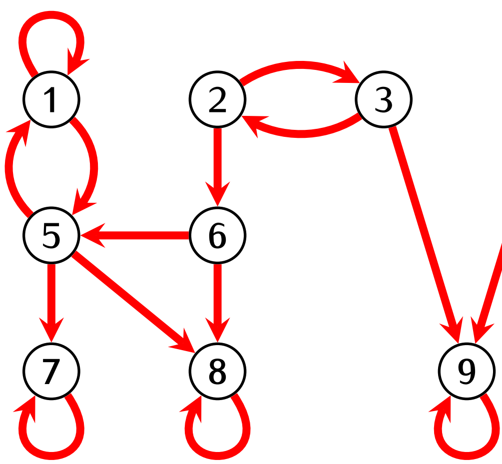
Definition

Consider a Markov chain with states $S_1, \dots, S_N$ and the transition matrix $P = (p_{ij})$.

- A state S_i is *absorbing* if $p_{ii} = 1$
- The Markov chain is *absorbing* if for each state there is a non-zero probability that the state will transition to an absorbing state after some number of steps.

Example.



non-absorbing states

absorbing states

Transition matrix:

$$P = \begin{array}{cc} & \begin{array}{ccccccccc} 7 & 8 & 9 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 7\\8\\9\\1\\2\\3\\4\\5\\6 \end{array} & \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & S \\ 0 & 0 & 1 & \\ \hline 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & Q \\ 0 & 0 & 0 & \end{array}\right] \end{array}$$

Transition matrix of an absorbing Markov chain

Note: By reordering states of an absorbing Markov chain, so that absorbing states come before non-absorbing ones, we can write the transition matrix in the following form:

$$P = \begin{array}{cc}
\begin{array}{cc} \overbrace{\qquad}^{\substack{k \\ \text{absorbing}}} & \overbrace{\qquad}^{\substack{n-k \\ \text{non-abs.}}} \end{array} & \\
\begin{bmatrix} \;I_k\; & \;S\; \\ \;0\; & \;Q\; \end{bmatrix} & \begin{array}{l} \big\} \; k \text{ absorbing} \\ \big\} \; n-k \text{ non-abs} \end{array}
\end{array}$$

where: I_k = $k \times k$ identity matrix $= \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

0 = $(n-k) \times k$ zero matrix

S = $k \times (n-k)$ matrix with some non-zero entries

Q = $(n-k) \times (n-k)$ <u>substochastic</u> matrix (sum of each column ≤ 1, and there is a column with sum < 1)

Block multiplication:

$$P^2 = \begin{bmatrix} I & S \\ 0 & Q \end{bmatrix}\begin{bmatrix} I & S \\ 0 & Q \end{bmatrix} = \begin{bmatrix} II + S0 & IS + SQ \\ 0 \cdot I + Q0 & 0S + QQ \end{bmatrix} = \begin{bmatrix} I & S + SQ \\ 0 & Q^2 \end{bmatrix}$$

In general:

$$P^n = \begin{bmatrix} I & S(I + Q + \dots + Q^{n-1}) \\ 0 & Q^n \end{bmatrix}$$

Long term behaviour:

$$\lim_{n \to \infty} P^n = \begin{bmatrix} I & S(I + Q + Q^2 + \dots) \\ 0 & \lim_{n \to \infty} Q^n \end{bmatrix}$$

Proposition

Consider an absorbing Markov chain with the transition matrix in the form

$$P = \begin{array}{c} \begin{array}{cc} \text{abs.} & \text{non-abs.} \end{array} \\ \left[\begin{array}{cc} I & S \\ 0 & Q \end{array}\right] \end{array} \begin{array}{l} \\ \text{abs.} \\ \text{non-abs.} \end{array}$$

Then the following hold:

- $\lim_n Q^n = 0$
- The infinite series $I + Q + Q^2 + \dots$ converges.
- $I + Q + Q^2 + \dots = (I - Q)^{-1}$

Proof of the last statement:

Enough to show:

$$(I - Q)(I + Q + Q^2 + \dots) = I$$

We have:

$$(I - Q)(I + Q + Q^2 + \dots) = (I - \cancel{Q}) + (\cancel{Q} - \cancel{Q^2}) + (\cancel{Q^2} - \cancel{Q^3}) + \dots = I$$

Definition

For an absorbing Markov chain the matrix

$$(I - Q)^{-1} = I + Q + Q^2 + \dots$$

is called the *fundamental matrix* of the Markov chain.

Corollary

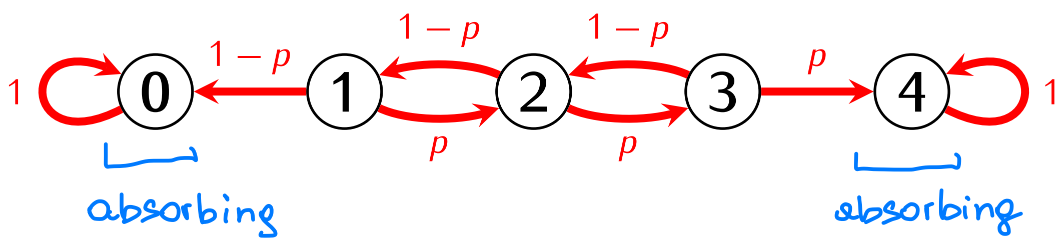
For an absorbing Markov chain the transition matrix

$$P = \begin{array}{c@{}l} \begin{array}{cc} \text{abs.} & \text{non-abs.} \end{array} & \\ \left[\begin{array}{cc} I & S \\ 0 & Q \end{array}\right] & \begin{array}{l} \text{abs.} \\ \text{non-abs.} \end{array} \end{array}$$

we have:

$$\lim_n P^n = \begin{bmatrix} I & S(I-Q)^{-1} \\ 0 & 0 \end{bmatrix}$$

Example. The gambling model:



Assume $p = \frac{1}{4}$

$$P = \begin{array}{c} \\ 0 \\ 4 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 4 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cc|ccc} 1 & 0 & 3/4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/4 \\ \hline 0 & 0 & 0 & 3/4 & 0 \\ 0 & 0 & 1/4 & 0 & 3/4 \\ 0 & 0 & 0 & 1/4 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} S \\ \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} Q \end{array}$$

<u>The fundamental matrix</u>:

$$(I - Q)^{-1} = \begin{bmatrix} 1 & -3/4 & 0 \\ -1/4 & 1 & -3/4 \\ 0 & -1/4 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 13/10 & 6/5 & 9/10 \\ 2/5 & 8/5 & 6/5 \\ 1/10 & 2/5 & 13/10 \end{bmatrix}$$

$$S(I-Q)^{-1} = \begin{bmatrix} 3/4 & 0 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \cdot (I-Q)^{-1}$$

$$= \begin{array}{c} \\ 0 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 39/40 & 9/10 & 27/40 \\ 1/40 & 1/10 & 13/40 \end{bmatrix} \end{array}$$