12 | All Groups Are Homotopy Groups

Recall that van Kampen's Theorem implies that for any group G we can find a space X such that $\pi_1(X) \cong G$. The goal of this chapter is to extend this result to higher homotopy groups. Since all groups $\pi_n(X)$ with $n \geq 2$ are abelian (3.4), we will show that the following holds:

12.1 Theorem. *For any abelian group G and any $n \geq 2$ there exists a space X such that $\pi_n(X) \cong G$. Moreover, such space X can be constructed in such way, that X is a CW complex and $X^{(n-1)} = *$.*

For every abelian group G there exists an epimorphism $\varphi\colon \bigoplus_{i\in I} \mathbb{Z} \to G$ for some set I. Indeed, it is enough to take $I = G$, the set of elements of the group G. Then we can define φ by $\varphi(e_g) = g$, where e_g is the generator of the copy of $\mathbb{Z} \subseteq \bigoplus_{h\in G} \mathbb{Z}$ indexed by g. Given such a homomorphism φ we get $G \cong \bigoplus_{i\in I} \mathbb{Z}/\ker(\varphi)$.

Based on this, in order to prove Theorem 12.1 it will suffice to show that:

1) for any set I and $n \geq 2$ there exists a space X such that $\pi_n(X) \cong \bigoplus_{i\in I} \mathbb{Z}$.
2) for any subgoup $H \subseteq \bigoplus_{i\in I} \mathbb{Z}$ and any $n \geq 2$ there exists a space X such that $\pi_n(X) \cong \bigoplus_{i\in I} \mathbb{Z}/H$.

12.2 Lemma. *Let $\{(X_i, \bar{x}_i)\}_{i\in I}$ be a family of pointed Hasdorff spaces. Let $X = \bigvee_{i\in I} X_i$, and let $* \in X$ denote the basepoint. For $k \in I$ let $r_k\colon X \to X_k$ be the retraction map. Then for any $n \geq 2$ we an epimorphism $\varphi\colon \pi_n(X, *) \to \bigoplus_{i\in I} \pi_n(X_i, \bar{x}_i)$ given by $\varphi([\omega]) = \sum_{i\in I} r_{i*}([\omega])$.*

Proof. For each $k \in I$ let $j_k\colon X_k \to X$ be the inclusion map. We have a homomorphism

$$\psi := \bigoplus_{i\in I} j_{i*}\colon \bigoplus_{i\in I} \pi_n(X_i, \bar{x}_i) \to \pi_n(X, *)$$

The retractions r_i define a map

$$\varphi := \prod_{i\in I} r_{i*}\colon \pi_n(X, *) \to \prod_{i\in I} \pi_n(X_i, \bar{x}_i)$$

We claim that $\operatorname{Im}(\varphi) \subseteq \bigoplus_{i\in I}\pi_n(X_i,\bar{x}_i) \subseteq \prod_{i\in I}\pi_n(X_i,\bar{x}_i)$. Indeed, if $\omega\colon (I^n,\partial I^n) \to (X,*)$ is a map representing an element $[\omega] \in \pi_n(X,*)$, then, by compactness of I^n, we have $\omega(I^n)\cap X_i \neq *$ for finitely many $i \in I$ only, and so $r_{i*}([\omega]) \neq 0$ for finitely many $i \in I$. Thus $\varphi([\omega]) \subseteq \bigoplus_{i\in I}\pi_n(X_i,\bar{x}_i)$. It follows that we can consider φ as a homomorphism $\pi_n(X,*) \to \bigoplus_{i\in I}\pi_n(X_i,\bar{x}_i)$.

Since $r_ij_i = \mathrm{id}_{X_i}$ for all $i \in I$, and $r_{i'}j_i$ is the constant map for all $i \neq i'$, it follows that $\varphi\psi$ is the identity homomorphism, and so φ is onto. $\square$

12.3 Note. In general, the epimorphism φ in Lemma 12.2 is not an isomorphism. For example, recall (5.11) that for $n \geq 2$ we have $\pi_n(S^1\vee S^n) \cong \pi_n(\bigvee_{i\in\mathbb{Z}} S^n)$. By Lemma 12.2 we get an epimorphism

$$\pi_n(S^1\vee S^n) \cong \pi_n(\textstyle\bigvee_{i\in\mathbb{Z}} S^n) \to \bigoplus_{i\in\mathbb{Z}}\mathbb{Z}$$

which shows that the group $\pi_n(S^1\vee S^n)$ is not finitely generated. Therefore $\pi_n(S^1\vee S^n) \ncong \pi_n(S^1)\oplus \pi_n(S^n) \cong \mathbb{Z}$.

