

## 5 | Some Computations

**5.1 Proposition.** *If  $X$  is a contractible space then  $\pi_n(X) = 0$  for all  $n \geq 0$ .*

*Proof.* Since  $X \simeq *$  thus  $\pi_n(X) \cong \pi_n(*) = 0$ . □

**5.2 Proposition.** *If  $X$  is a relative CW-complex,  $X^{(n)}$  is the  $n$ -skeleton of  $X$ , and  $x_0 \in X^{(n)}$ , then the homomorphism  $i_*: \pi_k(X^{(n)}, x_0) \rightarrow \pi_k(X, x_0)$  induced by the inclusion map  $i: X^{(n)} \hookrightarrow X$  is an isomorphism for  $k < n$  and an epimorphism for  $k = n$ .*

*Proof.* We can assume that  $x_0 \in X^{(0)}$ . Consider  $S^k$  as a CW complex with a 0-cell  $s_0 \in S^k$ . By the Cellular Approximation Theorem 2.11 any map  $\omega: (S^k, s_0) \rightarrow (X, x_0)$  is homotopic (relative to the basepoint) to cellular map  $\omega': (S^k, s_0) \rightarrow (X, x_0)$ . If  $k \leq n$  then  $\omega'(S^k) \subseteq X^{(n)}$ , so  $\omega'$  represents an element of  $\pi_k(X^{(n)}, x_0)$  such that  $i_*([\omega']) = [\omega]$ . This shows that  $i_*$  is an epimorphism for  $k \leq n$ .

Next, take  $[\omega_0], [\omega_1] \in \pi_k(X^{(n)}, x_0)$ . We can assume that the maps  $\omega_0, \omega_1: (S^k, s_0) \rightarrow (X^{(n)}, x_0)$  are cellular. If  $i_*([\omega_0]) = i_*([\omega_1])$  then there is a homotopy  $h: S^k \times [0, 1] \rightarrow X$ . Using the Cellular Approximation Theorem 2.11 again, we can assume that this homotopy is a cellular map. Since  $\dim S^k \times [0, 1] = k + 1$ , we obtain that if  $k < n$  then  $h(S^k \times [0, 1]) \rightarrow X^{(n)}$ . Thus  $h$  gives a homotopy between  $\omega_0$  and  $\omega_1$  in  $X^{(n)}$ . Therefore  $[\omega_0] = [\omega_1] \in \pi_k(X^{(n)}, x_0)$ . This shows that  $i_*$  is a monomorphism for  $k < n$ . □

**5.3 Corollary.** *If  $k < n$  then  $\pi_k(S^n) = 0$*

*Proof.* A sphere  $S^n$  can be given a CW-complex structure with one 0-cell and one  $n$ -cell. By Proposition 5.2 for  $k < n$  we get

$$\pi_k(S^n) \cong \pi_k((S^n)^{(k)}) \cong \pi_k(*) = 0$$

□

**5.4 Definition.** A space  $X$  is  $n$ -connected if  $\pi_k(X) = 0$  for all  $k \leq n$ .

Corollary 5.3 can be restated by saying that the sphere  $S^n$  is  $(n - 1)$ -connected.

**5.5 Proposition.** *For any space  $X$  and  $n \geq 0$  the following conditions are equivalent:*

- 1)  $X$  is  $n$ -connected.
- 2) For any  $k \leq n$  and any map  $\varphi: S^k \rightarrow X$  there exists a map  $\bar{\varphi}: D^{k+1} \rightarrow X$  such that  $\bar{\varphi}|_{S^k} = \varphi$ .

*Proof.* Follows from Proposition 3.7. □

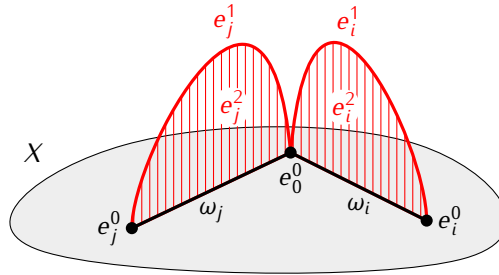
By Proposition 5.2 if  $X$  is a CW complex that has only one 0-cell and no  $k$ -cells for  $k \leq n$  (i.e.  $X^{(n)} = *$ ) then  $X$  is  $n$ -connected. One can show that the opposite is also true, up to a homotopy equivalence:

**5.6 Proposition.** *If  $X$  is an  $n$ -connected CW complex, then there exists a CW complex  $Y$  such that  $X \simeq Y$  and  $Y^{(n)} = *$ .*

*Proof.* We will show inductively that for any  $k = 0, \dots, n$  there exists a CW complex  $Y_k$  such that  $X \simeq Y_k$  and  $Y_k^{(k)} = *$ .

