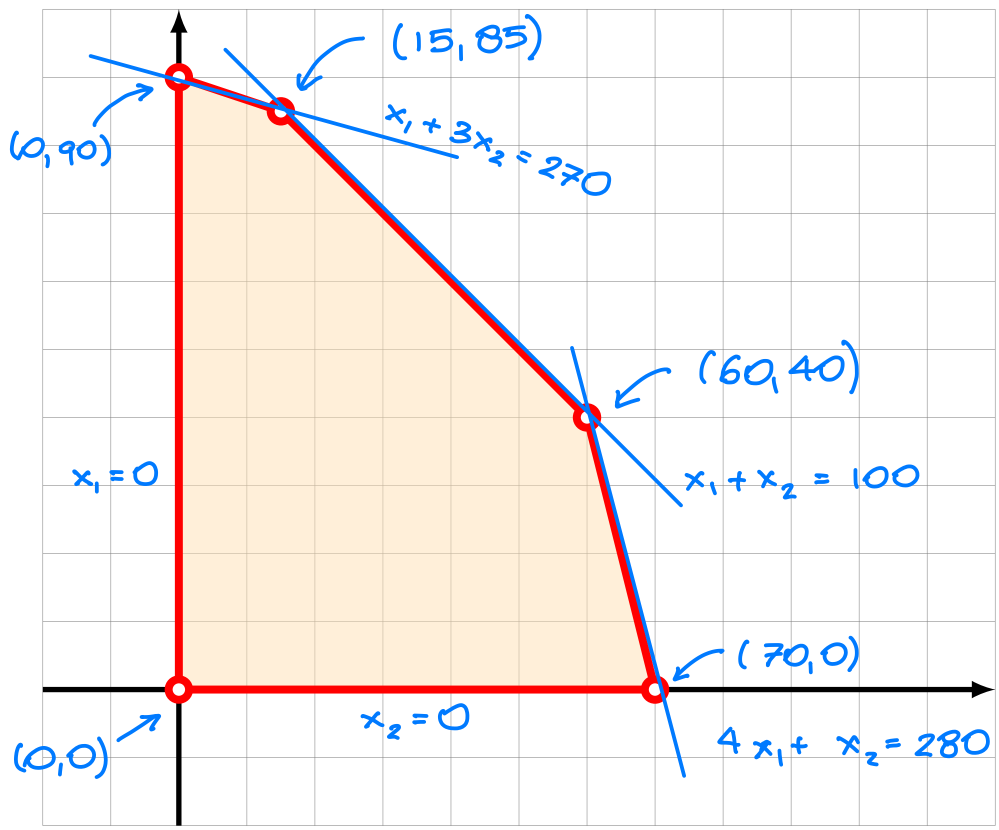


**Upshot.** A linear program can be solved as follows:

- Find coordinates of all extreme points of the feasible region.
- Compute the value of the objective function at each extreme point.
- The point with the biggest value is the maximum, the point with the smallest value is the minimum.

**Back to Example 1:**

$$f(x_1, x_2) = 200x_1 + 100x_2$$




$f(0,0) = 0$

$f(70,0) = 14{,}000$

$f(0,90) = 9{,}000$

$f(15,85) = 11{,}500$

$f(60,40) = 16{,}000$

↑ maximum

**Problem.** In practical applications there are too many extreme points to compute all of them.