

9 | Exact Puppe Sequence

Recall that a sequence of pointed sets

$$(S_2, s_2) \xrightarrow{f} (S_1, s_1) \xrightarrow{g} (S_0, s_0)$$

is *exact at* S_1 if $f(S_2) = g^{-1}(s_0)$.

For pointed spaces (X, x_0) and (Y, y_0) let $[X, Y]_*$ denote the set of pointed homotopy classes of maps $X \rightarrow Y$. This is a pointed set, with the basepoint represented by the constant function $c_{y_0}: X \rightarrow Y$, $c_{y_0}(x) = y_0$ for all $x \in X$.

9.1 Definition. A pointed space (X, x_0) is well-pointed if the pair (X, x_0) has the homotopy extension property.

9.2 Definition. A sequence of maps of spaces

$$(X_0, x_0) \xrightarrow{f_0} (X_1, x_1) \xrightarrow{f_1} (X_2, x_2)$$

is *exact at* X_1 if for any well-pointed space (Y, y_0) the sequence pointed sets

$$[Y, X_0]_* \xrightarrow{f_{0*}} [Y, X_1]_* \xrightarrow{f_{1*}} [Y, X_2]_*$$

is exact at $[Y, X_1]_*$.

9.3 Proposition. If $p: E \rightarrow B$ is a Hurewicz fibration, $e_0 \in E$, $b_0 = p(e_0) \in B$, $F = p^{-1}(b_0)$, and $i: F \rightarrow E$ is the inclusion map then the sequence $(F, e_0) \xrightarrow{i} (E, e_0) \xrightarrow{p} (B, b_0)$ is exact at E .

Proof. Exercise. □

Let $f: (X, x_0) \rightarrow (Y, y_0)$ be any pointed map. Consider the sequence

$$\text{hofib } f \xrightarrow{i(f)} X \xrightarrow{f} Y$$

where

$$i(f): \text{hofib}_{y_0} f = \{(x, \omega) \in X \times PY \mid f(x) = \omega(0), \omega(1) = y_0\} \longrightarrow X$$

is given by $i(f)(x, \omega) = x$. Since this sequence is homotopy equivalent to a sequence given by a Hurewicz fibration, it is exact at X . We can continue this construction inductively, by taking consecutive homotopy fibers:

$$\dots \longrightarrow \text{hofib } i^3(f) \xrightarrow{i^4(f)} \text{hofib } i^2(f) \xrightarrow{i^3(f)} \text{hofib } i(f) \xrightarrow{i^2(f)} \text{hofib } f \xrightarrow{i(f)} X \xrightarrow{f} Y \quad (*)$$

In this way we obtain a sequence which is exact at all spaces. As it turns out, this sequence has a more convenient description. The starting point for it is the following fact:

9.4 Proposition. *Let $f: X \rightarrow Y$ be a map and $y_0 \in Y$. Then the map $i(f): \text{hofib}_{y_0} f \rightarrow X$ is a Hurewicz fibration.*

Proof. Exercise. □

9.5 Corollary. *For any map of pointed spaces $f: (X, x_0) \rightarrow (Y, y_0)$ we have a commutative diagram*

$$\begin{array}{ccccc} \text{hofib } i(f) & \xrightarrow{i^2(f)} & \text{hofib } f & \xrightarrow{i(f)} & X \xrightarrow{f} Y \\ g \uparrow \simeq & \nearrow j & & & \\ \Omega Y & & & & \end{array}$$

Proof. We have

$$i(f)^{-1}(x_0) = \{(x_0, \omega) \in X \times PY \mid \omega(0) = f(x_0) = y_0, \omega(1) = y_0\} \cong \Omega Y$$

Thus ΩY can be identified with the fiber of $i(f)$ over y_0 , and the map $j: \Omega Y \rightarrow \text{hofib } f$, $j(\omega) = (x_0, \omega)$ with the inclusion of the fiber. By Proposition 9.4 and Corollary 8.18 we obtain a homotopy equivalence $g: \Omega Y \rightarrow \text{hofib } i(f)$ such that the above diagram commutes. □

9.6 Note. The homotopy equivalence in Corollary 9.5 can be explicitly described as follows. Up to a homeomorphism we have

$$\text{hofib } i(f) = \{(\omega, \tau) \in PX \times PY \mid f\omega(0) = \tau(0), \omega(1) = y_0, \tau(1) = x_0\}$$

Then $i^2(f): \text{hofib } i(f) \rightarrow \text{hofib } f$ is given by $(\omega, \tau) \mapsto (\omega(0), \tau)$ and $g(\tau) = (c_{x_0}, \tau)$.

