Example 1. A farmer plans to plant two types of crops on a 100 acre farm: C_1 and C_2. It costs

- \$100 and 4 hour of labor to grow 1 acre of C_1
- \$300 and 1 hour of labor to grow 1 acre of C_1.

Each acre of C_1 will bring \$200 profit, and each acre of C_2 will bring \$100 profit. The farmer can spend up to \$27,000 on the production costs and up to 280 hours of labor. How many acres of each crop should be planted to maximize the profit?

Mathematical formulation (linear program)

decision variables

Unknown: x_1 = the number of acres of C_1
x_2 = the number of acres of C_2

We want to maximize the profit function:

the objective function

$$z = 200x_1 + 100x_2$$

Restrictions:

constraints

$$x_1 + x_2 \leq 100$$
$$100x_1 + 300x_2 \leq 27{,}000$$
$$4x_1 + 1 \cdot x_2 \leq 280$$

nonnegativity constraints

$$x_1 \geq 0$$
$$x_2 \geq 0$$

Example 2. A company manufacturing widgets has 2 factories and 3 warehouses. The cost of sending 1000 widgets from each factory to each warehouse is as follows:

	W_1	W_2	W_3
F_1	5	5	3
F_2	6	4	1

The factory $\mathbf{F_1}$ can produce up to 10,000 widgets per week and $\mathbf{F_2}$ up to 7,000 widgets per week. The warehouses must receive exactly 8,000 widgets per week for $\mathbf{W_1}$, 5,000 widgets per week for $\mathbf{W_2}$, and 2,000 widgets per week for $\mathbf{W_3}$.

How many widgets should be shipped each week from each factory to each warehouse to minimize the shipping costs?

Linear program

Decision variables:

x_{ij} = the number of widgets shipped from F_i to W_j
$(i = 1, 2, \ j = 1, 2, 3)$

The objective function to minimize:

$$z = 5x_{11} + 5x_{12} + 3x_{13} + 6x_{21} + 4x_{22} + 1 \cdot x_{23}$$

Constraints:

$$x_{11} + x_{12} + x_{13} \leq 10{,}000$$
$$x_{21} + x_{22} + x_{23} \leq 7{,}000$$
$$x_{11} + x_{21} = 8{,}000$$
$$x_{12} + x_{22} = 5{,}000$$
$$x_{13} + x_{23} = 2{,}000$$
$$x_{ij} \geq 0$$

The general form of a linear program

For the decision variables $x_1, \ldots, x_n$ find the minimum (or the maximum) of the objective function

$$z = c_1 x_1 + \ldots + c_n x_n$$

Subject to the constraints:

$$a_{i1}x_1 + \ldots + a_{in}x_n \begin{matrix} \leq \\ = \\ \geq \end{matrix} b_i$$

for $i = 1, \ldots, m$, and possibly $x_j \geq 0$ for $j = 1, \ldots, n$.

Sidenote: The growth of linear functions

Example: $f(x_1, x_2) = 3x_1 + 2x_2$ $\quad f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$

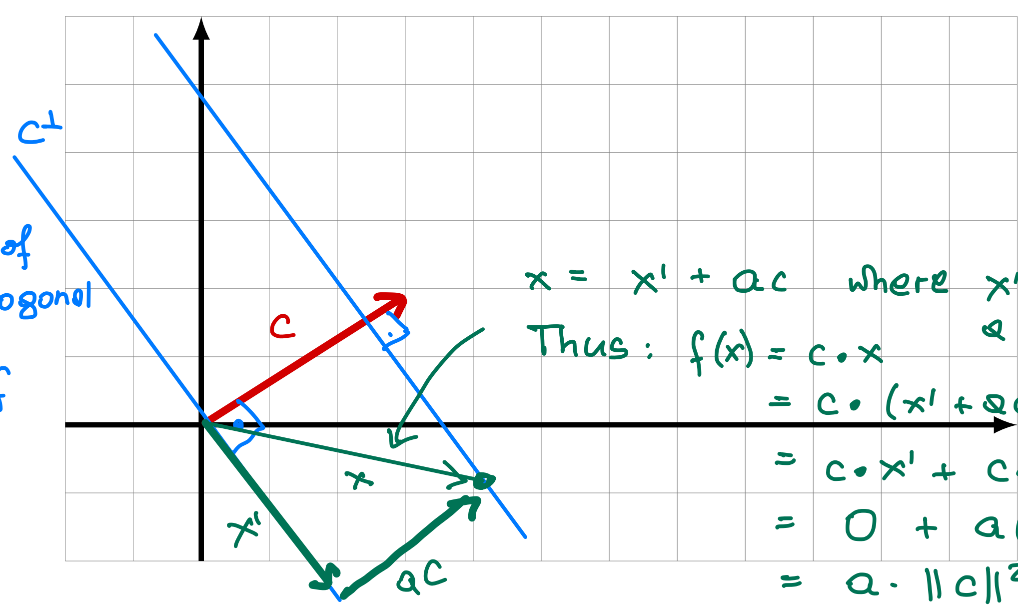
Denote: $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $c = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

