15 | Weak Equivalences and Homology

The main goal of this chapter is to show that the following holds:

15.1 Theorem. *If $f\colon X \to Y$ is a weak equivalence then the induced homomorphisms $f_*\colon H_i(X) \to H_i(Y)$ and $f^*\colon H^i(X) \to H^i(Y)$ are isomorphisms for all $i \geq 0$.*

15.2 Brief review of homological algebra.

- A *chain complex* C_* consists of a sequence of abelian groups and group homomorphisms
$$\dots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots$$
such that $\partial_n \partial_{n+1} = 0$ for all n. The homomorphisms ∂_n are called *differentials*.
- The *n-th homology group* of a chain complex C_* is the group $H_n(C_*) = \ker \partial_n / \operatorname{Im} \partial_{n+1}$.
- A *chain map* $f_*\colon C_* \to D_*$ is a sequence of homomophisms $f_n\colon C_n \to D_n$ such that $\partial_n f_n = f_{n-1}\partial_n$.
- A chain map $f_*\colon C_* \to D_*$ induces homomorphisms of homology groups $f_*\colon H_n(C_*) \to H_n(D_*)$.
- If $f_*, g_*\colon C_* \to D_*$ are chain maps, then a *chain homotopy* $s_*\colon C_* \to D_*$ from f_* to g_* is a sequence of homomorphisms $s_n\colon C_n \to D_{n+1}$ such that $f_n - g_n = \partial_{n+1}s_n + s_{n-1}\partial_n$
- If there exists a chain homotopy between chain maps $f_*, g_*\colon C_* \to D_*$ then f_* and g_* induce the same homorphism between homology groups $H_*(C_*) \to H_*(D_*)$.

15.3 Brief review of singular homology.

- A *singular chain complex* of a topological space X is a chain complex $C_*(X)$ such that $C_n(X)$ is the free abelian group generated by all singular simplices $\sigma\colon \Delta^n \to X$.
- Differentials in $C_*(X)$ are defined using face maps $d^i_n\colon \Delta^{n-1} \to \Delta^n$ for $i = 0, \dots, n$: $\partial_n \sigma = \sum_{i=0}^n (-1)^i \sigma d^i_n$.

- *Singular homology groups* of a space X are homology groups of $C_*(X)$: $H_n(X) := H_n(C_*(X))$.
- Any map of spaces $f\colon X \to Y$ defines a chain map of singular chain complexes $f_*\colon C_*(X) \to C_*(Y)$ given by $f_*(\sigma) = f\sigma$ for a singular simplex $\sigma\colon \Delta^n \to X$. This induces a homomorphism of homology groups $f_*\colon H_*(X) \to H_*(Y)$.
- For a space X let $i_0, i_1\colon X \to X \times [0,1]$ denote the inclusions $i_0(x) = (x,0)$ and $i_1(x) = (x,1)$. There exists a chain homotopy $s^X_*\colon C_*(X) \to C_{*+1}(X \times [0,1])$ from i_{0*} to i_{1*}.
- The chain homotopy s^X_* can be used to show that if $f, g\colon X \to Y$ are homotopic maps then they induce that same homomorphism of homology groups $H_*(X) \to H_*(Y)$.

Proof of Theorem 15.1. Assume first that $X \subseteq Y$ and that $f\colon X \hookrightarrow Y$ is the inclusion map. Notice that in this case $C_*(X)$ is a subcomplex of $C_*(Y)$ and the chain map $f_*\colon C_*(X) \to C_*(Y)$ is an inclusion.

We will associate to each singular simplex $\sigma\colon \Delta^n \to Y$ a homotopy $h^\sigma\colon \Delta^n \times [0,1] \to Y$ such that:

1) $h^\sigma_0 = \sigma$ and $h^\sigma_1(\Delta^n) \subseteq X$.
2) If $\sigma(\Delta^n) \subseteq X$ then $h^\sigma_t = \sigma$ for all $t \in [0,1]$.
3) $h^{\sigma d^i_n} = h^\sigma(d^i_n \times \mathrm{id}_{[0,1]})$

The homotopies h^σ will be constructed by induction with respect to n. For $n = 0$ giving a simplex $\sigma\colon \Delta^0 = \{*\} \to X$ is the same a giving a point $\sigma(*) = y \in Y$. Since f is a weak equivalence, there exist a path $h^\sigma\colon \Delta^0 \times [0,1] \to Y$ such that $h^\sigma(*,0) = y$ and $h^\sigma(*,1) \in X$. If $y \in X$ take h^σ to be the constant path.

