Example. Maximize

$$z = 3x_1 + x_2$$

subject to:

$$\begin{aligned} -x_1 + x_2 &\leq 1 \\ x_1 &\leq 3 \\ 2x_1 + x_2 &\leq 7 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$

Convert to the equality form:

Constraints: $-x_1 + x_2 + s_1 = 1$
$x_1 + s_2 = 3$
$2x_1 + x_2 + s_3 = 7$
$x_1, x_2, s_1, s_2, s_3 \geq 0$

augmented matrix

free variables, basic variables

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 3 \\ 2 & 1 & 0 & 0 & 1 & 7 \end{array}\right]$$

We want to maximize

$$z = 3x_1 + x_2 + \underbrace{0s_1 + 0s_2 + 0s_3}$$

coefficients of basic variables are all 0.

Note: By setting the free variables to 0 we get a basic feasible solution:

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ s_1 = 1 \\ s_2 = 3 \\ s_3 = 7 \end{cases}$$

this gives : $z = 0$

Goal: Look for other basic feasible solutions that make the value of z larger.

Simplex tableau:

$$\begin{array}{l|ccccc|c|l}
 & x_1 & x_2 & s_1 & s_2 & s_3 & & \\
\hline
\text{constraints} & -1 & 1 & 1 & 0 & 0 & 1 & s_1 \\
 & 1 & 0 & 0 & 1 & 0 & 3 & s_2 \\
 & 2 & 1 & 0 & 0 & 1 & 7 & s_3 \\
\hline
\text{the objective function} & 3 & 1 & 0 & 0 & 0 & z &
\end{array}$$

The current basic feasible solution:

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ s_1 = 1 \\ s_2 = 3 \\ s_3 = 7 \end{cases} \mapsto \quad z = 0$$

The pivot step:

1) We can increase the value of z by increasing either x_1 or x_2. Let's try to increase x_2, keep $x_1 = 0$.

2) Change of x_2 will affect the values of the basic variables s_1, s_2, s_3. These variables must stay ≥ 0, which restricts possible values of x_2.

Since by assumption $x_1 = 0$ we have:

$$x_2 + s_1 = 1 \Rightarrow s_1 = 1 - x_2 \geq 0 \quad \text{so: } x_2 \leq 1$$

$$0 \cdot x_2 + s_2 = 3 \Rightarrow s_2 = 3 \geq 0 \quad \text{- no restriction on } x_2$$

$$x_2 + s_3 = 7 \Rightarrow s_3 = 7 - x_2 \geq 0 \quad \text{so: } x_2 \leq 7$$

All these conditions are satisfied if $x_2 \leq 1$. For the biggest increase of z we set $x_2 = 1$ which makes $s_1 = 0$. Since we want free variables to have value 0, and basic variables to have non-zero values, we want to make x_2 into a basic variable and make s_1 free.

$$
\begin{array}{ccccc|c}
x_1 & x_2 & s_1 & s_2 & s_3 & \\
\hline
-1 & \boxed{1} & \boxed{1} & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 3 \\
2 & 1 & 0 & 0 & 1 & 7 \\
\hline
3 & 1 & 0 & 0 & 0 & z
\end{array}
\quad \longrightarrow \quad
\begin{array}{ccccc|cc}
\overset{\text{free}}{x_1} & x_2 & \overset{\text{free}}{s_1} & s_2 & s_3 & & \\
\hline
-1 & 1 & 1 & 0 & 0 & 1 & x_2 \\
1 & 0 & 0 & 1 & 0 & 3 & s_2 \\
3 & 0 & -1 & 0 & 1 & 6 & s_3 \\
\hline
4 & 0 & -1 & 0 & 0 & z-1 &
\end{array}
$$

The new basic feasible solution

$$
\begin{array}{ll}
\text{free} & \left\{ \begin{array}{l} x_1 = 0 \\ s_1 = 0 \end{array} \right. \\
\text{basic} & \left\{ \begin{array}{l} x_2 = 1 \\ s_2 = 3 \\ s_3 = 6 \end{array} \right.
\end{array}
\longmapsto \quad z - 1 = 0 \quad (\text{or} \quad z = 1)
$$

