13 | Proof of the Excision Theorem

Based on Tammo tom Dieck, *Algebraic Topology* sec. 6.9.

The goal of this section is to give a proof a the Excision Theorem. For reference, we bring up again its statement:

11.4 **Excision Theorem.** *Let X be a space and $X_1, X_2 \subseteq X$ be open such that $X = X_1 \cup X_2$. Assume that*

- $(X_1, X_1 \cap X_2)$ *is m-connected*
- $(X_2, X_1 \cap X_2)$ *is n-connected*

for some $m, n \geq 0$. Then for any $x_0 \in X_1 \cap X_2$ the homomorphism

$$i_* \colon \pi_k(X_1, X_1 \cap X_2, x_0) \to \pi_k(X, X_2, x_0)$$

induced by the inclusion map, is an isomorphism for $1 \leq k < m + n$ and it is onto for $k = m + n$.

13.1 Cubical subdivisions. The proof of Theorem 11.4 will involve working with certain subdivisions of cubes I^n. Here we set some terminology and notation related to such subdivisions.

Let $N \geq 1$ be some fixed integer. For $j = 0, \dots, N$ denote $c_j = \frac{j}{N}$. Also, let $\delta = \frac{1}{N}$. The numbers c_j define a subdivision of the interval $I = [0, 1]$ into subintervals $[c_j, c_{j+1}] = [c_j, c_j + \delta]$. More generally, an n-dimensional cube I^n has a subdivision into subcubes of the form

$$\begin{aligned} C_{j_1, \dots, j_n} &= [c_{j_1}, c_{j_1+1}] \times [c_{j_2}, c_{j_j+1}] \\ &= [c_{j_1}, c_{j_1} + \delta] \times [c_{j_2}, c_{j_2} + \delta] \times \dots \times [c_{j_n}, c_{j_n} + \delta] \end{aligned}$$

for some $0 \leq j_1, \dots, j_n \leq N - 1$. We will call this the N-cubical subdivision of I^n. This subdivision defines a CW complex structure on I^n. An m-dimensional cell in I^n is an m-dimensional subcube

$$C = [c_{j_1}, c_{j_1} + \epsilon_1] \times [c_{j_2}, c_{j_2} + \epsilon_2] \times \ldots \times [c_{j_n}, c_{j_n} + \epsilon_n]$$

where $\epsilon_i = \delta$ for m values of the index i and $\epsilon_i = 0$ otherwise. We will denote by $I^n(m)$ the m-skeleton of I^n with this cell structure.

Let $C_{j_1,\ldots,j_n}$ be an n-dimensional subcube:

$$C_{j_1,\ldots,j_n} = \{(t_1, \ldots, t_n) \in I^n \mid c_{j_i} \leq t_i \leq c_{j_i} + \delta\}$$

For $0 \leq p \leq N$ we will denote by $S_p C_{j_1,\ldots,j_n}$ and $L_p C_{j_1,\ldots,j_n}$ the subspaces of $C_{j_1,\ldots,j_n}$ given by

$$S_p C_{j_1,\ldots,j_n} = \{(t_1, \ldots, t_n) \in C_{j_1,\ldots,j_n} \mid c_{j_i} < t_i < c_{j_i} + \tfrac{\delta}{2} \text{ for at least } p \text{ coordinates } t_i\}$$
$$L_p C_{j_1,\ldots,j_n} = \{(t_1, \ldots, t_n) \in C_{j_1,\ldots,j_n} \mid c_{j_i} + \tfrac{\delta}{2} < t_i < c_k + \delta \text{ for at least } p \text{ coordinates } t_i\}$$

Also, denote

$$S_p = \bigcup_{j_1,\ldots,j_n} S_p C_{j_1,\ldots,j_n} \qquad L_p = \bigcup_{j_1,\ldots,j_n} L_p C_{j_1,\ldots,j_n}$$

