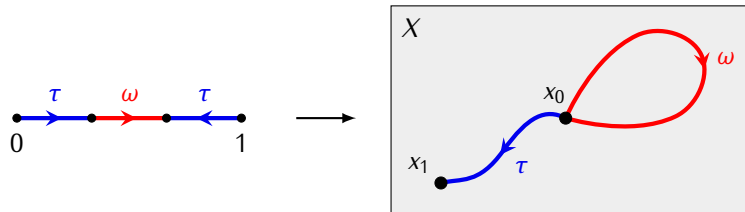


4 | Dependence on The Basepoint

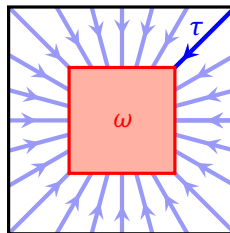
Let X be a space, and let $x_0, x_1 \in X$. Recall that any path $\tau: [0, 1] \rightarrow X$ such that $\tau(0) = x_0$ and $\tau(1) = x_1$ defines an isomorphism of fundamental groups

$$s_\tau: \pi_1(X, x_1) \rightarrow \pi_1(X, x_0)$$

given by $s_\tau([\omega]) = [\tau * \omega * \bar{\tau}]$, where $\bar{\tau}$ is obtained from τ by reverting orientation.



In a similar way, given a path $\tau: [0, 1] \rightarrow X$ with $\tau(0) = x_0$ and $\tau(1) = x_1$ we can define a map $s_\tau: \pi_n(X, x_1) \rightarrow \pi_n(X, x_0)$. To do this, given a map $\omega: (I^n, \partial I^n) \rightarrow (X, x_1)$, define a map $\omega_\tau: (I^n, \partial I^n) \rightarrow (X, x_0)$ as follows:



The smaller cube is mapped by ω and each radial ray joining the boundaries of the larger and smaller cube is mapped by the path τ .

Let $\pi_1(X, x_0, x_1)$ denote the set of homotopy classes of paths $\tau: [0, 1] \rightarrow X$ such that $\tau(0) = x_0$ and $\tau(1) = x_1$, with homotopies preserving the endpoints.

4.1 Lemma. Let $\omega, \omega': (I^n, \partial I^n) \rightarrow (X, x_1)$ be maps such that $\omega \simeq \omega' \text{ (rel } \partial I^n)$, and let $\tau, \tau': [0, 1] \rightarrow X$ be paths such that $\tau(0) = \tau'(0) = x_0$, $\tau(1) = \tau'(1) = x_1$ and $\tau \simeq \tau' \text{ (rel } \{0, 1\})$. Then $\omega_\tau \simeq \omega'_{\tau'} \text{ (rel } \partial I^n)$.

Equivalently, if $[\omega] = [\omega'] \in \pi_n(X, x_1)$ and $[\tau] = [\tau'] \in \pi_1(X, x_0, x_1)$ then $[\omega_\tau] = [\omega'_{\tau'}] \in \pi_n(X, x_0)$

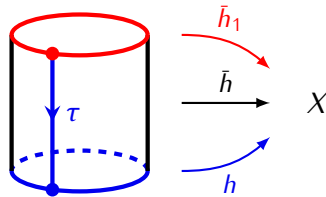
Proof. Exercise. □

4.2 Note. The homotopy class $[\omega_\tau]$ can be also described as follows. Consider the homotopy $h: \partial I^n \times [0, 1] \rightarrow X$ given by $h(x, t) = \tau(1 - t)$. Since the pair $(I^n, \partial I^n)$ has the homotopy extension property, we can extend h to a homotopy $\bar{h}: I^n \times [0, 1] \rightarrow X$ such that $\bar{h}_0 = \omega$. The map \bar{h}_1 defines an element $[\bar{h}_1] \in \pi_n(X, x_0)$. This element does not depend on the choice of the extension \bar{h} (exercise), and we have $[\bar{h}_1] = [\omega_\tau]$.

