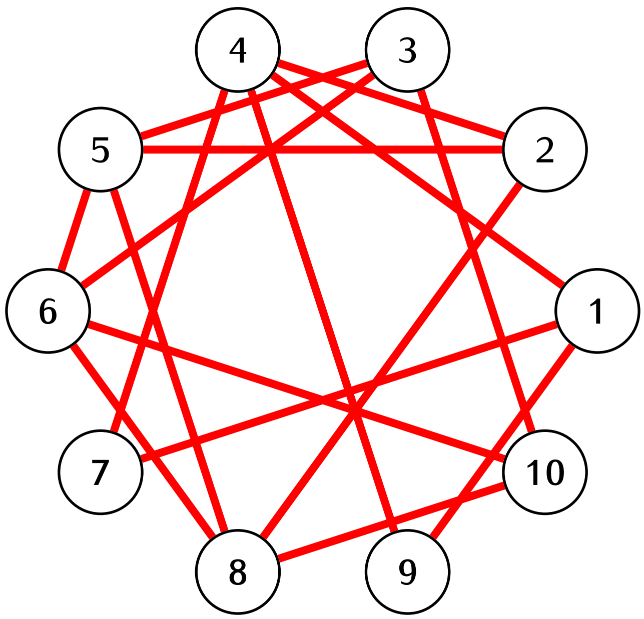
Note. You can use Python for computations needed to complete this assignment. See the Jupyter notebook provided with this assignment for a demonstration of some tools that may be useful.

1. Consider the following graph:



a) Compute the algebraic connectivity of this graph.

b) Apply the spectral partitioning algorithm to divide the vertices of this graph into two sets, with 5 vertices in each set. Explain your computations.

Redraw the graph in a way that better shows the partitioning (i.e. with vertices in each set grouped closer together). Try to clean up the picture so that possibly few edges intersect.

Does the partitioning computed using the spectral partitioning algorithm minimize the number of edges joining the two sets, or can you find a different partitioning that has fewer connecting edges?

c) Repeat part b) but partition the graph into sets with 4 and 6 vertices.

2. Assume that an $n \times n$ matrix A is invertible and orthogonally diagonalizable. Show that the inverse matrix A^{-1} is also orthogonally diagonalizable.

3. Let $q\colon \mathbb{R}^4 \to \mathbb{R}$ be a quadratic form given by

$$\begin{aligned} q\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) &= \\ &= 18x_1^2 + 18x_2^2 + 18x_3^2 + 18x_4^4 \\ &+ 6x_1x_2 + 22x_1x_3 - 4x_1x_4 - 4x_2x_3 \\ &+ 22x_2x_4 + 6x_3x_4 \end{aligned}$$

a) Find a symmetric matrix A that represents this quadratic form. That is it satisfies

$$q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$$

for any vector $\mathbf{x} \in \mathbb{R}^4$.

b) Is this this quadratic form positive definite, positive semidefinite, negative definite, negative semidefinite, or indefinite? Justify your answer.

c) Find vectors $\mathbf{v}$ and $\mathbf{w}$ such that $||\mathbf{v}|| = ||\mathbf{w}|| = 1$ and such that $q(\mathbf{v})$ has the maximum value among all vectors of length 1, while $q(\mathbf{w})$ has the minimum value. What are these maximum and minimum values? Justify your answer.

4. Let A, B be $n \times n$ symmetric matrices such that all eigenvalues of A and B are positive. Show that all eigenvalues of the matrix $A + B$ are also positive.

Hint. Consider quadratic forms.

5. Let A be $n \times n$ invertible symmetric matrix, and let A^{-1} be its inverse. Show that if the quadratic form $q_A(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$ is positive definite then the quadratic form $q_{A^{-1}}(\mathbf{x}) = \mathbf{x}^T A^{-1} \mathbf{x}$ is also positive definite.