## 9 Exact Puppe Sequence

Recall that a sequence of pointed sets

$$(S_2, s_2) \xrightarrow{f} (S_1, s_1) \xrightarrow{g} (S_0, s_0)$$

is exact at  $S_1$  if  $f(S_2) = g^{-1}(s_0)$ .

For pointed spaces  $(X, x_0)$  and  $(Y, y_0)$  let  $[X, Y]_*$  denote the set of pointed homotopy classes of maps  $X \to Y$ . This is a pointed set, with the basepoint represented by the constant function  $c_{y_0} \colon X \to Y$ ,  $c_{y_0}(x) = y_0$  for all  $x \in X$ .

- **9.1 Definition.** A pointed space  $(X, x_0)$  is well-pointed if the pair  $(X, x_0)$  has the homotopy extension property.
- **9.2 Definition.** A sequence of maps of spaces

$$(X_0, x_0) \xrightarrow{f_0} (X_1, x_1) \xrightarrow{f_1} (X_2, x_2)$$

is exact at  $X_1$  is for any well-pointed space  $(Y, y_0)$  the sequence pointed sets

$$[Y, X_0]_* \xrightarrow{f_{0*}} [Y, X_1]_* \xrightarrow{f_{1*}} [Y, X_2]_*$$

is exact at  $[Y, X_1]_*$ .

**9.3 Proposition.** If  $p: E \to B$  is a Hurewicz fibration,  $e_0 \in E$ ,  $b_0 = p(e_0) \in B$   $F = p^{-1}(b_0)$ , and  $i: F \to E$  is the inclusion map then the sequence  $(F, e_0) \xrightarrow{i} (E, e_0) \xrightarrow{p} (B, b_0)$  is exact at E.

*Proof.* Exercise.

Let  $f:(X,x_0)\to (Y,y_0)$  be any pointed map. Since the sequence

$$\mathsf{hofib}\, f \xrightarrow{i(f)} X \xrightarrow{f} Y$$

is homotopy equivalent to a sequence given by a Hurewicz fibration, it is exact at X. We can continue this construction inductively, by continuing to take homotopy fibers:

$$\dots \longrightarrow \mathsf{hofib}\; i^3(f) \xrightarrow{i^4(f)} \mathsf{hofib}\; i^2(f) \xrightarrow{i^3(f)} \mathsf{hofib}\; i(f) \xrightarrow{i^2(f)} \mathsf{hofib}\; f \xrightarrow{i(f)} X \xrightarrow{f} Y \tag{*}$$

In this way we obtain a sequence which is exact at all spaces. As it turns out, this sequence has a more convenient description. The starting point for it is the following fact:

**9.4 Proposition.** Let  $f: X \to Y$  be a map and  $y_0 \in Y$ . Then the map

$$i(f)$$
: hofib <sub>$u_0$</sub>   $f = \{(x, \omega) \in X \times PY \mid f(x) = \omega(0), \ \omega(1) = y_0\} \longrightarrow Y$ 

given by  $i(f)(x, \omega) = x$  is a Hurewicz fibration.

**9.5 Corollary.** For any map of pointed spaces  $f:(X,x_0)\to (Y,y_0)$  we have a commutative diagram

Proof. We have

$$i(f)^{-1}(x_0) = \{(x_0, \omega) \in X \times PY \mid \omega(0) = f(x_0) = y_0, \ \omega(1) = y_0\} \cong \Omega Y$$

Thus  $\Omega Y$  can be identified with the fiber of i(f) over  $y_0$ , and the map  $j \colon \Omega Y \to \text{hofib } f$ ,  $j(\omega) = (x_0, \omega)$  with the inclusion of the fiber. By Proposition 9.4 and Corollary 8.18 we obtain a homotopy equivalence  $g \colon \Omega Y \to \text{hofib } i(f)$  such that the above diagram commutes.

**9.6 Note.** The homotopy equivalence in Corollary 9.5 can be explicitly described as follows. Up to a homeomorphism we have

hofib 
$$i(f) = \{(\omega, \tau) \in PX \times PY \mid f\omega(0) = \tau(0), \ \omega(1) = y_0, \ \tau(1) = y_0\}$$

Then  $i^2(f)$ : hofib  $i(f) \to \text{hofib } f$  is given by  $(\omega, \tau) = (\omega(0), \tau)$  and  $g(\tau) = (c_{x_0}, \tau)$ .

