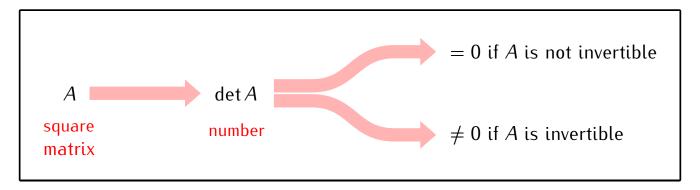
## MTH 461

## 12. Review: Determinants



• **Cofactor expansion.** If *A* is an  $n \times n$  matrix

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

then for any  $1 \le j \le n$  we have

$$\det A = (-1)^{1+j} a_{1j} \cdot \det A_{1j} + (-1)^{2+j} a_{2j} \cdot \det A_{2j} \cdots \cdots \cdots + (-1)^{n+1} a_{nj} \cdot \det A_{nj}$$

where  $A_{ij}$  is the matrix obtained by deleting the  $i^{th}$  row and  $j^{th}$  column of A.

## Example.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$1^{st} \text{ column expansion:}$$

$$\det A = (-1)^{l+1} \cdot 1 \cdot \det \begin{bmatrix} 5 & 6 \\ 8 & q \end{bmatrix} + (-1)^{2+l} \cdot 4 \cdot \det \begin{bmatrix} 2 & 3 \\ 8 & q \end{bmatrix} + (-1)^{l+1} \cdot 7 \cdot \det \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$$
and the second expansion:

2<sup>nd</sup> column expansion:  

$$\det A = (-1)^{1+2} \cdot 2 \cdot \det \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} + (-1)^{2+2} \cdot 5 \cdot \det \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix} + (-1)^{3+2} \cdot 8 \cdot \det \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$$

• Cramer's Rule: If A is an  $n \times n$  invertible matrix then

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

where  $C_{ij} = (-1)^{i+j} \det A_{ij}$ 

## **Proposition**

Consider a matrix equation:

$$A\mathbf{x} = \mathbf{b}$$

if A is an invertible matrix,  $\det A = \pm 1$  and all entries of A and  $\mathbf{b}$  are integers, then the solution of this equation consists of integers.

Proof: If A is an invertible matrix then 
$$A \times = b$$

gives:
$$A'A \times = A'b$$

$$I \times = A'b$$

The identity metrix
$$\times = A'b$$

By Cramer's rule, entries of A' are of the form

This gives:

x = A b vector of integers
matrix of integers

Thus all entries of x are integers.