Recall: If A is an $m \times n$ matrix then

$$A \cdot \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}}_{\text{vector in } \mathbb{R}^n} = \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}}_{\text{vector in } \mathbb{R}^m}$$

Example:

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}_{2\times 3 \text{ matrix}} \cdot \underbrace{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}}_{\text{vector in } \mathbb{R}^3} = 2 \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 1 \cdot \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 9 \end{bmatrix}}_{\text{vector in } \mathbb{R}^2}$$

Definition

If A is an $m \times n$ matrix then the function

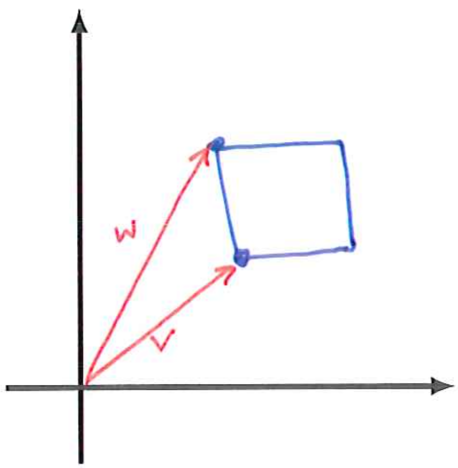
$$T_A \colon \mathbb{R}^n \to \mathbb{R}^m$$

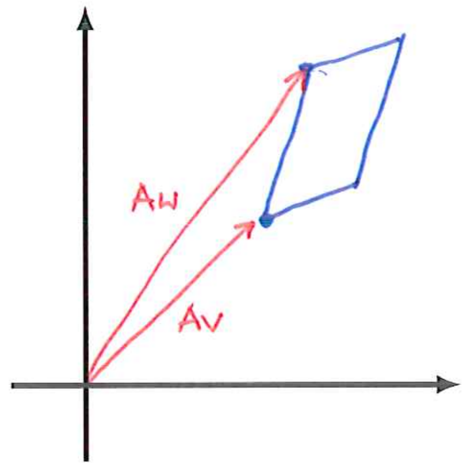
given by $T_A(\mathrm{v}) = A\mathrm{v}$ is called the *matrix transformation associated to A*.

Geometric interpretation of matrix transformations $\mathbb{R}^2 \to \mathbb{R}^2$

