14 | Weak Equivalences

14.1 Definition. Let $0 \leq n \leq \infty$. A map $f\colon X \to Y$ is an *n-equivalence* if the induced homomorphism $f_*\colon \pi_i(X, x_0) \to \pi_i(Y, f(x_0))$ is an isomorphism for $0 \leq i < n$ and it is an epimorphism for $i = n$ for all $x_0 \in X$. A map f is a *weak (homotopy) equivalence* if it is an ∞-equivalence.

Recall that for a map $f\colon X \to Y$ the mapping cylinder of f is the space

$$M_f = (X \times [0,1] \sqcup Y)/{\sim}$$

where $(x, 0) \sim f(x)$ for all $x \in X$. We will consider X as a subspace of M_f by identifying it with $X \times \{1\}$.

14.2 Proposition. *Given a map $f\colon X \to Y$ the following conditions are equivalent:*

1) *f is an n-equivalence.*
2) *For $k \leq n$, given any commutative diagram*

$$\begin{tikzcd} S^{k-1} \arrow[r, "\varphi"] \arrow[d, hook] & Y \arrow[d, "f"] \\ D^k \arrow[r, "\psi"'] \arrow[ur, dashed, "\overline{\psi}"] & Z \end{tikzcd}$$

there exists a map $\overline{\psi}\colon D^k \to Y$ such that $\overline{\psi}|_{S^{k-1}} = \varphi$ and $f\overline{\psi} \simeq \psi$ (rel S^{k-1}).

2) *For $k \leq n$, given any diagram*

$$\begin{tikzcd} S^{k-1} \arrow[r, "\varphi"] \arrow[d, hook] & Y \arrow[d, "f"] \\ D^k \arrow[r, "\psi"'] \arrow[ur, dashed, "\overline{\psi}"] & Z \end{tikzcd}$$

and a homotopy $\Phi\colon f\varphi \simeq \psi|_{S^{k-1}}$ there exists a map $\overline{\psi}\colon D^k \to Y$ and a homotopy $\overline{\Phi}\colon f\overline{\psi} \simeq \psi$ such that $\overline{\psi}|_{S^{n-1}} = \varphi$ and $\overline{\Phi}|_{S^{k-1}\times[0,1]} = \Phi$.

3) The pair (M_f, X) is n-connected.

Proof. Exercise. □

14.3 Proposition. *1) If $f, g\colon X \to Y$ are maps such that $f \simeq g$ and f is an n-equivalence then so is g.*

2) If $f\colon X \to Y$, $g\colon Y \to Z$, and any two of the maps f, g, gf are weak equivalences, then so is the third map.

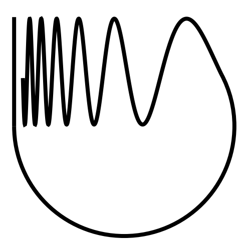
3) Every homotopy equivalence is a weak equivalence.

Proof. Exercise. □

One of the main goals of this chapter will be the proof of the following fact:

14.4 Theorem. *If X, Y are CW complexes then any weak equivalence $f\colon X \to Y$ is a homotopy equivalence.*

14.5 Note. Theorem 14.4 does not hold in general for spaces that are not CW complexes. For example, let W be the Warsaw circle (shown below). Since $\pi_i(W) = 0$ for all i, the constant map $W \to *$ is a weak equivalence. However, it is not a homotopy equivalence.



The proof Theorem 14.4 will use the following fact:

14.6 Proposition. *Assume that we have a diagram*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, hook] & Y \arrow[d, "f"] \\ X \arrow[r, "h"'] \arrow[ur, dashed, "\overline{h}"] & Z \end{tikzcd}$$

where (X, A) is a relative CW complex such that $\dim(X \smallsetminus A) \leq n$ *for some* $n \leq \infty$*, and* $f\colon Y \to Z$ *is an* n*-equivalence. Assume also that* $\Phi\colon A \times [0,1] \to Z$ *is a homotopy such that* $\Phi\colon h|_A \simeq gf$*. Then there exists a map* $\bar{h}\colon X \to Y$ *and a homotopy* $\bar{\Phi}\colon X \times [0,1] \to Z$ *such that* $\bar{h}|_A = g$*,* $\bar{\Phi}\colon h \simeq f\bar{h}$ *and* $\bar{\Phi}|_{A\times[0,1]} = \Phi$*.*

