

21 | Serre classes

The motivation for this chapter is to show that the following holds.

21.1 Theorem. *The homotopy groups $\pi_n(S^m)$ are finitely generated for all $n, m \geq 1$.*

This will follow from a more general result that will be stated in terms of Serre classes.

21.2 Definition. A *Serre class* is a non-empty collection \mathcal{C} of abelian groups satisfying the property that if

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is a short exact sequence of abelian groups then $B \in \mathcal{C}$ if and only if $A, C \in \mathcal{C}$

We will say that a Serre class \mathcal{C} is a *Serre ring* if in addition it satisfies that if $A, B \in \mathcal{C}$ then $A \otimes B \in \mathcal{C}$ and $\text{Tor}(A, B) \in \mathcal{C}$.

We will also say that a Serre class is *acyclic* if for every group $A \in \mathcal{C}$ we have $H_q(K(A, 1)) \in \mathcal{C}$ for all $q > 0$.

21.3 Proposition. *Let \mathcal{C} is a Serre class. The following hold:*

- 1) $0 \in \mathcal{C}$.
- 2) If $A \in \mathcal{C}$ and $A' \cong A$ then $A' \in \mathcal{C}$.
- 3) If $B \subseteq A$ then $A \in \mathcal{C}$ if and only if $B, A/B \in \mathcal{C}$.
- 4) If $A \rightarrow B \rightarrow C$ is an exact sequence and $A, C \in \mathcal{C}$ then $B \in \mathcal{C}$.
- 5) If $0 = A_{-1} \subseteq A_1 \subseteq A_2 \subseteq \dots \subseteq A_n$ then $A_n \in \mathcal{C}$ if and only if $A_i/A_{i-1} \in \mathcal{C}$ for all i .

Proof. Exercise. □

21.4 Proposition. *Let \mathcal{C} is a Serre ring. If X is a path connected space such that $H_q(X) \in \mathcal{C}$ for all $0 < q < p$ then $H_p(X; G) \in \mathcal{C}$ for any group $G \in \mathcal{C}$.*

Proof. By the Universal Coefficient Theorem we have

$$H_p(X; G) \cong (H_p(X) \otimes G) \oplus \text{Tor}(H_{p-1}(X), G)$$

This immediately gives that $H_p(X; G) \in \mathcal{C}$ for $p \geq 2$. For $p = 0$ we have $H_0(X; G) \cong G \in \mathcal{C}$ while for $p = 1$ we obtain $H_1(X; G) \cong H_1(X) \otimes G \in \mathcal{C}$. \square

21.5 Proposition. *All of the following are acyclic Serre rings:*

- \mathcal{C}_{fin} = the class of all finite abelian groups.
- \mathcal{C}_{fg} = the class of all finitely generated abelian groups.
- \mathcal{C}_{tor} = the class of all torsion abelian groups.
- \mathcal{C}_p = the class of all p -torsion abelian groups for a given prime p .

21.6 Theorem. *Let $F \rightarrow E \xrightarrow{p} B$ be a Serre fibration with a simply connected space B , and let \mathcal{C} be a Serre ring. If for two of the spaces F, E, B the homology groups $H_q(-)$ are in \mathcal{C} for all $q > 0$ then the same holds for the third space.*

Proof. There are three cases to consider.

Case 1: $H_q(F), H_q(B) \in \mathcal{C}$ for all $q > 0$.

Consider the Serre spectral sequence of the fibration p . We have $E_{p,q}^2 = H_p(B, H_q(F))$ so by Proposition 21.4 we get that $E_{p,q}^2 \in \mathcal{C}$ for all $(p, q) \neq (0, 0)$. Next, since groups $E_{p,q}^3$ are obtained by taking quotients of subgroups of the groups $E_{p,q}^2$, we get that $E_{p,q}^3 \in \mathcal{C}$ for all $(p, q) \neq (0, 0)$. Inductively, we obtain that $E_{p,q}^r \in \mathcal{C}$ for all $r \geq 2$ and $(p, q) \neq (0, 0)$, and so also $E_{p,q}^\infty \in \mathcal{C}$ for $(p, q) \neq (0, 0)$. For $q > 0$ the groups $H_q(E)$ admit a finite filtration such filtration quotients are isomorphic to groups $E_{p,q}^\infty$ with $(p, q) \neq 0$. This implies that $H_q(E) \in \mathcal{C}$.

Case 2: $H_q(F), H_q(E) \in \mathcal{C}$ for all $q > 0$.

Since all groups $E_{p,q}^\infty$ are quotients of subgroups of $H_{p+q}(E)$, we have $E_{p,q}^\infty \in \mathcal{C}$ for all $(p, q) \neq (0, 0)$. We will show that $E_{p,q}^2 \in \mathcal{C}$ for $(p, q) \neq (0, 0)$ by induction with respect to p . For $p = 0$ this holds since $E_{0,q}^2 \cong H_q(F)$. Assume that it also holds for $E_{i,q}^2$ for all $i < p$. It follows that $E_{i,q}^r$ for all $i < p$ and all $r \geq 2$.

