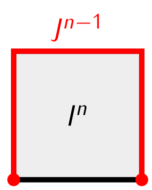
6 | Relative Homotopy Groups

6.1 Notation. Let $X \subseteq A_1 \subseteq A_2$ and $Y \subseteq B_1 \subseteq B_2$. By a map $f\colon (X, A_1, A_2) \to (Y, B_1, B_2)$ we will understand a map $f\colon X \to Y$ such that $f(A_i) \subseteq B_i$ for $i = 1, 2$. A homotopy of such maps is a homotopy $h\colon X \times [0, 1] \to Y$ such that $h_t(A_i) \subseteq B_i$ for $i = 1, 2$ and all $t \in [0, 1]$.

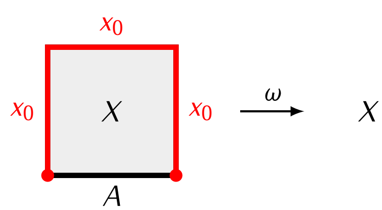
6.2 Notation. For $n \geq 1$ let J^{n-1} denote the subspace of $I^n = I^{n-1} \times I$ given by

$$J^{n-1} = I^{n-1} \times \{1\} \cup \partial I^{n-1} \times I$$

We have: $I^n \subseteq \partial I^n \subseteq J^{n-1}$.



6.3 Definition/Proposition. Let $x_0 \in A \subseteq X$. For $n \geq 2$, the *n-th relative homotopy group* of (X, A, x_0) is the group $\pi_n(X, A, x_0)$ whose elements are homotopy classes of maps $\omega\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$.

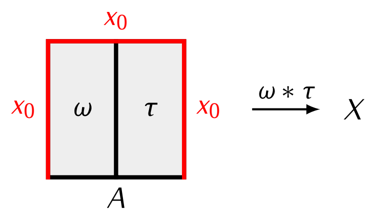


Multiplication in $\pi_n(X, A, x_0)$ is defined as follows. If $\omega, \tau\colon (I^n, \partial I^n) \to (X, x_0)$ then

$$[\omega] \cdot [\tau] = [\omega * \tau]$$

where $\omega * \tau\colon (I^n, \partial I^n) \to (X, x_0)$ is given by

$$(\omega * \tau)(s_1, s_2, \dots, s_n) = \begin{cases} \omega(2s_1, s_2, \dots, s_n) & \text{for } s_1 \in [0, \frac{1}{2}] \\ \tau(2s_1 - 1, s_2, \dots, s_n) & \text{for } s_1 \in [\frac{1}{2}, 1] \end{cases}$$



The trivial element of $\pi_n(X, x_0)$ is the homotopy class of the constant map $c_{x_0}\colon I^n \to X$. Also, for $[\omega] \in \pi_1(X, x_0)$ we have $[\omega]^{-1} = [\overline{\omega}]$ where $\overline{\omega}\colon (I^n, \partial I^n) \to (X, x_0)$ is given by

$$\overline{\omega}(s_1, s_2, \dots, s_n) = (1 - s_1, s_2, \dots, s_n)$$

By a similar argument as in the case of absolute homotopy groups (3.4) we obtain:

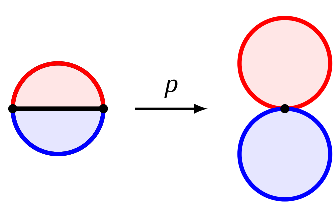
6.4 Theorem. *If $n \geq 3$ then the group $\pi_n(X, A, x_0)$ is abelian for any pointed pair (X, A, x_0).*

6.5 Note. A part of Definition 6.3 makes sense also for $n = 1$. In this case we have $\partial I^1 = \{0, 1\}$ and $J^0 = \{1\}$. Giving map $(I^1, \partial I_1, J^0) \to (X, A, x_0)$ is the same a defining a path in X that starts at x_0 and ends in A. Homotopy classes of such paths form the set $\pi_1(X, A, x_0)$. In general, this set does not have a group structure, but it has a basepoint defined by the constant path $c_{x_0}\colon I^1 \to X$ such that $c_{x_0}(I^1) = x_0$.

