

# 17 | Hurewicz Theorem

Hurewicz homomorphism is a map that connects homotopy and homology groups. Recall that  $H_n(S^n) \cong \mathbb{Z}$ . We will denote by  $\gamma_n$  a chosen generator of  $H_n(S^n)$ . Given an element  $[\varphi: (S^n, s_0) \rightarrow (X, x_0)] \in \pi_n(X, x_0)$  consider the homomorphism  $\varphi_*: H_*(S^n) \rightarrow H_*(X)$ . This homomorphism depends only on the homotopy class of  $\varphi$ .

**17.1 Definition.** The *Hurewicz homomorphism* is a function

$$h: \pi_n(X, x_0) \rightarrow H_n(X)$$

given by  $h([\varphi]) = \varphi_*(\gamma_n)$ .

**17.2 Proposition.** For any function  $f: X \rightarrow Y$  the following diagram commutes:

$$\begin{array}{ccc} \pi_n(X, x_0) & \xrightarrow{f_*} & \pi_n(Y, f(x_0)) \\ \downarrow h & & \downarrow h \\ H_n(X) & \xrightarrow{f_*} & H_n(Y) \end{array}$$

*Proof.* For  $[\varphi] \in \pi_n(X, x_0)$  we have

$$hf_*([\varphi]) = h([f\varphi]) = (f\varphi)_*(\gamma_n) = f_*\varphi_*(\gamma_n) = f_*h([\varphi])$$

□

**17.3 Proposition.** The *Hurewicz homomorphism* is a group homomorphism.

*Proof.* Let  $\varphi, \psi: (S^n, s_0) \rightarrow (X, x_0)$  where  $n \geq 1$ . Recall that the element  $[\varphi] \cdot [\psi] \in \pi_n(X, x_0)$  is the homotopy class of the map

$$S^n \xrightarrow{p} S^n \vee S^n \xrightarrow{\varphi \vee \psi} X$$

where  $p: S^n \rightarrow S^n \vee S^n$  is the pinch map. Let  $r_1, r_2: S^n \vee S^n \rightarrow S^n$  be the retractions of  $S^n \vee S^n$  onto the first and, respectively, the second copy of  $S^n$ . We have a commutative diagram

$$\begin{array}{ccccc}
 H_n(S^n) & \xrightarrow{p_*} & H_n(S^n \vee S^n) & \xrightarrow{(\varphi \vee \psi)_*} & H_n(X) \\
 & \searrow \text{id}_* \oplus \text{id}_* & \downarrow \cong \downarrow r_{1*} \oplus r_{2*} & \nearrow \varphi_* + \psi_* & \\
 & & H_n(S^n) \oplus H_n(S^n) & & 
 \end{array}$$

This gives:

$$h([\varphi] \cdot [\psi]) = ((\varphi \vee \psi)p)_*(\gamma_n) = (\varphi_* + \psi_*)(\text{id}_* \oplus \text{id}_*)(\gamma_n) = \varphi_*(\gamma_n) + \psi_*(\gamma_n) = h([\varphi]) + h([\psi])$$

□

**17.4 Hurewicz Isomorphism Theorem.** Let  $X$  be a space such that for some  $n \geq 2$  we have  $\pi_i(X) = 0$  for  $i < n$ . Then  $H_i(X) = 0$  for  $0 < i < n$  and the Hurewicz homomorphism

$$h: \pi_n(X, x_0) \rightarrow H_n(X)$$

is an isomorphism.

*Proof.* Assume first that  $X = S^n$ . We have  $H_i(S^n) = 0$  for  $0 < i < n$ . In degree  $n$  the Hurewicz homomorphism is a map  $h: \mathbb{Z} \cong \pi_n(S^n) \rightarrow H_n(S^n) \cong \mathbb{Z}$ . The group  $\pi_n(S^n)$  is generated by the homotopy class of the identity map  $\text{id}_{S^n}: S^n \rightarrow S^n$  (11.10). We have  $h([\text{id}_{S^n}]) = \text{id}_{S^n*}(\gamma_n) = \gamma_n$ . Therefore  $h$  maps a generator of  $\pi_n(S^n)$  to a generator of  $H^n(S^n)$ , and so it is an isomorphism.

