# 5 | Some Computations

**5.1 Proposition.** *If $X$ is a contractible space then $\pi_n(X) = 0$ for all $n \geq 0$.*

*Proof.* Since $X \simeq *$ thus $\pi_n(X) \overset{\simeq}{=} \pi_n(*) = 0$. □

**5.2 Proposition.** *If $X$ is a relative CW-complex, $X^{(n)}$ is the $n$-skeleton of $X$, and $x_0 \in X^{(n)}$, then the homomorphism $i_*\colon \pi_k(X^{(n)}, x_0) \to \pi_k(X, x_0)$ induced by the inclusion map $i\colon X^{(n)} \hookrightarrow X$ is an isomorphism for $k < n$ and an epimorphism for $k = n$.*

*Proof.* We can assume that $x_0 \in X^{(0)}$. Consider $S^k$ as a CW complex with a 0-cell $s_0 \in S^k$. By the Cellular Approximation Theorem 2.11 any map $\omega\colon (S^k, s_0) \to (X, x_0)$ is homotopic (relative to the basepoint) to cellular map $\omega'\colon (S^k, s_0) \to (X, x_0)$. If $k \leq n$ then $\omega'(S^k) \subseteq X^{(n)}$, so $\omega'$ represents an element of $\pi_k(X^{(n)}, x_0)$ such that $i_*([\omega']) = [\omega]$. This shows that $i_*$ is an epimorphism for $k \leq n$.

Next, take $[\omega_0], [\omega_1] \in \pi_k(X^{(n)}, x_0)$. We can assume that the maps $\omega_0, \omega_1\colon (S^k, s_0) \to (X^{(n)}, x_0)$ are cellular. If $i_*([\omega_0]) = i_*([\omega_1])$ then there is a homotopy $h\colon S^k \times [0, 1] \to X$. Using the Cellular Approximation Theorem 2.11 again, we can assume that this homotopy is a cellular map. Since $\dim S^k \times [0, 1] = k + 1$, we obtain that if $k < n$ then $h(S^k \times [0, 1]) \to X^{(n)}$. Thus $h$ gives a homotopy between $\omega_0$ and $\omega_1$ in $X^{(n)}$. Therefore $[\omega_0] = [\omega_1] \in \pi_k(X^{(n)}, x_0)$. This shows that $i_*$ is a monomorphism for $k < n$. □

**5.3 Corollary.** *If $k < n$ then $\pi_k(S^n) = 0$*

*Proof.* A sphere $S^n$ can be given a CW-complex structure with one 0-cell and one $n$-cell. Then by Proposition 5.2 for $k < n$ we have an epimorphism

$$\pi_k((S^n)^{(k)}) \to \pi_k(S^n)$$

Since $(S^n)^{(k)} = *$, thus $\pi_k((S^n)^{(k)}) = 0$ and so $\pi_k(S^n) = 0$. □

**5.4 Definition.** A space $X$ is *$n$-connected* if $\pi_k(X) = 0$ for all $k \leq n$.

Corollary 5.3 can be restated by saying that the sphere $S^n$ is $(n-1)$-connected.

**5.5 Proposition.** *For any space $X$ and $n \geq 0$ the following conditions are equivalent:*

1) *$X$ is $n$-connected.*
2) *For any $k \leq n$ and any map $\varphi\colon S^k \to X$ there exists a map $\overline{\varphi}\colon D^{k+1} \to X$ such that $\overline{\varphi}|_{S^k} = \varphi$.*

*Proof.* Follows from Proposition 3.7. □

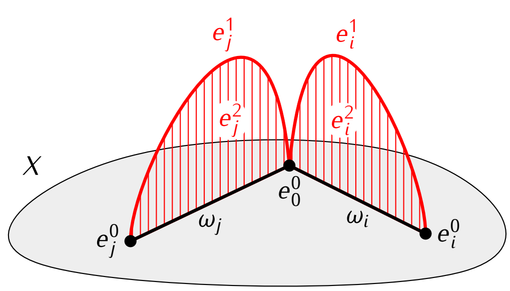
By Proposition 5.2 if $X$ is a CW complex that has only one 0-cell and no $k$-cells for $k \leq n$ (i.e. $X^{(n)} = *$) then $X$ is $n$-connected. One can show that the opposite is also true, up to a homotopy equivalence:

**5.6 Proposition.** *If $X$ is an $n$-connected CW complex, then there exists a CW complex $Y$ such that such that $X \simeq Y$ and $Y^{(n)} = *$.*

*Proof.* We will show inductively that for any $k = 0, \dots, n$ there exists a CW complex $Y_k$ such that $X \simeq Y_k$ and $Y_k^{(k)} = *$.