Proof. By induction on skeleta of (X, A), using Proposition 14.2. □

As a special case of Proposition 14.6 we obtain:

14.7 Corollary. *Assume that we have a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, hook] & Y \arrow[d, "f"] \\ X \arrow[r, "h"'] \arrow[ur, dashed, "\bar{h}"] & Z \end{tikzcd}$$

where (X, A) be a relative CW complex such that $\dim(X \smallsetminus A) \leq n$ *for some* $n \leq \infty$*, and* $f\colon Y \to Z$ *is an* n*-equivalence. Then there exists a map* $\bar{h}\colon X \to Y$ *such that* $\bar{h}|_A = g$ *and* $f\bar{h} \simeq h$ (rel A)*.*

Recall that by $[X, Y]$ we denote the set of homotopy classes of maps $X \to Y$. A map $f\colon Y \to Z$ induces a map of sets $f_*\colon [X, Y] \to [X, Z]$ given by $f_*[\varphi] = [f\varphi]$.

14.8 Corollary. *Let* $f\colon Y \to Z$ *be an* n*-equivalence for some* $n \leq \infty$*. For any CW complex* X *the map*

$$f_*\colon [X, Y] \to [X, Z]$$

is a bijection if $\dim X \leq n - 1$ *and it is onto if* $\dim X \leq n$*.*

Proof. The onto part follows from Corollary 14.7 with $A = \varnothing$. It remains to show that f_* is 1-1 if $\dim X \leq n-1$. Assume then that for some $\varphi_0, \varphi_1\colon X \to Y$ there is a homotopy $h\colon X \times [0,1] \to Z$ such that $h_0 = f\varphi_0$ and $h_1 = f\varphi_1$. This gives a commutative diagram

$$\begin{tikzcd} X \times \{0,1\} \arrow[r, "\varphi_0 \sqcup \varphi_1"] \arrow[d, hook, "i"'] & Y \arrow[d, "f"] \\ X \times [0,1] \arrow[r, "h"'] \arrow[ur, dashed, "\bar{h}"] & Z \end{tikzcd}$$

Consider the relative CW complex $(X \times [0,1], X \times \{0,1\})$. Since $\dim X \times [0,1] \leq n$, using Corollary 14.7 again we obtain that there exists $\bar{h}\colon X \times [0,1] \to Y$ which is homotopy between φ_0 and φ_1. □

Proof of Theorem 14.4. Let $f\colon X \to Y$ be a weak equivalence of CW complexes. By Corollary 14.8, the map

$$f_*\colon [Y, X] \to [Y, Y]$$

is a bijection. Therefore, there exists $g\colon Y \to X$ such that $f_*[g] = [\mathrm{id}_Y]$. Equivalently, $fg \simeq \mathrm{id}_Y$. Next, consider the bijection

$$f_*\colon [X, X] \to [X, Y]$$

We have $f_*[gf] = [fgf] = [f] = f_*[\mathrm{id}_X]$, which gives $[gf] = [\mathrm{id}_X]$, or equivalently $gf \simeq \mathrm{id}_X$. Therefore f is a homotopy equivalence with a homotopy inverse g. □

We have seen before (5.12) that two CW complexes X, Y that have isomorphic homotopy groups need not be homotopy equivalent. The issue is, that even if $\pi_i(X) \overset{\cong}{=} \pi_i(Y)$ for all $i \geq 0$, there may be no map $X \to Y$ which induces such isomorphisms. However, in two cases homotopy groups alone are enough to determine the homotopy type of a CW complex: for contractible spaces and for Eilenberg-MacLane spaces.

14.9 Proposition. *If X is a CW complex such that $\pi_i(X) = 0$ for all $i \geq 0$ then $X \simeq *$.*

Proof. The constant map $X \to *$ is weak equivalence, so by Theorem 14.4 it is a homotopy equivalence. □

14.10 Proposition. *Let X_1, X_2 be Eilenberg-MacLane spaces of type $K(G, n)$. That is, X_1, X_2 are path connected CW complexes such that*

$$\pi_i(X_k) \cong \begin{cases} G & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

for $k = 1, 2$. Then $X_1 \simeq X_2$.

