

12 | All Groups Are Homotopy Groups

Recall that van Kampen's Theorem implies that for any group G we can find a space X such that $\pi_1(X) \cong G$. The goal of this chapter is to extend this result to higher homotopy groups. Since all groups $\pi_n(X)$ with $n \geq 2$ are abelian (3.4), we will show that the following holds:

12.1 Theorem. *For any abelian group G and any $n \geq 2$ there exists a space X such that $\pi_n(X) \cong G$. Moreover, such space X can be constructed in such way, that X is a CW complex and $X^{(n-1)} = *$.*

For every abelian group G there exists an epimorphism $\varphi: \bigoplus_{i \in I} \mathbb{Z} \rightarrow G$ for some set I . Indeed, it is enough to take $I = G$, the set of elements of the group G . Then we can define φ by $\varphi(e_g) = g$, where e_g is the generator of the copy of $\mathbb{Z} \subseteq \bigoplus_{h \in G} \mathbb{Z}$ indexed by g . Given such a homomorphism φ we get $G \cong \bigoplus_{i \in I} \mathbb{Z} / \ker(\varphi)$.

Based on this, in order to prove Theorem 12.1 it will suffice to show that:

- 1) for any set I and $n \geq 2$ there exists a space X such that $\pi_n(X) \cong \bigoplus_{i \in I} \mathbb{Z}$.
- 2) for any subgroup $H \subseteq \bigoplus_{i \in I} \mathbb{Z}$ and any $n \geq 2$ there exists a space X such that $\pi_n(X) \cong \bigoplus_{i \in I} \mathbb{Z} / H$.

12.2 Lemma. *Let $\{(X_i, \bar{x}_i)\}_{i \in I}$ be a family of pointed Hasdorff spaces. Let $X = \bigvee_{i \in I} X_i$, and let $*$ $\in X$ denote the basepoint. For $k \in I$ let $r_k: X \rightarrow X_k$ be the retraction map. Then for any $n \geq 2$ we an epimorphism $\varphi: \pi_n(X, *) \rightarrow \bigoplus_{i \in I} \pi_n(X_i, \bar{x}_i)$ given by $\varphi([\omega]) = \sum_{i \in I} r_{i*}([\omega])$.*

Proof. For each $k \in I$ let $j_k: X_k \rightarrow X$ be the inclusion map. We have a homomorphism

$$\psi := \bigoplus_{i \in I} j_{i*}: \bigoplus_{i \in I} \pi_n(X_i, \bar{x}_i) \rightarrow \pi_n(X, *)$$

The retractions r_i define a map

$$\varphi := \prod_{i \in I} r_{i*}: \pi_n(X, *) \rightarrow \prod_{i \in I} \pi_n(X_i, \bar{x}_i)$$

We claim that $\text{Im}(\varphi) \subseteq \bigoplus_{i \in I} \pi_n(X_i, \bar{x}_i) \subseteq \prod_{i \in I} \pi_n(X_i, \bar{x}_i)$. Indeed, if $\omega: (I^n, \partial I^n) \rightarrow (X, *)$ is a map representing an element $[\omega] \in \pi_n(X, *)$, then, by compactness of I^n , we have $\omega(I^n) \cap X_i \neq *$ for finitely many $i \in I$ only, and so $r_{i*}([\omega]) \neq 0$ for finitely many $i \in I$. Thus $\varphi([\omega]) \in \bigoplus_{i \in I} \pi_n(X_i, \bar{x}_i)$. It follows that we can consider φ as a homomorphism $\pi_n(X, *) \rightarrow \bigoplus_{i \in I} \pi_n(X_i, \bar{x}_i)$.

Since $r_i j_i = \text{id}_{X_i}$ for all $i \in I$, and $r_{i'} j_i$ is the constant map for all $i \neq i'$, it follows that $\varphi \psi$ is the identity homomorphism, and so φ is onto. \square

12.3 Note. In general, the epimorphism φ in Lemma 12.2 is not an isomorphism. For example, recall (5.11) that for $n \geq 2$ we have $\pi_n(S^1 \vee S^n) \cong \pi_n(\bigvee_{i \in \mathbb{Z}} S^n)$. By Lemma 12.2 we get an epimorphism

$$\pi_n(S^1 \vee S^n) \cong \pi_n(\bigvee_{i \in \mathbb{Z}} S^n) \rightarrow \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}$$

which shows that the group $\pi_n(S^1 \vee S^n)$ is not finitely generated. Therefore $\pi_n(S^1 \vee S^n) \not\cong \pi_n(S^1) \oplus \pi_n(S^n) \cong \mathbb{Z}$.

