

# 3 | Higher Homotopy Groups

## 3.1 Notation.

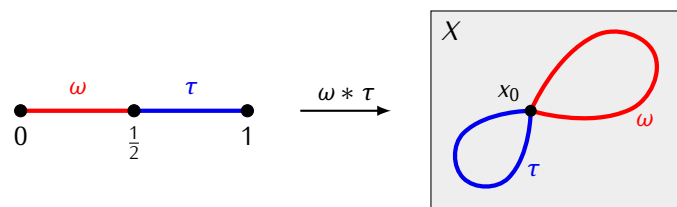
$$I^n = \{(s_1, \dots, s_n) \mid s_i \in [0, 1], i = 1, \dots, n\}$$

$$\partial I^n = \{(s_1, \dots, s_n) \in I^n \mid s_i \in \{0, 1\} \text{ for some } i\}$$

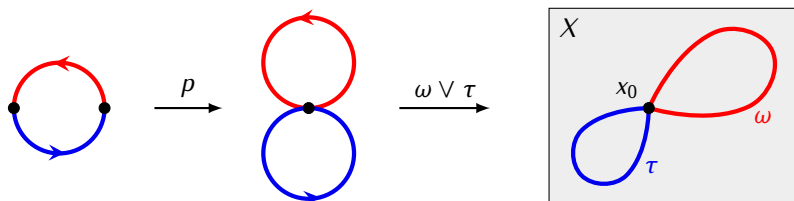
$$D^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_i x_i^2 \leq 1\}$$

$$S^{n-1} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_i x_i^2 = 1\}$$

Recall that the fundamental group  $\pi_1(X, x_0)$  of a pointed space  $(X, x_0)$  is the group whose elements are homotopy classes of maps  $\omega : (I, \partial I) \rightarrow (X, x_0)$ . Multiplication is given by concatenation of such maps.



Alternatively,  $\pi_1(X, x_0)$  can be described as a group whose elements are homotopy classes of maps  $\omega : (S^1, s_0) \rightarrow (X, x_0)$ . In this setting, the multiplication in  $\pi_1(X, x_0)$  is defined using the pinch map  $p : S^1 \rightarrow S^1 \vee S^1$ :



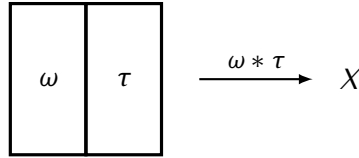
This construction can be generalized to define higher homotopy groups.

**3.2 Definition/Proposition.** Let  $(X, x_0)$  be a pointed space. For  $n \geq 1$  the  $n$ -th homotopy group of  $(X, x_0)$  is the group  $\pi_n(X, x_0)$  whose elements are homotopy classes of maps  $\omega: (I^n, \partial I^n) \rightarrow (X, x_0)$ . Multiplication in  $\pi_n(X, x_0)$  is defined as follows. If  $\omega, \tau: (I^n, \partial I^n) \rightarrow (X, x_0)$  then

$$[\omega] \cdot [\tau] = [\omega * \tau]$$

where  $\omega * \tau: (I^n, \partial I^n) \rightarrow (X, x_0)$  is given by

$$(\omega * \tau)(s_1, s_2, \dots, s_n) = \begin{cases} \omega(2s_1, s_2, \dots, s_n) & \text{for } s_1 \in [0, \frac{1}{2}] \\ \tau(2s_1 - 1, s_2, \dots, s_n) & \text{for } s_1 \in [\frac{1}{2}, 1] \end{cases}$$



The trivial element of  $\pi_n(X, x_0)$  is the homotopy class of the constant map  $c_{x_0}: I^n \rightarrow X$ . Also, for  $[\omega] \in \pi_1(X, x_0)$  we have  $[\omega]^{-1} = [\bar{\omega}]$  where  $\bar{\omega}: (I^n, \partial I^n) \rightarrow (X, x_0)$  is given by

