Definition

A *homogenous vector equation* is a vector equation of the form

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

(i.e. with the zero vector as the vector of constants).

Note: A homogenous equation always has at least one, trivial solution: $x_1 = 0, x_2 = 0, \ldots, x_p = 0$

This leaves two possibilities for homogenous equations:

1. only one solution e.g.:

$$x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

2. infinitely many solutions e.g.:

$$x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}-1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Definition

Let $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is *linearly independent* if the homogenous equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

has only one, trivial solution $x_1 = 0, \ldots, x_p = 0$. Otherwise the set is *linearly dependent.*

e.g. the set $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ is linearly independent

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Theorem

Let $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$. If the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent then the equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{w}$$

has exactly one solution for any vector $\mathbf{w} \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p)$.

If the set is linearly dependent then this equation has infinitely many solutions for any $\mathbf{w} \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p)$.

Proof: Assume that $\{v_1, \ldots, v_p\}$ is linearly dependent so we have

$$c_1v_1 + \ldots + c_pv_p = 0$$

where $c_i \neq 0$ for some i.

If $d_1v_1 + \ldots + d_pv_p = w$

then $(c_1+d_1)v_1 + \ldots + (c_p+d_p)v_p = 0 + w = w$

Thus the equation $x_1v_1 + \ldots + x_pv_p = w$ has two different solutions:

$$\begin{cases} x_1 = d_1 \\ \vdots \\ x_p = d_p \end{cases} \quad \text{and} \quad \begin{cases} x_1 = c_1 + d_1 \\ \vdots \\ x_p = c_p + d_p \end{cases}$$

Conversely: if $\{v_1, \ldots, v_p\}$ is linearly independent and $x_1v_1 + \ldots + x_pv_p = w$ has two solutions:

$$c_1v_1 + \ldots + c_pv_p = w$$
$$d_1v_1 + \ldots + d_pv_p = w$$

then: $(c_1 - d_1)v_1 + \ldots (c_p - d_p)v_p = w - w = 0$

By linear independence we get: $(c_1 - d_1) = 0, \ldots, (c_p - d_p) = 0$

so: $c_1 = d_1, \ldots, c_p = d_p$.

Example. Let

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \quad v_2 = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 3 \\ -12 \end{bmatrix}$$

Check is the set $\{v_1, v_2, v_3\}$ is linearly independent.

Solution:

We need to solve:

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$$

aug. matrix:

$$[v_1\ v_2\ v_3\ |\ 0] = \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ -2 & 4 & -12 & 0 \end{array}\right] \xrightarrow[\text{red.}]{\text{row}} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & \underbrace{0}_{\uparrow \text{free}} & 0 \end{array}\right] \end{array}$$

Solutions:

$$\begin{cases} x_1 = -4x_3 \\ x_2 = x_3 \\ x_3 = \text{free} \end{cases}$$

- infinitely many solutions, so the set $\{v_1, v_2, v_3\}$ is not linearly independent

Some properties of linearly (in)dependent sets

1) A set consisting of one vector $\{v_1\}$ is linearly dependent if and only if $v_1 = \mathbf{0}$.

- if $v_1 \neq 0$ then $x_1 v_1 = 0$ has only one solution $x_1 = 0$, so $\{v_1\}$ is lin. indep.

- if $v_1 = 0$ then $x_1 v_1 = 0$ holds for any value of x_1, so $\{v_1\}$ is lin. dependent.

2) A set consisting of two vectors $\{v_1, v_2\}$ is linearly dependent if and only if one vector is a scalar multiple of the other.

- if $\{v_1, v_2\}$ is lin. dependent then

$$c_1 v_1 + c_2 v_2 = 0$$

for some c_1, c_2 s.t. either $c_1 \neq 0$ or $c_2 \neq 0$. Say $c_1 \neq 0$. Then:

$$c_1 v_1 = -c_2 v_2$$

$$v_1 = \left(-\frac{c_2}{c_1}\right) \cdot v_2$$

So v_2 is a multiple of v_1.

3) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a set of p vectors in $\mathbb{R}^n$ and $p > n$ then this set is linearly dependent.

We need to show that if $p > n$ then

$$x_1 v_1 + \ldots + x_p v_p = 0$$

has more than one solution.

E.g.:

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Big\downarrow \text{ augmented matrix}$$

$$\begin{array}{cc} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ n = 2 \text{ rows } \left\{ \right. & \left[\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right] \\ & \underbrace{\hspace{6em}}_{p = 3 \text{ columns}} \end{array}$$

since $p > n$ one of the column will not contain a leading one so it will give a free variable.

Upshot: how to find the number of solutions of a vector equation