19 Serre classes

19.1 Definition. A *Serre class* is a non-empty collection ${\mathfrak C}$ of abelian groups satisfying the property that if

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is a short exact sequence of abelian groups then $B \in \mathcal{C}$ if and only if $A, C \in \mathcal{C}$

We will say that a Serre class \mathcal{C} is a *Serre ring* if it in addition satisfies that if $A, B \in \mathcal{C}$ then $A \otimes B \in \mathcal{C}$ and $Tor(A, B) \in \mathcal{C}$.

- **19.2 Proposition.** Let $\mathcal C$ is a Serre class. The following hold:
 - 1) $0 \in \mathbb{C}$.
 - 2) If $A \in \mathcal{C}$ and $A' \cong A$ then $A' \in \mathcal{C}$.
 - 3) If $B \subseteq A$ then $A \in \mathcal{C}$ if and only if $B, A/B \in \mathcal{C}$.
 - 4) If $A_1, \ldots, A_n \in \mathbb{C}$ then $A_1 \oplus \ldots \oplus A_n \in \mathbb{C}$.
 - 5) If $A \to B \to C$ is an exact sequence and $A, C \in \mathcal{C}$ then $B \in \mathcal{C}$.

Proof. Exercise. □

19.3 Proposition. Let $\mathbb C$ is a Serre ring. If X is a path connected space such that $H_q(X) \in \mathbb C$ for all q > 0 and $H_q(X; G) \in \mathbb C$ for any group $G \in \mathbb C$.

Proof. By the Universal Coefficient Theorem we have

$$H_q(X; G) \cong (H_q(X) \otimes G) \oplus \text{Tor}(H_{q-1}(X), G)$$

For q = 0 this gives $H_0(X; G) \cong G$. For q = 1 we get $H_1(X; G) \cong H_1(X) \otimes G$.

- **19.4 Example.** All of the following are Serre rings:
 - C_{fin} = the class of all finite abelian groups.

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- \bullet $\, \mathcal{C}_{fg} =$ the class of all finitely generated abelian groups.
- \bullet $\mathcal{C}_{tor} =$ the class of all torsion abelian groups.
- **19.5 Definition.** Let \mathcal{C} be a Serre class. We say that homomorphism of abelian groups $f \colon G \to H$ is an *isomorphism mod* \mathcal{C} if $\operatorname{Ker} f \in \mathcal{C}$ and $\operatorname{Coker} f := H/f(G) \in \mathcal{C}$
- **19.6 Theorem.** Let \mathcal{C} be a Serre ring. If X is a simply connected space then the following conditions are equivalent:
 - 1) $\pi_n(X) \in \mathcal{C}$ for all $n \geq 1$
 - 2) $H_n(X) \in \mathcal{C}$ for all $n \geq 1$
- **19.7 Corollary.** The homotopy groups $\pi_n(S^m)$ are finitely generated for all $n, m \ge 1$.