4.3 Note. Recall that elements of $\pi_n(X, x_1)$ can be alternatively defined as pointed homotopy classes of maps $\omega: (S^n, s_0) \rightarrow (X, x_1)$. In this setting, for $[\tau] \in \pi_1(X, x_0, x_1)$ the element $[\omega_\tau] \in \pi_n(X, x_0)$ can be described using a similar approach as in (4.2). Given such ω and τ we can define a function

$$h: (S^n \times \{0\}) \cup (\{s_0\} \times [0, 1]) \rightarrow X$$

so that $h(s, 0) = \omega(s)$ and $h(s_0, t) = \tau(1 - t)$. Since the pair (S^n, s_0) has the homotopy extension property, thus h can be extended to a homotopy $\bar{h}: S^n \times [0, 1] \rightarrow X$. One can check that the pointed homotopy class of the map $\bar{h}_1: (S^n, s_0) \rightarrow (X, x_0)$ does not depend on the choice of the extension \bar{h} . We set: $[\omega_\tau] = [\bar{h}_1] \in \pi_n(X, x_0)$.



4.4 Definition. Given $[\tau] \in \pi_1(X, x_0, x_1)$ let

$$s_{[\tau]}: \pi_n(X, x_1) \rightarrow \pi_n(X, x_0)$$

denote the function given by $s_{[\tau]}([\omega]) = [\omega_\tau]$.

4.5 Proposition. 1) For any $[\tau] \in \pi_1(X, x_0, x_1)$ the function $s_{[\tau]}$ is a group homomorphism.

2) If $[\tau] \in \pi_1(X, x_0, x_1)$ and $[\sigma] \in \pi_1(X, x_1, x_2)$ then

$$S_{[\tau*\sigma]} = S_{[\tau]} \circ S_{[\sigma]}: \pi_n(X, x_2) \rightarrow \pi_n(X, x_0)$$

3) If $c_{x_0}: [0, 1] \rightarrow X$ is the constant path, $c_{x_0}(t) = x_0$ for all $t \in [0, 1]$, then $S_{[c_{x_0}]}: \pi_n(X, x_0) \rightarrow \pi_n(X, x_0)$ is the identity homomorphism.

Proof. Exercise. □

4.6 Corollary. Let X be a space and let $x_0, x_1 \in X$. For any path $\tau: [0, 1] \rightarrow X$ be a path such that $\tau(0) = x_0$, $\tau(1) = x_1$ the homomorphism $S_{[\tau]}: \pi_n(X, x_1) \rightarrow \pi_n(X, x_0)$ is an isomorphism.

Proof. Let $\bar{\tau}$ be the inverse of τ . This defines homomorphisms

$$S_{[\tau]}: \pi_n(X, x_1) \xrightarrow{\sim} \pi_n(X, x_0): S_{[\bar{\tau}]}$$

We will show that $S_{[\bar{\tau}]} = S_{[\tau]}^{-1}$. Indeed, by Proposition 4.5 we have

$$S_{[\bar{\tau}]} \circ S_{[\tau]} = S_{[\bar{\tau}*\tau]} = S_{[c_{x_0}]} = \text{id}_{\pi_n(X, x_1)}$$

Analogously, $S_{[\tau]} \circ S_{[\bar{\tau}]} = \text{id}_{\pi_n(X, x_0)}$. □

Corollary 4.6 implies that if x_0, x_1 are in the same path connected component of X then $\pi_n(X, x_0) \cong \pi_n(X, x_1)$. On the other hand, if points $x_0, x_1 \in X$ belong to different path connected components of X , then in general there is no relationship between $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$.