Applying Corollary 9.5 iteratively to the sequence (*) we get homotopy equivalences

$$\begin{aligned}
 \mathrm{hofib} \, i(f) &\xleftarrow{\simeq} \Omega Y \\
 \mathrm{hofib} \, i^2(f) &\xleftarrow{\simeq} \Omega X \\
 \mathrm{hofib} \, i^3(f) &\xleftarrow{\simeq} \Omega \mathrm{hofib} \, f \\
 \mathrm{hofib} \, i^4(f) &\xleftarrow{\simeq} \Omega \mathrm{hofib} \, i(f) \simeq \Omega^2 Y \\
 \mathrm{hofib} \, i^5(f) &\xleftarrow{\simeq} \Omega \mathrm{hofib} \, i^2(f) \simeq \Omega^2 X \\
 \dots &\quad \dots \quad \dots \quad \dots \quad \dots
 \end{aligned}$$

Moreover, one can check that the following diagram commutes up to homotopy:

$$\begin{array}{ccccccccccccccc}
 \dots & \longrightarrow & \mathrm{hofib} \, i^4(f) & \xrightarrow{i^5(f)} & \mathrm{hofib} \, i^3(f) & \xrightarrow{i^4(f)} & \mathrm{hofib} \, i^2(f) & \xrightarrow{i^3(f)} & \mathrm{hofib} \, i(f) & \xrightarrow{i^2(f)} & \mathrm{hofib} \, f & \xrightarrow{i(f)} & X & \xrightarrow{f} & Y \\
 & & \uparrow \wr & & \uparrow \wr & & \uparrow \wr & & \uparrow \wr & & \uparrow \parallel & & \uparrow \parallel & & \uparrow \parallel \\
 \dots & \longrightarrow & \Omega^2 Y & \xrightarrow{\Omega j} & \Omega \mathrm{hofib} \, f & \xrightarrow{\Omega i(f)} & \Omega X & \xrightarrow{\Omega f} & \Omega Y & \xrightarrow{j} & \mathrm{hofib} \, f & \xrightarrow{i(f)} & X & \xrightarrow{f} & Y
 \end{array} \quad (**)$$

Since the upper row of this diagram is exact, the same is true for the lower row.

9.7 Definition. The sequence in the lower row of the diagram (**) is called the *Puppe exact sequence* associated to the map f .

As a consequence, for any map of pointed spaces $f: (X, x_0) \rightarrow (Y, y_0)$ and any well-pointed space (Z, z_0) we obtain a long exact sequence of sets:

$$\begin{aligned}
 \dots \xrightarrow{\Omega^2 f_*} [Z, \Omega^2 Y]_* &\xrightarrow{\Omega j_*} [Z, \Omega \mathrm{hofib} \, f]_* \xrightarrow{\Omega i(f)_*} [Z, \Omega X]_* \xrightarrow{\Omega f_*} [Z, \Omega Y]_* \\
 &\xrightarrow{j_*} [Z, \mathrm{hofib} \, f]_* \xrightarrow{i(f)_*} [Z, X]_* \xrightarrow{f_*} [Z, Y]_* \quad (\boxtimes)
 \end{aligned}$$

9.8 Note. For any pointed space (X, x_0) and $n \geq 1$ the loop space $\Omega^n X$ is quipped with a multiplication map $\mu: \Omega^n X \times \Omega^n X \rightarrow \Omega^n X$ given by concatenation of loops. For any pointed space (Z, z_0) this defines a multiplication

$$\mu_*: [Z, \Omega^n X]_* \times [Z, \Omega^n X]_* \rightarrow [Z, \Omega^n X]_*$$

given by $\mu_*([\varphi], [\psi]) = [\mu \circ (\varphi \times \psi)]$. This equips the set $[Z, \Omega^n X]_*$ with a group structure. Moreover, for $n \geq 2$ the multiplication μ commutes up to homotopy, and in effect $[Z, \Omega^n X]_*$ becomes an abelian group.

As a result the exact sequence (\boxtimes) becomes an exact sequence of groups starting at $[Z, \Omega Y]_*$ and its groups are abelian starting with $[Z, \Omega^2 Y]_*$.

9.9 Loop spaces and suspensions. There is a different way of interpreting group structures appearing in the sequence (\boxtimes) , which uses suspensions of a space in place of loop spaces.

9.10 Definition. Let X be a space. The *unreduced suspension* of X is the space

$$SX = X \times [0, 1] / (X \times \{0, 1\})$$

9.11 Note. Any map $f: X \rightarrow Y$ defines a map $Sf: SX \rightarrow SY$ given by $Sf([x, t]) = [f(x), t]$. This map is called the suspension of f . In this way we obtain the suspension functor

$$S: \mathbf{Top} \rightarrow \mathbf{Top}$$

This functor preserves homotopy classes of maps: if $f, g: X \rightarrow Y$ and $f \simeq g$ then $Sf \simeq Sg$.

