Definition

A matrix A is *totally unimodular* if every square matrix obtained by removing some rows and columns of A has determinant 0, 1, or -1.

Note: 1) If A is totally unimodular then all entries of A are equal to $0, 1, -1$.

2) Not every matrix with entries $1, 0, -1$ is totally unimodular.

e.g: $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $\det A = 2$

Proposition

Consider a linear program of the equality form: maximize

$$z = c_1 x_1 + \ldots + c_n x_n$$

subject to constraints:

$$a_{i1} x_1 + \ldots + a_{in} x_n = b_i$$

for $i = 1, \ldots, m$ and $x_j \geq 0$ for $j = 1, \ldots, n$.

If the the coefficient matrix

$$A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix}$$

is totally unimodular and $b_i \in \mathbb{Z}$ for $i = 1, \ldots, m$ then values of $x_1, \ldots, x_n$ for every basic feasible solution of the linear program are integers. In particular, if the linear program has a maximum, then there is a solution with integer values which gives this maximum.

Proof of Proposition.

Denote $A = \underbrace{[v_1 \; v_2 \; \cdots \; v_n]}_{\text{columns of } A}$. Without loss of generality assume that we have a basic feasible solution where $x_1, \dots, x_m$ are basic variables and $x_{m+1}, \dots, x_n$ are free. Then we have $x_{m+1} = \dots = x_n = 0$, so we only need to show that $x_1, \dots, x_m$ are integers.

We have:

$$A \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}_{= b}$$

so: $v_1 x_1 + v_2 x_2 + \dots + \underbrace{v_m x_m + v_{m+1} x_{m+1} + \dots + v_n x_n}_{= 0} = b$

Thus $x_1 v_1 + \dots + x_m v_m = b$

Let A_B be the $m \times m$ matrix with columns $v_1, \dots, v_m$:

$$A_B = [v_1 \; v_2 \; \cdots \; v_m]$$

This gives: $A_B \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = b$

We have:

1) since $x_1, \dots, x_m$ are basic variables, the columns $v_1, \dots, v_m$ are linearly independent. This means that A_B is an invertible matrix. Thus $\det A_B \neq 0$.

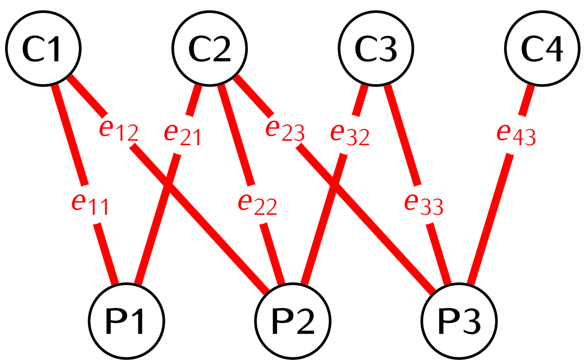
2) since A is totally unipotent we must have $\det A_B = \pm 1$ and all entries of A_B are integers.

Th gives that $x_1, \dots, x_m$ are integers.

Proposition

For any bipartite graph $G = (V_1 \cup V_2, E)$ the incidence matrix of G is totally unimodular.

Example.



$$\underbrace{\begin{array}{c} \\ C1 \\ C2 \\ C3 \\ C4 \\ P1 \\ P1 \\ P3 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_{11} & e_{12} & e_{21} & e_{22} & e_{23} & e_{32} & e_{33} & e_{43} \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}}_{\shortparallel \atop A}$$

<u>Induction:</u>

Assume that all submatrices A of size $n \times n$ have determinant $0, 1$ or -1.

Let B - submatrix of A of size $(n+1) \times (n+1)$

<u>Consider three cases:</u>

1) B has a column consisting of zeros.
Then $\det B = 0$ — ok

2) B has a column that contains only one entry equal to 1.
Then by cofactor expansion with respect to this column we get $\det B = \pm \det(B')$ where B' is an $n \times n$ matrix, so $\det B' = 0, 1$ or -1.
Thus $\det B = 0, 1$ or -1.

3) Every column of B contains two entries equal to 1.
Then rows of B are linearly dependent:
(sum of rows corresp. to C_i's) $-$ (sum of rows corresp. to P_i's) $= 0$
Thus B is not an invertible matrix, and $\det B = 0$ - ok.

Proposition

If A is a totally unimodular matrix and B is a matrix obtained by appending to A by a column that that has only one non-zero entry equal to 1, then B is totally unimodular.

$$B = \left[\begin{array}{c|c} & 0 \\ & \vdots \\ & 0 \\ A & 1 \\ & 0 \\ & \vdots \\ & 0 \end{array}\right]$$

Corollary

The simplex method always gives a solution to the assignment problem that consists of integers.

<u>Proof</u>: The constraints of the assignment problem are of the form $Ax=b$ where A is totally unimodular matrix and $b = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$. Thus all basic feasible solutions of this problem consist of integers.