12.4 Proposition. Let $\{(X_i, \bar{x}_i)\}_{i \in I}$ be a family of pointed CW-complexes. Given $n \geq 1$, assume that each complex X_i is n -connected. Then the homomorphism $\varphi: \pi_m(\bigvee_{i \in I} X_i, *) \rightarrow \bigoplus_{i \in I} \pi_m(X_i, \bar{x}_i)$ is an isomorphism for $m \leq 2n$.

Proof. For each CW complex X_i we can assume that \bar{x}_i is a 0-cell of X_i . Also, by Proposition 5.6 we can assume that X_i has no other 0-cells, and no k -cells for $k \leq n-1$.

By Proposition 12.4 φ is onto. It will suffice to show that $\ker \varphi = 0$ for $m \leq 2n$.

Assume first, that the set I is finite, so $\bigvee_{i \in I} X_i = X_1 \vee \dots \vee X_k$ for some $k \geq 0$. Take the product $X_1 \times \dots \times X_k$. The inclusion maps $\psi_j: X_j \rightarrow X_1 \times \dots \times X_k$ given by $\psi_j(x) = (\bar{x}_1, \dots, \bar{x}_{j-1}, x, \bar{x}_{j+1}, \dots, \bar{x}_k)$ define an embedding $\psi: X_1 \vee \dots \vee X_k \rightarrow X_1 \times \dots \times X_k$. This gives a commutative diagram:

$$\begin{array}{ccc} \pi_m(X_1 \vee \dots \vee X_k) & \xrightarrow{\psi_*} & \pi_m(X_1 \times \dots \times X_k) \\ \downarrow \varphi & & \downarrow \cong \\ \bigoplus_{j=1}^k \pi_{k+1}(X_j) & \xrightarrow{=} & \prod_{j=1}^k \pi_{k+1}(X_j) \end{array}$$

This shows that φ is a monomorphism if and only if ψ_* is one. If X_1, \dots, X_k are finite CW complexes, then the space $X_1 \times \dots \times X_k$ also has the structure of a CW complex, with cells given by products $e_1 \times \dots \times e_k$ where e_i is a cell in X_i . All cells of $X_1 \times \dots \times X_k$ that are not contained in $X_1 \vee \dots \vee X_k$ have dimension $2n+2$ or higher, so $X_1 \vee \dots \vee X_k$ is the $(2n+1)$ -skeleton of $X_1 \times \dots \times X_k$. Thus, by Proposition 5.2, ψ_* is an isomorphism for all $m \leq 2n$.

Next, assume that the set I is infinite, and let $\omega: (I^m, \partial I^m) \rightarrow (\bigvee_{i \in I} X_i, *)$ be a map such that $\varphi([\omega]) = 0$. By compactness of I^m we have $\omega(I^m) \cap X_i \neq *$ for finitely many $i \in I$ only. Thus we can consider ω as

¹This uses the fact that if $X_i \simeq X'_i$ for all $i \in I$ then $\bigvee_{i \in I} X_i \simeq \bigvee_{i \in I} X'_i$. This holds for well-pointed, path connected spaces.

a map $\omega: (I^m, \partial I^m) \rightarrow (X_{i_1} \vee \dots \vee X_{i_k}, *)$ for some $i_1, \dots, i_k \in I$. Since $\varphi([\omega]) = 0$, the homomorphism $\pi_m(X_{i_1} \vee \dots \vee X_{i_k}) \rightarrow \bigoplus_{j=1}^k \pi_{k+1}(X_{i_j})$ also maps $[\omega]$ to 0. By the finite case this means that $[\omega] = 0$. \square

12.5 Note. The proof of Proposition 12.4 uses the fact that if X and Y are CW complexes, then $X \times Y$ has the structure of a CW complex with cells given by products of cells in X and Y . An issue with this statement is that the topology induced on $X \times Y$ by this cell structure (where a set $U \subseteq X \times Y$ is open if and only if its intersection with each cell is an open subset of the cell) need not be the same as the product topology on $X \times Y$. The topology induced by the cell structure on $X \times Y$ is called the compactly generated topology. Let $X \times_{cg} Y$ denote the product taken with this topology, and let $X \times Y$ denote the product taken with the product topology. Every open set in $X \times Y$ is also open in $X \times_{cg} Y$, so the identity map $\text{id}: X \times_{cg} Y \rightarrow X \times Y$ is continuous. Moreover, this map induces an isomorphism of homotopy groups $\pi_n(X \times_{cg} Y) \xrightarrow{\cong} \pi_n(X \times Y)$ for all n . For this reason this change of topology does not affect the proof of Proposition 12.4.

