# 9 | Exact Puppe Sequence

Recall that a sequence of pointed sets

$$(S_2, s_2) \xrightarrow{f} (S_1, s_1) \xrightarrow{g} (S_0, s_0)$$

is *exact at* $S_1$ if $f(S_2) = g^{-1}(s_0)$.

For pointed spaces $(X, x_0)$ and $(Y, y_0)$ let $[X, Y]_*$ denote the set of pointed homotopy classes of maps $X \to Y$. This is a pointed set, with the basepoint represented by the constant function $c_{y_0} \colon X \to Y$, $c_{y_0}(x) = y_0$ for all $x \in X$.

**9.1 Definition.** A pointed space $(X, x_0)$ is well-pointed if the pair $(X, x_0)$ has the homotopy extension property.

**9.2 Definition.** A sequence of maps of spaces

$$(X_0, x_0) \xrightarrow{f_0} (X_1, x_1) \xrightarrow{f_1} (X_2, x_2)$$

is *exact at* $X_1$ is for any well-pointed space $(Y, y_0)$ the sequence pointed sets

$$[Y, X_0]_* \xrightarrow{f_{0*}} [Y, X_1]_* \xrightarrow{f_{1*}} [Y, X_2]_*$$

is exact at $[Y, X_1]_*$.

**9.3 Proposition.** *If $p \colon E \to B$ is a Hurewicz fibration, $e_0 \in E$, $b_0 = p(e_0) \in B$ $F = p^{-1}(b_0)$, and $i \colon F \to E$ is the inclusion map then the sequence $(F, e_0) \xrightarrow{i} (E, e_0) \xrightarrow{p} (B, b_0)$ is exact at $E$.*

*Proof.* Exercise. □

Let $f \colon (X, x_0) \to (Y, y_0)$ be any pointed map. Consider the sequence

$$\operatorname{hofib} f \xrightarrow{i(f)} X \xrightarrow{f} Y$$

where

$$i(f)\colon \operatorname{hofib}_{y_0} f = \{(x, \omega) \in X \times PY \mid f(x) = \omega(0),\ \omega(1) = y_0\} \longrightarrow X$$

is given by $i(f)(x, \omega) = x$. Since this sequence is homotopy equivalent to a sequence given by a Hurewicz fibration, it is exact at $X$. We can continue this construction inductively, by taking consecutive homotopy fibers:

$$\ldots \longrightarrow \operatorname{hofib} i^3(f) \xrightarrow{i^4(f)} \operatorname{hofib} i^2(f) \xrightarrow{i^3(f)} \operatorname{hofib} i(f) \xrightarrow{i^2(f)} \operatorname{hofib} f \xrightarrow{i(f)} X \xrightarrow{f} Y \tag{$*$}$$

In this way we obtain a sequence which is exact at all spaces. As it turns out, this sequence has a more convenient description. The starting point for it is the following fact:

**9.4 Proposition.** *Let $f\colon X \to Y$ be a map and $y_0 \in Y$. Then the map $i(f)\colon \operatorname{hofib}_{y_0} f \longrightarrow X$ is a Hurewicz fibration.*

*Proof.* Exercise. □

**9.5 Corollary.** *For any map of pointed spaces $f\colon (X, x_0) \to (Y, y_0)$ we have a commutative diagram*

$$\begin{tikzcd}
\operatorname{hofib} i(f) \arrow[r, "i^2(f)"] & \operatorname{hofib} f \arrow[r, "i(f)"] & X \arrow[r, "f"] & Y \\
\Omega Y \arrow[u, "g", "\simeq"'] \arrow[ur, "j"'] & & &
\end{tikzcd}$$

*Proof.* We have

$$i(f)^{-1}(x_0) = \{(x_0, \omega) \in X \times PY \mid \omega(0) = f(x_0) = y_0,\ \omega(1) = y_0\} \cong \Omega Y$$

Thus $\Omega Y$ can be identified with the fiber of $i(f)$ over $y_0$, and the map $j\colon \Omega Y \to \operatorname{hofib} f$, $j(\omega) = (x_0, \omega)$ with the inclusion of the fiber. By Proposition 9.4 and Corollary 8.18 we obtain a homotopy equivalence $g\colon \Omega Y \to \operatorname{hofib} i(f)$ such that the above diagram commutes. □

**9.6 Note.** The homotopy equivalence in Corollary 9.5 can be explicitly described as follows. Up to a homeomorphism we have

$$\operatorname{hofib} i(f) = \{(\omega, \tau) \in PX \times PY \mid f\omega(0) = \tau(0),\ \omega(1) = y_0,\ \tau(1) = x_0\}$$

Then $i^2(f)\colon \operatorname{hofib} i(f) \to \operatorname{hofib} f$ is given by $(\omega, \tau) = (\omega(0), \tau)$ and $g(\tau) = (c_{x_0}, \tau)$.