Choose a 0-cell  $e_0^0 \in X$ . Since  $\pi_0(X) = 0$ , the space  $X$  is path connected. Thus for any 0-cell  $e_i^0$  we can select a path  $\omega_i: [0, 1] \rightarrow X$  such that  $\omega(0) = e_0^0$  and  $\omega(1) = e_i^0$ . By the Cellular Approximation Theorem 2.11, we can assume that  $\omega_i$  is a path in  $X^{(1)}$ . We construct a new CW complex  $Y_0''$  by attaching cells to  $X$  as follows.

- 1) First, for each 0-cell  $e_i^0$  we attach to  $X$  a 1-cell  $e_i^1$  using the attaching map  $\varphi_i: S^0 = \{-1, 1\} \rightarrow X$  such that  $\varphi_i(-1) = e_0^0$  and  $\varphi_i(1) = e_i^0$ . Let  $Y_0' = X \cup \bigcup_i e_i^1$  be the CW complex obtained in this way.
- 2) In  $Y_0'$  each 0-cell  $e_i^0$  is connected to  $e_0^0$  by two different paths:  $\omega_i$ , and a path  $\tau_i$  that traverses the new cell  $e_i^1$ . For each  $i$  we attach a 2-cell  $e_i^2$  using an attaching map  $\psi_i: S^1 \rightarrow Y_0'$  that send the lower half circle to  $\omega_i$  and the upper half circle to  $\tau_i$ . Let  $Y_0'' = Y_0' \cup \bigcup_i e_i^2$ .



Notice that  $X$  is a deformation retract of  $Y_0''$ , so the inclusion map  $j: X \hookrightarrow Y_0''$  is a homotopy equivalence. Also  $A = X^{(0)} \cup \bigcup_i e_i^1$  is a contractible subcomplex of  $Y_0''$ . By Proposition 2.15, the quotient map

$q: Y_0'' \rightarrow Y_0''/A$  is a homotopy equivalence. Since  $Y_0''/A$  has a CW complex structure with only one 0-cell we can take  $Y_0 = Y_0''/A$ .

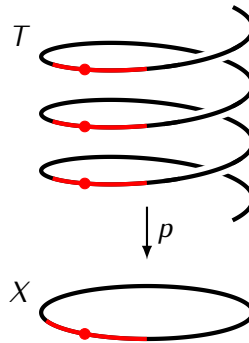
Next, assume that for some  $k \leq n$  we have already constructed a CW complex  $Y_{k-1}$  such that  $X \simeq Y_{k-1}$  and  $Y_{k-1}^{(k-1)} = *$ . This means that the  $k$ -skeleton of  $Y_{k-1}$  is given by  $Y_{k-1}^{(k)} = \bigvee_i S^k$ , with one copy of  $S^k$  for each  $k$ -cell  $e_i^k$  in  $Y_{k-1}$ . Let  $\varphi_j: S^k \hookrightarrow \bigvee_i S^k \subseteq Y_{k-1}$  be the inclusion of the  $j$ -th copy of  $S^k$ . Since  $\pi_k(Y_{k-1}) \cong \pi_k(X) = 0$ , each map  $\varphi_i$  extends to a map  $\omega_i: D^{k+1} \rightarrow Y_{k-1}$ . We construct a new CW complex  $Y_k''$  by attaching cells to  $Y_{k-1}$  as follows.

- 1) First, for each  $i$  we attach a  $(k+1)$ -cell  $e_i^{k+1}$  using  $\varphi_i: S^k \rightarrow Y_{k-1}$  as the attaching map. Let  $Y_k' = Y_{k-1} \cup \bigcup_i e_i^{k+1}$  be the CW complex obtained in this way.
- 2) For each  $i$  we have now two maps  $D^{k+1} \rightarrow Y_k'$ : the map  $\omega_i$ , and the characteristic map  $\tau_i$  of the cell  $e_i^{k+1}$ . Using these maps we attach, for each  $i$ , a  $(k+2)$ -cell  $e_i^{k+2}$ , using an attaching map  $\psi_i: S^{k+1} \rightarrow Y_k'$  that sends the lower hemisphere of  $S^{k+1}$  to  $\omega_i$  and the upper hemisphere to  $\tau_i$ . Let  $Y_k'' = Y_k' \cup \bigcup_i e_i^{k+2}$ .

