

# 19 | Serre classes

**19.1 Definition.** A *Serre class* is a non-empty collection  $\mathcal{C}$  of abelian groups satisfying the property that if

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is a short exact sequence of abelian groups then  $B \in \mathcal{C}$  if and only if  $A, C \in \mathcal{C}$

We will say that a Serre class  $\mathcal{C}$  is a *Serre ring* if it in addition satisfies that if  $A, B \in \mathcal{C}$  then  $A \otimes B \in \mathcal{C}$  and  $\text{Tor}(A, B) \in \mathcal{C}$ .

**19.2 Proposition.** Let  $\mathcal{C}$  is a Serre class. The following hold:

- 1)  $0 \in \mathcal{C}$ .
- 2) If  $A \in \mathcal{C}$  and  $A' \cong A$  then  $A' \in \mathcal{C}$ .
- 3) If  $B \subseteq A$  then  $A \in \mathcal{C}$  if and only if  $B, A/B \in \mathcal{C}$ .
- 4) If  $A_1, \dots, A_n \in \mathcal{C}$  then  $A_1 \oplus \dots \oplus A_n \in \mathcal{C}$ .
- 5) If  $A \rightarrow B \rightarrow C$  is an exact sequence and  $A, C \in \mathcal{C}$  then  $B \in \mathcal{C}$ .

*Proof.* Exercise. □

**19.3 Proposition.** Let  $\mathcal{C}$  is a Serre ring. If  $X$  is a path connected space such that  $H_q(X) \in \mathcal{C}$  for all  $q > 0$  and  $H_q(X; G) \in \mathcal{C}$  for any group  $G \in \mathcal{C}$ .

*Proof.* By the Universal Coefficient Theorem we have

$$H_q(X; G) \cong (H_q(X) \otimes G) \oplus \text{Tor}(H_{q-1}(X), G)$$

For  $q = 0$  this gives  $H_0(X; G) \cong G$ . For  $q = 1$  we get  $H_1(X; G) \cong H_1(X) \otimes G$ . □

**19.4 Example.** All of the following are Serre rings:

- $\mathcal{C}_{\text{fin}}$  = the class of all finite abelian groups.

- $\mathcal{C}_{\text{fg}}$  = the class of all finitely generated abelian groups.
- $\mathcal{C}_{\text{tor}}$  = the class of all torsion abelian groups.

**19.5 Definition.** Let  $\mathcal{C}$  be a Serre class. We say that homomorphism of abelian groups  $f: G \rightarrow H$  is an *isomorphism mod  $\mathcal{C}$*  if  $\text{Ker } f \in \mathcal{C}$  and  $\text{Coker } f := H/f(G) \in \mathcal{C}$

**19.6 Theorem.** Let  $\mathcal{C}$  be a Serre ring. If  $X$  is a simply connected space then the following conditions are equivalent:

- 1)  $\pi_n(X) \in \mathcal{C}$  for all  $n \geq 1$
- 2)  $H_n(X) \in \mathcal{C}$  for all  $n \geq 1$

**19.7 Corollary.** The homotopy groups  $\pi_n(S^m)$  are finitely generated for all  $n, m \geq 1$ .