Applying Corollary 9.5 iteratively to the sequence $(*)$ we get homotopy equivalences

$$
\begin{aligned}
&\operatorname{hofib} i(f) \xleftarrow{\simeq} \Omega Y\\
&\operatorname{hofib} i^2(f) \xleftarrow{\simeq} \Omega X\\
&\operatorname{hofib} i^3(f) \xleftarrow{\simeq} \Omega \operatorname{hofib} f\\
&\operatorname{hofib} i^4(f) \xleftarrow{\simeq} \Omega \operatorname{hofib} i(f) \simeq \Omega^2 Y\\
&\operatorname{hofib} i^5(f) \xleftarrow{\simeq} \Omega \operatorname{hofib} i^2(f) \simeq \Omega^2 X\\
&\dots \quad \dots \quad \dots \quad \dots \quad \dots
\end{aligned}
$$

Moreover, one can check that the following diagram commutes up to homotopy:

$$
\begin{tikzcd}
\dots \arrow[r] & \operatorname{hofib} i^4(f) \arrow[r, "i^5(f)"] & \operatorname{hofib} i^3(f) \arrow[r, "i^4(f)"] & \operatorname{hofib} i^2(f) \arrow[r, "i^3(f)"] & \operatorname{hofib} i(f) \arrow[r, "i^2(f)"] & \operatorname{hofib} f \arrow[r, "i(f)"] & X \arrow[r, "f"] & Y \\
\dots \arrow[r] & \Omega^2 Y \arrow[r, "\Omega j"'] \arrow[u, "\simeq"] & \Omega \operatorname{hofib} f \arrow[r, "\Omega i(f)"'] \arrow[u, "\simeq"] & \Omega X \arrow[r, "\Omega f"'] \arrow[u, "\simeq"] & \Omega Y \arrow[r, "j"'] \arrow[u, "\simeq"] & \operatorname{hofib} f \arrow[r, "i(f)"'] \arrow[u, "="] & X \arrow[r, "f"'] \arrow[u, "="] & Y \arrow[u, "="]
\end{tikzcd}
\qquad (**)
$$

Since the upper row of this diagram is exact, the same is true for the lower row.

**9.7 Definition.** The sequence in the lower row of the diagram $(**)$ is called the *Puppe exact sequence* associated to the map $f$.

As a consequence, for any map of pointed spaces $f\colon (X, x_0) \to (Y, y_0)$ and any well-pointed space $(Z, z_0)$ we obtain a long exact sequence of sets:

$$
\begin{aligned}
\dots \xrightarrow{\Omega^2 f_*} [Z, \Omega^2 Y]_* \xrightarrow{\Omega j_*} [Z, \Omega \operatorname{hofib} f]_* \xrightarrow{\Omega i(f)_*} [Z, \Omega X]_* \xrightarrow{\Omega f_*} [Z, \Omega Y]_* \\
\xrightarrow{j_*} [Z, \operatorname{hofib} f]_* \xrightarrow{i(f)_*} [Z, X]_* \xrightarrow{f_*} [Z, Y]_* \quad (\maltese)
\end{aligned}
$$

**9.8 Note.** For any pointed space $(X, x_0)$ and $n \geq 1$ the loop space $\Omega^n X$ is quipped with a multiplication map $\mu\colon \Omega^n X \times \Omega^n X \to \Omega^n X$ given by concatenation of loops. For any pointed space $(Z, z_0)$ this defines a multiplication

$$\mu_*\colon [Z, \Omega^n X]_* \times [Z, \Omega^n X]_* \to [Z, \Omega^n X]_*$$

given by $\mu_*([\varphi], [\psi]) = [\mu \circ (\varphi \times \psi)]$. This equips the set $[Z, \Omega^n X]_*$ with a group structure. Moreover, for $n \geq 2$ the multiplication $\mu$ commutes up to homotopy, and in effect $[Z, \Omega^n X]_*$ becomes an abelian group.

As a result the exact sequence $(\maltese)$ becomes an exact sequence of groups starting at $[Z, \Omega Y]_*$ and its groups are abelian starting with $[Z, \Omega^2 Y]_*$

**9.9 Loop spaces and suspensions.** There is a different way of interpreting group structures appearing in the sequence (✠), which uses suspensions of a space in place of loop spaces.

**9.10 Definition.** Let $X$ be a space. The *unreduced suspension* of $X$ if the space

$$SX = X \times [0,1]/(X \times \{0,1\})$$

**9.11 Note.** Any map $f\colon X \to Y$ defines a map $Sf\colon SX \to SY$ given by $Sf([x,t]) = [f(x), t]$. This map is called the suspension of $f$. In this way we obtain the suspension functor

$$S\colon \mathbf{Top} \to \mathbf{Top}$$

This functor preserves homotopy classes of maps: if $f, g\colon X \to Y$ and $f \simeq g$ then $Sf \simeq Sg$.

