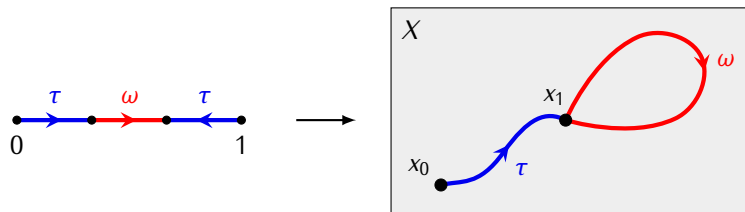


## 4 | Dependence on The Basepoint

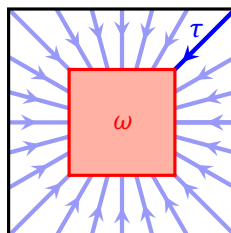
Let  $X$  be a space, and let  $x_0, x_1 \in X$ . Recall that any path  $\tau: [0, 1] \rightarrow X$  such that  $\tau(0) = x_0$  and  $\tau(1) = x_1$  defines an isomorphism of fundamental groups

$$s_\tau: \pi_1(X, x_1) \rightarrow \pi_1(X, x_0)$$

given by  $s_\tau([\omega]) = [\tau * \omega * \bar{\tau}]$ , where  $\bar{\tau}$  is obtained from  $\tau$  by reverting orientation.



In a similar way, given a path  $\tau: [0, 1] \rightarrow X$  with  $\tau(0) = x_0$  and  $\tau(1) = x_1$  we can define a map  $s_\tau: \pi_n(X, x_1) \rightarrow \pi_n(X, x_0)$ . To do this, given a map  $\omega: (I^n, \partial I^n) \rightarrow (X, x_1)$ , define a map  $\omega_\tau: (I^n, \partial I^n) \rightarrow (X, x_0)$  as follows:



The smaller cube is mapped by  $\omega$  and each radial ray joining the boundaries of the larger and smaller cube is mapped by the path  $\tau$ .

Let  $\pi_1(X, x_0, x_1)$  denote the set of homotopy classes of paths  $\tau: [0, 1] \rightarrow X$  such that  $\tau(0) = x_0$  and  $\tau(1) = x_1$ , with homotopies preserving the endpoints.

**4.1 Lemma.** Let  $\omega, \omega': (I^n, \partial I^n) \rightarrow (X, x_1)$  be maps such that  $\omega \simeq \omega' \text{ (rel } \partial I^n)$ , and let  $\tau, \tau': [0, 1] \rightarrow X$  be paths such that  $\tau(0) = \tau'(0) = x_0$ ,  $\tau(1) = \tau'(1) = x_1$  and  $\tau \simeq \tau' \text{ (rel } \{0, 1\})$ . Then  $\omega_\tau \simeq \omega'_{\tau'} \text{ (rel } \partial I^n)$ .

Equivalently, if  $[\omega] = [\omega'] \in \pi_n(X, x_1)$  and  $[\tau] = [\tau'] \in \pi_1(X, x_0, x_1)$  then  $[\omega_\tau] = [\omega'_{\tau'}] \in \pi_n(X, x_0)$

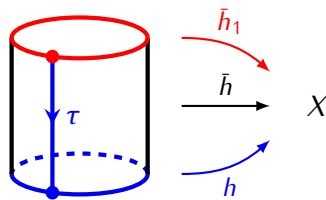
*Proof.* Exercise. □

**4.2 Note.** The homotopy class  $[\omega_\tau]$  can be also described as follows. Consider the homotopy  $h: \partial I^n \times [0, 1] \rightarrow X$  given by  $h(x, t) = \tau(1 - t)$ . Since the pair  $(I^n, \partial I^n)$  has the homotopy extension property, we can extend  $h$  to a homotopy  $\bar{h}: I^n \times [0, 1] \rightarrow X$  such that  $\bar{h}_0 = \omega$ . The map  $\bar{h}_1$  defines an element  $[\bar{h}_1] \in \pi_n(X, x_0)$ . This element does not depend on the choice of the extension  $\bar{h}$  (exercise), and we have  $[\bar{h}_1] = [\omega_\tau]$ .

