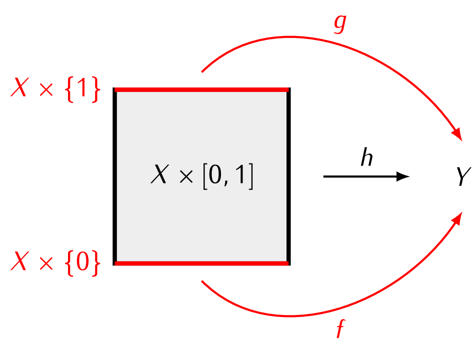
1 | Review: Homotopies

1.1 Definition. Let $f, g\colon X \to Y$ be continuous functions. A *homotopy* between f and g is a continuous function $h\colon X \times [0,1] \to Y$ such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$:



If such homotopy exists then we say that the functions f and g are *homotopic* and we write $f \simeq g$. We will also write $h\colon f \simeq g$ to indicate that h is a homotopy between f and g.

1.2 Note. Given a homotopy $h\colon X \times [0,1] \to Y$ it will be convenient denote by $h_t\colon X \to Y$ the function defined by $h_t(x) = h(x,t)$. If $h\colon f \simeq g$ then $h_0 = f$ and $h_1 = g$.

1.3 Definition. Let X be a space and let $A \subseteq X$. If $f, g\colon X \to Y$ are functions such that $f|_A = g|_A$ then we say that f and g are *homotopic relative to* A if there exists a homotopy $h\colon X \times [0,1] \to Y$ such that $h_0 = f$, $h_1 = g$ and $h_t|_A = f|_A = g|_A$ for all $t \in [0,1]$. In such case we write $f \simeq g$ (rel A).

1.4 Definition. A map $f\colon X \to Y$ is a *homotopy equivalence* if there exists a map $g\colon Y \to X$ such that $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$. If such maps exist we say that the spaces X and Y are *homotopy equivalent* and we write $X \simeq Y$.

1.5 Note. If f and g are maps as in Definition 1.4 then we say that g is a *homotopy inverse* of f.

1.6 Definition. If X is a space such that $X \simeq *$ then we say that X is a *contractible space*.

1.7 Definition. A subspace $A \subseteq X$ is a *deformation retract* of a space X if there exists a homotopy $h\colon X \times [0,1] \to X$ such that

1) $h_0 = \mathrm{id}_X$
2) $h_t|_A = \mathrm{id}_A$ for all $t \in [0,1]$
3) $h_1(x) \in A$ for all $x \in X$

In such case we say that h is a *deformation retraction* of X onto A.

1.8 Proposition. *If $A \subseteq X$ is a deformation retract of X then $A \simeq X$.*

1.9 Note. Let X, X' be spaces, $A \subseteq X$, $A' \subseteq X'$. By a map $f\colon (X, A) \to (X', A')$ we will understand a function $f\colon X \to X'$ such that $f(A) \subseteq A'$. A homotopy of such maps is a homotopy $h\colon X \times [0,1] \to X'$ such that $h_t(A) \subseteq A'$ for each $t \in [0,1]$.

We will also use a variant of this for triples of spaces. If $B \subseteq A \subseteq X$ and $B' \subseteq A' \subseteq X'$ then a map $f\colon (X, A, B) \to (X', A', B')$ is a function $f\colon X \to X'$ such that $f(A) \subseteq A'$ and $f(B) \subseteq B'$. A homotopy h of such maps satisfies the same conditions on every level h_t.