13.2 Lemma. *Consider I^n with the N-cubical subdivision for some $N > 0$. Assume that $A, B \subseteq I^n$ are closed, disjoint sets, such that $A \cap I^n(p) = \varnothing$ for some $p \leq n$. There exists a homotopy $\Phi\colon I^n \times [0,1] \to I^n$ satisfying the following conditions:*

(i) $\Phi(C \times [0,1]) \subseteq C$ *for each subcube (of any dimension) in* I^n*.*

(ii) $\Phi_0 = \mathrm{id}_{I^n}$*.*

(iii) $\Phi_1^{-1}(A) \subseteq S_{p+1}$ *and* $\Phi_1^{-1}(B) = B$*.*

Also, there exists a homotopy $\Psi\colon I^n \times [0,1] \to I^n$ that satisfies (i) and (ii) and

(iii') $\Psi_1^{-1}(A) \subseteq L_{p+1}$ *and* $\Psi_1^{-1}(B) = B$*.*

Proof. Let $\varphi\colon [0,1] \times [0,1] \to [0,1]$ be a homotopy defined as follows:

$$\varphi(t, s) = (1 - s)t + s \cdot \min(c_j + \delta, 2t - c_j)$$

for $t \in [c_j, c_j + \delta]$. This is a homotopy between the identity map on $[0,1]$ and a map that on each subinterval $[c_j, c_j + \delta]$ sends $[c_j + \frac{\delta}{2}, c_j + \delta]$ to the point $c_j + \delta$ and stretches $[c_j, c_j + \frac{\delta}{2}]$ linearly to $[c_j, c_j + \delta]$. Define $\widetilde{\Phi}\colon I^n \times [0,1] \to I^n$ by

$$\widetilde{\Phi}((t_1, \ldots, t_n), s) = (\varphi(t_1, s), \ldots, \varphi(t_n, s))$$

The homotopy $\widetilde{\Phi}$ satisfies conditions (i) and (ii). Moreover, $\widetilde{\Phi}_1(t_1, \ldots, t_n) \notin I^n(p)$ if and only if $(t_1, \ldots, t_n) \in S_{p+1}$. Since $A \cap I^n(p) = \varnothing$ this gives $\widetilde{\Phi}_1^{-1}(A) \subseteq S_{p+1}$. Let $\varrho\colon I^n \to [0,1]$ be a function such that $\varrho(A) = 1$ and $\varrho(B) = 0$. Define $\Phi\colon I^n \times [0,1] \to I^n$ by

$$\Phi(x, s) = \widetilde{\Phi}(x, s\varrho(x))$$

Then $\Phi_1^{-1}(A) = \widetilde{\Phi}_1^{-1}(A) \subseteq S_p$ and $\Phi_1^{-1}(B) = \widetilde{\Phi}_0^{-1}(B) = B$

The homotopy Ψ can be obtained analogously.

$\square$

13.3 Corollary. *Consider the cube I^n with the N-cubical subdivision for some $N \geq 1$. Assume that $A, B \subseteq I^n$ are closed, disjoint sets, such that $A \cap I^n(p) = \varnothing$ and $B \cap I^n(q) = \varnothing$ for some $p, q \leq n$. There exists a homotopy $\Lambda\colon I^n \times [0,1] \to I^n$ satisfying the following conditions:*

(i) $\Lambda(C \times [0,1]) \subseteq C$ *for each subcube (of any dimension) in* I^n.
(ii) $\Lambda_0 = \mathrm{id}_{I^n}$.
(iii) $\Lambda_1^{-1}(A) \subseteq S_{p+1}$ *and* $\Lambda_1^{-1}(B) \subseteq L_{q+1}$.

