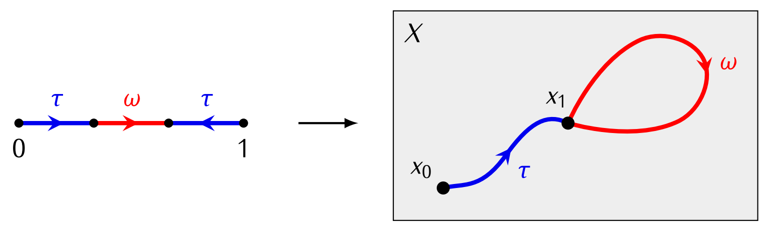
4 | Dependence on The Basepoint

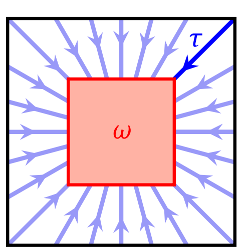
Let X be a space, and let $x_0, x_1 \in X$. Recall that any path $\tau\colon [0,1] \to X$ such that $\tau(0) = x_0$ and $\tau(1) = x_1$ defines a isomorphism of fundamental groups

$$s_\tau\colon \pi_1(X, x_1) \to \pi_1(X, x_0)$$

given by $s_\tau([\omega]) = [\tau * \omega * \bar{\tau}]$, where $\bar{\tau}$ is obtained from τ by reverting orientation.



In a similar way, given a path $\tau\colon [0,1] \to X$ with $\tau(0) = x_0$ and $\tau(1) = x_1$ we can define a map $s_\tau\colon \pi_n(X, x_1) \to \pi_n(X, x_0)$. To do this, given a map $\omega\colon (I^n, \partial I^n) \to (X, x_1)$, define a map $\omega_\tau\colon (I^n, \partial I^n) \to (X, x_0)$ as follows:



The smaller cube is mapped by ω and each radial ray joining the boundaries of the larger and smaller cube is mapped by the path τ.

Let $\pi_1(X, x_0, x_1)$ denote the set of homotopy classes of paths $\tau\colon [0,1] \to X$ such that $\tau(0) = x_0$ and $\tau(1) = x_1$, with homotopies preserving the endpoints.

4.1 Lemma. *Let $\omega, \omega'\colon (I^n, \partial I^n) \to (X, x_1)$ be maps such that $\omega \simeq \omega'$ (rel ∂I^n), and let $\tau, \tau'\colon [0,1] \to X$ be paths such that $\tau(0) = \tau'(0) = x_0$, $\tau(1) = \tau'(1) = x_1$ and $\tau \simeq \tau'$ (rel $\{0,1\}$). Then $\omega_\tau \simeq \omega'_{\tau'}$ (rel ∂I^n).*

Equivalently, if $[\omega] = [\omega'] \in \pi_n(X, x_1)$ and $[\tau] = [\tau'] \in \pi_1(X, x_0, x_1)$ then $[\omega_\tau] = [\omega'_{\tau'}] \in \pi_n(X, x_0)$

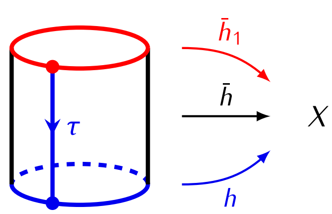
Proof. Exercise. □

4.2 Note. The homotopy class $[\omega_\tau]$ can be also described as follows. Consider the homotopy $h\colon \partial I^n \times [0,1] \to X$ given by $h(x,t) = \tau(1-t)$. Since the pair $(I^n, \partial I^n)$ has the homotopy extension property, we can extend h to a homotopy $\bar{h}\colon I^n \times [0,1] \to X$ such that $\bar{h}_0 = \omega$. The map $\bar{h}_1$ defines an element $[\bar{h}_1] \in \pi_n(X, x_0)$. This element does not depend on the choice of the extension $\bar{h}$ (exercise), and we have $[\bar{h}_1] = [\omega_\tau]$.

