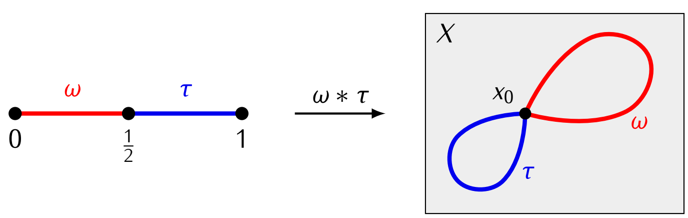
3 | Higher Homotopy Groups

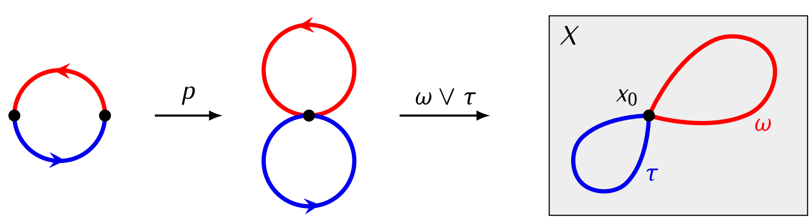
3.1 Notation.

$$I^n = \{(s_1, \dots, s_n) \mid s_i \in [0,1], i = 1, \dots, n\}$$
$$\partial I^n = \{(s_1, \dots, s_n) \in I^n \mid s_i \in \{0,1\} \text{ for some } i\}$$
$$D^n = \{(x_1, \dots x_n) \in \mathbb{R}^n \mid \sum_i x_i \leq 1\}$$
$$S^{n-1} = \{(x_1, \dots x_n) \in \mathbb{R}^n \mid \sum_i x_i = 1\}$$

Recall that the fundamental group $\pi_1(X, x_0)$ of a pointed space (X, x_0) is the group whose elements are homotopy classes of maps $\omega : (I, \partial I) \to (X, x_0)$. Multiplication is given by concatenation of such maps.



Alternatively, $\pi_1(X, x_0)$ can be described as a group whose elements are homotopy classes of maps $\omega \colon (S^1, s_0) \to (X, x_0)$. In this setting, the multiplication in $\pi_1(X, x_0)$ is defined using the pinch map $p \colon S^1 \to S^1 \vee S^1$:



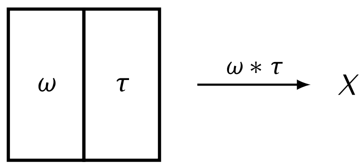
This construction can be generalized to define higher homotopy groups.

3.2 Definition/Proposition. Let (X, x_0) be a pointed space. For $n \geq 1$ the *n-th homotopy group* of (X, x_0) is the group $\pi_n(X, x_0)$ whose elements are homotopy classes of maps $\omega\colon (I^n, \partial I^n) \to (X, x_0)$. Multiplication in $\pi_n(X, x_0)$ is defined as follows. If $\omega, \tau\colon (I^n, \partial I^n) \to (X, x_0)$ then

$$[\omega] \cdot [\tau] = [\omega * \tau]$$

where $\omega * \tau\colon (I^n, \partial I^n) \to (X, x_0)$ is given by

$$(\omega * \tau)(s_1, s_2, \dots, s_n) = \begin{cases} \omega(2s_1, s_2, \dots, s_n) & \text{for } s_1 \in [0, \frac{1}{2}] \\ \tau(2s_1 - 1, s_2, \dots, s_n) & \text{for } s_1 \in [\frac{1}{2}, 1] \end{cases}$$



The trivial element of $\pi_n(X, x_0)$ is the homotopy class of the constant map $c_{x_0}\colon I^n \to X$. Also, for $[\omega] \in \pi_1(X, x_0)$ we have $[\omega]^{-1} = [\overline{\omega}]$ where $\overline{\omega}\colon (I^n, \partial I^n) \to (X, x_0)$ is given by

