

# 7 | Fibrations

**7.1 Definition.** A map  $p: E \rightarrow B$  has the *homotopy lifting property* for a space  $X$  if for any commutative diagram of the form

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\bar{f}} & E \\ \downarrow i & \nearrow \bar{h} & \downarrow p \\ X \times [0, 1] & \xrightarrow{h} & B \end{array}$$

there exists a map  $\bar{h}: X \times [0, 1] \rightarrow E$  such that  $\bar{h}i = \bar{f}$  and  $p\bar{h} = h$ .

In the setting of Definition 7.1 we will say that  $\bar{h}$  is a lift of  $h$  beginning at  $\bar{f}$ .

**7.2 Definition.** A map  $p: E \rightarrow B$  is

- a *Hurewicz fibration* if it has the homotopy lifting property for any space  $X$ .
- a *Serre fibration* if it has the homotopy lifting property for any CW complex  $X$ .

**7.3 Note.** Every Hurewicz fibration is a Serre fibration.

**7.4 Example.** For any spaces  $B, F$  the projection map  $\text{pr}_B: B \times F \rightarrow B$  is a Hurewicz fibration. Indeed, assume that we have a commutative diagram

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\bar{f}} & B \times F \\ \downarrow i & & \downarrow \text{pr}_B \\ X \times [0, 1] & \xrightarrow{h} & B \end{array}$$

Let  $\text{pr}_F: B \times F \rightarrow F$  be the projection onto  $F$ . We can define  $\bar{h}: X \times [0, 1] \rightarrow B \times F$  by

$$\bar{h}(x, t) = (h(x, t), \text{pr}_F \bar{f}(x, 0))$$

**7.5 Example.** Every covering map  $p: E \rightarrow B$  is a Hurewicz fibration.

**7.6 Definition.** Let  $A \subseteq X$ . A map  $p: E \rightarrow B$  has the *relative homotopy lifting property* for the pair  $(X, A)$  if for any commutative diagram of the form

$$\begin{array}{ccc} X \times \{0\} \cup A \times [0, 1] & \xrightarrow{\bar{f}} & E \\ \downarrow i & \nearrow \bar{h} & \downarrow p \\ X \times [0, 1] & \xrightarrow{h} & B \end{array}$$

there exists a map  $\bar{h}: X \times [0, 1] \rightarrow E$  such that  $\bar{h}i = \bar{f}$  and  $p\bar{h} = h$ .

**7.7 Theorem.** Let  $p: E \rightarrow B$  be a map. The following conditions are equivalent:

- 1)  $p$  is a Serre fibration;
- 2)  $p$  has the homotopy lifting property for  $D^n$  for all  $n \geq 0$ ;
- 3)  $p$  has the relative homotopy lifting property for  $(D^n, S^{n-1})$  for all  $n \geq 0$ ;
- 4)  $p$  has the relative homotopy lifting property for all relative CW-complexes  $(X, A)$ .

*Proof.* 1)  $\Rightarrow$  2) Obvious.

2)  $\Rightarrow$  3) Assume that we have a diagram

$$\begin{array}{ccc} D^n \times \{0\} \cup S^{n-1} \times [0, 1] & \xrightarrow{\bar{f}} & E \\ \downarrow & \nearrow \bar{h} & \downarrow p \\ D^n \times [0, 1] & \xrightarrow{h} & B \end{array}$$

We want to show that the map  $\bar{h}$  exists.

We can construct a homeomorphism  $\varphi: D^n \times [0, 1] \rightarrow D^n \times [0, 1]$  such that  $\varphi(D^n \times \{0\}) = D^n \times \{0\} \cup S^{n-1} \times [0, 1]$ . This gives a commutative diagram

$$\begin{array}{ccccc} D^n \times \{0\} & \xrightarrow{\varphi} & D^n \times \{0\} \cup S^{n-1} \times [0, 1] & \xrightarrow{\bar{f}} & E \\ \downarrow & & \downarrow & \nearrow \bar{h}' & \downarrow p \\ D^n \times [0, 1] & \xrightarrow{\varphi} & D^n \times [0, 1] & \xrightarrow{h} & B \end{array}$$

The map  $h': D^n \times [0, 1] \rightarrow E$  exists by 2). Then we can take  $\bar{h} = h'\varphi^{-1}$ .

