21 | Serre Spectral Sequence

The Serre spectral sequence is a special case of the spectral sequence associated to a filtration described in Theorem 20.9.

21.1 Definition. Let $p\colon E \to B$ is a Serre fibration where B is a connected CW complex. Let

$$\varnothing = B^{(-1)} \subseteq B^{(0)} \subseteq \dots B$$

be the filtration of B by skeleta. Taking $E^p := p^{-1}(B^{(k)})$ we obtain a filtration of the space E:

$$\varnothing = E^{-1} \subseteq E^0 \subseteq \dots \subseteq E$$

The *Serre spectral sequence* of the fibration p is the spectral sequence associated to this filtration.

By Theorem 20.9 we get that $E^1_{p,q} = H_{p+q}(E^p, E^{p-1})$ and that E^r_{**} converges to $H_*(E)$. The advantage of the Serre spectral sequence is that we can explicitly describe its second page:

21.2 Theorem. *Let E^r_{**} be the Serre spectral sequence of a fibration $p\colon E \to B$. Let $F = p^{-1}(b_0)$ for some $b_0 \in B$. If the space B is simply connected then $E^2_{p,q} \cong H_p(B, H_q(F))$.*

While we will skip the proof of this result, it is useful to point out that the assumption that B is simply connected is needed in order to obtain a canonical identification between fibers of p taken over different points. Assume for a moment p is a Hurewicz fibration and that $b_0, b_1 \in B$, Let $F_i = p^{-1}(b_i)$

for $i = 0, 1$. Given a path $\omega\colon [0,1] \to B$ such that $\omega(0) = b_0$ $\omega(1) = b_1$, consider the diagram

$$\begin{tikzcd} F_0 \times \{0\} \arrow[r, "i_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ F_0 \times [0,1] \arrow[r, "\omega\,\mathrm{pr}"'] \arrow[ur, dashed, "h"] & B \end{tikzcd}$$

where $\mathrm{pr}\colon F_0 \times [0,1] \to [0,1]$ is the projection map and $i_0\colon F_0 \to E$ is the inclusion. A lift h of $\omega\,\mathrm{pr}$ gives a homotopy in E between the map i_0 and a certain map $h_1\colon F_0 \to F_1$. One can show that this map h_1 is a homotopy equivalence and that its homotopy class depends only on the homotopy class of the path ω (relative its endpoints). If the space B is simply connected, all paths joining b_0 and b_1 are homotopic, so the homotopy class of h_1 is uniquely defined. In particular, we obtain canonical isomorphisms of homology groups $h_{1*}\colon H_q(F_0) \xrightarrow{\cong} H_q(F_1)$. If p is a Serre fibration, we can use the same argument, but in order to get the lift h we replace F_0 by its CW approximation.

We have seen already one application of the Serre spectral sequence in Theorem 19.6. Here is another one:

21.3 Proposition. *Let $S^k \to S^m \xrightarrow{p} S^n$ be a homotopy fibration sequence with $n \geq 1$. Then $k = n-1$ and $m = 2n-1$.*

Proof. If $n = 1$ then the long exact sequence of homotopy groups shows that we must have $m = 1$ and $k = 0$. Assume then that $n \geq 2$. Consider the Serre spectral sequence of this fibration. Its second page $E^2_{p,q} \cong H_p(S^n, H_q(S^k))$ has only four non-zero terms, all isomorphic to $\mathbb{Z}$:

$$\begin{array}{c|cc} k & E^2_{0,k} & E^2_{n,k} \\ & & \\ 0 & E^2_{0,0} & E^2_{n,0} \\ \hline & 0 & n \end{array}$$

All differentials originating and terminating at $E^r_{0,0}$ and $E^r_{k,n}$ are trivial, so $E^2_{,0} = E^\infty_{0,0}$ and $E^2_{n,k} = E^\infty_{n,k}$. The page E^∞_{**} can have non-zero terms $E^\infty_{p,q}$ only if $(p,q) = (0,0)$ or $p + q = m$. It follows that $k + n = m$. The terms $E^2_{0,k}$ and $E^2_{n,0}$ must kill each other, so they must be connected by a differential. This is possible only if $k = n - 1$. Taken together these observations imply that p is a fibration sequence of the form $S^{n-1} \to S^{2n-1} \xrightarrow{p} S^n$. □

Hopf bundles give examples of fibration sequences $S^{n-1} \to S^{2n-1} \to S^n$ for $n = 1, 2, 4, 8$. A theorem of Adams implies that these are the only fibration sequences where all three spaces are spheres.