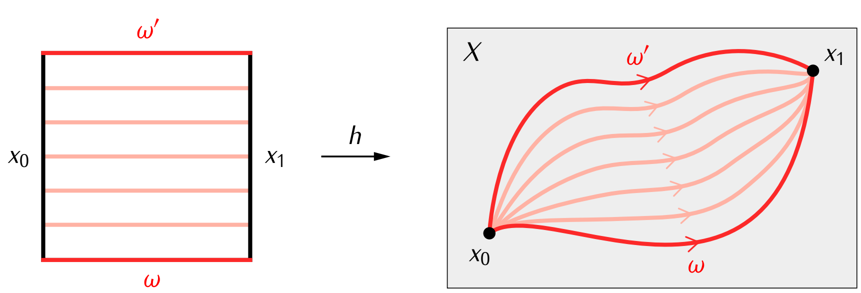
3 | The Fundamental Group

3.1 Definition. A *pointed topological space* is a pair (X, x_0), where X is a topological space and $x_0 \in X$. We say that x_0 is the *basepoint* of X. Given two pointed spaces (X, x_0) and (Y, y_0) a *basepoint preserving map* $f\colon (X, x_0) \to (Y, y_0)$ is a continuous function $f\colon X \to Y$ such that $f(x_0) = y_0$.

3.2 Definition. Let $\omega, \omega'\colon [0,1] \to X$ be paths such that $\omega(0) = \omega'(0) = x_0$ and $\omega(1) = \omega'(1) = x_1$ for some $x_0, x_1 \in X$. We say that the paths ω and ω' are *path homotopic* if for every $t \in [0,1]$ there exists a path $h_t\colon [0,1] \to X$ such that:

1) $h_t(0) = x_0$, and $h_t(1) = x_1$ for all $t \in [0,1]$
2) $h_0 = \omega$, and $h_1 = \omega'$
3) the function $h\colon [0,1] \times [0,1] \to X$ given by $h(s,t) = h_t(s)$ is continuous.

In this case we write $\omega \simeq \omega'$ and we say that h is a *path homotopy* between ω and ω'.



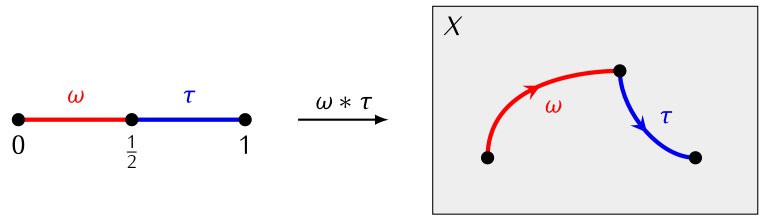
3.3 Lemma. *Let X be a space and let $x_0, x_1 \in X$. Path homotopy defines an equivalence relation on the set of paths in X that start at x_0 and terminate at x_1.*

Proof. Exercise. $\square$

3.4 Definition. For a path ω we will denote by $[\omega]$ the equivalence class of ω taken with respect to the equivalence relation given by path homotopy. We will say that $[\omega]$ is the *homotopy class* of ω.

3.6 Definition. Let $\omega, \tau\colon [0,1] \to X$ be paths such that $\omega(1) = \tau(0)$. The *concatenation* of ω and τ is the path $\omega * \tau\colon [0,1] \to X$ given by

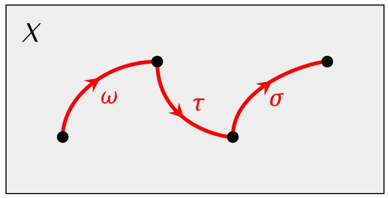
$$(\omega * \tau)(s) = \begin{cases} \omega(2s) & \text{for } s \in [0, \frac{1}{2}] \\ \tau(2s-1) & \text{for } s \in [\frac{1}{2}, 1] \end{cases}$$



3.7 Proposition. *Let ω ,τ be paths in X such that $\omega(1) = \tau(0)$. If ω', τ' are paths such that $\omega \simeq \omega'$ and $\tau \simeq \tau'$ then $\omega * \tau \simeq \omega' * \tau'$.*

3.8 Lemma. *If ω, τ, σ are paths in a space X such that $\omega(1) = \tau(0)$ and $\tau(1) = \sigma(0)$ then*

$$([\omega] \cdot [\tau]) \cdot [\sigma] = [\omega] \cdot ([\tau] \cdot [\sigma])$$



3.9 Lemma. *Let X be a space, and let $x_0 \in X$. Let $c_{x_0}\colon [0,1] \to X$ denote the constant path at the point x_0: $c_{x_0}(s) = x_0$ for all $t \in [0,1]$. If ω is a path in X such that $\omega(0) = x_0$ then $[c_{x_0}] \cdot [\omega] = [\omega]$. Also, if τ is a path such that $\tau(1) = x_0$ then $[\tau] \cdot [c_{x_0}] = [\tau]$.*

3.10 Lemma. *Let ω be a path in a space X such that $\omega(0) = x_0$ and $\omega(1) = x_1$. We have:*

$$[\omega] \cdot [\overline{\omega}] = [c_{x_0}], \qquad [\overline{\omega}] \cdot [\omega] = [c_{x_1}]$$

3.11 Proposition. *Let X be a topological space and let $x_0 \in X$. The set $\pi_1(X, x_0)$ taken with the multiplication given by*

$$[\omega] \cdot [\tau] = [\omega * \tau]$$

for $[\omega], [\tau] \in \pi_1(X, x_0)$ is a group. The trivial element in this group is the homotopy class of the constant path $[c_{x_0}]$, and for $[\omega] \in \pi_1(X, x_0)$ we have $[\omega]^{-1} = [\overline{\omega}]$.

3.12 Definition. Let (X, x_0) be a pointed space. The group $\pi_1(X, x_0)$ is called the *fundamental group* of (X, x_0).

3.13 Lemma. *Let* $f\colon (X, x_0) \to (Y, y_0)$ *be a map of pointed spaces. If* $\omega\colon [0,1] \to X$ *is a loop in* X *based at* x_0 *then* $f \circ \omega\colon [0,1] \to Y$ *is a loop in* Y *based at* y_0*. Moreover, if* ω' *is another loop in* X *based at* x_0 *such that* $\omega \simeq \omega'$ *then* $f \circ \omega \simeq f \circ \omega'$*.*

3.14 Proposition. *If* $f\colon (X, x_0) \to (Y, y_0)$ *is a map of pointed spaces then the function* $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ *is a group homomorphism.*

3.15 Corollary. *The assignments* $(X, x_0) \mapsto \pi_1(X, x_0)$ *and* $f \mapsto f_*$ *define a functor*

$$\pi_1\colon \mathbf{Top}_* \to \mathbf{Gr}$$

3.16 Corollary. *If* $(X, x_0), (Y, y_0)$ *are pointed spaces and* $f\colon X \to Y$ *is a homeomorphism such that* $f(x_0) = y_0$*, then* $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ *is an isomorphism.*

3.18 Note. Alternative construction of the fundamental group.