17 | Hurewicz Theorem

Hurewicz homomorphism is a map that connects homotopy and homology groups. Recall that $H_n(S^n) \cong \mathbb{Z}$ We will denote by γ_n a chosen generator of $H_n(S^n)$. Given an element $[\varphi\colon (S^n, s_0) \to (X, x_0)] \in \pi_n(X, x_0)$ consider the homomorphism $\varphi_*\colon H_*(S^n) \to H_n(X)$. This homomorphism depends only on the homotopy class of φ.

17.1 Definition. The *Hurewicz homomorphism* is a function

$$h\colon \pi_n(X, x_0) \to H_n(X)$$

given by $h([\varphi]) = \varphi_*(\gamma_n)$.

17.2 Proposition. *For any function $f\colon X \to Y$ the following diagram commutes:*

$$\begin{tikzcd}
\pi_n(X, x_0) \arrow[r, "f_*"] \arrow[d, "h"'] & \pi_n(Y, f(x_0)) \arrow[d, "h"] \\
H_n(X) \arrow[r, "f_*"'] & H_n(Y)
\end{tikzcd}$$

Proof. For $[\varphi] \in \pi_n(X, x_0)$ we have

$$hf_*([\varphi]) = h([f\varphi]) = (f\varphi)_*(\gamma_n) = f_*\varphi_*(\gamma_n) = f_*h([\varphi])$$

□

17.3 Proposition. *The Hurewicz homorphism is a group homomorphism.*

Proof. Let $\varphi, \psi\colon (S^n, s_0) \to (X, x_0)$ where $n \geq 1$. Recall that the element $[\varphi] \cdot [\psi] \in \pi_n(X, x_0)$ is the homotopy class of the map

$$S^n \xrightarrow{p} S^n \vee S^n \xrightarrow{\varphi \vee \psi} X$$

where $p\colon S^n \to S^n \vee S^n$ is the pinch map. Let $r_1, r_2\colon S^n \vee S^n \to S^n$ be the retractions of $S^n \vee S^n$ onto the first and, respectively, the second copy of S^n. We have a commutative diagram

$$\begin{tikzcd} H_n(S^n) \arrow[r, "p_*"] \arrow[dr, "\mathrm{id}_* \oplus \mathrm{id}_*"'] & H_n(S^n \vee S^n) \arrow[r, "(\varphi \vee \psi)_*"] \arrow[d, "r_{1*} \oplus r_{2*}", "\cong"'] & H_n(X) \\ & H_n(S^n) \oplus H_n(S^n) \arrow[ur, "\varphi_* + \psi_*"'] & \end{tikzcd}$$

This gives:

$$h([\varphi] \cdot [\psi]) = ((\varphi \vee \psi)p)_*(\gamma_n) = (\varphi_* + \psi_*)(\mathrm{id}_* \oplus \mathrm{id}_*)(\gamma_n) = \varphi_*(\gamma_n) + \psi_*(\gamma_n) = h([\varphi]) + h([\psi])$$

$\square$

17.4 Hurewicz Isomorphism Theorem. *Let X be a space such that for some $n \geq 2$ we have $\pi_i(X) = 0$ for $i < n$. Then $H_i(X) = 0$ for $0 < i < n$ and the Hurewicz homomorphism*

$$h\colon \pi_n(X, x_0) \to H_n(X)$$

is an isomorphism.

Proof. Assume first that $X = S^n$. We have $H_i(S^n) = 0$ for $0 < i < n$. In degree n the Hurewicz homomorphism is a map $h\colon \mathbb{Z} \cong \pi_n(S^n) \to H_n(S^n) \cong \mathbb{Z}$. The group $\pi_n(S^n)$ is generated by the homotopy class of the identity map $\mathrm{id}_{S^n}\colon S^n \to S^n$ (11.10). We have $h([\mathrm{id}_{S^n}]) = \mathrm{id}_{S^n*}(\gamma_n) = \gamma_n$. Therefore h maps a generator of $\pi_n(S^n)$ to a generator of $H^n(S^n)$, and so it is an isomorphism.

Next, assume that $X = \bigvee_{i\in I} S^n$. Again, in this case $H_i(\bigvee_{i\in I} S^n) = 0$ for $0 < i < n$. Also, the retraction maps $r_i\colon \bigvee_{i\in I} S^n \to S^n$ give a commutative diagram

$$\begin{tikzcd} \pi_n(\bigvee_{i\in I} S^n) \arrow[r, "\bigoplus r_{i*}", "\cong"'] \arrow[d, "h"'] & \bigoplus_{i\in I} \pi_n(S^n) \arrow[d, "\bigoplus_{i\in I} h", "\cong"'] \\ H_n(\bigvee_{i\in I} S^n) \arrow[r, "\cong", "\bigoplus r_{i*}"'] & \bigoplus_{i\in I} H_n(S^n) \end{tikzcd}$$

The the map $\bigoplus_{i\in I} h$ is an isomorphism by the previous case, so the left vertical map h is also an isomorphism.