Next, assume that  $X = \bigvee_{i \in I} S^n$ . Again, in this case  $H_i(\bigvee_{i \in I} S^n) = 0$  for  $0 < i < n$ . Also, the retraction maps  $r_i: \bigvee_{i \in I} S^n \rightarrow S^n$  give a commutative diagram

$$\begin{array}{ccc}
 \pi_n(\bigvee_{i \in I} S^n) & \xrightarrow[\cong]{\bigoplus r_{i*}} & \bigoplus_{i \in I} \pi_n(S^n) \\
 \downarrow h & & \downarrow \bigoplus_{i \in I} h \\
 H_n(\bigvee_{i \in I} S^n) & \xrightarrow[\bigoplus r_{i*}]{\cong} & \bigoplus_{i \in I} H_n(S^n)
 \end{array}$$

The map  $\bigoplus_{i \in I} h$  is an isomorphism by the previous case, so the left vertical map  $h$  is also an isomorphism.

For the next step, assume that  $X$  is an arbitrary CW complex with  $\pi_i(X) = 0$  for  $i < n$ . By Proposition 5.6 we can assume that  $X^{(n-1)} = *$ , which gives  $H_i(X) = 0$  for  $0 < i < n$ .

Let  $j: X^{(n+1)} \hookrightarrow X$  be the inclusion of the  $(n+1)$ -skeleton of  $X$ . By Proposition 17.2 we have a commutative diagram

$$\begin{array}{ccc} \pi_n(X^{(n+1)}) & \xrightarrow{j_*} & \pi_n(X) \\ \downarrow h & \cong & \downarrow h \\ H_n(X^{(n+1)}) & \xrightarrow{j_*} & H_n(X) \end{array}$$

The upper homomorphism  $j_*$  is an isomorphism by Proposition 5.2, and the lower  $j_*$  is an isomorphism by properties of homology groups. As a consequence, it is enough to show that  $h: \pi_n(X^{(n+1)}) \rightarrow H_n(X^{(n+1)})$  is an isomorphism.

Since  $X^{(n-1)} = *$ , it follows that  $X^{(n)} = \bigvee_{i \in I} S^n$  and  $X^{(n+1)} = X^{(n)} \cup \bigcup_{k \in K} e_k^{(n+1)}$  where  $\{e_k^{(n+1)}\}_{k \in K}$  are  $(n+1)$ -cells of  $X$ . Let  $\varphi_k: S^n \rightarrow X^{(n)}$  be the attaching map of the cell  $e_k^{(n+1)}$ , and let  $i: X^{(n)} \hookrightarrow X^{(n+1)}$  denote the inclusion map. We have a commutative diagram

$$\begin{array}{ccccccc} \pi_n(\bigvee_{k \in K} S^n) & \xrightarrow{(\bigvee_{k \in K} \varphi_k)_*} & \pi_n(X^{(n)}) & \xrightarrow{i_*} & \pi_n(X^{(n+1)}) & \longrightarrow & 0 \\ \downarrow h \cong & & \downarrow h \cong & & \downarrow h & & \downarrow \cong \\ H_n(\bigvee_{k \in K} S^n) & \xrightarrow{(\bigvee_{k \in K} \varphi_k)_*} & H_n(X^{(n)}) & \xrightarrow{i_*} & H_n(X^{(n+1)}) & \longrightarrow & H_{n-1}(\bigvee_{k \in K} S^n) \end{array}$$

The upper row of this diagram is exact by Proposition 12.7, and the lower row is exact by the long homology sequence associated to the map  $\bigvee_{k \in K} \varphi_k$ . By the Five Lemma we obtain that  $h: \pi_n(X^{(n+1)}) \rightarrow H_n(X^{(n+1)})$  is an isomorphism.

