Assume that we have an absorbing Markov chain with

- absorbing states $S_1, \dots, S_M$
- non-absorbing states $S_{M+1}, \dots, S_N$
- the transition matrix

$$\begin{array}{cccc} & \text{abs.} & \text{non-abs.} & \\ P = & \left[\begin{matrix} I \\ 0 \end{matrix}\right. & \left.\begin{matrix} S \\ Q \end{matrix}\right] & \begin{matrix} \text{abs.} \\ \text{non-abs.} \end{matrix} \end{array}$$

Questions:

1) If the chain starts in an non-absorbing state S_i, how many steps it will take on the average before it transitions to an absorbing state?

2) If the chain starts in an non-absorbing state S_i, how many times, on the average, it will visit a non-absorbing state S_j before being absorbed?

Note: If we can answer 2), then we can answer 1):

suppose that if we start at S_i then before being absorbed we visit on the average

S_M - n_M times
S_{M+1} - n_{M+1} times
$\vdots$
S_N - n_N times

Then it takes on the average $n_M + n_{M+1} + \dots + n_M$ steps to be absorbed.

Question: What does "on the average" mean?

Random variables and expected values

Example. In a certain game a player can:

- loose \$2 with the probability 0.5
- win \$2 with the probability 0.3
- win \$10 with the probability 0.2

How much will the player win per game on the average?

Answer:

$$\underbrace{(-2)\cdot 0.5 + 2\cdot 0.3 + 10\cdot 0.2}_{\text{the expected value of winnings in the game}} = -1 + 0.6 + 2 = 1.6$$

Definition

A *discrete random variable* X consists of

- A set of values (outcomes) $v_i \in \mathbb{R}$ for $i = 1, 2, \ldots$.
- Probabilities p_i that X assumes each of the values v_i:

$$P(X = v_i) = p_i$$

We have $0 \leq p_i \leq 1$ and $\sum p_i = 1$.

Example:

Values of X : $-2, 2, 10$

$$P(X=-2) = 0.5$$
$$P(X=2) = 0.3$$
$$P(X=10) = 0.2$$

Definition

If X is a discrete random variable with values $v_i \in \mathbb{R}$ then the *expected value* of X is the number

$$E[X] = \sum_i v_i P(X = v_i)$$

For the random variable from the previous example:

$$E[X] = (-2) \cdot P(X=-2) + 2 \cdot P(X=2) + 10 \cdot P(X=10)$$
$$= -2 \cdot 0.5 + 2 \cdot 0.3 + 10 \cdot 0.2 = 1.6$$

Note:

If X_1 - random variables with values $v_1, \dots, v_N$

X_2 - random variables with values $w_1, \dots, w_M$

then $X_1 + X_2$ - random variable with values $v_i + w_j$ for all i, j.

Definition

If $X_1, \dots, X_m$ are discrete random variables with values in $\mathbb{R}$ then

$$E[X_1 + \dots + X_m] = E[X_1] + \dots + E[X_m]$$

Back to absorbing Markov chains

Recall: We have an absorbing Markov chain with

- absorbing states $S_1, \ldots, S_M$
- non-absorbing states $S_{M+1}, \ldots, S_N$
- the transition matrix

$$P = \begin{array}{c} \\ \\ \end{array} \begin{array}{cc} \text{abs.} & \text{non-abs.} \\ \left[\begin{array}{c} I \\ 0 \end{array}\right. & \left.\begin{array}{c} S \\ Q \end{array}\right] \end{array} \begin{array}{l} \\ \text{abs.} \\ \text{non-abs.} \end{array}$$

Question. If the chain starts in an non-absorbing state S_i, how many times, on the average, it will visit a non-absorbing state S_j before being absorbed?

<u>Define:</u> X = the random variable for the number of times that the Markov chain starting at S_i will visit S_j before being absorbed

Values of X: $0, 1, \ldots$

<u>We want:</u>

$E[X] = 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2) + \ldots$

<u>Problem:</u> Difficult to compute $P(X=n)$.

Simplification.

