10 | Excision

One of the main properties of homology groups is excision. It can stated as follows:

10.1 Theorem. *Let X be a space and $X_1, X_2 \subseteq X$ be sets such that $X = \operatorname{Int} X_1 \cup \operatorname{Int} X_2$ where $\operatorname{Int} X_i$ is the interior of X_i in X. Then the map of pairs $i\colon (X_1, X_1 \cap X_2) \to (X, X_2)$ induces an isomorphism*

$$i_*\colon H_n(X_1, X_1 \cap X_2) \xrightarrow{\cong} H_n(X, X_2)$$

for all $n \geq 0$.

The same property does not holds in general for homotopy groups. However, it does hold under some extra assumptions. In order to make this precise we will need a definition.

10.2 Definition. Let $A \subseteq X$ and let $0 \leq n \leq \infty$. The pair (X, A) is *n-connected* if the map $\pi_0(A) \to \pi_0(X)$ is onto and $\pi_k(X, A, x_0) = \{1\}$ for all $x_0 \in A$ and all $1 \leq k \leq n$.

10.3 Proposition. *Let $A \subseteq X$. The following conditions are equivalent.*

1) *(X, A) is n-connected.*
2) *The homomorphism $i_*\colon \pi_k(A, x_0) \to \pi_k(X, x_0)$ induced by the inclusion map $i\colon A \hookrightarrow X$ is an isomorphism for all $x_0 \in A$ and all $k < n$ and it is an epimorphism for $k = n$.*
3) *For $k \leq n$, any map $(I^k, \partial I^k) \to (X, A)$ is homotopic relative to ∂I^k to a map $I^k \to A$.*

Proof. Exercise. □

10.4 Excision Theorem. *Let X be a space and $X_1, X_2 \subseteq X$ are open sets such that $X_1, X_2 \subseteq X$ be sets such that $X = \operatorname{Int} X_1 \cup \operatorname{Int} X_2$ where $\operatorname{Int} X_i$ is the interior of X_i in X. Assume that*

- *$(X_1, X_1 \cap X_2)$ is m-connected*
- *$(X_2, X_1 \cap X_2)$ is n-connected*

for some $m, n \geq 0$. Then for any $x_0 \in X_1 \cap X_2$ the homomorphism

$$i_*\colon \pi_k(X_1, X_1 \cap X_2, x_0) \to \pi_k(X, X_2, x_0)$$

induced by the inclusion map, is an isomorphism for $1 \leq k < m + n$ and it is onto for $k = m + n$.

In this chapter we will explore some consequences Theorem 10.4, an we will return to its proof in Chapter 12.

10.5 Proposition. *Let (X, A) be a pair with the homotopy extension property and let $q\colon X \to X/A$ be the quotient map. Let $x_0 \in A$ and $* = q(A) \in X/A$. If (X, A) is m-connected and the space A is n-connected for some $m, n \geq 0$ then the homomorphism*

$$q_*\colon \pi_k(X, A, x_0) \to \pi_k(X/A, *, *) = \pi_k(X/A, *)$$

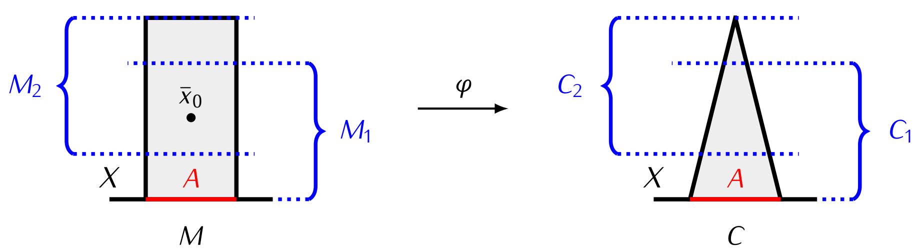
is an isomorphism for $k \leq m + n$ and it is an epimorphism for $k = m + n + 1$.

