8 | From Maps to Fibrations

As we have seen any fibration $F \to E \xrightarrow{p} B$ has the associated long exact sequence

$$\cdots \to \pi_n(F) \longrightarrow \pi_n(E) \xrightarrow{p_*} \pi_n(B) \to \dots$$

that relates the homotopy groups of the spaces B, E, and F. The main goal of this chapter is to show that this approach to computing homotopy groups can be used with an arbitrary map $f\colon X \to Y$ taken in place of a fibration p. We will show that the following holds:

8.1 Theorem. *Given any map $f\colon X \to Y$ there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "g_f", "\simeq"'] \arrow[dr, "f"'] & & E_f \arrow[dl, "p_f"] \\ & Y & \end{tikzcd}$$

such that $p_f\colon E_f \to Y$ is a Hurewicz fibration and $g\colon E_f \to X$ is a homotopy equivalence.

For $x_0 \in X$ and $e_0 = g_f(x_0) \in E_f$ we will get $\pi_n(X, x_0) \cong \pi_n(E_f, e_0)$ for all $n \geq 0$. In this way, the long exact sequence of a fibration gives an exact sequence

$$\cdots \to \pi_n(F, e_0) \longrightarrow \pi_n(X, x_0) \xrightarrow{f_*} \pi_n(Y, y_0) \to \dots$$

where $F = p_f^{-1}(y_0)$.

8.2 Mapping spaces. For spaces X, Y, let $\mathrm{Map}(X, Y)$ denote the set of all continuous functions $X \to Y$. For $A \subseteq X$ and $U \subseteq Y$ let $P(A, U) \subseteq \mathrm{Map}(X, Y)$ be the set

$$P(A, U) = \{f \in \mathrm{Map}(X, Y) \mid f(A) \subseteq U\}$$

8.3 Definition. The compact-open topology on $\mathrm{Map}(X, Y)$ is the topology with subbasis given by all sets of the form $P(A, U)$ where $A \subseteq X$ is compact and $U \subseteq Y$ is open.

Let X, Y, Z be spaces. For a function $\varphi\colon Z \to \mathrm{Map}(X, Y)$ denote by $\varphi^\sharp\colon Z \times X \to Y$ the function given by $\varphi^\sharp(z, x) = \varphi(z)(x)$. We will say that $\varphi^\sharp$ is the *adjoint* of φ.

8.4 Theorem. *If X is a locally compact Hausdorff space, then the compact-open topology on* $\mathrm{Map}(X, Y)$ *is the unique topology with the property that a map $\varphi\colon Z \to \mathrm{Map}(X, Y)$ is continuous if and only if $\varphi^\sharp\colon Z \times X \to Y$ is continuous.*

8.5 All mapping spaces below are equipped with the compact-open topology. The following properties hold:

1) The evaluation map $\mathrm{ev}\colon \mathrm{Map}(X, Y) \times X \to Y$ given by $\mathrm{ev}(f, x) = f(x)$ is continuous.
2) In particular for every $x_0 \in X$ the map $\mathrm{ev}_{x_0}\colon \mathrm{Map}(X, Y) \to Y$, $\mathrm{ev}_{x_0}(f) = f(x_0)$ is continuous.
3) If $\{*\}$ is a one point space, then the map $\mathrm{ev}_*\colon \mathrm{Map}(\{*\}, Y) \to Y$ is a homeomorphism.
4) For any continuous function $f\colon X \to Y$ and any space Z the induced function $f_*\colon \mathrm{Map}(Z, X) \to \mathrm{Map}(Z, Y)$ given by $f_*(g) = f \circ g$ is continuous.
5) For any continuous function $f\colon X \to Y$ and any space Z the induced function $f^*\colon \mathrm{Map}(Y, Z) \to \mathrm{Map}(Y, X)$ given by $f^*(g) = g \circ f$ is continuous.
6) If Y is a locally compact Hausdorff space, then for any spaces X and Y the map $F\colon \mathrm{Map}(X, Y) \times \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$ given by $F(f, g) = g \circ f$ is continuous.
7) If Y is a locally compact Hausdorff space and X is a Hausdorff space then for any space Z the map $\mathrm{adj}\colon \mathrm{Map}(X, \mathrm{Map}(Y, Z)) \to \mathrm{Map}(X \times Y, Z)$ given by $\mathrm{adj}(\varphi) = \varphi^\sharp$ is a homeomorphism.

