## 15 | Weak Homotopy Type

A complication with studying weak equivalences is that two spaces can be related via a chain of weak equivalences even when there is no direct weak equivalence between them. For example, take  $X,Y\subseteq\mathbb{R}$  where X consist of all rational numbers and  $Y=\{\frac{1}{n}\mid n=1,2,\dots\}\cup\{0\}$ . Since every path connected component of X and Y consists of a single point,  $\pi_0(X)$  and  $\pi_0(Y)$  are countable sets and all higher homotopy groups are trivial. A weak equivalence  $X\to Y$  would need to be a continuous bijection in order to induce a bijection  $\pi_0(X)\to\pi_0(Y)$ . However, one can check that there is no such continuous bijection. By the same argument, there is no weak equivalence  $Y\to X$ . On the other hand, if we take the set of integers  $\mathbb Z$  with the discrete topology, then any bijections  $\mathbb Z\to X$  and  $\mathbb Z\to Y$  are continuous functions and they are weak equivalences. Thus the spaces X and Y are related by a chain of weak equivalences:

$$X \leftarrow \mathbb{Z} \rightarrow Y$$

This motivates the following definition:

**15.1 Definition.** Spaces X and Y are weakly equivalent (or have the same weak homotopy type) if they can be connected by a zigzag of weak equivalences

$$X = Z_0 \rightarrow Z_1 \leftarrow Z_2 \rightarrow \ldots \leftarrow Z_{n-1} \rightarrow Z_n = Y$$

**15.2 Proposition.** If X, Y are CW complexes then they are weakly equivalent if and only if they are homotopy equivalent.

*Proof.* Assume that X, Y are connected by a zigzag of n weak equivalences:

$$X = Z_0 \xrightarrow{f_1} Z_1 \xleftarrow{f_2} Z_2 \to \ldots \leftarrow Z_{n-1} \xrightarrow{f_n} Z_n = Y \tag{*}$$

We will show that  $X \simeq Y$  by induction with respect to n. If n = 1, then we have a weak equivalence  $X = Z_0 \to Z_1 = Y$ , which by Theorem 14.4 is a homotopy equivalence.

Assume that the statement is true for any zigzag consisting of n-1 or fewer weak equivalences and that X, Y are connected by a sequence (\*). By Corollary 14.8 the map  $f_{2*}\colon [X,Z_2]\to [X,Z_1]$  is a bijection. This means that there exists a map  $g\colon X\to Z_2$  such that  $f_2g\simeq f_1$ . By Proposition 14.3 the map g is a weak equivalence. Thus we obtain a zigzag of weak equivalences of the form:

$$X \xrightarrow{g} Z_2 \xrightarrow{f_3} Z_3 \leftarrow \ldots \leftarrow Z_{n-1} \xrightarrow{f_n} Z_n = Y$$

By the inductive assumption  $X \simeq Y$ .

For spaces that are not CW complexes, the study of their weak homotopy type can be simplified using the notion of a CW approximation.

**15.3 Definition.** A *CW approximation* of a space *X* is a CW complex *Y* together with a weak equivalence  $f: Y \to X$ .

More generally, a *CW approximation* of a pair (X, A) is a relative CW complex (Y, A) together with a weak equivalence  $f: Y \to X$  such that  $f|_A = \mathrm{id}_A$ .

Notice that a CW approximation of a space X is the same as a CW approximation of the pair  $(X, \emptyset)$ . We will show that the following holds:

**15.4 Theorem.** Any pair (X, A) has a CW approximation. Moreover, any two CW approximations for such a pair are homotopy equivalent.

**15.5 Corollary.** Spaces X, Y are weakly eqivalent if and only if there exists a space Z and weak equivalences  $X \leftarrow Z \rightarrow Y$ .

*Proof.* If such a space Z exists, then by definition X and Y are weakly equivalent. Conversely, assume that we have a zigzag of weak equivalences connecting X and Y:

$$X = Z_0 \xrightarrow{f_1} Z_1 \xleftarrow{f_2} Z_2 \rightarrow \ldots \leftarrow Z_{n-1} \xrightarrow{f_n} Z_n = Y$$

We can extend it to

$$X' \stackrel{g_X}{\rightarrow} X = Z_0 \stackrel{f_1}{\rightarrow} Z_1 \stackrel{f_2}{\leftarrow} Z_2 \rightarrow \ldots \leftarrow Z_{n-1} \stackrel{f_n}{\rightarrow} Z_n = Y \stackrel{g_Y}{\leftarrow} Y'$$

where  $g_X \colon X' \to X$  and  $g_Y \colon Y' \to Y$  are CW approximations of X and Y, respectively. By Proposition 15.2 there exists a homotopy equivalence  $h \colon X' \to Y'$ . Thus we obtain a diagram of weak equivalences:  $X \stackrel{g_X}{\longleftrightarrow} X' \stackrel{g_{Yh}}{\longleftrightarrow} Y$ .

*Proof of Theorem 15.4.* Assume first that X is a path connected space. For  $n=0,1,\ldots$  we will construct relative CW complexes  $(Y^{(n)},A)$  and maps  $f^{(n)}\colon Y^{(n)}\to X$  such that

- 1)  $Y^{(n)}$  is obtained from  $Y^{(n-1)}$  by attaching n-cells.
- 2)  $f^{(0)}|_A = \mathrm{id}_A$  and  $f^{(n)}|_{V^{(n-1)}} = f^{(n-1)}$
- 3)  $f_*^{(n)} : \pi_i(Y^{(n)}) \to \pi_i(X)$  is an isomorphism for i < n and epimorphism for i = n.

Then the map  $\bigcup_n f^{(n)} \colon \bigcup_n Y^{(n)} \to X$  will give a CW approximation of (X, A).

