Recall:

Definition

Let A be an $n \times n$ matrix. If $\mathbf{v} \in \mathbb{R}^n$ is a non-zero vector and λ is a scalar such that
$$A\mathbf{v} = \lambda\mathbf{v}$$
then we say that

- λ is an *eigenvalue* of A

- $\mathbf{v}$ is an *eigenvector* of A corresponding to λ.

Example.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $A \cdot v = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3v$

Thus v is an eigenvector of A corresponding to the eigenvalue $\lambda = 3$.

$w = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ $A \cdot w = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 2 \end{bmatrix} = -2 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = 1 \cdot w$

So w is an eigenvector of A corresponding to the eigenvalue $\lambda = 1$.

Note. Eigenvectors corresponding to a given eigenvalue λ form a subspace of $\mathbb{R}^n$ which is called the *eigenspace* corresponding to the eigenvalue λ.

Computation of eigenvalues

Notation. $I_n :=$ the $n \times n$ identity matrix.

Definition

If A is an $n \times n$ matrix then

$$P(\lambda) = \det(A - \lambda I_n)$$

is a polynomial of degree n. $P(\lambda)$ is called the *characteristic polynomial* of the matrix A.

Proposition

If A is a square matrix then

$$\text{eigenvalues of } A = \text{roots of } P(\lambda)$$

Example.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$P(\lambda) = \det\left(\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right) = \det \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

$$= -\lambda^3 + 7\lambda^2 - 11\lambda + 5$$

$$\left(\text{Eigenvalues of } A \right) = \left(\text{roots of } P(\lambda) \right) = \left(\lambda_1 = 1, \lambda_2 = 5 \right)$$

Computation of eigenvectors

> ### Proposition
>
> If λ is an eigenvalue of an $n \times n$ matrix A then
>
> $$\begin{Bmatrix} \text{eigenvectors of } A \\ \text{corresponding to } \lambda \end{Bmatrix} = \begin{Bmatrix} \text{vectors in} \\ \text{Nul}(A - \lambda I_n) \end{Bmatrix}$$

Example.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$