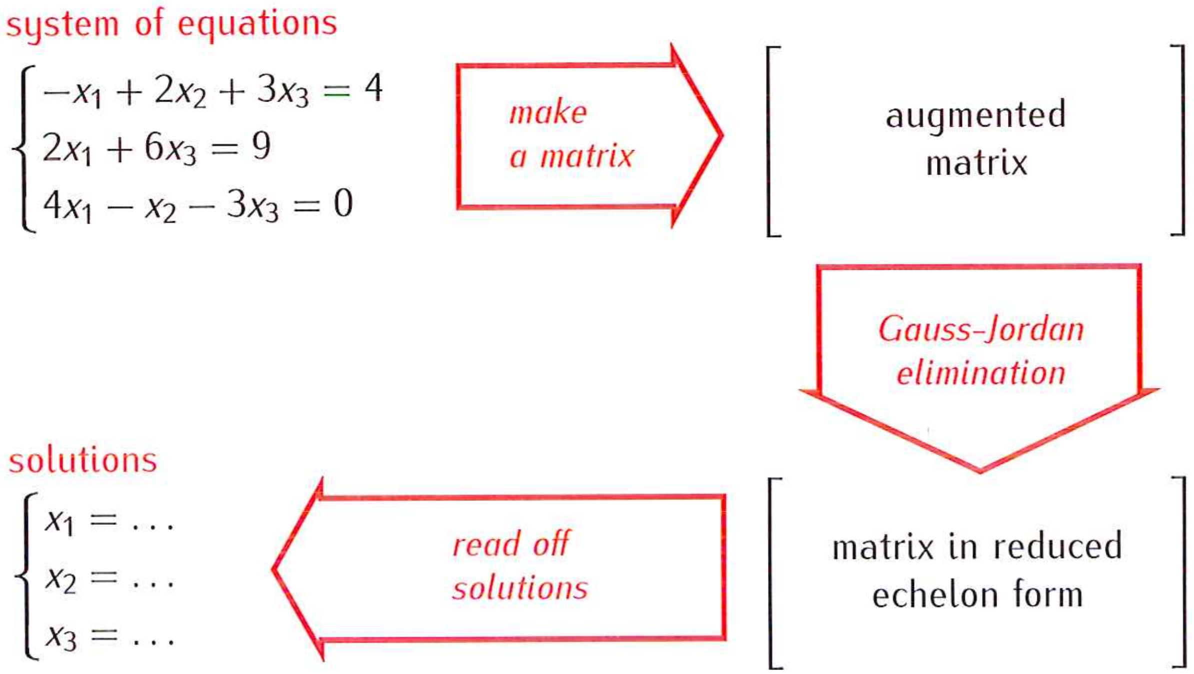
Recall:

How to solve a system of linear equations



system of equations

$$\begin{cases} -x_1 + 2x_2 + 3x_3 = 4 \\ 2x_1 + 6x_3 = 9 \\ 4x_1 - x_2 - 3x_3 = 0 \end{cases}$$

