19 | Spectral Sequences

The goal of this chapter is to give the definition of a spectral sequence and give some indication how they are used. Explanation in which circumstances spectral sequences occur will be left for later.

19.1 Definition. A *bigraded* abelian group G_{**} is a collection of abelian groups $G_{p,q}$ for $p, q \in \mathbb{Z}$.

19.2 Definition. A *(first quadrant, homological) spectral sequence* (E^r_{**}, d^r) is a sequence of bigraded abelian groups E^r_{**} for $r = 1, 2, \dots$ such that:

1) $E^r_{p,q} = 0$ if $p < 0$ or $q < 0$.
2) Each E^r_{**} is equipped with homomorphisms (*differentials*)

$$d^r \colon E^r_{p,q} \to E^r_{p-r,q+r-1}$$

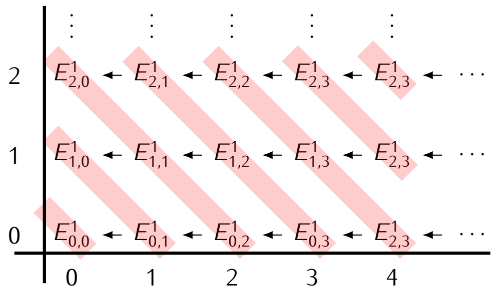
   satisfying $d^r d^r = 0$.

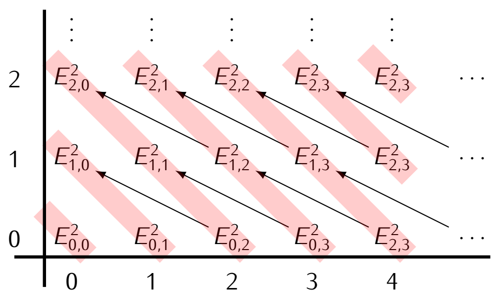
3) For each $r \geq 0$ we have $E^{r+1}_{p,q} \cong H_{p,q}(E^r_{**})$ where

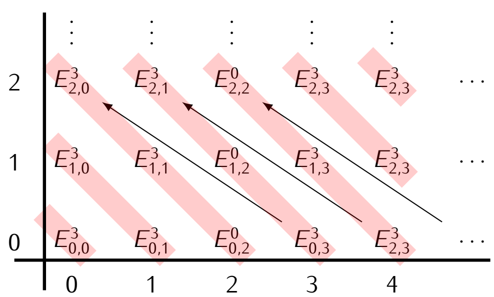
$$H_{p,q}(E^r_{**}) = \frac{\mathrm{Ker}(d^r \colon E^r_{p,q} \to E^r_{p-r,q+r-1})}{\mathrm{Im}(d^r \colon E^r_{p+r,q-r+1} \to E^r_{p,q})}$$

19.3 Note. The bigraded group E^r_{**} is called the *r-th page* of the spectral sequence.

Below are pictures of the first three pages of a spectral sequence. Notice that the differentials d^r always go between groups $E^r_{p,q}$ where $p + q = n$ for some n and groups where $p + q = n - 1$.







Since all groups $E^r_{p,q}$ with negative p or q are trivial, the differentials d^r originating at $E^r_{p,q}$ are trivial for $r > p$. Likewise, the differentials d^r terminating at $E^r_{p,q}$ are trivial if $r > q + 1$. As a consequence, for $r \leq$ max(p+1, q+2) we get

$$E^r_{p,q} = E^{r+1}_{p,q} = E^{r+2}_{p,q} = \dots$$

For each p, q let $E^\infty_{p,q}$ denote this recurring group. These groups form a bigraded group E^∞_{**}.

In typical applications of spectral sequences, E^∞_{**} is related to some object of interest, e.g. homology groups of some space. This is done as follows. We start with a graded abelian group H_* i.e. a collection of abelian groups H_n for $n \in \mathbb{Z}$. A *filtration* of H_* is a sequence of graded subgrops:

$$0 = F_{-1}H_* \subseteq F_0H_* \subseteq F_1H_* \subseteq \ldots \subseteq H_*$$

such that $\bigcup_{p=0}^{\infty} F_pH_* = H_*$.

19.4 Definition. We say that a spectral sequence (E^r_{**}, d^r) *converges* to a graded group H_* if there exists a filtration of H_* such that

$$E^{\infty}_{p,q} \cong F_pH_{p+q}/F_{p-1}H_{p+q}$$

for all p, q.