Choose a 0-cell $e^0_0 \in X$. Since $\pi_0(X) = 0$, the space $X$ is path connected. Thus for any 0-cell $e^0_i$ we can select a path $\omega_i\colon [0,1] \to X$ such that $\omega(0) = e_0$ and $\omega(1) = e^0_i$. By the Cellular Approximation Theorem 2.11, we can assume that $\omega_i$ is a path in $X^{(1)}$. We construct a new CW complex $Y''_0$ by attaching cells to $X$ as follows.

1) First, for each 0-cell $e^0_i$ we attach to $X$ a 1-cell $e^1_i$ using the attaching map $\varphi_i\colon S^0 = \{-1, 1\} \to X$ such that $\varphi_i(-1) = e^0_0$ and $\varphi_i(1) = e^0_i$. Let $Y'_0 = X \cup \bigcup_i e^1_i$ be the CW complex obtained in this way.
2) In $Y'_0$ each 0-cell $e^0_i$ is connected to $e^0_0$ by two different paths: $\omega_i$, and a path $\tau_i$ that traverses the new cell $e^1_i$. For each $i$ we attach a 2-cell $e^2_i$ using an attaching map $\psi_i\colon S^1 \to Y'_0$ that send the lower half circle to $\omega_i$ and the upper half circle to $\tau_i$. Let $Y''_0 = Y'_0 \cup \bigcup_i e^2_i$.




Notice that $X$ is a deformation retract of $Y''_0$, so the inclusion map $j\colon X \hookrightarrow Y''_0$ is a homotopy equivalence. Also $A = X^{(0)} \cup \bigcup_i e^1_i$ is a contractible subcomplex of $Y''_0$. By Proposition 2.15, the quotient map

$q\colon Y_0'' \to Y_0''/A$ is a homotopy equivalence. Since $Y_0''/A$ has a CW complex structure with only one 0-cell we can take $Y_0 = Y_0''/A$.

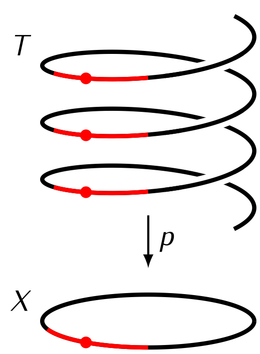
Next, assume that for some $k \leq n$ we have already constructed a CW complex $Y_{k-1}$ such that $X \simeq Y_{k-1}$ and $Y_{k-1}^{(k-1)} = *$. This means that the $k$-skeleton of $Y_{k-1}$ is given by $Y_{k-1}^{(k)} = \bigvee_i S^k$, with one copy of $S^k$ for each $k$-cell $e_i^k$ in $Y_{k-1}$. Let $\varphi_j\colon S^k \hookrightarrow \bigvee_i S^k \subseteq Y_{k-1}$ be the inclusion of the $j$-th copy of $S^k$. Since $\pi_k(Y_{k-1}) \cong \pi_k(X) = 0$, each map $\varphi_i$ extends to a map $\omega_i\colon D^{k+1} \to Y_{k-1}$. We construct a new CW complex $Y_k''$ by attaching cells to $Y_{k-1}$ as follows.

1) First, for each $i$ we attach a $(k+1)$-cell $e_i^{k+1}$ using $\varphi_i\colon S^k \to Y_{k-1}$ as the attaching map. Let $Y_k' = Y_{k-1} \cup \bigcup_i e_i^{k+1}$ be the CW complex obtained in this way.
2) For each $i$ we have now two maps $D^{k+1} \to Y_k'$: the map $\omega_i$, and the characteristic map $\tau_i$ of the cell $e_i^{k+1}$. Using these maps we attach, for each $i$, a $(k+2)$-cell $e_i^{k+2}$, using an attaching map $\psi_i\colon S^{k+1} \to Y_k'$ that sends the lower hemisphere of $S^{k+1}$ to $\omega_i$ and the upper hemisphere to $\tau_i$. Let $Y_k'' = Y_k' \cup \bigcup_i e_i^2$.