$$\overline{\omega}(s_1, s_2, \dots, s_n) = (1 - s_1, s_2, \dots, s_n)$$

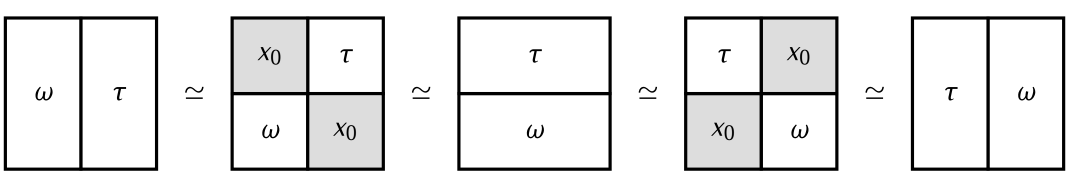
3.3 Note. A part of Definition 3.2 makes sense also for $n = 0$. In this case we have $I^0 = \{*\}$ and $\partial I^0 = \varnothing$. We define $\pi_0(X, x_0)$ as the set of homotopy classes of maps $\omega\colon (I^0, \partial I^0) \to (X, x_0)$. Giving such a map is the same as selecting a point $\omega(*) = x_\omega \in X$. Giving a homotopy of such maps is equivalent to giving a path between the corresponding points. Thus two points x_ω and x_τ represent the same element of $\pi_0(X, x_0)$ if they belong to the same path connected component. In other words, we get

$$\pi_0(X, x_0) \cong \left(\begin{matrix}\text{path connected} \\ \text{components of } X\end{matrix}\right)$$

The trivial element of $\pi_0(X, x_0)$ is given by the map $c_{x_0}\colon I^0 \to X$ such that $c_{x_0}(*) = x_0$. This corresponds to the path connected component of x_0 in X. In this way $\pi_0(X, x_0)$ becomes a pointed set. There is no multiplication defined in $\pi_0(X, x_0)$.

3.4 Theorem. *For $n \geq 2$ then the group $\pi_n(X, x_0)$ is abelian for any pointed space (X, x_0).*

Pictorial proof. A homotopy $\omega * \tau \simeq \tau * \omega$ can be depicted as follows:



The shaded squares in the pictures are mapped to the basepoint $x_0 \in X$. □

A more rigorous proof can be obtained using the following fact.

3.5 Eckmann–Hilton Theorem. *Let M be a set equipped with two binary operations*

$$\circ \colon M \times M \to M, \quad \bullet \colon M \times M \to M$$

Assume that there exist elements $1_\circ, 1_\bullet \in M$ such that $m \circ 1_\circ = 1_\circ \circ = m$ and $m \bullet 1_\bullet = 1_\bullet \bullet = m$ for all $m \in M$. Assume also, that for any $m_1, m_2, n_1, n_2 \in M$ we have

$$(m_1 \circ m_2) \bullet (n_1 \circ n_2) = (m_1 \bullet n_1) \circ (m_2 \bullet n_2)$$

Then for any $m, n \in M$ we have $m \circ n = m \bullet n$, and $m \circ n = n \circ m$.

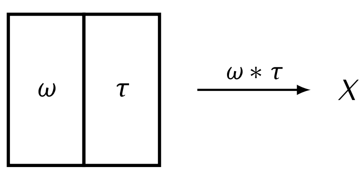
Proof. Exercise. □

Proof of Theorem 3.4. Recall that multiplication in $\pi_n(X, x_0)$ is defined by If $\omega, \tau \colon (I^n, \partial I^n) \to (X, x_0)$ then

$$[\omega] \cdot [\tau] = [\omega * \tau]$$

where $\omega * \tau \colon (I^n, \partial I^n) \to (X, x_0)$ is given by

$$(\omega * \tau)(s_1, s_2, \dots, s_n) = \begin{cases} \omega(2s_1, s_2, \dots, s_n) & \text{for } s_1 \in [0, \frac{1}{2}] \\ \tau(2s_1 - 1, s_2, \dots, s_n) & \text{for } s_1 \in [\frac{1}{2}, 1] \end{cases}$$

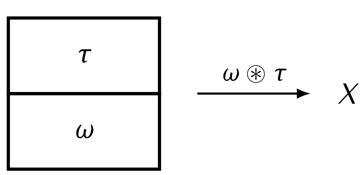


Since $n \geq 2$, we can also define a multiplication in $\pi_n(X, x_0)$ by

$$[\omega] \odot [\tau] = [\omega \circledast \tau]$$

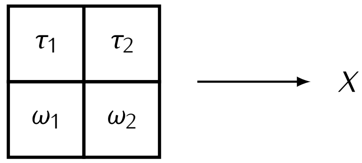
where

$$(\omega \circledast \tau)(s_1, s_2, \dots, s_n) = \begin{cases} \omega(s_1, 2s_2, \dots, s_n) & \text{for } s_2 \in [0, \frac{1}{2}] \\ \tau(s_1, 2s_2 - 1, \dots, s_n) & \text{for } s_2 \in [\frac{1}{2}, 1] \end{cases}$$