4.3 Note. Recall that elements of $\pi_n(X, x_1)$ can be alternatively defined as pointed homotopy classes of maps $\omega\colon (S^n, s_0) \to (X, x_1)$. In this setting, for $[\tau] \in \pi_1(X, x_0, x_1)$ the element $[\omega_\tau] \in \pi_n(X, x_0)$ can be described using a similar approach as in (4.2). Given such ω and τ we can define a function

$$h\colon (S^n \times \{0\}) \cup (\{s_0\} \times [0,1]) \to X$$

so that $h(s,0) = \omega(s)$ and $h(s_0,t) = \tau(1-t)$. Since the pair (S^n, s_0) has the homotopy extension property, thus h can be extended to a homotopy $\bar{h}\colon S^n \times [0,1] \to X$. One can check that the pointed homotopy class of the map $\bar{h}_1\colon (S^n, s_0) \to (X, x_0)$ does not depend on the choice of the extension $\bar{h}$. We set: $[\omega_\tau] = [h_1] \in \pi_n(X, x_0)$.



4.4 Definition. Given $[\tau] \in \pi_1(X, x_0, x_1)$ let

$$s_{[\tau]}\colon \pi_n(X, x_1) \to \pi_n(X, x_0)$$

denote the function given by $s_{[\tau]}([\omega]) = [\omega_\tau]$.

4.5 Proposition. *1) For any $[\tau] \in \pi_1(X, x_0, x_1)$ the function $s_{[\tau]}$ is a group homomorphism.*

2) If $[\tau] \in \pi_1(X, x_0, x_1)$ and $[\sigma] \in \pi_1(X, x_1, x_2)$ then

$$s_{[\tau*\sigma]} = s_{[\tau]} \circ s_{[\sigma]} \colon \pi_n(X, x_2) \to \pi_n(X, x_0)$$

3) If $c_{x_0} \colon [0,1] \to X$ is the constant path, $c_{x_0}(t) = x_0$ for all $t \in [0,1]$, then $s_{[c_{x_0}]} \colon \pi_n(X, x_0) \to \pi_n(X, x_0)$ is the identity homomorphism.

Proof. Exercise. □

4.6 Corollary. *Let X be a space and let $x_0, x_1 \in X$. For any path $\tau \colon [0,1] \to X$ be a path such that $\tau(0) = x_0$, $\tau(1) = x_1$ the homomorphism $s_{[\tau]} \colon \pi_n(X, x_1) \to \pi_n(X, x_0)$ is an isomorphism.*

Proof. Let $\overline{\tau}$ be the inverse of τ. This defines homomorphisms

$$s_{[\tau]} \colon \pi_n(X, x_1) \leftrightarrows \pi_n(X, x_0) \colon s_{[\overline{\tau}]}$$

We will show that $s_{[\overline{\tau}]} = s_{[\tau]}^{-1}$. Indeed, by Proposition 4.5 we have

$$s_{[\overline{\tau}]} \circ s_{[\tau]} = s_{[\overline{\tau}*\tau]} = s_{[c_{x_0}]} = \mathrm{id}_{\pi_n(X, x_1)}$$

Analogously, $s_{[\tau]} \circ s_{[\overline{\tau}]} = \mathrm{id}_{\pi_n(X, x_0)}$. □

Corollary 4.6 implies that if x_0, x_1 are in the same path connected component of X then $\pi_n(X, x_0) \cong \pi_n(X, x_1)$. On the other hand, if points $x_0, x_1 \in X$ belong to different path connected components of X, then in general there is no relationship between $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$.