Since all differentials terminating at $E_{p,0}^r$ are trivial, for each r we have an exact sequence

$$E_{p,0}^{r+1} \rightarrow E_{p,0}^r \xrightarrow{d^r} E_{p-r,r-1}^r$$

By assumption $E_{p-r,r-1}^r \in \mathcal{C}$, so if $E_{p,0}^{r+1} \in \mathcal{C}$ then the same is true for $E_{p,0}^r$. Since $E_{p,q}^{p+1} = E_{p,q}^\infty \in \mathcal{C}$, arguing inductively over decreasing values of r we obtain that $E_{p,0}^r \in \mathcal{C}$ for all $r \geq 2$. In particular, $H_p(B) = E_{p,0}^2 \in \mathcal{C}$. Using Proposition 21.4 we obtain that $E_{p,q}^2 = H_p(B, H_q(F)) \in \mathcal{C}$ for all $q \geq 0$.

Case 3: $H_q(B), H_q(E) \in \mathcal{C}$ for all $q > 0$.

This is similar to case 2. □

21.7 Proposition. *If \mathcal{C} is an acyclic Serre ring then for every $A \in \mathcal{C}$ and $n \geq 1$ we have $H_q(K(A, n)) \in \mathcal{C}$.*

Proof. We argue by induction with respect to n . The case $n = 1$ holds by definition of acyclicity of a Serre class. Assume that the statement is true for some $n \geq 1$. For $A \in \mathcal{C}$ consider the homotopy fibration sequence $K(A, n) = \Omega K(A, n+1) \rightarrow * \rightarrow K(A, n+1)$. Since $H_q(K(A, n)), H_q(*) \in \mathcal{C}$ for all $q > 0$, by Theorem 21.6 we obtain that $H_q(K(A, n+1)) \in \mathcal{C}$. □

21.8 Theorem. *Let \mathcal{C} be an acyclic Serre ring. If X is a simply connected space then the following conditions are equivalent:*

- 1) $\pi_n(X) \in \mathcal{C}$ for all $n \geq 1$
- 2) $H_n(X) \in \mathcal{C}$ for all $n \geq 1$

The proof of Theorem 21.8 will make use of the notion of Postnikov sections:

21.9 Definition. Let X be a path connected space. The n -th Postnikov section of X is a space X_n together with a map $f: X \rightarrow X_n$ such that

- 1) $f_*: \pi_q(X) \rightarrow \pi_q(X_n)$ is an isomorphism for $q \leq n$
- 2) $\pi_q(X) = 0$ for $q > n$.

The n -th Postnikov section of a space X can be constructed glueing to X cells in dimensions $n+1$ and higher to kill all homotopy groups above $\pi_n(X)$. The map $f: X \rightarrow X_n$ is then given by the inclusion.

Proof of Theorem 21.8.

1) \Rightarrow 2) Let X_n denote the n -th Postnikov section of X . By Theorem 16.4 we have $H_q(X) \cong H_q(X_n)$ for all $q < n$, so it will be enough to show that $H_q(X_n) \in \mathcal{C}$ for all $n, q > 0$. We will prove this by induction with respect to n . For $n = 2$ we have $P_2 = K(\pi_2(X), 2)$, so the statement holds by Proposition 21.7. Assume that it also holds for some $n \geq 2$. Notice that we have a fibration sequence

$$K(\pi_{n+1}, n+1) \rightarrow X_{n+1} \rightarrow X_n$$

Using Proposition 21.7 again we obtain that $H_q(K(\pi_{n+1}, n+1)) \in \mathcal{C}$ for $q > 0$, so using Theorem 21.6 we obtain that $H_q(X_{n+1}) \in \mathcal{C}$ for $q > 0$.

2) \Rightarrow 1) We will show that $\pi_n(X) \in \mathcal{C}$ by induction with respect to n . Since X is simply connected, for $n = 2$ by the Hurewicz Isomorphism Theorem we get $\pi_2(X) \cong H_2(X) \in \mathcal{C}$. Next, assume that $\pi_q(X) \in \mathcal{C}$ for all $q \leq n$ and consider the fibration sequence

$$\text{hofib } f \rightarrow X \rightarrow X_n$$

where X_n is the n -th Postnikov section of X . Notice that

$$\pi_q(\text{hofib } f) = \begin{cases} 0 & \text{if } q \leq n \\ \pi_q(X) & \text{if } q > n \end{cases}$$

Since $\pi_q(X_n) \in \mathcal{C}$ for all q , thus by part 1) \Rightarrow 2) we get that $H_q(X_n) \in \mathcal{C}$ for all $q > 0$. By assumption also $H_q(X) \in \mathcal{C}$ for $q > 0$. Therefore using Theorem 21.6 we obtain that $H_q(\text{hofib } f) \in \mathcal{C}$ for $q > 0$. Since $\text{hofib } f$ is n -connected, by the Hurewicz Isomorphism Theorem we obtain $H_{n+1}(\text{hofib } f) \cong \pi_{n+1}(\text{hofib } f) \cong \pi_{n+1}(X)$. Therefore $\pi_{n+1}(X) \in \mathcal{C}$.

□