18 | Cohomology via Homotopy

Recall (12.9) that for an abelian group G by $K(G, n)$ we denote the Eilenberg-MacLane space such that $\pi_n(K(G, n)) \cong G$. We will also denote by $K(G, 0)$ the discrete space consisting of elements of the group G. Notice that for every n we have a weak equivalence

$$K(G, n) \xrightarrow{\simeq} \Omega K(G, n+1) \xrightarrow{\simeq} \Omega^2 K(G, n+2)$$

For any pointed CW complex X this induces a bijection of sets of pointed homotopy classes

$$[X, K(G, n)]_* \xrightarrow{\cong} [X, \Omega^2 K(G, n+2)]_*$$

Since $[X, \Omega^2 K(G, n+2)]_*$ has a natural structure of an abelian group (9.8), we obtain in this way an abelian group structure on $[X, K(G, n)]_*$.

The main goal of this chapter is to show that the following holds:

18.1 Theorem. *Let G be an abelian group.*

1) For any pointed CW complex X and $n \geq 0$ there exists an isomorphism

$$T_X \colon [X, K(G, n)]_* \xrightarrow{\cong} \widetilde{H}^n(X; G)$$

where $\widetilde{H}^n(X; G)$ is the n-th reduced singular cohomology group of X with coefficients in G.

2) These isomorphisms are natural. That is, if $f \colon X \to Y$ is a map of pointed CW complexes then the following diagram commutes:

$$\begin{tikzcd}
{[X, K(G, n)]_*} \arrow[d, "T_X"', "\cong"] & {[Y, K(G, n)]_*} \arrow[l, "f^*"'] \arrow[d, "\cong"', "T_Y"] \\
\widetilde{H}^n(X; G) & \widetilde{H}^n(Y; G) \arrow[l, "f^*"]
\end{tikzcd}$$

18.2 Note. Let $\varphi\colon X \to K(G,n)$ be a pointed map. By part 2) of Theorem 18.1 we obtain a commutative diagram

$$\begin{tikzcd}
{[X, K(G,n)]_*} \arrow[d, "T_X"', "\cong"] & {[K(G,n), K(G,n)]_*} \arrow[l, "\varphi^*"'] \arrow[d, "\cong"', "T_{K(G,n)}"] \\
\widetilde{H}^n(X;G) & \widetilde{H}^n(K(G,n);G) \arrow[l, "\varphi^*"]
\end{tikzcd}$$

This gives:

$$T_X([\varphi]) = T_X(\varphi^*([\mathrm{id}_{K(G,n)}])) = \varphi^* T_{K(G,n)}([\mathrm{id}_{K(G,n)}])$$

This implies that for any pointed CW complex X the bijection T_X is determined by the cohomology class $\alpha_n = T_{K(G,n)}([\mathrm{id}_{K(G,n)}]) \in \widetilde{H}^n(K(g,n);G)$. This class is called the *fundamental class*.

18.3 Note. An unpointed version of Theorem 18.1 also holds: for any CW complex X there exists a natural isomorphism $T_X\colon [X, K(G,n)] \xrightarrow{\cong} H^n(X;G)$. This can be derived from Theorem 18.1 as follows. For a CW complex X let X_+ denote the space obtained by adding one 0-cell to X: $X_+ = X \sqcup \{+\}$. We consider $+$ as the basepoint of X_+. We have bijections $[X, K(G,n)] \cong [X_+, K(G,n)]_*$ and $H^n(X;G) \cong \widetilde{H}^n(X_+;G)$. Thus if $[X_+, K(G,n)]_* \cong \widetilde{H}^n(X_+;G)$ then $[X, K(G,n)] \cong H^n(X;G)$.

