Question. Consider a Markov chain with

- states $S_1, \ldots, S_N$
- a transition matrix P
- state vectors $X_0, X_1, \ldots$ $X_n = PX_{n-1} = P^n X_0$

What can we say about X_n when n is large?

Example. The weather model:

$$P = \begin{array}{c c} & \begin{array}{cc} R & S \end{array} \\ \begin{array}{c} R \\ S \end{array} & \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix} \end{array}$$

$X_0 = \begin{array}{cc} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{array}{c} R \\ S \end{array} \end{array}$ for n large: $X_n \approx \begin{array}{cc} \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} & \begin{array}{c} R \\ S \end{array} \end{array}$

$X_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ —— " —— $X_n \approx \begin{array}{cc} \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} & \begin{array}{c} R \\ S \end{array} \end{array}$

$X_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ —— " —— $X_n \approx \begin{array}{cc} \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} & \begin{array}{c} R \\ S \end{array} \end{array}$

<u>Note:</u>

$$P^n \approx \begin{array}{c c} & \begin{array}{cc} R & S \end{array} \\ \begin{array}{c} R \\ S \end{array} & \begin{bmatrix} 0.2 & 0.2 \\ 0.8 & 0.8 \end{bmatrix} \end{array}$$ for large n

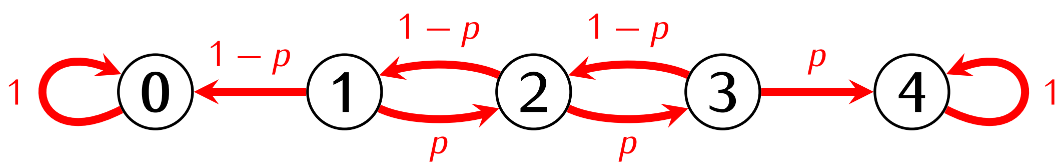
(converges, all colums the same)

For any $X_0 = \begin{bmatrix} p \\ q \end{bmatrix}$ we have:

$p+q=1$

$$X_n = P^n X_0 \approx \begin{bmatrix} 0.2 & 0.2 \\ 0.8 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} p + \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} q = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}$$

Example. The gambling model:



$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1-p & 0 & 0 & 0 \\ 0 & 0 & 1-p & 0 & 0 \\ 0 & p & 0 & 1-p & 0 \\ 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & p & 1 \end{bmatrix} \end{array}$$

For $p = 0.5$:

$$P^n \approx \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0.75 & 0.5 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0.25 & 0.5 & 0.25 & 1 \end{bmatrix} \end{array}$$

(converges, different columns)

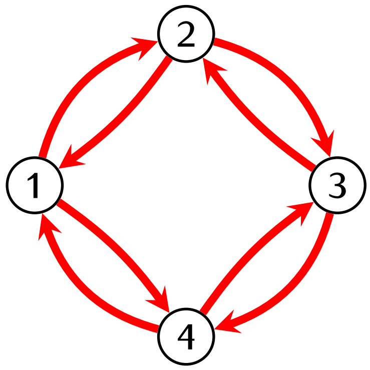
Note:

This means that $X_n = P^n X_0$ will depend on the choice of the vector X_0:

$$X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow X_n = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow X_n = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0 \\ 0.5 \end{bmatrix}$$

Example. Random walk on a circular network:



$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix} \end{array}$$

A random walker can return to the starting position after an even number of steps only. Thus P^n will have 0 on the main diagonal if n is odd, and non-zero numbers if n is even.

<u>Upshot</u>: In this case P^n does not converge.

The steady-state vector

Assume that a Markov chain that starts with some state vector X_0 converges to some vector:

$$\lim_{n\to\infty} X_n = Y$$

We have: $X_n = P^n X_0$

so:

$$\lim_{n\to\infty} P^n X_0 = Y$$

$$\lim_{n\to\infty} P(P^{n-1}X_0) = Y$$

$$\parallel$$

$$P\Big(\underbrace{\lim_{n\to\infty} P^{n-1}X_0}_{Y}\Big) = Y$$

We get: $PY = Y$

Definition

If P is a stochastic matrix then the *steady-state vector* of P is a probability vector Y such that $PY = Y$.

Note: Equivalently: Y is an eigenvector of P corresponding to the eigenvalue 1

Upshot: If P is the transition matrix of some Markov chain $X_0, X_1, \dots,$ and $\lim_{n\to\infty} X_n = Y$ then Y is a steady state vector of P.

