18 | Cohomology via Homotopy

Recall (12.9) that for an abelian group G by $K(G, n)$ we denote the Eilenberg-MacLane space such that $\pi_n(K(G, n)) \cong G$. We will also denote by $K(G, 0)$ the discrete space consisting of elements of the group G. Notice that for every n we have a weak equivalence

$$K(G, n) \xrightarrow{\simeq} \Omega K(G, n+1) \xrightarrow{\simeq} \Omega^2 K(G, n+2)$$

For any pointed CW complex this induces a bijection of sets of pointed homotopy classes

$$[X, K(G, n)]_* \xrightarrow{\cong} [X, \Omega^2 K(G, n+2)]_*$$

Since $[X, \Omega^2 K(G, n+2)]_*$ has a natural structure of an abelian group (9.8), we obtain in this way an abelian group structure on $[X, K(G, n)]_*$.

The main goal of this chapter is to show that the following holds:

18.1 Theorem. *Let G be an abelian group.*

1) For any pointed CW complex X and $n \geq 0$ there exists an isomorphism

$$T_X \colon [X, K(G, n)]_* \xrightarrow{\cong} \widetilde{H}^n(X; G)$$

where $\widetilde{H}^n(X; G)$ is the n-th reduced singular cohomology group of X with coefficients in G.

2) These isomorphisms are natural. That is, if $f \colon X \to Y$ is a map of pointed CW complexes then the following diagram commutes:

$$\begin{tikzcd}
[X, K(G, n)]_* \arrow[d, "T_X"', "\cong"] & [Y, K(G, n)]_* \arrow[l, "f^*"'] \arrow[d, "\cong"', "T_Y"] \\
\widetilde{H}^n(X; G) & \widetilde{H}^n(Y; G) \arrow[l, "f^*"]
\end{tikzcd}$$

18.2 Note. Let $\varphi\colon X \to K(G,n)$ be a pointed map. Part 2) of Theorem 18.1 gives a commutative diagram:

$$\begin{tikzcd}
{[X, K(G,n)]_*} \arrow[d, "T_X"', "\cong"] & {[K(G,n), K(G,n)]_*} \arrow[l, "\varphi^*"'] \arrow[d, "\cong"', "T_{K(G,n)}"] \\
\widetilde{H}^n(X; G) & \widetilde{H}^n(K(G,n); G) \arrow[l, "\varphi^*"]
\end{tikzcd}$$

This gives:

$$T_X([\varphi]) = T_X(\varphi^*([\mathrm{id}_{K(G,n)}])) = \varphi^* T_{K(G,n)}([\mathrm{id}_{K(G,n)}])$$

This implies that for any pointed CW complex X the bijection T_X is determined by the cohomology class $\alpha_n = T_{K(G,n)}([\mathrm{id}_{K(G,n)}]) \in \widetilde{H}^n(K(g,n); G)$ This class is called the *fundamental class*.

18.3 Note. An unpointed version of Theorem 18.1 also holds: for any CW complex X there exists a natural isomorphism $T_X\colon [X, K(G,n)] \xrightarrow{\cong} H^n(X; G)$. This can be derived from Theorem 18.1 as follows. For a CW complex X let X_+ denote the space obtained by adding one 0-cell to X: $X_+ = X \sqcup \{+\}$. We consider $+$ as the basepoint of X_+. Then we have bijections $[X, K(G,n)] \cong [X_+, K(G,n)]_*$ and $H^n(X; G) \cong \widetilde{H}^n(X_+; G)$. Thus if $[X_+, K(G,n)]_* \cong \widetilde{H}^n(X_+; G)$ then $[X, K(G,n)] \cong H^n(X; G)$.

The proof of Theorem 18.1 will proceed as follows. First, we will define the notion of a generalized cohomology theory, which consists of a sequence of functors $\{h^n\}_{n\in\mathbb{Z}}$ from the category of pointed CW complexes to the category of abelian groups that satisfy certain axioms. We will show that both assignments $X \mapsto \widetilde{H}^n(X; G)$ and $X \mapsto [X, K(G,n)]_*$ are generalized cohomology theories. Then, we will prove that if $\{h_n\}$ is any generalized cohomology theory such that $h^n(S^0) \cong \widetilde{H}^n(S^0; G)$ for all n, then for every pointed CW complex X and every n there is a natural isomorphism $h^n(X) \to H^n(X; G)$. Since the cohomology theory defined by Eilenberg-MacLane space satisfies this property, Theorem 18.1 will follow.