In this section we assume that we are working with a data matrix

$$A = \begin{bmatrix} X_1 & X_2 & \dots & X_M \end{bmatrix}$$

which has been demeaned. That is $m_{X_i} = 0$, or equivalently $X_i = \widetilde{X}_i$ for $i = 1, \dots, M$.

Example.

A data matrix with demeaned exam scores:

$$A = \begin{array}{r} \\ \text{Aly} \\ \text{Bob} \\ \text{Chen} \\ \text{Deb} \\ \text{Emma} \\ \text{Finn} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Ex 1} & \text{Ex 2} & \text{Ex 3} \end{array} \\ \begin{bmatrix} -24 & 1 & -40 \\ -3 & -2 & -6 \\ 29 & 5 & 17 \\ 26 & -2 & 9 \\ -43 & 5 & 30 \\ 15 & -7 & -10 \end{bmatrix} \end{array}$$

Definition

Let $A = \begin{bmatrix} X_1 & \dots & X_M \end{bmatrix}$ be a demeaned data matrix.

- The *1st principal axis* of A is a vector

$$\mathbf{u}_1 = \begin{bmatrix} c_1 \\ \vdots \\ c_M \end{bmatrix}$$

such that $||\mathbf{u}_1|| = 1$ and the variance of the vector

$$Y_1 = A\mathbf{u}_1 = c_1 X_1 + \ldots + c_M X_M$$

is the largest possible.

- The vector Y_1 is called the *1st principal component* of A.

Proposition

Given a demeaned data matrix $A = \begin{bmatrix} X_1 & \dots & X_M \end{bmatrix}$ the 1$^{\text{st}}$ principal axis of A is a vector

$$\mathbf{u}_1 = \begin{bmatrix} c_1 \\ \vdots \\ c_M \end{bmatrix}$$

such that $||\mathbf{u}_1|| = 1$ and $\mathbf{u}_1$ is an eigenvector of the covariance matrix C_A corresponding to the largest eigenvalue of this matrix.

Moreover, if $Y_1 = A\mathbf{u}_1$ is the 1$^{\text{st}}$ principal component of A then $\mathrm{Var}(Y_1) = \lambda_1$ where λ_1 is the largest eigenvalue of the covariance matrix C_A.