We have: $f(x) = \underset{\text{the dot product}}{c \bullet x}$

$c^\perp$

the set of vectors orthogonal to c

$x \in c^\perp$ iff $c \bullet x = 0$



$x = x' + ac$ where $x' \in c^\perp$, $a \in \mathbb{R}$

Thus: $f(x) = c \bullet x$
$= c \bullet (x' + ac)$
$= c \bullet x' + c \bullet (ac)$
$= 0 + a(c \bullet c)$
$= a \cdot \|c\|^2$

Upshot: The values of the function increase as we move in the direction of the vector c, and decrease as we move in the opposite direction.

Back to Example 1

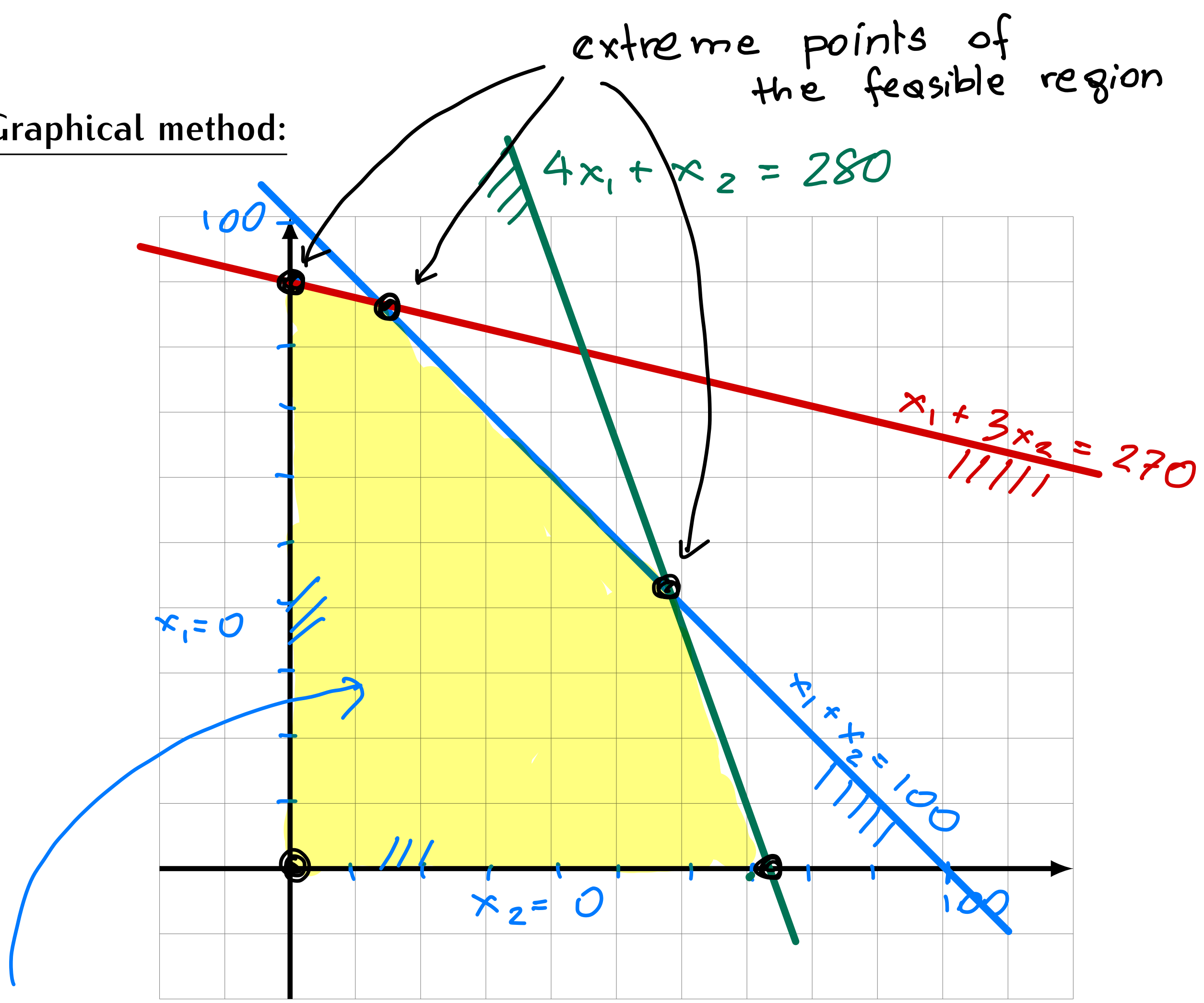
Maximize:

$$f(x_1, x_2) = 200x_1 + 100x_2$$

subject to the constraints:

$$\begin{aligned} x_1 + x_2 &\leq 100 \\ 100x_1 + 300x_2 &\leq 27000 \qquad x_1 + 3x_2 \leq 270 \\ 4x_1 + x_2 &\leq 280 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

Graphical method:

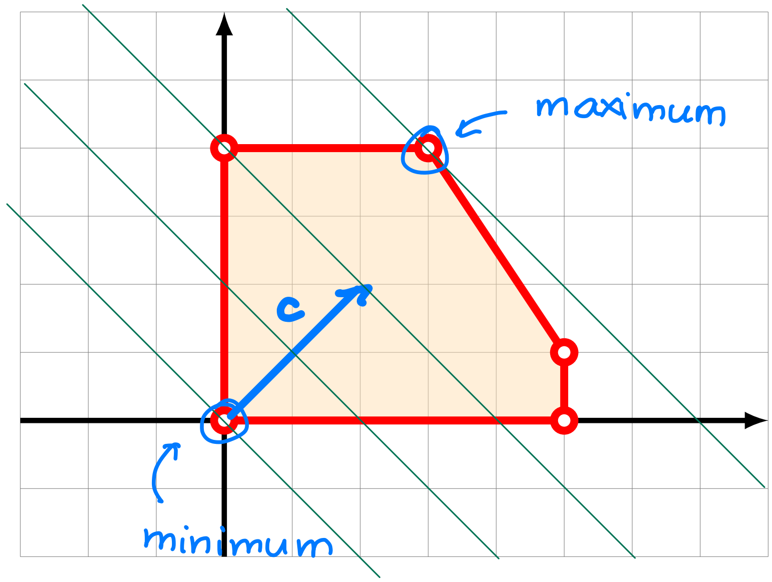


feasible region: consists of points (x_1, x_2) that satisfy all constraints. Such points are called feasible solutions

Fact

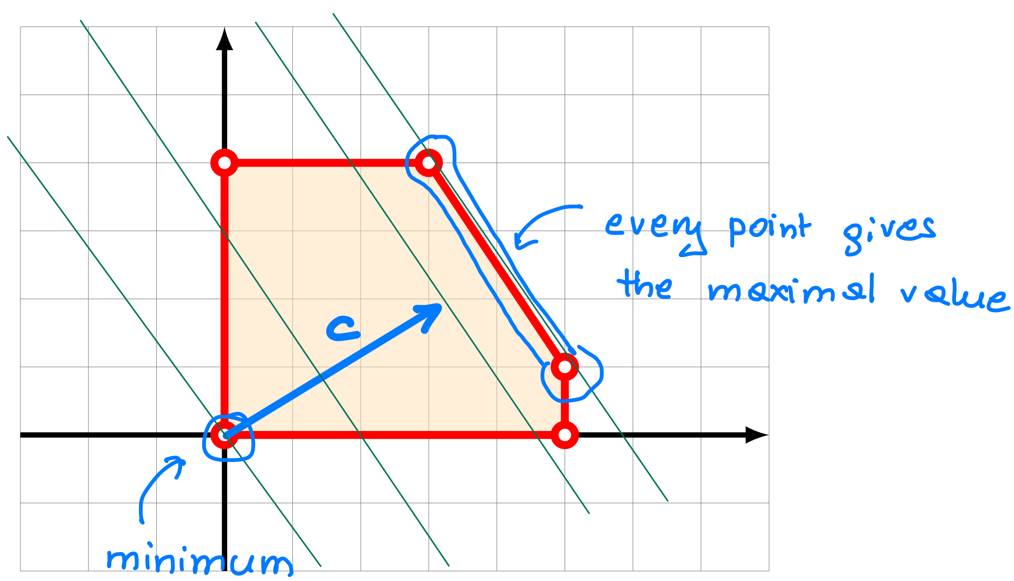
The maximum and the minimum of the objective function, if it exists, always occurs in one of extreme points of the feasible region.

