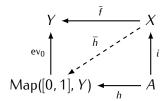
9 Cofibrations

9.1 Definition. A map $i: A \to X$ has the *homotopy extension property* for a space Y if for any commutative diagram of the form



there exists a map $\bar{h}: X \times [0,1] \to E$ such that $\bar{h}i = \bar{f}$ and $p\bar{h} = h$. Here $\text{ev}_0: \text{Map}([0,1], Y) \to Y$ is the evaluation at 0 map: $\text{ev}_0(\omega) = \omega(0)$.

Equivalently, $i: A \to X$ has the homotopy extension property for Y if given any map $\bar{f}: X \to Y$ and a homotopy $h^{\sharp}: A \times [0,1] \to Y$ such that $h_0^{\sharp} = \bar{f}i$ we can find a homotopy $\bar{h}^{\sharp}: X \times [0,1]$, such that $\bar{h}_0^{\sharp} = \bar{f}$ and $\bar{h}^{\sharp}(i(a),t) = h^{\sharp}(a,t)$ for all $(a,t) \in A \times [0,1]$.

In this setting we will say that \bar{h}^{\sharp} is an extension of h^{\sharp} beginning at \bar{f} .

- **9.2 Definition.** A map $i: A \to X$ is a *cofibration* if it has the homotopy extension property for any space Y.
- **9.3 Example.** By Theorem 2.14 if (X, A) is a relative CW complex then the inclusion $i: A \hookrightarrow X$ is a cofibration.

Recall that the mapping cylinder of a map $f: X \to Y$ is the quotient space

$$M_f = (X \times [0,1] \sqcup Y)/\sim$$

where $(x,0) \sim f(x)$ for all $x \in X$. We have a map $s_f \colon M_f \to Y \times [0,1]$ such that $s_f(x,t) = (f(x),t)$ for

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 $(x, t) \in X \times [0, 1] \text{ and } f(y) = (y, 0) \text{ for } y \in Y.$

9.4 Proposition. For a map $i: A \rightarrow X$ the following conditions are equivalent:

- 1) The map i is a cofibration.
- 2) The map i has the homotopy extension property for the space M_i
- 3) There exists a map $r_f: X \times [0,1] \to M_i$ such that $r_f s_f = id_{M_i}$

Proof. Exercise.

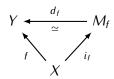
9.5 Corollary. If $i: A \to X$ is a cofibration then i is an embedding.

Proof. Exercise. Use condition 3) in Proposition 9.4.

9.6 Proposition. Given any map $f: X \to Y$ the map $i_f: X \to M_i$ given by $i_f(x) = (x, 1)$ is a cofibration.

Proof. Exercise.

9.7 Note. Given a map $f: X \to Y$, let $d_f: M_f \to Y$ be the strong deformation retraction. As a consequence of Proposition 9.6, we have a commutative diagram



where i_f is a cofibration.