Finally, let  $X$  be an arbitrary space with  $\pi_i(X) = 0$  for  $i < n$ . Let  $f: Y \rightarrow X$  be a CW approximation of  $X$  (15.3). Using Theorem 16.1 and the previous case we get  $H_i(X) \cong H_i(Y) = 0$  for  $0 < i < n$ .

By Proposition 17.2 we have a commutative diagram

$$\begin{array}{ccc} \pi_n(Y) & \xrightarrow{f_*} & \pi_n(X) \\ \downarrow h \cong & & \downarrow h \\ H_n(Y) & \xrightarrow{f_*} & H_n(X) \end{array}$$

Since  $f$  is a weak equivalence, the upper homomorphism  $f_*$  is an isomorphism by definition, and the lower  $f_*$  is an isomorphism by Theorem 16.1. Also, since  $Y$  is a CW complex the left vertical map is an isomorphism by the previous case. Therefore  $h: \pi_n(X) \rightarrow H_n(X)$  is an isomorphism.  $\square$

**17.5 Inverse Hurewicz Isomorphism Theorem.** *Let  $X$  be a simply connected space, and let  $H_i(X) = 0$  for  $1 \leq i < n$  for some  $n \geq 2$ . Then  $\pi_i(X) = 0$  for  $i < n$  and the Hurewicz homomorphism  $h: \pi_n(X) \rightarrow H_n(X)$  is an isomorphism.*

*Proof.* Exercise. □

Since all homology groups  $H_i(X)$  are abelian but the fundamental group  $\pi_1(X)$  need not be abelian, in general the Hurewicz homomorphism  $h: \pi_1(X) \rightarrow H_1(X)$  is not an isomorphism. However, a version of Theorem 17.4 still holds with the following modification. Recall that if  $G$  is a group then the commutator of  $G$  is the subgroup  $[G, G] \subseteq G$  generated by all elements of the form  $ghg^{-1}h^{-1}$  for  $g, h \in G$ . The commutator is a normal subgroup of  $G$ , and the quotient group  $G^{\text{ab}} := G/[G, G]$  is an abelian group. The group  $G^{\text{ab}}$  is called the abelianization of  $G$ .

If  $H$  is an abelian group then any homomorphism  $\varphi: G \rightarrow H$  defines a unique homomorphism  $\bar{\varphi}: G^{\text{ab}} \rightarrow H$  such that  $\varphi = \bar{\varphi}\eta$  where  $\eta: G \rightarrow G^{\text{ab}}$  is the quotient homomorphism. Also, if  $\psi: G \rightarrow H$  is a homomorphism of arbitrary groups, then  $\psi([G, G]) \subseteq [H, H]$ , and so  $\psi$  induces a homomorphism of abelianizations  $\psi^{\text{ab}}: G^{\text{ab}} \rightarrow H^{\text{ab}}$ .

**17.6 Theorem.** *Let  $X$  be a path connected space and let  $h: \pi_1(X, x_0) \rightarrow H_1(X)$  be the Hurewicz homomorphism. Then the induced homomorphism  $\bar{h}: \pi_1(X, x_0)^{\text{ab}} \rightarrow H_1(X)$  is an isomorphism.*

The proof will use the following algebraic fact.

**17.7 Lemma.** *Consider a sequence of group homomorphisms*

$$G \xrightarrow{\varphi} H \xrightarrow{\psi} K$$

*such that  $\psi$  is onto and  $\ker \psi = N_H(\text{Im } \varphi)$  where  $N_H(\text{Im } \varphi)$  is the normalizer of  $\text{Im } \varphi$  in  $H$ . Then the induced sequence*

$$G^{\text{ab}} \xrightarrow{\varphi^{\text{ab}}} H^{\text{ab}} \xrightarrow{\psi^{\text{ab}}} K^{\text{ab}} \rightarrow 0$$

*is exact.*

*Proof.* Exercise. □

*Proof of Theorem 17.6.* Take  $X = S^1$ . As in the proof of Theorem 17.4 we obtain that  $h: \pi_1(S^1) \rightarrow H_1(S^1)$  is an isomorphism. Also, since  $\pi_1(S^1) \cong \mathbb{Z}$  is an abelian group, thus  $\pi_1(S^1) \cong \pi_1(S^1)^{\text{ab}}$  and, up to this isomorphism,  $\bar{h}$  coincides with  $h$ .