Proof. Recall (12.14) that we can construct an Eilenberg-MacLane space X_0 of the type $K(G, n)$ such that $X_0^{(n-1)} = *$. It will be enough to show that for any other Eileberg-MacLane space Y of the same type there exists a weak equivalence $X_0 \to Y$. Indeed, by Theorem 14.4 this will give $X_0 \simeq Y$, and applying it to the spaces X_1 and X_2 we will obtain $X_1 \simeq X_0 \simeq X_2$.

Let then X_0, Y be Eilenberg-MacLane spaces of type $K(G, n)$ such that $X_0^{(n-1)} = *$. We can assume that the 0-cell $* \in X_0$ is the basepoint of X_0, and let $y_0 \in Y$ be a basepoint in Y. Let $\varphi\colon \pi_n(X_0, *) \to \pi_n(Y, y_0)$ be an isomorphism of groups. We will construct a map $f\colon (X_0, *) \to (Y, y_0)$ such that $f_* = \varphi$. To do this, notice that $X_0^{(n)} = \bigvee_{i \in I} S^n$. For $k \in I$ let $j_k\colon S^n \hookrightarrow X_0^n$ be the inclusion of the k-th copy of S^n. Let $[ij_k] \in \pi_n(X_0, *)$ be the element represented by $S^n \overset{j_k}{\hookrightarrow} X_0^{(n)} \overset{i}{\hookrightarrow} X_0$, and let $\omega_k\colon S^n \to Y$ be a map such that $[\omega_k] = \varphi([ij_k])$. Define $f_n\colon X_0^{(n)} \to Y$ by $f_n = \bigvee_{k \in I} \omega_k$.

Assume that we can extend f_n to some map $f\colon X_0 \to Y$. Then f induces a homomorphism $f_*\colon \pi_n(X_0, *) \to \pi_n(Y, y_0)$ such that

$$f_*([ij_k]) = [\omega_k] = \varphi([ij_k]) \tag{$*$}$$

for all $k \in I$. By Corollary 12.6 the elements $[j_k]$ generate the group $\pi_n(X_0^{(n)}, *)$, and by Proposition 5.2 the homomorphism $i_*\colon \pi_n(X_0^{(n)}, *) \to \pi_n(X_0, *)$ is onto. Therefore elements $[ij_k]$ generate $\pi_n(X_0, *)$. As a consequence, the equation $(*)$ implies that $f_*([\tau]) = \varphi([\tau])$ for all $[\tau] \in \pi_n(X_0, *)$. It follows that $f_*\colon \pi_i(X_0, *) \to \pi_i(Y, y_0)$ is an isomorphism for $i = n$ and since all other homotopy groups of X_0 and Y are trivial, f_* is an isomorphism for all $i \neq n$ as well. Therefore f is a weak equivalence.

An extension of $f_0\colon X_0^{(n)} \to Y$ to $f\colon X_0 \to Y$ can be constructed by induction with respect to skeleta of X_0. Assume that for some $m \geq n$ we have a map $f_m\colon X_0^{(m)} \to Y$ that extends f_n. Then $X_0^{(m+1)} = X_0^{(m)} \cup \bigcup_{j\in J} e_j^{m+1}$ for some $(m+1)$-cells e_j. Let $\varphi_j\colon S^m \to X^{(m)}$ be the attaching map of e_j^{m+1}, and let $\overline{\varphi}_j\colon D^{m+1} \to X^{(m)}$ be the characteristic map. Since $\pi_m(Y) = 0$, the map $f_m\varphi_j$ extends to $\psi_j\colon D^{m+1} \to Y$ We define $f_{m+1}\colon X_0^{(m+1)} \to Y$ by

$$f_{m+1}(x) = \begin{cases} f_m(x) & \text{if } x \in X^{(m)} \\ \psi_j(\overline{\varphi}_j^{-1}(x)) & \text{if } x \in e_j \end{cases}$$

$\square$

Using similar arguments as in the proof of Proposition 14.10 we can obtain:

14.11 Proposition. *Let $K(G, n)$, $K(H, n)$ be Eilenberg-MacLane spaces for some groups G, H and $n \geq 1$. For any homomorphism of groups $\varphi\colon \pi_n(K(G, n), x_0) \to \pi_n(K(H, n), y_0)$ there exists a map $f\colon (K(G, n), x_0) \to (K(H, n), y_0)$ such that $f_* = \varphi$.*

Proof. Exercise. $\square$