12.6 Corollary. For any set I and any $n \geq 2$ we have an isomorphism

$$\pi_n(\bigvee_{i \in I} S^n) \cong \bigoplus_{i \in I} \mathbb{Z}$$

Moreover, the group $\pi_n(\bigvee_{i \in I} S^n)$ is generated by elements $[j_k]$ for $k \in I$ where $j_k: S^n \hookrightarrow \bigvee_{i \in I} S^n$ is the inclusion of the k -th copy of S^n .

12.7 Proposition. Let (X, x_0) be a simply connected space, and let $\varphi_i: (S^n, s_0) \rightarrow (X, x_0)$ be maps representing elements of $\pi_n(X, x_0)$ for some $n \geq 2$. Consider the space $Y = X \cup \bigcup_i e_i^{n+1}$ obtained by attaching $(n+1)$ -cells to X using φ_i as the attaching maps. If $j: X \hookrightarrow Y$ is the inclusion map, then the induced homomorphism

$$j_*: \pi_k(X, x_0) \rightarrow \pi_k(Y, x_0)$$

is an isomorphism for $k < n$ and an epimorphism for $k = n$. Moreover, $\ker(j_*: \pi_n(X, x_0) \rightarrow \pi_n(Y, x_0))$ is the subgroup of $\pi_n(X, x_0)$ generated by the elements $[\varphi_i]$.

Proof. We can consider the pair (Y, X) as a relative CW complex with the n -skeleton given by X . Then j_* is an isomorphism for $k < n$ and epimorphism for $k = n$ by Proposition 5.2.

It remains to verify the statement about the kernel of j_* for $k = n$. Consider the exact sequence of the pair (Y, X) :

$$\dots \rightarrow \pi_{n+1}(Y, X) \xrightarrow{\partial} \pi_n(X) \xrightarrow{j_*} \pi_n(Y) \rightarrow \pi_n(Y, X) \rightarrow \dots$$

We have $\ker j_* = \text{Im } \partial$. By assumption, the space X is 1-connected, so from Theorem 11.5 we obtain that the quotient map $q: Y \rightarrow Y/X$ induces an isomorphism

$$q_*: \pi_{n+1}(Y, X) \xrightarrow{\cong} \pi_{n+1}(Y/X) \cong \pi_{n+1}(\bigvee_i S^{n+1}) \cong \bigoplus_i \mathbb{Z}$$

This implies that $\pi_{n+1}(Y, X)$ is generated by homotopy classes of maps $\bar{\varphi}_i: D^{n+1} \rightarrow Y$ which are the characteristic maps of the cells e_i^{n+1} . The boundary homomorphism is given by $\partial[\bar{\varphi}_i] = [\varphi_i]$. Therefore $\text{Im } \partial = \ker j_*$ is the subgroup of $\pi_n(X)$ generated by the elements $[\varphi_i]$. \square

Proposition 12.7 can be generalized to non-simply connected spaces as follows. Recall (4.14) that higher homotopy groups admit the action of the fundamental group:

$$\begin{aligned} \pi_1(X, x_0) \times \pi_n(X, x_0) &\rightarrow \pi_n(X, x_0) \\ ([\tau], [\omega]) &\mapsto [\tau] \odot [\omega] \end{aligned}$$

We have:

12.8 Proposition. *Let (X, x_0) be a space which is connected, locally path connected, and semi-locally simply connected. Let $\varphi_i: (S^n, s_0) \rightarrow (X, x_0)$ be maps representing elements of $\pi_n(X, x_0)$ for some $n \geq 2$. Consider the space $Y = X \cup \bigcup_i e_i^{n+1}$ obtained by attaching $(n+1)$ -cells to X using φ_i as the attaching maps. If $j: X \hookrightarrow Y$ is the inclusion map, then the induced homomorphism*

$$j_*: \pi_k(X, x_0) \rightarrow \pi_k(Y, x_0)$$

is an isomorphism for $k \leq n$ and an epimorphism for $k = n$. Moreover, $\ker(j_: \pi_n(X, x_0) \rightarrow \pi_n(Y, x_0))$ is the subgroup of $\pi_n(X, x_0)$ generated by the elements $[\omega] \odot [\varphi_i]$ for all $[\omega] \in \pi_1(X, x_0)$.*