As before, we observe that  $Y_{k-1}$  is a deformation retract of  $Y_k''$ , and that  $A = Y_{k-1}^{(k)} \cup \bigcup_i e_i^k$  is a contractible subcomplex of  $Y_k''$ . Therefore we obtain a  $X \simeq Y_{k-1} \simeq Y_k'' \simeq Y_k''/A$ . It remains to notice that the space  $Y_k = Y_k''/A$  has a CW-complex structure such that  $Y_k^{(k)} = *$ .

□

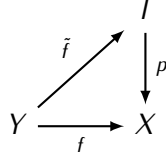
**5.7 Homotopy groups and coverings.** Recall that covering of a space  $X$  is a map  $p: T \rightarrow X$  which is locally homeomorphic to the projection map  $\text{pr}_1: U \times D \rightarrow U$  for some discrete space  $D$ .



Recall also, that one of the main properties of coverings is the following fact:

**5.8 Theorem (Lifting Criterion).** Let  $p: T \rightarrow X$  be a covering, let  $x_0 \in X$  and let  $\tilde{x}_0 \in p^{-1}(x_0)$ . Assume that  $Y$  is a connected and locally path connected space and let  $y_0 \in Y$ . A map  $f: (Y, y_0) \rightarrow (X, x_0)$

has a lift  $\tilde{f}: (Y, y_0) \rightarrow (T, \tilde{x}_0)$  if and only if  $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(T, \tilde{x}_0))$ .



Moreover, if a lift  $\tilde{f}$  exists, then it is unique.

Recall that for any covering  $p: (T, \tilde{x}_0) \rightarrow (X, x_0)$  the induced homomorphism  $p_*: \pi_1(T, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$  is a monomorphism. Using Theorem 5.8 we can generalize this as follows:

**5.9 Proposition.** *If  $p: T \rightarrow X$  is a covering,  $x_0 \in X$  and  $\tilde{x}_0 \in p^{-1}(x_0)$ , then the induced homomorphism*

$$p_*: \pi_n(T, \tilde{x}_0) \rightarrow \pi_n(X, x_0)$$

*is an isomorphism for all  $n > 1$ .*

*Proof.* Let  $n > 1$  and  $\omega: (S^n, s_0) \rightarrow (X, x_0)$  represents an element of  $\pi_n(X, x_0)$ . Since  $\pi_1(S^n) = 0$ , by Theorem 5.8 there exists a map  $\tilde{\omega}: (S^n, s_0) \rightarrow (T, \tilde{x}_0)$  such that  $p\tilde{\omega} = \omega$ . This shows that  $p_*: \pi_n(T, \tilde{x}_0) \rightarrow \pi_n(X, x_0)$  is onto.

Next, assume that  $\omega_0, \omega_1: (S^n, s_0) \rightarrow (T, \tilde{x}_0)$  are maps such that  $p_*([\omega_0]) = p_*([\omega_1])$ . This means that there exists a basepoint preserving homotopy  $h: S^n \times [0, 1] \rightarrow X$ , such that  $h_0 = p\omega_0$ ,  $h_1 = p\omega_1$ . Since  $S^n \times [0, 1] \simeq S^n$  we have  $\pi_1(S^n \times [0, 1]) \cong \pi_1(S^n) = 0$ . Thus by Theorem 5.8, there exists a homotopy  $\tilde{h}: S^n \times [0, 1] \rightarrow T$  such that  $p\tilde{h} = h$  and  $\tilde{h}(s_0, 0) = \tilde{x}_0$ . Using the uniqueness of lifts, one can check that  $\tilde{h}_0 = \omega_0$  and  $\tilde{h}_1 = \omega_1$ , and that the homotopy  $\tilde{h}$  preserves the basepoint (exercise). It follows that  $[\omega_0] = [\omega_1]$  in  $\pi_1(T, \tilde{x}_0)$ . Therefore  $p_*$  is a monomorphism.

□

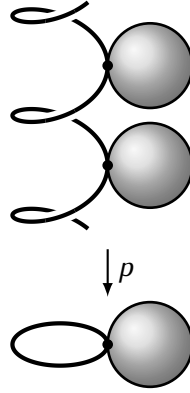
**5.10 Example.**  $\pi_n(S^1) = 0$  for all  $n > 1$ .

Indeed, universal covering of  $S^1$  is given by a map  $p: \mathbb{R} \rightarrow S^1$ . Since  $\mathbb{R}$  is a contractible space, by Proposition 5.9 for  $n > 1$  we obtain

$$\pi_n(S^1) \cong \pi_n(\mathbb{R}) = 0$$

**5.11 Example.** If  $m > 1$  then  $\pi_n(S^1 \vee S^m) \cong \pi_n(\bigvee_{i \in \mathbb{Z}} S^m)$  for all  $n > 1$ .

