15 | Weak Homotopy Type

A complication with studying weak equivalences is that two spaces can be related via a chain of weak equivalences even when there is no direct weak equivalence between them. For example, take $X,Y\subseteq\mathbb{R}$ where X consist of all rational numbers and $Y=\{\frac{1}{n}\mid n=1,2,\dots\}\cup\{0\}$. Since every path connected component of X and Y consists of a single point, $\pi_0(X)$ and $\pi_0(Y)$ are countable sets and all higher homotopy groups are trivial. A weak equivalence $X\to Y$ would need to be a continuous bijection in order to induce a bijection $\pi_0(X)\to\pi_0(Y)$. However, one can check that there is no such continuous bijection. By the same argument, there is no weak equivalence $Y\to X$. On the other hand, if we take the set of integers $\mathbb Z$ with the discrete topology, then any bijections $\mathbb Z\to X$ and $\mathbb Z\to Y$ are continuous functions and they are weak equivalences. Thus the spaces X and Y are related by a chain of weak equivalences:

$$X \leftarrow \mathbb{Z} \rightarrow Y$$

This motivates the following definition:

15.1 Definition. Spaces X and Y are weakly equivalent (or have the same weak homotopy type) if they can be connected by a zigzag of weak equivalences

$$X = Z_0 \rightarrow Z_1 \leftarrow Z_2 \rightarrow \ldots \leftarrow Z_{n-1} \rightarrow Z_n = Y$$

15.2 Proposition. If X, Y are CW complexes then they are weakly equivalent if and only if they are homotopy equivalent.

Proof. Assume that X, Y are connected by a zigzag of n weak equivalences:

$$X = Z_0 \xrightarrow{f_1} Z_1 \xleftarrow{f_2} Z_2 \to \ldots \leftarrow Z_{n-1} \xrightarrow{f_n} Z_n = Y \tag{*}$$

We will show that $X \simeq Y$ by induction with respect to n. If n = 1, then we have a weak equivalence $X = Z_0 \to Z_1 = Y$, which by Theorem 14.4 is a homotopy equivalence.

Assume that the statement is true for any zigzag consisting of n-1 or fewer weak equivalences and that X, Y are connected by a sequence (*). By Corollary 14.8 the map $f_{2*}: [X,Z_2] \to [X,Z_1]$ is a bijection. This means that there exists a map $g: X \to Z_2$ such that $f_2g \simeq f_1$. By Proposition 14.3 the map g is a weak equivalence. Thus we obtain a zigzag of weak equivalences of the form:

$$X \xrightarrow{g} Z_2 \xrightarrow{f_3} Z_3 \leftarrow \ldots \leftarrow Z_{n-1} \xrightarrow{f_n} Z_n = Y$$

By the inductive assumption $X \simeq Y$.

For spaces that are not CW complexes, the study of their weak homotopy type can be simplified using the notion of a CW approximation.

15.3 Definition. A *CW approximation* of a space *X* is a CW complex *Y* together with a weak equivalence $f: Y \to X$.

More generally, a *CW approximation* of a pair (X, A) is a relative CW complex (Y, A) together with a weak equivalence $f: Y \to X$ such that $f|_A = \mathrm{id}_A$.

Notice that a CW approximation of a space X is the same as a CW approximation of the pair (X, \varnothing) . We will show that the following holds:

- **15.4 Theorem.** Any pair (X, A) has a CW approximation. Moreover, any two CW approximations for such a pair are homotopy equivalent.
- **15.5 Corollary.** Spaces X, Y are weakly eqivalent if and only if there exists a space Z and weak equivalences $X \leftarrow Z \rightarrow Y$.

Proof. If such a space Z exists, then by definition X and Y are weakly equivalent. Conversely, assume that we have a zigzag of weak equivalences connecting X and Y:

$$X = Z_0 \xrightarrow{f_1} Z_1 \xleftarrow{f_2} Z_2 \rightarrow \ldots \leftarrow Z_{n-1} \xrightarrow{f_n} Z_n = Y$$

We can extend it to

$$X' \stackrel{g_X}{\rightarrow} X = Z_0 \stackrel{f_1}{\rightarrow} Z_1 \stackrel{f_2}{\leftarrow} Z_2 \rightarrow \ldots \leftarrow Z_{n-1} \stackrel{f_n}{\rightarrow} Z_n = Y \stackrel{g_Y}{\leftarrow} Y'$$

where $g_X \colon X' \to X$ and $g_Y \colon Y' \to Y$ are CW approximations of X and Y, respectively. By Proposition 15.2 there exists a homotopy equivalence $h \colon X' \to Y'$. Thus we obtain a diagram of weak equivalences: $X \stackrel{g_X}{\longleftrightarrow} X' \stackrel{g_Yh}{\longleftrightarrow} Y$.

Proof of Theorem 15.4. Assume first that X is a path connected space. For $n=0,1,\ldots$ we will construct relative CW complexes $(Y^{(n)},A)$ and maps $f^{(n)}\colon Y^{(n)}\to X$ such that

- 1) $Y^{(n)}$ is obtained from $Y^{(n-1)}$ by attaching n-cells.
- 2) $f^{(0)}|_A = \mathrm{id}_A$ and $f^{(n)}|_{Y^{(n-1)}} = f^{(n-1)}$
- 3) $f_*^{(n)} : \pi_i(Y^{(n)}) \to \pi_i(X)$ is an isomorphism for i < n and epimorphism for i = n.