Next, take  $X = \bigvee_{i \in I} S^1$  and let  $r_i: \bigvee_{i \in I} S^1 \rightarrow S^1$  be retraction maps. We have a commutative diagram

$$\begin{array}{ccc} \pi_1(\bigvee_{i \in I} S^1) & \xrightarrow{\bigoplus r_{i*}} & \bigoplus_{i \in I} \pi_1(S^1) \\ \downarrow h & & \downarrow \cong \bigoplus_{i \in I} h \\ H_n(\bigvee_{i \in I} S^1) & \xrightarrow[\bigoplus r_{i*}]{} & \bigoplus_{i \in I} H_n(S^1) \end{array}$$

The upper map  $\bigoplus r_{i*}$  essentially coincides with the abelianization of  $\pi_1(\bigvee_{i \in I} S^1)$ , and the map  $\bigoplus_{i \in I} h$  coincides, up to an isomorphism, with  $\bar{h}: \pi_1(\bigvee_{i \in I} S^1)^{\text{ab}} \rightarrow H_1(\bigvee_{i \in I} S^1)$ . It remains to notice that  $\bigoplus_{i \in I} h$  is an isomorphism by the previous case.

As in the proof of Theorem 17.4, it remains to consider the case where  $X$  is a 2-dimensional CW complex of the form  $X = \bigvee_{i \in I} S^1 \cup \bigcup_{k \in K} e_k^2$ . Let  $\varphi_k: S^1 \rightarrow \bigvee_{i \in I} S^1$  be the attaching map of the cell  $e_k^2$ . Denote  $\psi := \bigvee_{k \in K} \varphi_k: \bigvee_{k \in K} S^1 \rightarrow \bigvee_{i \in I} S^1$ . Also, let  $j: \bigvee_{i \in I} S^1 \hookrightarrow X$  be the inclusion of the 1-skeleton of  $X$ . We have a sequence of group homomorphisms

$$\pi_1\left(\bigvee_{k \in K} S^1\right) \xrightarrow{\psi_*} \pi_1\left(\bigvee_{i \in I} S^1\right) \xrightarrow{j_*} \pi_1(X)$$

By van Kampen's Theorem the homomorphism  $j_*$  is onto and  $\ker j_* = N_{\pi_1(\bigvee_{i \in I} S^1)}(\text{Im } \psi_*)$ . Consider the commutative diagram

$$\begin{array}{ccccccc} \pi_1\left(\bigvee_{k \in K} S^1\right)^{\text{ab}} & \xrightarrow{\psi_*} & \pi_1\left(\bigvee_{i \in I} S^1\right)^{\text{ab}} & \xrightarrow{i_*} & \pi_1(X)^{\text{ab}} & \longrightarrow & 0 \\ \bar{h} \downarrow \cong & & \bar{h} \downarrow \cong & & \downarrow \bar{h} & & \downarrow \cong \\ H_1\left(\bigvee_{i \in I} S^1\right) & \xrightarrow{\psi_*} & H_1\left(\bigvee_{i \in I} S^1\right) & \xrightarrow{i_*} & H_1(X) & \longrightarrow & 0 \end{array}$$

The upper row is exact by Lemma 17.7 and the lower row is exact by the long exact homology sequence associated to  $\psi_*$ . Using the Five Lemma we obtain that  $\bar{h}: \pi_1(X)^{\text{ab}} \rightarrow H_1(X)$  is an isomorphism.  $\square$

**17.8 Relative Hurewicz Homomorphism.** Recall that  $H_n(D^n, S^{n-1}) \cong \mathbb{Z}$ . Let  $\bar{y}_n$  denote a chosen generator of  $H_n(D^n, S^{n-1})$ . Given a pointed pair  $(X, A, x_0)$ , and an element  $[\varphi: (D^n, S^{n-1}, s_0) \rightarrow (X, A, x_0)] \in \pi_n(X, A, x_0)$  consider the function  $\varphi_*: H_n(D^n, S^{n-1}) \rightarrow H_n(X, A)$ .

