

16 | Weak Equivalences and Homology

The main goal of this chapter is to show that the following holds:

16.1 Theorem. *If $f: X \rightarrow Y$ is a weak equivalence then the induced homomorphisms $f_*: H_i(X) \rightarrow H_i(Y)$ and $f^*: H^i(X) \rightarrow H^i(Y)$ are isomorphisms for all $i \geq 0$.*

16.2 Brief review of homological algebra.

- A *chain complex* C_* consists of a sequence of abelian groups and group homomorphisms

$$\dots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \dots$$

such that $\partial_n \partial_{n+1} = 0$ for all n . The homomorphisms ∂_n are called *differentials*.

- The *n-th homology group* of a chain complex C_* is the group $H_n(C_*) = \ker \partial_n / \text{Im } \partial_{n+1}$.
- A *chain map* $f_*: C_* \rightarrow D_*$ is a sequence of homomorphisms $f_n: C_n \rightarrow D_n$ such that $\partial_n f_n = f_{n-1} \partial_n$.
- A chain map $f_*: C_* \rightarrow D_*$ induces homomorphisms of homology groups $f_*: H_n(C_*) \rightarrow H_n(D_*)$.
- If $f_*, g_*: C_* \rightarrow D_*$ are chain maps, then a *chain homotopy* $s_*: C_* \rightarrow D_*$ from f_* to g_* is a sequence of homomorphisms $s_n: C_n \rightarrow D_{n+1}$ such that $f_n - g_n = \partial_{n+1} s_n + s_{n-1} \partial_n$.
- If there exists a chain homotopy between chain maps $f_*, g_*: C_* \rightarrow D_*$ then f_* and g_* induce the same homomorphism between homology groups $H_*(C_*) \rightarrow H_*(D_*)$.

16.3 Brief review of singular homology.

- A *singular chain complex* of a topological space X is a chain complex $C_*(X)$ such that $C_n(X)$ is the free abelian group generated by all singular simplices $\sigma: \Delta^n \rightarrow X$.
- Differentials in $C_*(X)$ are defined using face maps $d_n^i: \Delta^{n-1} \rightarrow \Delta^n$ for $i = 0, \dots, n$: $\partial_n \sigma = \sum_{i=0}^n (-1)^i \sigma d_n^i$.

- Singular homology groups of a space X are homology groups of $C_*(X)$: $H_n(X) := H_n(C_*(X))$.
- Any map of spaces $f: X \rightarrow Y$ defines a chain map of singular chain complexes $f_*: C_*(X) \rightarrow C_*(Y)$ given by $f_*(\sigma) = f\sigma$ for a singular simplex $\sigma: \Delta^n \rightarrow X$. This induces a homomorphism of homology groups $f_*: H_*(X) \rightarrow H_*(Y)$.
- For a space X let $i_0, i_1: X \rightarrow X \times [0, 1]$ denote the inclusions $i_0(x) = (x, 0)$ and $i_1(x) = (x, 1)$. There exists a chain homotopy $s_*^X: C_*(X) \rightarrow C_{*+1}(X \times [0, 1])$ from i_{0*} to i_{1*} .
- The chain homotopy s_*^X can be used to show that if $f, g: X \rightarrow Y$ are homotopic maps then they induce the same homomorphism of homology groups $H_*(X) \rightarrow H_*(Y)$.

Proof of Theorem 16.1. Assume first that $X \subseteq Y$ and that $f: X \hookrightarrow Y$ is the inclusion map. Notice that in this case $C_*(X)$ is a subcomplex of $C_*(Y)$ and the chain map $f_*: C_*(X) \rightarrow C_*(Y)$ is an inclusion.

