10 | Cofibrations

10.1 Definition. A map $i\colon A \to X$ has the *homotopy extension property* for a space Y if for any commutative diagram of the form

$$\begin{tikzcd}
Y & X \arrow[l, "\bar{f}"'] \arrow[dl, dashed, "\bar{h}"'] \\
\mathrm{Map}([0,1], Y) \arrow[u, "\mathrm{ev}_0"] & A \arrow[l, "h"] \arrow[u, "i"']
\end{tikzcd}$$

there exists a map $\bar{h}\colon X \times [0,1] \to E$ such that $\bar{h}i = \bar{f}$ and $p\bar{h} = h$. Here $\mathrm{ev}_0\colon \mathrm{Map}([0,1], Y) \to Y$ is the evaluation at 0 map: $\mathrm{ev}_0(\omega) = \omega(0)$.

Equivalently, $i\colon A \to X$ has the homotopy extension property for Y if given any map $\bar{f}\colon X \to Y$ and a homotopy $h^\sharp\colon A \times [0,1] \to Y$ such that $h^\sharp_0 = \bar{f}i$ we can find a homotopy $\bar{h}^\sharp\colon X \times [0,1]$, such that $\bar{h}^\sharp_0 = \bar{f}$ and $\bar{h}^\sharp(i(a), t) = h^\sharp(a, t)$ for all $(a, t) \in A \times [0,1]$.

In this setting we will say that $\bar{h}^\sharp$ is an extension of $h^\sharp$ beginning at $\bar{f}$.

10.2 Definition. A map $i\colon A \to X$ is a *cofibration* if it has the homotopy extension property for any space Y. In such case we also say that the space $X/i(A)$ is the *cofiber* of i.

10.3 Example. By Theorem 2.14 if (X, A) is a relative CW complex then the inclusion $i\colon A \hookrightarrow X$ is a cofibration.

Recall that the mapping cylinder of a map $f\colon X \to Y$ is the quotient space

$$M_f = (X \times [0,1] \sqcup Y)/\!\sim$$

where $(x, 0) \sim f(x)$ for all $x \in X$. We have a map $s_f\colon M_f \to Y \times [0,1]$ such that $s_f(x,t) = (f(x), t)$ for

$(x, t) \in X \times [0, 1]$ and $f(y) = (y, 0)$ for $y \in Y$.

10.4 Proposition. *For a map $i: A \to X$ the following conditions are equivalent:*

1) *The map i is a cofibration.*
2) *The map i has the homotopy extension property for the space M_i*
3) *There exists a map $r_f: X \times [0, 1] \to M_i$ such that $r_f s_f = \mathrm{id}_{M_i}$*

Proof. Exercise. □

10.5 Corollary. *If $i: A \to X$ is a cofibration then i is an embedding.*

Proof. Exercise. Use condition 3) in Proposition 10.4. □

10.6 Proposition. *Given any map $f: X \to Y$ the map $i_f: X \to M_f$ given by $i_f(x) = (x, 1)$ is a cofibration.*

Proof. Exercise. □

10.7 Note. Given a map $f: X \to Y$, let $d_f: M_f \to Y$ be the strong deformation retraction. As a consequence of Proposition 10.6, we have a commutative diagram

$$\begin{tikzcd} Y & & M_f \arrow[ll, "d_f"', "\simeq"] \\ & X \arrow[ul, "f"] \arrow[ur, "i_f"'] & \end{tikzcd}$$

where i_f is a cofibration. A homotopy inverse of d_f is given by the inclusion map $j_f: Y \to M_f$.

10.8 Note. Recall that the mapping cone of a map $f: X \to Y$ is the space $C_f = M_f/X \times \{1\}$. The space C_f is the cofiber of the cofibration $i_f: X \to M_f$.

10.9 Coexact Puppe sequence. The construction of the coexact Puppe sequence of a map is dual to the construction of the exact Puppe sequence given in Chapter 9.