12.4 Proposition. *Let $\{(X_i,\bar{x}_i)\}_{i\in I}$ be a family of pointed CW-complexes. Given $n\geq 1$, assume that each complex X_i is n-connected. Then the homomorphism $\varphi\colon \pi_m(\bigvee_{i\in I}X_i,*) \to \bigoplus_{i\in I}\pi_m(X_i,\bar{x}_i)$ is an isomorphism for $m \leq 2n$.*

Proof. For each CW complex X_i we can assume that $\bar{x}_i$ is a 0-cell of X_i. Also, by Proposition 5.6 we can assume that X_i has no other 0-cells, and no k-cells for $k \leq n$ [1].

By Proposition 12.4 φ is onto. It will suffice to show that $\ker\varphi = 0$ for $m \leq 2n$.

Assume first, that the set I is finite, so $\bigvee_{i\in I}X_i = X_1\vee\cdots\vee X_k$ for some $k\geq 0$. Take the product $X_1\times\ldots\times X_k$. The inclusion maps $\psi_j\colon X_j \to X_1\times\ldots\times X_k$ given by $\psi_j(x) = (\bar{x}_1,\ldots,\bar{x}_{j-1},x,\bar{x}_{j+1},\ldots\bar{x}_k)$ define an embedding $\psi\colon X_1\vee\ldots\vee X_k \to X_1\times\ldots\times X_k$. This gives a commutative diagram:

$$\begin{tikzcd}
\pi_m(X_1\vee\ldots\vee X_k) \arrow[r, "\psi_*"] \arrow[d, "\varphi"'] & \pi_m(X_1\times\ldots\times X_k) \arrow[d, "\cong"'] \\
\bigoplus_{j=1}^k \pi_{k+1}(X_i) \arrow[r, "="] & \prod_{j=1}^k \pi_{k+1}(X_i)
\end{tikzcd}$$

This shows that φ is a monomorphism if any only if ψ_* is one. If $X_1,\ldots,X_k$ are finite CW complexes, then the space $X_1\times\cdots\times X_k$ also has the structure of a CW complex, with cells given by products $e_1\times\cdots\times e_k$ where e_i is a cell in X_i. All cells of $X_1\times\cdots\times X_k$ that are not contained in $X_1\vee\cdots\vee X_k$ have dimension $2n+2$ or higher, so $X_1\vee\cdots\vee X_k$ is the $(2n+1)$-skeleton of $X_1\times\cdots\times X_k$. Thus, by Proposition 5.2, ψ_* is an isomorphism for all $m\leq 2n$.

Next, assume that the set I is infinite, and let $\omega\colon (I^m,\partial I^m) \to (\bigvee_{i\in I}X_i,*)$ be a map such that $\varphi([\omega]) = 0$. By compactness of I^m we have $\omega(I^m)\cap X_i \neq *$ for finitely many $i\in I$ only. Thus we can consider ω as

[1]This uses the fact that if $X_i \simeq X_i'$ for all $i\in I$ then $\bigvee_{i\in I}X_i \simeq \bigvee X'_{i\in I}$. This holds for well-pointed, path connected spaces.

a map $\omega\colon (I^m, \partial I^m) \to (X_{i_1} \vee \ldots \vee X_{i_k}, *)$ for some $i_1, \ldots, i_k \in I$. Since $\varphi([\omega]) = 0$, the homomorphism $\pi_m(X_{i_1} \vee \ldots \vee X_{i_k}) \to \bigoplus_{j=1}^{k} \pi_{k+1}(X_{i_j})$ also maps $[\omega]$ to 0. By the finite case this means that $[\omega] = 0$. $\square$

12.5 Note. The proof of Proposition 12.4 uses the fact that if X and Y are CW complexes, then $X \times Y$ has the structure of a CW complex with cells given by products of cells in X and Y. An issue with this statement is that the topology induced on $X \times Y$ by this cell structure (where a set $U \subseteq X \times Y$ is open if and only if its intersection with each cell is an open subset of the cell) need not be the same as the product topology on $X \times Y$. The topology induced by the cell structure on $X \times Y$ is called the compactly generated topology. Let $X \times_{cg} Y$ denote the product taken with this topology, and let $X \times Y$ denote the product taken with the product topology. Every open set in $X \times Y$ is also open in $X \times_{cg} Y$, so the identity map $\mathrm{id}\colon X \times_{cg} Y \to X \times Y$ is continuous. Moreover, this map induces an isomorphism of homotopy groups $\pi_n(X \times_{cg} Y) \xrightarrow{\cong} \pi_n(X \times Y)$ for all n. For this reason this change of topology does not affect the proof of Proposition 12.4.