4.7 Proposition. *Let X be a space, $x_0 \in X$, and let X_0 be the path connected component of X such that $x_0 \in X_0$. Then the inclusion map $i \colon X_0 \hookrightarrow X$ induces an isomorphism*

$$i_* \colon \pi_n(X_0, x_0) \xrightarrow{\cong} \pi_n(X, x_0)$$

Proof. Since I^n is path connected, for any map $\omega \colon (I^n, \partial I^n) \to (X, x_0)$ we have $\omega(I^n) \subseteq X_0$. This shows that i_* is onto. Also, if $h \colon I^n \times [0,1] \to X$ is a homotopy $h \colon \omega \simeq \omega'$ where $\omega, \omega' \colon I^n \to X_0$ then, since $I^n \times [0,1]$ is path connected, we have $h(I^n \times [0,1]) \subseteq X_0$. It implies that i_* is 1-1. □

4.8 Note. Given a path connected space X we will sometimes write $\pi_n(X)$ to denote the n-th homotopy group of X taken with respect to some unspecified basepoint of X. By Corollary 4.6 this will not create problems as long as we are interested in the isomorphism type of the fundamental group only.

Similarly as for the fundamental group we have:

4.9 Proposition. *Let $f, g\colon X \to Y$ be homotopic maps and let $h\colon f \simeq g$. For $x_0 \in X$ let τ be the path in Y given by $\tau(t) = h(x_0, t)$. The following diagram commutes:*

$$\begin{tikzcd} & \pi_n(Y, g(x_0)) \arrow[d, "s_{[\tau]}", "\cong"'] \\ \pi_n(X, x_0) \arrow[ur, "g_*"] \arrow[r, "f_*"'] & \pi_n(Y, f(x_0)) \end{tikzcd}$$

Proof. Exercise. □

4.10 Note. Proposition 4.9 implies, in particular, that if $f, g\colon (X, x_0) \to (Y, y_0)$ are maps of pointed spaces and $f \simeq g$ (rel $\{x_0\}$) then $f_* = g_*$.

4.11 Corollary. *If $f, g\colon X \to Y$ are maps such that $f \simeq g$ then the homomorphism $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is an isomorphism (or it is trivial or 1-1 or onto) if and only if the homomorphism $g_*\colon \pi_n(X, x_0) \to \pi_n(Y, g(x_0))$ has the same property.*

4.12 Proposition. *If $f\colon X \to Y$ is a homotopy equivalence then for any $x_0 \in X$ the homomorphism $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is an isomorphism.*

Proof. Let $g\colon Y \to X$ be a homotopy inverse of f. Consider the sequence of homomorphisms

$$\pi_n(X, x_0) \xrightarrow{f_*} \pi_n(Y, f(x_0)) \xrightarrow{g_*} \pi_n(X, gf(x_0)) \xrightarrow{f_*} \pi_n(Y, fgf(x_0))$$

Composition of the first two homomorphisms satisfies $g_*f_* = (gf)_*$. Since $gf \simeq \mathrm{id}_X$ and id_{X*} is an isomorphism, by Corollary 4.11 we obtain that g_*f_* is an isomorphism. This implies in particular that g_* is onto. Similarly, composing the last two homomorphisms we obtain $f_*g_* = (fg)_*$ and since $fg \simeq \mathrm{id}_Y$ we get that f_*g_* is an isomorphism. This means that g_* is 1-1. As a consequence g_* is an isomorphism. It follows that the first homomorphism f_* is a composition of two isomorphisms: $f_* = g_*^{-1}(g_*f_*)$, and so f_* is an isomorphism. □

4.13 Corollary. *If X, Y are path connected spaces and $X \simeq Y$ then $\pi_n(X, x_0) \overset{\cong}{=} \pi_n(Y, y_0)$ for any $x_0 \in X$, $y_0 \in Y$.*

4.14 The action of π_1. If $[\tau] \in \pi_1(X, x_0)$ then $s_{[\tau]}$ is an isomorphism

$$s_{[\tau]}\colon \pi_n(X, x_0) \to \pi_n(X, x_0)$$

Denote $[\tau] \odot [\omega] := s_{[\tau]}(\omega)$.

