

16 | Hurewicz Theorem

Hurewicz homomorphism is a map that connects homotopy and homology groups. Recall that $H_n(S^n) \cong \mathbb{Z}$. We will denote by γ_n a chosen generator of $H_n(S^n)$. Given an element $[\varphi: (S^n, s_0) \rightarrow (X, x_0)] \in \pi_n(X, x_0)$ consider the homomorphism $\varphi_*: H_*(S^n) \rightarrow H_*(X)$. This homomorphism depends only on the homotopy class of φ .

16.1 Definition. The *Hurewicz homomorphism* is a function

$$h: \pi_n(X, x_0) \rightarrow H_n(X)$$

given by $h([\varphi]) = \varphi_*(\gamma_n)$.

16.2 Proposition. For any function $f: X \rightarrow Y$ the following diagram commutes:

$$\begin{array}{ccc} \pi_n(X, x_0) & \xrightarrow{f_*} & \pi_n(Y, f(x_0)) \\ \downarrow h & & \downarrow h \\ H_n(X) & \xrightarrow{f_*} & H_n(Y) \end{array}$$

Proof. For $[\varphi] \in \pi_n(X, x_0)$ we have

$$hf_*([\varphi]) = h([f\varphi]) = (f\varphi)_*(\gamma_n) = f_*\varphi_*(\gamma_n) = f_*h([\varphi])$$

□

16.3 Proposition. The Hurewicz homomorphism is a group homomorphism.

Proof. Let $\varphi, \psi: (S^n, s_0) \rightarrow (X, x_0)$ where $n \geq 1$. Recall that the element $[\varphi] \cdot [\psi] \in \pi_n(X, x_0)$ is the homotopy class of the map

$$S^n \xrightarrow{p} S^n \vee S^n \xrightarrow{\varphi \vee \psi} X$$

where $p: S^n \rightarrow S^n \vee S^n$ is the pinch map. Let $r_1, r_2: S^n \vee S^n \rightarrow S^n$ be the retractions of $S^n \vee S^n$ onto the 1st and, respectively, second copy of S^n . We have a commutative diagram

$$\begin{array}{ccccc}
 H_n(S^n) & \xrightarrow{p_*} & H_n(S^n \vee S^n) & \xrightarrow{(\varphi \vee \psi)_*} & H_n(X) \\
 & \searrow \text{id}_* \oplus \text{id}_* & \downarrow \cong \downarrow r_{1*} \oplus r_{2*} & \nearrow \varphi_* + \psi_* & \\
 & & H_n(S^n) \oplus H_n(S^n) & &
 \end{array}$$

This gives:

$$h([\varphi] \cdot [\psi]) = ((\varphi \vee \psi)p)_*(\gamma_n) = (\varphi_* + \psi_*)(\text{id}_* \oplus \text{id}_*)(\gamma_n) = \varphi_*(\gamma_n) + \psi_*(\gamma_n) = h([\varphi]) + h([\psi])$$

□

16.4 Hurewicz Isomorphism Theorem. Let X be a space such that for some $n \leq 2$ we have $\pi_i(X) = 0$ for $i < n$. Then $H_i(X) = 0$ for $0 < i < n$ and the Hurewicz homomorphism

$$h: \pi_n(X, x_0) \rightarrow H_n(X)$$

is an isomorphism.

Proof. Assume first that $X = S^n$. We have $H_i(S^n) = 0$ for $0 < i < n$. In degree n the Hurewicz homomorphism is a map $h: \mathbb{Z} \cong \pi_n(S^n) \rightarrow H_n(S^n) \cong \mathbb{Z}$. The group $\pi_n(S^n)$ is generated by the homotopy class of the identity map $\text{id}_{S^n}: S^n \rightarrow S^n$ (10.17). We have $h([\text{id}_{S^n}]) = \text{id}_{S^n*}(\gamma_n) = \gamma_n$. Therefore h maps a generator of $\pi_n(S^n)$ to a generator of $H^n(S^n)$, and so it is an isomorphism.

Next, assume that $X = \bigvee_{i \in I} S^n$. Again, in this case $H_i(\bigvee_{i \in I} S^n) = 0$ for $0 < i < n$. Also, the retraction maps $r_i: \bigvee_{i \in I} S^n \rightarrow S^n$ give a commutative diagram

$$\begin{array}{ccc}
 \pi_n(\bigvee_{i \in I} S^n) & \xrightarrow[\cong]{\bigoplus r_{i*}} & \bigoplus_{i \in I} \pi_n(S^n) \\
 \downarrow h & & \downarrow \bigoplus_{i \in I} h \\
 H_n(\bigvee_{i \in I} S^n) & \xrightarrow[\cong]{\bigoplus r_{i*}} & \bigoplus_{i \in I} H_n(S^n)
 \end{array}$$

The map $\bigoplus_{i \in I} h$ is an isomorphism by the previous case, so the left vertical map h is also an isomorphism.