Define random variables X_n $n = 0, 1, \ldots$

$$X_n = \begin{cases} 1 & \text{if the Markov chain that started at } S_i \\ & \text{is in the state } S_j \text{ after } n \text{ steps} \\ 0 & \text{otherwise} \end{cases}$$

Note: $X = X_0 + X_1 + X_2 + \ldots$

so: $E[X] = E[X_0 + X_1 + \ldots] = E[X_0] + E[X_1] + \ldots$

$$P^n = \underbrace{\left[\begin{array}{c|c} I & * \\ \hline 0 & Q^n \end{array}\right]}_{}$$ – n-step transition matrix

$Q^n = [q_{ij}^{(n)}]$ (Note: $Q^0 = I$)

$E[X_n] = 1 \cdot q_{ji}^{(n)} + 0 \cdot (1 - q_{ji}^{(n)}) = q_{ji}^{(n)}$

This gives:

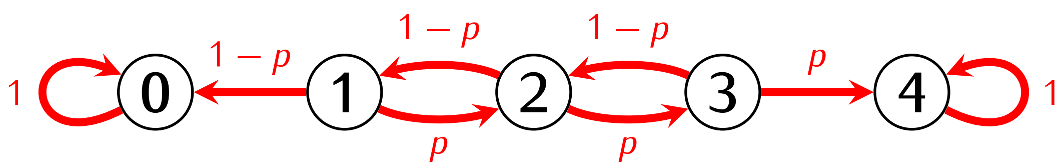
$$E[X] = \underbrace{E[X_0] + E[X_1] + E[X_2] + \ldots}_{\text{the } ji\text{-th entry of the matrix}}$$

$$Q^0 + Q^1 + Q^2 + \ldots$$

$$= I + Q^1 + Q^2 + \ldots$$

$$= (I - Q)^{-1}$$ ← the fundamental matrix

Example. The gambling model (with $p \neq 0, 1$):



$$P = \begin{array}{c} \\ 0 \\ 4 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 4 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1-p & 0 & 0 \\ 0 & 1 & 0 & 0 & p \\ 0 & 0 & 0 & 1-p & 0 \\ 0 & 0 & p & 0 & 1-p \\ 0 & 0 & 0 & p & 0 \end{bmatrix} \end{array}$$

Take $p = \frac{1}{4}$:

$$P = \begin{array}{c} \\ 0 \\ 4 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 4 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cc|ccc} 1 & 0 & 3/4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/4 \\ \hline 0 & 0 & 0 & 3/4 & 0 \\ 0 & 0 & 1/4 & 0 & 3/4 \\ 0 & 0 & 0 & 1/4 & 0 \end{array}\right] \end{array} \leftarrow Q$$

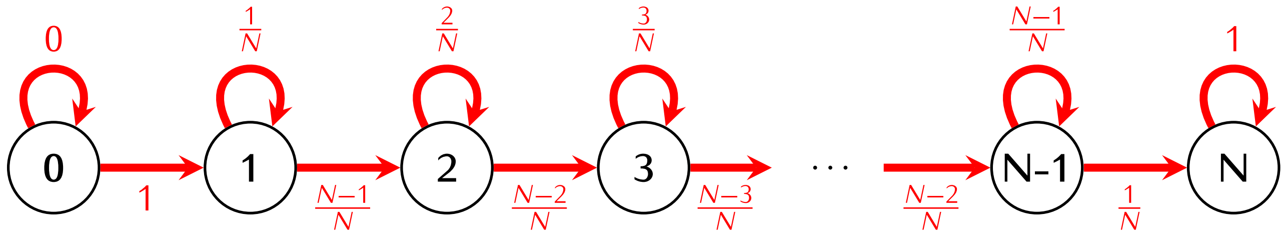
$$(I - Q)^{-1} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 13/10 & 12/10 & 9/10 \\ 4/10 & 16/10 & 12/10 \\ 1/10 & 4/10 & 13/10 \end{bmatrix} \end{array}$$

sum

$$\begin{array}{ccc} \begin{bmatrix} 18/10 & 32/10 & 34/10 \end{bmatrix} \\ 1 \quad\quad 2 \quad\quad 3 \end{array}$$

the number of steps it will take on the average to get an absorbing state from each of non-absorbing states.

