

## 22 | Serre classes

The motivation for this chapter is to show that the following holds.

**22.1 Theorem.** *The homotopy groups  $\pi_n(S^m)$  are finitely generated for all  $n, m \geq 1$ .*

This will follow from a more general result that will be stated in terms of Serre classes.

**22.2 Definition.** A *Serre class* is a non-empty collection  $\mathcal{C}$  of abelian groups satisfying the property that if

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is a short exact sequence of abelian groups then  $B \in \mathcal{C}$  if and only if  $A, C \in \mathcal{C}$

We will say that a Serre class  $\mathcal{C}$  is a *Serre ring* if in addition it satisfies that if  $A, B \in \mathcal{C}$  then  $A \otimes B \in \mathcal{C}$  and  $\text{Tor}(A, B) \in \mathcal{C}$ .

We will also say that a Serre class is *acyclic* if for every group  $A \in \mathcal{C}$  we have  $H_q(K(A, 1)) \in \mathcal{C}$  for all  $q > 0$ .

**22.3 Proposition.** *Let  $\mathcal{C}$  is a Serre class. The following hold:*

- 1)  $0 \in \mathcal{C}$ .
- 2) If  $A \in \mathcal{C}$  and  $A' \cong A$  then  $A' \in \mathcal{C}$ .
- 3) If  $B \subseteq A$  then  $A \in \mathcal{C}$  if and only if  $B, A/B \in \mathcal{C}$ .
- 4) If  $A \rightarrow B \rightarrow C$  is an exact sequence and  $A, C \in \mathcal{C}$  then  $B \in \mathcal{C}$ .
- 5) If  $0 = A_{-1} \subseteq A_1 \subseteq A_2 \subseteq \dots \subseteq A_n$  then  $A_n \in \mathcal{C}$  if and only if  $A_i/A_{i-1} \in \mathcal{C}$  for all  $i$ .

*Proof.* Exercise. □

**22.4 Proposition.** *Let  $\mathcal{C}$  is a Serre ring. If  $X$  is a path connected space such that  $H_q(X) \in \mathcal{C}$  for all  $0 < q < p$  then  $H_p(X; G) \in \mathcal{C}$  for any group  $G \in \mathcal{C}$ .*

*Proof.* By the Universal Coefficient Theorem we have

$$H_p(X; G) \cong (H_p(X) \otimes G) \oplus \text{Tor}(H_{p-1}(X), G)$$

This immediately gives that  $H_p(X; G) \in \mathcal{C}$  for  $p \geq 2$ . For  $p = 0$  we gave  $H_0(X; G) \cong G \in \mathcal{C}$  while for  $p = 1$  we obtain  $H_1(X; G) \cong H_1(X) \otimes G \in \mathcal{C}$ .  $\square$

**22.5 Proposition.** *All of the following are acyclic Serre rings:*

- $\mathcal{C}_{fin}$  = the class of all finite abelian groups.
- $\mathcal{C}_{fg}$  = the class of all finitely generated abelian groups.
- $\mathcal{C}_{tor}$  = the class of all torsion abelian groups.
- $\mathcal{C}_p$  = the class of all  $p$ -torsion abelian groups for a given prime  $p$ .

**22.6 Theorem.** *Let  $F \rightarrow E \xrightarrow{p} B$  be a Serre fibration with a simply connected space  $B$ , and let  $\mathcal{C}$  be a Serre ring. If for two of the spaces  $F, E, B$  the homology groups  $H_q(-)$  are in  $\mathcal{C}$  for all  $q > 0$  then the same holds for the third space.*

*Proof.* There are three cases to consider.

*Case 1:*  $H_q(F), H_q(B) \in \mathcal{C}$  for all  $q > 0$ .

Consider the Serre spectral sequence of the fibration  $p$ . We have  $E_{p,q}^2 = H_p(B, H_q(F))$  so by Proposition 22.4 we get that  $E_{p,q}^2 \in \mathcal{C}$  for all  $(p, q) \neq (0, 0)$ . Next, since groups  $E_{p,q}^3$  are obtained by taking quotients of subgroups of the groups  $E_{p,q}^2$ , we get that  $E_{p,q}^3 \in \mathcal{C}$  for all  $(p, q) \neq (0, 0)$ . Inductively, we obtain that  $E_{p,q}^r \in \mathcal{C}$  for all  $r \geq 2$  and  $(p, q) \neq (0, 0)$ , and so also  $E_{p,q}^\infty \in \mathcal{C}$  for  $(p, q) \neq (0, 0)$ . For  $q > 0$  the groups  $H_q(E)$  admit a finite filtration such filtration quotients are isomorphic to groups  $E_{p,q}^\infty$  with  $(p, q) \neq 0$ . This implies that  $H_q(E) \in \mathcal{C}$ .

*Case 2:*  $H_q(F), H_q(E) \in \mathcal{C}$  for all  $q > 0$ .

Since all groups  $E_{p,q}^\infty$  are quotients of subgroups of  $H_{p+q}(E)$ , we have  $E_{p,q}^\infty \in \mathcal{C}$  for all  $(p, q) \neq (0, 0)$ . We will show that  $E_{p,q}^2 \in \mathcal{C}$  for  $(p, q) \neq (0, 0)$  by induction with respect to  $p$ . For  $p = 0$  this holds since  $E_{0,q}^2 \cong H_q(F)$ . Assume that it also holds for  $E_{i,q}^2$  for all  $i < p$ . It follows that  $E_{i,q}^r$  for all  $i < p$  and all  $r \geq 2$ .

