20 | Spectral Sequence From a Filtration

The goal of this chapter is to describe a construction of a spectral sequence associated to a filtration of a chain complex. By a chain complex we will mean here a non-negatively graded chain complex, i.e. a chain complex of abelian group

$$\ldots \to C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \ldots$$

such that $C_n = 0$ for $n < 0$.

20.1 Definition. Let C_* be a chain complex. A *filtration* of C_* is a sequence of subcomplexes

$$0 = F_{-1}C_* \subseteq F_0C_* \subseteq \ldots \subseteq C_*$$

such that $\bigcup_p F_pC_* = C_*$. The filtration is *first quadrant* if $H_p(F_qC_*/F_{q-1}C_*) = 0$ for $p < q$.

20.2 Example. Let X be a CW complex. The filtration of X with respect to the skeleta

$$\varnothing = X^{(-1)} \subseteq X^{(1)} \subseteq X^{(2)} \subseteq \ldots \subseteq X$$

defines a filtration of the singular chain complex of X:

$$0 = C_*(X^{(-1)}) \subseteq C_*(X^{(1)}) \subseteq C_*(X^{(2)}) \subseteq \ldots \subseteq C_*(X)$$

Since $H_p(C_*(X^{(q)}), C_*(X^{(q-1)})) \cong H_p(X^{(q)}, X^{(q-1)}) = 0$ for $p < q$, so this is a first quadrant filtration.

20.3 Note. A filtration $\{F_pC_*\}$ of a chain complex C_* induces a filtration of homology groups of C_*

$$0 = F_{-1}H_n(C_*) \subseteq F_1H_n(C_*) \subseteq \ldots \subseteq H_n(C_*)$$

where $F_pH_n(C_*) := \mathrm{Im}(H_n(F_pC_*)) \to H_n(C_*))$. Since $\bigcup F_pC_* = C_*$ we have $\bigcup_p F_pH_n(C_*) = H_n(C_*)$.

Assume that we are given a chain complex C_* with differentials $\partial\colon C_n \to C_{n-1}$, and that $\{F_pC_*\}$ is a filtration of C_*. Denote $E^0_{p,q} := F_pC_{p+q}/F_{p-1}C_{p+q}$. We will consider subgroups $B^\infty_{p,q}, Z^\infty_{p,q} \subseteq E^0_{p,q}$ defined as follows:

$$Z^\infty_{p,q} = \{[x] \in E^0_{p,q} \mid \partial z = 0 \in C_{p+q-1} \text{ for some } z \in [x]\}$$
$$B^\infty_{p,q} = \{[x] \in E^0_{p,q} \mid \partial b \in [x] \text{ for some } b \in C_{p+q+1}\}$$

We have $B^\infty_{p,q} \subseteq Z^\infty_{p,q}$. Define $E^\infty_{p,q} := Z^\infty_{p,q}/B^\infty_{p,q}$.

20.4 Proposition. *$E^\infty_{p,q} \cong F_pH_{p+q}(C_*)/F_{p-1}H_{p+q}(C_*)$.*

Proof. Exercise. □

The spectral sequence we are constructing will introduce intermediate stages $E^0_{p,q}$ between $E^0_{p,q}$ and $E^\infty_{p,q}$ such that each stage is closer approximation of $E^\infty_{p,q}$. More precisely, for $r = 1, 2, \dots$ define:

$$Z^r_{p,q} = \{[x] \in E^0_{p,q} \mid \partial z \in F_{p-r}C_{p+q-1} \text{ for some } z \in [x]\}$$
$$B^r_{p,q} = \{[x] \in E^0_{p,q} \mid \partial b \in [x] \text{ for some } b \in F_{p+r-1}C_{p+q+1}\}$$

We have inclusions

$$B^1_{p,q} \subseteq B^2_{p,1} \subseteq \dots \subseteq B^\infty_{p,q} \subseteq Z^\infty_{p,q} \subseteq \dots \subseteq Z^2_{p,q} \subseteq Z^1_{p,q}$$

Define: $E^r_{p,q} := Z^r_{p,q}/B^r_{p,q}$.

20.5 Proposition. *In the setting described above we have*

1) $B^\infty_{p,q} = \bigcup_r B^r_{p,q}$ and $Z^\infty_{p,q} = \bigcap_r Z^r_{p,q}$.
2) $E^1_{p,q} \cong H_{p+q}(F_pC_*/F_{p-1}C_*)$.

Proof. Exercise. □

20.6 Note. Since $F_pC_* = 0$ if $p < 0$, we get that $E^1_{p,q} = 0$ for $p < 0$. If F_pC_* is a first quadrant filtration, then we also get $E^1_{p,q} = 0$ for $q < 0$.

