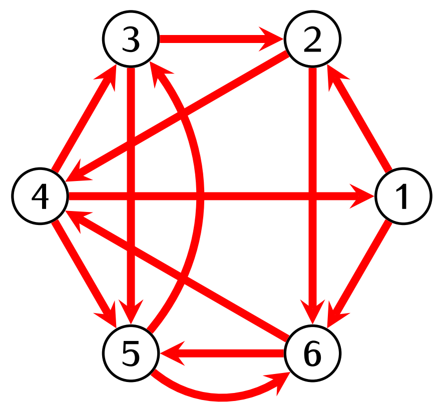
1. Consider the following directed graph:



a) Compute the number of paths consisting of exactly 10 edges, which start at the vertex 1 and end at the vertex 2.

b) Compute the number of paths consisting of exactly 10 edges, which start at the vertex 1 and end at the vertex 2 and do not pass through the vertex 3. Explain your reasoning.

c) Compute the number of paths consisting of exactly 10 edges, which start at the vertex 1 and end at the vertex 2 and pass through vertex 3 at least once. Explain your reasoning.

2. A factory stamps its products with serial numbers. Each serial number consists of a sequence of 9 digits $d_1d_2d_3\dots d_9$ such that $1 \leq d_i \leq 5$ for each i. Moreover, a serial number cannot contain any of the following tuples of numbers: 11, 22, 25, 52, 34, 43. Thus, for example, the sequence 231452331 is not allowed, since it contains 52. Compute the total number of all possible serial numbers. Explain your reasoning.

3. The Laplacian of a certain undirected, simple graph G with vertices $1, 2, \ldots, 8$ has the following linearly independent eigenvectors for the eigenvalue $\lambda = 0$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

The Laplacian has no additional eigenvectors for $\lambda = 0$ that are linearly independent from the above vectors.

The first entry of each of these vectors correspond to the vertex 1, the second entry to the vertex 2 etc.

a) For which pairs of vertices of this graph there exists a path joining these vertices? Explain your reasoning.

b) Draw all possible graphs whose Laplacian has eigenvectors for $\lambda = 0$ indicated above.

4. An undirected simple graph is k*-regular* if every vertex of G has degree k. Show that if G is a k-regular graph and A is the adjacency matrix of G then k is an eigenvalue of A and that it is the largest eigenvalue of this matrix.

Hint. Show that $\mathbf{v}$ is an eigenvector of A if and only if it is an eigenvector of the Laplacian L of G.

5. Let L be the Laplacian of an undirected simple graph. Show that if

$$\mathbf{u} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \neq \mathbf{0}$$

is an eigenvector of L corresponding to the eigenvalue $\lambda \neq 0$ then there exist $1 \leq i, j \leq N$ such that $x_i < 0$ and $x_j > 0$.

Hint. Recall that every symmetric matric is orthogonally diagonalizable. A consequence of this is that if $\mathbf{u}$, $\mathbf{v}$ are eigenvectors of such a matrix corresponding to different eigenvalues then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors.