Definition

A matrix A is *totally unimodular* if every square matrix obtained by removing some rows and columns of A has determinant 0, 1, or -1.

Proposition

Consider a linear program of the equality form: maximize

$$z = c_1 x_1 + \ldots + c_n x_n$$

subject to constraints:

$$a_{i1} x_1 + \ldots + a_{in} x_n = b_i$$

for $i = 1, \ldots, m$ and $x_j \geq 0$ for $j = 1, \ldots, n$.

If the the coefficient matrix

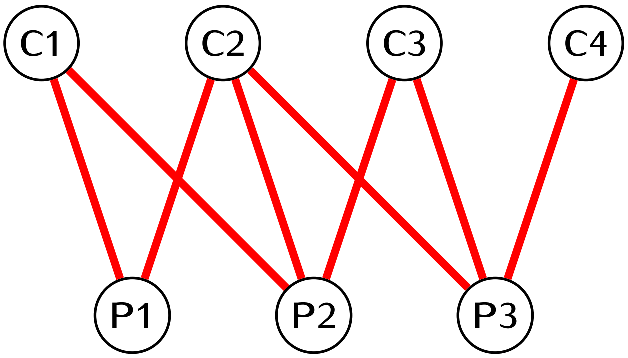
$$A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix}$$

is totally unimodular and $b_i \in \mathbb{Z}$ for $i = 1, \ldots, m$ then values of $x_1, \ldots, x_n$ for every basic feasible solution of the linear program are integers. In particular, if the linear program has a maximum, then there is a solution with integer values which gives this maximum.

Proposition

For any bipartite graph $G = (V_1 \cup V_2, E)$ the incidence matrix of G is totally unimodular.

Example.



$$\begin{array}{c} C1 \\ C2 \\ C3 \\ C4 \\ P1 \\ P2 \\ P3 \end{array} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Proposition

If A is a totally unimodular matrix and B is a matrix obtained by appending to A by a column that that has only one non-zero entry equal to 1, then B is totally unimodular.

Corollary

The simplex method always gives a solution to the assignment problem that consists of integers.