<u>Recall:</u>

1) If A is an $m \times n$ matrix then the function

$$T_A \colon \mathbb{R}^n \to \mathbb{R}^m$$

defined by $T_A(\mathrm{v}) = A\mathrm{v}$ is called the matrix transformation associated to A.

2) A function $T \colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if
(ii) $T(\mathrm{u} + \mathrm{v}) = T(\mathrm{u}) + T(\mathrm{v})$
(ii) $T(c\mathrm{v}) = cT(\mathrm{v})$

3) Every matrix transformation is a linear transformation.

4) Every linear transformation $T \colon \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation:

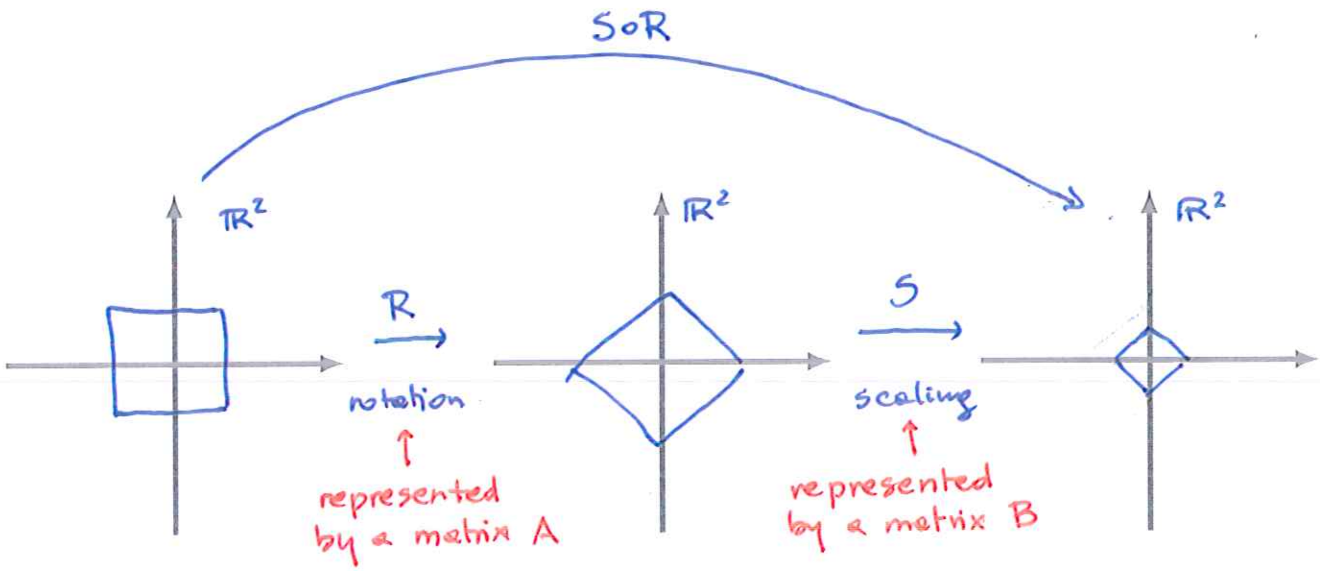
$$T(\mathrm{v}) = A\mathrm{v}$$

where

$$A = \left[\ T(\mathbf{e}_1)\ \ T(\mathbf{e}_2)\ \ \ \dots\ \ T(\mathbf{e}_n)\ \right]$$

The matrix A is called the standard matrix of T.

Composition of linear transformations



$$v \longrightarrow \underset{\displaystyle R(v)}{\overset{\displaystyle Av}{\shortparallel}} \longrightarrow \underset{\displaystyle S(R(v))}{\overset{\displaystyle B(Av)}{\shortparallel}}$$

Question:

Is there a matrix C such that

$$S \circ R(v) = Cv$$

(or equivalently: $Cv = B(Av)$) ?

Theorem

If $S\colon \mathbb{R}^n \to \mathbb{R}^m$ and $T\colon \mathbb{R}^m \to \mathbb{R}^k$ are linear transformation then the composition

$$T \circ S\colon \mathbb{R}^n \to \mathbb{R}^k$$

is also a linear transformation.