The pivot step:

The free variable we can increase are x_1 and s_1.
Increasing s_1 will decrease the value of z,
but increasing x_1 will increase z.
Thus we will increase the value of x_1 while keeping $x_2, s_2, s_3 \geq 0$.
(Note: we keep $s_1 = 0$)

$$-x_1 + x_2 = 1 \Rightarrow x_2 = 1 + x_1 \geq 0 \quad \text{- no restriction on } x_1$$

$$x_1 + s_2 = 3 \Rightarrow s_2 = 3 - x_1 \geq 0 \quad \underline{\text{so}}: \quad x_1 \leq 3$$

$$3x_1 + s_3 = 6 \Rightarrow s_3 = 6 - 3x_1 \geq 0 \quad \underline{\text{so}} \quad 3x_1 \leq 6, \quad x_1 \leq 2$$

The biggest value of x_1 satisfying all conditions is $x_1 = 2$.
Then $s_3 = 0$, so s_3 becomes a free variable and x_1 becomes basic.

$$\begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & \\ \hline -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 3 \\ \boxed{3} & 0 & -1 & 0 & \boxed{1} & 6 \\ \hline 4 & 0 & -1 & 0 & 0 & z-1 \end{array} \quad \left(\tfrac{1}{3}\right)\cdot \text{ (row 3)} \longrightarrow \quad \begin{array}{ccccc|c} x_1 & x_2 & s_1 & s_2 & s_3 & \\ \hline -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 3 \\ \boxed{1} & 0 & -\frac{1}{3} & 0 & \frac{1}{3} & 2 \\ \hline 4 & 0 & -1 & 0 & 0 & z-1 \end{array}$$

(1), (-1), (-4)

The new basic feasible solution:

free
$$s_1 = 0, \quad s_3 = 0$$
basic
$$x_2 = 3, \quad s_2 = 1, \quad x_1 = 2$$

$$\longmapsto \quad z - 9 = 0 \quad (\text{or } z = 9)$$

free

$$\begin{array}{ccccc|cc} x_1 & x_2 & s_1 & s_2 & s_3 & & \\ \hline 0 & 1 & 2/3 & 0 & 1/3 & 3 & x_2 \\ 0 & 0 & 1/3 & 1 & -1/3 & 1 & s_2 \\ 1 & 0 & -1/3 & 0 & 1/3 & 2 & x_1 \\ \hline 0 & 0 & 1/3 & 0 & -4/3 & z-9 & \end{array}$$

The pivot step:

We can increase z by increasing s_1 (and keeping $s_3 = 0$)

We have:

$$x_2 + \tfrac{2}{3}s_1 = 3 \Rightarrow x_2 = 3 - \tfrac{2}{3}s_1 \geq 0 \quad \underline{\text{so:}} \quad \tfrac{2}{3}s_1 \leq 3, \quad s_1 \leq \tfrac{9}{2}$$

$$\tfrac{1}{3}s_1 + s_2 = 1 \Rightarrow s_2 = 1 - \tfrac{1}{3}s_1 \geq 0 \quad \underline{\text{so:}} \quad \tfrac{1}{3}s_1 \leq 1, \quad s_1 \leq 3$$

$$x_1 - \tfrac{1}{3}s_1 = 2 \Rightarrow x_1 = 2 + \tfrac{1}{3}s_1 \geq 0 \quad \text{no restiction on } s_1$$

The largest value of s_1 satisfying all conditions is $s_1 = 3$. Then $s_2 = 0$, so s_2 becomes a free variable and s_1 basic

3.

x_1	x_2	s_1	s_2	s_3	
0	1	$2/3$	0	$1/3$	3
0	0	$\boxed{1/3}$	$\boxed{1}$	$-1/3$	1
1	0	$-1/3$	0	$1/3$	2
0	0	$1/3$	0	$-4/3$	$z-9$

