

8 | From Maps to Fibrations

As we have seen any fibration $F \rightarrow E \xrightarrow{p} B$ has the associated long exact sequence

$$\cdots \rightarrow \pi_n(F) \longrightarrow \pi_n(E) \xrightarrow{p_*} \pi_n(B) \rightarrow \cdots$$

that relates the homotopy groups of the spaces B , E , and F . The main goal of this chapter is to show that this approach to computing homotopy groups can be used with an arbitrary map $f: X \rightarrow Y$ taken in place of a fibration p . We will show that the following holds:

8.1 Theorem. *Given any map $f: X \rightarrow Y$ there exists a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{g_f} & E_f \\ & \searrow f & \swarrow p_f \\ & Y & \end{array}$$

such that $p_f: E_f \rightarrow Y$ is a Hurewicz fibration and $g: E_f \rightarrow X$ is a homotopy equivalence.

For $x_0 \in X$ and $e_0 = g_f(x_0) \in E_f$ we will get $\pi_n(X, x_0) \cong \pi_n(E_f, e_0)$ for all $n \geq 0$. In this way, the long exact sequence of a fibration gives an exact sequence

$$\cdots \rightarrow \pi_n(F, e_0) \longrightarrow \pi_n(X, x_0) \xrightarrow{f_*} \pi_n(Y, y_0) \rightarrow \cdots$$

where $F = p_f^{-1}(y_0)$.

8.2 Mapping spaces. For spaces X, Y , let $\text{Map}(X, Y)$ denote the set of all continuous functions $X \rightarrow Y$. For $A \subseteq X$ and $U \subseteq Y$ let $P(A, U) \subseteq \text{Map}(X, Y)$ be the set

$$P(A, U) = \{f \in \text{Map}(X, Y) \mid f(A) \subseteq U\}$$

8.3 Definition. The compact-open topology on $\text{Map}(X, Y)$ is the topology with subbasis given by all sets of the form $P(A, U)$ where $A \subseteq X$ is compact and $U \subseteq Y$ is open.

Let X, Y, Z be spaces. For a function $\varphi: Z \rightarrow \text{Map}(X, Y)$ denote by $\varphi^\sharp: Z \times X \rightarrow Y$ the function given by $\varphi^\sharp(z, x) = \varphi(z)(x)$. We will say that φ^\sharp is the *adjoint* of φ .

8.4 Theorem. If X is a locally compact Hausdorff space, then the compact-open topology on $\text{Map}(X, Y)$ is the unique topology with the property that a map $\varphi: Z \rightarrow \text{Map}(X, Y)$ is continuous if and only if $\varphi^\sharp: Z \times X \rightarrow Y$ is continuous.

8.5 All mapping spaces below are equipped with the compact-open topology. The following properties hold:

- 1) The evaluation map $\text{ev}: \text{Map}(X, Y) \times X \rightarrow Y$ given by $\text{ev}(f, x) = f(x)$ is continuous.
- 2) In particular for every $x_0 \in X$ the map $\text{ev}_{x_0}: \text{Map}(X, Y) \rightarrow Y$, $\text{ev}_{x_0}(f) = f(x_0)$ is continuous.
- 3) If $\{*\}$ is a one point space, then the map $\text{ev}_*: \text{Map}(\{*\}, Y) \rightarrow Y$ is a homeomorphism.
- 4) For any continuous function $f: X \rightarrow Y$ and any space Z the induced function $f_*: \text{Map}(Z, X) \rightarrow \text{Map}(Z, Y)$ given by $f_*(g) = f \circ g$ is continuous.
- 5) For any continuous function $f: X \rightarrow Y$ and any space Z the induced function $f^*: \text{Map}(Y, Z) \rightarrow \text{Map}(Y, X)$ given by $f^*(g) = g \circ f$ is continuous.
- 6) If Y is a locally compact Hausdorff space, then for any spaces X and Y the map $F: \text{Map}(X, Y) \times \text{Map}(Y, Z) \rightarrow \text{Map}(X, Z)$ given by $F(f, g) = g \circ f$ is continuous.

From now on, all mapping spaces will taken with the compact-open topology.

8.6 Example. Let X be a locally compact space and let $f, g: X \rightarrow Y$. Giving a map $\omega: [0, 1] \rightarrow \text{Map}(X, Y)$ such that $\omega(0) = f$ and $\omega(1) = g$ is equivalent to giving a homotopy $\omega^\sharp: X \times [0, 1] \rightarrow Y$ between f and g . In effect, homotopy classes of maps $X \rightarrow Y$ correspond to path connected components of the space $\text{Map}(X, Y)$.