18.4 Example. For any CW complex X we have:

- $[X, S^1] \cong H^1(X, \mathbb{Z})$
- $[X, \mathbb{CP}^\infty] \cong H^2(X, \mathbb{Z})$
- $[X, \mathbb{RP}^\infty] \cong H^2(X, \mathbb{Z}/2)$

The proof of Theorem 18.1 will proceed as follows. First, we will define a general notion of a cohomology theory, which consists of a sequence of functors $\{h^n\}_{n\in\mathbb{Z}}$ from the category of pointed CW complexes to the category of abelian groups satisfying certain axioms. We will show that both assignments $X \mapsto \widetilde{H}^n(X;G)$ and $X \mapsto [X, K(G,n)]_*$ satisfy this definition. Then we will prove that if $\{h^n\}$ is any generalized cohomology theory such that $h^n(S^0) \cong \widetilde{H}^n(S^0;G)$ for all n, then for every pointed CW complex X and every n there is a natural isomorphism $h^n(X) \to H^n(X;G)$. Since the cohomology theory defined by Eilenberg-MacLane space satisfies this property, Theorem 18.1 will follow.

18.5 Definition. Let $\mathbf{CW}_*$ denote the category of pointed CW complexes and basepoint preserving maps. A *(reduced) cohomology theory* consists of:

- A sequence contravariant functors $\{h^n\colon \mathbf{CW}_* \to \mathbf{Ab}\}_{n\in\mathbb{Z}}$.
- For every $X \in \mathbf{CW}_*$ and every $n \in \mathbb{Z}$ a natural isomorphism $\Sigma\colon h^n(X) \to h^{n+1}(\Sigma X)$. Naturality

means that for any map $f\colon X \to Y$ we have a commutative diagram

$$\begin{tikzcd} h^n(Y) \arrow[r, "\Sigma", "\cong"'] \arrow[d, "f^*"'] & h^{n+1}(\Sigma Y) \arrow[d, "\Sigma f^*"] \\ h^n(X) \arrow[r, "\cong", "\Sigma"'] & h^{n+1}(\Sigma X) \end{tikzcd}$$

Moreover, the following axioms are satisfied:

- (**Homotopy axiom**) If $f, g\colon X \to Y$ are maps such that $f \simeq g$ then $f^* = g^*\colon h^n(Y) \to h^n(X)$ for all n.
- (**Exactness axiom**) For any pair (X, A) where $A \subseteq X$ is a subcomplex, $i\colon A \hookrightarrow X$ is the inclusion and $q\colon X \to X/A$ is the quotient map, the following sequence is exact:

$$h^n(A) \xleftarrow{i^*} h^n(X) \xleftarrow{q^*} h^n(X/A)$$

- (**Wedge axiom**) For any family of pointed CW complexes $\{X_i\}_{i\in I}$ the inclusion maps $X_j \hookrightarrow \bigvee_{i\in I} X_i$ induce isomorphisms $h^n\left(\bigvee_{i\in I} X_i\right) \xrightarrow{\cong} \prod_{i\in I} h^n(X_i)$ for all n.

18.6 Some consequences of the axioms.

- $h^n(*) = 0$ for all n.
- For any pair (X, A) where $A \subseteq X$ is a subcomplex, there is a long exact sequence

$$\ldots \longleftarrow h^n(A) \xleftarrow{i^*} h^n(X) \xleftarrow{q^*} h^n(X/A) \xleftarrow{\delta} h^{n-1}(A) \longleftarrow \ldots$$

The map $\delta\colon h^{n-1}(A) \to h^n(X/A)$ is the composition of the suspension isomorphism $\Sigma\colon h^{n-1}(A) \to h^n(\Sigma A)$, the homomorphism induced by the quotient map $C_i \to C_i/X \cong \Sigma A$, where C_i is the cone of the inclusion $i\colon A \hookrightarrow X$, and the isomorphism induced by the homotopy equivalence $X/A \xrightarrow{\simeq} C_i$.

18.7 Example. Given an abelian group G, consider the reduced singular cohomology functors $X \mapsto \widetilde{H}^n(X; G)$. For $n < 0$ set $\widetilde{H}^n(X; G) = 0$ for all X. Then the functors $\{\widetilde{H}^n(-; G)\}$ define a cohomology theory.

