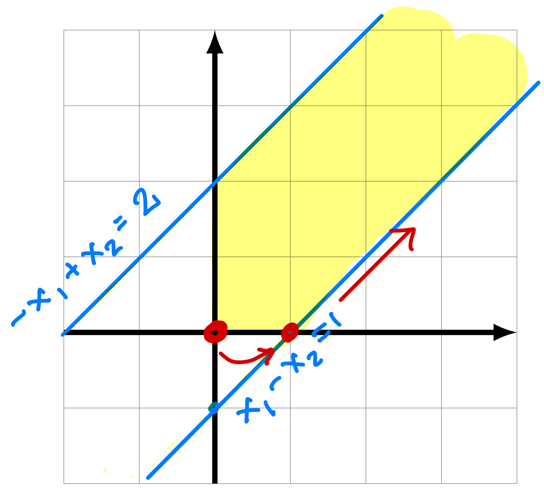
Exception handling: unboundedness

Example. Maximize

$$z = x_1$$

subject to:

$$\begin{aligned} x_1 - x_2 &\leq 1 \\ -x_1 + x_2 &\leq 2 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$



How to detect unboundedness using the simplex method:

The equality form:

$$x_1 - x_2 + s_1 = 1$$
$$-x_2 + x_2 + s_2 = 2$$
$$z = x_1 + 0x_2 + 0 \cdot s_1 + 0 \cdot s_2$$

Tableau:

x_1	x_2	s_1	s_2		
1	-1	1	0	1	s_1
-1	1	0	1	2	s_2
1	0	0	0	z	

The basic feasible solution

free: $x_1 = 0$, $x_2 = 0$

basic: $s_1 = 1$, $s_2 = 2$

$z = 0$

Pivot step: increase x_1, keep $x_2 = 0$

$x_1 + s_1 = 1 \Rightarrow s_1 = 1 - x_1 \geq 0$ so: $x_1 \leq 1$

$-x_1 + s_2 = 2 \Rightarrow s_2 = 2 + x_2 \geq 0$ - no restriction

Upshot: x_1 becomes basic $(x_1 = 1)$

s_1 becomes free

x_1	x_2	s_1	s_2		
$\boxed{1}$	-1	1	0	1	s_1
-1	1	0	1	2	s_2
1	0	0	0	z	

(1) (−1)

$\longrightarrow$

x_1	x_2	s_1	s_2		
1	-1	1	0	1	x_1
0	0	1	1	3	s_2
0	1	-1	0	$z-1$	

The new basic feasible solution:

free: $x_2 = 0$, $s_1 = 0$

basic: $x_1 = 1$, $s_2 = 3$

The pivot step: increase x_2, keep $s_1 = 0$.

$$x_1 - x_2 = 1 \Rightarrow x_1 = 1 + x_2 \geq 0 \quad \text{- no restrictions}$$

$$s_2 = 3 \geq 0 \quad \text{- no restrictions}$$

Upshot: We can make z as large as we want by increasing x_2 - there is no maximum

x_1	$\boxed{x_2}$	s_1	s_2		
1	-1	1	0	1	x_1
0	0	1	1	3	s_2
0	1	-1	0	$z-1$	

Note: Unboundedness happens if we have a free variable with a positive coeff. in the objective function and all constraint coefficients ≤ 0.

Exception handling: degeneracy

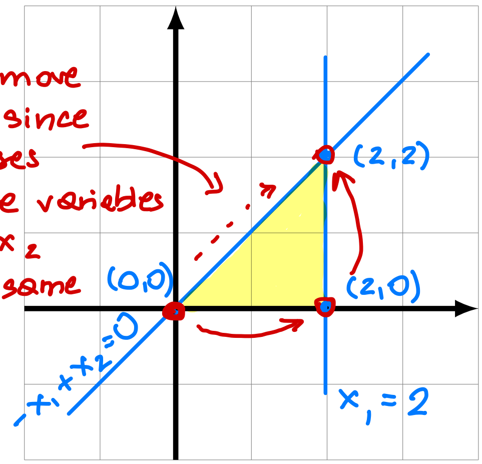
Example. Maximize

$$z = x_2$$

subject to:

$$\begin{aligned} -x_1 + x_2 &\leq 0 \\ x_1 &\leq 2 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$

We can't move this way since it increases both free variables x_1 and x_2 at the same time.



The equality form:

$$-x_1 + x_2 + s_1 = 0$$
$$x_1 + s_2 = 2$$
$$z = 0x_1 + 1\cdot x_2 + 0s_1 + 0s_2$$

Tableau:

x_1	x_2	s_1	s_2		
-1	1	1	0	0	s_1
1	0	0	1	2	s_2
0	1	0	0	z	

The basic feasible solution:

free: $x_1 = 0$, $x_2 = 0$

basic: $s_1 = 0$, $s_2 = 2$

← a degenerate solution: one of the basic variables is 0.

The pivot step: increase x_2, keep $x_1 = 0$.

$x_2 + s_1 = 0 \Rightarrow s_1 = -x_2 \geqslant 0$ so $x_2 = 0$ ← can't increase it

The only remaining option: increase x_1 even though this will not change the value of z:

$-x_1 + s_1 = 0 \Rightarrow s_1 = x_1 \geqslant 0$ – no restrictions

$x_1 + s_2 = 2 \Rightarrow s_2 = 2 - x_1 \geqslant 0$ so: $x_1 \leqslant 2$

Upshot: x_1 becomes basic
s_2 becomes free

A degenerate pivot step

Degeneracy continued...

$$(1)\quad \begin{array}{cccc|c|c} x_1 & x_2 & s_1 & s_2 & & \\ \hline -1 & 1 & 1 & 0 & 0 & s_1 \\ \boxed{1} & 0 & 0 & 1 & 2 & s_2 \\ \hline 0 & 1 & 0 & 0 & z & \end{array} \longrightarrow \begin{array}{cccc|c|c} x_1 & x_2 & s_1 & s_2 & & \\ \hline 0 & 1 & 1 & 1 & 2 & s_1 \\ 1 & 0 & 0 & 1 & 2 & x_1 \\ \hline 0 & 1 & 0 & 0 & z & \end{array}$$

The new basic feasible solution:

free: $x_2 = 0$, $s_2 = 0$

basic: $s_1 = 2$, $x_1 = 2$

The pivot step: increase x_2, keep $s_2 = 0$

$x_2 + s_1 = 2 \Rightarrow s_1 = 2 - x_2 \geq 0$ so: $x_2 \leq 2$

$x_1 = 2 \geq 0$ – no restrictions

Upshot: x_2 becomes basic, $(x_2 = 2)$

s_1 becomes free

$$(-1)\quad \begin{array}{cccc|c|c} x_1 & x_2 & s_1 & s_2 & & \\ \hline 0 & \boxed{1} & 1 & 1 & 2 & s_1 \\ 1 & 0 & 0 & 1 & 2 & x_1 \\ \hline 0 & 1 & 0 & 0 & z & \end{array} \longrightarrow \begin{array}{cccc|c} x_1 & x_2 & s_1 & s_2 & \\ \hline 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 1 & 2 \\ \hline 0 & 0 & -1 & -1 & z-2 \end{array}$$

all coeff. ≤ 0 so we reached the maximum

Maximum at $x_1 = 2, x_2 = 2$

$z = 2$

Upshot: Degenerate basic feasible solutions can lead to pivot steps that do not increase the objective function