From now on, all mapping spaces will taken with the compact-open topology.

8.6 Example. Let X be a locally compact space and let $f, g\colon X \to Y$. Giving a map $\omega\colon [0, 1] \to \mathrm{Map}(X, Y)$ such that $\omega(0) = f$ and $\omega(1) = g$ is equivalent to giving a homotopy $\omega^\sharp\colon X \times [0, 1] \to Y$ between f and g. In effect, homotopy classes of maps $X \to Y$ correspond to path connected components of the space $\mathrm{Map}(X, Y)$.

8.7 Example. Let X be a space. The *path space* of X is the space $PX = \mathrm{Map}([0, 1], X)$.

For $x_0 \in X$ consider the subspace of PX given by

$$\Omega_{x_0}X = \{\omega \in PX \mid \omega(0) = \omega(1) = x_0\}$$

This space is called the *loop space* of X based at x_0. Denote by $c_{x_0} \in \Omega_{x_0}X$ the constant loop $c_{x_0}(t) = x_0$ for all $t \in [0, 1]$.

Notice that every element $\omega \in \Omega_{x_0}X$ represents an element of $\pi_1(X, x_0)$. Similarly as in Example 8.6 we also obtain that path connected components of $\Omega_{x_0}X$ correspond to homotopy classes of loops in X. In this way, the assignment $[\omega] \mapsto [\omega^\sharp]$ gives a bijection

$$\pi_0(\Omega_{x_0}X, c_{x_0}) \xrightarrow{\cong} \pi_1(X, x_0)$$

Concatenation of loops defines a map $\Omega_{x_0}X \times \Omega_{x_0}X \to \Omega_{x_0}X$ which, in turn, induces a map

$$\pi_0(\Omega_{x_0}X, c_{x_0}) \times \pi_0(\Omega_{x_0}X, c_{x_0}) \to \pi_0(\Omega_{x_0}X, c_{x_0})$$

This defines a group structure on $\pi_0(\Omega_{x_0}X, c_{x_0})$ such that the bijection $\pi_0(\Omega_{x_0}X, c_{x_0}) \cong \pi_1(X, x_0)$ becomes an isomorphism of groups.

Generalizing this, any element of $\pi_n(\Omega_{x_0}, c_{x_0})$ is represented by a map $\omega\colon (I^n, \partial I^n) \to (\Omega_{x_0}X, c_{x_0})$. The adjoint of ω is a map $\omega^\sharp\colon I^n \times [0, 1] = I^{n+1} \to X$ such that $\omega^\sharp(\partial I^{n+1}) = x_0$. In other words, we obtain a map $\omega^\sharp\colon (I^{n+1}, \partial I^{n+1}) \to (X, x_0)$. It is easy to verify that maps $\omega_1, \omega_2\colon (I^n, \partial I^n) \to (\Omega_{x_0}X, c_{x_0})$ are homotopic if and only of their adjoints $\omega_1^\sharp$, $\omega_2^\sharp$ are homotopic. Thus the correspondence $[\omega] \mapsto [\omega^\sharp]$ defines a bijection

$$\pi_n(\Omega_{x_0}X, c_{x_0}) \xrightarrow{\cong} \pi_{n+1}(X, x_0)$$

One can check that this is an isomorphism of groups.

8.8 Note. A map of pointed spaces $f\colon (X, x_0) \to (Y, y_0)$ induces a map of loop spaces $\Omega f\colon \Omega_{x_0}X \to \Omega_{y_0}Y$. In this way we obtain a functor

$$\Omega\colon \mathbf{Top}_* \to \mathbf{Top}_*$$

8.9 Example. Let $x_0 \in A \subseteq X$. Denote

$$P(X, A, x_0) = \{\omega\colon [0, 1] \to X \mid \omega(0) \in A,\ \omega(1) = x_0\}$$

Similarly as in Example 8.7, one can check that for any map $\omega\colon (I^n, \partial I^n) \to (P(X, A, x_0), c_{x_0})$ the adjoint $\omega^\sharp\colon I^{n+1} \to X$ represents an element $[\omega^\sharp] \in \pi_{n+1}(X, A, x_0)$. The assignment $[\omega] \to [\omega^\sharp]$ gives an isomorphism

$$\pi_n(P(X, A, x_0)) \xrightarrow{\cong} \pi_{n+1}(X, A, x_0)$$

for any $n \geq 1$.

