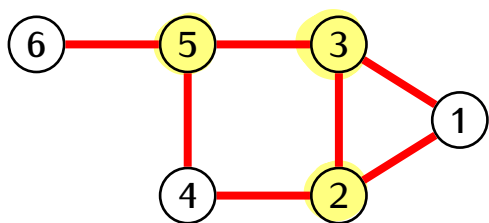


Definition

A *vertex cover* of a graph G is a set S of vertices of G such that every edge of G has at least one end in S .

A *minimum vertex cover* of G is a vertex cover such that there is no vertex cover with a smaller number of vertices.

Example.



$2, 3, 5$ is a minimum vertex cover

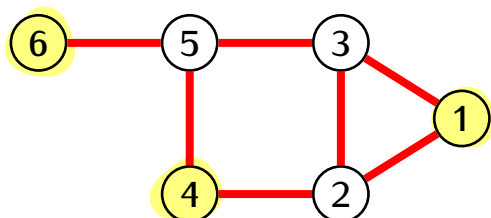
$1, 2, 5$ is another minimum cover

Definition

An *independent set* of a graph G is a set S of vertices of G such that there is no edge between any two elements of S .

A *maximum independent set* of G is an independent set such that there is no independent set with a larger number of vertices.

Example.



$1, 4, 6$ is a maximum independent set

$1, 3, 6$ is another maximum independent set

Proposition

Let $G = (V, E)$ be an undirected graph.

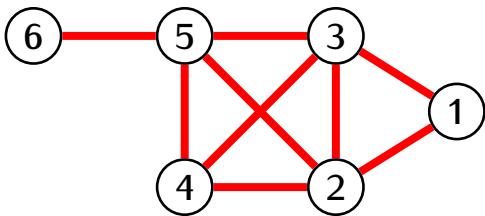
- 1) A set $S \subseteq V$ is independent if and only if the set $V \setminus S$ is vertex cover of G .
- 2) A set $S \subseteq V$ is a maximum independent set if and only if the set $V \setminus S$ is a minimum vertex cover of G .

Definition

Let G be an undirected graph. A *clique* is a set S of vertices of G such that any two vertices are connected by an edge.

A *maximum clique* of G is a clique such that there is no clique with a bigger number of vertices.

Example.



Some cliques:

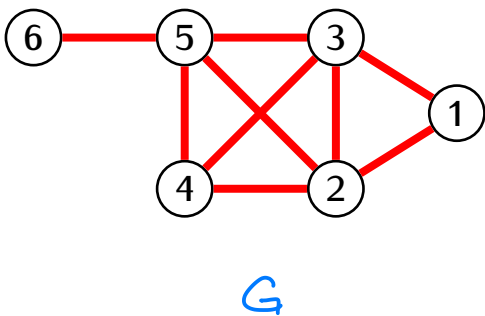
$\{5, 6\}$, $\{1, 2, 3\}$, $\{2, 3, 4, 5\}$

$\{2, 3, 4, 5\}$ is a maximum clique

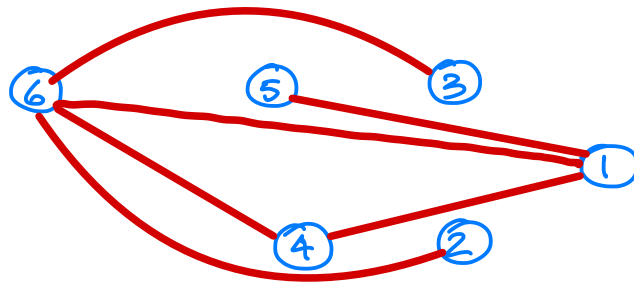
Definition

Let G be a simple graph. A complement of G is a graph G' such that G' has the same vertices as G , and two vertices are connected by an edge in G' if and only if there is no edge between them in G .

Example.



G



G'

Proposition

Let G be a simple graph and let G' be its complement. A set S of vertices G is a clique in G if and only if S is an independent set in G' .

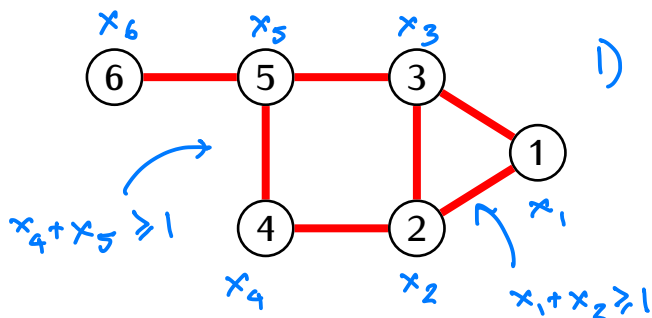
Corollary

Let $G = (V, E)$ be a simple graph, let G' be its complement and let S be a set of vertices of G . The following conditions are equivalent:

- 1) The set S is a maximum clique in G .
- 2) The set S is a maximum independent set in G' .
- 3) The set $V \setminus S$ is a minimum vertex cover in G' .

Problem. Given a graph $G = (V, E)$ find a minimum vertex cover of G .

Integer program reformulation:



1) Decision variables: $x_v, v \in V$
 one variable for each vertex,

$$x_v = \begin{cases} 1 & \text{if } v \text{ is in the cover} \\ 0 & \text{otherwise} \end{cases}$$

2) Objective function to minimize: $z = \sum_{v \in V} x_v$

3) Constraints: $x_v + x_w \geq 1$ if v, w are vertices connected by an edge (one constraint per edge)
 This means that each edge has at least one end in the cover.

Also: $0 \leq x_v \leq 1, x_v \in \mathbb{Z}$

Note: The constraints of this problem are of the form

$A^T \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \geq \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ where A is the incidence matrix of the graph G .

An approximated solution of the minimum vertex cover problem:

Proposition

Assume that each minimum vertex cover of a graph G consists of N vertices. Let S_{LP} be a vertex cover selected using the solution of the linear program as described above. Then

$$|S_{LP}| \leq 2N$$