18.8 Example. For an abelian group G, let $h^n_G(X) = [X, K(G, n)]_*$. For $n < 0$ we set $K(G, n) = *$. Then the functors $\{h^n_G\}$ form a cohomology theory. To define the suspension isomorphism

$$\Sigma\colon h^n_G(X) = [X, K(G, n)]_* \longrightarrow [\Sigma X, K(G, n+1)]_* = h^{n+1}_G(\Sigma X)$$

choose a week equivalence $\varphi_n\colon K(G, n) \to \Omega K(G, n{+}1)$. This induces an isomorphism $\varphi_n^*\colon [X, K(G, n)]_* \to [X, \Omega K(G+1, n)]_*$. Then we compose it with the adjunction isomorphism $[X, \Omega K(G+1, n)]_* \xrightarrow{\cong} [\Sigma X, K(G+1, n)]_*$

It is obvious that $\{h^n_G\}$ satisfies the homotopy axiom. The exactness axiom is also satisfied by Proposition 10.12. The wedge axiom holds since for any family of well-pointed spaces $\{X_i\}_{i\in I}$ and any pointed space Z, inclusion maps induce a bijection $[\bigvee_{i\in I} X_i, Z]_* \to \prod_{i\in I}[X_i, Z]_*$.

Notice that the only property of the spaces $K(G, n)$ used in Example 18.8 is that for each n there exist a weak homotopy equivalence $\varphi_n\colon K(G, n) \to \Omega K(G, n+1)$. This motivates the following definition.

18.9 Definition. An *Ω-spectrum* $(K_n, \varphi_n)_{n\in\mathbb{Z}}$ is a sequence of pointed spaces K_n and weak homotopy equivalences $\varphi_n\colon K_n \xrightarrow{\cong} \Omega K_{n+1}$.

By the same argument as in Example 18.8 we obtain:

18.10 Proposition. *Every Ω-spectrum $(K_n, \varphi_n)_{n\in\mathbb{Z}}$ defines a cohomology theory $\{h^n\}_{n\in\mathbb{Z}}$ given by $h^n(X) = [X, K_n]_*$.*

18.11 Definition. A cohomology theory $\{h^n\}$ satisfies the *dimension axiom* if $h^n(S^0) = 0$ for $n \neq 0$.

18.12 Theorem. *Let $\{h^n_1\}_{n\in\mathbb{Z}}$ and $\{h^n_2\}_{n\in\mathbb{Z}}$ be cohomology theories that satisfy the dimension axiom and such that $h^0_1(S^0) \cong h^0_2(S^0)$. Then for each pointed CW complex there exists natural isomorphism $T_X\colon h^n_1(X) \xrightarrow{\cong} h^n_2(X)$. Naturality means that each pointed map $f\colon X \to Y$ gives a commutative diagram*

$$\begin{tikzcd}
h^*_1(Y) \arrow[r, "f^*"] \arrow[d, "T_Y"', "\cong"] & h^*_1(X) \arrow[d, "T_X", "\cong"'] \\
h^*_2(Y) \arrow[r, "f^*"'] & h^*_2(X)
\end{tikzcd}$$

Proof of Theorem 18.1. For the reduced singular cohomology theory we have

$$\widetilde{H}^n(S^0; G) \cong \begin{cases} G & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Also,

$$[S^0, K(G, n)]_* \cong \pi_0(K(G, n)) \cong \begin{cases} G & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Therefore we can apply Theorem 18.12. □

The proof of Theorem 18.12 will require some preparation.

18.13 Lemma. *Let $\{h^n\}$ be a cohomology theory satisfying the dimension axiom. Then:*

1) $h^q(\bigvee_{i\in I} S^n) = 0$ *for* $q \neq n$.

2) For any CW complex X *the inclusion of the* n*-th skeleton* $j\colon X^{(n)} \hookrightarrow X$ *induces an isomorphism* $j^*\colon h^q(X) \xrightarrow{\cong} h^q(X^{(n)})$ *for all* $q < n$ *Also,* $h^q(X^{(n)}) = 0$ *for* $q > n$.