Since all differentials terminating at  $E_{p,0}^r$  are trivial, for each  $r$  we have an exact sequence

$$E_{p,0}^{r+1} \rightarrow E_{p,0}^r \xrightarrow{d^r} E_{p-r,r-1}^r$$

By assumption  $E_{p-r,r-1}^r \in \mathcal{C}$ , so if  $E_{p,0}^{r+1} \in \mathcal{C}$  then the same is true for  $E_{p,0}^r$ . Since  $E_{p,q}^{p+1} = E_{p,q}^\infty \in \mathcal{C}$ , arguing inductively over decreasing values of  $r$  we obtain that  $E_{p,0}^r \in \mathcal{C}$  for all  $r \geq 2$ . In particular,  $H_p(B) = E_{p,0}^2 \in \mathcal{C}$ . Using Proposition 22.4 we obtain that  $E_{p,q}^2 = H_p(B, H_q(F)) \in \mathcal{C}$  for all  $q \geq 0$ .

*Case 3:*  $H_q(B), H_q(E) \in \mathcal{C}$  for all  $q > 0$ .

This is similar to case 2. □

**22.7 Proposition.** *If  $\mathcal{C}$  is an acyclic Serre ring then for every  $A \in \mathcal{C}$  and  $n \geq 1$  we have  $H_q(K(A, n)) \in \mathcal{C}$ .*

*Proof.* We argue by induction with respect to  $n$ . The case  $n = 1$  holds by definition of acyclicity of a Serre class. Assume that the statement is true for some  $n \geq 1$ . For  $A \in \mathcal{C}$  consider the homotopy fibration sequence  $K(A, n) = \Omega K(A, n+1) \rightarrow * \rightarrow K(A, n+1)$ . Since  $H_q(K(A, n)), H_q(*) \in \mathcal{C}$  for all  $q > 0$ , by Theorem 22.6 we obtain that  $H_q(K(A, n+1)) \in \mathcal{C}$ . □

**22.8 Theorem.** *Let  $\mathcal{C}$  be an acyclic Serre ring. If  $X$  is a simply connected space then the following conditions are equivalent:*

- 1)  $\pi_n(X) \in \mathcal{C}$  for all  $n \geq 1$
- 2)  $H_n(X) \in \mathcal{C}$  for all  $n \geq 1$

The proof of Theorem 22.8 will make use of the notion of Postnikov sections:

**22.9 Definition.** Let  $X$  be a path connected space. The  $n$ -th Postnikov section of  $X$  is a space  $X_n$  together with a map  $f: X \rightarrow X_n$  such that

- 1)  $f_*: \pi_q(X) \rightarrow \pi_q(X_n)$  is an isomorphism for  $q \leq n$
- 2)  $\pi_q(X) = 0$  for  $q > n$ .

The  $n$ -th Postnikov section of a space  $X$  can be constructed glueing to  $X$  cells in dimensions  $n+1$  and higher to kill all homotopy groups above  $\pi_n(X)$ . The map  $f: X \rightarrow X_n$  is then given by the inclusion.

*Proof of Theorem 22.8.*

1)  $\Rightarrow$  2) Let  $X_n$  denote the  $n$ -th Postnikov section of  $X$ . By Theorem 16.4 we have  $H_q(X) \cong H_q(X_n)$  for all  $q < n$ , so it will be enough to show that  $H_q(X_n) \in \mathcal{C}$  for all  $n, q > 0$ . We will prove this by induction with respect to  $n$ . For  $n = 2$  we have  $P_2 = K(\pi_2(X), 2)$ , so the statement holds by Proposition 22.7. Assume that it also holds for some  $n \geq 2$ . Notice that we have a fibration sequence

$$K(\pi_{n+1}, n+1) \rightarrow X_{n+1} \rightarrow X_n$$

Using Proposition 22.7 again we obtain that  $H_q(K(\pi_{n+1}, n+1)) \in \mathcal{C}$  for  $q > 0$ , so using Theorem 22.6 we obtain that  $H_q(X_{n+1}) \in \mathcal{C}$  for  $q > 0$ .

2)  $\Rightarrow$  1) We will show that  $\pi_n(X) \in \mathcal{C}$  by induction with respect to  $n$ . Since  $X$  is simply connected, for  $n = 2$  by the Hurewicz Isomorphism Theorem we get  $\pi_2(X) \cong H_2(X) \in \mathcal{C}$ . Next, assume that  $\pi_q(X) \in \mathcal{C}$  for all  $q \leq n$  and consider the fibration sequence

$$\text{hofib } f \rightarrow X \rightarrow X_n$$

where  $X_n$  is the  $n$ -th Postnikov section of  $X$ . Notice that

$$\pi_q(\text{hofib } f) = \begin{cases} 0 & \text{if } q \leq n \\ \pi_q(X) & \text{if } q > n \end{cases}$$

Since  $\pi_q(X_n) \in \mathcal{C}$  for all  $q$ , thus by part 1)  $\Rightarrow$  2) we get that  $H_q(X_n) \in \mathcal{C}$  for all  $q > 0$ . By assumption also  $H_q(X) \in \mathcal{C}$  for  $q > 0$ . Therefore using Theorem 22.6 we obtain that  $H_q(\text{hofib } f) \in \mathcal{C}$  for  $q > 0$ . Since  $\text{hofib } f$  is  $n$ -connected, by the Hurewicz Isomorphism Theorem we obtain  $H_{n+1}(\text{hofib } f) \cong \pi_{n+1}(\text{hofib } f) \cong \pi_{n+1}(X)$ . Therefore  $\pi_{n+1}(X) \in \mathcal{C}$ .

□