For the next step, assume that X is an arbitrary CW complex with $\pi_i(X) = 0$ for $i < n$. By Proposition 5.6 we can assume that $X^{(n-1)} = *$, which gives $H_i(X) = 0$ for $0 < i < n$.

Let $j\colon X^{(n+1)} \hookrightarrow X$ be the inclusion of the $(n+1)$-skeleton of X. By Proposition 17.2 we have a commutative diagram

$$\begin{tikzcd}
\pi_n(X^{(n+1)}) \arrow[r, "j_*", "\cong"'] \arrow[d, "h"'] & \pi_n(X) \arrow[d, "h"] \\
H_n(X^{(n+1)}) \arrow[r, "\cong", "j_*"'] & H_n(X)
\end{tikzcd}$$

The upper homomorphism j_* is an isomorphism by Proposition 5.2, and the lower j_* is an isomorphism by properties of homology groups. As a consequence, it is enough to show that $h\colon \pi_n(X^{(n+1)}) \to H_n(X^{(n+1)})$ is an isomorphism.

Since $X^{(n-1)} = *$, it follows that $X^{(n)} = \bigvee_{i\in I} S^n$ and $X^{(n+1)} = X^{(n)} \cup \bigcup_{k\in K} e_k^{(n+1)}$ where $\{e_k^{(n+1)}\}_{k\in K}$ are $(n+1)$-cells of X. Let $\varphi_k\colon S^n \to X^{(n)}$ be the attaching map of the cell e_k^{n+1}, and let $i\colon X^{(n)} \hookrightarrow X^{(n+1)}$ denote the inclusion map. We have a commutative diagram

$$\begin{tikzcd}
\pi_n(\bigvee_{k\in K} S^n) \arrow[r, "(\bigvee_{k\in K}\varphi_k)_*"] \arrow[d, "h"', "\cong"] & \pi_n(X^{(n)}) \arrow[r, "i_*"] \arrow[d, "h"', "\cong"] & \pi_n(X^{(n+1)}) \arrow[r] \arrow[d, "h"] & 0 \arrow[d, "\cong"] \\
H_n(\bigvee_{k\in K} S^n) \arrow[r, "(\bigvee_{k\in K}\varphi_k)_*"'] & H_n(X^{(n)}) \arrow[r, "i_*"'] & H_n(X^{(n+1)}) \arrow[r] & H_{n-1}(\bigvee_{k\in K} S^n)
\end{tikzcd}$$

The upper row of this diagram is exact by Proposition 12.7, and the lower row is exact by the long homology sequence associated to the map $\bigvee_{k\in K}\varphi_k$. By the Five Lemma we obtain that $h\colon \pi_n(X^{(n+1)}) \to H_n(X^{(n+1)})$ is an isomorphism.

Finally, let X be an arbitary space with $\pi_i(X) = 0$ for $i < n$. Let $f\colon Y \to X$ be a CW approximation of X (15.3). Using Theorem 16.1 and the previous case we get $H_i(X) \cong H_i(Y) = 0$ for $0 < i < n$.

By Proposition 17.2 we have a commutative diagram

$$\begin{tikzcd}
\pi_n(Y) \arrow[r, "f_*", "\cong"'] \arrow[d, "h"', "\cong"] & \pi_n(X) \arrow[d, "h"] \\
H_n(Y) \arrow[r, "\cong", "f_*"'] & H_n(X)
\end{tikzcd}$$

Since f is a weak equivalence, the upper homomorphism f_* is an isomorphism by definition, and the lower f_* is an isomorphism by Theorem 16.1. Also, since Y is a CW complex the left vertical map is an isomorphism by the previous case. Therefore $h\colon \pi_n(X) \to H_n(X)$ is an isomorphism. $\square$

17.5 Inverse Hurewicz Isomorphism Theorem. *Let X be a simply connected space, and let $H_i(X) = 0$ for $1 \leq i < n$ for some $n \geq 2$. Then $\pi_i(X) = 0$ for $i < n$ and the Hurewicz homomorphism $h\colon \pi_n(X) \to H_n(X)$ is an isomorphism.*

Proof. Exercise. ☐

Since all homology groups $H_i(X)$ are abelian but the fundamental group $\pi_1(X)$ need not be abelian, in general the Hurewicz homomorphism $h\colon \pi_1(X) \to H_1(X)$ is not an isomorphism. However, a version of Theorem 17.4 still holds with the following modification. Recall that if G is a group then the commutator of G is the subgroup $[G, G] \subseteq G$ generated by all elements of the form $ghg^{-1}h^{-1}$ for $g, h \in G$. The commutator is a normal subgroup of G, and the quotient group $G^{\mathrm{ab}} := G/[G, G]$ is an abelian group. The group G^{ab} is called the abelianization of G.

