A (square matrix) $\longrightarrow$ $\det A$ (number)

- $= 0$ if A is not invertible
- $\neq 0$ if A is invertible

- **Cofactor expansion.** If A is an $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

then for any $1 \leq j \leq n$ we have

$$\begin{aligned} \det A = & \quad (-1)^{1+j} a_{1j} \cdot \det A_{1j} \\ & + \ (-1)^{2+j} a_{2j} \cdot \det A_{2j} \\ & \dots \qquad \dots \qquad \dots \qquad \dots \\ & + \ (-1)^{n+1} a_{nj} \cdot \det A_{nj} \end{aligned}$$

where A_{ij} is the matrix obtained by deleting the i^{th} row and j^{th} column of A.

Example.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

1^{st} column expansion:

$$\det A = (-1)^{1+1} \cdot 1 \cdot \det \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} + (-1)^{2+1} \cdot 4 \cdot \det \begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix} + (-1)^{3+1} \cdot 7 \cdot \det \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$$

2^{nd} column expansion:

$$\det A = (-1)^{1+2} \cdot 2 \cdot \det \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} + (-1)^{2+2} \cdot 5 \cdot \det \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix} + (-1)^{3+2} \cdot 8 \cdot \det \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$$

- **Cramer's Rule:** If A is an $n \times n$ invertible matrix then

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

where $C_{ij} = (-1)^{i+j} \det A_{ij}$

Proposition

Consider a matrix equation:

$$A\mathbf{x} = \mathbf{b}$$

if A is an invertible matrix, $\det A = \pm 1$ and all entries of A and $\mathbf{b}$ are integers, then the solution of this equation consists of integers.

Proof: If A is an invertible matrix then

$$Ax = b$$

gives:

$$A^{-1}Ax = A^{-1}b$$

$$\underbrace{I}_{\text{the identity matrix}} x = A^{-1}b$$

$$x = A^{-1}b$$

By Cramer's rule, entries of A^{-1} are of the form

$$\underbrace{\frac{1}{\det A}}_{\substack{=\pm 1 \\ \text{since } \det A = \pm 1}} \cdot (-1)^{i+j} \underbrace{\det A_{ij}}_{\substack{\text{integer, since all} \\ \text{entries of } A \text{ are integers}}}$$

This gives: $x = \underbrace{A^{-1}}_{\text{matrix of integers}} \underbrace{b}_{\text{vector of integers}}$

Thus all entries of x are integers.