We will associate to each singular simplex $\sigma: \Delta^n \rightarrow Y$ a homotopy $h^\sigma: \Delta^n \times [0, 1] \rightarrow Y$ such that:

- 1) $h_0^\sigma = \sigma$ and $h_1^\sigma(\Delta^n) \subseteq X$.
- 2) If $\sigma(\Delta^n) \subseteq X$ then $h_t^\sigma = \sigma$ for all $t \in [0, 1]$.
- 3) $h^{\sigma d_n^i} = h^\sigma(d_n^i \times \text{id}_{[0,1]})$

The homotopies h^σ will be constructed by induction with respect to n . For $n = 0$ giving a simplex $\sigma: \Delta^0 = \{*\} \rightarrow X$ is the same as giving a point $\sigma(*) = y \in Y$. Since f is a weak equivalence, there exist a path $h^\sigma: \Delta^0 \times [0, 1] \rightarrow Y$ such that $h^\sigma(*, 0) = y$ and $h^\sigma(*, 1) \in X$. If $y \in X$ take h^σ to be the constant path.

Assume that we have already constructed homotopies h^τ satisfying 1)–3) for all $\tau: \Delta^k \rightarrow Y$ with $k < n$, and let $\sigma: \Delta^n \rightarrow Y$. If $\sigma(\Delta^n) \subseteq X$, define h^σ using condition 2). Otherwise, let $\partial\Delta^n := \bigcup_{i=0}^n d_n^i(\Delta^{n-1}) \subseteq \Delta^n$. Since homotopies $h^{\sigma d_n^i}$ are already defined, condition 3) determines a map $h: \Delta^n \times \{0\} \cup \partial\Delta^n \times [0, 1] \rightarrow Y$ such that $h_0 = \sigma$ and $h_1(\partial\Delta^n) \subseteq X$. The pair $(\Delta^n, \partial\Delta^n)$ is a relative CW complex, so by Proposition 14.6 we can extend h to a homotopy $h^\sigma: \Delta^n \times [0, 1] \rightarrow Y$ such that $h_0^\sigma = \sigma$ and $h_1^\sigma(\Delta^n) \subseteq X$.

Define a map $\varphi_*: C_*(Y) \rightarrow C_*(X)$ by $\varphi(\sigma) = h_1^\sigma$. Condition 3) implies that φ_* is a chain map. Also, by condition 2) we obtain $\varphi_* f_* = \text{id}_{C_*(X)}$. Finally, a chain homotopy $\Phi_*: C_*(Y) \rightarrow C_*(Y)$ between $f_* \varphi_*$ and $\text{id}_{C_*(Y)}$ can be obtained as follows. Given a singular simplex $\sigma: \Delta^n \rightarrow Y$ the homotopy h^σ induces a chain map $h_*^\sigma: C_*(\Delta^n \times [0, 1]) \rightarrow C_*(Y)$. We also have a chain homotopy $s_*^{\Delta^n}: C_*(\Delta^n) \rightarrow C_*(\Delta^n \times [0, 1])$. The identity map $\text{id}_{\Delta^n}: \Delta^n \rightarrow \Delta^n$ is a singular simplex in $C_n(\Delta^n)$. We set $\varphi(\sigma) = h_*^\sigma s_*^{\Delta^n}(\text{id}_{\Delta^n})$.

For a general weak equivalence $f: X \rightarrow Y$ consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow i & \nearrow r \\ & M_f & \end{array}$$

where M_f is the mapping cylinder of f , i is the inclusion map and r is the retraction. Since f is a weak equivalence and r is a homotopy equivalence, thus i is a weak equivalence. Therefore, by the argument above, i induces an isomorphism on homology groups $i_*: H_*(X) \xrightarrow{\cong} H_*(M_f)$. Also, since every homotopy equivalence induces an isomorphism on homology, thus we get an isomorphism $r_*: H_*(M_f) \xrightarrow{\cong} H_*(Y)$. Therefore $f_* = r_* i_*: H_*(X) \rightarrow H_*(Y)$ is an isomorphism.

The statement that a weak equivalence induces an isomorphism of cohomology groups follows from the Universal Coefficients Theorem, which implies that if a map $f: X \rightarrow Y$ gives an isomorphism on homology, then it also induces an isomorphism on cohomology. \square

Using the same arguments as in the proof of Theorem 16.1, this theorem can be generalized as follows:

16.4 Theorem. *If $f: X \rightarrow Y$ is an n -equivalence for some $n \geq 1$ then $f_*: H_i(X) \rightarrow H_i(Y)$ is an isomorphism for all $i < n$ and it is an epimorphism for $i = n$.*