4.15 Definition. For $n \geq 0$ the *action of* $\pi_1(X, x_0)$ *on* $\pi_n(X, x_0)$ is the map

$$\begin{aligned} \pi_1(X, x_0) \times \pi_n(X, x_0) &\to \pi_n(X, x_0) \\ ([\tau]\,,\ [\omega]) &\mapsto [\tau] \odot [\omega] \end{aligned}$$

4.16 Note. By Proposition 4.5, for any $[\tau], [\tau'] \in \pi_1(X, x_0)$ and $\omega, \omega' \in \pi_n(X, x_0)$ we have:

- $[\tau] \odot ([\omega] \cdot [\omega']) = ([\tau] \odot [\omega]) \cdot ([\tau \odot [\omega'])$
- $([\tau] \cdot [\tau']) \odot [\omega] = [\tau] \odot ([\tau'] \odot [\omega])$
- $[c_{x_0}] \odot [\omega] = [\omega]$ where $[c_{x_0}] \in \pi_1(X, x_0)$ is the trivial element.
- $[\tau] \odot [c_{x_0}] = [c_{x_0}]$ where $[c_{x_0}] \in \pi_n(X, x_0)$ is the trivial element.

4.17 Proposition. *For any map $f\colon (X, x_0) \to (Y, y_0)$ the following diagram commutes:*

$$\begin{tikzcd} \pi_1(X, x_0) \times \pi_n(X, x_0) \arrow[r, "\odot"] \arrow[d, "f_* \times f_*"'] & \pi_n(X, x_0) \arrow[d, "f_*"] \\ \pi_1(Y, y_0) \times \pi_n(Y, y_0) \arrow[r, "\odot"'] & \pi_n(Y, y_0) \end{tikzcd}$$

Proof. Exercise. □

4.18 Example. The action of $\pi_1(X, x_0)$ on $\pi_1(X, x_0)$ if given by conjugation:

$$[\tau] \odot [\omega] = [\tau] \cdot [\omega] \cdot [\tau]^{-1}$$

4.19 Definition. A path connected space X is *n-simple* if for some $x_0 \in X$ the action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$ is trivial: $[\tau] \odot [\omega] = [\omega]$ for all $[\tau] \in \pi_1(X, x_0)$ and $[\omega] \in \pi_n(X, x_0)$. A path connected space is *simple* if it is n-simple for all $n \geq 1$.

The following fact implies that n-simplicity of a space X does not depend on the choice of a basepoint $x_0 \in X$:

4.20 Proposition. *Let X be a space, let $x_0, x_1 \in X$, and let $\tau\colon [0, 1] \to X$ be a path such that $\tau(0) = x_0$ and $\tau(1) = x_1$. Then the following diagram commutes:*

$$\begin{tikzcd} \pi_1(X, x_1) \times \pi_n(X, x_1) \arrow[r, "\odot"] \arrow[d, "s_{[\tau]} \times s_{[\tau]}"'] & \pi_n(X, x_1) \arrow[d, "s_{[\tau]}"] \\ \pi_1(X, x_0) \times \pi_n(X, x_0) \arrow[r, "\odot"'] & \pi_n(X, x_0) \end{tikzcd}$$

Proof. Exercise. □

For spaces X, Y let $[X, Y]$ denote the set of homotopy classes of maps $X \to Y$. Notice that for any space X and any n we have a map of sets

$$\phi\colon \pi_n(X, x_0) \to [S^n, X]$$

which maps the pointed homotopy class of map $\omega\colon (S^n, s_0) \to (X, x_0)$ to the unpointed homotopy class of the same map.