12.6 Corollary. *For any set I and any $n \geq 2$ we have an isomorphism*

$$\pi_n(\textstyle\bigvee_{i\in I} S^n) \cong \bigoplus_{i\in I} \mathbb{Z}$$

Moreover, the group $\pi_n(\bigvee_{i\in I} S^n)$ is generated by elements $[j_k]$ for $k \in I$ where $j_k\colon S^n \hookrightarrow \bigvee_{i\in I} S^n$ is the inclusion of the k-th copy of S^n.

12.7 Proposition. *Let (X, x_0) be a simply connected space, and let $\varphi_i\colon (S^n, s_0) \to (X, x_0)$ be maps representing elements of $\pi_n(X, x_0)$ for some $n \geq 2$. Consider the space $Y = X \cup \bigcup_i e_i^{n+1}$ obtained by attaching $(n+1)$-cells to X using φ_i as the attaching maps. If $j\colon X \hookrightarrow Y$ is the inclusion map, then the induced homomorphism*

$$j_*\colon \pi_k(X, x_0) \to \pi_k(Y, x_0)$$

is an isomorphism for $k < n$ and an epimorphism for $k = n$. Moreover, $\ker(j_*\colon \pi_n(X, x_0) \to \pi_n(Y, x_0))$ *is the subgroup of $\pi_n(X, x_0)$ generated by the elements $[\varphi_i]$.*

Proof. We can consider the pair (Y, X) as a relative CW complex with the n-skeleton given by X. Then j_* is an isomorphism for $k < n$ and epimorphism for $k = n$ by Proposition 5.2.

It remains to verify the statement about the kernel of j_* for $k = n$. Consider the exact sequence of the pair (Y, X):

$$\cdots \to \pi_{n+1}(Y, X) \xrightarrow{\partial} \pi_n(X) \xrightarrow{j_*} \pi_n(Y) \to \pi_n(Y, X) \to \ldots$$

We have $\ker j_* = \operatorname{Im} \partial$. By assumption, the space X is 1-connected, so from Theorem 11.5 we obtain that the quotient map $q\colon Y \to Y/X$ induces an isomorphism

$$q_*\colon \pi_{n+1}(Y, X) \xrightarrow{\cong} \pi_{n+1}(Y/X) \cong \pi_{n+1}(\textstyle\bigvee_i S^{n+1}) \cong \oplus_i \mathbb{Z}$$

This implies that $\pi_{n+1}(Y, X)$ is generated by homotopy classes of maps $\overline{\varphi}_i\colon D^{n+1} \to Y$ which are the characteristic maps of the cells e_i^{n+1}. The boundary homorphism is given by $\partial[\overline{\varphi}_i] = [\varphi_i]$. Therefore $\operatorname{Im} \partial = \ker j_*$ is the subgroup of $\pi_n(X)$ generated by the elements $[\varphi_i]$. $\square$

Proposition 12.7 can be generalized to non-simply connected spaces as follows. Recall (4.14) that higher homotopy groups admit the action of the fundamental group:

$$\begin{array}{rcl} \pi_1(X, x_0) \times \pi_n(X, x_0) & \to & \pi_n(X, x_0) \\ ([\tau]\ ,\ [\omega]) & \mapsto & [\tau] \odot [\omega] \end{array}$$

We have:

12.8 Proposition. *Let (X, x_0) be a space which is connected, locally path connected, and semi-locally simply connected. Let $\varphi_i\colon (S^n, s_0) \to (X, x_0)$ be maps representing elements of $\pi_n(X, x_0)$ for some $n \geq 2$. Consider the space $Y = X \cup \bigcup_i e_i^{n+1}$ obtained by attaching $(n+1)$-cells to X using φ_i as the attaching maps. If $j\colon X \hookrightarrow Y$ is the inclusion map, then the induced homomorphism*

$$j_*\colon \pi_k(X, x_0) \to \pi_k(Y, x_0)$$

is an isomorphism for $k \leq n$ and an epimorphism for $k = n$. Moreover, $\ker(j_*\colon \pi_n(X, x_0) \to \pi_n(Y, x_0))$ is the subgroup of $\pi_n(X, x_0)$ generated by the elements $[\omega] \odot [\varphi_i]$ for all $[\omega] \in \pi_1(X, x_0)$.