As before, we observe that $Y_{k-1}$ is a deformation retract of $Y_k''$, and that $A = Y_{k-1}^{(k)} \cup \bigcup_i e_i^k$ is a contractible subcomplex of $Y_k''$. Therefore we obtain a $X \simeq Y_{k-1} \simeq Y_k'' \simeq Y_k''/A$. It remains to notice that the space $Y_k = Y_k''/A$ has a CW-complex structure such that $Y_k^{(k)} = *$.

□

**5.7 Homotopy groups and coverings.** Recall that covering of a space $X$ is a map $p\colon T \to X$ which is locally homeomorphic to the projection map $\mathrm{pr}_1\colon U \times D \to U$ for some discrete space $D$.




Recall also, that one of the main properties of coverings is the following fact:

**5.8 Theorem (Lifting Criterion).** *Let $p\colon T \to X$ be a covering, let $x_0 \in X$ and let $\tilde{x}_0 \in p^{-1}(x_0)$. Assume that $Y$ is a connected and locally path connected space and let $y_0 \in Y$. A map $f\colon (Y, y_0) \to (X, x_0)$*

*has a lift* $\tilde{f}\colon (Y, y_0) \to (T, \tilde{x}_0)$ *if and only if* $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(T, \tilde{x}_0))$.

$$\begin{tikzcd} & T \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

*Moreover, if a lift* $\tilde{f}$ *exists, then it is unique.*

Recall that for any covering $p\colon (T, \tilde{x}_0) \to (X, x_0)$ the induced homomorphism $p_*\colon \pi_1(T, \tilde{x}_0) \to \pi_1(X, x_0)$ is a monomorphism. Using Theorem 5.8 we can generalize this as follows:

**5.9 Proposition.** *If* $p\colon T \to X$ *is a covering,* $x_0 \in X$ *and* $\tilde{x}_0 \in p^{-1}(x_0)$*, then the induced homomorphism*

$$p_*\colon \pi_n(T, \tilde{x}_0) \to \pi_n(X, x_0)$$

*is an isomorphism for all* $n > 1$.

*Proof.* Let $n > 1$ and $\omega\colon (S^n, s_0) \to (X, x_0)$ represents an element of $\pi_n(X, x_0)$. Since $\pi_1(S^n) = 0$, by Theorem 5.8 there exists a map $\widetilde{\omega}\colon (S^n, s_0) \to (T, \tilde{x}_0)$ such that $p\widetilde{\omega} = \omega$. This shows that $p_*\colon \pi_n(T, \tilde{x}_0) \to \pi_n(X, x_0)$ is onto.

Next, assume that $\omega_0, \omega_1\colon (S^n, s_0) \to (T, \tilde{x}_0)$ are maps such that $p_*([\omega_0]) = p_*([\omega_1])$. This means that there exists a basepoint preserving homotopy $h\colon S^n \times [0, 1] \to X$, such that $h_0 = p\omega_0$, $h_1 = p\omega_1$. Since $S^n \times [0, 1] \simeq S^n$ we have $\pi_1(S^n \times [0, 1]) \cong \pi_1(S^n) = 0$. Thus by Theorem 5.8, there exists a homotopy $\widetilde{h}\colon S^n \times [0, 1] \to T$ such that $p\widetilde{h} = h$ and $\widetilde{h}(s_0, 0) = \tilde{x}_0$. Using the uniqueness of lifts, one can check that $\widetilde{h}_0 = \omega_0$ and $\widetilde{h}_1 = \omega_1$, and that the homotopy $\widetilde{h}$ preserves the basepoint (exercise). It follows that $[\omega_0] = [\omega_1]$ in $\pi_1(T, \tilde{x}_0)$. Therefore $p_*$ is a monomorphism.

□

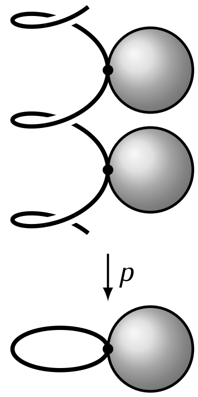
**5.10 Example.** $\pi_n(S^1) = 0$ for all $n > 1$.