To see this, notice that the universal covering of  $S^1 \vee S^m$  is the space  $\tilde{X}$  obtained by attaching copies of  $S^m$  at all integer points of the real line:



The space  $\tilde{X}$  can be given the structure of a CW complex, such that the real line  $\mathbb{R}$  is its subcomplex. Since  $\mathbb{R} \simeq \{*\}$ , by Theorem 2.14 we have  $\tilde{X} \simeq \tilde{X}/\mathbb{R} \cong \bigvee_{i \in \mathbb{Z}} S^m$ . Therefore for  $n > 1$  we obtain

$$\pi_n(S^1 \vee S^m) \cong \pi_n(\tilde{X}) \cong \pi_n(\bigvee_{i \in \mathbb{Z}} S^m)$$

**5.12 Note.** Example 5.11 can be used to show that if  $X, Y$  are spaces such that  $\pi_n(X) \cong \pi_n(Y)$  for all  $n \geq 0$ , then this does not imply that  $X \simeq Y$ .

Take, for example,  $X = S^1 \vee S^m$  for some  $m > 1$ , and let  $Y = S^1 \vee S^m \vee S^m$ . These spaces are not homotopy equivalent, since they have different homology groups:  $H_m(X; \mathbb{Z}) \cong \mathbb{Z}$  and  $H_m(Y; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ .

On the other hand, since both these spaces are path connected, we have  $\pi_0(X) \cong \pi_0(Y) \cong \{*\}$ . Also, since  $\pi_1(S^m) = 0$ , thus by van Kampen's theorem we get  $\pi_1(X) \cong \pi_1(S^1) \cong \pi_1(Y)$ .

The universal covering space  $\tilde{Y}$  of  $Y$  is the space obtained by attaching  $S^m \vee S^m$  at all integer points of  $\mathbb{R}$ . Using the same argument as in Example 5.11, we obtain  $\tilde{Y} \simeq \bigvee_{i \in \mathbb{Z}} (S^m \vee S^m) \cong \bigvee_{i \in \mathbb{Z}} S^m$ . Therefore for  $n \geq 2$  we have

$$\pi_n(X) \cong \pi_n(\bigvee_{i \in \mathbb{Z}} S^m) \cong \pi_n(Y)$$

**5.13 Theorem.** For a family  $(X_i, \bar{x}_i)_{i \in I}$  be a family of pointed spaces there is an isomorphism

$$\pi_n \left( \prod_{i \in I} X_i, (\bar{x}_i)_{i \in I} \right) \cong \prod_{i \in I} \pi_n(X_i, \bar{x}_i)$$

*Proof.* For  $j \in I$  let  $p_j: \prod_{i \in I} X_i \rightarrow X_j$  denote the projection onto the  $j$ -th factor. The induced homomorphisms  $p_{j*}$  define a homomorphism:

$$\prod_{i \in I} p_{i*}: \pi_n \left( \prod_{i \in I} X_i, (\bar{x}_i)_{i \in I} \right) \rightarrow \prod_{i \in I} \pi_n(X_i, \bar{x}_i)$$

To obtain a homomorphism going in the opposite direction, let  $([\omega_i])_{i \in I}$  be an element of  $\prod_{i \in I} \pi_n(X_i, \bar{x}_i)$ . Then each  $\omega_i$  is a map  $\omega_i: (S^n, s_0) \rightarrow (X_i, \bar{x}_i)$ . Take the product map

$$\prod_{i \in I} \omega_i: (S^n, s_0) \rightarrow \left( \prod_i X_i, \bar{x}_i \right)$$

One can check that the assignment  $([\omega_i])_{i \in I} \mapsto [\prod_{i \in I} \omega_i]$  gives a well-defined homomorphism

$$g: \prod_{i \in I} \pi_n(X_i, \bar{x}_i) \rightarrow \pi_n \left( \prod_{i \in I} X_i, (\bar{x}_i)_{i \in I} \right)$$

and that the compositions  $g \circ \prod_{i \in I} p_{i*}$  and  $\prod_{i \in I} p_{i*} \circ g$  are identity homomorphisms (exercise).  $\square$

**5.14 Example.** Since  $\pi_1(S^1) \cong \mathbb{Z}$  and  $\pi_n(S^1) = 0$  for  $n > 1$ , thus for any set  $I$  we have

$$\pi_n \left( \prod_{i \in I} S^1 \right) \cong \begin{cases} \prod_{i \in I} \mathbb{Z} & \text{for } n = 1 \\ 0 & \text{for } n > 1 \end{cases}$$