4 | Dependence on the Basepoint

4.1 Proposition. *Let X be a space, let $x_0 \in X$, and let $Y \subseteq X$ be the path connected component of x_0. If $i\colon Y \to X$ is the inclusion map then the induced homomorphism*

$$i_*\colon \pi_1(Y, x_0) \to \pi_1(X, x_0)$$

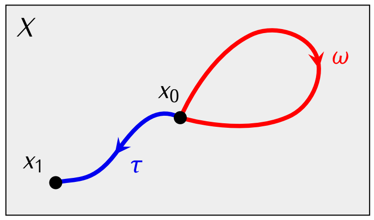
is an isomorphism of groups.

Proof. Exercise. $\square$

4.2 Proposition. *Let X be a space and let $x_0, x_1 \in X$. For any path $\tau\colon [0,1] \to X$ such that $\tau(0) = x_0$ and $\tau(1) = x_1$ the function*

$$s_\tau\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$$

given by $s_\tau([\omega]) = [\bar{\tau} * \omega * \tau]$ is an isomorphism of groups.



Proof. Exercise. □

4.3 Corollary. *If X is a space and $x_0, x_1 \in X$ are points than belong to the same path connected component of X then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.*

4.6 Proposition. *Let $x_0, x_1 \in X$ and let $f \colon X \to Y$ be a continuous function. Given a path τ in X such that $\tau(0) = x_0$ and $\tau(1) = x_1$ consider the isomorphisms $s_\tau \colon \pi_1(X, x_0) \to \pi_1(X, x_1)$ and $s_{f\tau} \colon \pi_1(Y, f(x_0)) \to \pi_1(Y, f(x_1))$ defined as in Proposition 4.2. Then following diagram commutes:*

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "f_*"] \arrow[d, "s_\tau"', "\cong"] & \pi_1(Y, f(x_0)) \arrow[d, "\cong"', "s_{f\tau}"] \\
\pi_1(X, x_1) \arrow[r, "f_*"'] & \pi_1(Y, f(x_1))
\end{tikzcd}$$

Proof. Exercise. □

4.7 Corollary. *Let X be a path connected space, $x_0, x_1 \in X$, and let $f \colon X \to Y$ be a continuous function. The homomorphism $f_* \colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism (or is the trivial homomorphism or is 1-1 or onto) if and only if the homomorphism $f_* \colon \pi_1(X, x_1) \to \pi_1(Y, f(x_1))$ has the same property.*

Fundamental groupoid $\Pi_1(X)$

4.10 Corollary. *The assignments $X \mapsto \Pi_1(X)$ and $f \mapsto f_*$ define a functor*

$$\Pi_1 \colon \mathbf{Top} \to \mathbf{Cat}$$

from the category of unpointed topological spaces to the category of small categories

Proof. Exercise. $\square$