Proof. Let $j\colon A \hookrightarrow X$ be the inclusion map. Let M denote the the mapping cylinder of j:

$$M = (A \times [0, 1] \sqcup X)/\sim$$

where $(x, 0) \sim x$ for all $x \in A$. Also, let $C = M/(A \times \{1\})$ be the mapping cone of j. In other words, C is obtained by attaching the cone $CA = A \times [0, 1]/(A \times \{1\})$ to X.

Take the quotient map $\varphi\colon M \to C$. Denote by $M_1, M_2 \subseteq M$ the subspaces of M given by $M_1 = X \cup A \times [0, \frac{3}{4}]$ and $M_2 = A \times [\frac{1}{4}, 1]$, and let $C_i = \varphi(M_i)$ for $i = 1, 2$. Also, let $\bar{x}_0 = (x_0, \frac{1}{2}) \in M_1 \cap M_2$.



Let $r\colon M \to X$ be the retraction map, and let $s\colon C \to X/A$ be the map that sends the cone $CA \subseteq C$ to the point $* \in X/A$. Both r and s are homotopy equivalences. For s this follows from Proposition 2.15 using the fact that since (X, A) has the homotopy extension property, then (C, CA) also has this property.

For any $k \geq 1$ the following diagram commutes:

$$\begin{tikzcd}
\pi_k(X, A, x_0) \arrow[r, "q_*"] & \pi_k(X/A, *, *) \\
\pi_k(M, M_2, \bar{x}_0) \arrow[r, "\varphi_*"] \arrow[u, "r_*", "\cong"'] & \pi_k(C, C_2, \varphi(\bar{x}_0)) \arrow[u, "\cong", "s_*"'] \\
\pi_k(M_1, M_1 \cap M_2, \bar{x}_0) \arrow[r, "k_*", "\cong"'] \arrow[u, "i_*", "\cong"'] & \pi_k(C_1, C_1 \cap C_2, \varphi(\bar{x}_0)) \arrow[u, "i'_*"]
\end{tikzcd}$$

The homomorphisms i_*, i'_* and k_* are induced by inclusions. Since $i\colon (M_1, M_1 \cap M_2) \to (M_j, M_1 \cap M_2)$ is a homotopy equivalence and $k\colon (M_1, M_1 \cap M_2) \to (C_1, C_1 \cap C_2)$ is a homeomorphism, i_* and k_* are isomorphisms. It follows that q_* is an isomorphism or epimorphism if and only if i'_* has the same property.

From the above diagram we also obtain that $\pi_k(C_1, C_1 \cap C_2, \varphi(\bar{x}_0)) \cong \pi_k(X, A, x_0)$ for all k, so $(C_1, C_1 \cap C_2)$ is m-connected. Also, since C_2 is a conctractible space, from the long exact sequence of the pair $(C_2, C_1 \cap C_2)$ we get

$$\pi_k(C_2, C_1 \cap C_2, \varphi(\bar{x}_0)) \cong \pi_{k-1}(C_1 \cap C_2, \varphi(\bar{x}_0)) \cong \pi_{k-1}(A, x_0)$$

Since by assumption A is n-connected, thus $(C_2, C_1 \cap C_2)$ is $(n+1)$-connected. By the Excision Theorem 10.4 we obtain that i'_* (and thus also q_*) is an isomorphism for $k \leq m + n$ and an epimorphism for $k = m + n + 1$. □

10.6 Definition. Let X be a space. The *unreduced suspension* of X if the space

$$SX = X \times [0, 1]/(X \times \{0, 1\})$$

10.7 Note. Any map $f\colon X \to Y$ defines a map $Sf\colon SX \to SY$ given by $Sf([x, t]) = [f(x), t]$. This map is called the suspension of f. In this way we obtain the suspension functor

$$S\colon \mathbf{Top} \to \mathbf{Top}$$

This functor preserves homotopy classes of maps: if $f, g\colon X \to Y$ and $f \simeq g$ then $Sf \simeq Sg$.

10.8 Example. For a sphere S^n we have $SS^n \cong S^{n+1}$.

10.9 Definition. Let (X, x_0) be a pointed space. The *reduced suspension* of X is the pointed space

$$\Sigma X = X \times [0, 1]/(X \times \{0, 1\} \cup \{x_0\} \times [0, 1])$$

or equivalently $\Sigma X = SX/\{[x_0, t] \mid t \in [0, 1]\}$. The basepoint in ΣX is given by $[x_0, 0] \in \Sigma X$. We will denote it by $\bar{x}_0$.