As a next step, assume that X is an arbitrary CW complex with $\pi_i(X) = 0$ for $i < n$. By Proposition 5.6 we can assume that $X^{(n-1)} = *$, which gives $H_i(X) = 0$ for $0 < i < n$.

Let $j: X^{(n+1)} \hookrightarrow X$ be the inclusion of the $(n+1)$ -skeleton of X . By Proposition 16.2 we have a commutative diagram

$$\begin{array}{ccc} \pi_n(X^{(n+1)}) & \xrightarrow{j_*} & \pi_n(X) \\ \downarrow h & \cong & \downarrow h \\ H_n(X^{(n+1)}) & \xrightarrow{j_*} & H_n(X) \end{array}$$

The upper homomorphism j_* is an isomorphism by Proposition 5.2, and the lower j_* is an isomorphism by properties of homology groups. As a consequence, it is enough to show that $h: \pi_n(X^{(n+1)}) \rightarrow H_n(X^{(n+1)})$ is an isomorphism.

Since $X^{(n-1)} = *$, it follows that $X^{(n)} = \bigvee_{i \in I} S^n$ and $X^{(n+1)} = X^{(n)} \cup \bigcup_{k \in K} e_k^{(n+1)}$ where $\{e_k^{(n+1)}\}_{k \in K}$ are $(n+1)$ -cells of X . Let $\varphi_k: S^n \rightarrow X^{(n)}$ be the attaching map of the cell $e_k^{(n+1)}$, and let $i: X^{(n)} \hookrightarrow X^{(n+1)}$ denote the inclusion map. We have a commutative diagram

$$\begin{array}{ccccccc} \pi_n(\bigvee_{k \in K} S^n) & \xrightarrow{(\bigvee_{k \in K} \varphi_k)_*} & \pi_n(X^{(n)}) & \xrightarrow{i_*} & \pi_n(X^{(n+1)}) & \longrightarrow & 0 \\ \downarrow h \cong & & \downarrow h \cong & & \downarrow h & & \downarrow \cong \\ H_n(\bigvee_{k \in K} S^n) & \xrightarrow{(\bigvee_{k \in K} \varphi_k)_*} & H_n(X^{(n)}) & \xrightarrow{i_*} & H_n(X^{(n+1)}) & \longrightarrow & H_{n-1}(\bigvee_{k \in K} S^n) \end{array}$$

The upper row of this diagram is exact by Proposition 11.7, and the lower row is exact by the long homology sequence associated to the map $\bigvee_{k \in K} \varphi_k$. By the Five Lemma we obtain that $h: \pi_n(X^{(n+1)}) \rightarrow H_n(X^{(n+1)})$ is an isomorphism.

Finally, let X be an arbitrary space with $\pi_i(X) = 0$ for $i < n$. Let $f: Y \rightarrow X$ be a CW approximation of X (14.3). Using Theorem 15.1 and the previous case we get $H_i(X) \cong H_i(Y) = 0$ for $0 < i < n$.

By Proposition 16.2 we have a commutative diagram

$$\begin{array}{ccc} \pi_n(Y) & \xrightarrow{f_*} & \pi_n(X) \\ \downarrow h \cong & & \downarrow h \\ H_n(Y) & \xrightarrow{f_*} & H_n(X) \end{array}$$

Since f is a weak equivalence, the upper homomorphism f_* is an isomorphism by definition, and the lower f_* is an isomorphism by Theorem 15.1. Also, since Y is a CW complex the left vertical map is an isomorphism by the previous case. Therefore $h: \pi_n(X) \rightarrow H_n(X)$ is an isomorphism. \square

16.5 Inverse Hurewicz Isomorphism Theorem. *Let X be a simply connected space, and let $H_i(X) = 0$ for $1 \leq i < n$ for some $n \geq 2$. Then $\pi_i(X) = 0$ for $i < n$ and the Hurewicz homomorphism $h: \pi_n(X) \rightarrow H_n(X)$ is an isomorphism.*

Proof. Exercise. □

Since all homology groups $H_i(X)$ are abelian but the fundamental group $\pi_1(X)$ need not be abelian, in general the Hurewicz homomorphism $h: \pi_1(X) \rightarrow H_1(X)$ is not an isomorphism. However, a version of Theorem 16.4 still holds with the following modification. Recall, that if G is a group then the commutator of G is the subgroup $[G, G] \subseteq G$ generated by all elements of the form $ghg^{-1}h^{-1}$ for $g, h \in G$. The commutator is a normal subgroup of G , and the quotient group $G^{\text{ab}} := G/[G, G]$ is an abelian group. The group G^{ab} is called the abelianization of G .