3)  $\Rightarrow$  4) Let  $(X, A)$  be a relative complex, and assume that we have a commutative diagram

$$\begin{array}{ccc} X \times \{0\} \cup A \times [0, 1] & \xrightarrow{\bar{f}} & E \\ \downarrow & \nearrow \bar{h} & \downarrow p \\ X \times [0, 1] & \xrightarrow{h} & B \end{array}$$

We want to show that the map  $\bar{h}$  exists.

Assume that  $X$  is obtained by attaching a single  $n$ -dimensional cell  $e^n$  to  $A$  using an attaching map  $\varphi: S^{n-1} \rightarrow A$ , i.e.  $X = A \cup_{\varphi} e^n$ . Let  $\bar{\varphi}: D^n \rightarrow X$  be the characteristic map of  $e^n$  (2.2). Then the above diagram can be extended as follow:

$$\begin{array}{ccccc} D^n \times \{0\} \cup S^{n-1} \times [0, 1] & \xrightarrow{\bar{\varphi} \times \{0\} \cup \varphi \times [0, 1]} & X \times \{0\} \cup A \times [0, 1] & \xrightarrow{\bar{f}} & E \\ \downarrow & & \downarrow & \nearrow h' & \downarrow p \\ D^n \times [0, 1] & \xrightarrow{\bar{\varphi} \times [0, 1]} & X \times [0, 1] & \xrightarrow{h} & B \end{array}$$

The map  $h'$  exists by 3). Since  $X \times [0, 1]$  is a quotient space of  $A \times [0, 1] \sqcup D^n \times [0, 1]$ , the map

$$\bar{f} \sqcup h': A \times [0, 1] \sqcup D^n \times [0, 1] \rightarrow E$$

defines the desired map  $\bar{h}: X \times [0, 1] \rightarrow E$ . The general statement can be obtained from here by induction with respect to cell attachments.

4)  $\Rightarrow$  1) Let  $X$  be a CW complex and  $A = \emptyset$ . Then the relative lifting property for  $(X, A)$  is the same as the lifting property for  $X$ .  $\square$

**7.8 Note.** Property 3) in Theorem 7.7 can be equivalently stated as follows. Given a cube  $I^n$ , let  $K$  be a subset of  $\partial I^n$  consisting of all but one face of  $I^n$ . Then for any commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{\bar{f}} & E \\ \downarrow & \nearrow \bar{h} & \downarrow p \\ I^n & \xrightarrow{h} & B \end{array}$$

there exists a map  $\bar{h}: I^n \rightarrow E$  such that this diagram commutes.

**7.9 Lemma.** Let  $p: E \rightarrow B$  be a Serre fibration. Let  $e_0 \in E$  and  $b_0 \in B$  be points such that  $p(e_0) = b_0$ , and let  $F = p^{-1}(b_0)$ . For any  $n \geq 1$  the map  $p: (E, F, e_0) \rightarrow (B, b_0, b_0)$  induces an isomorphism of homotopy groups

$$p_*: \pi_n(E, F, e_0) \longrightarrow \pi_n(B, b_0, b_0) = \pi_n(B, b_0)$$

*Proof.* To check that  $p_*: \pi_n(E, F, e_0) \rightarrow \pi_n(B, b_0)$  is onto, take a map  $\omega: (I^n, \partial I^n) \rightarrow (B, b_0)$ . By the relative homotopy lifting property for  $(I^{n-1}, \partial I^{n-1})$ , we can find a map  $\bar{\omega}: I^n \rightarrow E$  such that  $\bar{\omega}(J^{n-1}) = e_0$  and  $p\bar{\omega} = \omega$ .

$$\begin{array}{ccc} J^{n-1} = I^{n-1} \times \{1\} \cup \partial I^{n-1} \times [0, 1] & \xrightarrow{c_{e_0}} & E \\ \downarrow & \nearrow \bar{\omega} & \downarrow p \\ I^n = I^{n-1} \times [0, 1] & \xrightarrow{\omega} & B \end{array}$$

Then  $\bar{\omega}$  represents an element of  $\pi_n(E, F, e_0)$ , and  $p_*([\bar{\omega}]) = [\omega]$ .