10.10 Definition. A pointed space (X, x_0) is well pointed if the pair (X, x_0) has the homotopy extension property.

10.11 Note. If (X, x_0) is a well pointed space, then Proposition 2.15 implies that the quotient map $SX \to \Sigma X$ is a homotopy equivalence. In particular, for any basepoint $x_0 \in S^n$ we have $\Sigma S^n \simeq SS^n \cong S^{n+1}$. One can show that we actually have a homemorphism $\Sigma S^n \cong S^{n+1}$

10.12 Note. Any map $f\colon (X, x_0) \to (Y, y_0)$ of pointed spaces, defines a map $\Sigma f\colon \Sigma X \to \Sigma Y$ given by $\Sigma f([x, t]) = [f(x), t]$. This defines the suspension functor

$$\Sigma\colon \mathbf{Top}_* \to \mathbf{Top}_*$$

Similarly as for the unreduced suspension, the reduced suspension preserves homotopy classes: if $f, g\colon (X, x_0) \to (Y, y_0)$ are maps of pointed spaces and $f \simeq g$ then $\Sigma f \simeq \Sigma g$.

Let (X, x_0) be a pointed space and let $\omega\colon (I^n, \partial I^n) \to (X, x_0)$ represent an element $[\omega] \in \pi_n(X, x_0)$. Consider the map $\Sigma'\omega\colon I^{n+1} \to \Sigma X$ obtained the composition

$$\Sigma'\omega\colon I^{n+1} = I^n \times [0, 1] \xrightarrow{q} \Sigma I^n \xrightarrow{\Sigma\omega} \Sigma X$$

where q is the quotient map. One can check that $\Sigma'\omega$ represents an element of $\pi_{n+1}(\Sigma X, \bar{x}_0)$.

10.13 Definition/Proposition. The assignment $[\omega] \mapsto [\Sigma'\omega]$ defines a homomorphism of groups

$$\Sigma_*\colon \pi_n(X, x_0) \to \pi_{n+1}(\Sigma X, \bar{x}_0)$$

which is called the *suspension homomorphism*.

Proof. The function Σ_* is well defined since the suspension functor preserves homotopy classes of maps. It remains to check that Σ_* is a group homomorphism (exercise). □

10.14 Freudenthal Suspension Theorem. *Let (X, x_0) be a well pointed, n-connected space. Then the suspension homomorphism*

$$\Sigma_*\colon \pi_k(X, x_0) \to \pi_{k+1}(\Sigma X, \bar{x}_0)$$

is an isomorphism for $k \leq 2n$ and it is an epimorphism for $k = 2n + 1$.

Proof. First, let $CX = X \times [0, 1]/X \times \{1\}$ be the cone on X. Identifying X with $X \times \{0\}$ we can consider it as a subspace of CX. Since CX is a contractible space, in the long exact sequence of the pair (CX, X) the homomorphism $\partial\colon \pi_{k+1}(CX, X, x_0) \to \pi_k(X, x_0)$ is an isomorphism for all $k \geq 0$. Let ∂^{-1} be the inverse isomorphism.

One can check (exercise) that if (X, x_0) is a well pointed space, then for any $k \geq 0$ the following diagram commutes:

$$\begin{tikzcd}
\pi_k(X, x_0) \arrow[r, "\Sigma_*"] \arrow[d, "\partial^{-1}"', "\cong"] & \pi_{k+1}(\Sigma X, \bar{x}_0) \\
\pi_{k+1}(CX, X, x_0) \arrow[r, "q_*"'] & \pi_{k+1}(CX/X, \bar{x}_0) \arrow[u, "\cong", "q'_*"']
\end{tikzcd}$$

Here q_* and q'_* are induced by the quotient maps $q\colon CX \to CX/X$ and $q'\colon CX/X = SX \to \Sigma X$.