4.7 Proposition. Let X be a space, $x_0 \in X$, and let X_0 be the path connected component of X such that $x_0 \in X_0$. Then the inclusion map $i: X_0 \hookrightarrow X$ induces an isomorphism

$$i_*: \pi_n(X_0, x_0) \xrightarrow{\cong} \pi_n(X, x_0)$$

Proof. Since I^n is path connected, for any map $\omega: (I^n, \partial I^n) \rightarrow (X, x_0)$ we have $\omega(I^n) \subseteq X_0$. This shows that i_* is onto. Also, if $h: I^n \times [0, 1] \rightarrow X$ is a homotopy $h: \omega \simeq \omega'$ where $\omega, \omega': I^n \rightarrow X_0$ then, since $I^n \times [0, 1]$ is path connected, we have $h(I^n \times [0, 1]) \subseteq X_0$. It implies that i_* is 1-1. □

4.8 Note. Given a path connected space X we will sometimes write $\pi_n(X)$ to denote the n -th homotopy group of X taken with respect to some unspecified basepoint of X . By Corollary 4.6 this will not create problems as long as we are interested in the isomorphism type of the fundamental group only.

Similarly as for the fundamental group we have:

4.9 Proposition. Let $f, g: X \rightarrow Y$ be homotopic maps and let $h: f \simeq g$. For $x_0 \in X$ let τ be the path in Y given by $\tau(t) = h(x_0, t)$. The following diagram commutes:

$$\begin{array}{ccc} & & \pi_n(Y, g(x_0)) \\ & \nearrow f_* & \downarrow \cong \quad s_{[\tau]} \\ \pi_n(X, x_0) & & \pi_n(Y, f(x_0)) \\ & \searrow g_* & \end{array}$$

Proof. Exercise. □

4.10 Note. Proposition 4.9 implies, in particular, that if $f, g: (X, x_0) \rightarrow (Y, y_0)$ are maps of pointed spaces and $f \simeq g$ (rel $\{x_0\}$) then $f_* = g_*$.

4.11 Corollary. If $f, g: X \rightarrow Y$ are maps such that $f \simeq g$ then the homomorphism $f_*: \pi_n(X, x_0) \rightarrow \pi_n(Y, f(x_0))$ is an isomorphism (or it is trivial or 1-1 or onto) if and only if the homomorphism $g_*: \pi_n(X, x_0) \rightarrow \pi_n(Y, g(x_0))$ has the same property.

4.12 Proposition. If $f: X \rightarrow Y$ is a homotopy equivalence then for any $x_0 \in X$ the homomorphism $f_*: \pi_n(X, x_0) \rightarrow \pi_n(Y, f(x_0))$ is an isomorphism.

Proof. Let $g: Y \rightarrow X$ be a homotopy inverse of f . Consider the sequence of homomorphisms

$$\pi_n(X, x_0) \xrightarrow{f_*} \pi_n(Y, f(x_0)) \xrightarrow{g_*} \pi_n(X, gf(x_0)) \xrightarrow{f_*} \pi_n(Y, fgf(x_0))$$

Composition of the first two homomorphisms satisfies $g_*f_* = (gf)_*$. Since $gf \simeq \text{id}_X$ and id_{X*} is an isomorphism, by Corollary 4.11 we obtain that g_*f_* is an isomorphism. This implies in particular that g_* is onto. Similarly, composing the last two homomorphisms we obtain $f_*g_* = (fg)_*$ and since $fg \simeq \text{id}_Y$ we get that f_*g_* is an isomorphism. This means that g_* is 1-1. As a consequence g_* is an isomorphism. It follows that the first homomorphism f_* is a composition of two isomorphisms: $f_* = g_*^{-1}(g_*f_*)$, and so f_* is an isomorphism. □

4.13 Corollary. If X, Y are path connected spaces and $X \simeq Y$ then $\pi_n(X, x_0) \cong \pi_n(Y, y_0)$ for any $x_0 \in X, y_0 \in Y$.

4.14 The action of π_1 . If $[\tau] \in \pi_1(X, x_0)$ then $s_{[\tau]}$ is an isomorphism

$$s_{[\tau]}: \pi_n(X, x_0) \rightarrow \pi_n(X, x_0)$$

Denote $[\tau] \odot [\omega] := s_{[\tau]}(\omega)$.

