1. Let $S^n \to E \xrightarrow{p} B$ be a Serre fibration where B is a simply connected space. Use the Serre spectral sequence to show that there exists an exact sequence of the form

$$\dots \to H_q(E) \to H_q(B) \to H_{q-n-1}(B) \to H_{q-1}(E) \to H_{q-1}(B) \to \dots$$

This exact sequence is called the *Gysin sequence*.

2. Recall that the complex projective space $\mathbb{CP}^n$ fits into a fibration sequence $S^1 \to S^{2n} \to \mathbb{CP}^n$. Use this to show that

$$H_q(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & \text{if } q \text{ is even and } q \leq 2n \\ 0 & \text{otherwise} \end{cases}$$

3. Let $F \to E \xrightarrow{p} S^n$ be a Serre fibration for some $n \geq 2$. Use the Serre spectral sequence to show that there exists an exact sequence of the form

$$\dots \to H_q(F) \to H_q(E) \to H_{q-n}(F) \to H_{q-1}(F) \to H_{q-1}(E) \to \dots$$

This exact sequence is called the *Wang sequence*.

4. Let $f\colon S^2 \to S^2$ be a map of degree 2. Compute homology groups of the homotopy fiber of f.

5. Let $F \to E \xrightarrow{f} B$ be a Serre fibration where B is a simply connected space. Consider the Serre spectral sequence $E^r_{*,*}$ of this fibration. Notice that since all differentials terminating at groups $E^r_{p,0}$ are trivial, we obtain

$$H_p(B) = E^2_{p,0} \subseteq E^3_{p,0} \subseteq \dots \subseteq E^{p+1}_{p,0} = E^\infty_{p,0} = H_p(E)/F_{p-1}H_p(E)$$

Show that the resulting homomorphism $H_p(B) \hookrightarrow H_p(E)/F_{p-1}H_p(E) \to H_p(E)$ coincides with the homomorphism $f_*\colon H_p(E) \to H_p(B)$.

6. Let $F \to E \xrightarrow{f} B$ be a Serre fibration where B is a simply connected space. Consider the Serre spectral sequence $E^r_{*,*}$ of this fibration. Notice that since all differentials that start at groups $E^r_{0,q}$ are trivial, the group $E^{r+1}_{q,0}$ is a quotient group of $E^r_{q,0}$. As a result we obtain a sequence of quotient homomorphisms:

$$H_q(F) = E^2_{0,q} \to E^3_{0,q} \to \dots \to E^{q+2}_{0,q} = E^\infty_{0,q} = F_0H_q(E) \subseteq H_q(E)$$

Show that the resulting homomorphism $H_q(F) \to H_q(E)$ coincides with the homomorphism i_* induced by the inclusion of the fiber $i\colon F \hookrightarrow E$.