- **Determinants**

**1)** Computation:

- by cofactor expansion
- by row reduction

**2)** Properties:

- a matrix is invertible if and only if $\det A \neq 0$
- determinants and elementary row/column operations
- algebraic properties:

❶ $\det(AB) = \det(A)\det(B)$
❷ $\det(A^{-1}) = (\det A)^{-1}$
❸ $\det(A^T) = \det A$
❹ $\det(A+B) \neq \det A + \det B$

**3)** Cramer's rule. If $A$ is an $n \times n$ invertible matrix and $\mathbf{b} \in \mathbb{R}^n$ then the solution of the equation

$$A\mathbf{x} = \mathbf{b}$$

is given by

$$\mathbf{x} = \frac{1}{\det A}\begin{bmatrix} \det A_1(\mathbf{b}) \\ \vdots \\ \det A_n(\mathbf{b}) \end{bmatrix}$$

**4)** If $A$ is an $n \times n$ invertible matrix then is the matrix

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

where $C_{ij} = (-1)^{i+j}\det A_{ij}$

**5)** Geometric interpretation of determinants:

- determinants compute areas of parallelograms
- determinants measure how linear transformations change area
- the sign of a determinant indicates if a linear transformation preserves or reverses orientation

## • General vector spaces

**1)** Definition.

**2)** Examples:

- $\mathbb{R}^n$
- $\mathbb{P}$, $\mathbb{P}_n$ – vector spaces of polynomials
- $M_{m,n}(\mathbb{R})$ – the vector space of $m \times n$ matrices
- $\mathcal{F}(\mathbb{R})$, $C(\mathbb{R})$, $C^\infty(\mathbb{R})$ – vector spaces of functions $f\colon \mathbb{R} \to \mathbb{R}$ (all functions, continuous functions, smooth functions)

**3)** Subspace of a vector space:

- definition
- subspaces associated to an $m \times n$ matrix $A$:

$$\mathrm{Nul}(A) \subseteq \mathbb{R}^n$$
$$\mathrm{Col}(A) \subseteq \mathbb{R}^m$$

**4)** Linear transformations of vectors spaces:

- definition
- the image $\mathrm{Im}(T)$ and kernel $\mathrm{Ker}(T)$ of a linear transformation $T$

**5)** Basis of a vector space

- definition
- computation of bases of $\mathbb{R}^n$, $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$
- the standard bases of the vector spaces of polynomials ($\mathbb{P}$ and $\mathbb{P}_n$)

**6)** Coordinates of a vector relative to a basis

**7)** Dimension of a vector space:
- definition
- properties

## • Sample TRUE/FALSE questions and answers

Here is a sample of true/false questions. Questions of this type will be a part of the exam. In order to answer these questions you need to provide reasoning. Simply writing TRUE or FALSE as an answer will give you very little or no credit. In order to show that a statement is false, it suffices to give one example illustrating that it is false. In order to show that a statement is true, you need to provide a reasoning explaining why it is true in all instances - giving one example in this case will not suffice, since the statement may not work for some other examples.

*For each of the statements given below decide if it is true or false. If you decide that it is true justify your answer. If you think it is false give a counterexample.*

**a)** If $A$ is a $4 \times 4$ which contains a row of zeros then $\det A = 0$.

**b)** If $A$ is an invertible matrix and $A^T$ is the transpose of $A$ then $\det(AA^T) > 0$.

**c)** If $V$ is a vector space and $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a set of vectors in $V$ such that $\mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k) = V$ then $\dim V = k$.

**d)** If $V$ is a subspace of $\mathbb{R}^3$, and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ are vectors such that $\mathbf{v} + \mathbf{w} \in V$ then $\mathbf{v} \in V$ and $\mathbf{w} \in V$.

**Here are solutions to the sample TRUE/FALSE questions from the previous page. You should try to answer all questions by yourself before reading these solutions.**

**a)** TRUE. Assume for example that the second row of $A$ is the zero row:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Computing $\det A$ by cofactor expansion with respect to the second row we obtain:

$$\det A = 0 \cdot C_{21} + 0 \cdot C_{22} + 0 \cdot C_{23} + 0 \cdot C_{24} = 0$$

If the row of zeros is some other row, then we use cofactor expansion with respect to that row to get the same result.

**b)** TRUE. Since $\det A = \det A^T$ we have

$$\det(AA^T) = \det(A)\det(A^T) = (\det A)^2 \geq 0$$

Also, since $A$ is an invertible matrix we have $\det A \neq 0$, so $\det(AA^T) > 0$.

**c)** FALSE. Take e.g. $V = \mathbb{R}^2$ and let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Then $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \mathbb{R}^2$, but $\dim \mathbb{R}^2 = 2 \neq 3$.

**d)** FALSE. Take e.g. $V$ to be the subspace consisting of all vectors whose last coordinate is 0:

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix} \mid a_1, a_2 \in \mathbb{R} \right\}$$

Let $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. Then $\mathbf{v} + \mathbf{w} \in V$, but $\mathbf{v} \notin V$ and $\mathbf{w} \notin V$.