A - 2×2 matrix

$$T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$v \longmapsto Av$$

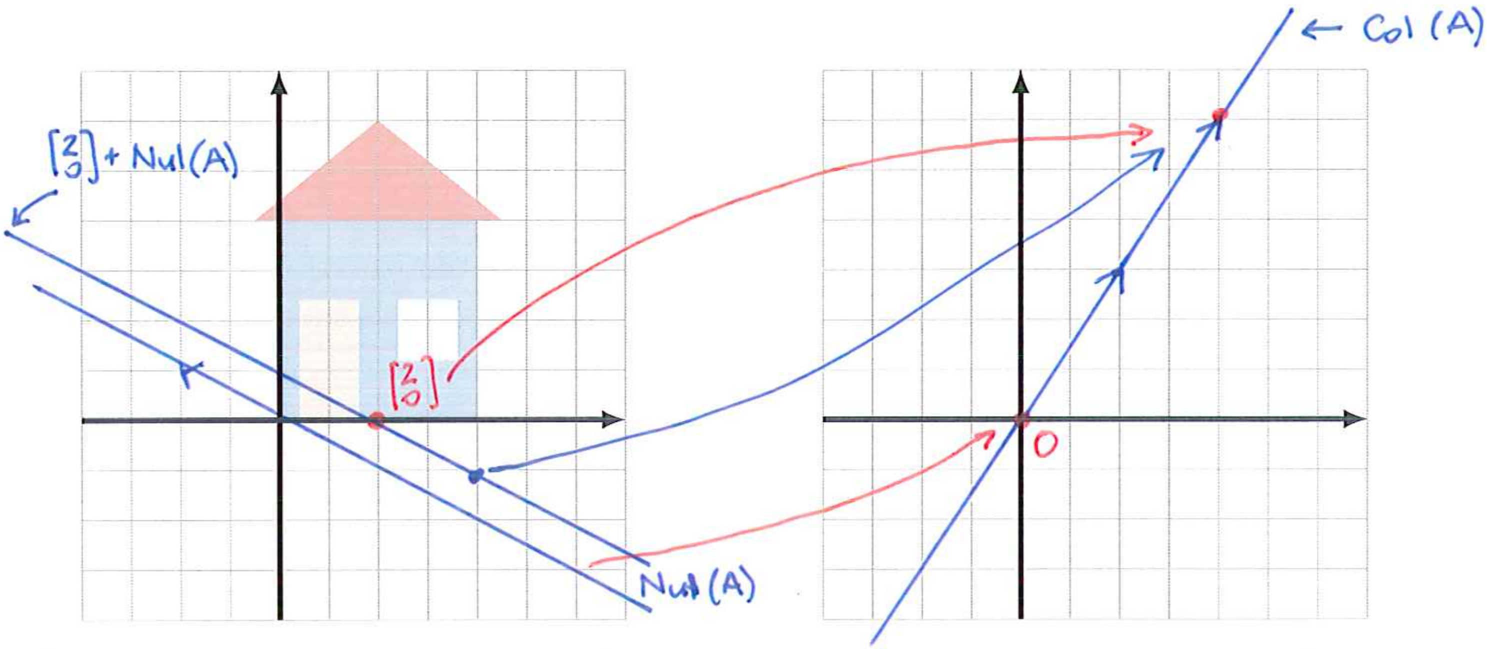




Null spaces, column spaces and matrix transformations

Example.

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \qquad T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$



$$\left(\text{Possible values of } T_A\right) = \left(\text{vectors of the form } Av\right)$$
$$= \left(\text{vectors in } \mathrm{Col}(A)\right)$$
$$= \mathrm{Span}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix}\right)$$

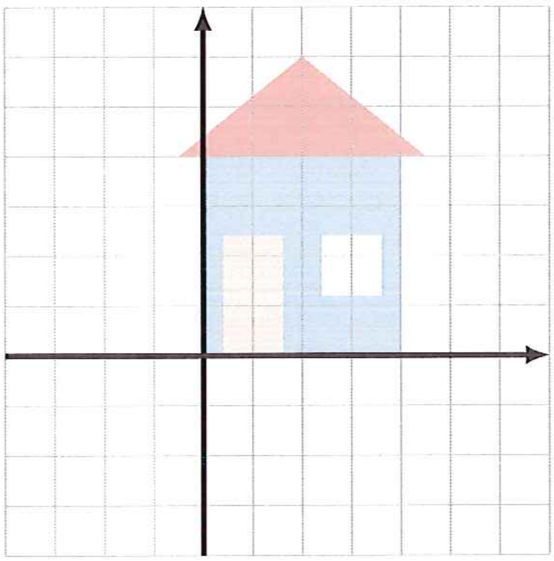
$$\left(\text{vectors } v \text{ s.t. } T_A(v) = 0\right) = \left(\text{vectors } v \text{ s.t. } Av = 0\right)$$
$$= \mathrm{Nul}(A)$$

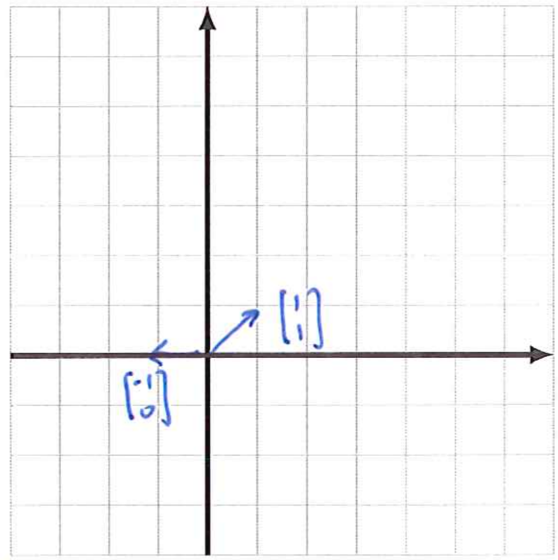
$$\left[\begin{array}{cc|c} 2 & 4 & 0 \\ 3 & 6 & 0 \end{array}\right] \xrightarrow{\text{row red.}} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} \qquad x = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