**4.3 Note.** Recall that elements of  $\pi_n(X, x_1)$  can be alternatively defined as pointed homotopy classes of maps  $\omega: (S^n, s_0) \rightarrow (X, x_1)$ . In this setting, for  $[\tau] \in \pi_1(X, x_0, x_1)$  the element  $[\omega_\tau] \in \pi_n(X, x_0)$  can be described using a similar approach as in (4.2). Given such  $\omega$  and  $\tau$  we can define a function

$$h: (S^n \times \{0\}) \cup (\{s_0\} \times [0, 1]) \rightarrow X$$

so that  $h(s, 0) = \omega(s)$  and  $h(s_0, t) = \tau(1 - t)$ . Since the pair  $(S^n, s_0)$  has the homotopy extension property, thus  $h$  can be extended to a homotopy  $\bar{h}: S^n \times [0, 1] \rightarrow X$ . One can check that the pointed homotopy class of the map  $\bar{h}_1: (S^n, s_0) \rightarrow (X, x_0)$  does not depend on the choice of the extension  $\bar{h}$ . We set:  $[\omega_\tau] = [\bar{h}_1] \in \pi_n(X, x_0)$ .



**4.4 Definition.** Given  $[\tau] \in \pi_1(X, x_0, x_1)$  let

$$s_{[\tau]}: \pi_n(X, x_1) \rightarrow \pi_n(X, x_0)$$

denote the function given by  $s_{[\tau]}([\omega]) = [\omega_\tau]$ .

**4.5 Proposition.** 1) For any  $[\tau] \in \pi_1(X, x_0, x_1)$  the function  $s_{[\tau]}$  is a group homomorphism.

2) If  $[\tau] \in \pi_1(X, x_0, x_1)$  and  $[\sigma] \in \pi_1(X, x_1, x_2)$  then

$$S_{[\tau*\sigma]} = S_{[\tau]} \circ S_{[\sigma]}: \pi_n(X, x_2) \rightarrow \pi_n(X, x_0)$$

3) If  $c_{x_0}: [0, 1] \rightarrow X$  is the constant path,  $c_{x_0}(t) = x_0$  for all  $t \in [0, 1]$ , then  $S_{[c_{x_0}]}: \pi_n(X, x_0) \rightarrow \pi_n(X, x_0)$  is the identity homomorphism.

*Proof.* Exercise. □

**4.6 Corollary.** Let  $X$  be a space and let  $x_0, x_1 \in X$ . For any path  $\tau: [0, 1] \rightarrow X$  be a path such that  $\tau(0) = x_0$ ,  $\tau(1) = x_1$  the homomorphism  $S_{[\tau]}: \pi_n(X, x_1) \rightarrow \pi_n(X, x_0)$  is an isomorphism.

*Proof.* Let  $\bar{\tau}$  be the inverse of  $\tau$ . This defines homomorphisms

$$S_{[\tau]}: \pi_n(X, x_1) \xrightarrow{\sim} \pi_n(X, x_0): S_{[\bar{\tau}]}$$

We will show that  $S_{[\bar{\tau}]} = S_{[\tau]}^{-1}$ . Indeed, by Proposition 4.5 we have

$$S_{[\bar{\tau}]} \circ S_{[\tau]} = S_{[\bar{\tau}*\tau]} = S_{[c_{x_0}]} = \text{id}_{\pi_n(X, x_1)}$$

Analogously,  $S_{[\tau]} \circ S_{[\bar{\tau}]} = \text{id}_{\pi_n(X, x_0)}$ . □

Corollary 4.6 implies that if  $x_0, x_1$  are in the same path connected component of  $X$  then  $\pi_n(X, x_0) \cong \pi_n(X, x_1)$ . On the other hand, if points  $x_0, x_1 \in X$  belong to different path connected components of  $X$ , then in general there is no relationship between  $\pi_n(X, x_0)$  and  $\pi_n(X, x_1)$ .

