9 | Cofibrations

9.1 Definition. A map $i\colon A \to X$ has the *homotopy extension property* for a space Y if for any commutative diagram of the form

$$\begin{tikzcd} Y & X \arrow[l, "\bar{f}"'] \arrow[dl, dashed, "\bar{h}"'] \\ \mathrm{Map}([0,1], Y) \arrow[u, "\mathrm{ev}_0"] & A \arrow[l, "h"] \arrow[u, "i"'] \end{tikzcd}$$

there exists a map $\bar{h}\colon X \times [0,1] \to E$ such that $\bar{h}i = \bar{f}$ and $p\bar{h} = h$. Here $\mathrm{ev}_0\colon \mathrm{Map}([0,1], Y) \to Y$ is the evaluation at 0 map: $\mathrm{ev}_0(\omega) = \omega(0)$.

Equivalently, $i\colon A \to X$ has the homotopy extension property for Y if given any map $\bar{f}\colon X \to Y$ and a homotopy $h^\sharp\colon A \times [0,1] \to Y$ such that $h^\sharp_0 = \bar{f}i$ we can find a homotopy $\bar{h}^\sharp\colon X \times [0,1]$, such that $\bar{h}^\sharp_0 = \bar{f}$ and $\bar{h}^\sharp(i(a), t) = h^\sharp(a, t)$ for all $(a, t) \in A \times [0,1]$.

In this setting we will say that $\bar{h}^\sharp$ is an extension of $h^\sharp$ beginning at $\bar{f}$.

9.2 Definition. A map $i\colon A \to X$ is a *cofibration* if it has the homotopy extension property for any space Y.

9.3 Example. By Theorem 2.14 if (X, A) is a relative CW complex then the inclusion $i\colon A \hookrightarrow X$ is a cofibration.

Recall that the mapping cylinder of a map $f\colon X \to Y$ is the quotient space

$$M_f = (X \times [0,1] \sqcup Y)/\sim$$

where $(x, 0) \sim f(x)$ for all $x \in X$. We have a map $s_f\colon M_f \to Y \times [0,1]$ such that $s_f(x,t) = (f(x), t)$ for

$(x,t) \in X \times [0,1]$ and $f(y) = (y,0)$ for $y \in Y$.

9.4 Proposition. *For a map $i\colon A \to X$ the following conditions are equivalent:*

1) *The map i is a cofibration.*
2) *The map i has the homotopy extension property for the space M_i*
3) *There exists a map $r_f\colon X \times [0,1] \to M_i$ such that $r_f s_f = \mathrm{id}_{M_i}$*

Proof. Exercise. □

9.5 Corollary. *If $i\colon A \to X$ is a cofibration then i is an embedding.*

Proof. Exercise. Use condition 3) in Proposition 9.4. □

9.6 Proposition. *Given any map $f\colon X \to Y$ the map $i_f\colon X \to M_i$ given by $i_f(x) = (x,1)$ is a cofibration.*

Proof. Exercise. □

9.7 Note. Given a map $f\colon X \to Y$, let $d_f\colon M_f \to Y$ be the strong deformation retraction. As a consequence of Proposition 9.6, we have a commutative diagram

$$\begin{tikzcd} Y & & M_f \arrow[ll, "d_f"', "\simeq"] \\ & X \arrow[ul, "f"] \arrow[ur, "i_f"'] & \end{tikzcd}$$

where i_f is a cofibration.