Proof. Take a homotopy Φ as in Lemma 13.2. Using the same lemma with $A = \Phi_1^{-1}(A)$ and $B = \Phi_1^{-1}(B) = B$ we obtain a homotopy Ψ that satisfies (i), (ii) and $\Psi_1^{-1}(\Phi_1^{-1}(A)) = \Phi_1^{-1}(A) \subseteq S_{p+1}$ and $\Psi_1^{-1}(\Phi_1^{-1}(B)) = \Psi_1^{-1}(B) \subseteq L_{q+1}$. The homotopy Λ can be then defined by

$$\Lambda(x,s) = \begin{cases} \Phi(x, 2s) & \text{for } s \leq \frac{1}{2} \\ \Psi(\Phi(x,1), 2s) & \text{for } s \geq \frac{1}{2} \end{cases}$$

$\square$

Proof of Theorem 11.4. Denote $X_0 = X_1 \cap X_2$. We will first show that the homomorphism

$$i_*\colon \pi_k(X_1, X_0, x_0) \to \pi_k(X, X_2, x_0)$$

is onto for $k \leq m+n$.

Assume then $k \leq m+n$ and let $\omega\colon I^k \to X$ be a map representing an element of $\pi_k(X, X_2, x_0)$. We have $\omega(I^{k-1} \times \{0\}) \subseteq X_2$ and $\omega((\partial I^k \times I) \cup (I^{k-1} \times \{1\})) = x_0$. We need to show that ω is homotopic through such maps to $\tau\colon I^k \to X$ such that $\tau(I^k) \subseteq X_1$ and $\tau(I^{k-1} \times \{0\}) \subseteq X_0$.

Consider I^k with a N-cubical subdivision such that for each subcube $C \subseteq I^k$ we have either $\omega(C) \subseteq X_1$ or $\omega(C) \subseteq X_2$. We claim that there exists a homotopy $h\colon \omega \simeq \omega_1$ such that

1) if $\omega(C) \subseteq X_0$ then $h(x,t) = \omega(x)$ for $(x,t) \in C \times [0,1]$
2) if $\omega(C) \subseteq X_i$ for $i = 1,2$ then $h(C \times [0,1]) \subseteq X_i$.
3) $\omega_1^{-1}(X_1 \smallsetminus X_0) \cap I^k(m) = \varnothing$
4) $\omega_1^{-1}(X_2 \smallsetminus X_0) \cap I^k(n) = \varnothing$.

The homotopy h can be constructed by induction with respect to skeleta of I^k. Let C^0 be a 0-dimensional subcube of I^k. If $\omega(C^0) \in X_0$ take $h|_{C^0 \times [0,1]}$ to be the constant map to the point $\omega(C^0)$ if $C^0 \in X_i \smallsetminus X_0$ for $i = 1,2$ take $h|_{C^0 \times [0,1]}$ to be a path in X_i that joins $\omega(C^0)$ with a point in X_0. Such path exists by the connectivity assumption on the pair (X_i, X_0). In effect we obtain a homotopy

$h\colon I^k(0)\times[0,1]\to X$ satisfying 1)–4). For the inductive step, assume that we already constructed a homotopy $h\colon I^k(r)\times[0,1]\to X$ for some $r\geq 0$, and let C^{r+1} be an $(r+1)$-dimensional cube. The homotopy h is already defined on ∂C^{r+1}. If $\omega(C^{r+1})\subseteq X_0$, we extend h to C^{r+1} using condition 1). If $\omega(C^{r+1})\subseteq X_1$ and $r+1\leq m$ then we can extend h to a homotopy $h\colon C^{r+1}\times[0,1]\to X_1$ such that $h_1(C^{r+1})\subseteq X^0$ by Proposition 11.3. We proceed analogously if $\omega(C^{r+1})\subseteq X_2$ and $r+1\leq k$. In all other cases we extend h to C^{r+1} in an arbitary way that satisfies condition 2).

To check that the resulting map $\omega_1=h_1\colon I^k\to X$ satisfies condition 3), let $C^r\subseteq I^k(m)$ be an r-dimensional subcube for some $r\leq m$. If $\omega(C^r)\subseteq X_1$ then $\omega_1(C^r)\subseteq X_0$ and if $\omega(C^r)\subseteq X_2$ then $\omega_1(C^r)\subseteq X_2$. Thus $\omega_1(C^r)\cap(X_1\smallsetminus X_0)=\varnothing$. Condition 4) is satisfied by the same argument.

