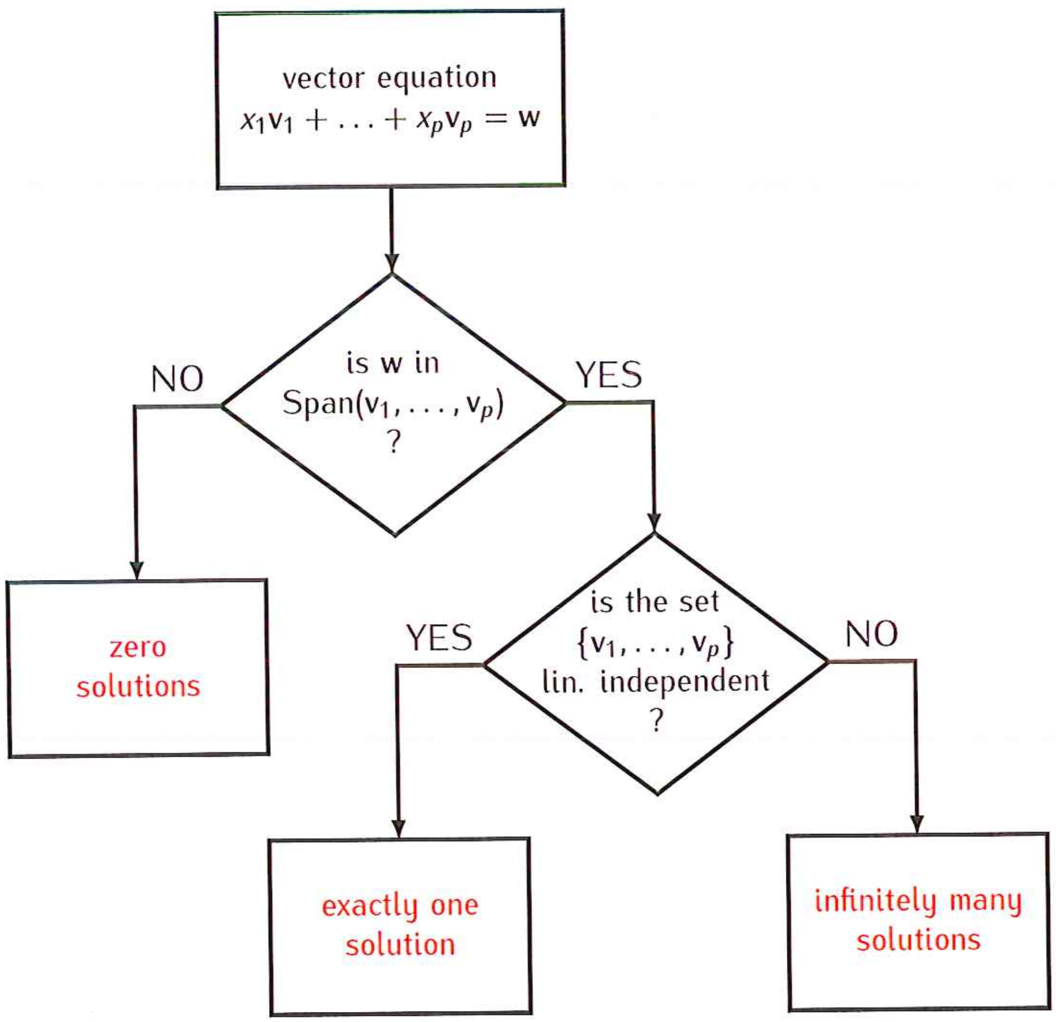
Recall:

1) $\text{Span}(\mathrm{v}_1, \ldots, \mathrm{v}_p) = \left\{ \begin{array}{c} \text{the set of all} \\ \text{linear combinations} \\ c_1\mathrm{v}_1 + \ldots + c_p\mathrm{v}_p \end{array} \right\}$

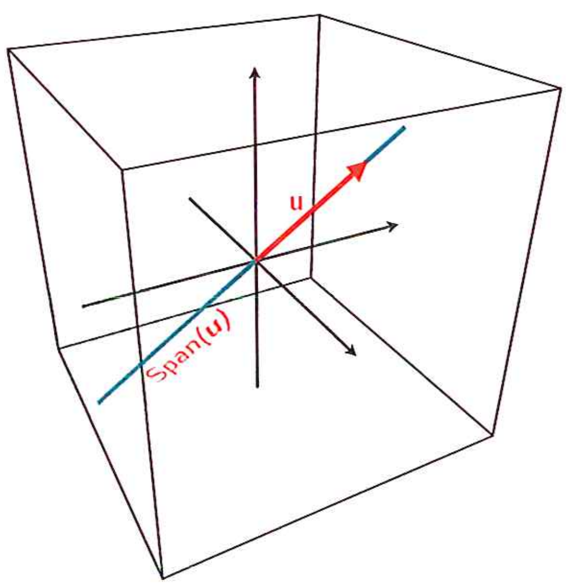
2) A set of vectors $\{\mathrm{v}_1, \ldots, \mathrm{v}_p\}$ is linearly independent if the equation