Results on existence spectral sequences typically say that there exists a spectral sequence for which we can say describe in some useful way groups $E^r_{p,q}$ for some fixed r, and that this sequence converges to some interesting graded group H_*. Here is one example of such statement:

19.5 Theorem. *Let $p\colon E \to B$ and let $F = p^{-1}(b_0)$ for some $b_0 \in B$. If the space B is simply connected then there exists a spectral sequence (E^r_{**}, d^r) such that*

$$E^2_{p,q} \cong H_p(B, H_q(F))$$

for all p, q, and which converges to $H_*(E)$.

The spectral sequence described in this theorem is called the *Serre spectral sequence* of the fibration p.

The next result is an example of how spectral sequences are used in computations.

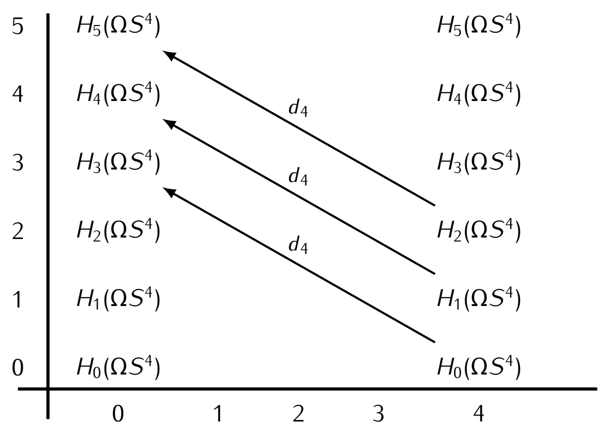
19.6 Theorem. *If $n \geq 2$ then*

$$H_m(\Omega S^n) \cong \begin{cases} \mathbb{Z} & \text{if } (n-1)|m \\ 0 & \text{otherwise} \end{cases}$$

Proof. The space ΩS^n is the fiber of a Serre fibration $p\colon P \to S^n$ with a contractible space P. Consider the Serre spectral sequence of this fibration. We have

$$E^2_{p,q} \cong H_p(S^n, H_q(\Omega S^n)) \cong \begin{cases} H_q(\Omega S^n) & \text{if } p = 0, n \\ 0 & \text{otherwise} \end{cases}$$

For example, for $n = 4$ the second page of this spectral sequence looks as follows:



All differentials in the spectral sequence are trivial, except, possibly $d^n\colon E^n_{p,q} \to E^n_{0,q+n-1}$. It follows that $E^2_{**} = E^n_{**}$ and $E^{n+1}_{**} = E^\infty_{**}$. The total space P of the fibration is contractible, so $H_0(P) = \mathbb{Z}$ and $H_p(P) = 0$ for $p > 0$. By Theorem 19.5 we have $E^\infty_{p,q} \cong F_pH_{p+q}(P)/F_{p-1}H_{p+q}(P)$ for some filtration $\{F_pH_*(P)\}$ of $H_*(P)$. It follows that

$$E^{n+1}_{p,q} = E^\infty_{p,q} \cong \begin{cases} \mathbb{Z} & \text{if } (p,q) = (0,0) \\ 0 & \text{otherwise} \end{cases}$$

Since $E^{n+1}_{p,q} \cong H_{p,q}(E^n_{**})$ we obtain that $H_0(\Omega S^n) \cong \mathbb{Z}$ and $H_p(\Omega S^n) = 0$ for $0 < p \leq n-2$. Also, all differentials d^n must be isomorphisms. This gives:

$$H_p(\Omega S^n) \cong H_{p+(n-1)}(\Omega S^n) \cong H_{p+2(n-1)}(\Omega S^n) \cong H_{p+3(n-1)}(\Omega S^n) \cong \dots$$

Taking $p = 0$ we obtain that $H_m(\Omega S^n) \cong \mathbb{Z}$ if $(n-1)|m$. In all other cases $H_m(\Omega S^n) \cong H_p(\Omega S^n)$ for some $0 < p \leq n-2$, and so it is a trivial group. $\square$

19.7 Note. The proof of Theorem 19.6 used the observation that all differentials d^r in the Serre spectral sequence of the fibration $p\colon P \to S^n$ were trivial for $r \geq n+1$. A situation like this appears frequently in computations involving spectral sequences, which motivates the next definition.

19.8 Definition. We say that a spectral sequence *collapses* at the page r_0 if all differentials d^r are trivial for $r \geq r_0$.

If a spectral sequence collapses at the page r_0 then we have $E^{r_0}_{p,q} = E^{r_0+1}_{p,q} = \dots = E^\infty_{p,q}$.