Proof. The only non-trivial part is the statement about $\ker j_*$. The conditions on the space X guarantee that it has a universal covering $p_X\colon \widetilde{X} \to X$. Let $p_X^{-1}(x_0) = \{\widetilde{x}_k\}_{k\in K}$ and let $\widetilde{\varphi}_{i,k}\colon S^n \to \widetilde{X}$ denote the lift of φ_i such that $\widetilde{\varphi}_{i,k}(s_0) = \widetilde{x}_k$. Let $\widetilde{Y} = \widetilde{X} \cup \bigcup_{i,j} e_{i,k}^{n+1}$ be the space obtained by attaching $(n+1)$-cells to $\widetilde{X}$ using $\varphi_{i,k}$ as attaching maps. The natural map $p_Y\colon \widetilde{Y} \to Y$ is a universal covering of Y. We get a commutative diagram:

$$\begin{tikzcd} \pi_n(\widetilde{X}, \widetilde{x}_0) \arrow[r, "\tilde{j}_*"] \arrow[d, "p_{X*}"', "\cong"] & \pi_n(\widetilde{Y}, \widetilde{x}_0) \arrow[d, "\cong"', "p_{Y*}"] \\ \pi_n(X, x_0) \arrow[r, "j_*"'] & \pi_n(Y, x_0) \end{tikzcd}$$

where $\tilde{j}\colon \widetilde{X} \to \widetilde{Y}$ is the inclusion and $\widetilde{x}_0 \in p_X^{-1}(x_0)$. Since p_{X*} and p_{Y*} are isomorphisms (5.9), we obtain that $\ker j_* = p_{X*}(\ker \tilde{j}_*)$.

For each $\widetilde{x}_k \in p^{-1}(x_0)$ let $\widetilde{\omega}_k$ be a path in $\widetilde{X}$ such that $\widetilde{\omega}_k(0) = \widetilde{x}_0$ and $\widetilde{\omega}_k(1) = \widetilde{x}_k$. Then for each $[\omega] \in \pi_1(X, x_0)$ we have $[\omega] = [p_X\widetilde{\omega}_k]$ for some k. Let $s_k\colon \pi_n(\widetilde{X}, \widetilde{x}_k) \to \pi_n(\widetilde{X}, \widetilde{x}_0)$ be the change of the basepoint isomorphism defined by $\widetilde{\omega}_k$ (4.4). Since $\widetilde{X}$ is simply connected, using Proposition 12.7 we obtain that $\ker \tilde{j}_*$ is generated by the elements $s_k[\widetilde{\varphi}_{i,k}]$ for all i, k. Thus $\ker j_*$ is generated by elements $p_{X*}s_k[\widetilde{\varphi}_{i,k}]$. It remains to notice that $p_{X*}s_k[\widetilde{\varphi}_{i,k}] = [p_X\omega_k] \odot [p_X\widetilde{\varphi}_{i,k}] = [p_X\omega_k] \odot [\varphi_i]$ (exercise).

□

Proof of Theorem 12.1. Given an abelian group G and $n \geq 2$, we can find a set I and an epimorphism

$$\Phi\colon \pi_n(\bigvee_{i\in I} S^n) \cong \bigoplus_{i\in I} \mathbb{Z} \to G$$

Let $\ker \Phi = \{[\varphi_k \colon S^n \to \bigvee_{i\in I} S^n]\}_{k\in K}$, and let X be the space obtained by attaching $(n+1)$-cells to $\bigvee_{i\in} S^n$ using the maps φ_i. By Proposition 12.7 we obtain $\pi_n(X) \cong \pi_n(\bigvee_{i\in I} S^n)/\ker \Phi \cong G$. □

12.9 Definition. Given a group G and an integer $n \geq 1$, an *Eilenberg-MacLane space* of the type $K(G, n)$ is a path connnected CW complex X such that

$$\pi_i(X) \cong \begin{cases} G & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

12.10 Note. Eilenberg-MacLane spaces are not uniquely defined, but as we will see later (14.10), they are unique up to homotopy equivalence. By abuse of notation we will write $X = K(G, n)$ to indicate that X has the type of $K(G, n)$.

