

# 14 | Weak Homotopy Type

A complication with studying weak equivalences is that two spaces can be related via a chain of weak equivalences even when there is no direct weak equivalence between them. For example, take  $X, Y \subseteq \mathbb{R}$  where  $X$  consist of all rational numbers and  $Y = \{\frac{1}{n} \mid n = 1, 2, \dots\} \cup \{0\}$ . Since every path connected component of  $X$  and  $Y$  consists of a single point,  $\pi_0(X)$  and  $\pi_0(Y)$  are countable sets and all higher homotopy groups are trivial. A weak equivalence  $X \rightarrow Y$  would need to be a continuous bijection in order to induce a bijection  $\pi_0(X) \rightarrow \pi_0(Y)$ . However, one can check that there is no such continuous bijection. By the same argument, there is no weak equivalence  $Y \rightarrow X$ . On the other hand, if we take the set of integers  $\mathbb{Z}$  with the discrete topology, then any bijections  $\mathbb{Z} \rightarrow X$  and  $\mathbb{Z} \rightarrow Y$  are continuous functions and they are weak equivalences. Thus the spaces  $X$  and  $Y$  are related by a chain of weak equivalences:

$$X \leftarrow \mathbb{Z} \rightarrow Y$$

This motivates the following definition:

**14.1 Definition.** Spaces  $X$  and  $Y$  are *weakly equivalent* (or have the same *weak homotopy type*) if they can be connected by a zigzag of weak equivalences

$$X = Z_0 \rightarrow Z_1 \leftarrow Z_2 \rightarrow \dots \leftarrow Z_{n-1} \rightarrow Z_n = Y$$

**14.2 Proposition.** If  $X, Y$  are CW complexes then they are weakly equivalent if and only if they are homotopy equivalent.

*Proof.* Assume that  $X, Y$  are connected by a zigzag of  $n$  weak equivalences:

$$X = Z_0 \xrightarrow{f_1} Z_1 \xleftarrow{f_2} Z_2 \rightarrow \dots \leftarrow Z_{n-1} \xrightarrow{f_n} Z_n = Y \quad (*)$$

We will show that  $X \simeq Y$  by induction with respect to  $n$ . If  $n = 1$ , then we have a weak equivalence  $X = Z_0 \rightarrow Z_1 = Y$ , which by Theorem 13.4 is a homotopy equivalence.

Assume that the statement is true for any zigzag consisting of  $n - 1$  or fewer weak equivalences and that  $X, Y$  are connected by a sequence  $(*)$ . By Corollary 13.8 the map  $f_{2*}: [X, Z_2] \rightarrow [X, Z_1]$  is a bijection. This means that there exists a map  $g: X \rightarrow Z_2$  such that  $f_2 g \simeq f_1$ . By Proposition 13.3 the map  $g$  is a weak equivalence. Thus we obtain a zigzag of weak equivalences of the form:

$$X \xrightarrow{g} Z_2 \xrightarrow{f_3} Z_3 \leftarrow \dots \leftarrow Z_{n-1} \xrightarrow{f_n} Z_n = Y$$

By the inductive assumption  $X \simeq Y$ . □

For spaces that are not CW complexes, the study of their weak homotopy type can be simplified using the notion of a CW approximation.

**14.3 Definition.** A *CW approximation* of a space  $X$  is a CW complex  $Y$  together with a weak equivalence  $f: Y \rightarrow X$ .

More generally, a *CW approximation* of a pair  $(X, A)$  is a relative CW complex  $(Y, A)$  together with a weak equivalence  $f: Y \rightarrow X$  such that  $f|_A = \text{id}_A$ .

Notice that a CW approximation of a space  $X$  is the same as a CW approximation of the pair  $(X, \emptyset)$ .