Assume that we have already constructed homotopies h^τ satisfying 1)–3) for all $\tau\colon \Delta^k \to Y$ with $k < n$, and let $\sigma\colon \Delta^n \to Y$. If $\sigma(\Delta^n) \subseteq X$, define h^σ using condition 2). Otherwise, let $\partial\Delta^n := \bigcup_{i=0}^n d^i_n(\Delta^{n-1}) \subseteq \Delta^n$. Since homotopies $h^{\sigma d^i_n}$ are already defined, condition 3) determines a map $h\colon \Delta^n \times \{0\} \cup \partial\Delta^n \times [0,1] \to Y$ such that $h_0 = \sigma$ and $h_1(\partial\Delta^n) \subseteq X$. The pair $(\Delta^n, \partial\Delta^n)$ is a relative CW complex, so by Proposition 13.6 we can extend h to a homotopy $h^\sigma\colon \Delta^n \times [0,1] \to Y$ such that $h^\sigma_0 = \sigma$ and $h^\sigma_1(\Delta^n) \subseteq X$.

Define a map $\varphi_*\colon C_*(Y) \to C_*(X)$ by $\varphi(\sigma) = h^\sigma_1$. Condition 3) implies that φ_* is a chain map. Also, by condition 2) we obtain $\varphi_* f_* = \mathrm{id}_{C_*(X)}$. Finally, a chain homotopy $\Phi_*\colon C_*(Y) \to C_*(Y)$ between $f_*\varphi_*$ and $\mathrm{id}_{C_*(Y)}$ can be obtained as follows. Given a singular simplex $\sigma\colon \Delta^n \to Y$ the homotopy h^σ induces a chain map $h^\sigma_*\colon C_*(\Delta^n \times [0,1]) \to C_*(Y)$. We also have a chain homotopy $s^{\Delta^n}_*\colon C_*(\Delta^n) \to C_*(\Delta^n \times [0,1])$. The identity map $\mathrm{id}_{\Delta^n}\colon \Delta^n \to \Delta^n$ is a singular simplex in $C_n(\Delta^n)$. We set $\varphi(\sigma) = h^\sigma_* s^{\Delta^n}_*(\mathrm{id}_{\Delta^n})$.

For a general weak equivalence $f\colon X \to Y$ consider the commutative diagram

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, hook, "i"'] & & Y \\
& M_f \arrow[ur, "r"'] &
\end{tikzcd}
$$

where M_f is the mapping cylinder of f, i is the inclusion map and r is the retraction. Since f is a weak equivalence and r is a homotopy equivalence, thus i is a weak equivalence. Therefore, by the argument above, i induces an isomorphism on homology groups $i_*\colon H_*(X) \xrightarrow{\cong} H_*(M_f)$. Also, since every homotopy equivalence induces an isomorphism on homology, thus we get an isomorphism $r_*\colon H_*(M_f) \xrightarrow{\cong} H_*(Y)$. Therefore $f_* = r_* i_*\colon H_*(X) \to H_*(Y)$ is an isomorphism.

The statement that a weak equivalence induces an isomorphism of cohomology groups follows from the Universal Coefficients Theorem, which implies that if a map $f\colon X \to Y$ gives an isomorphism on homology, then it also induces an isomorphism on cohomology. $\square$

Using the same arguments as in the proof of Theorem 15.1, this theorem can be generalized as follows:

15.4 Theorem. *If $f\colon X \to Y$ is an n-equivalence for some $n \geq 1$ then then $f_*\colon H_i(X) \to H_i(Y)$ is an isomorphism for all $i < n$ and it is an epimorphism for $i = n$.*