Indeed, universal covering of $S^1$ is given by a map $p\colon \mathbb{R} \to S^1$. Since $\mathbb{R}$ is a contractible space, by Proposition 5.9 for $n > 1$ we obtain

$$\pi_n(S^1) \cong \pi_n(\mathbb{R}) = 0$$

**5.11 Example.** If $m > 1$ then $\pi_n(S^1 \vee S^m) \cong \pi_n(\bigvee_{i \in \mathbb{Z}} S^m)$ for all $n > 1$.

To see this, notice that the universal covering of $S^1 \vee S^m$ is the space $\widetilde{X}$ obtained by attaching copies of $S^m$ at all integer points of the real line:




The space $\widetilde{X}$ can be given the structure of a CW complex, such that the real line $\mathbb{R}$ is its subcomplex. Since $\mathbb{R} \simeq \{*\}$, by Theorem 2.14 we have $\widetilde{X} \simeq \widetilde{X}/\mathbb{R} \cong \bigvee_{i\in\mathbb{Z}} S^m$. Therefore for $n > 1$ we obtain

$$\pi_n(S^1 \vee S^m) \cong \pi_n(\widetilde{X}) \cong \pi_n(\bigvee_{i\in\mathbb{Z}} S^m)$$

**5.12 Note.** Example 5.11 can be used to show that if $X$, $Y$ are spaces such that $\pi_n(X) \cong \pi_n(Y)$ for all $n \geq 0$, then this does not imply that $X \simeq Y$.

Take, for example, $X = S^1 \vee S^m$ for some $m > 1$, and let $Y = S^1 \vee S^m \vee S^m$. These spaces are not homotopy equivalent, since they have different homology groups: $H_m(X;\mathbb{Z}) \cong \mathbb{Z}$ and $H_m(Y;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}$.

On the other hand, since both these spaces are path connected, we have $\pi_0(X) \cong \pi_0(Y) \cong \{*\}$. Also, since $\pi_1(S^m) = 0$, thus by van Kampen's theorem we get $\pi_1(X) \cong \pi_1(S^1) \cong \pi_1(Y)$.

The universal covering space $\widetilde{Y}$ of $Y$ is the space obtained by attaching $S^m \vee S^m$ at all integer points of $\mathbb{R}$. Using the same argument as in Example 5.11, we obtain $\widetilde{Y} \simeq \bigvee_{i\in\mathbb{Z}}(S^m \vee S^m) \cong \bigvee_{i\in\mathbb{Z}} S^m$. Therefore for $n \geq 2$ we have

$$\pi_n(X) \cong \pi_n(\bigvee_{i\in\mathbb{Z}} S^m) \cong \pi_n(Y)$$

**5.13 Theorem.** *For a family $(X_i, \bar{x}_i)_{i\in I}$ be a family of pointed spaces there is an isomorphism*

$$\pi_n\left(\prod_{i\in I} X_i,\ (\bar{x}_i)_{i\in I}\right) \cong \prod_{i\in I} \pi_n(X_i, \bar{x}_i)$$

*Proof.* For $j \in I$ let $p_j\colon \prod_{i\in I} X_i \to X_j$ denote the projection onto the $j$-th factor. The induced homomorphisms $p_{j*}$ define a homomorphism:

$$\prod_{i\in I} p_{i*}\colon \pi_n\left(\prod_{i\in I} X_i,\ (\bar{x}_i)_{i\in I}\right) \to \prod_{i\in I} \pi_n(X_i, \bar{x}_i)$$

To obtain a homomorphism going in the opposite direction, let $([\omega_i])_{i\in I}$ be an element of $\prod_{i\in I}\pi_n(X_i, \bar{x}_i)$. Then each $\omega_i$ is a map $\omega_i\colon (S^n, s_0) \to (X_i, \bar{x}_i)$. Take the product map

$$\prod_{i\in I}\omega_i\colon (S^n, s_0) \to (\prod_i X_i, \bar{x}_i)$$

One can check that the assignment $([\omega_i])_{i\in I} \mapsto [\prod_{i\in I}\omega_i]$ gives a well-defined homomorphism

$$g\colon \prod_{i\in I}\pi_n(X_i, \bar{x}_i) \to \pi_n\left(\prod_{i\in I}X_i,\ (\bar{x}_i)_{i\in I}\right)$$

and that the compositions $g \circ \prod_{i\in I} p_{i*}$ and $\prod_{i\in I} p_{i*} \circ g$ are identity homomorphisms (exercise). $\square$

**5.14 Example.** Since $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_n(S^1) = 0$ for $n > 1$, thus for any set $I$ we have

$$\pi_n\left(\prod_{i\in I}S^1\right) \cong \begin{cases} \prod_{i\in I}\mathbb{Z} & \text{for } n = 1 \\ 0 & \text{for } n > 1 \end{cases}$$