As in Chapter 9 we will be interested here in pointed spaces and homotopy classes of maps that preserve basepoints. In this case we will use a slightly weakened version of a cofibration: a map of pointed spaces $i: (A, a_0) \to (X, x_0)$ is a cofibration if has the homotopy extension property for all pointed maps $(X, x_0) \to (Y, y_0)$ and pointed homotopies $A \times [0, 1] \to Y$. In this context we modify the constructions of the mapping mapping cylinder and the mapping cone as follows:

10.10 Definition. For a map of pointed spaces $f: (X, x_0) \to (Y, y_0)$ the *reduced mapping cylinder* of f is the space $\overline{M}_f = M_f/\{x_0\} \times [0, 1]$. The *reduced mapping cone* is the space $\overline{C}_f = \overline{M}_f/X \times \{1\}$.

The reduced mapping cylinder and mapping cone come with a natural choice of basepoints. As in (10.7) for any map $f\colon (X, x_0) \to (Y, y_0)$ we have a commutative diagram

$$\begin{tikzcd} Y & & \overline{M}_f \arrow[ll, "d_f"', "\simeq"] \\ & X \arrow[ul, "f"] \arrow[ur, "i_f"'] & \end{tikzcd}$$

where i_f is a pointed cofibration and d_f is a pointed homotopy equivalence. Also, $\overline{C}_f$ is the cofiber of i_f.

10.11 Definition. A sequence of maps of spaces

$$(X_0, x_0) \xrightarrow{f_0} (X_1, x_1) \xrightarrow{f_1} (X_2, x_2)$$

is *coexact at* X_1 is for any pointed space (Y, y_0) the sequence pointed sets

$$[X_2, Y]_* \xrightarrow{f_1^*} [X_1, Y]_* \xrightarrow{f_0^*} [X_0, Y]_*$$

is exact at $[X_1, Y]_*$.

10.12 Proposition. *If $i\colon A \to X$ is a cofibration, $q\colon X \to X/i(A)$ is the quotient map, $x_0 \in A$ then the sequence $(A, x_0) \xrightarrow{i} (X, i(x_0)) \xrightarrow{q} (X/A, qi(x_0))$ is coexact at X.*

For any map $f\colon (X, x_0) \to (Y, y_0)$ consider the sequence

$$X \xrightarrow{f} Y \xrightarrow{q(f)} \overline{C}_f$$

where $q(f)(y) = (y, 0)$. Since this sequence is homotopy equivalent to the cofibration sequence $X \xrightarrow{i_f} \overline{M}_f \longrightarrow \overline{C}_f$, is is coexact at Y. Continuing this construction inductively we obtain a coexact sequence

$$X \xrightarrow{f} Y \xrightarrow{q(f)} \overline{C}_f \xrightarrow{q^2(f)} \overline{C}_{q(f)} \xrightarrow{q^3(f)} \overline{C}_{q(f)} \xrightarrow{q^4(f)} \overline{C}_{q^3(f)} \longrightarrow \dots \qquad (*)$$

As in Chapter 9 our goal will be to show that this sequence admits a more convenient description. This will depend on two facts that dualize Propositon 9.4 and Corollary 8.18

10.13 Proposition. *For any map $f\colon (X, x_0) \to (Y, y_0)$ the map $q(f)\colon X \to \overline{C}_f$ is a cofibration.*

Proof. Exercise. □

10.14 Proposition. *If $f\colon (X, x_0) \to (Y, y_0)$ is a cofibration then the quotient map*

$$\overline{C}_f \to Y/f(X)$$

is a homotopy equivalence.

Proof. Exercise. □

Notice that $\overline{C}_f/q(f) \cong \Sigma X$, where ΣX is the reduced suspension of X. In this way we obtain:

10.15 Proposition. *For any map of pointed spaces $f\colon (X, x_0) \to (Y, y_0)$ we have a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "q(f)"] & \overline{C}_f \arrow[r, "q^2(f)"] \arrow[dr, "g"'] & \overline{C}_{q(f)} \arrow[d, "\simeq"] \\ & & & \Sigma X \end{tikzcd}$$

