

22 | Serre classes

22.1 Definition. A *Serre class* is a non-empty collection \mathcal{C} of abelian groups satisfying the property that if

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is a short exact sequence of abelian groups then $B \in \mathcal{C}$ if and only if $A, C \in \mathcal{C}$

We will say that a Serre class \mathcal{C} is a *Serre ring* if it in addition satisfies that if $A, B \in \mathcal{C}$ then $A \otimes B \in \mathcal{C}$ and $\text{Tor}(A, B) \in \mathcal{C}$.

22.2 Proposition. Let \mathcal{C} is a Serre class. The following hold:

- 1) $0 \in \mathcal{C}$.
- 2) If $A \in \mathcal{C}$ and $A' \cong A$ then $A' \in \mathcal{C}$.
- 3) If $B \subseteq A$ then $A \in \mathcal{C}$ if and only if $B, A/B \in \mathcal{C}$.
- 4) If $A_1, \dots, A_n \in \mathcal{C}$ then $A_1 \oplus \dots \oplus A_n \in \mathcal{C}$.
- 5) If $A \rightarrow B \rightarrow C$ is an exact sequence and $A, C \in \mathcal{C}$ then $B \in \mathcal{C}$.

Proof. Exercise. □

22.3 Proposition. Let \mathcal{C} is a Serre ring. If X is a path connected space such that $H_q(X) \in \mathcal{C}$ for all $q > 0$ and $H_q(X; G) \in \mathcal{C}$ for any group $G \in \mathcal{C}$.

Proof. By the Universal Coefficient Theorem we have

$$H_q(X; G) \cong (H_q(X) \otimes G) \oplus \text{Tor}(H_{q-1}(X), G)$$

For $q = 0$ this gives $H_0(X; G) \cong G$. For $q = 1$ we get $H_1(X; G) \cong H_1(X) \otimes G$. □

22.4 Example. All of the following are Serre rings:

- \mathcal{C}_{fin} = the class of all finite abelian groups.

- \mathcal{C}_{fg} = the class of all finitely generated abelian groups.
- \mathcal{C}_{tor} = the class of all torsion abelian groups.

22.5 Definition. Let \mathcal{C} be a Serre class. We say that homomorphism of abelian groups $f: G \rightarrow H$ is an *isomorphism mod \mathcal{C}* if $\text{Ker } f \in \mathcal{C}$ and $\text{Coker } f := H/f(G) \in \mathcal{C}$

22.6 Theorem. Let \mathcal{C} be a Serre ring. If X is a simply connected space then the following conditions are equivalent:

- 1) $\pi_n(X) \in \mathcal{C}$ for all $n \geq 1$
- 2) $H_n(X) \in \mathcal{C}$ for all $n \geq 1$

22.7 Corollary. The homotopy groups $\pi_n(S^m)$ are finitely generated for all $n, m \geq 1$.