15 | Weak Homotopy Type

A complication with studying weak equivalences is that two spaces can be related via a chain of weak equivalences even when there is no direct weak equivalence between them. For example, take $X, Y \subseteq \mathbb{R}$ where X consist of all rational numbers and $Y = \{\frac{1}{n} \mid n = 1, 2, \dots\} \cup \{0\}$. Since every path connected component of X and Y consists of a single point, $\pi_0(X)$ and $\pi_0(Y)$ are countable sets and all higher homotopy groups are trivial. A weak equivalence $X \to Y$ would need to be a continuous bijection in order to induce a bijection $\pi_0(X) \to \pi_0(Y)$. However, one can check that there is no such continuous bijection. By the same argument, there is no weak equivalence $Y \to X$. On the other hand, if we take the set of integers $\mathbb{Z}$ with the discrete topology, then any bijections $\mathbb{Z} \to X$ and $\mathbb{Z} \to Y$ are continuous functions and they are weak equivalences. Thus the spaces X and Y are related by a chain of weak equivalences:

$$X \leftarrow \mathbb{Z} \to Y$$

This motivates the following definition:

15.1 Definition. Spaces X and Y are *weakly equivalent* (or have the same *weak homotopy type*) if they can be connected by a zigzag of weak equivalences

$$X = Z_0 \to Z_1 \leftarrow Z_2 \to \dots \leftarrow Z_{n-1} \to Z_n = Y$$

15.2 Proposition. *If X, Y are CW complexes then they are weakly equivalent if and only if they are homotopy equivalent.*

Proof. Assume that X, Y are connected by a zigzag of n weak equivalences:

$$X = Z_0 \xrightarrow{f_1} Z_1 \xleftarrow{f_2} Z_2 \to \dots \leftarrow Z_{n-1} \xrightarrow{f_n} Z_n = Y \tag{$*$}$$

We will show that $X \simeq Y$ by induction with respect to n. If $n = 1$, then we have a weak equivalence $X = Z_0 \to Z_1 = Y$, which by Theorem 14.4 is a homotopy equivalence.

Assume that the statement is true for any zigzag consisting of $n-1$ or fewer weak equivalences and that X, Y are connected by a sequence ($*$). By Corollary 14.8 the map $f_{2*}\colon [X, Z_2] \to [X, Z_1]$ is a bijection. This means that there exists a map $g\colon X \to Z_2$ such that $f_2 g \simeq f_1$. By Proposition 14.3 the map g is a weak equivalence. Thus we obtain a zigzag of weak equivalences of the form:

$$X \xrightarrow{g} Z_2 \xrightarrow{f_3} Z_3 \leftarrow \ldots \leftarrow Z_{n-1} \xrightarrow{f_n} Z_n = Y$$

By the inductive assumption $X \simeq Y$. □

For spaces that are not CW complexes, the study of their weak homotopy type can be simplified using the notion of a CW approximation.

15.3 Definition. A *CW approximation* of a space X is a CW complex Y together with a weak equivalence $f\colon Y \to X$.

More generally, a *CW approximation* of a pair (X, A) is a relative CW complex (Y, A) together with a weak equivalence $f\colon Y \to X$ such that $f|_A = \mathrm{id}_A$.

Notice that a CW approximation of a space X is the same as a CW approximation of the pair $(X, \varnothing)$.

