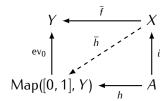
10.1 Definition. A map $i: A \to X$ has the *homotopy extension property* for a space Y if for any commutative diagram of the form



there exists a map $\bar{h}: X \times [0,1] \to E$ such that $\bar{h}i = \bar{f}$ and $p\bar{h} = h$. Here $\text{ev}_0: \text{Map}([0,1], Y) \to Y$ is the evaluation at 0 map: $\text{ev}_0(\omega) = \omega(0)$.

Equivalently, $i: A \to X$ has the homotopy extension property for Y if given any map $\bar{f}: X \to Y$ and a homotopy $h^{\sharp}: A \times [0,1] \to Y$ such that $h_0^{\sharp} = \bar{f}i$ we can find a homotopy $\bar{h}^{\sharp}: X \times [0,1]$, such that $\bar{h}_0^{\sharp} = \bar{f}$ and $\bar{h}^{\sharp}(i(a),t) = h^{\sharp}(a,t)$ for all $(a,t) \in A \times [0,1]$.

In this setting we will say that \bar{h}^{\sharp} is an extension of h^{\sharp} beginning at \bar{f} .

- **10.2 Definition.** A map $i: A \to X$ is a *cofibration* if it has the homotopy extension property for any space Y. In such case we also say that the space X/i(A) is the *cofiber* of i.
- **10.3 Example.** By Theorem 2.14 if (X, A) is a relative CW complex then the inclusion $i: A \hookrightarrow X$ is a cofibration.

Recall that the mapping cylinder of a map $f: X \to Y$ is the quotient space

$$M_f = (X \times [0,1] \sqcup Y)/\sim$$

where $(x,0) \sim f(x)$ for all $x \in X$. We have a map $s_f \colon M_f \to Y \times [0,1]$ such that $s_f(x,t) = (f(x),t)$ for

 $(x, t) \in X \times [0, 1] \text{ and } f(y) = (y, 0) \text{ for } y \in Y.$

10.4 Proposition. For a map $i: A \rightarrow X$ the following conditions are equivalent:

- 1) The map i is a cofibration.
- 2) The map i has the homotopy extension property for the space M_i
- 3) There exists a map $r_f: X \times [0,1] \to M_i$ such that $r_f s_f = id_{M_i}$

Proof. Exercise.

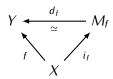
10.5 Corollary. If $i: A \to X$ is a cofibration then i is an embedding.

Proof. Exercise. Use condition 3) in Proposition 10.4. □

10.6 Proposition. Given any map $f: X \to Y$ the map $i_f: X \to M_f$ given by $i_f(x) = (x, 1)$ is a cofibration.

Proof. Exercise.

10.7 Note. Given a map $f: X \to Y$, let $d_f: M_f \to Y$ be the strong deformation retraction. As a consequence of Proposition 10.6, we have a commutative diagram



where i_f is a cofibration. A homotopy inverse of d_f is given by the inclusion map $j_f: Y \to M_f$.

10.8 Note. Recall that the mapping cone of a map $f: X \to Y$ is the space $C_f = M_f/X \times \{1\}$. The space C_f is the cofiber of the cofibration $i_f: X \to M_f$.

10.9 Coexact Puppe sequence. The construction of the coexact Puppe sequence of a map is dual to the construction of the exact Puppe sequence given in Chapter 9.

As in Chapter 9 we will be interested here in pointed spaces and homotopy classes of maps that preserve basepoints. In this case we will use a slightly weakened version of a cofibration: a map of pointed spaces $i: (A, a_0) \to (X, x_0)$ is a cofibration if has the homotopy extension property for all pointed maps $(X, x_0) \to (Y, y_0)$ and pointed homotopies $A \times [0, 1] \to Y$. In this context we modify the constructions of the mapping mapping cylinder and the mapping cone as follows:

10.10 Definition. For a map of pointed spaces $f:(X,x_0)\to (Y,y_0)$ the reduced mapping cylinder of f is the space $\overline{M}_f=M_f/\{x_0\}\times [0,1]$. The reduced mapping cone is the space $\overline{C}_f=\overline{M}_f/X\times \{1\}$.

The reduced mapping cylinder and mapping cone come with a natural choice of basepoints. As in (10.7) for any map $f: (X, x_0) \to (Y, y_0)$ we have a commutative diagram

$$Y \xrightarrow{\frac{d_f}{\simeq}} \overline{M}_f$$

where i_f is a pointed cofibration and d_f is a pointed homotopy equivalence. Also, \overline{C}_f is the cofiber of i_f .

10.11 Definition. A sequence of maps of spaces

$$(X_0, x_0) \xrightarrow{f_0} (X_1, x_1) \xrightarrow{f_1} (X_2, x_2)$$

is coexact at X_1 is for any pointed space (Y, y_0) the sequence pointed sets

$$[X_2, Y]_* \xrightarrow{f_1^*} [X_1, Y]_* \xrightarrow{f_0^*} [X_0, Y]_*$$

is exact at $[X_1, Y]_*$.

10.12 Proposition. If $i: A \to X$ is a cofibration, $q: X \to X/i(A)$ is the quotient map, $x_0 \in A$ then the sequence $(A, x_0) \xrightarrow{i} (X, i(x_0)) \xrightarrow{q} (X/A, qi(x_0))$ is coexact at X.