Proof: We need to check:

1) $T\circ S(u+v) = T\circ S(u) + T\circ S(v)$

2) $T\circ S(cu) = c\cdot(T\circ S(u))$

1) $T\circ S(u+v) = T(S(u+v)) = T(S(u)+S(v))$

since S is linear

$= T(S(u)) + T(S(v))$

since T is linear

$= T\circ S(u) + T\circ S(v)$

2) similar

Upshot. The function $T \circ S$ is represented by some matrix C:

$$T \circ S(\mathrm{v}) = C\mathrm{v}$$

Question. Let $S\colon \mathbb{R}^n \to \mathbb{R}^m$ and $T\colon \mathbb{R}^m \to \mathbb{R}^k$ be linear transformations, and let

- B is the standard matrix of S
- A is the standard matrix of T

What if the standard matrix of $T \circ S\colon \mathbb{R}^n \to \mathbb{R}^k$?

$$\begin{tikzcd}
 & w \arrow[r] & Aw \\
v \arrow[r] & Bv & \\
\mathbb{R}^n \arrow[r, "S"] \arrow[rr, bend right, "T\circ S"'] & \mathbb{R}^m \arrow[r, "T"] & \mathbb{R}^k \\
v \arrow[rr, bend right] & & Cv
\end{tikzcd}$$

C ← the standard matrix of $T\circ S$

We have:

$$\begin{aligned}
C &= [T\circ S(e_1) \quad T\circ S(e_2) \quad \cdots \quad T\circ S(e_n)] \\
&= [T(S(e_1)) \quad T(S(e_2)) \quad \cdots \quad T(S(e_n))] \\
&= [A\cdot(Be_1) \quad A\cdot(Be_2) \quad \cdots \quad A(Be_n)]
\end{aligned}$$

Note: If $B = [v_1 \ v_2 \ \cdots \ v_n]$ then $Be_1 = [v_1 \ v_2 \cdots v_n]\cdot\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix} = v_1$

In the same way: $Be_2 = v_2, \ldots, Be_n = v_n$

We get:

$$C = [Av_1 \ Av_2 \ldots \ Av_n]$$

where $v_1, v_2, \ldots, v_n$ - columns of B.

Definition

Let

- A be an $k \times m$ matrix
- $B = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$ be an $m \times n$ matrix

Then $A \cdot B$ is an $k \times n$ matrix given by

$$A \cdot B = \begin{bmatrix} Av_1 & Av_2 & \dots & Av_n \end{bmatrix}$$

Note: If $S: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $T: \mathbb{R}^m \longrightarrow \mathbb{R}^k$ are linear transformations

B = the standard matrix of S

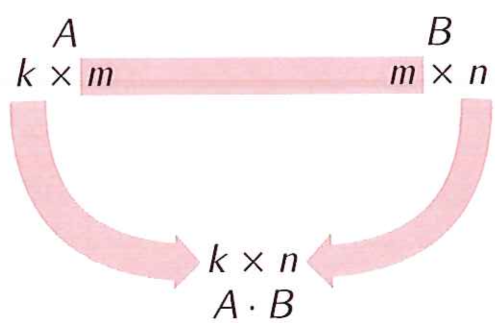
A = the standard matrix of T

then AB is the standard matrix of $T \circ S$:

$$T \circ S(v) = (AB)v$$

Note. The product $A \cdot B$ is defined only if

(number of columns of A) $=$ (number of rows of B)



Example.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} \qquad B = \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & -1 & 2 & 1 \\ 4 & 5 & 1 & 0 \\ 1 & 2 & 3 & 1 \end{bmatrix} \end{array} = [v_1 \ v_2 \ v_3 \ v_4]$$

2×3 $\qquad$ 3×4

AB is defined and it is a 2×4 matrix

$$AB = [Av_1 \ Av_2 \ Av_3 \ Av_4]$$

$$Av_1 = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} = 0 \cdot \begin{bmatrix} 0 \\ 3 \end{bmatrix} + 4 \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 1 \cdot \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 21 \end{bmatrix}$$

$$Av_2 = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix} = (-1) \begin{bmatrix} 0 \\ 3 \end{bmatrix} + 5 \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 2 \cdot \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 9 \\ 27 \end{bmatrix}$$

$$Av_3 = \quad \dots \qquad \dots \quad = \begin{bmatrix} 7 \\ 25 \end{bmatrix}$$

$$Av_4 = \quad \dots \qquad \dots \quad = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

We obtain:

$$AB = \begin{bmatrix} 6 & 9 & 7 & 2 \\ 21 & 27 & 25 & 8 \end{bmatrix}$$