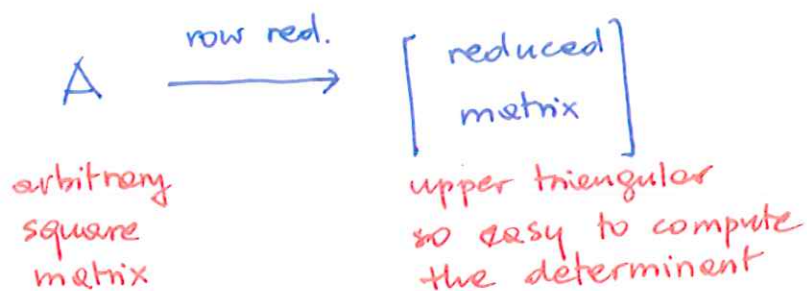


Recall: If A is an upper triangular matrix:

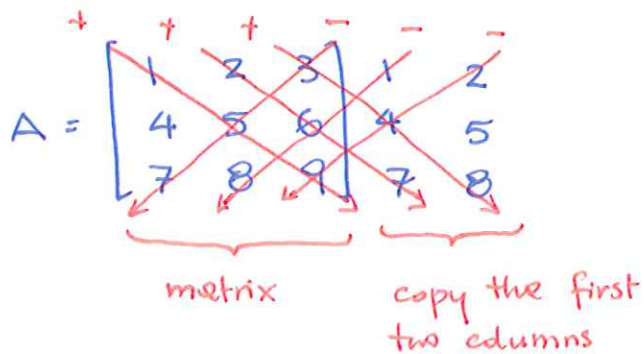
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

then $\det A = a_{11} \cdot a_{22} \cdot \dots \cdot a_{nn}$.

Note. If A is a square matrix then the reduced echelon form of A is always upper triangular.



Note: Here is a direct way of computing the determinant of a 3×3 matrix:



$$\det A = (1 \cdot 5 \cdot 9) + (2 \cdot 6 \cdot 7) + (3 \cdot 4 \cdot 8) \\ - (3 \cdot 5 \cdot 7) - (1 \cdot 6 \cdot 8) - (2 \cdot 4 \cdot 9)$$

Warning: This works for 3×3 matrices only.

Theorem

Let A and B be $n \times n$ matrices.

1) If B is obtained from A by interchanging two rows (or two columns) then

$$\det B = -\det A$$

2) If B is obtained from A by multiplying one row (or one column) of A by a scalar k then

$$\det B = k \cdot \det A$$

2) If B is obtained from A by adding a multiple of one row of A to another row (or adding a multiple of one column to another column) then

$$\det B = \det A$$

Example.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 7 \\ 2 & 5 & 1 \end{bmatrix}$$

$$\det A = 1 \cdot C_{11} + 2 \cdot C_{12} + 3 \cdot C_{13}$$

$$C_{11} = (-1)^{1+1} \det \begin{bmatrix} 0 & 7 \\ 5 & 1 \end{bmatrix}, C_{12} = \dots$$

$$B = \begin{bmatrix} 4 \cdot 1 & 4 \cdot 2 & 4 \cdot 3 \\ 1 & 0 & 7 \\ 2 & 5 & 1 \end{bmatrix}$$

$$\det B = 4 \cdot 1 \cdot C_{11} + 4 \cdot 2 \cdot C_{12} + 4 \cdot 3 \cdot C_{13}$$

$$C_{11} = (-1)^{1+1} \det \begin{bmatrix} 0 & 7 \\ 5 & 1 \end{bmatrix}, C_{12} = \dots$$

So:

$$\begin{aligned} \det B &= 4 \cdot (1 \cdot C_{11} + 2 \cdot C_{12} + 3 \cdot C_{13}) \\ &= 4 \cdot \det A \end{aligned}$$

Computation of determinants via row reduction

Idea. To compute $\det A$, row reduce A to the echelon form. Keep track how the determinant changes at each step of the row reduction process.

Example. Compute $\det A$ where

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 4 & 0 & 10 \\ 3 & 4 & 1 & 7 \\ -2 & 5 & 3 & 0 \end{bmatrix}$$

$$\begin{aligned} \det \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 4 & 0 & 10 \\ 3 & 4 & 1 & 7 \\ -2 & 5 & 3 & 0 \end{bmatrix} &= (-1) \cdot \det \begin{bmatrix} 2 & 4 & 0 & 10 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 1 & 7 \\ -2 & 5 & 3 & 0 \end{bmatrix} \cdot \left(\frac{1}{2}\right) \\ &= \textcircled{2} \cdot (-1) \cdot \det \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 1 & 7 \\ -2 & 5 & 3 & 0 \end{bmatrix} \cdot \begin{matrix} (-3) \\ 2 \end{matrix} \quad \begin{matrix} \uparrow \\ \text{reciprocal of } \frac{1}{2} \end{matrix} \\ &= 2 \cdot (-1) \cdot \det \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & -2 & 1 & -8 \\ 0 & 9 & 3 & 10 \end{bmatrix} \cdot \begin{matrix} 2 \\ (-9) \end{matrix} \\ &= 2 \cdot (-1) \cdot \det \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -15 & -17 \end{bmatrix} \cdot 3 \quad \begin{matrix} \uparrow \\ \cdot 3 \end{matrix} \\ &= 2 \cdot (-1) \cdot \det \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & -23 \end{bmatrix} \quad \text{upper triangular} \\ &= 2 \cdot (-1) \cdot (1 \cdot 1 \cdot 5 \cdot (-23)) \\ &= \underline{\underline{230}} \end{aligned}$$

Theorem

If A is a square matrix then A is invertible if and only if $\det A \neq 0$

Recall: A is invertible if and only if its reduced echelon form is the identity matrix.

Proof:

$$\det A = (\text{some non-zero number}) \cdot \det \begin{bmatrix} \text{reduced} \\ \text{echelon} \\ \text{form} \\ \text{of } A \end{bmatrix}$$

1) if A is invertible then

$$\det A = (\text{some non-zero number}) \cdot \det I \neq 0$$

↑
the identity matrix

2) if A is not invertible then

$$\begin{bmatrix} \text{reduced} \\ \text{form} \\ \text{of } A \end{bmatrix} = \text{upper triangular matrix with a zero on the main diagonal}$$

(e.g. $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$)

This gives:

$$\det \begin{bmatrix} \text{reduced} \\ \text{form} \\ \text{of } A \end{bmatrix} = 0$$

so: $\det A = 0.$

Further properties of determinants

- 1) $\det(A^T) = \det A$
- 2) $\det(AB) = (\det A) \cdot (\det B)$
- 3) $\det(A^{-1}) = (\det A)^{-1}$

Note. In general $\det(A + B) \neq \det A + \det B$.

↑
example:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\det A = 0$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\det B = 0$$

$$\det(A+B) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$