Systems of linear equations

$$\begin{cases} 2x_1 + 6x_2 - 6x_3 - 2x_4 = -4 \\ 4x_2 - 8x_3 = \ 4 \\ 2x_1 + 7x_2 - 8x_3 = -1 \end{cases}$$

matrix equation $\longrightarrow$

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 2 & 6 & -6 & -2 \\ 0 & 4 & -8 & 0 \\ 2 & 7 & -8 & 0 \end{bmatrix} \end{array} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ -1 \end{bmatrix}$$

$$A \cdot x = b$$

Row reduction

$$\left[\begin{array}{cccc|c} 2 & 6 & -6 & -2 & -4 \\ 0 & 4 & -8 & 0 & 4 \\ 2 & 7 & -8 & 0 & -1 \end{array}\right] \cdot\left(\tfrac{1}{2}\right) \longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & -3 & -1 & -2 \\ 0 & 4 & -8 & 0 & 4 \\ 2 & 7 & -8 & 0 & -1 \end{array}\right] (-2) \longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & -3 & -1 & -2 \\ 0 & 4 & -8 & 0 & 4 \\ 0 & 1 & -2 & 2 & 3 \end{array}\right] \cdot\left(\tfrac{1}{4}\right)$$

augmented matrix

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & -3 & -1 & -2 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 1 & -2 & 2 & 3 \end{array}\right] (-1) \longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & -3 & -1 & -2 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 2 \end{array}\right] \cdot\left(\tfrac{1}{2}\right) \longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & -3 & -1 & -2 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] (1)$$

matrix in echelon form

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & -3 & 0 & -1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] (-3) \longrightarrow \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & -4 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \end{array}$$

matrix in the reduced echelon form

Solutions:

$$\begin{cases} x_1 = -4 - 3x_3 \\ x_2 = 1 + 2x_3 \\ x_4 = 1 \\ x_3 = \text{free} \end{cases}$$

x_3 is a free variable

x_1, x_2, x_4 are basic variables

Note.

- A consistent system of equations with free variables has infinitely many solutions
- Once we fix values of the free variables, the values of the basic variables are uniquely determined.

Pivoting

Pivoting is an operation that lets us modify which variables are basic and which are free.

$$\begin{cases} x_1 = -4 - 3x_3 \\ x_2 = 1 + 2x_3 \\ x_4 = 1 \\ x_3 = \text{free} \end{cases}$$

$\mapsto$ $3x_3 = -4 - x_1 \Rightarrow x_3 = -\frac{4}{3} - \frac{1}{3}x_1$

$x_2 = 1 + 2\left(-\frac{4}{3} - \frac{1}{3}x_1\right) = -\frac{5}{3} - \frac{2}{3}x_1$

$x_4 = 1$

$$\begin{cases} x_3 = -\frac{4}{3} - \frac{1}{3}x_1 \\ x_2 = -\frac{5}{3} - \frac{2}{3}x_1 \\ x_4 = 1 \\ x_1 = \text{free} \end{cases}$$

x_2, x_3, x_4 - basic variables

x_1 - free variable

In matrix terms:

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} \boxed{1} & 0 & \boxed{3} & 0 & -4 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \cdot \left(\tfrac{1}{3}\right) \\ \begin{array}{cccc} \underbrace{\quad}_{\text{make free}} & & \underbrace{\quad}_{\text{make basic}} & \end{array} \end{array} \longrightarrow \left[\begin{array}{cccc|c} \frac{1}{3} & 0 & 1 & 0 & -\frac{4}{3} \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] (2)$$

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} \frac{1}{3} & 0 & 1 & 0 & -\frac{4}{3} \\ \frac{2}{3} & 1 & 0 & 0 & -\frac{5}{3} \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \end{array} \rightarrow \begin{cases} x_3 = -\frac{4}{3} - \frac{1}{3}x_1 \\ x_2 = -\frac{5}{3} - \frac{2}{3}x_1 \\ x_4 = 1 \\ x_1 = \text{free} \end{cases}$$

Note.

- The columns of the matrix corresponding to basic variables are linearly independent.

e.g.:

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 2 & 6 & -6 & -2 & -4 \\ 0 & 4 & -8 & 0 & 4 \\ \underbrace{2}_{v_1} & \underbrace{7}_{v_2} & \underbrace{-8}_{v_3} & \underbrace{0}_{v_4} & -1 \end{array}\right]$$

The variables x_1, x_2, x_3 cannot be all basic at the same time, since their corresponding columns are not linearly independent: $v_3 = 3v_1 - 2v_2$

- The number basic (and free variables) does not depend on which variables are basic and which are free.

(number of lin. indep. columns in a matrix)
= (the dimension of the column space of the matrix)
= (the rank of the matrix)

Definition

We will say that an $m \times n$ matrix is in a *basic form* if it contains m columns that correspond to the columns of the $m \times m$ identity matrix.

e.g.:

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & -4 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \qquad \left[\begin{array}{cccc|c} \frac{1}{3} & 0 & 1 & 0 & -\frac{4}{3} \\ \frac{2}{3} & 1 & 0 & 0 & -\frac{5}{3} \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Note. If a matrix A is in the basic form then in a matrix equation $A\mathbf{x} = \mathbf{b}$ the columns of A corresponding the columns of the identity matrix give basic variables, and the other columns correspond to free variables.