$$\bar{\omega}(s_1, s_2, \dots, s_n) = \omega(1 - s_1, s_2, \dots, s_n)$$

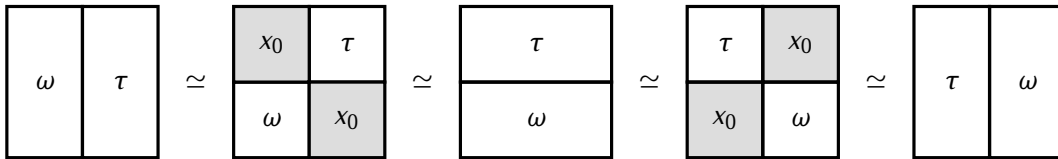
**3.3 Note.** A part of Definition 3.2 makes sense also for  $n = 0$ . In this case we have  $I^0 = \{*\}$  and  $\partial I^0 = \emptyset$ . We define  $\pi_0(X, x_0)$  as the set of homotopy classes of maps  $\omega: (I^0, \partial I^0) \rightarrow (X, x_0)$ . Giving such a map is the same as selecting a point  $\omega(*) = x_\omega \in X$ . Giving a homotopy of such maps is equivalent to giving a path between the corresponding points. Thus two points  $x_\omega$  and  $x_\tau$  represent the same element of  $\pi_0(X, x_0)$  if they belong to the same path connected component. In other words, we get

$$\pi_0(X, x_0) \cong \left( \begin{array}{c} \text{path connected} \\ \text{components of } X \end{array} \right)$$

The trivial element of  $\pi_0(X, x_0)$  is given by the map  $c_{x_0}: I^0 \rightarrow X$  such that  $c_{x_0}(*) = x_0$ . This corresponds to the path connected component of  $x_0$  in  $X$ . In this way  $\pi_0(X, x_0)$  becomes a pointed set. There is no multiplication defined in  $\pi_0(X, x_0)$ .

**3.4 Theorem.** For  $n \geq 2$  then the group  $\pi_n(X, x_0)$  is abelian for any pointed space  $(X, x_0)$ .

*Pictorial proof.* A homotopy  $\omega * \tau \simeq \tau * \omega$  can be depicted as follows:



The shaded squares in the pictures are mapped to the basepoint  $x_0 \in X$ . □

A more rigorous proof can be obtained using the following fact.

**3.5 Eckmann–Hilton Theorem.** *Let  $M$  be a set equipped with two binary operations*

$$\circ: M \times M \rightarrow M, \quad \bullet: M \times M \rightarrow M$$

*Assume that there exist elements  $1_\circ, 1_\bullet \in M$  such that  $m \circ 1_\circ = 1_\circ \circ m = m$  and  $m \bullet 1_\bullet = 1_\bullet \bullet m = m$  for all  $m \in M$ . Assume also, that for any  $m_1, m_2, n_1, n_2 \in M$  we have*

$$(m_1 \circ m_2) \bullet (n_1 \circ n_2) = (m_1 \bullet n_1) \circ (m_2 \bullet n_2)$$

*Then for any  $m, n \in M$  we have  $m \circ n = m \bullet n$ , and  $m \circ n = n \circ m$ .*

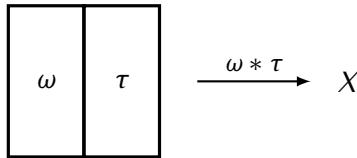
*Proof.* Exercise. □

*Proof of Theorem 3.4.* Recall that multiplication in  $\pi_n(X, x_0)$  is defined by If  $\omega, \tau: (I^n, \partial I^n) \rightarrow (X, x_0)$  then

$$[\omega] \cdot [\tau] = [\omega * \tau]$$

where  $\omega * \tau: (I^n, \partial I^n) \rightarrow (X, x_0)$  is given by

$$(\omega * \tau)(s_1, s_2, \dots, s_n) = \begin{cases} \omega(2s_1, s_2, \dots, s_n) & \text{for } s_1 \in [0, \frac{1}{2}] \\ \tau(2s_1 - 1, s_2, \dots, s_n) & \text{for } s_1 \in [\frac{1}{2}, 1] \end{cases}$$

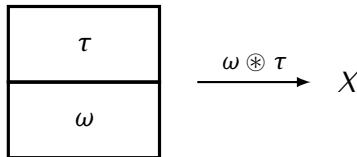


Since  $n \geq 2$ , we can also define a multiplication in  $\pi_n(X, x_0)$  by

$$[\omega] \odot [\tau] = [\omega \circledast \tau]$$

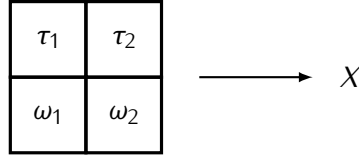
where

$$(\omega \circledast \tau)(s_1, s_2, \dots, s_n) = \begin{cases} \omega(s_1, 2s_2, \dots, s_n) & \text{for } s_2 \in [0, \frac{1}{2}] \\ \tau(s_1, 2s_2 - 1, \dots, s_n) & \text{for } s_2 \in [\frac{1}{2}, 1] \end{cases}$$