We will show that the following holds:

15.4 Theorem. *Any pair (X, A) has a CW approximation. Moreover, any two CW approximations for such a pair are homotopy equivalent.*

15.5 Corollary. *Spaces X, Y are weakly eqivalent if and only if there exists a space Z and weak equivalences $X \leftarrow Z \to Y$.*

Proof. If such a space Z exists, then by definition X and Y are weakly equivalent. Conversely, assume that we have a zigzag of weak equivalences connecting X and Y:

$$X = Z_0 \xrightarrow{f_1} Z_1 \xleftarrow{f_2} Z_2 \to \ldots \leftarrow Z_{n-1} \xrightarrow{f_n} Z_n = Y$$

We can extend it to

$$X' \xrightarrow{g_X} X = Z_0 \xrightarrow{f_1} Z_1 \xleftarrow{f_2} Z_2 \to \ldots \leftarrow Z_{n-1} \xrightarrow{f_n} Z_n = Y \xleftarrow{g_Y} Y'$$

where $g_X\colon X' \to X$ and $g_Y\colon Y' \to Y$ are CW approximations of X and Y, respectively. By Proposition 15.2 there exists a homotopy equivalence $h\colon X' \to Y'$. Thus we obtain a diagram of weak equivalences: $X \xleftarrow{g_X} X' \xrightarrow{g_Y h} Y$. □

Proof of Theorem 15.4. Assume first that X is a path connected space. For $n = 0, 1, \ldots$ we will construct relative CW complexes $(Y^{(n)}, A)$ and maps $f^{(n)}\colon Y^{(n)} \to X$ such that

1) $Y^{(n)}$ is obtained from $Y^{(n-1)}$ by attaching n-cells.
2) $f^{(0)}|_A = \mathrm{id}_A$ and $f^{(n)}|_{Y^{(n-1)}} = f^{(n-1)}$
3) $f^{(n)}_*\colon \pi_i(Y^{(n)}) \to \pi_i(X)$ is an isomorphism for $i < n$ and epimorphism for $i = n$.

Then the map $\bigcup_n f^{(n)}\colon \bigcup_n Y^{(n)} \to X$ will give a CW approximation of (X, A).

Let $\{A_i\}_{i\in I}$ be path connected components of A. Also, let $x_0 \in X$. For each $i \in I$ choose a point $a_i \in A_i$. Let $(Y^{(1)}, A)$ be a 1-dimensional relative CW complex obtained by:

- adding to A a single 0-cell e^0;
- for each $i \in I$ adding to $A \cup e^0$ a 1-cell e^1_i attached to the points e_0 and a_i.
- for each element $[\tau\colon (S^1, s_0) \to (X, x_0)] \in \pi_1(X, x_0)$ attaching to the resulting space a circle S^1_τ, by identifying s_0 with e^0.

Since X is path connected, for each $i \in I$ there is a path $\omega_i\colon [0, 1] \to X$ such that $\omega_i(0) = x_0$ and $\omega_i(1) = a_i$. Take a map $f^{(1)}\colon Y^{(1)} \to X$ such that $f^{(1)}(x) = x$ for all $x \in A$, $f^{(1)}(e^0) = x_0$. Also, $f^{(1)}$ maps each cell e^1_i using the path ω_i, and each circle S^1_τ using the map τ. Notice that $f^{(1)}_*\colon \pi_i(Y^{(1)}, e_0) \to \pi_i(X, x_0)$ is a bijection for $i = 0$ and it is onto for $i = 1$.

Next, assume that for $i = 1, \dots, n$ we already constructed spaces $Y^{(i)}$ and maps $f^{(i)}\colon Y^{(i)} \to X$ satisfying conditions 1)–3). Take the epimorphism $f^{(n)}_*\colon \pi_n(Y^{(n)}, e^0) \to \pi_n(X, x_0)$. Let $\overline{Y}^{(n+1)}$ denote the space obtained by attaching to $Y^{(n)}$ an $(n+1)$-cell e^{n+1}_ω for each element $[\omega\colon (S^n, s_0) \to (Y^{(n)}, e^0)] \in \ker f^{(n)}_*$, using ω as the attaching map. Since $[f^{(n)}\omega] = 0$ in $\pi_n(X, x_0)$, the map $f^{(n)}\omega\colon S^n \to X$ can be extended to a map $D^{n+1} \to X$. We can use this to extend $f^{(n)}$ to a map $\bar{f}^{(n+1)}\colon \overline{Y}^{(n+1)} \to X$. Subsequently, take $Y^{(n+1)}$ to be the space obtained by attaching to $\overline{Y}^{(n+1)}$ a sphere $S^{(n+1)}_\tau$ for each $[\tau\colon (S^{n+1}, s_0) \to (X, x_0)] \in \pi_{n+1}(X, x_0)$, by identifying s_0 with e^0. Extend $\bar{f}^{(n+1)}$ to $f^{(n+1)}\colon Y^{(n+1)} \to X$, mapping $S^{(n+1)}_\tau$ using τ.