**4.7 Proposition.** Let  $X$  be a space,  $x_0 \in X$ , and let  $X_0$  be the path connected component of  $X$  such that  $x_0 \in X_0$ . Then the inclusion map  $i: X_0 \hookrightarrow X$  induces an isomorphism

$$i_*: \pi_n(X_0, x_0) \xrightarrow{\cong} \pi_n(X, x_0)$$

*Proof.* Since  $I^n$  is path connected, for any map  $\omega: (I^n, \partial I^n) \rightarrow (X, x_0)$  we have  $\omega(I^n) \subseteq X_0$ . This shows that  $i_*$  is onto. Also, if  $h: I^n \times [0, 1] \rightarrow X$  is a homotopy  $h: \omega \simeq \omega'$  where  $\omega, \omega': I^n \rightarrow X_0$  then, since  $I^n \times [0, 1]$  is path connected, we have  $h(I^n \times [0, 1]) \subseteq X_0$ . It implies that  $i_*$  is 1-1. □

**4.8 Note.** Given a path connected space  $X$  we will sometimes write  $\pi_n(X)$  to denote the  $n$ -th homotopy group of  $X$  taken with respect to some unspecified basepoint of  $X$ . By Corollary 4.6 this will not create problems as long as we are interested in the isomorphism type of the fundamental group only.

Similarly as for the fundamental group we have:

**4.9 Proposition.** Let  $f, g: X \rightarrow Y$  be homotopic maps and let  $h: f \simeq g$ . For  $x_0 \in X$  let  $\tau$  be the path in  $Y$  given by  $\tau(t) = h(x_0, t)$ . The following diagram commutes:

$$\begin{array}{ccc}
 & & \pi_n(Y, g(x_0)) \\
 & \nearrow g_* & \downarrow \cong \quad s_{[\tau]} \\
 \pi_n(X, x_0) & & \\
 & \searrow f_* & \downarrow \\
 & & \pi_n(Y, f(x_0))
 \end{array}$$

*Proof.* Exercise. □

**4.10 Note.** Proposition 4.9 implies, in particular, that if  $f, g: (X, x_0) \rightarrow (Y, y_0)$  are maps of pointed spaces and  $f \simeq g$  (rel  $\{x_0\}$ ) then  $f_* = g_*$ .

**4.11 Corollary.** If  $f, g: X \rightarrow Y$  are maps such that  $f \simeq g$  then the homomorphism  $f_*: \pi_n(X, x_0) \rightarrow \pi_n(Y, f(x_0))$  is an isomorphism (or it is trivial or 1-1 or onto) if and only if the homomorphism  $g_*: \pi_n(X, x_0) \rightarrow \pi_n(Y, g(x_0))$  has the same property.

**4.12 Proposition.** If  $f: X \rightarrow Y$  is a homotopy equivalence then for any  $x_0 \in X$  the homomorphism  $f_*: \pi_n(X, x_0) \rightarrow \pi_n(Y, f(x_0))$  is an isomorphism.

*Proof.* Let  $g: Y \rightarrow X$  be a homotopy inverse of  $f$ . Consider the sequence of homomorphisms

$$\pi_n(X, x_0) \xrightarrow{f_*} \pi_n(Y, f(x_0)) \xrightarrow{g_*} \pi_n(X, gf(x_0)) \xrightarrow{f_*} \pi_n(Y, fgf(x_0))$$

Composition of the first two homomorphisms satisfies  $g_*f_* = (gf)_*$ . Since  $gf \simeq \text{id}_X$  and  $\text{id}_{X*}$  is an isomorphism, by Corollary 4.11 we obtain that  $g_*f_*$  is an isomorphism. This implies in particular that  $g_*$  is onto. Similarly, composing the last two homomorphisms we obtain  $f_*g_* = (fg)_*$  and since  $fg \simeq \text{id}_Y$  we get that  $f_*g_*$  is an isomorphism. This means that  $g_*$  is 1-1. As a consequence  $g_*$  is an isomorphism. It follows that the first homomorphism  $f_*$  is a composition of two isomorphisms:  $f_* = g_*^{-1}(g_*f_*)$ , and so  $f_*$  is an isomorphism. □

**4.13 Corollary.** If  $X, Y$  are path connected spaces and  $X \simeq Y$  then  $\pi_n(X, x_0) \cong \pi_n(Y, y_0)$  for any  $x_0 \in X, y_0 \in Y$ .

**4.14 The action of  $\pi_1$ .** If  $[\tau] \in \pi_1(X, x_0)$  then  $s_{[\tau]}$  is an isomorphism

$$s_{[\tau]}: \pi_n(X, x_0) \rightarrow \pi_n(X, x_0)$$

Denote  $[\tau] \odot [\omega] := s_{[\tau]}(\omega)$ .