make a matrix

augmented matrix

Gauss-Jordan elimination

matrix in reduced echelon form

read off solutions

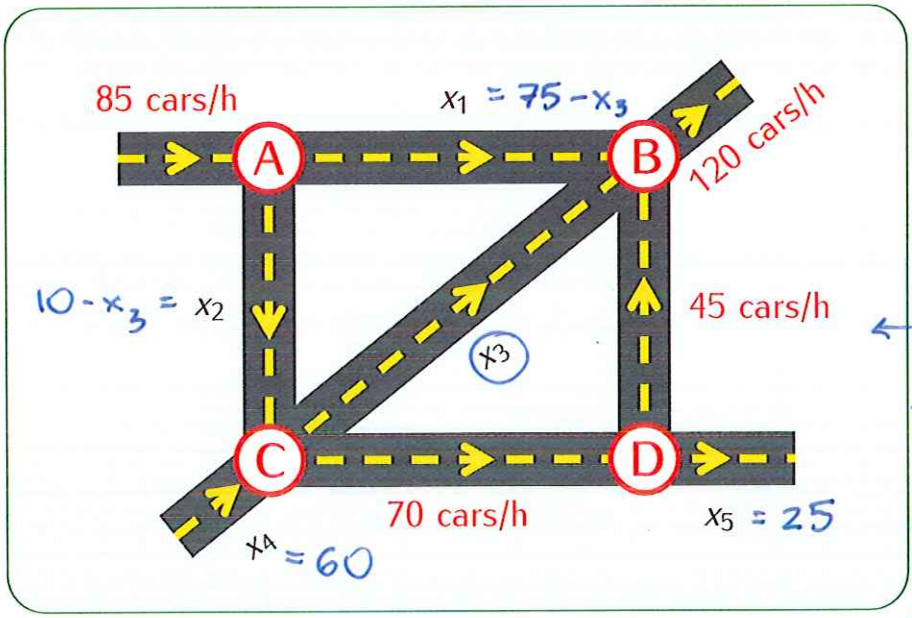
solutions

$$\begin{cases} x_1 = \dots \\ x_2 = \dots \\ x_3 = \dots \end{cases}$$

Next: Some applications of systems of linear equations:

- Computations of traffic flow.
- Balancing chemical equations.
- Google PageRank.

Computations of traffic flow



← In order to get full information about the flow of traffic we would need to measure the flow x_3.

Problem. Find the flow rate of cars on each segment of streets.

Note:

- flow into an intersection = flowout of that intersection
- total flow in = total flow out

$$\begin{array}{lccc} & \text{IN} & = & \text{OUT} \\ \text{total :} & 85 + x_4 & = & 120 + x_5 \\ \text{@A :} & 85 & = & x_1 + x_2 \\ \text{@B :} & x_1 + x_3 + 45 & = & 120 \\ \text{@C :} & x_2 + x_4 & = & x_3 + 70 \\ \text{@D :} & 70 & = & x_5 + 45 \end{array} \Bigg| \Rightarrow \left\{ \begin{array}{l} x_4 - x_5 = 35 \\ x_1 + x_2 = 85 \\ x_1 + x_3 = 75 \\ x_2 - x_3 + x_4 = 70 \\ x_5 = 25 \end{array} \right.$$

augmented matrix:

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 0 & 0 & 1 & -1 & 35 \\ 1 & 1 & 0 & 0 & 0 & 85 \\ 1 & 0 & 1 & 0 & 0 & 75 \\ 0 & 1 & -1 & 1 & 0 & 70 \\ 0 & 0 & 0 & 0 & 1 & 25 \end{array}\right] \end{array} \xrightarrow[\text{reduction}]{\text{row}} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} \textcircled{1} & 0 & 1 & 0 & 0 & 75 \\ 0 & \textcircled{1} & -1 & 0 & 0 & 10 \\ 0 & 0 & 0 & \textcircled{1} & 0 & 60 \\ 0 & 0 & 0 & 0 & \textcircled{1} & 25 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \rightarrow \left\{ \begin{array}{l} x_1 = 75 - x_3 \\ x_2 = 10 + x_3 \\ x_3 = \text{free} \\ x_4 = 60 \\ x_5 = 25 \end{array} \right.$$

Balancing chemical equations

Burning propane:

$$\boxed{\overset{1}{\cancel{x_1}}C_3H_8 + \overset{5}{\cancel{x_2}}O_2 \longrightarrow \overset{3}{\cancel{x_3}}CO_2 + \overset{4}{\cancel{x_4}}H_2O}$$

Note:

- The numbers x_1, x_2, x_3, x_4 are integers.
- The number of atoms of each element on the left side is the same as the number of atoms of that element on the right side.