Let $f\colon X \to Y$ be a map, and let PY be the path space of Y. Define

$$E_f = \{(x, \omega) \in X \times PY \mid f(x) = \omega(0)\} \subseteq X \times PY$$

We have a map $r_f\colon PX \to E_f$ give by $r_f(\omega) = (\omega(0), f\omega))$

8.10 Proposition. *For a map $f\colon X \to Y$ the following conditions are equivalent:*

1) *The map f is a Hurewicz fibration.*
2) *The map f has the homotopy lifting property for the space E_f*
3) *There exists a map $s_f\colon E_f \to PX$ such that $r_f s_f = \mathrm{id}_{E_f}$*

Proof. 1) $\Rightarrow$ 2) Obvious.

2) $\Rightarrow$ 3) Consider the following commutative diagram:

$$\begin{tikzcd} E_f \times \{0\} \arrow[r, "\bar{k}"] \arrow[d, hook] & X \arrow[d, "f"] \\ E_f \times [0,1] \arrow[r, "g"'] \arrow[ur, dashed, "\bar{g}"] & Y \end{tikzcd} \tag{$*$}$$

Here $\bar{k}((x,\omega),0) = x$ and $g((x,\omega),t) = \omega(t)$. By 2) there exists a homotopy $\bar{g}$ that commutes this the rest of the diagram. Take $s_f = \bar{g}^\sharp$, the adjoint of $\bar{g}$.

3) $\Rightarrow$ 1) Assume that we have the following commutative diagram and we want to show that a homotopy lift $\bar{h}$ exists:

$$\begin{tikzcd} Z \times \{0\} \arrow[r, "\bar{d}"] \arrow[d, hook] & X \arrow[d, "f"] \\ Z \times [0,1] \arrow[r, "h"'] \arrow[ur, dashed, "\bar{h}"] & Y \end{tikzcd}$$

For $z \in Z$ let $\omega_z\colon [0,1] \to Y$ be the path given by $\omega_z(t) = h(z,t)$. Define a map $u\colon Z \to E_f$ by $u(z) = (\bar{k}(z,0), \omega_z)$. Notice that, in the notation of diagram ($*$) we have $\bar{d} = \bar{k}(u \times \mathrm{id}_{\{0\}})$ and $h = g(u \times \mathrm{id}_{[0,1]})$. As a consequence, we can take $\bar{h} = \bar{g}(u \times \mathrm{id}_{[0,1]})$. $\square$

Proof of Theorem 8.1. Let $f\colon X \to Y$ be a map. As before, define

$$E_f = \{(x,\omega) \in X \times PY \mid f(x) = \omega(0)\}$$

Let $g_f\colon X \to E_f$ be given by $g_f(x) = (x, c_{f(x)})$ where $c_{f(x)}\colon [0,1] \to Y$ is the constant path at $f(x)$. Also, let $p_f\colon E_f \to Y$ be given by $p_f(x,\omega) = \omega(1)$. We have $f = p_f g_f$.

We will show that g_f is a homotopy equivalence with the homotopy inverse given by the projection map $\mathrm{pr}\colon E_f \to X$, $\mathrm{pr}(x,\omega) = x$. We have $\mathrm{pr}\, g_f = \mathrm{id}_X$. The composition $g_f\,\mathrm{pr}\colon E_f \to E_f$ is given by $g_f\,\mathrm{pr}(x,\omega) = (x, c_x)$. A homotopy $h\colon g_f\,\mathrm{pr} \simeq \mathrm{id}_{E_f}$ is defined by $h((x,\omega),t) = (x, \omega_t(x))$, where $\omega_t\colon [0,1] \to Y$, $\omega_t(s) = \omega(ts)$.