**9.12 Example.** For a sphere $S^n$ we have $SS^n \cong S^{n+1}$.

**9.13 Definition.** Let $(X, x_0)$ be a pointed space. The *reduced suspension* of $X$ is the pointed space

$$\Sigma X = X \times [0,1]/(X \times \{0,1\} \cup \{x_0\} \times [0,1])$$

or equivalently $\Sigma X = SX/\{[x_0, t] \mid t \in [0,1]\}$. The basepoint in $\Sigma X$ is given by $[x_0, 0] \in \Sigma X$.

**9.14 Note.** If $(X, x_0)$ is a well-pointed space, then Proposition 2.15 implies that the quotient map $SX \to \Sigma X$ is a homotopy equivalence. In particular, for any basepoint $x_0 \in S^n$ we have $\Sigma S^n \simeq SS^n \cong S^{n+1}$. One can show that actually there is a homeomorphism $\Sigma S^n \cong S^{n+1}$

**9.15 Note.** Any map $f\colon (X, x_0) \to (Y, y_0)$ of pointed spaces, defines a map $\Sigma f\colon \Sigma X \to \Sigma Y$ given by $\Sigma f([x,t]) = [f(x), t]$. This defines the suspension functor

$$\Sigma\colon \mathbf{Top}_* \to \mathbf{Top}_*$$

Similarly as for the unreduced suspension, the reduced suspension preserves homotopy classes: if $f, g\colon (X, x_0) \to (Y, y_0)$ are maps of pointed spaces and $f \simeq g$ then $\Sigma f \simeq \Sigma g$.

Let $X$ be a Hausdorff space. By properties of mapping spaces (8.5) the adjunction map $\mathrm{adj}(\omega) = \omega^\sharp$ defines a homeomorphism $\mathrm{adj}\colon \mathrm{Map}(X \times [0,1], Y) \to \mathrm{Map}(X, PY)$. Let $(X, x_0)$ and $(Y, y_0)$ be pointed spaces. Consider $\Omega_{y_0} Y$ as a subspace of $PY$ and let $\mathrm{Map}_*(X, \Omega_{y_0} Y)$ be the subspace of $\mathrm{Map}(X, PY)$ consisting of basepoint preserving maps. Then adj restricts to a homeomorphism between this subspace and the subspace of $\mathrm{Map}(X \times [0,1], Y)$ consisting of all maps $f\colon X \times [0,1] \to Y$ such that $f(X \times \{0,1\} \cup \{x_0\} \times [0,1]) = y_0$. Such maps are in a bijective correspondence with basepoint preserving maps $\Sigma X \to Y$. In this way we obtain a homeomorphism

$$\mathrm{adj}\colon \mathrm{Map}_*(\Sigma X, Y) \xrightarrow{\cong} \mathrm{Map}_*(X, \Omega Y)$$

On the level of homotopy classes of maps this gives a bijection

$$\mathrm{adj}\colon [\Sigma X, Y]_* \xrightarrow{\cong} [X, \Omega Y]_*$$

The set of the right hand side has a group structure induced by concatenation of loops. A group structure on the left hand side can be defined using the pinch map $\Sigma X \to \Sigma X \vee \Sigma X$. In this way the above bijection becomes an isomorphism of groups.

As a result, the exact sequence (✠) can be equivalently written as

$$\ldots \xrightarrow{f_*} [\Sigma^2 Z, Y]_* \xrightarrow{j_* \,\mathrm{adj}} [\Sigma Z, \mathrm{hofib}\, f]_* \xrightarrow{i(f)_*} [\Sigma Z, X]_* \xrightarrow{f_*} [\Sigma Z, Y]_* \xrightarrow{j_* \,\mathrm{adj}} [Z, \mathrm{hofib}\, f]_* \xrightarrow{i(f)_*} [Z, X]_* \xrightarrow{f_*} [Z, Y]_*$$

Consider this sequence with $Z = S^0$. Since $\Sigma^n S^0 \cong S^n$ we obtain

$$\ldots \xrightarrow{f_*} [S^2, Y]_* \xrightarrow{j_* \,\mathrm{adj}} [S^1, \mathrm{hofib}\, f]_* \xrightarrow{i(f)_*} [S^1, X]_* \xrightarrow{f_*} [S^1, Y]_* \xrightarrow{j_* \,\mathrm{adj}} [S^0, \mathrm{hofib}\, f]_* \xrightarrow{i(f)_*} [S^0, X]_* \xrightarrow{f_*} [S^0, Y]_*$$

Since $[S^n, Y]_* = \pi_n(Y)$ we recover the long exact sequence from Corollary 8.12.