4.21 Proposition. *Let X be a path connected space, and let $n \geq 1$. The following conditions are equivalent:*

1) *X is n-simple.*
2) *For any $x_0, x_1 \in X$, $[\tau], [\sigma] \in \pi_1(X, x_0, x_1)$ and $[\omega] \in \pi_n(X, x_1)$ we have $s_{[\tau]}([\omega]) = s_{[\sigma]}([\omega])$. Thus there is a canonical isomorphism $\pi_n(X, x_1) \xrightarrow{\cong} \pi_n(X, x_0)$.*
3) *For any $x_0 \in X$ the map $\phi\colon \pi_n(X, x_0) \to [S^n, X]$ is a bijection. Therefore any (unpointed) map $f\colon S^n \to X$ defines a unique element of $\pi_n(X, x_0)$.*

Proof. 1) ⇒ 2) Let $[\tau], [\sigma] \in \pi_1(X, x_0, x_1)$ and $[\omega] \in \pi_n(X, x_1)$. Since $[\overline{\tau} * \sigma] \in \pi_1(X, x_1)$, by 1) we obtain

$$s_{[\overline{\tau}]}s_{[\sigma]}([\omega]) = s_{[\overline{\tau}*\sigma]}([\omega]) = [\omega]$$

Also, since $s_{[\overline{\tau}]}$ is the inverse isomorphism of $s_{[\tau]}$ we get

$$s_{[\sigma]}([\omega]) = s_{[\tau]}s_{[\overline{\tau}]}s_{[\sigma]}([\omega]) = s_{[\tau]}([\omega])$$

2) ⇒ 1) Let $[\tau], [c_{x_0}] \in \pi_1(X, x_0)$, where $[c_{x_0}]$ is the trivial elemment. By 2) we have

$$s_{[\tau]}([\omega]) = s_{[c_{x_0}]}([\omega]) = [\omega]$$

for any $[\omega] \in \pi_n)(X, x_0)$. Therefore X is n-simple.

1) ⇒ 3) The map ϕ is always onto. Indeed, take any map $\omega\colon S^n \to X$. Since X is path connected, there exists a path $\tau\colon [0, 1] \to X$ such that $\tau(0) = x_0$ and $\tau(1) = \omega(s_0)$. Consider the map

$$h\colon (S^n \times \{0\}) \cup (\{s_0\} \times [0, 1]) \to X$$

so that $h(s, 0) = \omega(s)$ and $h(s_0, t) = \tau(1 - t)$. The pair (S^n, s_0) has the homotopy extension property, so h can be extended to a homotopy $\bar{h}\colon S^n \times [0, 1] \to X$. The for the map h_1 we have $h_1(s_0) = x_0$, so $[h_1] \in \pi_n(X, x_0)$. Also, h is homotopic to $h_0 = \omega$. Therefore we have $\phi([h_1]) = [\omega]$.

To show that ϕ is 1-1, we will use the description of $s_{[\tau]}$ in terms of maps from spheres given in Note 4.2. Given two elements $[\omega_0], [\omega_1] \in \pi_n(X, x_0)$ assume that $\phi([\omega_0]) = \phi[\omega_1]$. This means that there exists a homotopy $h\colon S^n \times [0,1] \to X$ such that $h_0 = \omega_0$ and $h_1 = \omega_1$. Let $\tau\colon [0,1] \to X$ be a path given by $\tau(t) = h(s_0, t)$. Then $[\tau] \in \pi_1(X, x_0)$, and by (4.2) we have

$$[\omega_1] = [\bar{\tau}] \odot [\omega_0] = s_{[\bar{\tau}]}([\omega_0])$$

By 1) we have $s_{[\bar{\tau}]}([\omega_0]) = [\omega_0]$. Thus $[\omega_1] = [\omega_0] \in \pi_n(X, x_0)$.

3) $\Rightarrow$ 1) Let $[\tau] \in \pi_1(X, x_0)$, $[\omega] \in \pi_n(X, x_0)$. Let $\omega_\tau\colon (S^n, s_0) \to (X, x_0)$ be some map such that $[\omega_\tau] = s_{[\tau]}([\omega])$. By (4.2) the maps ω_τ and ω are freely homotopic, i.e. $\phi([\omega_\tau]) = \phi([\omega])$. By assumption ϕ is 1-1, thus we obtain

$$[\omega] = [\omega_\tau] = s_{[\tau]}([\omega]) = [\tau] \odot [\omega]$$

in $\pi_n(X, x_0)$.

$\square$