If H is an abelian group then any homomorphism $\varphi: G \rightarrow H$ defines a unique homomorphism $\bar{\varphi}: G^{\text{ab}} \rightarrow H$ such that $\varphi = \bar{\varphi}\eta$ where $\eta: G \rightarrow G^{\text{ab}}$ is the quotient homomorphism. Also, if $\psi: G \rightarrow H$ is a homomorphism of arbitrary groups, then $\psi([G, G]) \subseteq [H, H]$, and so ψ induces a homomorphism of abelianizations $\psi^{\text{ab}}: G^{\text{ab}} \rightarrow H^{\text{ab}}$.

16.6 Theorem. *Let X be a path connected space and let $h: \pi_1(X, x_0) \rightarrow H_1(X)$ be the Hurewicz homomorphism. Then the induced homomorphism $\bar{h}: \pi_1(X, x_0)^{\text{ab}} \rightarrow H_1(X)$ is an isomorphism.*

The proof will use the following algebraic fact.

16.7 Lemma. *Consider a sequence of group homomorphisms*

$$G \xrightarrow{\varphi} H \xrightarrow{\psi} K$$

such that ψ is onto and $\ker \psi = N_H(\text{Im } \varphi)$ where $N_H(\text{Im } \varphi)$ is the normalizer of $\text{Im } \varphi$ in H . Then the induced sequence

$$G^{\text{ab}} \xrightarrow{\varphi^{\text{ab}}} H^{\text{ab}} \xrightarrow{\psi^{\text{ab}}} K^{\text{ab}} \rightarrow 0$$

is exact.

Proof. Exercise. □

Proof of Theorem 16.6. Take $X = S^1$. As in the proof of Theorem 16.4 we obtain that $h: \pi_1(S^1) \rightarrow H_1(S^1)$ is an isomorphism. Also, since $\pi_1(S^1) \cong \mathbb{Z}$ is an abelian group, thus $\pi_1(S^1) \cong \pi_1(S^1)^{\text{ab}}$ and, up to this isomorphism, \bar{h} coincides with h .

Next, take $X = \bigvee_{i \in I} S^1$ and let $r_i: \bigvee_{i \in I} S^1 \rightarrow S^1$ be retraction maps. We have a commutative diagram

$$\begin{array}{ccc} \pi_1(\bigvee_{i \in I} S^1) & \xrightarrow{\bigoplus r_{i*}} & \bigoplus_{i \in I} \pi_1(S^1) \\ \downarrow h & & \downarrow \cong \bigoplus_{i \in I} h \\ H_n(\bigvee_{i \in I} S^1) & \xrightarrow[\bigoplus r_{i*}]{} & \bigoplus_{i \in I} H_n(S^1) \end{array}$$

The upper map $\bigoplus r_{i*}$ essentially coincides with the abelianization of $\pi_1(\bigvee_{i \in I} S^1)$, and the map $\bigoplus_{i \in I} h$ coincides, up to an isomorphism, with $\bar{h}: \pi_1(\bigvee_{i \in I} S^1)^{\text{ab}} \rightarrow H_1(\bigvee_{i \in I} S^1)$. It remains to notice that $\bigoplus_{i \in I} h$ is an isomorphism by the previous case.

As in the proof of Theorem 16.4, it remains to consider the case where X is a 2-dimensional CW complex of the form $X = \bigvee_{i \in I} S^1 \cup \bigcup_{k \in K} e_k^2$. Let $\varphi_k: S^1 \rightarrow \bigvee_{i \in I} S^1$ be the attaching map of the cell e_k^{n+1} . Denote $\psi := \bigvee_{k \in K} \varphi_k: \bigvee_{k \in K} S^1 \rightarrow \bigvee_{i \in I} S^1$. Also, let $j: \bigvee_{i \in I} S^1 \hookrightarrow X$ be the inclusion of the 1-skeleton of X . We have a sequence of group homomorphisms

$$\pi_1\left(\bigvee_{k \in K} S^1\right) \xrightarrow{\psi_*} \pi_1\left(\bigvee_{i \in I} S^1\right) \xrightarrow{j_*} \pi_1(X)$$