$$\begin{array}{lccc} & \text{LEFT} & = & \text{RIGHT} \\ C: & 3x_1 & = & x_3 \\ H: & 8x_1 & = & 2x_4 \\ O: & 2x_2 & = & 2x_3 + x_4 \end{array} \Rightarrow \begin{cases} 3x_1 - x_3 = 0 \\ 8x_1 - x_4 = 0 \\ 2x_2 - 2x_3 - x_4 = 0 \end{cases}$$

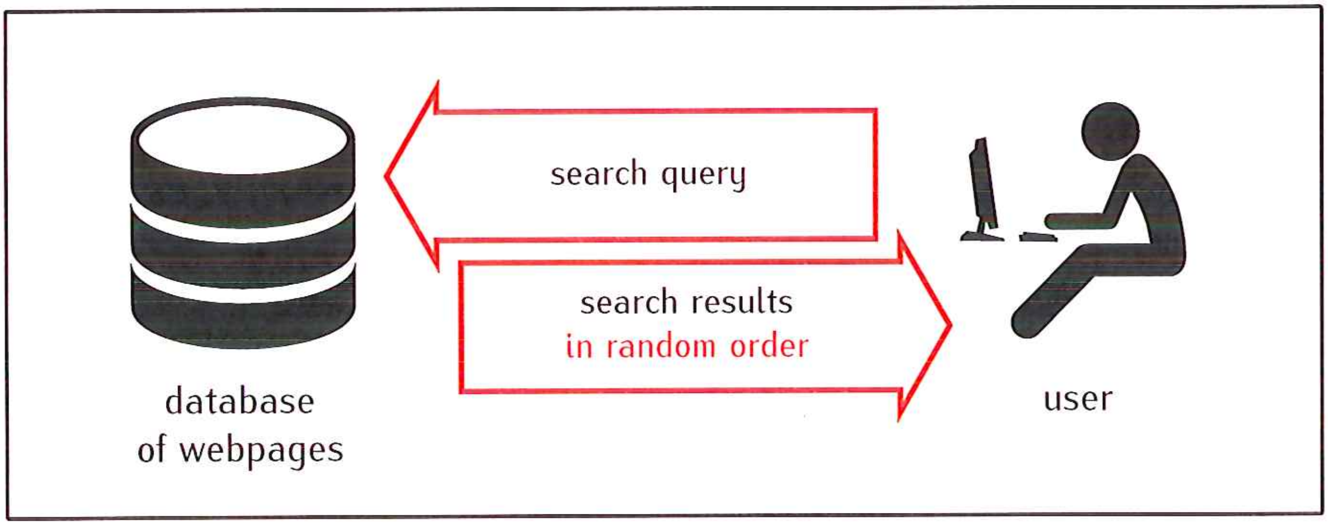
homogenous system (i.e. zeros only on the right hand side)

$$\left[\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ 3 & 0 & -1 & 0 & 0 \\ 8 & 0 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right] \xrightarrow{\text{row red}} \left[\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ 1 & 0 & 0 & -1/4 & 0 \\ 0 & 1 & 0 & -5/4 & 0 \\ 0 & 0 & 1 & \underbrace{-3/4}_{\text{free}} & 0 \end{array}\right]$$

$$\begin{cases} x_1 = \frac{1}{4}x_4 \\ x_2 = \frac{5}{4}x_4 \\ x_3 = \frac{3}{4}x_4 \\ x_4 = \text{free} \end{cases} \qquad \boxed{\begin{array}{c}\text{set} \\ x_4 = 4\end{array}} \rightarrow \begin{cases} x_1 = 1 \\ x_2 = 5 \\ x_3 = 3 \\ x_4 = 1 \end{cases}$$

Google PageRank

Early search engines:



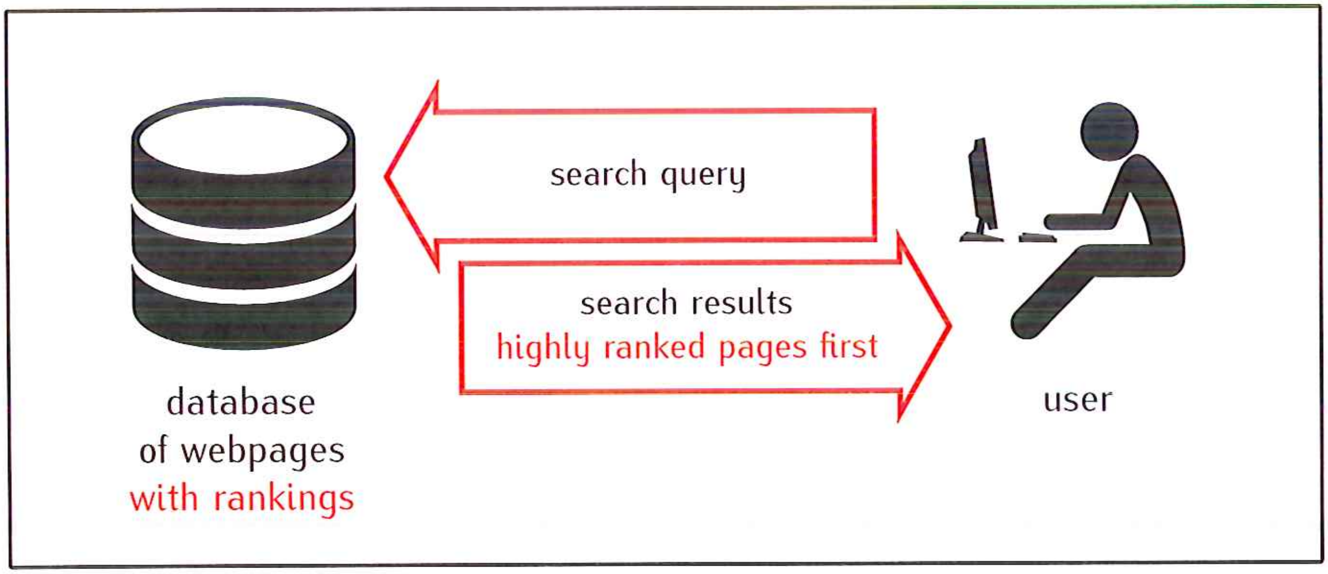
search query

search results
in random order

database
of webpages

user

Google search engine:



search query

search results
highly ranked pages first

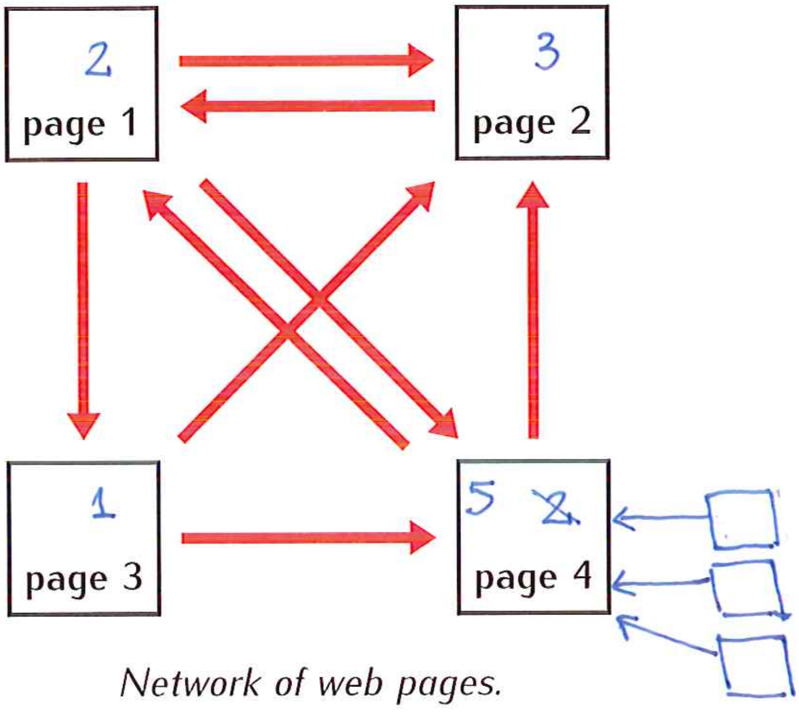
database
of webpages
with rankings

user

How to rank webpages?