Example. The weather model:

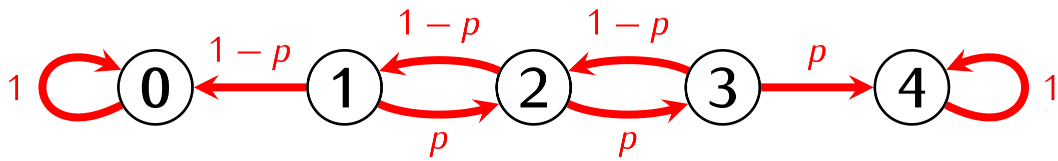
$$P = \begin{array}{c c} & \begin{array}{cc} \text{R} & \text{S} \end{array} \\ \begin{array}{c} \text{R} \\ \text{S} \end{array} & \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix} \end{array}$$

$$(\text{eigensp. of } \lambda = 1) = \text{Nul}(P - 1I) = \text{Nul}\left(\begin{bmatrix} -0.4 & 0.1 \\ 0.4 & -0.1 \end{bmatrix}\right)$$

$$= \text{Span}\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right)$$

Thus $y = \begin{bmatrix} 1/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}$ is the only steady state vector of P

Example. The gambling model (with $p \neq 0, 1$):



$$
P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
\begin{bmatrix}
1 & 1-p & 0 & 0 & 0 \\
0 & 0 & 1-p & 0 & 0 \\
0 & p & 0 & 1-p & 0 \\
0 & 0 & p & 0 & 0 \\
0 & 0 & 0 & p & 1
\end{bmatrix}
\end{array}
$$

(eigensp. of $\lambda = 1$) $= \mathrm{Nul}(P - 1I)$

$$
= \mathrm{Nul}\left(\begin{bmatrix} 0 & 1-p & 0 & 0 & 0 \\ 0 & -1 & 1-p & 0 & 0 \\ 0 & p & -1 & 1-p & 0 \\ 0 & 0 & 0 & p & 0 \end{bmatrix}\right)
$$

$$
= \mathrm{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \left\{ \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \\ b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}
$$

<u>Steady state vectors</u>

$$
y = \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \\ 1-a \end{bmatrix} \qquad 0 \leq a \leq 1
$$

Proposition

If P is a stochastic matrix then P has a steady-state vector.

Lemma

If A is a square matrix then λ is an eigenvalue of A if and only if λ is an eigenvalue of A^T.

Proof of Lemma

eigenvalues of A = roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$

eigenvalues of A^T = roots of the characteristic polynomial $q(\lambda) = \det(A^T - \lambda I)$

$= \det((A - \lambda I)^T)$

Recall: for any square matrix B we have:

$$\det B = \det B^T$$

so: $p(\lambda) = q(\lambda)$ $\blacksquare$

Proof of Proposition:

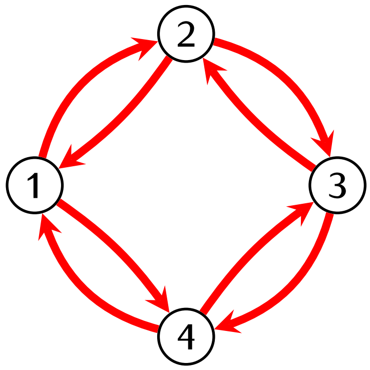
P - stochastic $\Rightarrow$ sums of colums $= 1$

$\Rightarrow$ sums of rows of $P^T = 1$

$$\Rightarrow P^T \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Thus the vector $v = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ is an eigenvector of P^T corresp. to eigenvalue $\lambda = 1$. By Lemma $\lambda = 1$ is an eigenvalue of P too.

Example. Random walk on a circular network:



$$P = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1/2 & 0 & 1/2 \\ 2 & 1/2 & 0 & 1/2 & 0 \\ 3 & 0 & 1/2 & 0 & 1/2 \\ 4 & 1/2 & 0 & 1/2 & 0 \end{array}$$

Note:

$$P \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

so $Y = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$ is a steady state vector of P.

However, in general, for an arbitrary vector X_0 the limit $\lim_{n\to\infty} P^n X_0$ does not exist.

Definition

A stochastic matrix P is *regular* if there is $N \geq 0$ such that all entries of P^N are positive.

Perron-Frobenius Theorem

If P is a regular stochastic matrix then:

- There exists only one steady state vector Y of P
- For any probability vector X we have

$$\lim_n P^n X = Y$$