

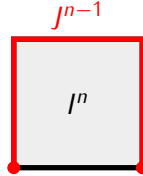
## 6 | Relative Homotopy Groups

**6.1 Notation.** Let  $X \subseteq A_1 \subseteq A_2$  and  $Y \subseteq B_1 \subseteq B_2$ . By a map  $f: (X, A_1, A_2) \rightarrow (Y, B_1, B_2)$  we will understand a map  $f: X \rightarrow Y$  such that  $f(A_i) \subseteq B_i$  for  $i = 1, 2$ . A homotopy of such maps is a homotopy  $h: X \times [0, 1] \rightarrow Y$  such that  $h_t(A_i) \subseteq B_i$  for  $i = 1, 2$  and all  $t \in [0, 1]$ .

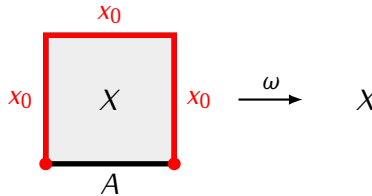
**6.2 Notation.** For  $n \geq 1$  let  $J^{n-1}$  denote the subspace of  $I^n = I^{n-1} \times I$  given by

$$J^{n-1} = I^{n-1} \times \{1\} \cup \partial I^{n-1} \times I$$

We have:  $I^n \subseteq \partial I^n \subseteq J^{n-1}$ .



**6.3 Definition/Proposition.** Let  $x_0 \in A \subseteq X$ . For  $n \geq 2$ , the  $n$ -th relative homotopy group of  $(X, A, x_0)$  is the group  $\pi_n(X, A, x_0)$  whose elements are homotopy classes of maps  $\omega: (I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)$ .

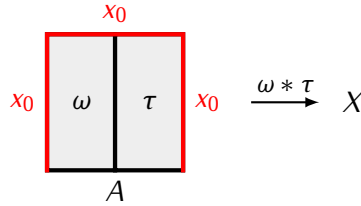


Multiplication in  $\pi_n(X, A, x_0)$  is defined as follows. If  $\omega, \tau: (I^n, \partial I^n) \rightarrow (X, x_0)$  then

$$[\omega] \cdot [\tau] = [\omega * \tau]$$

where  $\omega * \tau: (I^n, \partial I^n) \rightarrow (X, x_0)$  is given by

$$(\omega * \tau)(s_1, s_2, \dots, s_n) = \begin{cases} \omega(2s_1, s_2, \dots, s_n) & \text{for } s_1 \in [0, \frac{1}{2}] \\ \tau(2s_1 - 1, s_2, \dots, s_n) & \text{for } s_1 \in [\frac{1}{2}, 1] \end{cases}$$



The trivial element of  $\pi_n(X, x_0)$  is the homotopy class of the constant map  $c_{x_0}: I^n \rightarrow X$ . Also, for  $[\omega] \in \pi_1(X, x_0)$  we have  $[\omega]^{-1} = [\bar{\omega}]$  where  $\bar{\omega}: (I^n, \partial I^n) \rightarrow (X, x_0)$  is given by

$$\bar{\omega}(s_1, s_2, \dots, s_n) = (1 - s_1, s_2, \dots, s_n)$$

By a similar argument as in the case of absolute homotopy groups (3.4) we obtain:

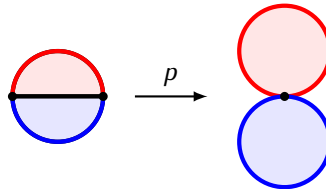
**6.4 Theorem.** *If  $n \geq 3$  then the group  $\pi_n(X, A, x_0)$  is abelian for any pointed pair  $(X, A, x_0)$ .*

**6.5 Note.** A part of Definition 6.3 makes sense also for  $n = 1$ . In this case we have  $\partial I^1 = \{0, 1\}$  and  $J^0 = \{1\}$ . Giving map  $(I^1, \partial I^1, J^0) \rightarrow (X, A, x_0)$  is the same as defining a path in  $X$  that starts at  $x_0$  and ends in  $A$ . Homotopy classes of such paths form the set  $\pi_1(X, A, x_0)$ . In general, this set does not have a group structure, but it has a basepoint defined by the constant path  $c_{x_0}: I^1 \rightarrow X$  such that  $c_{x_0}(I^1) = x_0$ .

