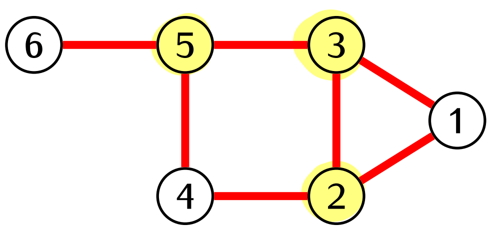
Definition

A *vertex cover* of a graph G is a set S of vertices of G such that every edge of G has at least one end in S.

A *minimum vertex cover* of G is a vertex cover such that there is no vertex cover with a smaller number of vertices.

Example.



2, 3, 5 is a minimum vertex cover

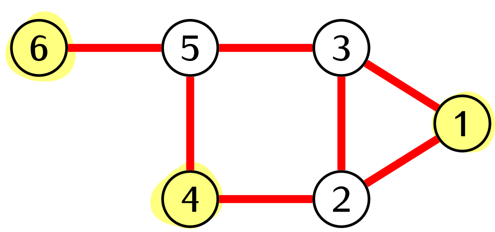
1, 2, 5 is another minimum cover

Definition

An *independent set* of a graph G is a set S of vertices of G such that there is no edge between any two elements of S.

A *maximum independent set* of G is an independent set such that there is no independent set with a larger number of vertices.

Example.



1, 4, 6 is a maximum independent set

1, 3, 6 is another maximum independent set

Proposition

Let $G = (V, E)$ be an undirected graph.

1) A set $S \subseteq V$ is independent if and only if the set $V \setminus S$ is vertex cover of G.

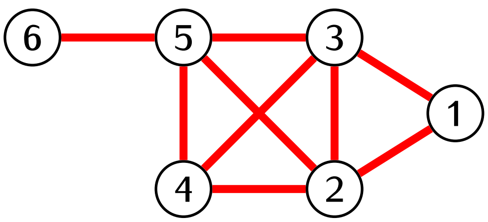
2) A set $S \subseteq V$ is a maximum independent set if and only if the set $V \setminus S$ is a minimum vertex cover of G.

Definition

Let G be an undirected graph. A *clique* is a set S of vertices of G such that any two vertices are connected by an edge.

A *maximum clique* of G is a clique such that there is no clique with a bigger number of vertices.

Example.



Some cliques:

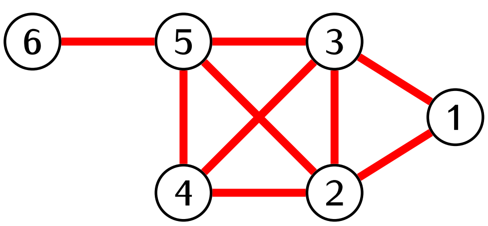
$\{5,6\}$, $\{1,2,3\}$, $\{2,3,4,5\}$

$\{2,3,4,5\}$ is a maximum clique

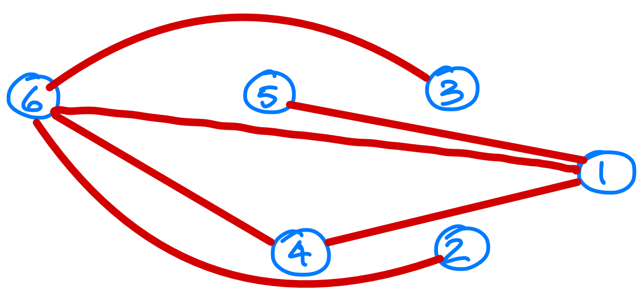
Definition

Let G be a simple graph. A complement of G is a graph G' such that G' has the same vertices as G, and two vertices are connected by an edge in G' if and only there is no edge between them in G.

Example.



G



G'

Proposition

Let G be a simple graph and let G' be its complement. A set S of vertices G is a clique in G if and only if S is an independent set in G'.

Corollary

Let $G = (V, E)$ be a simple graph, let G' be its complement and let S be a set of vertices of G. The following conditions are equivalent:

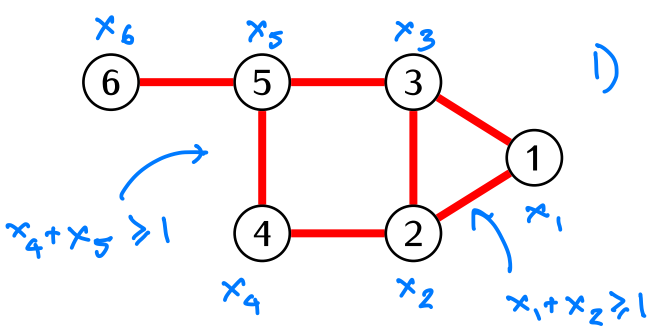
1) The set S is a maximum clique in G.

2) The set S is a maximum independent set in G'.

3) The set $V \setminus S$ is a minimum vertex cover in G'.

Problem. Given a graph $G = (V, E)$ find a minimum vertex cover of G.

Integer program reformulation:



1) Decision variables: x_v, $v \in V$

one variable for each vertex,

$$x_v = \begin{cases} 1 & \text{if } v \text{ is in the cover} \\ 0 & \text{otherwise} \end{cases}$$

2) Objective function to minimize: $z = \sum_{v \in V} x_v$

3) Constraints: $x_v + x_w \geq 1$ if v, w are vertices connected by an edge (one constraint per edge). This means that each edge has at least one end in the cover.

Also: $0 \leq x_v \leq 1$, $x_v \in \mathbb{Z}$

Note: The constraints of this problem are of the form

$$A^T \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \geq \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

where A is the incidence matrix of the graph G.

An approximated solution of the minimum vertex cover problem:

- LP relaxation: drop the assumption that $x_v \in \mathbb{Z}$.
- Solve the relaxed program. This will give a solution consisting of some numbers $0 \leq x_v \leq 1$
- Define $S_{LP} = \{ v \in V \mid x_v \geq \frac{1}{2} \}$

Note: 1) S_{LP} is a vertex cover since every edge has at least one vertex v with $x_v \geq \frac{1}{2}$:

$x_v + x_w \geq 1$ so $x_v \geq \frac{1}{2}$ or $x_w \geq \frac{1}{2}$ (or both)

2) S_{LP} need not be a minimum vertex cover.

Proposition

Assume that each minimum vertex cover of a graph G consists of N vertices. Let S_{LP} be a vertex cover selected using the solution of the linear program as described above. Then

$$|S_{\mathrm{LP}}| \leq 2N$$

Proof: Let $\{\bar{x}_v\}_{v \in V}$ be a solution of the integer program

$\{\tilde{x}_v\}_{v \in V}$ be a solution of the relaxed program

Since vertices v such that $\bar{x}_v = 1$ form a minimum vertex cover, we have:

$$\sum_{v \in V} \bar{x}_v = N$$

We also have:
$$\sum_{v \in V} \tilde{x}_v \leq \sum_{v \in V} \bar{x}_v$$

(since minimum of the relaxed program can't be bigger than the minimum of the integer program)

This gives:

$$|S_{LP}| = \sum_{v \in S_{LP}} 1 \overset{\text{since } v \in S_{LP} \text{ if } \tilde{x}_v \geq \frac{1}{2}}{\leq} \sum_{v} 2 \cdot \tilde{x}_v \leq 2 \sum_{v} \bar{x}_v = 2N$$