4.15 Definition. For $n \geq 0$ the *action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$* is the map

$$\begin{aligned} \pi_1(X, x_0) \times \pi_n(X, x_0) &\rightarrow \pi_n(X, x_0) \\ ([\tau], [\omega]) &\mapsto [\tau] \odot [\omega] \end{aligned}$$

4.16 Note. By Proposition 4.5, for any $[\tau], [\tau'] \in \pi_1(X, x_0)$ and $\omega, \omega' \in \pi_n(X, x_0)$ we have:

- $\tau \odot ([\omega] \cdot [\omega']) = (\tau \odot [\omega]) \cdot ([\tau \odot [\omega']])$
- $([\tau] \cdot [\tau']) \odot [\omega] = [\tau] \odot ([\tau'] \odot [\omega])$
- $[c_{x_0}] \odot [\omega] = [\omega]$ where $[c_{x_0}] \in \pi_1(X, x_0)$ is the trivial element.
- $[\tau] \odot [c_{x_0}] = [c_{x_0}]$ where $[c_{x_0}] \in \pi_n(X, x_0)$ is the trivial element.

4.17 Proposition. For any map $f: (X, x_0) \rightarrow (Y, y_0)$ the following diagram commutes:

$$\begin{array}{ccc} \pi_1(X, x_0) \times \pi_n(X, x_0) & \xrightarrow{\odot} & \pi_n(X, x_0) \\ f_* \times f_* \downarrow & & \downarrow f_* \\ \pi_1(Y, y_0) \times \pi_n(Y, y_0) & \xrightarrow{\odot} & \pi_n(Y, y_0) \end{array}$$

Proof. Exercise. □

4.18 Example. The action of $\pi_1(X, x_0)$ on $\pi_1(X, x_0)$ is given by conjugation:

$$[\tau] \odot [\omega] = [\tau] \cdot [\omega] \cdot [\tau]^{-1}$$

4.19 Definition. A path connected space X is *n-simple* if for some $x_0 \in X$ the action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$ is trivial: $[\tau] \odot [\omega] = [\omega]$ for all $[\tau] \in \pi_1(X, x_0)$ and $[\omega] \in \pi_n(X, x_0)$. A path connected space is *simple* if it is *n-simple* for all $n \geq 1$.

The following fact implies that *n-simplicity* of a space X does not depend on the choice of a basepoint $x_0 \in X$:

4.20 Proposition. Let X be a space, let $x_0, x_1 \in X$, and let $\tau: [0, 1] \rightarrow X$ be a path such that $\tau(0) = x_0$ and $\tau(1) = x_1$. Then the following diagram commutes:

$$\begin{array}{ccc} \pi_1(X, x_1) \times \pi_n(X, x_1) & \xrightarrow{\odot} & \pi_n(X, x_1) \\ s_{[\tau]} \times s_{[\tau]} \downarrow & & \downarrow s_{[\tau]} \\ \pi_1(X, x_0) \times \pi_n(X, x_0) & \xrightarrow{\odot} & \pi_n(X, x_0) \end{array}$$

Proof. Exercise. □

For spaces X, Y let $[X, Y]$ denote the set of homotopy classes of maps $X \rightarrow Y$. Notice that for any space X and any n we have a map of sets

$$\phi: \pi_n(X, x_0) \rightarrow [S^n, X]$$

which maps the pointed homotopy class of map $\omega: (S^n, s_0) \rightarrow (X, x_0)$ to the unpointed homotopy class of the same map.

4.21 Proposition. *Let X be a path connected space, and let $n \geq 1$. The following conditions are equivalent:*

- 1) X is n -simple.
- 2) For any $x_0, x_1 \in X$, $[\tau], [\sigma] \in \pi_1(X, x_0, x_1)$ and $[\omega] \in \pi_n(X, x_1)$ we have $s_{[\tau]}([\omega]) = s_{[\sigma]}([\omega])$. Thus there is a canonical isomorphism $\pi_n(X, x_1) \xrightarrow{\cong} \pi_n(X, x_0)$.
- 3) For any $x_0 \in X$ the map $\phi: \pi_n(X, x_0) \rightarrow [S^n, X]$ is a bijection. Therefore any (unpointed) map $f: S^n \rightarrow X$ defines a unique element of $\pi_n(X, x_0)$.