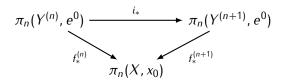
Let  $\{A_i\}_{i\in I}$  be path connected components of A. Also, let  $x_0 \in X$ . For each  $i \in I$  choose a point  $a_i \in A_i$ . Let  $(Y^{(1)}, A)$  be a 1-dimensional relative CW complex obtained by:

- adding to A a single 0-cell  $e^0$ ;
- for each  $i \in I$  adding to  $A \cup e^0$  a 1-cell  $e_i^1$  attached to the points  $e_0$  and  $a_i$ .
- for each element  $[\tau: (S^1, s_0) \to (X, x_0)] \in \pi_1(X, x_0)$  attaching to the resulting space a circle  $S^1_{\tau}$ , by identifying  $s_0$  with  $e^0$ .

Since X is path connected, for each  $i \in I$  there is a path  $\omega_i \colon [0,1] \to X$  such that  $\omega_i(0) = x_0$  and  $\omega_i(1) = a_i$ . Take a map  $f^{(1)} \colon Y^{(1)} \to X$  such that  $f^{(1)}(x) = x$  for all  $x \in A$ ,  $f^{(1)}(e^0) = x_0$ . Also,  $f^{(1)}$  maps each cell  $e^1_i$  using the path  $\omega_i$ , and each circle  $S^1_{\tau}$  using the map  $\tau$ . Notice that  $f^{(1)}_* \colon \pi_i(Y^{(1)}, e_0) \to \pi_i(X, x_0)$  is a bijection for i = 0 and it is onto for i = 1.

Next, assume that for  $i=1,\ldots,n$  we already constructed spaces  $Y^{(i)}$  and maps  $f^{(i)}\colon Y^{(i)}\to X$  satisfying conditions 1)-3). Take the epimorphism  $f_*^{(n)}\colon \pi_n(Y^{(n)},e^0)\to \pi_n(X,x_0)$ . Let  $\overline{Y}^{(n+1)}$  denote the space obtained by attaching to  $Y^{(n)}$  an (n+1)-cell  $e_\omega^{n+1}$  for each element  $[\omega\colon (S^n,s_0)\to (Y^{(n)},e^0)]\in \ker f_*^{(n)}$ , using  $\omega$  as the attaching map. Since  $[f^{(n)}\omega]=0$  in  $\pi_n(X,x_0)$ , the map  $f^{(n)}\omega\colon S^n\to X$  can be extended to a map  $D^{n+1}\to X$ . We can use this to extend  $f^{(n)}$  to a map  $\overline{f}^{(n+1)}\colon \overline{Y}^{(n+1)}\to X$ . Subsequently, take  $Y^{(n+1)}$  to be the space obtained by attaching to  $\overline{Y}^{(n+1)}$  a sphere  $S_\tau^{(n+1)}$  for each  $[\tau\colon (S^{n+1},s_0)\to (X,x_0)]\in \pi_{n+1}(X,x_0)$ , by identifying  $s_0$  with  $e^0$ . Extend  $\overline{f}^{(n+1)}$  to  $f^{(n+1)}\colon Y^{(n+1)}\to X$ , mapping  $S_\tau^{(n+1)}$  using  $\tau$ .

We have a commutative diagram



where  $i: Y^{(n)} \hookrightarrow Y^{(n+1)}$  is the inclusion map. Since  $f_*^{(n)}$  is onto, thus so is  $f_*^{(n+1)}$ . Also, by construction  $\ker f^{(n+1)} = 0$ . Therefore  $f_*^{(n+1)}: \pi_i(Y^{(n+1)}, e^0) \to \pi_i(X, x_0)$  is an isomorphism for  $i \leq n$  and it is an epimorphism for i = n+1.

Next, assume that X is not path connected and let  $\{X_i\}_{i\in I}$  be path connected components of X. Construct a CW approximation  $Y_i$  for each pair  $(X_i,A\cap X_i)$ , using the procedure described above. Then a CW approximation of (X,A) can be obtained by taking the quotient space  $A\sqcup \bigsqcup_{i\in I} Y_i/\sim$ , where the relation  $\sim$  identifies points of  $X_i\cap A\subseteq Y_i$  with the corresponding points of  $X_i$ .

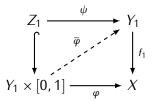
Finally, assume that for i = 1, 2 a map  $f_i: (Y_i, A) \to (X, A)$  is a CW approximation of (X, A). This gives

a commutative diagram



By Corollary 14.7 there exists  $g: Y_1 \to Y_2$  such that g(x) = x for all  $x \in A$  and  $f_2g \simeq f_1$  (rel A) By the same argument, there exists  $h: Y_2 \to Y_1$  such that h(x) = x for all  $x \in A$  and  $f_1h \simeq f_2$  (rel A). This shows that there exists a map  $\varphi: Y_1 \times [0,1] \to X$  which gives a homotopy  $f_1 \simeq f_1hg$  (rel A).

Consider the space  $Z_1 = Y_1 \times \{0,1\} \cup A \times [0,1] \subseteq Y_1 \times [0,1]$ . Then  $(Y_1 \times [0,1], Z_1)$  is a relative CW complex. We have a commutative diagram



where

$$\psi(y, t) = \begin{cases} y & \text{if } t < 1\\ hg(y) & \text{if } t = 1 \end{cases}$$

Using Corollary 14.7 again, we obtain that there exists  $\overline{\varphi}\colon Y_1\times [0,1]\to X$ , which gives a homotopy  $\mathrm{id}_{Y_1}\simeq hg$  (rel A). Analogously, we obtain that  $\mathrm{id}_{Y_2}\simeq gh$  (rel A).