Applying Corollary 9.5 iteratively to the sequence (\*) we get homotopy equivalences

hofib 
$$i(f) \stackrel{\simeq}{\longleftarrow} \Omega Y$$
  
hofib  $i^2(f) \stackrel{\simeq}{\longleftarrow} \Omega X$   
hofib  $i^3(f) \stackrel{\simeq}{\longleftarrow} \Omega$  hofib  $f$   
hofib  $i^4(f) \stackrel{\simeq}{\longleftarrow} \Omega$  hofib  $i(f) \simeq \Omega^2 Y$   
hofib  $i^5(f) \stackrel{\simeq}{\longleftarrow} \Omega$  hofib  $i^2(f) \simeq \Omega^2 X$ 

Moreover, one can check that the following diagram commutes up to homotopy:

Since the upper row of this diagram is exact, the same is true for the lower row.

**9.7 Definition.** The sequence in the lower row of the diagram (\*\*) is called the *Puppe exact sequence* associated to the map f.

As a consequence, for any map of pointed spaces  $f:(X,x_0)\to (Y,y_0)$  and any well-pointed space  $(Z,z_0)$  we obtain a long exact sequence of sets:

$$\dots \xrightarrow{\Omega^2 f_*} [Z, \Omega^2 Y]_* \xrightarrow{\Omega j_*} [Z, \Omega \text{ hofib } f]_* \xrightarrow{\Omega i(f)_*} [Z, \Omega X]_* \xrightarrow{\Omega f_*} [Z, \Omega Y]_*$$

$$\xrightarrow{j_*} [Z, \text{ hofib } f]_* \xrightarrow{i(f)_*} [Z, X]_* \xrightarrow{f_*} [Z, Y]_* \quad (\maltese)$$

**9.8 Note.** For any pointed space  $(X, x_0)$  and  $n \ge 1$  the loop space  $\Omega^n X$  is quipped with a multiplication map  $\mu \colon \Omega^n X \times \Omega^n X \to \Omega^n X$  given by concatenation of loops. For any pointed space  $(Z, z_0)$  this defines a multiplication

$$\mu_* \colon [Z, \Omega^n X]_* \times [Z, \Omega^n X]_* \to [Z, \Omega^n X]_*$$

given by  $\mu_*([\varphi], [\psi]) = [\mu \circ (\varphi \times \psi)]$ . This equips the set  $[Z, \Omega^n X]_*$  with a group structure. Moreover, for  $n \ge 2$  the multiplication  $\mu$  commutes up to homotopy, and in effect  $[Z, \Omega^n X]_*$  becomes an abelian group.

As a result the exact sequence ( $\mathbb{Y}$ ) becomes an exact sequence of groups starting at  $[Z, \Omega Y]_*$  and its groups are abelian starting with  $[Z, \Omega^2 Y]_*$ 

- **9.9 Loop spaces and suspensions.** There is a different way of interpreting group structures appearing in the sequence  $(\mathbb{H})$ , which uses suspensions of a space in place of loop spaces.
- **9.10 Definition.** Let *X* be a space. The *unreduced suspension* of *X* if the space

$$SX = X \times [0, 1]/(X \times \{0, 1\})$$

**9.11 Note.** Any map  $f: X \to Y$  defines a map  $Sf: SX \to SY$  given by Sf([x, t]) = [f(x), t]. This map is called the suspension of f. In this way we obtain the suspension functor

$$S: \mathsf{Top} \to \mathsf{Top}$$

This functor preserves homotopy classes of maps: if  $f, g: X \to Y$  and  $f \simeq g$  then  $Sf \simeq Sg$ .

- **9.12 Example.** For a sphere  $S^n$  we have  $SS^n \cong S^{n+1}$ .
- **9.13 Definition.** Let  $(X, x_0)$  be a pointed space. The reduced suspension of X is the pointed space

$$\Sigma X = X \times [0, 1]/(X \times \{0, 1\} \cup \{x_0\} \times [0, 1])$$

or equivalently  $\Sigma X = SX/\{[x_0, t] \mid t \in [0, 1]\}$ . The basepoint in  $\Sigma X$  is given by  $[x_0, 0] \in \Sigma X$ .