Proof. 1) This follows from the isomorphisms

$$h^q(\bigvee_{i\in I} S^n) \cong \prod_{i\in I} h^q(S^n) \cong \prod_{i\in I} h^q(\Sigma^n S^0) \cong \prod_{i\in I} h^{q-n}(S^0)$$

2) For finite-dimensional CW complexes this can be proved by induction on skeleta of X, using cofibration sequences $X^{(k-1)} \hookrightarrow X^{(k)} \to \bigvee S^k$. This can be generalized to the case $\dim X = \infty$ using the infinite telescope construction (see e.g. Hatcher, *Algebraic Topology* pp. 138–139), which gives a cofibration sequence $\bigvee_k X^{(k)} \to X \to \bigvee_k \Sigma X^{(k)}$. $\square$

For abelian groups G, H let $\mathrm{Hom}(G, H)$ denote the set of homomorphisms $G \to H$. This set has a group structure with addition defined by $(\varphi + \psi)(g) = \varphi(g) + \psi(g)$ for $\varphi, \psi \in \mathrm{Hom}(G, H)$.

18.14 Proposition. *Let* $\{h^n\}$ *be a cohomology theory. For a pointed CW complex* X *and* $n \geq 1$ *consider the map*

$$\Phi\colon \pi_n(X) \to \mathrm{Hom}(h^q(X), h^q(S^n))$$

that sends an element $[\varphi\colon S^n \to X] \in \pi_n(X)$ *to the induced homomorphism* $\varphi^*\colon h^q(X) \to h^q(S^n)$*. Then the map* Φ *is a homomorphism of groups.*

Proof. The constant map $S^n \to X$ induces the trivial homomorphism $h^q(X) \to h^q(S^n)$, so Φ preserves trivial elements. Let $[\varphi], [\psi] \in \pi_n(X)$. The element $[\varphi]\cdot[\psi] \in \pi_n(X)$ is the homotopy class of the map

$$S^n \xrightarrow{p} S^n \vee S^n \xrightarrow{\varphi\vee\psi} S^n$$

where p is the pinch map. We need to show that $p^*(\varphi \vee \psi)^* = \varphi^* + \psi^*\colon h^q(X) \to h^q(S^n)$. This follows from commutativity of the following diagram:

$$\begin{tikzcd}
h^q(X) \arrow[r, "(\varphi\vee\psi)^*"] \arrow[dr, bend right, "\varphi^*\times\psi^*"'] & h^q(S^n \vee S^n) \arrow[r, "p^*"] \arrow[d, "\cong"] & h^q(S^n) \\
& h^q(S^n) \times h^q(S^n) \arrow[ur, bend right, "\mu"'] &
\end{tikzcd}$$

Here the isomorphism $h^q(S^n \vee S^n) \to h^q(S^n) \times h^q(S^n)$ is induced by the inclusion maps and μ is given by $\mu(x, y) = x + y$. $\square$

18.15 Corollary. *Let $\{h_1^n\}$, $\{h_2^n\}$ be cohomology theories and let $T\colon h_1^q(S^n) \to h_2^q(S^n)$ be an arbitrary homomorphism. Then for any map $f\colon S^n \to S^n$ the following diagram commutes:*

$$\begin{tikzcd} h_1^q(S^n) \arrow[r, "f^*"] \arrow[d, "T"'] & h_1^q(S^n) \arrow[d, "T"] \\ h_2^q(S^n) \arrow[r, "f^*"'] & h_2^q(S^n) \end{tikzcd} \tag{$*$}$$

Proof. Using Proposition 18.14 we obtain that homotopy classes of maps f for which the diagram $(*)$ commutes form a subgroup of $\pi_n(S^n)$. Since the homotopy class of the identity map $\mathrm{id}_{S^n}\colon S^n \to S^n$ belongs to this subgroup, the subgroup contains all elements of $\pi_n(S^n)$. $\square$

Let $\{h^n\}$ be a cohomology theory and let X be a CW complex. For $n \geq 0$ consider the map

$$\varphi_n\colon X^{(n+1)}/X^{(n)} \to \Sigma X^{(n)} \to \Sigma(X^{(n)}/X^{(n-1)})$$

