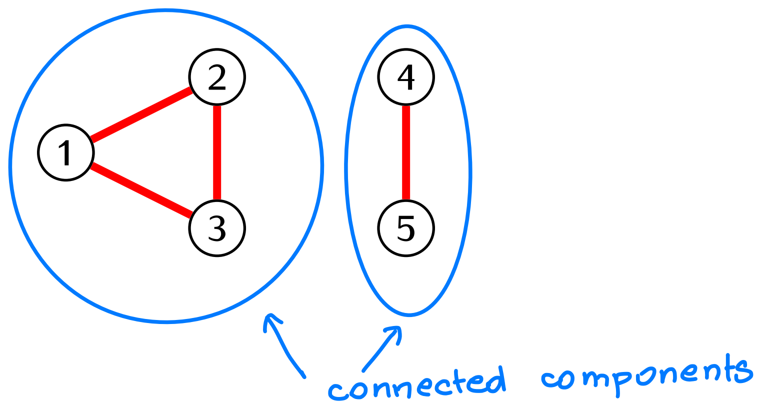
Note. From now on all graphs are simple, undirected unless it is indicated otherwise.

Definition

A graph is *connected* if any two vertices can be joined by a path.

A *connected component* of a graph is a maximal subgraph that is connected.



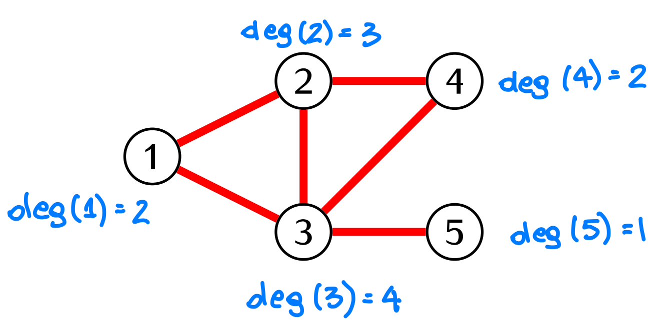
Goal:

- How to check if a graph is connected?
- If a graph is not connected, how to count its connected components?

Definition

If i is a vertex of a graph then the *degree* of i is the number

$$\deg(i) = (\text{the number of edges attached to } i)$$



Definition

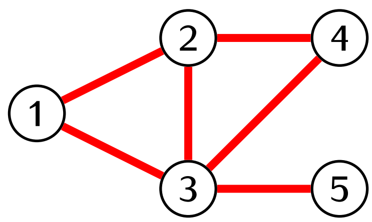
Let G be a graph with vertices $1, 2, \ldots, N$. The *Laplacian* of G is a matrix

$$L = D - A$$

where

- A is the adjacency matrix of A
- D is a diagonal matrix with degrees of vertices on the diagonal.

Example.



$$L = \underbrace{\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 2 & & & & \\ & 3 & & 0 & \\ & & 4 & & \\ & 0 & & 2 & \\ & & & & 1 \end{bmatrix} \end{array}}_{\substack{D \\ \text{the degree matrix}}} - \underbrace{\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}}_{\substack{A \\ \text{the adjacency matrix}}} = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Note: The sum of each row of the Laplacian is 0.

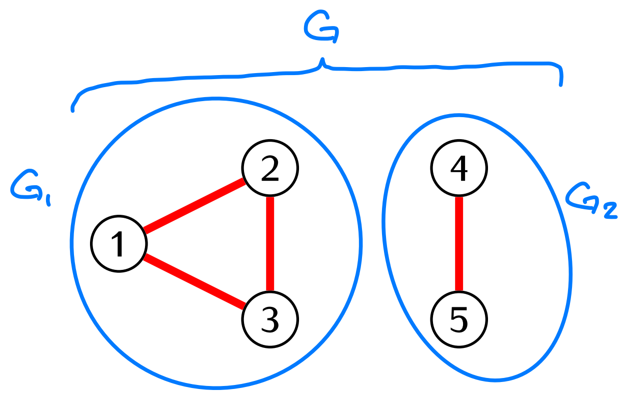
This gives:

$$L \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Upshot: 1) $\lambda = 0$ is an eigenvalue of L

2) The vector $v = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ is an eigenvector corresponding to this eigenvalue.

Example.



Laplacian of G_1

$$L = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 2 & & & & \\ & 2 & & 0 & \\ & & 2 & & \\ & 0 & & 1 & \\ & & & & 1 \end{bmatrix} \end{array} - \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

Laplacian of G_2

__Note:__

vertices of G_1

$$L \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

vertices of G_2

$$L \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus L has two linearly independent eigenvectors corresponding to the eigenvalue $\lambda = 0$.