It remains to verify that  $p_*$  is 1-1. Assume that  $\omega_0, \omega_1: (I^n, \partial I^n, J^{n-1}) \rightarrow (E, F, e_0)$  be maps such that  $p_*([\omega_0]) = p_*([\omega_1])$ . Then there exists a homotopy  $h: I^n \times I \rightarrow B$  with  $h_0 = p\omega_0$  and  $h_1 = p\omega_1$ , and such that  $h(\partial I^n \times [0, 1]) = b_0$ . Take the subset  $K \subseteq I^n \times I$  given by

$$K = I^n \times \{0, 1\} \cup J^{n-1} \times [0, 1]$$

Notice that  $K$  consists of all faces of the cube  $I^{n+1} = I^n \times [0, 1]$ , except for the face  $I^{n-1} \times \{0\} \times [0, 1]$ . Define  $\bar{f}: K \rightarrow E$  by

$$\bar{f}(x) = \begin{cases} \omega_0(x) & \text{for } x \in I^n \times \{0\} \\ \omega_1(x) & \text{for } x \in I^n \times \{1\} \\ e_0 & \text{for } x \in J^{n-1} \times [0, 1] \end{cases}$$

By (7.8) we can find a map  $\bar{h}: I^{n+1} \rightarrow E$  such that  $\bar{h}|_K = \bar{f}$  and  $p\bar{h} = h$ . Such map  $\bar{h}$  gives a homotopy between  $\omega_0$  and  $\omega_1$ . Therefore  $[\omega_0] = [\omega_1]$  in  $\pi_n(E, F, e_0)$ .  $\square$

**7.10 Theorem.** Let  $p: E \rightarrow B$  be a Serre fibration. Let  $e_0 \in E$  and  $b_0 \in B$  be such that  $p(e_0) = b_0$ , and let  $F = p^{-1}(b_0)$ . Let  $i: F \rightarrow E$  be the inclusion map. For any  $n \geq 1$  define a homomorphism  $\partial: \pi_n(B, b_0) \rightarrow \pi_{n-1}(F, e_0)$  given by

$$\partial: \pi_n(B, b_0) \xrightarrow{p_*^{-1}} \pi_n(E, F, e_0) \xrightarrow{\partial} \pi_{n-1}(F, e_0)$$

Then the following sequence is exact:

$$\begin{aligned} \dots \longrightarrow \pi_{n+1}(B, b_0) &\xrightarrow{\partial} \pi_n(F, e_0) \xrightarrow{i_*} \pi_n(E, e_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial} \dots \\ &\dots \xrightarrow{p_*} \pi_1(B, b_0) \xrightarrow{\partial} \pi_0(F, e_0) \xrightarrow{i_*} \pi_0(E, e_0) \xrightarrow{p_*} \pi_0(B, b_0) \end{aligned}$$

*Proof.* Exactness in almost all places follows from the exactness of the long exact sequence of the triple  $(E, F, e_0)$ , and the commutativity of the following diagram:

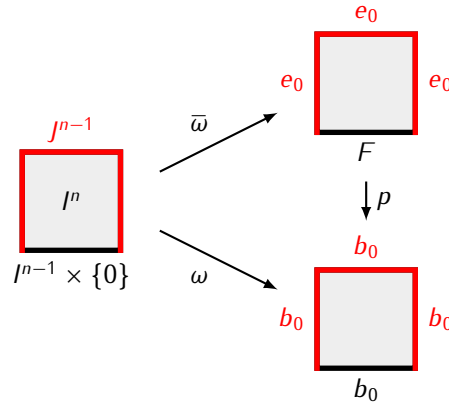
$$\begin{array}{ccccccccccc}
 \dots & \longrightarrow & \pi_n(F, e_0) & \xrightarrow{i_*} & \pi_n(E, e_0) & \xrightarrow{p_*} & \pi_n(B, b_0) & \xrightarrow{\partial} & \pi_{n-1}(F, e_0) & \xrightarrow{i_*} & \pi_{n-1}(E, e_0) & \longrightarrow & \dots \\
 & & \parallel \uparrow & & \parallel \uparrow & & p_* \uparrow \cong & & \parallel \uparrow & & \parallel \uparrow & & \\
 \dots & \longrightarrow & \pi_n(F, e_0) & \xrightarrow{i_*} & \pi_n(E, e_0) & \xrightarrow{j_*} & \pi_n(E, F, e_0) & \xrightarrow{\partial} & \pi_{n-1}(F, e_0) & \xrightarrow{i_*} & \pi_{n-1}(E, e_0) & \longrightarrow & \dots
 \end{array}$$

Since the long exact sequence of  $(E, F, x_0)$  ends at  $\pi_0(E, e_0)$ , exactness of the sequence

$$\pi_0(F, e_0) \xrightarrow{i_*} \pi_0(E, e_0) \xrightarrow{p_*} \pi_0(B, b_0)$$

needs to be checked separately (exercise).  $\square$

**7.11 Note.** The map  $\partial: \pi_n(B, b_0) \rightarrow \pi_{n-1}(F, e_0)$  can be described directly as follows. Take a map  $\omega: (I^n, \partial I^n) \rightarrow (B, x_0)$ . Since  $p: E \rightarrow B$  is a Serre fibration, by (7.8) we can find  $\bar{\omega}: I^n \rightarrow E$  such that  $p\bar{\omega} = \omega$ , and  $\bar{\omega}(J^{n-1}) = e_0$ . Then  $\partial([\omega]) = [\bar{\omega}|_{I^{n-1} \times \{0\}}]$ .



**7.12 Example.** Consider the product fibration  $\text{pr}_B: B \times F \rightarrow B$ . For  $b_0 \in B$  we have  $\text{pr}_B^{-1}(b_0) = \{b_0\} \times F \cong F$ . This for  $f_0 \in F$  the exact sequence looks as follows:

$$\dots \rightarrow \pi_{n+1}(B, b_0) \xrightarrow{\partial} \pi_n(F, f_0) \xrightarrow{i_*} \pi_n(B \times F, (b_0, f_0)) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(F, f_0) \xrightarrow{i_*} \dots$$

The projection map  $\text{pr}_F: B \times F \rightarrow F$  induces homomorphisms  $\text{pr}_{F*}: \pi_n(B \times F, (b_0, f_0)) \rightarrow \pi_n(F, f_0)$  such that  $\text{pr}_{F*} i_* = \text{id}_{\pi_n(F, f_0)}$ . This means that  $\text{Im } \partial = \text{Ker } i_* = 0$ . Therefore for each  $n \geq 1$  we obtain a split short exact sequence

$$0 \longrightarrow \pi_n(F, f_0) \xrightleftharpoons[\text{pr}_{F*}]{i_*} \pi_n(B \times F, (b_0, f_0)) \xrightarrow{p_*} \pi_n(B, b_0) \longrightarrow 0$$

This shows that  $\pi_n(B \times F, (b_0, f_0)) \cong \pi_n(B, b_0) \times \pi_n(F, f_0)$ , which is a special case of the product formula (5.13).