Since (X, x_0) is well pointed, the map q' is a homotopy equivalence, and thus q'_* is an isomorphism. It follows that Σ_* is an isomorphism or epimorphism if and only if this holds for q_*. Since X is n-connected and CX is contractible, the pair (CX, X) is $n+1$-connected. Therefore, by Proposition 10.5, q_* is an isomorphism for $k+1 \leq 2n+1$ (or $k \leq 2n$) and an epimorphism for $k+1 = 2n+2$ (i.e. $k = 2n+1$)

□

Since the sphere S^n is $(n-1)$-connected, by Theorem 10.14 we obtain:

10.15 Corollary. *The suspension homomorphism*

$$\Sigma_*\colon \pi_k(S^n) \to \pi_{k+1}(\Sigma S^n) \cong \pi_{k+1}(S^{n+1})$$

is an isomorphism for $k \leq 2n-2$ and an epimorphism for $k = 2n-1$.

10.16 Corollary. *For any $n \geq 1$ we have $\pi_n(S^n) \cong \mathbb{Z}$.*

Proof. We argue by induction with respect to n. We already know that $\pi_1(S^1) \cong \mathbb{Z}$. Also, by Theorem 7.26 we have $\pi_2(S^2) \cong \mathbb{Z}$.

Next, assume that $\pi_n(S^n) \cong \mathbb{Z}$ for some $n \geq 2$. In such case $2n-2 \geq n$, so by Corollary 10.15 we obtain $\mathbb{Z} \cong \pi_n(S^n) \cong \pi_{n+1}(S^{n+1})$. □

10.17 Note. 1) By Corollary 10.15 the suspension homomorphism $\Sigma_*\colon \pi_n(S^n) \to \pi_{n+1}(S^{n+1})$ is an isomorphism for all $n \geq 2$. By the same corollary $\Sigma_*\colon \pi_1(S^1) \to \pi_2(S^2)$ is onto, and since every epimorphism $\mathbb{Z} \to \mathbb{Z}$ is an isomorphism, it follows that this is an isomorphism as well.

2) The generator of the group $\pi_n(S^n)$ is represented by the identity map $\mathrm{id}\colon S^n \to S^n$. For $n = 1$ it follows from the direct computation of $\pi_1(S^1)$, and for $n > 1$ it holds since the suspension isomorphism maps the homotopy class of $\mathrm{id}_{S^{n-1}}$ to the homotopy class of id_{S^n}

10.18 Corollary. *$\pi_3(S^2) \cong \mathbb{Z}$ and the generator of $\pi_3(S^2)$ is given by the homotopy class of the Hopf bundle map (7.25).*

Proof. The long exact sequence of the Hopf fibration $S^1 \to S^3 \xrightarrow{p} S^2$ gives an exact sequence:

$$0 = \pi_3(S^1) \longrightarrow \pi_3(S^3) \xrightarrow{p_*} \pi_3(S^2) \xrightarrow{\partial} \pi_2(S^1) = 0$$

Therefore p_* is an isomorphism and so $\pi_3(S^2) \cong \pi_3(S^3) \cong \mathbb{Z}$. Also, since $[\mathrm{id}_{S^3}]$ is a generator of $\pi_3(S^3)$, thus $p_*([\mathrm{id}_{S^3}]) = [p]$ is a generator of $\pi_3(S^2)$. $\square$

10.19 Note. Notice that since $\pi_2(S^1) = 0$, the suspension homomorphism $\Sigma_* \colon \pi_2(S^1) \to \pi_3(S^2)$ is not an isomorphism.

10.20 Corollary. *For $n \geq 1$ the group $\pi_{n+1}(S^n)$ is cyclic.*

Proof. We have $\pi_2(S^1) = 0$ and $\pi_3(S^2) \cong \mathbb{Z}$. By Corollary 10.15 the suspension homomorphism $\mathbb{Z} \cong \pi_3(S^2) \to \pi_4(S^3)$ is onto, so $\pi_4(S^3)$ is a cyclic group. By the same corollary we have $\pi_{n+1}(S^n) \cong \pi_{n+2}(S^{n+1})$ for all $n \geq 3$. $\square$