It remains to show that $p_f\colon E_f \to Y$ is a Hurewicz fibration. To see this, assume that we have a commutative diagram

$$\begin{tikzcd} Z \times \{0\} \arrow[r, "\bar{k}"] \arrow[d, hook] & E_f \arrow[d, "p_f"] \\ Z \times [0,1] \arrow[r, "h"'] \arrow[ur, dashed, "\bar{h}"] & Y \end{tikzcd}$$

Denote $\bar{k}(z) = (x_z, \omega_z)$. By commutativity of the diagram we have $\omega_z(1) = h_0(z)$. Then the lift $\bar{h}$ can be defined by

$$\bar{h}(z,t) = (x_z, \tau_{z,t} * \omega_z)$$

where $\tau_{z,t} * \omega_z$ is the concatenation of ω_z with the path $\tau_{z,t}\colon [0,1] \to Y$ given by $\tau_{z,t}(s) = h(z, st)$. $\square$

8.11 Definition. Let $f\colon X \to Y$ be a map, and let $p_f\colon E_f \to Y$ be the Hurewicz fibration associated to f, as in Theorem 8.1. The *homotopy fiber* of f over a point $y_0 \in Y$ is the space

$$\mathrm{hofib}_{y_0} f = p_f^{-1}(y_0)$$

Explicitly:

$$\mathrm{hofib}_{y_0} f = \{(x, \omega) \in X \times PY \mid f(x) = \omega(0),\ \omega(1) = y_0\}$$

8.12 Corollary. *Let $f\colon (X, x_0) \to (Y, y_0)$ be map of pointed spaces. Denote $v_0 = (x_0, c_{y_0}) \in \mathrm{hofib}_{y_0} f$. We have a long exact sequence of homotopy groups*

$$\dots \longrightarrow \pi_{n+1}(Y, y_0) \longrightarrow \pi_n(\mathrm{hofib}_{x_0} f, v_0) \xrightarrow{i(f)_*} \pi_n(X, x_0) \xrightarrow{f_*} \pi_n(Y, y_0) \longrightarrow \dots$$
$$\dots \xrightarrow{f_*} \pi_1(Y, y_0) \longrightarrow \pi_0(\mathrm{hofib}_{x_0} f, v_0) \xrightarrow{i(f)_*} \pi_0(X, x_0) \xrightarrow{f_*} \pi_0(Y, y_0)$$

Here the map

$$i(f)\colon \mathrm{hofib}_{y_0} f = \{(x, \omega) \in X \times PY \mid f(x) = \omega(0),\ \omega(1) = y_0\} \longrightarrow Y$$

is given by $i(f)(x, \omega) = x$.

8.13 Example. Given a space X and $x_0 \in X$, consider a map $f\colon \{*\} \to X$, $f(*) = x_0$. Then

$$\begin{aligned} \mathrm{hofib}_{x_0} f &= \{(*, \omega) \in \{*\} \times PX \mid \omega(0) = x_0 = \omega(1)\} \\ &\cong \{\omega \in PX \mid \omega(0) = x_0 = \omega(1)\} \\ &= \Omega_{x_0} X \end{aligned}$$

Since $\pi_n(\{*\}) = 0$ for all $n \geq 0$ the exact sequence becomes

$$\dots \longrightarrow 0 \longrightarrow \pi_n(X, x_0) \longrightarrow \pi_{n-1}(\Omega_{x_0} X, c_{x_0}) \longrightarrow 0 \longrightarrow \dots \longrightarrow 0 \longrightarrow \pi_1(X, x_0) \longrightarrow \pi_0(\Omega_{x_0} X, c_{x_0}) \longrightarrow 0$$

where $c_{x_0} \in \Omega_{x_0}X$ is the constant loop at x_0. This recovers the isomorophisms $\pi_n(\Omega_{x_0}X, c_{x_0}) \cong \pi_{n+1}(X, x_0)$, which we obtained in Example 8.7

Notice that if x_1 belongs to a different path connected component of X than x_0, then $\mathrm{hofib}_{x_1} f = \varnothing$.