**4.15 Definition.** For  $n \geq 0$  the *action* of  $\pi_1(X, x_0)$  on  $\pi_n(X, x_0)$  is the map

$$\begin{aligned} \pi_1(X, x_0) \times \pi_n(X, x_0) &\rightarrow \pi_n(X, x_0) \\ ([\tau], [\omega]) &\mapsto [\tau] \odot [\omega] \end{aligned}$$

**4.16 Note.** By Proposition 4.5, for any  $[\tau], [\tau'] \in \pi_1(X, x_0)$  and  $\omega, \omega' \in \pi_n(X, x_0)$  we have:

- $[\tau] \odot ([\omega] \cdot [\omega']) = ([\tau] \odot [\omega]) \cdot ([\tau] \odot [\omega'])$
- $([\tau] \cdot [\tau']) \odot [\omega] = [\tau] \odot ([\tau'] \odot [\omega])$
- $[c_{x_0}] \odot [\omega] = [\omega]$  where  $[c_{x_0}] \in \pi_1(X, x_0)$  is the trivial element.
- $[\tau] \odot [c_{x_0}] = [c_{x_0}]$  where  $[c_{x_0}] \in \pi_n(X, x_0)$  is the trivial element.

**4.17 Proposition.** For any map  $f: (X, x_0) \rightarrow (Y, y_0)$  the following diagram commutes:

$$\begin{array}{ccc} \pi_1(X, x_0) \times \pi_n(X, x_0) & \xrightarrow{\odot} & \pi_n(X, x_0) \\ f_* \times f_* \downarrow & & \downarrow f_* \\ \pi_1(Y, y_0) \times \pi_n(Y, y_0) & \xrightarrow{\odot} & \pi_n(Y, y_0) \end{array}$$

*Proof.* Exercise. □

**4.18 Example.** The action of  $\pi_1(X, x_0)$  on  $\pi_1(X, x_0)$  is given by conjugation:

$$[\tau] \odot [\omega] = [\tau] \cdot [\omega] \cdot [\tau]^{-1}$$

**4.19 Definition.** A path connected space  $X$  is *n-simple* if for some  $x_0 \in X$  the action of  $\pi_1(X, x_0)$  on  $\pi_n(X, x_0)$  is trivial:  $[\tau] \odot [\omega] = [\omega]$  for all  $[\tau] \in \pi_1(X, x_0)$  and  $[\omega] \in \pi_n(X, x_0)$ . A path connected space is *simple* if it is *n-simple* for all  $n \geq 1$ .

The following fact implies that *n-simplicity* of a space  $X$  does not depend on the choice of a basepoint  $x_0 \in X$ :

**4.20 Proposition.** Let  $X$  be a space, let  $x_0, x_1 \in X$ , and let  $\tau: [0, 1] \rightarrow X$  be a path such that  $\tau(0) = x_0$  and  $\tau(1) = x_1$ . Then the following diagram commutes:

$$\begin{array}{ccc} \pi_1(X, x_1) \times \pi_n(X, x_1) & \xrightarrow{\odot} & \pi_n(X, x_1) \\ s_{[\tau]} \times s_{[\tau]} \downarrow & & \downarrow s_{[\tau]} \\ \pi_1(X, x_0) \times \pi_n(X, x_0) & \xrightarrow{\odot} & \pi_n(X, x_0) \end{array}$$

*Proof.* Exercise. □

For spaces  $X, Y$  let  $[X, Y]$  denote the set of homotopy classes of maps  $X \rightarrow Y$ . Notice that for any space  $X$  and any  $n$  we have a map of sets

$$\phi: \pi_n(X, x_0) \rightarrow [S^n, X]$$

which maps the pointed homotopy class of map  $\omega: (S^n, s_0) \rightarrow (X, x_0)$  to the unpointed homotopy class of the same map.

**4.21 Proposition.** *Let  $X$  be a path connected space, and let  $n \geq 1$ . The following conditions are equivalent:*

- 1)  $X$  is  $n$ -simple.
- 2) For any  $x_0, x_1 \in X$ ,  $[\tau], [\sigma] \in \pi_1(X, x_0, x_1)$  and  $[\omega] \in \pi_n(X, x_1)$  we have  $s_{[\tau]}([\omega]) = s_{[\sigma]}([\omega])$ . Thus there is a canonical isomorphism  $\pi_n(X, x_1) \xrightarrow{\cong} \pi_n(X, x_0)$ .
- 3) For any  $x_0 \in X$  the map  $\phi: \pi_n(X, x_0) \rightarrow [S^n, X]$  is a bijection. Therefore any (unpointed) map  $f: S^n \rightarrow X$  defines a unique element of  $\pi_n(X, x_0)$ .