Very simple ranking:

$$\text{ranking of a page} = \left(\begin{array}{c}\text{number of links}\\ \text{pointing to that page}\end{array}\right)$$



Network of web pages.

Problem. This is very easy to manipulate.

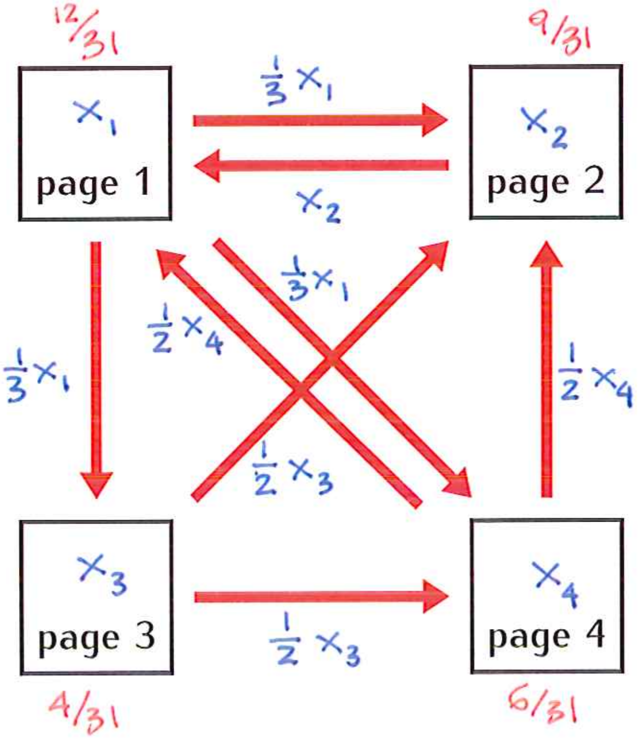
How to rank webpages?

Google PageRank: Links from highly ranked pages are worth more than links from lower ranked pages.

If:

- the rank of a page is x
- the page has n links to other pages

then each link from that page is worth x/n.



$$\begin{cases} x_1 = x_2 + \frac{1}{2}x_4 \\ x_2 = \frac{1}{3}x_1 + \frac{1}{2}x_3 + \frac{1}{2}x_4 \\ x_3 = \frac{1}{3}x_1 \\ x_4 = \frac{1}{3}x_1 + \frac{1}{2}x_3 \end{cases}$$

$\downarrow$ simplify

$$\begin{cases} x_1 - x_2 - \frac{1}{2}x_4 = 0 \\ -\frac{1}{3}x_1 + x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 = 0 \\ -\frac{1}{3}x_1 + x_3 = 0 \\ -\frac{1}{3}x_1 - \frac{1}{2}x_3 + x_4 = 0 \\ x_1 + x_2 + x_3 + x_4 = 1 \end{cases}$$

← This system has a trivial solution $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$

← Adding this equation eliminates the trivial solution

aug. matrix $\downarrow$

$$\left[\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ 1 & -1 & 0 & -1/2 & 0 \\ -1/3 & 1 & -1/2 & -1/2 & 0 \\ -1/3 & 0 & 1 & 0 & 0 \\ -1/3 & 0 & -1/2 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right] \xrightarrow{\text{row red.}} \left[\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ 1 & 0 & 0 & 0 & 12/31 \\ 0 & 1 & 0 & 0 & 9/31 \\ 0 & 0 & 1 & 0 & 4/31 \\ 0 & 0 & 0 & 1 & 6/31 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{solution}} \begin{cases} x_1 = 12/31 \\ x_2 = 9/31 \\ x_3 = 4/31 \\ x_4 = 6/31 \end{cases}$$