Linear Algebra	Calculus
$\mathbb{R}^n = \begin{pmatrix} \text{set of all column vectors} \\ \text{with } n \text{ entries} \end{pmatrix}$	$C^\infty(\mathbb{R}) = \begin{pmatrix} \text{set of all smooth} \\ \text{functions } f\colon \mathbb{R} \to \mathbb{R} \end{pmatrix}$
Column vectors can be added and multiplied by real numbers.	**Functions** can be added and multiplied by real numbers.
Linear transformation is a function $T\colon \mathbb{R}^n \to \mathbb{R}^m, \quad T(\mathbf{v}) = A\mathbf{v}$ It satisfies: • $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ • $T(c\mathbf{v}) = cT(\mathbf{v})$	**Differentiation** is a function $D\colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \quad D(f) = f'$ It satisfies: • $D(f+g) = D(f) + D(g)$ • $D(cf) = cD(f)$
Typical problem: given a vector $\mathbf{b}$ find all vectors $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{b}$ (i.e solve the equation $A\mathbf{x} = \mathbf{b}$).	**Typical problem:** given a function g find all functions f such that $D(f) = g$ (i.e find antiderivatives of g).
Fact: Such vectors x are of the form $\mathbf{x} = \mathbf{v}_0 + \mathbf{n}$ where: • $\mathbf{v}_0$ is some distinguished solution of $A\mathbf{x} = \mathbf{b}$; • $\mathbf{n} \in \mathrm{Nul}(A)$ (i.e. $\mathbf{n}$ is a solution of $A\mathbf{x} = \mathbf{0}$).	**Fact:** Such functions f are of the form $f = F + C$ where: • F is some distinguished antiderivative of g; • C is a constant function (i.e. C is a solution of $D(f) = 0$).

Definition

A *(real) vector space* is a set V together with two operations:

- addition

$$\begin{aligned} V \times V &\longrightarrow V \\ (\mathbf{u}, \ \mathbf{v}) &\longmapsto \mathbf{u} + \mathbf{v} \end{aligned}$$

- multiplication by scalars

$$\begin{aligned} \mathbb{R} \times V &\longrightarrow V \\ (c, \ \mathbf{v}) &\longmapsto c \cdot \mathbf{v} \end{aligned}$$

Moreover the following conditions must be satisfied:

1) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3) there is an element $\mathbf{0} \in V$ such that $\mathbf{0} + \mathbf{u} = \mathbf{u}$ for any $\mathbf{u} \in V$
4) for any $\mathbf{u} \in V$ there is an element $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7) $(cd)\mathbf{u} = c(d\mathbf{u})$
8) $1\mathbf{u} = \mathbf{u}$

Elements of V are called *vectors*.

Theorem

If V is a vectors space then:

1) $c \cdot \mathbf{0} = \mathbf{0}$ where $c \in \mathbb{R}$ and $\mathbf{0} \in V$ is the zero vector;

2) $0 \cdot \mathbf{u} = \mathbf{0}$ where $0 \in \mathbb{R}$, $\mathbf{u} \in V$ and $\mathbf{0}$ is the zero vector;

3) $(-1) \cdot \mathbf{u} = -\mathbf{u}$

Proof of 2) :

We have:

$$0 \cdot u = (0+0)\cdot u \overset{\text{by }(6)}{=} 0\cdot u + 0 \cdot u$$

This gives:

$$(0\cdot u) + (-(0\cdot u)) = \underbrace{(0\cdot u + 0\cdot u) + (-(0\cdot u))}$$

$$(0\cdot u) + (-(0\cdot u)) \overset{\text{by }(4)}{=} \mathbf{0}$$

$$(0\cdot u + 0\cdot u) + (-(0\cdot u)) \overset{\text{by }(2)}{=} (0\cdot u) + (0\cdot u + (-(0\cdot u))) \overset{\text{by }(4)}{=} (0\cdot u) + \mathbf{0} \overset{\text{by }(3)\text{ and }(1)}{=} 0\cdot u$$

So: $\mathbf{0} = 0 \cdot u$

Examples of vector spaces.

1) $\mathbb{R}^n = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \middle| \ a_1, \dots, a_n \in \mathbb{R} \right\}$

2) $\mathcal{F}(\mathbb{R}) = \{$the set of all functions $f: \mathbb{R} \to \mathbb{R}\}$

3) $\mathbb{P} = \{$the set of all polynomials of variable $t\}$

$= \{a_0 + a_1 t + \dots + a_m t^m \mid a_i \in \mathbb{R}, \ m \geq 0\}$

4) $M_{m,n}(\mathbb{R}) = \{$the set of all $m \times n$ matrices$\}$

$$= \left\{ \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \middle| \ a_{ij} \in \mathbb{R} \right\}$$

Defitnition

Let V be a vector space. A *subspace* of V is a subset $W \subseteq V$ such that

1) $\mathbf{0} \in W$
2) if $\mathbf{u}, \mathbf{v} \in W$ then $\mathbf{u} + \mathbf{v} \in W$
3) if $\mathbf{u} \in W$ and $c \in \mathbb{R}$ then $c\mathbf{u} \in W$.

Example.

Recall: $\mathbb{P}$ = the vector space of all polynomials.

Take $\mathbb{P}_n = \{$the set of polynomials of degree $\leq n\}$

$\mathbb{P}_n$ is a subspace of $\mathbb{P}$

Note:

Let S_3 = the set of polynomials of degree equal to $3\}$

S_3 is not a subspace of $\mathbb{P}$

E.g: $\left.\begin{array}{l} p(t) = 7 + t - 2t^2 + 3t^3 \\ q(t) = 5 - 4t + 2t^2 - 3t^3 \end{array}\right\}$ polynomials in S_3

$p(t) + q(t) = 12 - 3t\ \}$ polynomial of degree 1, not in S_3

Proposition: If V is a vector space, and $W \subseteq V$ is a subspace then W is itself a vector space.

Example.

Recall: $\mathcal{F}(\mathbb{R})$ = the vector space of all functions $f\colon \mathbb{R} \to \mathbb{R}$

<u>Some interesting subspaces of $\mathcal{F}(\mathbb{R})$:</u>

1) $C(\mathbb{R})$ = the subspace of all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$

2) $C^n(\mathbb{R})$ = the subspace of all functions $f\colon \mathbb{R} \to \mathbb{R}$ that are differentiable n or more times.

3) $C^\infty(\mathbb{R})$ = the subspace of all smooth functions $f\colon \mathbb{R} \to \mathbb{R}$ (i.e. functions that have derivatives of all orders: f', f'', f''', ...).

<u>Note</u>

Let $S = \left\{ \begin{array}{l} \text{the set of all functions } f\colon \mathbb{R} \to \mathbb{R} \\ \text{such that } f(t) \geq 0 \text{ for all } t \in \mathbb{R} \end{array} \right\}$

S is <u>not</u> a subspace of $\mathcal{F}(\mathbb{R})$

<u>E.g.</u> Take $f(t) = t^2$, then $f(t) \in S$

but $(-2)\cdot f(t) = -2\cdot t^2$ is not in S.

Note. If V is a vector space then:

1) the biggest subspace of V is V itself;
2) the smallest subspace of V is the subspace $\{0\}$ consisting of the zero vector only;
3) if a subspace of V contains a non-zero vector, then it contains infinitely many vectors.

Indeed: if W is a subspace of V and $u \in W$, $u \neq 0$ then for any $c \in \mathbb{R}$ we have $cu \in W$ and $c_1 u \neq c_2 u$ for $c_1 \neq c_2$.