Proof. The only non-trivial part is the statement about $\ker j_*$. The conditions on the space X guarantee that it has a universal covering $p_X: \tilde{X} \rightarrow X$. Let $p_X^{-1}(x_0) = \{\tilde{x}_k\}_{k \in K}$ and let $\tilde{\varphi}_{i,k}: S^n \rightarrow \tilde{X}$ denote the lift of φ_i such that $\tilde{\varphi}_{i,k}(s_0) = \tilde{x}_k$. Let $\tilde{Y} = \tilde{X} \cup \bigcup_{i,j} e_{i,k}^{n+1}$ be the space obtained by attaching $(n+1)$ -cells to \tilde{X} using $\tilde{\varphi}_{i,k}$ as attaching maps. The natural map $p_Y: \tilde{Y} \rightarrow Y$ is a universal covering of Y . We get a commutative diagram:

$$\begin{array}{ccc} \pi_n(\tilde{X}, \tilde{x}_0) & \xrightarrow{\tilde{j}_*} & \pi_n(\tilde{Y}, \tilde{x}_0) \\ p_{X*} \downarrow \cong & & \cong \downarrow p_{Y*} \\ \pi_n(X, x_0) & \xrightarrow{j_*} & \pi_n(Y, x_0) \end{array}$$

where $\tilde{j}: \tilde{X} \rightarrow \tilde{Y}$ is the inclusion and $\tilde{x}_0 \in p_X^{-1}(x_0)$. Since p_{X*} and p_{Y*} are isomorphisms (5.9), we obtain that $\ker j_* = p_{X*}(\ker \tilde{j}_*)$.

For each $\tilde{x}_k \in p_X^{-1}(x_0)$ let $\tilde{\omega}_k$ be a path in \tilde{X} such that $\tilde{\omega}_k(0) = \tilde{x}_0$ and $\tilde{\omega}_k(1) = \tilde{x}_k$. Then for each $[\omega] \in \pi_1(X, x_0)$ we have $[\omega] = [p_X \tilde{\omega}_k]$ for some k . Let $s_k: \pi_n(\tilde{X}, \tilde{x}_k) \rightarrow \pi_n(\tilde{X}, \tilde{x}_0)$ be the change of the basepoint isomorphism defined by $\tilde{\omega}_k$ (4.4). Since \tilde{X} is simply connected, using Proposition 12.7 we obtain that $\ker \tilde{j}_*$ is generated by the elements $s_k[\tilde{\varphi}_{i,k}]$ for all i, k . Thus $\ker j_*$ is generated by elements $p_{X*} s_k[\tilde{\varphi}_{i,k}]$. It remains to notice that $p_{X*} s_k[\tilde{\varphi}_{i,k}] = [p_X \omega_k] \odot [p_X \tilde{\varphi}_{i,k}] = [p_X \omega_k] \odot [\varphi_i]$ (exercise). □

Proof of Theorem 12.1. Given an abelian group G and $n \geq 2$, we can find a set I and an epimorphism

$$\Phi: \pi_n\left(\bigvee_{i \in I} S^n\right) \cong \bigoplus_{i \in I} \mathbb{Z} \rightarrow G$$

Let $\ker \Phi = \{[\varphi_k: S^n \rightarrow \bigvee_{i \in I} S^n]\}_{k \in K}$, and let X be the space obtained by attaching $(n+1)$ -cells to $\bigvee_{i \in I} S^n$ using the maps φ_i . By Proposition 12.7 we obtain $\pi_n(X) \cong \pi_n(\bigvee_{i \in I} S^n) / \ker \Phi \cong G$. \square

12.9 Definition. Given a group G and an integer $n \geq 1$, an *Eilenberg-MacLane space* of the type $K(G, n)$ is a path connected CW complex X such that

$$\pi_i(X) \cong \begin{cases} G & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

12.10 Note. Eilenberg-MacLane spaces are not uniquely defined, but as we will see later (14.10), they are unique up to homotopy equivalence. By abuse of notation we will write $X = K(G, n)$ to indicate that X has the type of $K(G, n)$.