**17.9 Definition.** The *relative Hurewicz homomorphism* is a function

$$h: \pi_n(X, A, x_0) \rightarrow H_n(X, A)$$

given by  $h([\varphi]) = \varphi_*(\bar{y}_n)$ .

**17.10 Proposition.** The relative Hurewicz homomorphism is a group homomorphism for  $n \geq 2$ .

**17.11 Relative Inverse Hurewicz Isomorphism Theorem.** Let  $(X, A)$  be a pair of simply connected CW complexes. If  $H_i(X, A) = 0$  for all  $0 < i < n$  for some  $n \geq 2$  then  $\pi_i(X, A) = 0$  for all  $i < n$  and the Hurewicz homomorphism  $h: \pi_n(X, A) \rightarrow H_n(X, A)$  is an isomorphism.

*Proof.* See tom Dieck, Theorem 20.1.3 p. 497. Uses commutativity of the diagram

$$\begin{array}{ccc} \pi_i(X, A) & \longrightarrow & \pi_i(X/A) \\ \downarrow h & & \downarrow h \\ H_i(X, A) & \longrightarrow & H_i(X/A) \end{array}$$

and the Inverse Hurewicz Theorem 17.5 applied to the space  $X/A$ .  $\square$

**17.12 Theorem.** *Let  $X, Y$  be simply connected CW complexes and let  $f: X \rightarrow Y$  be a map such that for some  $n \geq 2$  the homomorphism  $f_*: H_i(X) \rightarrow H_i(Y)$  is an isomorphism for  $i < n$  and epimorphism for  $i = n$ . Then  $f_*: \pi_i(X) \rightarrow \pi_i(Y)$  is an isomorphism for  $i < n$  and epimorphism for  $i = n$ .*

*Proof.* Let  $M_f$  be the mapping cylinder of  $f$ . The assumption about  $f$  is equivalent to the condition that  $H_i(M_f, X) = 0$  for  $i < n$ . By Theorem 17.11 this gives  $\pi_i(M_f, X) = 0$  for  $i < n$ . The statement then follows from the long exact sequence of homotopy groups of the pair  $(M_f, X)$ .  $\square$

**17.13 Corollary.** *Let  $f: X \rightarrow Y$  be a map of simply connected CW complexes such that  $f_*: H_i(X) \rightarrow H_i(Y)$  is an isomorphism for all  $i \geq 0$ . Then  $f$  is a homotopy equivalence.*

Let  $p_X: \tilde{X} \rightarrow X$  denote the universal cover of a space  $X$ . Given a map  $f: X \rightarrow Y$  we can find a map  $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$  such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ p_X \downarrow & & \downarrow p_Y \\ X & \xrightarrow{f} & Y \end{array}$$

**17.14 Theorem.** *Let  $f: X \rightarrow Y$  be a map of path connected CW complexes. If the homomorphisms  $f_*: \pi_1(X) \rightarrow \pi_1(Y)$  and  $\tilde{f}_*: H_i(\tilde{X}) \rightarrow H_i(\tilde{Y})$  for all  $i \geq 0$  are isomorphisms then  $f$  is a homotopy equivalence.*

*Proof.* By Theorem 17.12 the map  $\tilde{f}_*: \pi_i(\tilde{X}) \rightarrow \pi_i(\tilde{Y})$  is an isomorphism for all  $i \geq 0$ . Since  $p_{X*}: \pi_i(\tilde{X}) \rightarrow \pi_i(X)$  and  $p_{Y*}: \pi_i(\tilde{Y}) \rightarrow \pi_i(Y)$  are isomorphisms for  $i \geq 2$ , this gives that  $f_*: \pi_i(X) \rightarrow \pi_i(Y)$  is an isomorphism for  $i \geq 2$ . By assumption,  $f_*: \pi_1(X) \rightarrow \pi_1(Y)$  is an isomorphism as well, so  $f$  is a weak equivalence and thus a homotopy equivalence.  $\square$