9.12 Example. For a sphere S^n we have $SS^n \cong S^{n+1}$.

9.13 Definition. Let (X, x_0) be a pointed space. The *reduced suspension* of X is the pointed space

$$\Sigma X = X \times [0, 1] / (X \times \{0, 1\} \cup \{x_0\} \times [0, 1])$$

or equivalently $\Sigma X = SX / \{[x_0, t] \mid t \in [0, 1]\}$. The basepoint in ΣX is given by $[x_0, 0] \in \Sigma X$.

9.14 Note. If (X, x_0) is a well-pointed space, then Proposition 2.15 implies that the quotient map $SX \rightarrow \Sigma X$ is a homotopy equivalence. In particular, for any basepoint $x_0 \in S^n$ we have $\Sigma S^n \simeq SS^n \cong S^{n+1}$. One can show that actually there is a homeomorphism $\Sigma S^n \cong S^{n+1}$.

9.15 Note. Any map $f: (X, x_0) \rightarrow (Y, y_0)$ of pointed spaces, defines a map $\Sigma f: \Sigma X \rightarrow \Sigma Y$ given by $\Sigma f([x, t]) = [f(x), t]$. This defines the suspension functor

$$\Sigma: \mathbf{Top}_* \rightarrow \mathbf{Top}_*$$

Similarly as for the unreduced suspension, the reduced suspension preserves homotopy classes: if $f, g: (X, x_0) \rightarrow (Y, y_0)$ are maps of pointed spaces and $f \simeq g$ then $\Sigma f \simeq \Sigma g$.

Let X be a Hausdorff space. By properties of mapping spaces (8.5) the adjunction map $\text{adj}(\omega) = \omega^\#$ defines a homeomorphism $\text{adj}: \text{Map}(X \times [0, 1], Y) \rightarrow \text{Map}(X, PY)$. Let (X, x_0) and (Y, y_0) be pointed spaces. Consider $\Omega_{y_0} Y$ as a subspace of PY and let $\text{Map}_*(X, \Omega_{y_0} Y)$ be the subspace of $\text{Map}(X, PY)$ consisting of basepoint preserving maps. Then adj restricts to a homeomorphism between this subspace and the subspace of $\text{Map}(X \times [0, 1], Y)$ consisting of all maps $f: X \times [0, 1] \rightarrow Y$ such that $f(X \times \{0, 1\} \cup \{x_0\} \times [0, 1]) = y_0$. Such maps are in a bijective correspondence with basepoint preserving maps $\Sigma X \rightarrow Y$. In this way we obtain a homeomorphism

$$\text{adj}: \text{Map}_*(\Sigma X, Y) \xrightarrow{\cong} \text{Map}_*(X, \Omega Y)$$

On the level of homotopy classes of maps this gives a bijection

$$\text{adj}: [\Sigma X, Y]_* \xrightarrow{\cong} [X, \Omega Y]_*$$

The set of the right hand side has a group structure induced by concatenation of loops. A group structure on the left hand side can be defined using the pinch map $\Sigma X \rightarrow \Sigma X \vee \Sigma X$. In this way the above bijection becomes an isomorphism of groups.

As a result, the exact sequence (♣) can be equivalently written as

$$\dots \xrightarrow{f_*} [\Sigma^2 Z, Y]_* \xrightarrow{j_* \text{adj}} [\Sigma Z, \text{hofib } f]_* \xrightarrow{i(f)_*} [\Sigma Z, X]_* \xrightarrow{f_*} [\Sigma Z, Y]_* \xrightarrow{j_* \text{adj}} [Z, \text{hofib } f]_* \xrightarrow{i(f)_*} [Z, X]_* \xrightarrow{f_*} [Z, Y]_*$$

Consider this sequence with $Z = S^0$. Since $\Sigma^n S^0 \cong S^n$ we obtain

$$\dots \xrightarrow{f_*} [S^2, Y]_* \xrightarrow{j_* \text{adj}} [S^1, \text{hofib } f]_* \xrightarrow{i(f)_*} [S^1, X]_* \xrightarrow{f_*} [S^1, Y]_* \xrightarrow{j_* \text{adj}} [S^0, \text{hofib } f]_* \xrightarrow{i(f)_*} [S^0, X]_* \xrightarrow{f_*} [S^0, Y]_*$$

Since $[S^n, Y]_* = \pi_n(Y)$ we recover the long exact sequence from Corollary 8.12.