Applying Proposition 10.14 iteratively to the sequence (∗) we get homotopy equivalences

$$\begin{aligned} &\overline{C}_{q(f)} \xrightarrow{\simeq} \Sigma X \\ &\overline{C}_{q^2(f)} \xrightarrow{\simeq} \Sigma Y \\ &\overline{C}_{q^3(f)} \xrightarrow{\simeq} \Sigma \overline{C}_f \\ &\overline{C}_{q^4(f)} \xrightarrow{\simeq} \Sigma \overline{C}_{q(f)} \simeq \Sigma^2 X \\ &\overline{C}_{q^5(f)} \xrightarrow{\simeq} \Sigma \overline{C}_{q^2(f)} \simeq \Sigma^2 Y \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \end{aligned}$$

Moreover, one can check that the following diagram commutes up to homotopy:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "="'] & Y \arrow[r, "q(f)"] \arrow[d, "="'] & \overline{C}_f \arrow[r, "q^2(f)"] \arrow[d, "="'] & \overline{C}_{q(f)} \arrow[r, "q^3(f)"] \arrow[d, "\simeq"'] & \overline{C}_{q^2(f)} \arrow[r, "q^4(f)"] \arrow[d, "\simeq"'] & \overline{C}_{q^3(f)} \arrow[r, "q^5(f)"] \arrow[d, "\simeq"'] & \overline{C}_{q^4(f)} \arrow[r] \arrow[d, "\simeq"'] & \dots \\ X \arrow[r, "f"'] & Y \arrow[r, "q(f)"'] & \overline{C}_f \arrow[r, "g"'] & \Sigma X \arrow[r, "\Sigma f"'] & \Sigma Y \arrow[r, "\Sigma q(f)"'] & \Sigma \overline{C}_f \arrow[r, "\Sigma g"'] & \Sigma^2 X \arrow[r] & \dots \end{tikzcd} \tag{∗∗}$$

10.16 Definition. The sequence in the lower row of the diagram (∗∗) is called the *Puppe coexact sequence* associated to the map f.

As a consequence, for any map of pointed spaces $f\colon (X, x_0) \to (Y, y_0)$ and any pointed space (Z, z_0) we obtain a long exact sequence of sets:

$$[X, Z]_* \xleftarrow{f^*} [Y, Z]_* \xleftarrow{q(f)^*} [\overline{C}_f, Z]_* \xleftarrow{g^*} [\Sigma X, Z]_* \xleftarrow{\Sigma f^*} [\Sigma Y, Z]_* \xleftarrow{\Sigma q(f)^*} [\Sigma \overline{C}_f, Z]_* \xleftarrow{\Sigma g^*} [\Sigma^2 X, Z]_* \longleftarrow \dots \tag{✠}$$

Starting with $[\Sigma X, Z]_*$ the sets in this sequence have a group structure defined by the suspension, and all maps are homomorphisms of groups. Starting with $[\Sigma^2, Z]_*$ all groups are abelian.

10.17 Note. 1) Using the adjunction $\mathrm{adj}\colon [\Sigma X, Y]_* \xrightarrow{\cong} [X, \Omega Y]_*$ as in (9.15) we can rewrite the sequence (✠) in the form

$$[X, Z]_* \xleftarrow{f^*} [Y, Z]_* \xleftarrow{q(f)^*} [\overline{C}_f, Z]_* \xleftarrow{g^*} [X, \Omega Z]_* \xleftarrow{f^*} [Y, \Omega Z]_* \xleftarrow{q(f)^*} [\overline{C}_f, \Omega Z]_* \xleftarrow{g^*} [X, \Omega^2 Z]_* \longleftarrow \dots$$

In this setting, groups structures are induced the multiplication in loop spaces.

2) Assume that the map $f\colon (X, x_0) \to (Y, y_0)$ is a cofibration. Using Corollary 10.5 we can then assume that X is a subspace of Y and that f is the inclusion map. By Proposition 10.14 we have $\overline{C}_f \simeq Y/X$, so the above sequence can be written as

$$[X, Z]_* \xleftarrow{f^*} [Y, Z]_* \xleftarrow{q^*} [Y/X, Z]_* \xleftarrow{g^*} [X, \Omega Z]_* \xleftarrow{f^*} [Y, \Omega Z]_* \xleftarrow{q^*} [Y/X, \Omega Z]_* \xleftarrow{g^*} [X, \Omega^2 Z]_* \longleftarrow \dots$$

where $q\colon Y \to Y/X$ is the quotient map.