12.11 Example. $S^1 = K(\mathbb{Z}, 1)$.

12.12 Example. Recall that the n-dimensional real projective space $\mathbb{RP}^n$ is the quotient space of S^n obtained by identifying antipodal points: $\mathbb{RP}^n = S^n/\sim$ where $x \sim -x$ for all $x \in S^n$. The quotient map $q \colon S^n \to \mathbb{RP}^n$ is the 2-fold universal cover of $\mathbb{RP}^n$. It follows that

$$\pi_i(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z}/2 & \text{if } i = 1 \\ \pi_i(S^n) & i \geq 2 \end{cases}$$

Embeddings of spheres $S^1 \hookrightarrow S^2 \hookrightarrow \ldots$ induce embeddings of projective spaces $\mathbb{RP}^1 \hookrightarrow \mathbb{RP}^2 \hookrightarrow \ldots$. Take $S^\infty = \bigcup_{n=1}^\infty S^n$ and $\mathbb{RP}^\infty = \bigcup_{n=1}^\infty \mathbb{RP}^n$. The quotient map $q \colon S^\infty \to \mathbb{RP}^\infty$ is a 2-fold universal covering of $\mathbb{RP}^\infty$. Since S^∞ is a contractible space (2.18), we obtain

$$\pi_i(\mathbb{RP}^\infty) \cong \begin{cases} \mathbb{Z}/2 & \text{if } i = 1 \\ 0 & i\text{if } \geq 2 \end{cases}$$

Therefore $\mathbb{RP}^\infty = K(\mathbb{Z}/2, 1)$.

12.13 Example. Recall (7.21) that for a complex projective space the quotient map $p \colon S^{2n+1} \to \mathbb{CP}^n$ is a Serre fibration with the fiber S^1. The long exact sequence of this fibration gives

$$\pi_i(\mathbb{CP}^n) \cong \begin{cases} 0 & \text{if } i = 1 \\ \mathbb{Z} & \text{if } i = 2 \\ \pi_i(S^{2n+1}) & \text{if } i \geq 3 \end{cases}$$

The embedding maps $S^3 \hookrightarrow S^5 \hookrightarrow \ldots$ induce embeddings $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2 \hookrightarrow \ldots$. We again have $S^\infty = \bigcup_{n=1}^\infty S^{2n+1}$. Also, define $\mathbb{CP}^\infty = \bigcup_{n=1}^\infty \mathbb{CP}^n$. The map $p \colon S^\infty \to \mathbb{CP}^\infty$ is again a Serre fibration with fiber S^1. Since S^∞ is contractible, the long exact sequence of this fibration gives

$$\pi_i(\mathbb{CP}^\infty) \cong \begin{cases} 0 & \text{if } i = 1 \\ \mathbb{Z} & \text{if } i = 2 \\ 0 & \text{if } i \geq 3 \end{cases}$$

Thus $\mathbb{CP}^\infty = K(\mathbb{Z}, 2)$.

12.14 Proposition. *For any $n \geq 1$ and any group G (abelian if $n \geq 2$) there exists an Eilenberg-MacLane space $K(G, n)$. Moreover, it is possible to construct such space so that $K(G, n)^{(n-1)} = *$.*

Proof. By Theorem 12.1, if $n \geq 2$ then we can find a path connected CW complex (X_n, x_0) such that $X_n^{(n-1)} = *$ and

$$\pi_i(X_n, x_0) \cong \begin{cases} G & \text{if } i = n \\ 0 & \text{if } i < n \end{cases}$$

For $n = 1$ such CW complex can be constructed using van Kampen's theorem. Let X_{n+1} be the space obtained by attaching an $(n+2)$-cells to X_n using all possible maps $(S^{n+1}, s_0) \to (X_n, x_0)$. Then $X_n \subseteq X_{n+1}$, and using Proposition 5.2 we obtain

$$\pi_i(X_{n+1}, x_0) \cong \begin{cases} 0 & \text{if } i = n+1 \\ G & \text{if } i = n \\ 0 & \text{if } i < n \end{cases}$$

In the same way, for any $m > n + 1$ we can can inductively construct a space X_m such that X_m is obtained by attaching $(m + 1)$-cells to X_{m-1} and

$$\pi_i(X_m, x_0) \cong \begin{cases} 0 & \text{if } n < i \leq m \\ G & \text{if } i = n \\ 0 & \text{if } i < n \end{cases}$$

Then we can take $K(G, n) = \bigcup_{m=n}^{\infty} X_m$. $\square$

12.15 Corollary. *For any sequence of groups $G_1, G_2, \ldots$ such that G_i is abelian for $i \geq 2$, there exists a path connected CW complex X such that $\pi_i(X) \cong G_i$ for all $i \geq 1$.*

Proof. Take $X = \prod_{i=1}^{\infty} K(G_i, i)$. $\square$