*Proof.* 1)  $\Rightarrow$  2) Let  $[\tau], [\sigma] \in \pi_1(X, x_0, x_1)$  and  $[\omega] \in \pi_n(X, x_1)$ . Since  $[\bar{\tau} * \sigma] \in \pi_1(X, x_1)$ , by 1) we obtain

$$s_{[\bar{\tau}]}s_{[\sigma]}([\omega]) = s_{[\bar{\tau} * \sigma]}([\omega]) = [\omega]$$

Also, since  $s_{[\bar{\tau}]}$  is the inverse isomorphism of  $s_{[\tau]}$  we get

$$s_{[\sigma]}([\omega]) = s_{[\tau]}s_{[\bar{\tau}]}s_{[\sigma]}([\omega]) = s_{[\tau]}([\omega])$$

2)  $\Rightarrow$  1) Let  $[\tau], [c_{x_0}] \in \pi_1(X, x_0)$ , where  $[c_{x_0}]$  is the trivial element. By 2) we have

$$s_{[\tau]}([\omega]) = s_{[c_{x_0}]}([\omega]) = [\omega]$$

for any  $[\omega] \in \pi_n(X, x_0)$ . Therefore  $X$  is  $n$ -simple.

1)  $\Rightarrow$  3) The map  $\phi$  is always onto. Indeed, take any map  $\omega: S^n \rightarrow X$ . Since  $X$  is path connected, there exists a path  $\tau: [0, 1] \rightarrow X$  such that  $\tau(0) = x_0$  and  $\tau(1) = \omega(s_0)$ . Consider the map

$$h: (S^n \times \{0\}) \cup (\{s_0\} \times [0, 1]) \rightarrow X$$

so that  $h(s, 0) = \omega(s)$  and  $h(s_0, t) = \tau(1 - t)$ . The pair  $(S^n, s_0)$  has the homotopy extension property, so  $h$  can be extended to a homotopy  $\bar{h}: S^n \times [0, 1] \rightarrow X$ . Then for the map  $h_1$  we have  $h_1(s_0) = x_0$ , so  $[h_1] \in \pi_n(X, x_0)$ . Also,  $h$  is homotopic to  $h_0 = \omega$ . Therefore we have  $\phi([h_1]) = [\omega]$ .

To show that  $\phi$  is 1-1, we will use the description of  $s_{[\tau]}$  in terms of maps from spheres given in Note 4.2. Given two elements  $[\omega_0], [\omega_1] \in \pi_n(X, x_0)$  assume that  $\phi([\omega_0]) = \phi([\omega_1])$ . This means that there

exists a homotopy  $h: S^n \times [0, 1] \rightarrow X$  such that  $h_0 = \omega_0$  and  $h_1 = \omega_1$ . Let  $\tau: [0, 1] \rightarrow X$  be a path given by  $\tau(t) = h(s_0, t)$ . Then  $[\tau] \in \pi_1(X, x_0)$ , and by (4.2) we have

$$[\omega_1] = [\bar{\tau}] \odot [\omega_0] = s_{[\bar{\tau}]}([\omega_0])$$

By 1) we have  $s_{[\bar{\tau}]}([\omega_0]) = [\omega_0]$ . Thus  $[\omega_1] = [\omega_0] \in \pi_n(X, x_0)$ .

3)  $\Rightarrow$  1) Let  $[\tau] \in \pi_1(X, x_0)$ ,  $[\omega] \in \pi_n(X, x_0)$ . Let  $\omega_\tau: (S^n, s_0) \rightarrow (X, x_0)$  be some map such that  $[\omega_\tau] = s_{[\tau]}([\omega])$ . By (4.2) the maps  $\omega_\tau$  and  $\omega$  are freely homotopic, i.e.  $\phi([\omega_\tau]) = \phi([\omega])$ . By assumption  $\phi$  is 1-1, thus we obtain

$$[\omega] = [\omega_\tau] = s_{[\tau]}([\omega]) = [\tau] \odot [\omega]$$

in  $\pi_n(X, x_0)$ .

□