Notice that for any  $\omega_1, \omega_2, \tau_1, \tau_2: (I^n, \partial I^n) \rightarrow (X, x_0)$  we have

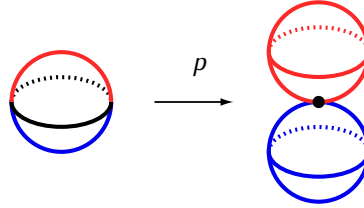
$$(\omega_1 * \omega_2) \circledast (\tau_1 * \tau_2) = (\omega_1 \circledast \tau_1) * (\omega_2 \circledast \tau_2)$$



The result follows from Theorem 3.5. □

**3.6 Alternative construction.** Just as for the fundamental group, higher homotopy groups can be also described using maps from spheres. Since  $I^n / \partial I^n \cong S^n$ , giving a map  $(I^n, \partial I^n) \rightarrow (X, x_0)$  is equivalent to giving a map  $(S^n, s_0) \rightarrow (X, x_0)$  for some basepoint  $s_0 \in S^n$ . Thus elements of  $\pi_n(X, x_0)$  can be described as homotopy classes of such maps.

To describe multiplication in  $\pi_n(X, x_0)$  in this setting, consider the pinch map  $p: S^n \rightarrow S^n \vee S^n$  that maps the upper hemisphere of  $S^n$  onto one copy of  $S^n \subseteq S^n \vee S^n$ , the lower hemisphere onto the second copy, and the equator of  $S^n$  to the basepoint of  $S^n \vee S^n$ :



Given two basepoint preserving maps  $\omega, \tau: (S^n, s_0) \rightarrow (X, x_0)$ , let  $\omega \vee \tau: S^n \vee S^n \rightarrow X$  be the function that maps the first copy of  $S^n$  using  $\omega$  and the second copy using  $\tau$ . Then we have

$$[\omega] \cdot [\tau] = [(\omega \vee \tau) \circ p]$$

The following fact is often useful:

**3.7 Proposition.** *A map  $\omega: (S^n, s_0) \rightarrow (X, x_0)$  represents the trivial element of  $\pi_n(X, x_0)$  if and only if there exists a map  $\omega': D^{n+1} \rightarrow X$  such that  $\omega'|_{S^n} = \omega$ .*

*Proof.* Exercise. □

**3.8 Functoriality.** Let  $f: (X, x_0) \rightarrow (Y, y_0)$  be a map of pointed spaces. For any  $\omega: (I^n, \partial I^n) \rightarrow (X, x_0)$ , composition with  $f$  gives a map  $f \circ \omega: (I^n, \partial I^n) \rightarrow (Y, y_0)$ . If  $\omega, \tau: (I^n, \partial I^n) \rightarrow (X, x_0)$  and  $\omega \simeq \tau$ , then  $f \circ \omega \simeq f \circ \tau$ . Therefore we get a well defined function

$$f_*: \pi_n(X, x_0) \rightarrow \pi_n(Y, y_0)$$

given by  $f_*([\omega]) = [f \circ \omega]$ . If  $n \geq 0$  then  $f_*$  is a homomorphism of groups. For maps  $f: (X, x_0) \rightarrow (Y, y_0)$  and  $g: (Y, y_0) \rightarrow (Z, z_0)$  we have  $(gf)_* = g_*f_*$ . Also, if  $\text{id}_X: (X, x_0) \rightarrow (X, x_0)$  is the identity map, then  $\text{id}_{X*}: \pi_n(X, x_0) \rightarrow \pi_n(X, x_0)$  is the identity homomorphism. This shows that the assignments  $(X, x_0) \rightarrow \pi_n(X, x_0)$  define functors:

$$\pi_0: \mathbf{Top}_* \rightarrow \mathbf{Set}_*$$

$$\pi_1: \mathbf{Top}_* \rightarrow \mathbf{Gr}$$

$$\pi_n: \mathbf{Top}_* \rightarrow \mathbf{Ab}$$

for  $n \geq 2$ , where  $\mathbf{Top}_*$  is the category of pointed topological spaces, and  $\mathbf{Set}_*$ ,  $\mathbf{Gr}$ ,  $\mathbf{Ab}$  are the categories of pointed sets, groups, and abelian groups, respectively.

As a consequence, if  $f: (X, x_0) \rightarrow (Y, y_0)$  is a homeomorphism, then  $f_*: \pi_n(X, x_0) \rightarrow \pi_n(Y, y_0)$  is an isomorphism for  $n \geq 0$ .