By van Kampen's Theorem we have $\pi_1(X) \cong \pi_1(\bigvee_{i \in I} S^1) / N_{\pi_1(\bigvee_{i \in I} S^1)}(\text{Im } \psi_*)$. Consider the commutative diagram

$$\begin{array}{ccccccc} \pi_1(\bigvee_{k \in K} S^1)^{\text{ab}} & \xrightarrow{\psi_*} & \pi_1(\bigvee_{i \in I} S^1)^{\text{ab}} & \xrightarrow{i_*} & \pi_1(X)^{\text{ab}} & \longrightarrow & 0 \\ \downarrow \bar{h} \cong & & \downarrow \bar{h} \cong & & \downarrow \bar{h} & & \downarrow \cong \\ H_1(\bigvee_{i \in I} S^1) & \xrightarrow{\psi_*} & H_1(\bigvee_{i \in I} S^1) & \xrightarrow{i_*} & H_1(X) & \longrightarrow & 0 \end{array}$$

The upper row is exact by Lemma 16.7 and the lower row by the long exact homology sequence associated to ψ_* . Using the Five Lemma we obtain that $\bar{h}: \pi_1(X)^{\text{ab}} \rightarrow H_1(X)$ is an isomorphism. \square

16.8 Relative Hurewicz Homomorphism. Recall that $H_n(D^n, S^{n-1}) \cong \mathbb{Z}$. Let \bar{y}_n denote a chosen generator of $H_n(D^n, S^{n-1})$. Given a pointed pair (X, A, x_0) , and an element $[\varphi: (D^n, S^{n-1}, s_0) \rightarrow (X, A, x_0)] \in \pi_n(X, A, x_0)$ consider the homomorphism $\varphi_*: H_n(D^n, S^{n-1}) \rightarrow H_n(X, A)$.

16.9 Definition. The *relative Hurewicz homomorphism* is a function

$$h: \pi_n(X, A, x_0) \rightarrow H_n(X, A)$$

given by $h([\varphi]) = \varphi_*(\bar{y}_n)$.

16.10 Relative Inverse Hurewicz Isomorphism Theorem. Let (X, A) be a pair of simply connected CW complexes. If $H_i(X, A) = 0$ for all $0 < i < n$ for some $n \geq 2$ then $\pi_i(X, A)$ for all $i < n$ and the Hurewicz homomorphism $h: \pi_n(X, A) \rightarrow H_n(X, A)$ is an isomorphism.

16.11 Theorem. Let X, Y be simply connected CW complexes and let $f: X \rightarrow Y$ be a map such that for some $n \geq 2$ the homomorphism $f_*: H_i(X) \rightarrow H_i(Y)$ is an isomorphism for $i < n$ and epimorphism for $i = n$. Then $f_*: \pi_i(X) \rightarrow \pi_i(Y)$ is an isomorphism for $i < n$ and epimorphism for $i = n$.

Proof. Let M_f be the mapping cylinder of f . The assumption about f is equivalent to the statement that $H_i(M_f, X) = 0$ for $i < n$. By Theorem 16.10 this gives $\pi_i(M_f, X) = 0$ for $i < n$. The statement then follows from the long exact sequence of homotopy groups of the pair (M_f, X) . \square

16.12 Theorem. *Let $f: X \rightarrow Y$ be a map of simply connected CW complexes such that $f_*: H_i(X) \rightarrow H_i(Y)$ is an isomorphism for all $i \geq 0$. Then f is a homotopy equivalence.*

Given a CW complex X let $p_X: \tilde{X} \rightarrow X$ denote the universal cover of X . Given a map $f: X \rightarrow Y$ we can find a map $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$ such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ p_X \downarrow & & \downarrow p_Y \\ X & \xrightarrow{f} & Y \end{array}$$

16.13 Theorem. *Let $f: X \rightarrow Y$ be a map of path connected CW complexes. If the homomorphisms $f_*: \pi_1(X) \rightarrow \pi_1(Y)$ and $\tilde{f}_*: H_i(\tilde{X}) \rightarrow H_i(\tilde{Y})$ for all $i \geq 0$ are isomorphisms then f is a homotopy equivalence.*

Proof. By Theorem 16.11 the map $\tilde{f}_*: \pi_i(\tilde{X}) \rightarrow \pi_i(\tilde{Y})$ is an isomorphism for all $i \geq 0$. Since $\pi_i(\tilde{X}) \cong \pi_i(X)$ and $\pi_i(\tilde{Y}) \cong \pi_i(Y)$ for $i \geq 2$ this gives that $f_*: \pi_i(X) \rightarrow \pi_i(Y)$ is an isomorphism for $i \geq 2$. By assumption, $f_*: \pi_1(X) \rightarrow \pi_1(Y)$ is an isomorphism as well, so f is a weak equivalence. \square