## 21 | Serre Spectral Sequence

The Serre spectral sequence is a special case of the spectral sequence associated to a filtration described in Theorem 20.9.

**21.1 Definition.** Let  $p: E \to B$  is a Serre fibration where B is a connected CW complex. Let

$$\varnothing = B^{(-1)} \subseteq B^{(0)} \subseteq \dots B$$

be the filtration of B by skeleta. Taking  $E^p := p^{-1}(B^{(k)})$  we obtain a filtration of the space E:

$$\emptyset = E^{-1} \subseteq E^0 \subseteq \ldots \subseteq E$$

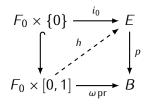
The Serre spectral sequence of the fibration p is the spectral sequence associated to this filtration.

By Theorem 20.9 we get that  $E_{p,q}^1 = H_{p+q}(E^p, E^{p-1})$  and that  $E_{**}^r$  converges to  $H_*(E)$ . The advantage of the Serre spectral sequence is that we can explicitly describe its second page:

**21.2 Theorem.** Let  $E_{**}^r$  be the Serre spectral sequence of a fibration  $p: E \to B$ . Let  $F = p^{-1}(b_0)$  for some  $b_0 \in B$ . If the space B is simply connected then  $E_{p,q}^2 \cong H_p(B, H_q(F))$ .

While we will skip the proof of this result, it is useful to point out that the assumption that B is simply connected is needed in order to obtain a canonical identification between fibers of p taken over different points. Assume for a moment p is a Hurewicz fibration and that  $b_0, b_1 \in B$ , Let  $F_i = p^{-1}(b_i)$ 

for i=0,1. Given a path  $\omega:[0,1]\to B$  such that  $\omega(0)=b_0$   $\omega(1)=b_1$ , consider the diagram

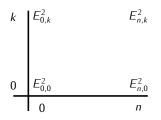


where pr:  $F_0 \times [0,1] \to [0,1]$  is the projection map and  $i_0 \colon F_0 \to E$  is the inclusion. A lift h of  $\omega$  pr gives a homotopy in E between the map  $i_0$  and a certain map  $h_1 \colon F_0 \to F_1$ . One can show that this map  $h_1$  is a homotopy equivalence and that its homotopy class depends only on the homotopy class of the path  $\omega$  (relative its endpoints). If the space B is simply connected, all paths joining  $b_0$  and  $b_1$  are homotopic, so the homotopy class of  $h_1$  is uniquely defined. In particular, we obtain canonical isomorphisms of homology groups  $h_{1*} \colon H_q(F_0) \xrightarrow{\cong} H_q(F_1)$ . If p is a Serre fibration, we can use the same argument, but in order to get the lift h we replace  $F_0$  by its CW approximation.

We have seen already one application of the Serre spectral sequence in Theorem 19.6. Here is another one:

**21.3 Proposition.** Let  $S^k \to S^m \xrightarrow{p} S^n$  be a homotopy fibration sequence with  $n \ge 1$ . Then k = n - 1 and m = 2n - 1.

*Proof.* If n=1 then the long exact sequence of homotopy groups shows that we must have m=1 and k=0. Assume then that  $n\geq 2$ . Consider the Serre spectral sequence of this fibration. Its second page  $E_{p,q}^2\cong H_p(S^n,H_q(S^k))$  has only four non-zero terms, all isomorphic to  $\mathbb{Z}$ :



All differentials originating and terminating at  $E_{0,0}^r$  and  $E_{k,n}^r$  are trivial, so  $E_{0,0}^2 = E_{0,0}^\infty$  and  $E_{n,k}^2 = E_{n,k}^\infty$ . The page  $E_{**}^\infty$  can have non-zero terms  $E_{p,q}^\infty$  only if (p,q)=(0,0) or p+q=m. It follows that k+n=m. The terms  $E_{0,k}^2$  and  $E_{n,0}^2$  must kill each other, so they must be connected by a differential. This is possible only if k=n-1. Taken together these observations imply that p is a fibration sequence of the form  $S^{n-1} \to S^{2n-1} \stackrel{p}{\to} S^n$ .

Hopf bundles give examples of fibration sequences  $S^{n-1} \to S^{2n-1} \to S^n$  for n = 1, 2, 4, 8. A theorem of Adams implies that these are the only fibration sequences where all three spaces are spheres.