$$x_1\mathrm{v}_1 + \ldots + x_p\mathrm{v}_p = \mathbf{0}$$

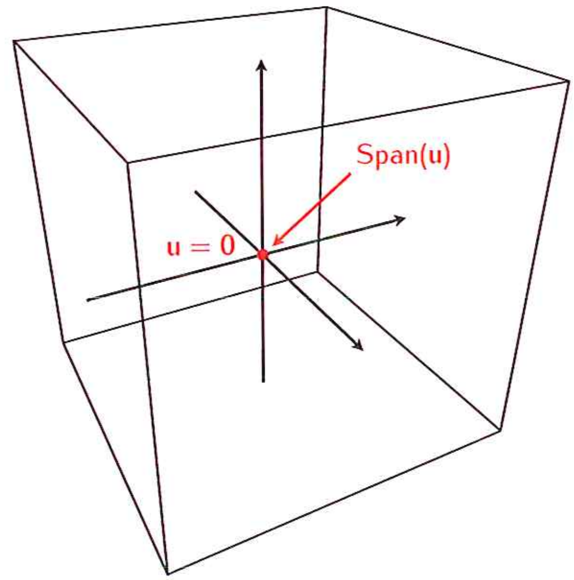
has only one, trivial solution $x_1 = 0, \ldots, x_p = 0$.



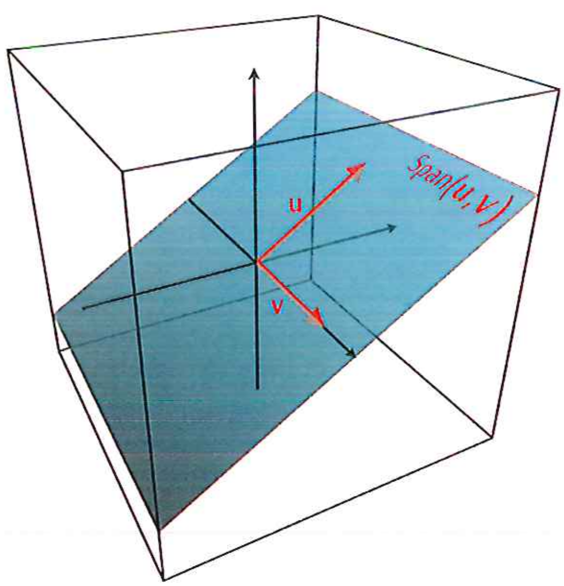
Linear independence vs. Span



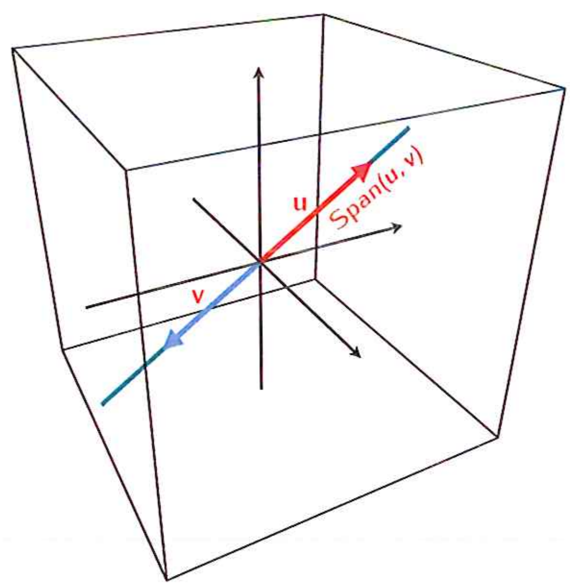
$\{\mathbf{u}\}$ linearly independent



$\{\mathbf{u}\}$ linearly dependent



$\{\mathbf{u}, \mathbf{v}\}$ linearly independent



$\{\mathbf{u}, \mathbf{v}\}$ linearly dependent

Theorem

Let $\{v_1, \ldots, v_p\}$ be a set of vectors in $\mathbb{R}^n$. The following conditions are equivalent:

1) The set $\{v_1, \ldots, v_p\}$ is linearly dependent.

2) For some v_i we have $v_i \in \mathrm{Span}(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_p)$.

3) For some v_i we have

$$\mathrm{Span}(v_1, \ldots, v_p) = \mathrm{Span}(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_p)$$

Example.

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The set $\{v_1, v_2, v_3\}$ is linearly dependent e.g.

$$2v_1 - v_2 + 0v_3 = 0$$

This gives:

$$(*) \quad v_2 = 2v_1 + 0v_3$$

So: $v_2 \in \mathrm{Span}(v_1, v_3)$

Also, assume that $w \in \mathrm{Span}(v_1, v_2, v_3)$

Then:

$$w = c_1v_1 + c_2v_2 + c_3v_3$$

but using $(*)$ we obtain:

$$w = c_1v_1 + c_2(2v_1 + 0v_3) + c_3v_3$$

$$= (c_1 + 2c_2)v_1 + c_3v_3$$

So: $w \in \mathrm{Span}(v_1, v_3)$.