8.7 Example. Let X be a space. The *path space* of X is the space $PX = \text{Map}([0, 1], X)$.

For $x_0 \in X$ consider the subspace of PX given by

$$\Omega_{x_0} X = \{\omega \in PX \mid \omega(0) = \omega(1) = x_0\}$$

This space is called the *loop space* of X based at x_0 . Denote by $c_{x_0} \in \Omega_{x_0} X$ the constant loop $c_{x_0}(t) = x_0$ for all $t \in [0, 1]$.

Notice that every element $\omega \in \Omega_{x_0} X$ represents an element of $\pi_1(X, x_0)$. Similarly as in Example 8.6 we also obtain that path connected components of $\Omega_{x_0} X$ correspond to homotopy classes of loops in X . In this way, the assignment $[\omega] \mapsto [\omega^\#]$ gives a bijection

$$\pi_0(\Omega_{x_0} X, c_{x_0}) \xrightarrow{\cong} \pi_1(X, x_0)$$

Concatenation of loops defines a map $\Omega_{x_0} X \times \Omega_{x_0} X \rightarrow \Omega_{x_0} X$ which, in turn, induces a map

$$\pi_0(\Omega_{x_0} X, c_{x_0}) \times \pi_0(\Omega_{x_0} X, c_{x_0}) \rightarrow \pi_0(\Omega_{x_0} X, c_{x_0})$$

This defines a group structure on $\pi_0(\Omega_{x_0} X, c_{x_0})$ such that the bijection $\pi_0(\Omega_{x_0} X, c_{x_0}) \cong \pi_1(X, x_0)$ becomes an isomorphism of groups.

Generalizing this, any element of $\pi_n(\Omega_{x_0} X, c_{x_0})$ is represented by a map $\omega: (I^n, \partial I^n) \rightarrow (\Omega_{x_0} X, c_{x_0})$. The adjoint of ω is a map $\omega^\#: I^n \times [0, 1] = I^{n+1} \rightarrow X$ such that $\omega^\#(\partial I^{n+1}) = x_0$. In other words, we obtain a map $\omega^\#: (I^{n+1}, \partial I^{n+1}) \rightarrow (X, x_0)$. It is easy to verify that maps $\omega_1, \omega_2: (I^n, \partial I^n) \rightarrow (\Omega_{x_0} X, c_{x_0})$ are homotopic if and only if their adjoints $\omega_1^\#, \omega_2^\#$ are homotopic. Thus the correspondence $[\omega] \mapsto [\omega^\#]$ defines a bijection

$$\pi_n(\Omega_{x_0} X, c_{x_0}) \xrightarrow{\cong} \pi_{n+1}(X, x_0)$$

One can check that this is an isomorphism of groups.

8.8 Example. Let $x_0 \in A \subseteq X$. Denote

$$P(X, A, x_0) = \{\omega: [0, 1] \rightarrow X \mid \omega(0) \in A, \omega(1) = x_0\}$$

Similarly as in Example 8.7, one can check that for any map $\omega: (I^n, \partial I^n) \rightarrow (P(X, A, x_0), c_{x_0})$ the adjoint $\omega^\#: I^{n+1} \rightarrow X$ represents an element $[\omega^\#] \in \pi_{n+1}(X, A, x_0)$. The assignment $[\omega] \mapsto [\omega^\#]$ gives an isomorphism

$$\pi_n(P(X, A, x_0)) \xrightarrow{\cong} \pi_{n+1}(X, A, x_0)$$

for any $n \geq 1$.

Let $f: X \rightarrow Y$ be a map, and let PY be the path space of Y . Define

$$E_f = \{(x, \omega) \in X \times PY \mid f(x) = \omega(0)\} \subseteq X \times PY$$

We have a map $r_f: PX \rightarrow E_f$ give by $r_f(\omega) = (\omega(0), f\omega)$

8.9 Proposition. *For a map $f: X \rightarrow Y$ the following conditions are equivalent:*

- 1) *The map f is a Hurewicz fibration.*
- 2) *The map f has the homotopy lifting property for the space E_f*
- 3) *There exists a map $s_f: E_f \rightarrow PX$ such that $r_f s_f = \text{id}_{E_f}$*

Proof. 1) \Rightarrow 2) Obvious.