<u>so</u>: $\mathrm{Nul}(A) = \mathrm{Span}\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right)$

Example.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \qquad T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$





$$\left(\text{Possible values of } T_A\right) = \mathrm{Col}(A) = \mathrm{Span}\left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\0\end{bmatrix}\right)$$

<u>Recall:</u> If an $m \times n$ matrix A has a pivot position in every row then $\mathrm{Col}(A) = \mathbb{R}^m$

In our case:

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \xrightarrow{\text{row red.}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \underline{\text{so:}}\ \mathrm{Col}(A) = \mathbb{R}^2$$

<u>Upshot:</u> Any vector in $\mathbb{R}^2$ is a value of T_A

<u>Nul(A):</u>

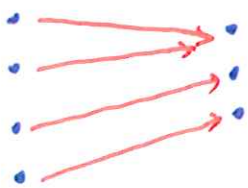
$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 0 & 0 \end{array}\right] \xrightarrow{\text{row red.}} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \end{array} \quad \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \quad \underline{\text{so:}}\ \mathrm{Nul}(A) = \left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$$

<u>This gives:</u> if $v \neq w$ then $T_A(v) \neq T_A(w)$.

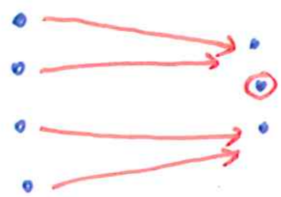
Recall:

A function $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is:

- *onto* if for each $\mathbf{b} \in \mathbb{R}^m$ there is $\mathrm{v} \in \mathbb{R}^m$ such that $F(\mathrm{v}) = \mathbf{b}$;

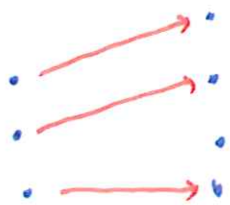


onto

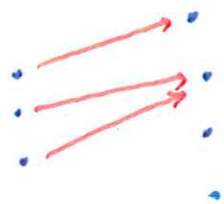


not onto

- *one-to-one* if for any $\mathrm{v}_1, \mathrm{v}_2$ such that $\mathrm{v}_1 \neq \mathrm{v}_2$ we have $F(\mathrm{v}_2) \neq F(\mathrm{v}_2)$.



one-to-one



not one-to-one

Proposition

Let A be an $m \times n$ matrix. The following conditions are equivalent:

1) The matrix transformation $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ is onto.
2) $\mathrm{Col}(A) = \mathbb{R}^m$.
3) The matrix A has a pivot position in every row.

Proposition

Let A be an $m \times n$ matrix. The following conditions are equivalent:

1) The matrix transformation $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one.
2) $\mathrm{Nul}(A) = \{0\}$.
3) The matrix A has a pivot position in every column.

Example. For the following 3×4 matrix A check if the matrix transformation $T_A \colon \mathbb{R}^4 \to R^3$ is onto and if it is one-to-one.

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -2 & -2 & 1 & -5 \\ 1 & 1 & -1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ -2 & -2 & 1 & -5 \\ 1 & 1 & -1 & 4 \end{bmatrix} \xrightarrow{\text{row red.}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

pivot position in every row $\Rightarrow$ $\mathrm{Col}(A) = \mathbb{R}^3$

<u>so</u>: T_A is onto

no pivot position in the second column $\Rightarrow$ $\mathrm{Nul}(A) \neq \{\mathbf{0}\}$

<u>so</u>: T_A is not one-to-one.

Example. For the following 3×3 matrix A check if the matrix transformation $T_A \colon \mathbb{R}^3 \to R^3$ is onto and if it is one-to-one.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 2 & 5 \end{bmatrix} \xrightarrow{\text{row red.}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

pivot position in every row $\Rightarrow$ $\mathrm{Col}(A) = \mathbb{R}^3$

<u>so</u>: T_A is onto

pivot position in every column $\Rightarrow$ $\mathrm{Nul}(A) = \{\mathbf{0}\}$

<u>so</u>: T_A is one-to-one.

Proposition

Let A be an $m \times n$ matrix. If the matrix transformation $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$ is both onto and one-to-one then we must have $m = n$ (i.e. A must be a square matrix).