- **9.14 Note.** If  $(X, x_0)$  is a well-pointed space, then Proposition 2.15 implies that the quotient map  $SX \to \Sigma X$  is a homotopy equivalence. In particular, for any basepoint  $x_0 \in S^n$  we have  $\Sigma S^n \cong S^{n+1}$ . One can show that actually there is a homeomorphism  $\Sigma S^n \cong S^{n+1}$
- **9.15 Note.** Any map  $f:(X,x_0)\to (Y,y_0)$  of pointed spaces, defines a map  $\Sigma f:\Sigma X\to \Sigma Y$  given by  $\Sigma f([x,t])=[f(x),t]$ . This defines the suspension functor

$$\Sigma \colon \mathsf{Top}_* \to \mathsf{Top}_*$$

Similarly as for the unreduced suspension, the reduced suspension preserves homotopy classes: if  $f,g:(X,x_0)\to (Y,y_0)$  are maps of pointed spaces and  $f\simeq g$  then  $\Sigma f\simeq \Sigma g$ .

Let X be a Hausdorff space. By properties of mapping spaces (8.5) the adjunction map  $\operatorname{adj}(\omega) = \omega^{\sharp}$  defines a homeomorphism  $\operatorname{adj} \colon \operatorname{Map}(X \times [0,1],Y) \to \operatorname{Map}(X,\operatorname{Map}([0,1],Y))$ . Let  $(X,x_0)$  and  $(Y,y_0)$  be pointed spaces. Consider  $\Omega_{y_0}Y$  as a subspace of  $\operatorname{Map}([0,1],Y)$  and let  $\operatorname{Map}_*(X,\Omega_{y_0}Y)$  be the subspace of  $\operatorname{Map}(X,\operatorname{Map}([0,1],Y))$  consisting of basepoint preserving maps. Then adj restricts to a homeomorphism between this subspace and the subspace of  $\operatorname{Map}(X \times [0,1],Y)$  consisting of all maps  $f\colon X \times [0,1] \to Y$  such that  $f(X \times \{0,1\} \cup \{x_0\} \times [0,1]) = y_0$ . Such maps are in a bijective correspondence with basepoint preserving maps  $\Sigma X \to Y$ . In this way we obtain a homeomorphism

$$adj: Map_*(\Sigma X, Y) \xrightarrow{\cong} Map_*(X, \Omega Y)$$

On the level of homotopy classes of maps this gives a bijection

$$\operatorname{adj}: [\Sigma X, Y]_* \xrightarrow{\cong} [X, \Omega Y]_*$$

The set of the right hand side has a group structure induced by concatenation of loops. A group structure on the left hand side can be defined using the pinch map  $\Sigma X \to \Sigma X \vee \Sigma X$ . In this way the above bijection becomes an isomorphism of groups.

As a result, the exact sequence  $(\mathbb{H})$  can be equivalently written as

$$\dots \xrightarrow{f_*} [\Sigma^2 Z, Y]_* \xrightarrow{j_* \text{ adj}} [\Sigma Z, \text{ hofib } f]_* \xrightarrow{i(f)_*} [\Sigma Z, X]_* \xrightarrow{f_*} [\Sigma Z, Y]_* \xrightarrow{j_* \text{ adj}} [Z, \text{ hofib } f]_* \xrightarrow{i(f)_*} [Z, X]_* \xrightarrow{f_*} [Z, Y]_*$$

Consider this sequence with  $Z = S^0$ . Since  $\Sigma^n S^0 \cong S^n$  we obtain

$$\dots \xrightarrow{f_*} [S^2, Y]_* \xrightarrow{j_* \text{ adj}} [S^1, \text{hofib } f]_* \xrightarrow{i(f)_*} [S^1, X]_* \xrightarrow{f_*} [S^1, Y]_* \xrightarrow{j_* \text{ adj}} [S^0, \text{hofib } f]_* \xrightarrow{i(f)_*} [S^0, X]_* \xrightarrow{f_*} [S^0, Y]_*$$

Since  $[S^n, Y]_* = \pi_n(Y)$  we recover the long exact sequence from Corollary 8.12.