For any map $f:(X,x_0)\to (Y,y_0)$ consider the sequence

$$X \xrightarrow{f} Y \xrightarrow{q(f)} \overline{C}_f$$

where q(f)(y) = (y, 0). Since this sequence is homotopy equivalent to the cofibration sequence $X \xrightarrow{i_f} \overline{M_f} \longrightarrow \overline{C_f}$, is is coexact at Y. Continuing this construction inductively we obtain a coexact sequence

$$X \xrightarrow{f} Y \xrightarrow{q(f)} \overline{C}_f \xrightarrow{q^2(f)} \overline{C}_{q(f)} \xrightarrow{q^3(f)} \overline{C}_{q(f)} \xrightarrow{q^4(f)} \overline{C}_{q^3(f)} \longrightarrow \dots$$
 (*)

As in Chapter 9 our goal will be to show that this sequence admits a more convenient description. This will depend on two facts that dualize Propositon 9.4 and Corollary 8.18

10.13 Proposition. For any map $f:(X,x_0)\to (Y,y_0)$ the map $q(f):X\to \overline{C}_f$ is a cofibration.

10.14 Proposition. If $f:(X,x_0)\to (Y,y_0)$ is a cofibration then the quotient map

$$\overline{C}_f \to Y/f(X)$$

is a homotopy equivalence.

Notice that $\overline{C}_f/q(f) \cong \Sigma X$, where ΣX is the reduced suspension of X. In this way we obtain:

10.15 Proposition. For any map of pointed spaces $f:(X,x_0)\to (Y,y_0)$ we have a commutative diagram

$$X \xrightarrow{f} Y \xrightarrow{q(f)} \overline{C}_f \xrightarrow{q^2(f)} \overline{C}_{q(f)}$$

$$\downarrow^{\simeq}$$

$$\Sigma X$$

Applying Proposition 10.14 iteratively to the sequence (*) we get homotopy equivalences

$$\overline{C}_{q(f)} \xrightarrow{\simeq} \Sigma X$$

$$\overline{C}_{q^{2}(f)} \xrightarrow{\simeq} \Sigma Y$$

$$\overline{C}_{q^{3}(f)} \xrightarrow{\simeq} \Sigma \overline{C}_{f}$$

$$\overline{C}_{q^{4}(f)} \xrightarrow{\simeq} \Sigma \overline{C}_{q(f)} \simeq \Sigma^{2} X$$

$$\overline{C}_{q^{5}(f)} \xrightarrow{\simeq} \Sigma \overline{C}_{q^{2}(f)} \simeq \Sigma^{2} Y$$
...

Moreover, one can check that the following diagram commutes up to homotopy:

$$X \xrightarrow{f} Y \xrightarrow{q(f)} \overline{C}_{f} \xrightarrow{q^{2}(f)} \overline{C}_{q(f)} \xrightarrow{q^{3}(f)} \overline{C}_{q^{2}(f)} \xrightarrow{q^{4}(f)} \overline{C}_{q^{3}(f)} \xrightarrow{q^{5}(f)} \overline{C}_{q^{4}(f)} \longrightarrow \dots$$

$$\parallel \downarrow \qquad \parallel \downarrow \qquad \parallel \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \downarrow \qquad \qquad \downarrow \qquad \qquad$$

10.16 Definition. The sequence in the lower row of the diagram (**) is called the *Puppe coexact sequence* associated to the map f.

As a consequence, for any map of pointed spaces $f:(X,x_0)\to (Y,y_0)$ and any pointed space (Z,z_0) we obtain a long exact sequence of sets:

$$[X,Z]_* \xleftarrow{f^*} [Y,Z]_* \xleftarrow{q(f)^*} [\overline{C}_f,Z]_* \xleftarrow{g^*} [\Sigma X,Z]_* \xleftarrow{\Sigma f^*} [\Sigma Y,Z]_* \xleftarrow{\Sigma q(f)^*} [\Sigma \overline{C}_f,Z]_* \xleftarrow{\Sigma g^*} [\Sigma^2 X,Z]_* \longleftarrow \dots \quad (\maltese)$$

Starting with $[\Sigma X, Z]_*$ the sets in this sequence have a group structure defined by the suspension, and all maps are homomorphisms of groups. Starting with $[\Sigma^2, Z]_*$ all groups are abelian.

10.17 Note. 1) Using the adjunction adj: $[\Sigma X, Y]_* \stackrel{\cong}{\longrightarrow} [X, \Omega Y]_*$ as in (9.15) we can rewrite the sequence (\color{L}) in the form

$$[X,Z]_* \xleftarrow{f^*} [Y,Z]_* \xleftarrow{q(f)^*} [\overline{C}_f,Z]_* \xleftarrow{g^*} [X,\Omega Z]_* \xleftarrow{f^*} [Y,\Omega Z]_* \xleftarrow{q(f)^*} [\overline{C}_f,\Omega Z]_* \xleftarrow{g^*} [X,\Omega^2 Z]_* \leftarrow \dots$$

In this setting, groups structures are induced the multiplication in loop spaces.

2) Assume that the map $f:(X,x_0)\to (Y,y_0)$ is a cofibration. Using Corollary 10.5 we can then assume that X is a subspace of Y and that f is the inclusion map. By Proposition 10.14 we have $\overline{C}_f\simeq Y/X$, so the above sequence can be written as

$$[X,Z]_* \xleftarrow{f^*} [Y,Z]_* \xleftarrow{q^*} [Y/X,Z]_* \xleftarrow{q^*} [X,\Omega Z]_* \xleftarrow{f^*} [Y,\Omega Z]_* \xleftarrow{q^*} [Y/X,\Omega Z]_* \xleftarrow{q^*} [X,\Omega^2 Z]_* \longleftarrow \dots$$

where $q: Y \to Y/X$ is the quotient map.