8.14 Example. The map $f\colon * \to X$ in Example 8.7 can be interpreted as an inclusion $\{x_0\} \hookrightarrow X$. Generalizing it, for $A \subseteq X$, consider the inclusion map $j\colon A \hookrightarrow X$. In this case we have

$$\begin{aligned} E_j &= \{(a, \omega) \in A \times PX \mid j(a) = \omega(0)\} \\ &\cong \{\omega \in P(X) \mid \omega(0) \in A\} \end{aligned}$$

For $x_0 \in X$ we get:

$$\begin{aligned} \mathrm{hofib}_{x_0} j &= \{\omega \in PX \mid \omega(0) \in A,\ \omega(1) = x_0\} \\ &= P(X, A, x_0) \end{aligned}$$

Recall (8.9) that we have isomorphisms $\pi_n(P(X, A, x_0), c_{x_0}) \xrightarrow{\cong} \pi_{n+1}(X, A, x_0)$. They fit into a commutative diagram of exact sequences:

$$\begin{tikzcd}
\dots \arrow[r] & \pi_{n+1}(X, A, x_0) \arrow[r, "\partial"] & \pi_n(A, x_0) \arrow[r, "j_*"] & \pi_n(X, x_0) \arrow[r] & \pi_n(X, A, x_0) \arrow[r] & \dots \\
\dots \arrow[r] & \pi_n(P(X, A, x_0), c_{x_0}) \arrow[r] \arrow[u, "\cong"] & \pi_n(E_j, c_{x_0}) \arrow[r, "p_{j*}"'] \arrow[u, "\cong", "g_{j*}"'] & \pi_n(X, x_0) \arrow[r, "\partial"'] \arrow[u, "="] & \pi_{n-1}(P(X, A, x_0), c_{x_0}) \arrow[r, "i_*"'] \arrow[u, "\cong"] & \dots
\end{tikzcd}$$

Here $g_j\colon E_j \to A$ and $p_j\colon E_j \to X$ are given by $g_j(\omega) = \omega(0)$ and $p_j(\omega) = \omega(1)$.

8.15 Definition. Consider a commutative diagram

$$\begin{tikzcd}
E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\
& B &
\end{tikzcd}$$

where p_1, p_2 are Hurewicz fibrations. The map f is a *fibrewise homotopy equivalence* if there exists a map $g\colon E_2 \to E_1$ such that $p_1 g = p_2$ and homotopies $h\colon gf \simeq \mathrm{id}_{E_1}$, $h'\colon fg \simeq \mathrm{id}_{E_2}$ such that $p_1 h_t = p_1$ and $p_2 h'_t = p_2$ for all $t \in [0, 1]$

8.16 Note. In the notation of Definition 8.15, if $f\colon E_1 \to E_2$ is a fibrewise homotopy equivalence then for any subspace $A \subseteq B$ the map $f|_{p_1^{-1}(A)}\colon p_1^{-1}(A) \to p_2^{-1}(A)$ is a homotopy equivalence. In particular, for any $b_0 \in B$ the map of fibers $f|_{p_1^{-1}(b_0)}\colon p^{-1}(b_0) \to p_2^{-1}(b_0)$ is a homotopy equivalence.

8.17 Proposition. *For a map $f\colon X \to Y$ consider the commutative diagram as in Theorem 8.1:*

$$\begin{tikzcd} X \arrow[rr, "g_f", "\simeq"'] \arrow[dr, "f"'] & & E_f \arrow[dl, "p_f"] \\ & Y & \end{tikzcd}$$

where $E_f = \{(x, \omega) \in X \times PY \mid f(x) = \omega(0)\}$, $p_f(x, \omega) = \omega(1)$ and $g_f(x) = (x, c_{f(x)})$.

If f is a Hurewicz fibration then g_f is a fiberwise homotopy equivalence.

Proof. Exercise. □

8.18 Corollary. *Let $f\colon X \to Y$ is a Hurewicz fibration and let $g_f\colon X \to E_f$ be given as in Proposition 8.17. Then for $y_0 \in Y$ the map*

$$g_f|_{f^{-1}(y_0)}\colon f^{-1}(y_0) \to p_f^{-1}(y_0) = \mathrm{hofib}_{y_0} f$$

is a homotopy equivalence.

Proof. It follows from Proposition 8.17 and Note 8.16. □