**6.6 Proposition.** *For any space  $X$  we have  $\pi_n(X, x_0, x_0) \cong \pi_n(X, x_0)$ .*

**6.7 Proposition.** *For any space  $X$  we have  $\pi_n(X, X, x_0) = 0$ .*

**6.8 Alternative construction.** Just as absolute homotopy groups we can describe in terms of maps from spheres, relative homotopy groups can be constructed using maps from discs. Let  $s_0 \in S^{n-1} \subseteq D^n$ . Elements of  $\pi_n(X, A, x_0)$  can be identified with homotopy classes of maps  $\omega: (D^n, S^{n-1}, s_0) \rightarrow (X, A, x_0)$ . For  $n \geq 2$ , multiplication in  $\pi_n(X, A, x_0)$  is induced by the pinch map  $p: D^n \rightarrow D^n \vee D^n$ , which collapses the equatorial subdisc  $D^{n-1} \subseteq D^n$  into a point.



6.9 For any  $n \geq 1$ , a map  $f: (X, A, x_0) \rightarrow (Y, B, y_0)$  induces a map

$$f_*: \pi_n(X, A, x_0) \rightarrow \pi_n(Y, B, y_0)$$

given by  $f_*([\omega]) = [f \circ \omega]$ . For  $n \geq 2$ , the map  $f_*$  is a homomorphism of groups. In this way we obtain functors

$$\pi_1: \mathbf{Top}_*^2 \rightarrow \mathbf{Set}_*$$

$$\pi_2: \mathbf{Top}_*^2 \rightarrow \mathbf{Gr}$$

$$\pi_n: \mathbf{Top}_*^2 \rightarrow \mathbf{Ab}$$

for  $n \geq 3$ , where  $\mathbf{Top}_*^2$  is the category of pointed pairs  $(X, A, x_0)$  as objects and maps of such pairs as morphisms.

**6.10 Proposition.** *If  $f, g: (X, A, x_0) \rightarrow (Y, B, y_0)$  are maps such that  $f \simeq g$  (as maps of pointed pairs) then  $f_* = g_*: \pi_n(X, A, x_0) \rightarrow \pi_n(Y, B, y_0)$  for all  $n \geq 1$ .*

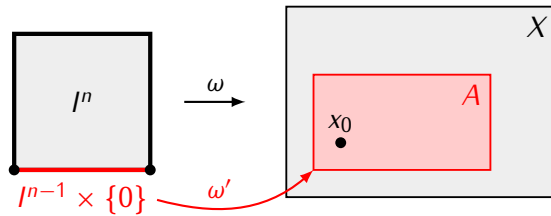
**6.11 Long exact sequence of a pair.** Consider a pointed pair  $(X, A, x_0)$ . The inclusion  $i: (A, x_0) \hookrightarrow (X, x_0)$  induces homomorphisms  $i_*: \pi_n(A, x_0) \rightarrow \pi_n(X, x_0)$  for  $n \geq 0$ . Also, the map of pointed pairs  $j: (X, x_0, x_0) \rightarrow (X, A, x_0)$  induces homomorphisms

$$j_*: \pi_n(X, x_0) = \pi_n(X, x_0, x_0) \rightarrow \pi_n(X, A, x_0)$$

for  $n \geq 1$ . We also have homomorphisms

$$\partial: \pi_n(X, A, x_0) \rightarrow \pi_{n-1}(A, x_0)$$

defined as follows. For  $\omega: (I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)$ , let  $\omega': (I^{n-1}, \partial I^{n-1}) \rightarrow (A, x_0)$  be the restriction of  $\omega$  to  $I^{n-1} \times \{0\}$ . Then  $\partial([\omega]) = [\omega']$ .



**6.12 Theorem.** *For any pointed pair  $(X, A, x_0)$  the following sequence is exact:*

$$\begin{aligned} \dots \longrightarrow \pi_{n+1}(X, A, x_0) \xrightarrow{\partial} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \dots \\ \dots \xrightarrow{j_*} \pi_1(X, A, x_0) \xrightarrow{\partial} \pi_0(A, x_0) \xrightarrow{i_*} \pi_0(X, x_0) \end{aligned}$$

*Proof.* Exercise. □

**6.13 Note.** The end of the exact sequence in Theorem 6.12 consists of maps of pointed sets. Given such maps

$$(S_2, s_2) \xrightarrow{f} (S_1, s_1) \xrightarrow{g} (S_0, s_0)$$

exactness at  $S_1$  means that  $f(S_2) = g^{-1}(s_0)$ .