Example. Toy collecting:



Question. How many steps, on the average it will take to collect all toys?

$$P = \begin{array}{c|c} & \begin{array}{cccccc} N & 0 & 1 & 2 \cdots & (N-2) & (N-1) \end{array} \\ \begin{array}{c} N \\ 0 \\ 1 \\ 2 \\ \vdots \\ N-2 \\ N-1 \end{array} & \left[\begin{array}{c|cccccc} 1 & 0 & 0 & \cdots & & 0 & 1/N \\ \hline 0 & 0 & & & & & \\ 0 & 1 & \frac{1}{N} & & & 0 & \\ 0 & & \frac{N-1}{N} & \frac{2}{N} & & & \\ \vdots & & & \frac{N-2}{N} & \ddots & & \\ 0 & & 0 & & \ddots & \frac{N-2}{N} & \\ 0 & & & & & \frac{2}{N} & \frac{N-1}{N} \end{array}\right] \end{array} \Bigg\} Q$$

$$I - Q = \begin{bmatrix} 1 & & & & & 0 \\ -1 & \frac{N-1}{N} & & & & \\ & -\frac{N-1}{N} & \frac{N-2}{N} & & & \\ & & -\frac{N-2}{N} & \ddots & & \\ & 0 & & \ddots & \frac{2}{N} & \\ & & & & -\frac{2}{N} & \frac{1}{N} \end{bmatrix}$$

$$(I - Q)^{-1} = \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & \cdots & N-1 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & & 1 \end{bmatrix} \end{array} \begin{array}{l} \cdot\ N/N \\ \cdot\ N/N-1 \\ \cdot\ N/N-2 \\ \vdots \\ \cdot\ N/1 \end{array}$$

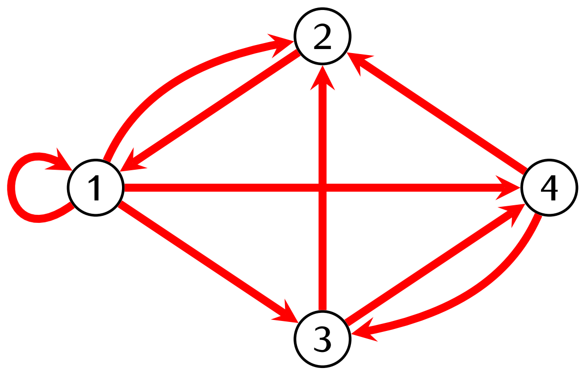
the average number of steps needed to reach the absorbing state starting with state 0:

$$S_N = \frac{N}{N} + \frac{N}{N-1} + \frac{N}{N-2} + \dots + \frac{N}{1}$$

Example: $S_5 \approx 11.4$, $S_{10} = 28.3$, $S_{20} = 71.9$, $S_{50} \approx 225.0$

Transit time

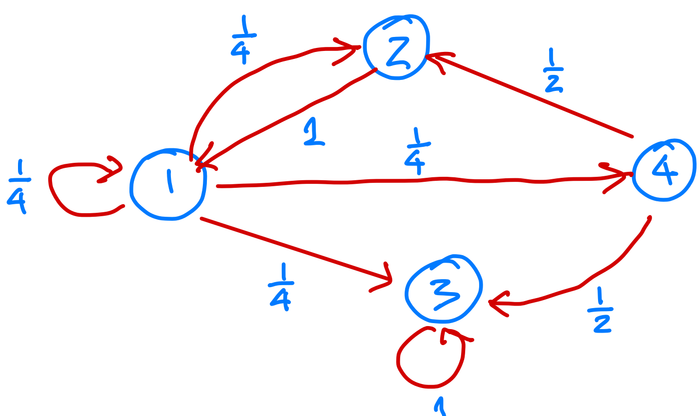
Example. Consider a random walk on the following directed network:



How many steps will it take on the average to get from node 2 to node 3?

Solution:

1) Make the node 3 absorbing:



2) Use the previous method to compute how many steps it will take to get from state 2 to an absorbing state.