We will show that the following holds:

**14.4 Theorem.** *Any pair  $(X, A)$  has a CW approximation. Moreover, any two CW approximations for such a pair are homotopy equivalent.*

**14.5 Corollary.** *Spaces  $X, Y$  are weakly equivalent if and only if there exists a space  $Z$  and weak equivalences  $X \leftarrow Z \rightarrow Y$ .*

*Proof.* If such a space  $Z$  exists, then by definition  $X$  and  $Y$  are weakly equivalent. Conversely, assume that we have a zigzag of weak equivalences connecting  $X$  and  $Y$ :

$$X = Z_0 \xrightarrow{f_1} Z_1 \xleftarrow{f_2} Z_2 \rightarrow \dots \leftarrow Z_{n-1} \xrightarrow{f_n} Z_n = Y$$

We can extend it to

$$X' \xrightarrow{g_X} X = Z_0 \xrightarrow{f_1} Z_1 \xleftarrow{f_2} Z_2 \rightarrow \dots \leftarrow Z_{n-1} \xrightarrow{f_n} Z_n = Y \xleftarrow{g_Y} Y'$$

where  $g_X: X' \rightarrow X$  and  $g_Y: Y' \rightarrow Y$  are CW approximations of  $X$  and  $Y$ , respectively. By Proposition 14.2 there exists a homotopy equivalence  $h: X' \rightarrow Y'$ . Thus we obtain a diagram of weak equivalences:  $X \xleftarrow{g_X} X' \xrightarrow{g_Y h} Y$ . □

*Proof of Theorem 14.4.* Assume first that  $X$  is a path connected space. For  $n = 0, 1, \dots$  we will construct relative CW complexes  $(Y^{(n)}, A)$  and maps  $f^{(n)}: Y^{(n)} \rightarrow X$  such that

- 1)  $Y^{(n)}$  is obtained from  $Y^{(n-1)}$  by attaching  $n$ -cells.
- 2)  $f^{(0)}|_A = \text{id}_A$  and  $f^{(n)}|_{Y^{(n-1)}} = f^{(n-1)}$
- 3)  $f_*^{(n)}: \pi_i(Y^{(n)}) \rightarrow \pi_i(X)$  is an isomorphism for  $i < n$  and epimorphism for  $i = n$ .

Then the map  $\bigcup_n f^{(n)}: \bigcup_n Y^{(n)} \rightarrow X$  will give a CW approximation of  $(X, A)$ .

Let  $\{A_i\}_{i \in I}$  be path connected components of  $A$ . Also, let  $x_0 \in X$ . For each  $i \in I$  choose a point  $a_i \in A_i$ . Let  $(Y^{(1)}, A)$  be a 1-dimensional relative CW complex obtained by:

- adding to  $A$  a single 0-cell  $e^0$ ;
- for each  $i \in I$  adding to  $A \cup e^0$  a 1-cell  $e_i^1$  attached to the points  $e_0$  and  $a_i$ .
- for each element  $[\tau: (S^1, s_0) \rightarrow (X, x_0)] \in \pi_1(X, x_0)$  attaching to the resulting space a circle  $S_\tau^1$ , by identifying  $s_0$  with  $e^0$ .

Since  $X$  is path connected, for each  $i \in I$  there is a path  $\omega_i: [0, 1] \rightarrow X$  such that  $\omega_i(0) = x_0$  and  $\omega_i(1) = a_i$ . Take a map  $f^{(1)}: Y^{(1)} \rightarrow X$  such that  $f^{(1)}(x) = x$  for all  $x \in A$ ,  $f^{(1)}(e^0) = x_0$ . Also,  $f^{(1)}$  maps each cell  $e_i^1$  using the path  $\omega_i$ , and each circle  $S_\tau^1$  using the map  $\tau$ . Notice that  $f_*^{(1)}: \pi_i(Y^{(1)}, e_0) \rightarrow \pi_i(X, x_0)$  is a bijection for  $i = 0$  and it is onto for  $i = 1$ .

Next, assume that for  $i = 1, \dots, n$  we already constructed spaces  $Y^{(i)}$  and maps  $f^{(i)}: Y^{(i)} \rightarrow X$  satisfying conditions 1)–3). Take the epimorphism  $f_*^{(n)}: \pi_n(Y^{(n)}, e^0) \rightarrow \pi_n(X, x_0)$ . Let  $\bar{Y}^{(n+1)}$  denote the space obtained by attaching to  $Y^{(n)}$  an  $(n+1)$ -cell  $e_\omega^{n+1}$  for each element  $[\omega: (S^n, s_0) \rightarrow (Y^{(n)}, e^0)] \in \ker f_*^{(n)}$ , using  $\omega$  as the attaching map. Since  $[f^{(n)}\omega] = 0$  in  $\pi_n(X, x_0)$ , the map  $f^{(n)}\omega: S^n \rightarrow X$  can be extended to a map  $D^{n+1} \rightarrow X$ . We can use this to extend  $f^{(n)}$  to a map  $\bar{f}^{(n+1)}: \bar{Y}^{(n+1)} \rightarrow X$ . Subsequently, take  $Y^{(n+1)}$  to be the space obtained by attaching to  $\bar{Y}^{(n+1)}$  a sphere  $S_\tau^{(n+1)}$  for each  $[\tau: (S^{n+1}, s_0) \rightarrow (X, x_0)] \in \pi_{n+1}(X, x_0)$ , by identifying  $s_0$  with  $e^0$ . Extend  $\bar{f}^{(n+1)}$  to  $f^{(n+1)}: Y^{(n+1)} \rightarrow X$ , mapping  $S_\tau^{(n+1)}$  using  $\tau$ .