If H is an abelian group then any homomorphism $\varphi\colon G \to H$ defines a unique homomorphism $\overline{\varphi}\colon G^{\mathrm{ab}} \to H$ such that $\varphi = \overline{\varphi}\eta$ where $\eta\colon G \to G^{\mathrm{ab}}$ is the quotient homomorphism. Also, if $\psi\colon G \to H$ is a homomorphism of arbitary groups, then $\psi([G, G]) \subseteq [H, H]$, and so ψ induces a homomorphism of abelianizations $\psi^{\mathrm{ab}}\colon G^{\mathrm{ab}} \to H^{\mathrm{ab}}$.

17.6 Theorem. *Let X be a path connected space and let $h\colon \pi_1(X, x_0) \to H_1(X)$ be the Hurewicz homomorphism. Then the induced homomorphism $\bar{h}\colon \pi_1(X, x_0)^{ab} \to H_1(X)$ is an isomorphism.*

The proof will use the following algebraic fact.

17.7 Lemma. *Consider a sequence of group homomorphisms*

$$G \xrightarrow{\varphi} H \xrightarrow{\psi} K$$

such that ψ is onto and $\ker\psi = N_H(\operatorname{Im}\varphi)$ where $N_H(\operatorname{Im}\varphi)$ is the normalizer of $\operatorname{Im}\varphi$ in H. Then the induced sequence

$$G^{ab} \xrightarrow{\varphi^{ab}} H^{ab} \xrightarrow{\psi^{ab}} K^{ab} \longrightarrow 0$$

is exact.

Proof. Exercise. ☐

Proof of Theorem 17.6. Take $X = S^1$. As in the proof of Theorem 17.4 we obtain that $h\colon \pi_1(S^1) \to H_1(S^1)$ is an isomorphism. Also, since $\pi_1(S^1) \cong \mathbb{Z}$ is an abelian group, thus $\pi_1(S^1) \cong \pi_1(S^1)^{\mathrm{ab}}$ and, up to this isomorphism, $\bar{h}$ coincides with h.

Next, take $X = \bigvee_{i\in I} S^1$ and let $r_i\colon \bigvee_{i\in I} S^1 \to S^1$ be retraction maps. We have a commutative diagram

$$\begin{tikzcd}
\pi_1(\bigvee_{i\in I} S^1) \arrow[r, "\bigoplus r_{i*}"] \arrow[d, "h"'] & \bigoplus_{i\in I} \pi_1(S^1) \arrow[d, "\bigoplus_{i\in I} h", "\cong"'] \\
H_n(\bigvee_{i\in I} S^1) \arrow[r, "\cong", "\bigoplus r_{i*}"'] & \bigoplus_{i\in I} H_n(S^1)
\end{tikzcd}$$

The upper map $\bigoplus r_{i*}$ essentially conincides with the abelianization of $\pi_1(\bigvee_{i\in I} S^1)$, and the map $\bigoplus_{i\in I} h$ coincides, up to an isomorphism, with $\bar{h}\colon \pi_1(\bigvee_{i\in I} S^1)^{\mathrm{ab}} \to H_1(\bigvee_{i\in I})$. It remains to notice that $\bigoplus_{i\in I} h$ is an isomorphism by the previous case.

As in the proof of Theorem 17.4, it remains to consider the case where X is a 2-dimensional CW complex of the form $X = \bigvee_{i\in I} S^1 \cup \bigcup_{k\in K} e^2_k$. Let $\varphi_k\colon S^1 \to \bigvee_{i\in I} S^1$ be the attaching map of the cell e^2_k. Denote $\psi := \bigvee_{k\in K} \varphi_k\colon \bigvee_{k\in K} S^1 \to \bigvee_{i\in I} S^1$ Also, let $j\colon \bigvee_{i\in I} S^1 \hookrightarrow X$ be the inclusion of the 1-skeleton of X. We have a sequence of group homomorphisms

$$\pi_1(\bigvee_{k\in K} S^1) \xrightarrow{\psi_*} \pi_1(\bigvee_{i\in I} S^1) \xrightarrow{j_*} \pi_1(X)$$

By van Kampen's Theorem the hoomorphism j_* is onto and $\ker j_* = N_{\pi_1(\bigvee_{i\in I} S^1)}(\operatorname{Im}\psi_*)$. Cosider the commutative diagram

$$\begin{tikzcd}
\pi_1(\bigvee_{k\in K} S^1)^{\mathrm{ab}} \arrow[r, "\psi_*"] \arrow[d, "\bar{h}"', "\cong"] & \pi_1(\bigvee_{i\in I} S^1)^{\mathrm{ab}} \arrow[r, "i_*"] \arrow[d, "\bar{h}"', "\cong"] & \pi_1(X)^{\mathrm{ab}} \arrow[r] \arrow[d, "\bar{h}"] & 0 \arrow[d, "\cong"] \\
H_1(\bigvee_{i\in I} S^1) \arrow[r, "\psi_*"'] & H_1(\bigvee_{i\in I} S^1) \arrow[r, "i_*"'] & H_1(X) \arrow[r] & 0
\end{tikzcd}$$