12.11 Example. $S^1 = K(\mathbb{Z}, 1)$.

12.12 Example. Recall that the n -dimensional real projective space \mathbb{RP}^n is the quotient space of S^n obtained by identifying antipodal points: $\mathbb{RP}^n = S^n / \sim$ where $x \sim -x$ for all $x \in S^n$. The quotient map $q: S^n \rightarrow \mathbb{RP}^n$ is the 2-fold universal cover of \mathbb{RP}^n . It follows that

$$\pi_i(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z}/2 & \text{if } i = 1 \\ \pi_i(S^n) & i \geq 2 \end{cases}$$

Embeddings of spheres $S^1 \hookrightarrow S^2 \hookrightarrow \dots$ induce embeddings of projective spaces $\mathbb{RP}^1 \hookrightarrow \mathbb{RP}^2 \hookrightarrow \dots$. Take $S^\infty = \bigcup_{n=1}^\infty S^n$ and $\mathbb{RP}^\infty = \bigcup_{n=1}^\infty \mathbb{RP}^n$. The quotient map $q: S^\infty \rightarrow \mathbb{RP}^\infty$ is a 2-fold universal covering of \mathbb{RP}^∞ . Since S^∞ is a contractible space (2.18), we obtain

$$\pi_i(\mathbb{RP}^\infty) \cong \begin{cases} \mathbb{Z}/2 & \text{if } i = 1 \\ 0 & \text{if } i \geq 2 \end{cases}$$

Therefore $\mathbb{RP}^\infty = K(\mathbb{Z}/2, 1)$.

12.13 Example. Recall (7.24) that for a complex projective space the quotient map $p: S^{2n+1} \rightarrow \mathbb{CP}^n$ is a Serre fibration with the fiber S^1 . The long exact sequence of this fibration gives

$$\pi_i(\mathbb{CP}^n) \cong \begin{cases} 0 & \text{if } i = 1 \\ \mathbb{Z} & \text{if } i = 2 \\ \pi_i(S^{2n+1}) & \text{if } i \geq 3 \end{cases}$$

The embedding maps $S^3 \hookrightarrow S^5 \hookrightarrow \dots$ induce embeddings $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2 \hookrightarrow \dots$. We again have $S^\infty = \bigcup_{n=1}^\infty S^{2n+1}$. Also, define $\mathbb{CP}^\infty = \bigcup_{n=1}^\infty \mathbb{CP}^n$. The map $p: S^\infty \rightarrow \mathbb{CP}^\infty$ is again a Serre fibration

with fiber S^1 . Since S^∞ is contractible, the long exact sequence of this fibration gives

$$\pi_i(\mathbb{CP}^\infty) \cong \begin{cases} 0 & \text{if } i = 1 \\ \mathbb{Z} & \text{if } i = 2 \\ 0 & \text{if } i \geq 3 \end{cases}$$

Thus $\mathbb{CP}^\infty = K(\mathbb{Z}, 2)$.

12.14 Proposition. *For any $n \geq 1$ and any group G (abelian if $n \geq 2$) there exists an Eilenberg-MacLane space $K(G, n)$. Moreover, it is possible to construct such space so that $K(G, n)^{(n-1)} = *$.*

Proof. By Theorem 12.1, if $n \geq 2$ then we can find a path connected CW complex (X_n, x_0) such that $X_n^{(n-1)} = *$ and

$$\pi_i(X_n, x_0) \cong \begin{cases} G & \text{if } i = n \\ 0 & \text{if } i < n \end{cases}$$

For $n = 1$ such CW complex can be constructed using van Kampen's theorem. Let X_{n+1} be the space obtained by attaching an $(n+2)$ -cells to X_n using all possible maps $(S^{n+1}, s_0) \rightarrow (X_n, x_0)$. Then $X_n \subseteq X_{n+1}$, and using Proposition 5.2 we obtain

$$\pi_i(X_{n+1}, x_0) \cong \begin{cases} 0 & \text{if } i = n+1 \\ G & \text{if } i = n \\ 0 & \text{if } i < n \end{cases}$$

In the same way, for any $m > n+1$ we can inductively construct a space X_m such that X_m is obtained by attaching $(m+1)$ -cells to X_{m-1} and

$$\pi_i(X_m, x_0) \cong \begin{cases} 0 & \text{if } n < i \leq m \\ G & \text{if } i = n \\ 0 & \text{if } i < n \end{cases}$$

Then we can take $K(G, n) = \bigcup_{m=n}^\infty X_m$. □

12.15 Corollary. *For any sequence of groups G_1, G_2, \dots such that G_i is abelian for $i \geq 2$, there exists a path connected CW complex X such that $\pi_i(X) \cong G_i$ for all $i \geq 1$.*

Proof. Take $X = \prod_{i=1}^\infty K(G_i, i)$. □