2) \Rightarrow 3) Consider the following commutative diagram:

$$\begin{array}{ccc}
 E_f \times \{0\} & \xrightarrow{\bar{k}} & X \\
 \downarrow & \nearrow \bar{g} & \downarrow f \\
 E_f \times [0, 1] & \xrightarrow{g} & Y
 \end{array} \quad (*)$$

Here $\bar{k}((x, \omega), 0) = x$ and $g((x, \omega), t) = \omega(t)$. By 2) there exists a homotopy \bar{g} that commutes this the rest of the diagram. Take $s_f = \bar{g}^\sharp$, the adjoint of \bar{g} .

3) \Rightarrow 1) Assume that we have the following commutative diagram and we want to show that a homotopy lift \bar{h} exists:

$$\begin{array}{ccc}
 Z \times \{0\} & \xrightarrow{\bar{d}} & X \\
 \downarrow & \nearrow \bar{h} & \downarrow f \\
 Z \times [0, 1] & \xrightarrow{h} & Y
 \end{array}$$

For $z \in Z$ let $\omega_z: [0, 1] \rightarrow Y$ be the path given by $\omega_z(t) = h(z, t)$. Define a map $u: Z \rightarrow E_f$ by $u(z) = (\bar{k}(z, 0), \omega_z)$. Notice that, in the notation of diagram (*) we have $\bar{d} = \bar{k}(u \times \text{id}_{\{0\}})$ and $h = g(u \times \text{id}_{[0, 1]})$. As a consequence, we can take $\bar{h} = \bar{g}(u \times \text{id}_{[0, 1]})$. \square

Proof of Theorem 8.1. Let $f: X \rightarrow Y$ be a map. As before, define

$$E_f = \{(x, \omega) \in X \times PY \mid f(x) = \omega(0)\} \subseteq X \times PY$$

Let $g_f: X \rightarrow E_f$ be given by $g_f(x) = (x, c_x)$ where $c_x: [0, 1] \rightarrow X$ is the constant path at x . Also, let $p_f: E_f \rightarrow Y$ be given by $p_f(x, \omega) = \omega(1)$. We have $f = p_f g_f$.

We will show that g_f is a homotopy equivalence with the homotopy inverse given by the projection map $\text{pr}: E_f \rightarrow X$, $\text{pr}(x, \omega) = x$. We have $\text{pr} g_f = \text{id}_X$. The composition $g_f \text{pr}: E_f \rightarrow E_f$ is given by $g_f \text{pr}(x, \omega) = (x, c_x)$. A homotopy $h: g_f \text{pr} \simeq \text{id}_{E_f}$ is defined by $h((x, \omega), t) = (x, \omega_t(x))$, where $\omega_t: [0, 1] \rightarrow Y$, $\omega_t(s) = \omega(ts)$.

It remains to show that $p_f: E_f \rightarrow Y$ is a Hurewicz fibration. To see this, assume that we have a commutative diagram

$$\begin{array}{ccc}
 Z \times \{0\} & \xrightarrow{\bar{k}} & E_f \\
 \downarrow & \nearrow \bar{h} & \downarrow p_f \\
 Z \times [0, 1] & \xrightarrow{h} & Y
 \end{array}$$

Denote $\bar{k}(z) = (x_z, \omega_z)$. By commutativity of the diagram we have $\omega_z(1) = h_0(z)$. Then the lift \bar{h} can be defined by

$$\bar{h}(z, t) = (x_z, \tau_{z,t} * \omega_z)$$

where $\tau_{z,t} * \omega_z$ is the concatenation of ω_z with the path $\tau_{z,t}: [0, 1] \rightarrow Y$ given by $\tau_{z,t}(s) = h(z, st)$. \square

8.10 Definition. Let $f: X \rightarrow Y$ be a map, and let $p_f: E_f \rightarrow Y$ be the Hurewicz fibration associated to f , as in Theorem 8.1. The *homotopy fiber* of f over a point $y_0 \in Y$ is the space

$$\text{hofib}_{y_0} f = p_f^{-1}(y_0)$$

Explicitly:

$$\text{hofib}_{y_0} f = \{(x, \omega) \in X \times PY \mid f(x) = \omega(0), \omega(1) = y_0\}$$