Then the map $\bigcup_n f^{(n)} \colon \bigcup_n Y^{(n)} \to X$ will give a CW approximation of (X, A).

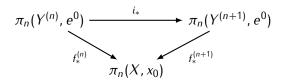
Let $\{A_i\}_{i\in I}$ be path connected components of A. Also, let $x_0 \in X$. For each $i \in I$ choose a point $a_i \in A_i$. Let $(Y^{(1)}, A)$ be a 1-dimensional relative CW complex obtained by:

- adding to A a single 0-cell e^0 ;
- for each $i \in I$ adding to $A \cup e^0$ a 1-cell e_i^1 attached to the points e_0 and a_i .
- for each element $[\tau: (S^1, s_0) \to (X, x_0)] \in \pi_1(X, x_0)$ attaching to the resulting space a circle S^1_{τ} , by identifying s_0 with e^0 .

Since X is path connected, for each $i \in I$ there is a path $\omega_i \colon [0,1] \to X$ such that $\omega_i(0) = x_0$ and $\omega_i(1) = a_i$. Take a map $f^{(1)} \colon Y^{(1)} \to X$ such that $f^{(1)}(x) = x$ for all $x \in A$, $f^{(1)}(e^0) = x_0$. Also, $f^{(1)}$ maps each cell e_i^1 using the path ω_i , and each circle S_{τ}^1 using the map τ . Notice that $f_*^{(1)} \colon \pi_i(Y^{(1)}, e_0) \to \pi_i(X, x_0)$ is a bijection for i = 0 and it is onto for i = 1.

Next, assume that for $i=1,\ldots,n$ we already constructed spaces $Y^{(i)}$ and maps $f^{(i)}\colon Y^{(i)}\to X$ satisfying conditions 1)-3). Take the epimorphism $f_*^{(n)}\colon \pi_n(Y^{(n)},e^0)\to \pi_n(X,x_0)$. Let $\overline{Y}^{(n+1)}$ denote the space obtained by attaching to $Y^{(n)}$ an (n+1)-cell e_ω^{n+1} for each element $[\omega\colon (S^n,s_0)\to (Y^{(n)},e^0)]\in \ker f_*^{(n)}$, using ω as the attaching map. Since $[f^{(n)}\omega]=0$ in $\pi_n(X,x_0)$, the map $f^{(n)}\omega\colon S^n\to X$ can be extended to a map $D^{n+1}\to X$. We can use this to extend $f^{(n)}$ to a map $\overline{f}^{(n+1)}\colon \overline{Y}^{(n+1)}\to X$. Subsequently, take $Y^{(n+1)}$ to be the space obtained by attaching to $\overline{Y}^{(n+1)}$ a sphere $S_\tau^{(n+1)}$ for each $[\tau\colon (S^{n+1},s_0)\to (X,x_0)]\in \pi_{n+1}(X,x_0)$, by identifying s_0 with e^0 . Extend $\overline{f}^{(n+1)}$ to $f^{(n+1)}\colon Y^{(n+1)}\to X$, mapping $S_\tau^{(n+1)}$ using τ .

We have a commutative diagram



where $i: Y^{(n)} \hookrightarrow Y^{(n+1)}$ is the inclusion map. Since $f_*^{(n)}$ is onto, thus so is $f_*^{(n+1)}$. Also, by construction $\ker f^{(n+1)} = 0$. Therefore $f_*^{(n+1)}: \pi_i(Y^{(n+1)}, e^0) \to \pi_i(X, x_0)$ is an isomorphism for $i \leq n$ and it is an epimorphism for i = n+1.

Next, assume that X is not path connected and let $\{X_i\}_{i\in I}$ be path connected components of X. Construct a CW approximation Y_i for each pair $(X_i,A\cap X_i)$, using the procedure described above. Then a CW approximation of (X,A) can be obtained by taking the quotient space $A\sqcup \bigsqcup_{i\in I} Y_i/\sim$, where the relation \sim identifies points of $X_i\cap A\subseteq Y_i$ with the corresponding points of X_i .

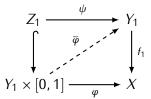
Finally, assume that for i = 1, 2 a map $f_i: (Y_i, A) \to (X, A)$ is a CW approximation of (X, A). This gives

a commutative diagram



By Corollary 14.7 there exists $g: Y_1 \to Y_2$ such that g(x) = x for all $x \in A$ and $f_2g \simeq f_1$ (rel A) By the same argument, there exists $h: Y_2 \to Y_1$ such that h(x) = x for all $x \in A$ and $f_1h \simeq f_2$ (rel A). This shows that there exists a map $\varphi: Y_1 \times [0,1] \to X$ which gives a homotopy $f_1 \simeq f_1hg$ (rel A).

Consider the space $Z_1 = Y_1 \times \{0,1\} \cup A \times [0,1] \subseteq Y_1 \times [0,1]$. Then $(Y_1 \times [0,1], Z_1)$ is a relative CW complex. We have a commutative diagram



where

$$\psi(y, t) = \begin{cases} y & \text{if } t < 1\\ hg(y) & \text{if } t = 1 \end{cases}$$

Using Corollary 14.7 again, we obtain that there exists $\overline{\varphi}\colon Y_1\times [0,1]\to X$, which gives a homotopy $\mathrm{id}_{Y_1}\simeq hg$ (rel A). Analogously, we obtain that $\mathrm{id}_{Y_2}\simeq gh$ (rel A).