$f(x) = c \cdot x$



Note. It may happen that the objective function assumes the minimum or the maximum in infinitely many points, but even then some of them will be extreme points.

$f(x) = c \cdot x$

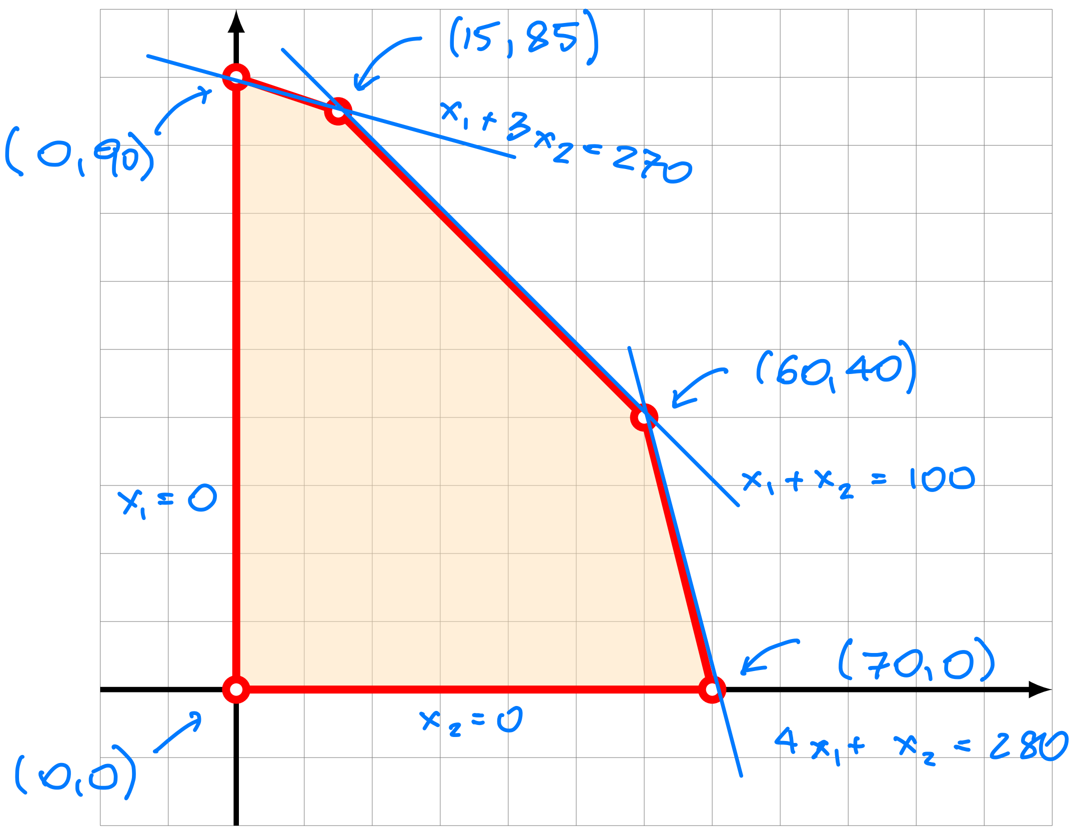


Upshot. A linear program can be solved as follows:

- Find coordinates of all extreme points of the feasible region.
- Compute the value of the objective function at each extreme point.
- The point with the biggest value is the maximum, the point with the smallest value is the minimum.

Back to Example 1:

$$f(x_1, x_2) = 200x_1 + 100x_2$$



$f(0,0) = 0$

$f(70,0) = 14{,}000$

$f(0,90) = 9{,}000$

$f(15,85) = 11{,}500$

$\underbrace{f(60,40) = 16{,}000}_{\uparrow \text{ maximum}}$

Problem. In practical applications there are too many extreme points to compute all of them.