8.11 Corollary. Let $f: X \rightarrow Y$ be map, $x_0 \in X$ and $y_0 = f(x_0) \in Y$. Denote $v_0 = (x_0, c_{y_0}) \in \text{hofib}_{y_0} f$. We have a long exact sequence of homotopy groups

$$\begin{aligned} \dots \longrightarrow \pi_{n+1}(Y, y_0) \longrightarrow \pi_n(\text{hofib}_{x_0} f, v_0) \longrightarrow \pi_n(X, x_0) \xrightarrow{f_*} \pi_n(Y, y_0) \longrightarrow \dots \\ \dots \xrightarrow{f_*} \pi_1(Y, y_0) \longrightarrow \pi_0(\text{hofib}_{x_0} f, v_0) \longrightarrow \pi_0(X, x_0) \xrightarrow{f_*} \pi_0(Y, y_0) \end{aligned}$$

8.12 Example. If $p: E \rightarrow B$ is a Hurewicz fibration, then for any $b_0 \in B$ we have a homotopy equivalence $\text{hofib}_{b_0} p \simeq p^{-1}(b_0)$ (exercise).

8.13 Example. Given a space X and $x_0 \in X$, consider a map $f: \{*\} \rightarrow X$, $f(*) = x_0$. Then

$$\begin{aligned} \text{hofib}_{x_0} f &= \{(*, \omega) \in \{*\} \times PX \mid \omega(0) = x_0 = \omega(1)\} \\ &\cong \{\omega \in PX \mid \omega(0) = x_0 = \omega(1)\} \\ &= \Omega_{x_0} X \end{aligned}$$

Since $\pi_n(\{*\}) = 0$ for all $n \geq 0$ the exact sequence becomes

$$\dots \longrightarrow 0 \longrightarrow \pi_n(X, x_0) \longrightarrow \pi_{n-1}(\Omega_{x_0} X, c_{x_0}) \longrightarrow 0 \longrightarrow \dots \longrightarrow 0 \longrightarrow \pi_1(X, x_0) \longrightarrow \pi_0(\Omega_{x_0} X, c_{x_0}) \longrightarrow 0$$

where $c_{x_0} \in \Omega_{x_0} X$ is the constant loop at x_0 . This recovers the isomorphisms $\pi_n(\Omega_{x_0} X, c_{x_0}) \cong \pi_{n+1}(X, x_0)$, which we obtained in Example 8.7

Notice that if x_1 belongs to a different path connected component of X than x_0 , then $\text{hofib}_{x_1} f = \emptyset$.

8.14 Example. The map $f: * \rightarrow X$ in Example 8.7 can be interpreted as an inclusion $\{x_0\} \hookrightarrow X$. Generalizing it, for $A \subseteq X$, consider the inclusion map $j: A \hookrightarrow X$. In this case we have

$$\begin{aligned} E_j &= \{(a, \omega) \in A \times PX \mid j(a) = \omega(0)\} \\ &\cong \{\omega \in P(X) \mid \omega(0) \in A\} \end{aligned}$$

For $x_0 \in X$ we get:

$$\begin{aligned} \text{hofib}_{x_0} j &= \{ \omega \in PX \mid \omega(0) \in A, \omega(1) = x_0 \} \\ &= P(X, A, x_0) \end{aligned}$$

Recall (8.8) that we have isomorphisms $\pi_n(P(X, A, x_0), c_{x_0}) \xrightarrow{\cong} \pi_{n+1}(X, A, x_0)$. They fit into a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & \pi_{n+1}(X, A, x_0) & \xrightarrow{\partial} & \pi_n(A, x_0) & \xrightarrow{j_*} & \pi_n(X, x_0) & \longrightarrow & \pi_n(X, A, x_0) & \longrightarrow & \dots \\ & & \uparrow \wr \parallel & & \uparrow \wr \parallel g_{j*} & & \uparrow \parallel & & \uparrow \wr \parallel & & \\ \dots & \longrightarrow & \pi_n(P(X, A, x_0), c_{x_0}) & \longrightarrow & \pi_n(E_j, c_{x_0}) & \xrightarrow{p_{j*}} & \pi_n(X, x_0) & \xrightarrow{\partial} & \pi_{n-1}(P(X, A, x_0), c_{x_0}) & \xrightarrow{i_*} & \dots \end{array}$$

Here $g_j: E_j \rightarrow A$ and $p_j: E_j \rightarrow X$ are given by $g_j(\omega) = \omega(0)$ and $p_j(\omega) = \omega(1)$.