In general, if a graph G with vertices $1, \dots, N$ has M connected components $G_1, \dots, G_M$ then then the Laplacian of G has at least M linearly independent eigenvectors $v_1, v_2, \dots, v_M$ corresponding to the eigenvalue $\lambda = 0$:

$$v_k = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \qquad \text{where} \quad x_i = \begin{cases} 1 & \text{if } i \in G_k \\ 0 & \text{otherwise} \end{cases}$$

Goal:

Proposition $*$

If L is the Laplacian of a graph G then

$$\begin{pmatrix} \text{the number of} \\ \text{connected components} \\ \text{of } G \end{pmatrix} = \begin{pmatrix} \text{the number of} \\ \text{linearly independent eigenvectors} \\ \text{of } L \text{ corresponding to } \lambda = 0 \end{pmatrix}$$

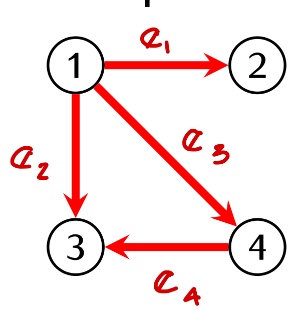
Definition

Let G be a directed graph with vertices $1, 2, \ldots, N$ and edges $e_1, e_2, \ldots, e_M$. The *edge incidence matrix* of G is an $N \times M$ matrix $B = (b_{ij})$ such that

- rows of B are labeled by vertices of G
- columns of B are labeled by edges of G
- the entries of B are given by

$$b_{ij} = \begin{cases} -1 & \text{if the edge } e_j \text{ starts at the vertex } i \\ +1 & \text{if the edge } e_j \text{ ends at the vertex } i \\ \ \ 0 & \text{otherwise} \end{cases}$$

Example.



$$B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \end{array}$$

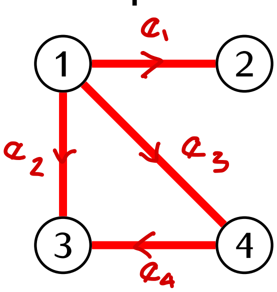
Lemma

Let

- G be a simple undirected graph
- L be the Laplacian of G
- B be the edge incidence matrix of G with the direction of edges selected in an arbitrary way.

Then $L = BB^T$.

Example.



$$BB^T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \end{array} \cdot \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \end{array} = L$$

Proof of Proposition $*$.

From before: If G has vertices $1, \dots, N$ and M connected components $G_1, \dots, G_M$ then the Laplacian of G has M linearly independent eigenvectors corresponding to $\lambda = 0$: $v_1, \dots, v_M$ where

$$v_k = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \qquad x_i = \begin{cases} 1 & \text{if } i \in G_k \\ 0 & \text{otherwise} \end{cases}$$

We need to show: there are no more linearly independent eigenvectors of L for $\lambda = 0$.

Enough to show: if $w = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$ is an eigenvector of L for $\lambda = 0$ then $x_i = x_j$ for any two vertices i, j that are connected by an edge.

Let B = the edge incidence matrix of G with some orientation of edges. By Lemma: $L = BB^T$

Claim: $Lv = 0$ if and only if $B^T v = 0$

Indeed: if $B^T v = 0$ then $Lv = BB^T v = B \cdot 0 = 0$

Conversely: if $Lv = 0$ then $v^T L v = 0$

so: $(v^T B)(B^T v) = 0$

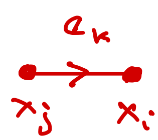
the dot product of $B^T v$ with itself

$$(B^T v)^T (B^T v) = 0$$

This gives: $B^T v = 0$.

It remains to notice that

$$B^T \cdot \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_N \end{bmatrix} = \begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_k \\ \vdots \end{array} \begin{bmatrix} \\ \\ \\ x_i - x_j \\ \\ \end{bmatrix} \leftarrow \text{if}$$



Thus $B^T v = 0$ if and only if $x_i = x_j$ for any vertices i, j connected by an edge.

Proposition

If B is a any matrix then all eigenvalues of the matrix $A = BB^T$ are greater or equal to 0.

<u>Proof:</u> Let $Av = \lambda v$, $v \neq 0$

Then $BB^Tv = \lambda v$

$$\underbrace{v^T B B^T v}_{\substack{\| \\ \underbrace{(B^Tv)^T(B^Tv)}_{\text{dot product of } B^Tv \text{ with itself}} \\ \| \\ \|B^Tv\|^2}} = v^T\lambda v = \lambda \underbrace{v^Tv}_{\text{dot product of } v \text{ with itself}} = \lambda \cdot \|v\|^2$$

<u>We obtain:</u>

$$\|B^Tv\|^2 = \lambda \|v\|^2$$

<u>so:</u>

$$\lambda = \frac{\|B^Tv\|^2}{\|v\|^2} \geqslant 0$$

Corollary

If L is the Laplacian of a graph G then all eigenvalues of L are greater or equal to 0.