Notice that for any $\omega_1, \omega_2, \tau_1, \tau_2 \colon (I^n, \partial I^n) \to (X, x_0)$ we have

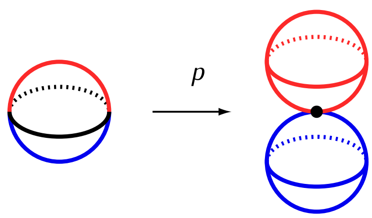
$$(\omega_1 * \omega_2) \circledast (\tau_1 * \tau_2) = (\omega_1 \circledast \tau_1) * (\omega_2 \circledast \tau_2)$$



The result follows from Theorem 3.5. $\square$

3.6 Alternative construction. Just as for the fundamental group, higher homotopy groups can be also described using maps from spheres. Since $I^n/\partial I^n \cong S^n$, giving a map $(I^n, \partial I^n) \to (X, x_0)$ is equivalent to giving a map $(S^n, s_0) \to (X, x_0)$ for some basepoint $s_0 \in S^n$. Thus elements of $\pi_n(X, x_0)$ can be described as homotopy classes of such maps.

To describe multiplication in $\pi_n(X, x_0)$ in this setting, consider the pinch map $p \colon S^n \to S^n \vee S^n$ that maps the upper hemisphere of S^n onto one copy of $S^n \subseteq S^n \vee S^n$, the lower hemisphere onto the second copy, and the equator of S^n to the basepoint of $S^n \vee S^n$:



Given two basepoint preserving maps $\omega, \tau \colon (S^n, s_0) \to (X, x_0)$, let $\omega \vee \tau \colon S^n \vee S^n \to X$ be the function that maps the first copy of S^n using ω and the second copy using τ. Then we have

$$[\omega] \cdot [\tau] = [(\omega \vee \tau) \circ p]$$

The following fact is often useful:

3.7 Proposition. *A map $\omega \colon (S^n, s_0) \to (X, x_0)$ represents the trivial element of $\pi_n(X, x_0)$ if and only if there exists a map $\omega' \colon D^{n+1} \to X$ such that $\omega'|_{S^n} = \omega$.*

Proof. Exercise. $\square$

3.8 Functoriality. Let $f \colon (X, x_0) \to (Y, y_0)$ be a map of pointed spaces. For any $\omega \colon (I^n, \partial I^n) \to (X, x_0)$, composition with f gives a map $f \circ \omega \colon (I^n, \partial I^n) \to (Y, y_0)$. If $\omega, \tau \colon (I^n, \partial I^n) \to (X, x_0)$ and $\omega \simeq \tau$, then $f \circ \omega \simeq f \circ \tau$. Therefore we get a well defined function

$$f_* \colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$$

given by $f_*([\omega]) = [f \circ \omega]$. If $n \geq 0$ then f_* is a homomorphism of groups. For maps $f\colon (X, x_0) \to (Y, y_0)$ and $g\colon (Y, y_0) \to (Z, z_0)$ we have $(gf)_* = g_* f_*$. Also, if $\mathrm{id}_X\colon (X, x_0) \to (X, x_0)$ is the identity map, then $\mathrm{id}_{X*}\colon \pi_n(X, x_0) \to \pi_n(X, x_0)$ is the identity homomorphism. This shows that the assignments $(X, x_0) \to \pi_n(X, x_0)$ define functors:

$$\begin{aligned} \pi_0\colon \mathbf{Top}_* &\to \mathbf{Set}_* \\ \pi_1\colon \mathbf{Top}_* &\to \mathbf{Gr} \\ \pi_n\colon \mathbf{Top}_* &\to \mathbf{Ab} \end{aligned}$$

for $n \geq 2$, where $\mathbf{Top}_*$ is the category of pointed topological spaces, and $\mathbf{Set}_*$, $\mathbf{Gr}$, $\mathbf{Ab}$ are the categories of pointed sets, groups, and abelian groups, respectively.

As a consequence, if $f\colon (X, x_0) \to (Y, y_0)$ is a homeomorphism, then $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is an isomorphism for $n \geq 0$.