**7.13 Example.** Let  $p: E \rightarrow B$  be a covering, let  $b_0 \in B$  and let  $e_0 \in p^{-1}(b_0)$ . The space  $F = p^{-1}(b_0)$  is discrete, so  $\pi_n(F) = 0$  for all  $n \geq 1$ . Therefore the exact sequence of the fibration becomes

$$\begin{aligned} \dots \longrightarrow 0 \longrightarrow \pi_n(E, e_0) \xrightarrow{p_*} \pi_n(B, b_0) \longrightarrow 0 \longrightarrow \dots \\ \dots \longrightarrow 0 \longrightarrow \pi_1(E, e_0) \xrightarrow{p_*} \pi_1(B, b_0) \xrightarrow{\partial} \pi_0(F, e_0) \xrightarrow{i_*} \pi_0(E, e_0) \xrightarrow{p_*} \pi_0(B, b_0) \end{aligned}$$

This shows that  $p_*: \pi_n(E, e_0) \rightarrow \pi_n(B, b_0)$  is an isomorphism for all  $n \geq 2$ , and it is a monomorphism for  $n = 1$ . This recovers the statement of Proposition 5.9.

The image of  $p_*: \pi_1(E, e_0) \rightarrow \pi_1(B, b_0)$  coincides with  $\text{Ker}(\partial: \pi_1(B, b_0) \rightarrow \pi_0(F, e_0))$ . By the definition of the map  $\partial$ , an element  $[\omega] \in \pi_1(B, b_0)$  is in  $\text{Ker} \partial$  if  $\omega: [0, 1] \rightarrow B$  has a lift  $\bar{\omega}: [0, 1] \rightarrow E$  such that  $\bar{\omega}(1) = e_0$  and  $\omega(1)$  is in the same path connected component of  $F$  as  $e_0$ . Since  $F$  is discrete, it means that  $\bar{\omega}(1) = e_0 = \bar{\omega}(0)$ . As a consequence, we obtain that  $\text{Im}(p_*: \pi_1(E, e_0) \rightarrow \pi_1(B, b_0))$  consists of elements  $[\omega] \in \pi_1(B, b_0)$  such that the lift of  $\omega$  ending at  $e_0$  is a loop.

**7.14 Theorem.** Let  $p: E \rightarrow B$  be map and let  $\{U_i\}_{i \in I}$  be an open cover of  $B$ . Assume that for each  $i \in I$  the map  $p_i: p^{-1}(U_i) \rightarrow U_i$ , which is the restriction of  $p$  is a Serre fibration. Then  $p$  is a Serre fibration.

**7.15 Note.** An analogous fact is true for Hurewicz fibrations, under the assumption that  $B$  is a paracompact space.

*Proof of Theorem 7.14.* See e.g. Hatcher *Algebraic Topology*, Proposition 4.48 p. 379. □

**7.16 Definition.** A map  $p: E \rightarrow B$  is a *fiber bundle* with fiber  $F$  if for every point  $b \in B$  there exists an open neighborhood  $b \in U \subseteq B$  and a homeomorphism  $h_U: p^{-1}(U) \rightarrow U \times F$  such that the following diagram commutes:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{h_U} & U \times F \\ & \searrow p \quad \swarrow \text{pr}_1 & \\ & U & \end{array}$$

Here  $\text{pr}_1: U \times F \rightarrow U$  is the projection map  $\text{pr}_1(x, y) = x$ .

**7.17 Proposition.** Every fiber bundle is a Serre fibration.

*Proof.* This follows from Theorem 7.14 and Example 7.4. □

**7.18 Example.** Every covering space  $p: E \rightarrow B$  is a fiber bundle whose fiber is a discrete space.

7.19 Example. **Mobius band**

7.20 Example. **Klein bottle**

7.21 Example. Consider  $S^{2n+1}$  as a subspace of the complex space  $\mathbb{C}^n$ :

$$S^{2n+1} = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid \sum_{i=0}^n \|z_i\|^2 = 1\}$$

In particular,  $S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$ . The  $n$ -dimensional complex projective space is the quotient space

$$\mathbb{CP}^n = S^{2n+1} / \sim$$

where  $(z_1, \dots, z_n) \sim \lambda(z_0, \dots, z_n)$  for all  $\lambda \in S^1$ . We will denote by  $[z_0, \dots, z_n] \in \mathbb{CP}^n$  the equivalence class of  $(z_0, \dots, z_n)$ . Let  $p: S^{2n+1} \rightarrow \mathbb{CP}^n$  be the quotient map  $p(z_0, \dots, z_n) = [z_0, \dots, z_n]$ .

