5 | First Computations

5.1 Proposition. *If $X = \{*\}$ is a space consisting of only one point then $\pi_1(X)$ is the trivial group.*

5.2 Proposition. *For any $n \geq 1$ the group $\pi_1(\mathbb{R}^n)$ is trivial.*

5.3 Proposition. *For any $n \geq 1$ the group $\pi_1(D^n)$ is trivial.*

5.5 Definition. A space X is *simply connected* if it is path connected and $\pi_1(X)$ is trivial.

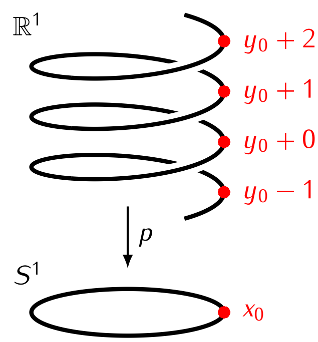
5.6 Proposition. *A space X is simply connected if and only if X is path connected and for any two paths $\omega, \tau\colon [0,1] \to X$ satisfying $\omega(0) = \tau(0)$ and $\omega(1) = \tau(1)$ we have $\omega \simeq \tau$.*

Proof. Exercise. $\square$

Goal:

5.7 Theorem. $\pi_1(S^1) \cong \mathbb{Z}$.

5.8 Definition. The *universal covering* of S^1 is the map $p\colon \mathbb{R}^1 \to S^1$ given by $p(s) = (\cos 2\pi s, \sin 2\pi s)$.

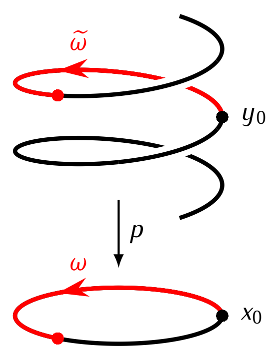


5.10 Definition. Let ω be a path in S^1. We say that a path $\widetilde{\omega}$ in $\mathbb{R}$ is a *lift* of ω if $p \circ \widetilde{\omega} = \omega$.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ {[0,1]} \arrow[ur, "\widetilde{\omega}"] \arrow[r, "\omega"'] & S^1 \end{tikzcd}$$

5.11 Proposition. *Let $p\colon \mathbb{R}^1 \to S^1$ be the universal covering of S^1, let $x_0 \in S^1$, and let $y_0 \in \mathbb{R}^1$ be a point such that $p(y_0) = x_0$.*

1) For any path $\omega\colon [0,1] \to S^1$ such that $\omega(0) = x_0$ there exists a lift $\widetilde{\omega}\colon [0,1] \to \mathbb{R}^1$ satisfying $\widetilde{\omega}(0) = y_0$. Moreover, such lift is unique.



2) Let $\omega, \tau\colon [0,1] \to S^1$ be paths such that $\omega(0) = \tau(0) = x_0$, $\omega(1) = \tau(1)$ and $\omega \simeq \tau$. If $\widetilde{\omega}$, $\widetilde{\tau}$ are lifts of ω, τ, respectively, such that $\widetilde{\omega}(0) = \widetilde{\tau}(0) = y_0$ then $\widetilde{\omega}(1) = \widetilde{\tau}(1)$ and $\widetilde{\omega} \simeq \widetilde{\tau}$.

5.12 Definition. Let $x_0 \in S^1$ and $y_0 \in \mathbb{R}$ be points such that $p(y_0) = x_0$. Let ω be a loop in S^1 based at x_0 and let $\widetilde{\omega}$ be the unique lift of ω such that $\widetilde{\omega}(0) = y_0$. The *degree* of ω is the integer $\deg(\omega)$ such that $\widetilde{\omega}(1) = y_0 + \deg(\omega)$.