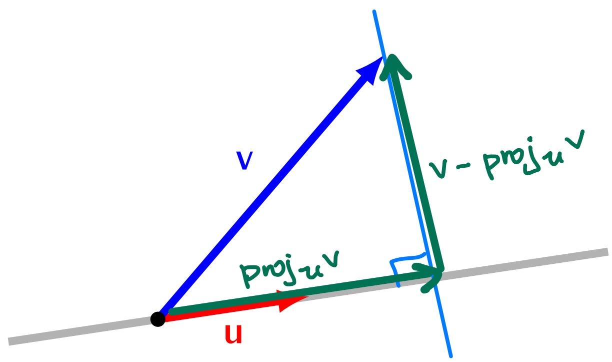
Definition

Given vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{u} \neq \mathbf{0}$, the *orthogonal projection* of $\mathbf{v}$ onto $\mathbf{u}$ is the vector $\text{proj}_{\mathbf{u}}\mathbf{v}$ such that

1) $\text{proj}_{\mathbf{u}}\mathbf{v} = c\mathbf{u}$ for some $c \in \mathbb{R}$
2) the vector $\mathbf{v} - \text{proj}_{\mathbf{u}}\mathbf{v}$ is orthogonal to $\mathbf{u}$.



Proposition

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $\mathbf{u} \neq \mathbf{0}$. For any $c \in \mathbb{R}$ we have

$$\text{dist}(v, \text{proj}_u v) = ||\mathbf{v} - \text{proj}_{\mathbf{u}}\mathbf{v}|| \leq ||\mathbf{v} - c\mathbf{u}|| = \text{dist}(v, cu)$$

Proof: $\|v - cu\|^2 = \|(v - \text{proj}_u v) - (cu - \text{proj}_u v)\|^2$

$$= \|v - \text{proj}_u v\|^2 + 2\cdot \underbrace{(v - \text{proj}_u v)\cdot(cu - \text{proj}_u v)}_{= 0} + \underbrace{\|cu - \text{proj}_u v\|^2}_{\geq 0}$$

$\leftarrow$ since $v - \text{proj}_u v$ is orthogonal to u

This gives: $\|v - cu\|^2 \geq \|v - \text{proj}_u v\|^2$

so: $\|v - cu\| \geq \|v - \text{proj}_u v\|$

Proposition

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $\mathbf{u} \neq \mathbf{0}$. Then $\text{proj}_{\mathbf{u}}\mathbf{v} = c\mathbf{u}$ where

$$c = \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}$$

In particular, if $||\mathbf{u}|| = 1$ then $\text{proj}_{\mathbf{u}}\mathbf{v} = c\mathbf{u}$ where

$$c = \mathbf{v} \cdot \mathbf{u}$$

Proof: By definition, if $\text{proj}_u v = cu$ then

$$u \cdot (v - \text{proj}_u v) = 0$$
$$u \cdot (v - cu) = 0$$
$$u \cdot v - c(u \cdot u) = 0$$
$$c = \frac{u \cdot v}{u \cdot u}$$

Example.

$$\mathbf{u} = \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$u \cdot u = \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = 1$

so $||u|| = 1$

$\text{proj}_u v = c \cdot u$

where $c = u \cdot v = \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = -\frac{1}{3}$

Corollary

If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $||\mathbf{u}|| = 1$ then

$$||\text{proj}_{\mathbf{u}}\mathbf{v}|| = |\mathbf{v} \cdot \mathbf{u}|$$