We will show that  $p: S^{2n+1} \rightarrow \mathbb{CP}^n$  is a fiber bundle with fiber  $S^1$ . Let  $b = [z_0, \dots, z_n] \in \mathbb{CP}^n$ . Choose  $0 \leq i \leq n$  such that  $z_i \neq 0$ , and take  $U_i = \{(w_0, \dots, w_n) \in \mathbb{CP}^n \mid w_i \neq 0\}$ . This set is an open neighborhood of  $b$  in  $\mathbb{CP}^n$ . We have

$$p^{-1}(U_i) = \{(w_0, \dots, w_n) \in S^{2n+1} \mid w_i \neq 0\}$$

Define a map  $h_i: p^{-1}(U_i) \rightarrow U_i \times S^1$  by  $h_i(w_0, \dots, w_n) = ([w_0, \dots, w_n], w_i / \|w_i\|)$ . This is a homeomorphism, with the inverse given by

$$h_i^{-1}([v_0, \dots, v_n], \lambda) = \frac{\|v_i\|}{v_i} \cdot \lambda \cdot (v_0, \dots, v_n)$$

Let  $n \geq 1$ . The long exact sequence of the bundle  $p: S^{2n+1} \rightarrow \mathbb{CP}^n$  has the form

$$\begin{aligned} \dots \longrightarrow \pi_m(S^1) \xrightarrow{i_*} \pi_m(S^{2n+1}) \xrightarrow{p_*} \pi_m(\mathbb{CP}^n) \xrightarrow{\partial} \pi_{m-1}(S^1) \longrightarrow \dots \\ \dots \longrightarrow \pi_2(S^{2n+1}) \xrightarrow{p_*} \pi_2(\mathbb{CP}^n) \xrightarrow{\partial} \pi_1(S^1) \xrightarrow{i_*} \pi_1(S^{2n+1}) \xrightarrow{p_*} \pi_1(\mathbb{CP}^n, b_0) \xrightarrow{\partial} \pi_0(S^1) = 0 \end{aligned}$$

Since  $\pi_m(S^1) = 0$  for  $m > 1$ , we obtain that  $\pi_m(\mathbb{CP}^n) \cong \pi_m(S^{2n+1})$  for  $m \geq 3$ . Also, since  $\pi_1(S^1) \cong \mathbb{Z}$  and  $\pi_m(S^{2n+1}) = 0$  for  $m < 2n + 1$ , thus  $\pi_2(\mathbb{CP}^n) \cong \pi_1(S^1) \cong \mathbb{Z}$  and  $\pi_1(\mathbb{CP}^n) = 0$ .

7.22 Example. As a special case of Example 7.21, take  $n = 1$ . In this case, we have a homeomorphism  $\mathbb{CP}^1 \cong S^2$ . To see this, define a map  $h: \mathbb{CP}^1 \setminus \{[1, 0]\} \rightarrow \mathbb{C}$  by  $h([z_0, z_1]) = \frac{z_0}{z_1}$ . This is a homeomorphism with the inverse given by  $h^{-1}(z) = \frac{1}{1+\|z\|} \cdot [z, 1]$ . Since  $S^2$  is homeomorphic to the one-point compactification of  $\mathbb{C}$ , i.e.  $S^2 \cong \mathbb{C} \cup \{\infty\}$ , the map  $h$  can be extended to a homeomorphism  $h: \mathbb{CP}^1 \rightarrow S^2$  by setting  $h([1, 0]) = \infty$ .

Under the identification  $\mathbb{CP}^1 \cong S^2$  the bundle  $S^1 \rightarrow S^3 \xrightarrow{p} \mathbb{CP}^1$  becomes  $S^1 \rightarrow S^3 \xrightarrow{p} S^2$ . This bundle is called the *Hopf bundle* (or the *Hopf fibration*).

Using the long exact sequence of the Hopf fibration we obtain:

**7.23 Theorem.**  $\pi_2(S^2) \cong \mathbb{Z}$ .