Next, consider the homotopy Λ as in Corollary 13.3 for the sets $A=\omega_1^{-1}(X_1\smallsetminus X_0)$ and $B=\omega_1^{-1}(X_2\smallsetminus X_0)$. The composition $\omega_1\Lambda\colon I^k\times I\to X$ gives a homotopy between ω_1 and a map ω_2 satisfying $\omega_2^{-1}(X_1\smallsetminus X_0)\subseteq S_{m+1}$ and $\omega_2^{-1}(X_2\smallsetminus X_0)\subseteq L_{n+1}$. Take the projection map $\mathrm{pr}\colon I^k\to I^{k-1}$, $\mathrm{pr}(t_1,\dots,t_{k-1},t_k)=(t_1,\dots,t_{k-1})$. We claim that the sets $\mathrm{pr}(S_{m+1})$ and $\mathrm{pr}(L_{n+1})$ are disjoint. Indeed, if $(t_1,t_2,\dots,t_{k-1})\in\mathrm{pr}(S_{m+1})\cap\mathrm{pr}(L_{n+1})$ then there are numbers $c_{j_1},\dots,c_{j_{k-1}}\in\{0,\frac{1}{N},\dots,\frac{N-1}{N}\}$ such that $c_{j_i}<t_i<c_{j_i}+\frac{\delta}{2}$ for at least m coordinates t_i and $c_{j_i}+\frac{\delta}{2}<t_i<c_{j_i}+\delta$ for at least n coordinates t_i. However, by assumption $k-1<m+n$, so this is impossible. As a consequence, the sets $\mathrm{pr}(\omega_2^{-1}(X_1\smallsetminus X_0))$ and $\mathrm{pr}(\omega_2^{-1}(X_2\smallsetminus X_0))$ are disjoint. We also have $\partial I^{k-1}\cap\mathrm{pr}(\omega_2^{-1}(X_2\smallsetminus X_0))=\varnothing$ so $\mathrm{pr}(\omega_2^{-1}(X_1\smallsetminus X_0))\cup\partial I^{k-1}$ and $\mathrm{pr}(\omega_2^{-1}(X_2\smallsetminus X_0))$ are disjoint, closed subsets of I^{k-1}. Take a function $\varrho\colon I^{k-1}\to[0,1]$ such that $\varrho(\mathrm{pr}(\omega_2^{-1}(X_1\smallsetminus X_0))\cup\partial I^{k-1})=1$ and $\varrho(\mathrm{pr}(\omega_2^{-1}(X_2\smallsetminus X_0)))=0$. Define a homotopy $h\colon I^k\times[0,1]\to X$ by

$$h((t_1,\dots,t_{k-1},t_k),s)=\omega_2(t_1,\dots,t_{k-1},(1-s)t_k+s\varrho(t_1,\dots,t_{k-1})t_k)$$

Then $h_0=\omega_2\simeq\omega$ while h_1 gives the desired map τ.

The argument that $i_*\colon\pi_k(X_1,X_0,x_0)\to\pi_k(X,X_2,x_0)$ is 1-1 for $k<m+n$ is analogous. In such case we start with two maps $\omega_0,\omega_1\colon I^k\to X_1$ representing two elements of $\pi_k(X_1,X_0,x_0)$. If these maps represent the same element in $\pi_k(X,X_2,x_0)$ then there exists $h\colon I^{k+1}=I^k\times I\to X$ such that $h|_{I^k\times\{i\}}=\omega_i$ for $i=0,1$ and that satisfies the appropriate conditions on the other faces of I^{k+1}. We want to show that h is homotopic to a map $h'\colon I^{k+1}\to X_1$. Since $k+1\leq m+n$ this can be done in the same way as above. $\square$