Let $d^n\colon h^n(X^{(n)}/X^{(n-1)}) \to h^{n+1}(X^{(n+1)}/X^{(n)})$ be a homomorphism given by the composition

$$d^n\colon h^n(X^{(n)}/X^{(n-1)}) \xrightarrow{\Sigma} h^{n+1}(\Sigma(X^{(n)}/X^{(n-1)})) \xrightarrow{\varphi_n^*} h^{n+1}(X^{(n+1)}/X^{(n)})$$

18.16 Proposition. *Let $\{h^n\}$ be a cohomology theory. For a CW complex X consider the maps*

$$h^{n-1}(X^{(n-1)}/X^{(n-2)}) \xrightarrow{d^{n-1}} h^n(X^{(n)}/X^{(n-1)}) \xrightarrow{d^n} h^{n+1}(X^{(n+1)}/X^{(n)})$$

Then $\mathrm{Im}(d^{n-1}) \subseteq \mathrm{Ker}(d^n)$. Moreover, of $\{h^n\}$ satisfies the dimension axiom then $h^n(X) \cong h^n(X^{(n+1)}) \cong \mathrm{Ker}(d^n)/\mathrm{Im}(d^{n-1})$.

Proof. Exercise. Use Lemma 18.13 and long exact sequences for the pairs $(X^{(n+1)}, X^{(n)})$, $(X^{(n)}, X^{(n-1)})$ and $(X^{(n-1)}, X^{(n-2)})$. $\square$

Proof of Theorem 18.12. We will construct natural isomorphisms $T_X\colon h_1^*(X) \to h_2^*(X)$ in a few steps.

1) We define $T_{S^n}\colon h_1^*(S^n) \to h_2^*(S^n)$ by induction with respect to n. By assumption we have isomorphisms $T_{S^0}\colon h_1^*(S^0) \to h_2^*(S^0)$. Assume that T_{S^n} is already defined for some n. Choose a homeomorphism $f_{n+1}\colon S^{n+1} \to \Sigma S^n$ and define $T_{S^{n+1}}$ so that the following diagram commutes:

$$\begin{tikzcd} h_1^*(S^n) \arrow[r, "f_n^*\Sigma", "\cong"'] \arrow[d, "T_{S^n}"', "\cong"] & h_1^*(S^{n+1}) \arrow[d, "T_{S^{n+1}}"] \\ h_2^*(S^n) \arrow[r, "\cong", "f_n^*\Sigma"'] & h_2^*(S^{n+1}) \end{tikzcd}$$

2) By the definition of a cohomology theory, for any set J inclusion maps $S^n \to \bigvee_{j\in J} S^n$ induce isomorphisms $h_i^*(\bigvee_{j\in J} S^n) \xrightarrow{\cong} \prod_{j\in J} h_i^*(S^n)$ Choose isomorphisms $T_{\bigvee_{j\in J} S^n}$ so that the following diagram commutes:

$$\begin{tikzcd}
h_1^*(\bigvee_{j\in J} S^n) \arrow[r, "\cong"'] \arrow[d, "T_{\bigvee_{j\in J} S^n}"', "\cong"] & \prod_{j\in J} h_1^*(S^n) \arrow[d, "\cong"', "\prod_{j\in J} T_{S^n}"] \\
h_2^*(\bigvee_{j\in J} S^n) \arrow[r, "\cong"] & \prod_{j\in J} h_2^*(S^n)
\end{tikzcd}$$

We claim that isomorphisms $T_{\bigvee_{j\in J} S^n}$ defined above are natural with respect to all maps $f\colon \bigvee_{j\in J} S^n \to \bigvee_{k\in K} S^n$. That is, for any such map the following diagram commutes:

$$\begin{tikzcd}
h_1^*(\bigvee_{k\in K} S^n) \arrow[r, "f^*"] \arrow[d, "T_{\bigvee_{k\in K} S^n}"', "\cong"] & h_1^*(\bigvee_{j\in J} S^n) \arrow[d, "\cong"', "T_{\bigvee_{j\in J} S^n}"] \\
h_2^*(\bigvee_{k\in K} S^n) \arrow[r, "f^*"'] & h_2^*(\bigvee_{j\in J} S^n)
\end{tikzcd}$$

Using the isomorphisms $h_i^*(\bigvee_{j\in J} S^n) \cong \prod_{j\in J} h_i^*(S^n)$ and compactness of spheres, this can be reduced (exercise) to checking that for any pointed map $f\colon S^n \to S^n$ we have $f^*T_{S^n} = T_{S^n}f_*$. This, however, follows from Corollary 18.15.

