18 | Serre classes

18.1 Definition. A *Serre class* is a non-empty collection $\mathcal{C}$ of abelian groups satisfying the property that if

$$0 \to A \to B \to C \to 0$$

is a short exact sequence of abelian groups then $B \in \mathcal{C}$ if and only if $A, C \in \mathcal{C}$

We will say that a Serre class $\mathcal{C}$ is a *Serre ring* if it in additon satisfies that if $A, B \in \mathcal{C}$ then $A \otimes B \in \mathcal{C}$ and $\mathrm{Tor}(A, B) \in \mathcal{C}$.

18.2 Proposition. *Let $\mathcal{C}$ is a Serre class. The following hold:*

1) $0 \in \mathcal{C}$.
2) *If $A \in \mathcal{C}$ and $A' \cong A$ then $A' \in \mathcal{C}$.*
3) *If $B \subseteq A$ then $A \in \mathcal{C}$ if and only if $B, A/B \in \mathcal{C}$.*
4) *If $A_1, \dots, A_n \in \mathcal{C}$ then $A_1 \oplus \dots \oplus A_n \in \mathcal{C}$.*
5) *If $A \to B \to C$ is an exact sequence and $A, C \in \mathcal{C}$ then $B \in \mathcal{C}$.*

Proof. Exercise. $\square$

18.3 Proposition. *Let $\mathcal{C}$ is a Serre ring. If X is a path connected space such that $H_q(X) \in \mathcal{C}$ for all $q > 0$ and $H_q(X; G) \in \mathcal{C}$ for any group $G \in \mathcal{C}$.*

Proof. By the Universal Coefficient Theorem we have

$$H_q(X; G) \cong (H_q(X) \otimes G) \oplus \mathrm{Tor}(H_{q-1}(X), G)$$

For $q = 0$ this gives $H_0(X; G) \cong G$. For $q = 1$ we get $H_1(X; G) \cong H_1(X) \otimes G$. $\square$

18.4 Example. All of the following are Serre rings:

- $\mathcal{C}_{\text{fin}}$ = the class of all finite abelian groups.

- $\mathcal{C}_{\mathrm{fg}}$ = the class of all finitely generated abelian groups.
- $\mathcal{C}_{\mathrm{tor}}$ = the class of all torsion abelian groups.

18.5 Definition. Let $\mathcal{C}$ be a Serre class. We say that homomorphism of abelian groups $f\colon G \to H$ is an *isomorphism mod* $\mathcal{C}$ if $\operatorname{Ker} f \in \mathcal{C}$ and $\operatorname{Coker} f := H/f(G) \in \mathcal{C}$

18.6 Theorem. *Let $\mathcal{C}$ be a Serre ring. If X is a simply connected space then the following conditions are equivalent:*

1) *$\pi_n(X) \in \mathcal{C}$ for all $n \geq 1$*
2) *$H_n(X) \in \mathcal{C}$ for all $n \geq 1$*

18.7 Corollary. *The homotopy groups $\pi_n(S^m)$ are finitely generated for all $n, m \geq 1$.*