6.6 Proposition. *For any space X we have $\pi_n(X, x_0, x_0) \cong \pi_n(X, x_0)$.*

6.7 Proposition. *For any space X we have $\pi_n(X, X, x_0) = 0$.*

6.8 Alternative construction. Just as absolute homotopy groups we can described in terms of maps from spheres, relative homotopy groups can be constructed using maps from discs. Let $s_0 \in S^{n-1} \subseteq D^n$. Elements of $\pi_n(X, A, x_0)$ can be identified with homotopy classes of maps $\omega\colon (D^n, S^{n-1}, s_0) \to (X, A, x_0)$. For $n \geq 2$, multiplication in $\pi_n(X, A, x_0)$ is induced by the pinch map $p\colon D^n \to D^n \vee D^n$, which collapses the equatorial subdisc $D^{n-1} \subseteq D^n$ into a point.



6.9 For any $n \geq 1$, a map $f\colon (X, A, x_0) \to (Y, B, y_0)$ induces a map

$$f_*\colon \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$$

given by $f_*([\omega]) = [f \circ \omega]$. For $n \geq 2$, the map f_* is a homomorphism of groups. In this way we obtain functors

$$\begin{aligned}
\pi_1\colon \mathbf{Top}^2_* &\to \mathbf{Set}_* \\
\pi_2\colon \mathbf{Top}^2_* &\to \mathbf{Gr} \\
\pi_n\colon \mathbf{Top}^2_* &\to \mathbf{Ab}
\end{aligned}$$

for $n \geq 3$, where $\mathbf{Top}^2_*$ is the category of pointed pairs (X, A, x_0) as objects and maps of such pairs as morphisms.

6.10 Proposition. *If $f, g\colon (X, A, x_0) \to (Y, B, y_0)$ are maps such that $f \simeq g$ (as maps of pointed pairs) then $f_* = g_*\colon \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$ for all $n \geq 1$.*

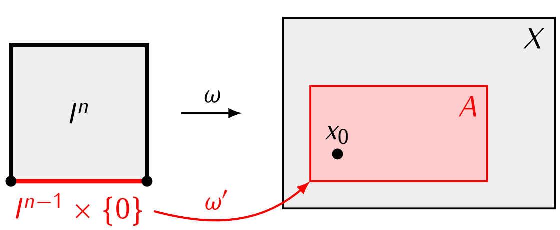
6.11 Long exact sequence of a pair. Consider a pointed pair (X, A, x_0). The inclusion $i\colon (A, x_0) \hookrightarrow (X, x_0)$ induces homomorphisms $i_*\colon \pi_n(A, x_0) \to \pi_n(X, x_0)$ for $n \geq 0$. Also, the map of pointed pairs $j\colon (X, x_0, x_0) \to (X, A, x_0)$ induces homomorphisms

$$j_*\colon \pi_n(X, x_0) = \pi_n(X, x_0, x_0) \to \pi_n(X, A, x_0)$$

for $n \geq 1$. We also have homomorphisms

$$\partial\colon \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$$

defined as follows. For $\omega\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$, let $\omega'\colon (I^{n-1}, \partial I^{n-1}) \to (A, x_0)$ be the restriction of ω to $I^{n-1} \times \{0\}$. Then $\partial([\omega]) = [\omega']$.



6.12 Theorem. *For any pointed pair (X, A, x_0) the following sequence is exact:*

$$\ldots \longrightarrow \pi_{n+1}(X, A, x_0) \xrightarrow{\partial} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \ldots$$
$$\ldots \xrightarrow{j_*} \pi_1(X, A, x_0) \xrightarrow{\partial} \pi_0(A, x_0) \xrightarrow{i_*} \pi_0(X, x_0)$$

Proof. Exercise. □

6.13 Note. The end of the exact sequence in Theorem 6.12 consists of maps of pointed sets. Given such maps

$$(S_2, s_2) \xrightarrow{f} (S_1, s_1) \xrightarrow{g} (S_0, s_0)$$

exactness at S_1 means that $f(S_2) = g^{-1}(s_0)$.