Example. Maximize

$$z = 3x_1 + x_2$$

subject to:

$$\begin{aligned} -x_1 + x_2 &\leq 1 \\ x_1 &\leq 3 \\ 2x_1 + x_2 &\leq 7 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$

Convert to the equality form:

Constraints:

$$\begin{aligned} -x_1 + x_2 + s_1 &= 1 \\ x_1 + s_2 &= 3 \\ 2x_1 + x_2 + s_3 &= 7 \\ x_1, x_2, s_1, s_2, s_3 &\geq 0 \end{aligned}$$

augmented matrix

free variables basic variables

$$\begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & s_3 & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 3 \\ 2 & 1 & 0 & 0 & 1 & 7 \end{array}\right]$$

We want to maximize

$$z = 3x_1 + x_2 + \underbrace{0s_1 + 0s_2 + 0s_3}$$

coefficients of basic variables are all 0.

Note: By setting the free variables to 0 we get a basic feasible solution:

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ s_1 = 1 \\ s_2 = 3 \\ s_3 = 7 \end{cases}$$

this gives : $z = 0$

Goal: Look for other basic feasible solutions that make the value of z larger.

Simplex tableau

$$\begin{array}{lccccc|cl}
 & x_1 & x_2 & s_1 & s_2 & s_3 & & \\
\text{constraints} & -1 & 1 & 1 & 0 & 0 & 1 & s_1 \\
 & 1 & 0 & 0 & 1 & 0 & 3 & s_2 \\
 & 2 & 1 & 0 & 0 & 1 & 7 & s_3 \\
\hline
\text{the objective function} & 3 & 1 & 0 & 0 & 0 & z &
\end{array}$$

The current basic feasible solution:

free $\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$

basic $\begin{cases} s_1 = 1 \\ s_2 = 3 \\ s_3 = 7 \end{cases}$

$\longmapsto \quad z = 0$

The pivot step:

1) We can increase the value of z by increasing either x_1 or x_2. Let's try to increase x_2, keep $x_1 = 0$.

2) Change of x_2 will affect the values of the basic variables s_1, s_2, s_3. These variables must stay ≥ 0, which restricts possible values of x_2.

Since by assumption $x_1 = 0$ we have:

$x_2 + s_1 = 1 \Rightarrow s_1 = 1 - x_2 \geq 0$ so: $x_2 \leq 1$

$0 \cdot x_2 + s_2 = 3 \Rightarrow s_2 = 3 \geq 0$ - no restriction on x_2

$x_2 + s_3 = 7 \Rightarrow s_3 = 7 - x_2 \geq 0$ so: $x_2 \leq 7$

All these conditions are satisfied if $x_2 \leq 1$.
For the biggest increase of z we set $x_2 = 1$ which makes $s_1 = 0$. Since we want free variables to have value 0, and basic variables to have non-zero values, we want to make x_2 into a basic variable and make s_1 free.

x_1	x_2	s_1	s_2	s_3	
-1	1	1	0	0	1
1	0	0	1	0	3
2	1	0	0	1	7
3	1	0	0	0	z

(-1) (-1)

$\longrightarrow$

free

x_1	x_2	s_1	s_2	s_3		
-1	1	1	0	0	1	x_2
1	0	0	1	0	3	s_2
3	0	-1	0	1	6	s_3
4	0	-1	0	0	$z-1$	

The new basic feasible solution

free: $x_1 = 0$, $s_1 = 0$

basic: $x_2 = 1$, $s_2 = 3$, $s_3 = 6$

$\longmapsto$ $z - 1 = 0$ (or $z = 1$)

The pivot step:

The free variables are x_1 and s_1.

Increasing s_1 will decrease z,

but increasing x_1 will increase z.

Thus we will increase x_1 while keeping $x_2, s_2, s_3 \geq 0$.

(Note: we keep $s_1 = 0$)

$-x_1 + x_2 = 1 \Rightarrow x_2 = 1 + x_1 \geq 0$ - no restriction on x_1

$x_1 + s_2 = 3 \Rightarrow s_2 = 3 - x_1 \geq 0$ so: $x_1 \leq 3$

$3x_1 + s_3 = 6 \Rightarrow s_3 = 6 - 3x_1 \geq 0$ so $3x_1 \leq 6$

$x_1 \leq 2$

The biggest value of x_1 satisfying all conditions is $x_1 = 2$

Then $s_3 = 0$, so s_3 becomes free and x_1 basic.

x_1	x_2	s_1	s_2	s_3	
-1	1	1	0	0	1
1	0	0	1	0	3
3	0	-1	0	1	6
4	0	-1	0	0	$z-1$

x_1	x_2	s_1	s_2	s_3	
0	1	$\frac{2}{3}$	0	$\frac{1}{3}$	3
0	0	$\frac{1}{3}$	1	$-\frac{1}{3}$	1
1	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	2
0	0	$\frac{1}{3}$	0	$-\frac{4}{3}$	$z - 9$

Geometric interpretation of the simplex method

Recall: Maximize

$$z = 3x_1 + x_2$$

subject to:

$$\begin{aligned} -x_1 + x_2 &\le 1 \\ x_1 &\le 3 \\ 2x_1 + x_2 &\le 7 \\ x_1,\ x_2 &\ge 0 \end{aligned}$$