The groups $E^r_{p,q}$ will form pages of our spectral sequence. In order to finish the construction we still need to specify differentials $d^r\colon E^r_{p,q} \to E^r_{p-r,q+r-1}$. This can be done as follow. By definition, every element of $E^r_{p,q} = Z^r_{p,q}/B^r_{p,q}$ is represented by $z \in F_pC_{p+q}$ such that $\partial z \in F_{p-r}C_{p+q}$. We set $d^r([z]) = [\partial z]$.

20.7 Proposition. *The function $d^r\colon E^r_{p,q} \to E^r_{p-r,q+r-1}$ is a well-defined homomorphism. Moreover, $d^rd^r = 0$ and $H_{p,q}(E^r_{**}, d^r) \cong E^{r+1}_{p,q}$.*

Proof. Exercise. □

Here is a result summarizing the above constructions:

20.8 Theorem. *Let C_* be a chain complex with a first quadrant filtration*

$$0 = F_{-1}C_* \subseteq F_0C_* \subseteq \ldots \subseteq C_*$$

such that $\bigcup_p F_pC_* = C_*$. Then there exists a first quadrant spectral sequence E^r_{**} such that

- $E^1_{p,q} = H_{p+q}(F_pC_*/F_{p-1}(C_*))$;
- *the sequence converges to $H_*(C_*)$.*

Applying this to the singular chain complex of a topological space we obtain:

20.9 Theorem. *Let X be a space with a filtration*

$$\varnothing = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \ldots \subseteq X$$

such that for every compact subset $A \subseteq X$ we have $A \subseteq X_p$ for some $p \geq 0$. Assume also that $H_p(X_q, X_{q-1}) = 0$ for $p < q$. Then there exists a first quadrant spectral sequence E^r_{**} such that

- $E^1_{p,q} = H_{p+q}(X_p, X_{p-1})$
- *The sequence converges to $H_*(X)$. More precisely,*

$$E^\infty_{p,q} = F_pH_{p+q}(X)/F_{p-1}H_{p+q}(X)$$

where $F_pH_n(X) = \mathrm{Im}(H_n(X_p) \to H_n(X))$.

Proof. The filtration of the space X induces a filtration of the singular chain complex of X:

$$0 = C_*(X_{-1}) \subseteq C_*(X_0) \subseteq C_*(X_1) \subseteq \ldots \subseteq C_*(X)$$

The condition on the compact sets in X implies that $\bigcup_p C_*(X_p) = C_*(X)$. Thus the statement follows from Theorem 20.8. □

20.10 Note. The differentials $d^1\colon E^1_{p,q} = H_{p+q}(X_p, X_{p-1}) \to H_{p+q-1}(X_{p-1}, X_{p-2}) = E^1_{p-1,q}$ can be more explicitly described as compositions

$$H_{p+q}(X_p, X_{p-1}) \xrightarrow{\delta} H_{p+q-1}(X_{p-1}) \to H_{p+q-1}(X_{p-1}, X_{p-2})$$

where δ is the boundary map from the homology long exact sequence of the pair (X_p, X_{p-1}).

20.11 Example. For a CW complex X consider the filtration of X by its skeleta:

$$\varnothing = X^{(-1)} \subseteq X^{(0)} \subseteq X^{(1)} \subseteq \ldots \subseteq X$$

In the spectral sequence associated to this filtration we have

$$E^1_{p,q} = H_{p+q}(X^{(p)}, X^{(p-1)}) = \begin{cases} H_p(X^{(p)}, X^{(p-1)}) & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

As a consequence the first page of the spectral sequence looks as follows:

$$\begin{array}{c|ccccccc}
2 & \vdots & & \vdots & & \vdots & & \\
1 & 0 & & 0 & & 0 & & \\
0 & H_0(X^{(0)}, X^{(-1)}) & \xleftarrow{d^1} & H_1(X^{(1)}, X^{(0)}) & \xleftarrow{d^1} & H_2(X^{(2)}, X^{(1)}) & \xleftarrow{d^1} & \cdots \\
\hline
 & 0 & & 1 & & 2 & &
\end{array}$$

The spectral sequence collapses at the second page, so $E^2_{p,q} \cong E^\infty_{p,q}$. We also have

$$E^\infty_{p,q} = \frac{\mathrm{Im}(H_{p+q}(X^{(p)}) \to H_{p+q}(X))}{\mathrm{Im}(H_{p+q}(X^{(p-1)}) \to H_{p+q}(X))} \cong \begin{cases} H_p(X) & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

As a consequence, singular homology groups of X are isomorphic to the homology groups of the chain complex given by the first row of E^1. This chain complex is the cellular chain complex of X.