$\longrightarrow$ $(-\frac{2}{3})$ $(\frac{1}{3})$ $(-\frac{1}{3})$

x_1	x_2	s_1	s_2	s_3	
0	1	$2/3$	0	$1/3$	3
0	0	$\boxed{1}$	3	-1	3
1	0	$-1/3$	0	$1/3$	2
0	0	$1/3$	0	$-4/3$	$z-9$

$\downarrow$

free

x_1	x_2	s_1	s_2	s_3		
0	1	0	-2	1	1	x_2
0	0	1	3	-1	3	s_1
1	0	0	1	0	3	x_1
0	0	0	-1	-1	$z-10$	

The new basic feasible solution

free $\begin{cases} s_2 = 0 \\ s_3 = 0 \end{cases}$

basic $\begin{cases} x_2 = 1 \\ s_1 = 3 \\ x_1 = 3 \end{cases}$

$\longmapsto$ $z - 10 = 0$ (or: $z = 10$)

Note: The objective function depends on s_2 and s_3 only, but increasing these variables will make z smaller. This means that we can't increase the value of z anymore, we reached the maximum.

The final solution of the linear program:

The maximum value is $z = 10$.
It is obtained for $x_1 = 3$, $x_2 = 1$.

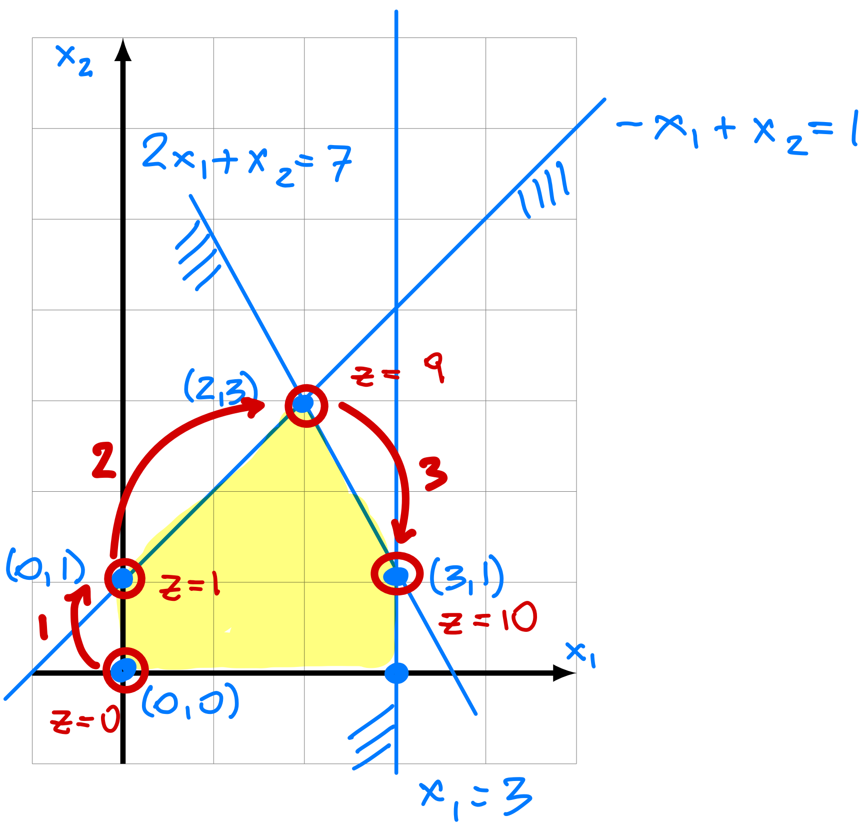
Geometric interpretation of the simplex method

Recall: Maximize

$$z = 3x_1 + x_2$$

subject to:

$$\begin{aligned} -x_1 + x_2 &\leq 1 \\ x_1 &\leq 3 \\ 2x_1 + x_2 &\leq 7 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$



Note: It would have been more efficient to increase x_1 instead of x_2 in the first pivot step. In the simplex method there are various pivoting rules which aim to decide which free variable should be used at each step to get to the solution as quickly as possible.