Proof. 1) \Rightarrow 2) Let $[\tau], [\sigma] \in \pi_1(X, x_0, x_1)$ and $[\omega] \in \pi_n(X, x_1)$. Since $[\bar{\tau} * \sigma] \in \pi_1(X, x_1)$, by 1) we obtain

$$s_{[\bar{\tau}]}s_{[\sigma]}([\omega]) = s_{[\bar{\tau} * \sigma]}([\omega]) = [\omega]$$

Also, since $s_{[\bar{\tau}]}$ is the inverse isomorphism of $s_{[\tau]}$ we get

$$s_{[\sigma]}([\omega]) = s_{[\tau]}s_{[\bar{\tau}]}s_{[\sigma]}([\omega]) = s_{[\tau]}([\omega])$$

2) \Rightarrow 1) Let $[\tau], [c_{x_0}] \in \pi_1(X, x_0)$, where $[c_{x_0}]$ is the trivial element. By 2) we have

$$[\tau] \odot [\omega] = s_{[\tau]}([\omega]) = s_{[c_{x_0}]}([\omega]) = [c_{x_0}] \odot [\omega] = [\omega]$$

for any $[\omega] \in \pi_n(X, x_0)$. Therefore X is n -simple.

1) \Rightarrow 3) The map ϕ is always onto. Indeed, take any map $\omega: S^n \rightarrow X$. Since X is path connected, there exists a path $\tau: [0, 1] \rightarrow X$ such that $\tau(0) = x_0$ and $\tau(1) = \omega(s_0)$. Consider the map

$$h: (S^n \times \{0\}) \cup (\{s_0\} \times [0, 1]) \rightarrow X$$

so that $h(s, 0) = \omega(s)$ and $h(s_0, t) = \tau(1 - t)$. Since the pair (S^n, s_0) has the homotopy extension property, thus h can be extended to a homotopy $\bar{h}: S^n \times [0, 1] \rightarrow X$. For the map h_1 we have $h_1(s_0) = x_0$, so $[h_1] \in \pi_n(X, x_0)$. Also, h is homotopic to $h_0 = \omega$. Therefore we have $\phi([h_1]) = [\omega]$.

To show that ϕ is 1-1, we will use the description of $s_{[\tau]}$ in terms maps from spheres given in Note 4.2. Given two elements $[\omega_0], [\omega_1] \in \pi_n(X, x_0)$ assume that $\phi([\omega_0]) = \phi([\omega_1])$. This means that there exists a

homotopy $h: S^n \times [0, 1] \rightarrow X$ such that $h_0 = \omega_0$ and $h_1 = \omega_1$. Let $\tau: [0, 1] \rightarrow X$ be a path given by $\tau(t) = h(s_0, t)$. Then $[\tau] \in \pi_1(X, x_0)$, and by (4.2) we have

$$[\omega_1] = [\bar{\tau}] \odot [\omega_0] = s_{[\bar{\tau}]}([\omega_0])$$

By 1) we have $s_{[\bar{\tau}]}([\omega_0]) = [\omega_0]$. Thus $[\omega_1] = [\omega_0] \in \pi_n(X, x_0)$.

3) \Rightarrow 1) Let $[\tau] \in \pi_1(X, x_0)$, $[\omega] \in \pi_n(X, x_0)$. Let $\omega_\tau: (S^n, s_0) \rightarrow (X, x_0)$ be some map such that $[\omega_\tau] = s_{[\tau]}([\omega])$. By (4.2) the maps ω_τ and ω are freely homotopic, i.e. $\phi([\omega_\tau]) = \phi([\omega])$. By assumption ϕ is 1-1, thus we obtain

$$[\omega] = [\omega_\tau] = s_{[\tau]}([\omega]) = [\tau] \odot [\omega]$$

in $\pi_n(X, x_0)$.

□