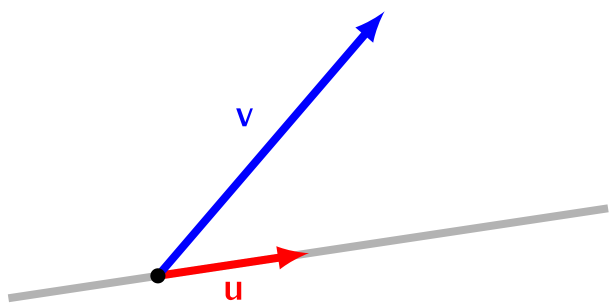
Definition

Given vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{u} \neq \mathbf{0}$, the *orthogonal projection* of $\mathbf{v}$ onto $\mathbf{u}$ is the vector $\mathrm{proj}_{\mathbf{u}}\mathbf{v}$ such that

1) $\mathrm{proj}_{\mathbf{u}}\mathbf{v} = c\mathbf{u}$ for some $c \in \mathbb{R}$
2) the vector $\mathbf{v} - \mathrm{proj}_{\mathbf{u}}\mathbf{v}$ is orthogonal to $\mathbf{u}$.



Proposition

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $\mathbf{u} \neq \mathbf{0}$. For any $c \in \mathbb{R}$ we have

$$||\mathbf{v} - \mathrm{proj}_{\mathbf{u}}\mathbf{v}|| \leq ||\mathbf{v} - c\mathbf{u}||$$

Proposition

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $\mathbf{u} \neq \mathbf{0}$. Then $\text{proj}_{\mathbf{u}}\mathbf{v} = c\mathbf{u}$ where

$$c = \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}$$

In particular, if $||\mathbf{u}|| = 1$ then $\text{proj}_{\mathbf{u}}\mathbf{v} = c\mathbf{u}$ where

$$c = \mathbf{v} \cdot \mathbf{u}$$

Example.

$$\mathbf{u} = \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Corollary

If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $||\mathbf{u}|| = 1$ then

$$||\text{proj}_{\mathbf{u}}\mathbf{v}|| = |\mathbf{v} \cdot \mathbf{u}|$$