We have a commutative diagram

$$\begin{tikzcd}
\pi_n(Y^{(n)}, e^0) \arrow[rr, "i_*"] \arrow[dr, "f^{(n)}_*"'] & & \pi_n(Y^{(n+1)}, e^0) \arrow[dl, "f^{(n+1)}_*"] \\
& \pi_n(X, x_0) &
\end{tikzcd}$$

where $i\colon Y^{(n)} \hookrightarrow Y^{(n+1)}$ is the inclusion map. Since $f^{(n)}_*$ is onto, thus so is $f^{(n+1)}_*$. Also, by construction $\ker f^{(n+1)} = 0$. Therefore $f^{(n+1)}_*\colon \pi_i(Y^{(n+1)}, e^0) \to \pi_i(X, x_0)$ is an isomorphism for $i \leq n$ and it is an epimorphism for $i = n + 1$.

Next, assume that X is not path connected and let $\{X_i\}_{i\in I}$ be path connected components of X. Construct a CW approximation Y_i for each pair $(X_i, A \cap X_i)$, using the procedure described above. Then a CW approximation of (X, A) can be obtained by taking the quotient space $A \sqcup \bigsqcup_{i\in I} Y_i/\sim$, where the relation $\sim$ identifies points of $X_i \cap A \subseteq Y_i$ with the corresponding points of A.

Finally, assume that for $i = 1, 2$ a map $f_i\colon (Y_i, A) \to (X, A)$ is a CW approximation of (X, A). This gives a commutative diagram

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, hook] & Y_2 \arrow[d, "f_2"] \\ Y_1 \arrow[r, "f_1"'] \arrow[ur, dashed, "g"] & X \end{tikzcd}$$

By Corollary 14.7 there exists $g\colon Y_1 \to Y_2$ such that $g(x) = x$ for all $x \in A$ and $f_2 g \simeq f_1$ (rel A) By the same argument, there exists $h\colon Y_2 \to Y_1$ such that $h(x) = x$ for all $x \in A$ and $f_1 h \simeq f_2$ (rel A). This shows that there exists a map $\varphi\colon Y_1 \times [0,1] \to X$ which gives a homotopy $f_1 \simeq f_1 hg$ (rel A).

Consider the space $Z_1 = Y_1 \times \{0,1\} \cup A \times [0,1] \subseteq Y_1 \times [0,1]$. Then $(Y_1 \times [0,1], Z_1)$ is a relative CW complex. We have a commutative diagram

$$\begin{tikzcd} Z_1 \arrow[r, "\psi"] \arrow[d, hook] & Y_1 \arrow[d, "f_1"] \\ Y_1 \times [0,1] \arrow[r, "\varphi"'] \arrow[ur, dashed, "\overline{\varphi}"] & X \end{tikzcd}$$

where

$$\psi(y,t) = \begin{cases} y & \text{if } t < 1 \\ hg(y) & \text{if } t = 1 \end{cases}$$

Using Corollary 14.7 again, we obtain that there exists $\overline{\varphi}\colon Y_1 \times [0,1] \to X$, which gives a homotopy $\mathrm{id}_{Y_1} \simeq hg$ (rel A). Analogously, we obtain that $\mathrm{id}_{Y_2} \simeq gh$ (rel A). $\square$