3) There are now two possible ways of obtaining an isomorphism $h_1^*(\Sigma\bigvee_{i\in I} S^n) \to h_2^*(\Sigma\bigvee_{i\in I} S^n)$. One is to use the suspension isomorphisms $\Sigma\colon h_k^*(\bigvee_{i\in I} S^n) \to h_k^{*+1}(\Sigma\bigvee_{i\in I} S^n)$ and the already defined isomorphism $T_{\bigvee_{i\in I} S^n}$. Another is to use the homeomorphism

$$\bigvee_{i\in I} S^{n+1} \xrightarrow{\bigvee f_{n+1}} \bigvee_{i\in I} \Sigma S^n \xrightarrow{\bigvee \Sigma j_i} \Sigma \bigvee_{i\in I} S^n$$

and the isomorphism $T_{\bigvee_{i\in I} S^{n+1}}$. Here $f_{n+1}\colon S^{n+1} \to \Sigma S^n$ is a homeomorphism and Σj_i is the suspension of the inclusion map $j_i\colon S^n \to \bigvee_{i\in I} S^n$. One can check that both these methods give the same isomorphism $T_{\Sigma\bigvee S^n}\colon h_1^*(\Sigma\bigvee_{i\in I} S^n) \to h_2^*(\Sigma\bigvee_{i\in I} S^n)$. Using naturality of isomorphisms $T_{\bigvee S^{n+1}}$ established in 2), we obtain that for any map $f\colon \bigvee_{j\in J} S^{n+1} \to \Sigma\bigvee_{k\in K} S^n$ we have a commutative diagram

$$\begin{tikzcd}
h_1^{*-1}(\bigvee_{k\in K} S^n) \arrow[r, "\Sigma", "\cong"'] \arrow[d, "T_{\bigvee_{k\in K} S^n}"', "\cong"] & h_1^*(\Sigma\bigvee_{k\in K} S^n) \arrow[r, "f^*"] \arrow[d, "T_{\Sigma\bigvee_{k\in K} S^n}"', "\cong"] & h_1^*(\bigvee_{j\in J} S^{n+1}) \arrow[d, "\cong"', "T_{\bigvee_{j\in J} S^{n+1}}"] \\
h_2^{*-1}(\bigvee_{k\in K} S^n) \arrow[r, "\cong", "\Sigma"'] & h_2^*(\Sigma\bigvee_{k\in K} S^n) \arrow[r, "f^*"'] & h_2^*(\bigvee_{j\in J} S^{n+1})
\end{tikzcd}$$

4) Let now X be an arbitrary pointed CW complex. By Proposition 18.16 for $k = 1, 2$ we have isomorphisms $h_k^n(X) \cong \operatorname{Ker}(d_k^n)/\operatorname{Im}(d_k^{n-1})$ where $d_k^n\colon h_k^n(X^{(n)}/X^{(n-1)}) \to h_k^{n+1}(X^{(n+1)}/X^{(n)})$. If we could

find isomorphisms $T_{X,n}\colon h_1^n(X^{(n)}/X^{(n-1)}) \to h_2^n(X^{(n)}/X^{(n-1)})$ such that $d_2^n T_{X,n} = T_{X,n+1} d_1^n$, then they would induce isomorphisms

$$T_X\colon h_1^n(X) \cong \operatorname{Ker}(d_1^n)/\operatorname{Im}(d_1^{n-1}) \longrightarrow \operatorname{Ker}(d_2^n)/\operatorname{Im}(d_2^{n-1}) \cong h_2^n(X)$$