We have a commutative diagram

$$\begin{array}{ccc}
 \pi_n(Y^{(n)}, e^0) & \xrightarrow{i_*} & \pi_n(Y^{(n+1)}, e^0) \\
 \searrow f_*^{(n)} & & \swarrow f_*^{(n+1)} \\
 & \pi_n(X, x_0) &
 \end{array}$$

where  $i: Y^{(n)} \hookrightarrow Y^{(n+1)}$  is the inclusion map. Since  $f_*^{(n)}$  is onto, thus so is  $f_*^{(n+1)}$ . Also, by construction  $\ker f^{(n+1)} = 0$ . Therefore  $f_*^{(n+1)}: \pi_i(Y^{(n+1)}, e^0) \rightarrow \pi_i(X, x_0)$  is an isomorphism for  $i \leq n$  and it is an epimorphism for  $i = n+1$ .

Next, assume that  $X$  is not path connected and let  $\{X_i\}_{i \in I}$  be path connected components of  $X$ . Construct a CW approximation  $Y_i$  for each pair  $(X_i, A \cap X_i)$ , using the procedure described above. Then a CW approximation of  $(X, A)$  can be obtained by taking the quotient space  $A \sqcup \bigsqcup_{i \in I} Y_i / \sim$ , where the relation  $\sim$  identifies points of  $X_i \cap A \subseteq Y_i$  with the corresponding points of  $A$ .

Finally, assume that for  $i = 1, 2$  a map  $f_i: (Y_i, A) \rightarrow (X, A)$  is a CW approximation of  $(X, A)$ . This gives

a commutative diagram

$$\begin{array}{ccc}
 A & \hookrightarrow & Y_2 \\
 \downarrow & \nearrow g & \downarrow f_2 \\
 Y_1 & \xrightarrow{f_1} & X
 \end{array}$$

By Corollary 13.7 there exists  $g: Y_1 \rightarrow Y_2$  such that  $g(x) = x$  for all  $x \in A$  and  $f_2 g \simeq f_1$  (rel  $A$ ). By the same argument, there exists  $h: Y_2 \rightarrow Y_1$  such that  $h(x) = x$  for all  $x \in A$  and  $f_1 h \simeq f_2$  (rel  $A$ ). This shows that there exists a map  $\varphi: Y_1 \times [0, 1] \rightarrow X$  which gives a homotopy  $f_1 \simeq f_1 h g$  (rel  $A$ ).

Consider the space  $Z_1 = Y_1 \times \{0, 1\} \cup A \times [0, 1] \subseteq Y_1 \times [0, 1]$ . Then  $(Y_1 \times [0, 1], Z_1)$  is a relative CW complex. We have a commutative diagram

$$\begin{array}{ccc}
 Z_1 & \xrightarrow{\psi} & Y_1 \\
 \downarrow & \nearrow \bar{\varphi} & \downarrow f_1 \\
 Y_1 \times [0, 1] & \xrightarrow{\varphi} & X
 \end{array}$$

where

$$\psi(y, t) = \begin{cases} y & \text{if } t < 1 \\ hg(y) & \text{if } t = 1 \end{cases}$$

Using Corollary 13.7 again, we obtain that there exists  $\bar{\varphi}: Y_1 \times [0, 1] \rightarrow X$ , which gives a homotopy  $\text{id}_{Y_1} \simeq hg$  (rel  $A$ ). Analogously, we obtain that  $\text{id}_{Y_2} \simeq gh$  (rel  $A$ ).  $\square$