The upper row is exact by Lemma 17.7 and the lower row is exact by the long exact homology sequence associated to ψ_*. Using the Five Lemma we obtain that $\bar{h}\colon \pi_1(X)^{\mathrm{ab}} \to H_1(X)$ is an isomorphism. $\square$

17.8 Relative Hurewicz Homomorphism. Recall that $H_n(D^n, S^{n-1}) \cong \mathbb{Z}$. Let $\bar{\gamma}_n$ denote a chosen generator of $H_n(D^n, S^{n-1})$. Given a pointed pair (X, A, x_0), and an element $[\varphi\colon (D^n, S^{n-1}, s_0) \to (X, A, x_0)] \in \pi_n(X, A, x_0)$ consider the function $\varphi_*\colon H_n(D^n, S^{n-1}) \to H_n(X, A)$.

17.9 Definition. The *relative Hurewicz homomorphism* is a function

$$h\colon \pi_n(X, A, x_0) \to H_n(X, A)$$

given by $h([\varphi]) = \varphi_*(\bar{\gamma}_n)$.

17.10 Proposition. *The relative Hurewicz homomorphism is a group homomorphism for $n \geq 2$.*

17.11 Relative Inverse Hurewicz Isomorphism Theorem. *Let (X, A) be a pair of simply connected CW complexes. If $H_i(X, A) = 0$ for all $0 < i < n$ for some $n \geq 2$ then $\pi_i(X, A) = 0$ for all $i < n$ and the Hurewicz homomorphism $h\colon \pi_n(X, A) \to H_n(X, A)$ is an isomorphism.*

Proof. See tom Dieck, Theorem 20.1.3 p. 497. Uses commutativity of the diagram

$$\begin{tikzcd} \pi_i(X,A) \arrow[r] \arrow[d, "h"'] & \pi_i(X/A) \arrow[d, "h"] \\ H_i(X,A) \arrow[r] & H_i(X/A) \end{tikzcd}$$

and the Inverse Hurewicz Theorem 17.5 applied to the space X/A. ☐

17.12 Theorem. *Let X, Y be simply connected CW complexes and let $f\colon X \to Y$ be a map such that for some $n \geq 2$ the homorphism $f_*\colon H_i(X) \to H_i(Y)$ is an isomorphism for $i < n$ and epimorphism for $i = n$. Then $f_*\colon \pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i < n$ and epimorphism for $i = n$.*

Proof. Let M_f be the mapping cylinder of f. The assumption about f is equivalent to the condition that $H_i(M_f, X) = 0$ for $i < n$. By Theorem 17.11 this gives $\pi_i(M_f, X) = 0$ for $i < n$. The statement then follows from the long exact sequence of homotopy groups of the pair (M_f, X). ☐

17.13 Corollary. *Let $f\colon X \to Y$ be a map of simply connected CW complexes such that $f_*\colon H_i(X) \to H_i(Y)$ is an isomorphism for all $i \geq 0$. Then f is a homotopy equivalence.*

Let $p_X\colon \widetilde{X} \to X$ denote the universal cover of a space X. Given a map $f\colon X \to Y$ we can find a map $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ such that the following diagram commutes:

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{f}"] \arrow[d, "p_X"'] & \widetilde{Y} \arrow[d, "p_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

17.14 Theorem. *Let $f\colon X \to Y$ be a map of path connected CW complexes. If the homomorphisms $f_*\colon \pi_1(X) \to \pi_1(Y)$ and $\widetilde{f}_*\colon H_i(\widetilde{X}) \to H_i(\widetilde{Y})$ for all $i \geq 0$ are isomorphisms then f is a homotopy equivalence.*

Proof. By Theorem 17.12 the map $\widetilde{f}_*\colon \pi_i(\widetilde{X}) \to \pi_i(\widetilde{Y})$ is an isomorphism for all $i \geq 0$. Since $p_{X*}\colon \pi_i(\widetilde{X}) \to \pi_i(X)$ and $p_{Y*}\colon \pi_i(\widetilde{Y}) \cong \pi_i(Y)$ are isomorphisms for $i \geq 2$, this gives that $f_*\colon \pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i \geq 2$. By assumption, $f_*\colon \pi_1(X) \to \pi_1(Y)$ is an isomorphism as well, so f is a weak equivalence and thus a homotopy equivalence. ☐