Example. A company needs to hire people for 5 different positions $P_1, \ldots, P_5$. There are 7 candidates $C_1, \ldots, C_7$ who interviewed for these positions. The table below shows the interview score (higher is better) how each person is qualified for each position. Blank entries indicate the score of 0 (i.e. a candidate is either not suitable or not interested in the corresponding position).

w_{11} w_{12} w_{14}

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
P_1	70	90		75	55		60
P_2	40	95	85			80	
P_3	50		75		70		65
P_4			60	80		35	
P_5		75		70		35	20

Which candidate should be offered which position so that the sum of scores of the assignment is the largest possible?

Integer program: x_{ij} - decision variable

$$x_{ij} = \begin{cases} 1 & \text{if candidate } C_j \text{ is selected for the position } P_i \\ 0 & \text{otherwise} \end{cases}$$

We want to maximize

$$z = \sum_{ij} w_{ij} x_{ij}$$

Constraints:

1) Select only one person for each position:
$$\sum_j x_{ij} = 1 \quad \text{for } i = 1, 2, \ldots, 5$$

2) Each person can get at most one position:
$$\sum_i x_{ij} \leq 1$$

3) $x_{ij} \leq 1$] ← not needed, follows from 1) and 4)

4) $x_{ij} \geq 0$

5) $x_{ij} \in \mathbb{Z}$

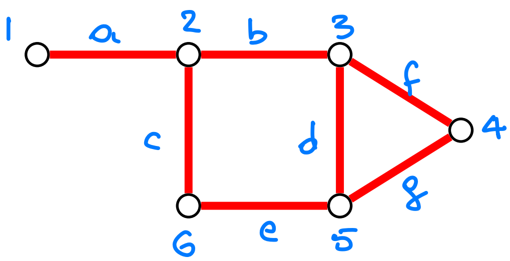
Goal: All basic feasible solutions of the assignment problem consist of integers.

Definition

A *graph* (or a *network*) is a pair $G = (V, E)$ where:

- V is the set of *vertices* (or *nodes*);
- E is the set of *edges*;
- each edge connects two vertices.

Example.

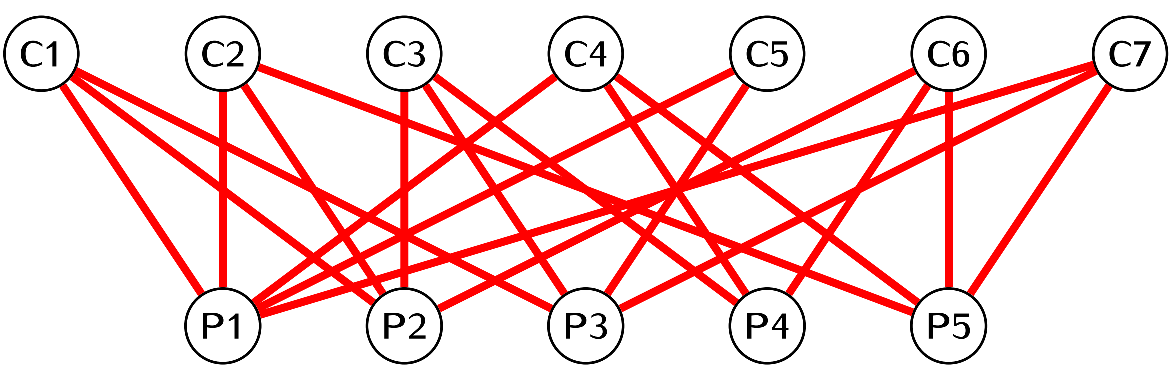


$G = (V, E)$
$V = \{1, 2, 3, 4, 5, 6\}$
$E = \{a, b, c, d, e, f, g\}$

Definition

A *bipartite graph* is a graph $G = (V, E)$ such the set of nodes is a union of two disjoint subsets $V = V_1 \cup V_2$ and that every edge connects some node in V_1 with some node in V_2.

Example. Bipartite graph for the assignment problem:

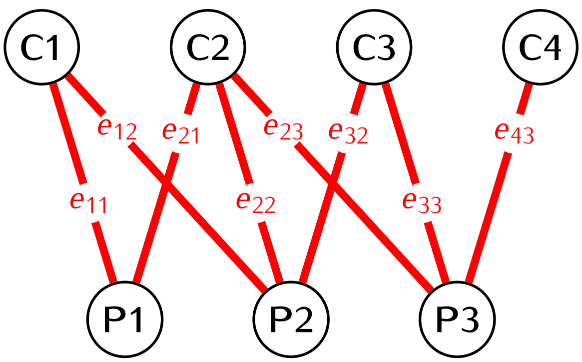


Definition

The *edge incidence matrix* of a graph $G = (V, E)$ is a matrix A such that:

- rows of A are labeled by vertices of G
- columns of A are labeled by edges of G
- the entry in the row of a vertex **v** and the column of an edge **e** is 1 if the edge **e** is attached to **v**; otherwise it is 0.

Example.



$$
\begin{array}{c} \\ C1 \\ C2 \\ C3 \\ C4 \\ P1 \\ P2 \\ P3 \end{array}
\begin{array}{c}
\begin{array}{cccccccc} e_{11} & e_{12} & e_{21} & e_{22} & e_{23} & e_{32} & e_{33} & e_{43} \end{array} \\
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1
\end{bmatrix}
\end{array}
$$

Note: In the assignment problem:

- the decision variat, $x_{ij} \longleftrightarrow$ edges e_{ij} of the graph
- there is one constraint for every vertex:

C1: $x_{11} + x_{12} \leq 1$
C2: $x_{21} + x_{22} \leq 1$
$\vdots$
P1: $x_{11} + x_{21} = 1$
$\vdots$
P3: $x_{13} + x_{33} + x_{43} = 1$

the equality form $\longrightarrow$

C_1: $x_{11} + x_{12} + s_1 = 1$
C_2: $x_{21} + x_{22} + s_2 = 1$
$\vdots$
P3: $x_{13} + x_{33} + x_{43} = 1$

Augmented matrix:

$$
\begin{array}{c} \\ C_1 \\ C_2 \\ \vdots \\ P2 \\ P3 \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{32} & x_{33} & x_{43} & s_1 & s_2 & s_3 & s_4 \end{array} \\
\left[\begin{array}{cccccccccccc|c}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
\vdots & & & & & & & & & & & & \vdots \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\end{array}
$$

the incidence matrix of the graph

some columns of the identity matrix