Such isomorphisms $T_{X,n}$ can be constructed as follows. For each n choose a homeomorphism $f_n\colon \bigvee_{i\in I_n} S^n \to X^{(n)}/X^{(n-1)}$. Then define $T_{X,n}$ so that the following diagram commutes:

$$\begin{tikzcd}
h_1^n(X^{(n)}/X^{(n-1)}) \arrow[r, "f_n^*", "\cong"'] \arrow[d, "T_{X,n}"'] & h_1^n(\bigvee_{i\in I_n} S^n) \arrow[d, "\cong"', "T_{\bigvee_{i\in I_n} S^n}"] \\
h_2^n(X^{(n)}/X^{(n-1)}) \arrow[r, "\cong", "f_n^*"'] & h_2^n(\bigvee_{i\in I_n} S^n)
\end{tikzcd}$$

Commutativity of isomorphisms $T_{X,n}$ with the maps d_k^n follows from commutativity of the following diagram:

$$\begin{tikzcd}
h_1^n(X^{(n)}/X^{(n-1)}) \arrow[r, "\Sigma", "\cong"'] \arrow[d, "f_n^*"', "\cong"] \arrow[rr, bend left, "d_1^n"] \arrow[dddd, bend right, "T_{X,n}"'] & h_1^{n+1}(\Sigma(X^{(n)}/X^{(n-1)})) \arrow[r, "\varphi_n^*"] \arrow[d, "\cong"', "(\Sigma f_n)^*"] & h_1^{n+1}(X^{(n+1)}/X^{(n)}) \arrow[d, "\cong"', "f_{n+1}^*"] \arrow[dddd, bend left, "T_{X,n+1}"] \\
h_1^n(\bigvee_{i\in I_n} S^n) \arrow[r, "\Sigma", "\cong"'] \arrow[d, "T_{\bigvee_{i\in I^n} S^n}"', "\cong"] & h_1^{n+1}(\Sigma \bigvee_{i\in I_n} S^n) \arrow[r, "((\Sigma f_n)^{-1}\varphi_n f_{n+1})^*"] \arrow[d, "T_{\Sigma\bigvee_{i\in I^n} S^n}"', "\cong"] & h_1^{n+1}(\bigvee_{i\in I_{n+1}} S^n) \arrow[d, "\cong"', "T_{\bigvee_{i\in I^{n+1}} S^{n+1}}"] \\
h_2^n(\bigvee_{i\in I_n} S^n) \arrow[r, "\Sigma", "\cong"'] & h_2^{n+1}(\Sigma \bigvee_{i\in I_n} S^n) \arrow[r, "((\Sigma f_n)^{-1}\varphi_n f_{n+1})^*"] & h_2^{n+1}(\bigvee_{i\in I_{n+1}} S^n) \\
h_2^n(X^{(n)}/X^{(n-1)}) \arrow[r, "\Sigma", "\cong"'] \arrow[u, "f_n^*"', "\cong"] \arrow[rr, bend right, "d_2^n"'] & h_2^{n+1}(\Sigma(X^{(n)}/X^{(n-1)})) \arrow[r, "\varphi_n^*"] \arrow[u, "\cong"', "(\Sigma f_n)^*"] & h_2^{n+1}(X^{(n+1)}/X^{(n)}) \arrow[u, "\cong"', "f_{n+1}^*"]
\end{tikzcd}$$

The maps φ_n are defined as in Proposition 18.16. The middle squares commute by 3).

To check that the isomorphisms T_X are natural with respect to maps $f\colon X \to Y$, notice that we can assume that f is cellular and so it induces homomorphisms $f^*\colon h_k^n(Y^{(n)}/Y^{(n-1)}) \to h_k^n(X^{(n)}/X^{(n-1)})$ which commute with the maps d_k^n. Then it remains check that $f^*T_{Y,n} = T_{X,n}f^*$. This can be verified using naturality of the isomorphisms $T_{\bigvee S^n}$ established in 2).

$\square$