Recall:

How to solve a system of linear equations

system of equations

$$\begin{cases} -x_1 + 2x_2 + 3x_3 = 4 \\ 2x_1 + 6x_3 = 9 \\ 4x_1 - x_2 - 3x_3 = 0 \end{cases}$$

make a matrix →

$\left[\text{augmented matrix} \right]$

"row reduction"
Gauss-Jordan elimination ↓

$\left[\text{matrix in reduced echelon form} \right]$

← *read off solutions*

solutions

$$\begin{cases} x_1 = \dots \\ x_2 = \dots \\ x_3 = \dots \end{cases}$$

- Every system of linear equations can be represented by a matrix
- Elementary row operations:
  - interchange of two rows
  - multiplication of a row by a non-zero number
  - addition of a multiple of one row to another row.
- Elementary row operations do not change solutions of systems of linear equations.

Definition

A matrix is in the *reduced echelon form* if:

- the non-zero entry of each row is a 1 ("a leading one");
- the leading one in each row is to the right of the leading one in the row above it;
- all entries above each leading one are 0.

leading ones

$$\begin{bmatrix} 1 & * & * & * & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

($*$ = any number)

Example

$$\begin{bmatrix} 1 & 0 & 4 & 0 & 7 & 0 \\ 0 & 1 & 5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 1 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

NOT reduced echelon form

Fact

If a system of linear equations is represented by a matrix in the reduced echelon form then it is easy to solve the system.

Example

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \begin{cases} x_1 + 3x_3 = 0 \\ x_1 + 7x_3 = 0 \\ x_4 = 0 \\ \boxed{0x_1 + 0x_2 + 0x_3 + 0x_4 = 1} \end{cases}$$

$0 = 1$

can't happen!

so: NO SOLUTIONS

Proposition

A matrix in the reduced echelon form represents an inconsistent system if and only if it contains a row of the form

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

i.e. with the leading one in the last column.

Example

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

no leading one in the column of x_3

$$\begin{cases} x_1 + 3x_3 = 0 \\ x_2 + 7x_3 = 0 \\ x_4 = 0 \\ 0 = 0 \end{cases}$$

$$\downarrow$$

$$\begin{cases} x_1 = -3x_3 \\ x_2 = -7x_3 \\ x_3 = \text{any number} \\ x_4 = 0 \end{cases}$$

x_3 is a free variable

$$\begin{cases} x_1 = -3x_3 \\ x_2 = -7x_3 \\ x_3 = \text{free} \\ x_4 = 0 \end{cases}$$

INFINITELY MANY SOLUTIONS

Note

In an augmented matrix in the reduced echelon form free variables correspond to columns of the coefficient matrix that do not contain leading ones.

Example

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 8 \end{array}\right]$$

$$\begin{cases} x_1 = 5 \\ x_2 = 6 \\ x_3 = 7 \\ x_4 = 8 \end{cases}$$

EXACTLY ONE SOLUTION

Note

A matrix in the reduced echelon represents a system of equations with exactly one solution if and only if it has a leading one in every column except for the last one.

Gauss-Jordan elimination process (= row reduction)

$$\left[\quad \text{matrix} \quad\right] \xrightarrow{\textit{elementary row operations}} \left[\quad \begin{array}{c}\text{matrix}\\ \text{in reduced}\\ \text{echelon form}\end{array} \quad\right]$$

1. Interchange rows, if necessary, to bring a non-zero element to the top of the first non-zero column of the matrix.

2. Multiply the first row so that its first non-zero entry becomes 1.

3. Add multiples of the first row to eliminate non-zero entries below the leading one.

4. Ignore the first row; apply steps 1-3 to the rest of the matrix.

5. Eliminate non-zero entries above all leading ones.

Example.

$$\begin{bmatrix} 0 & 4 & -8 & 0 & 4 \\ 2 & 6 & -6 & -2 & -4 \\ 2 & 7 & -8 & 0 & -1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 2 & 6 & -6 & -2 & -4 \\ 0 & 4 & -8 & 0 & 4 \\ 2 & 7 & -8 & 0 & -1 \end{bmatrix} \cdot \frac{1}{2}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 3 & -3 & -1 & -2 \\ 0 & 4 & -8 & 0 & 4 \\ 2 & 7 & -8 & 0 & -1 \end{bmatrix} \cdot (-2)$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 3 & -3 & -1 & -2 \\ 0 & 4 & -8 & 0 & 4 \\ 0 & 1 & -2 & 2 & 3 \end{bmatrix} \cdot \frac{1}{4}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 3 & -3 & -1 & -2 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 1 & -2 & 2 & 3 \end{bmatrix} \cdot (-1)$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 3 & -3 & -1 & -2 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix} \cdot \frac{1}{2}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & \boxed{3} & -3 & -1 & -2 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \cdot 1$$

matrix in the echelon form
(not reduced yet!)

$$\downarrow$$

$$\begin{bmatrix} 1 & \boxed{3} & -3 & 0 & -1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \cdot (-3)$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & -4 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

matrix in the reduced echelon form

(THE END)

How to solve systems of linear equations: example

$$\begin{cases} 4x_2 - 8x_3 = 4 \\ 2x_1 + 6x_2 - 6x_3 - 2x_4 = -4 \\ 2x_1 + 7x_2 - 8x_3 = -1 \end{cases}$$

$\xrightarrow{1}$

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & & \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 0 & 4 & -8 & 0 & 4 \\ 2 & 6 & -6 & -2 & -4 \\ 2 & 7 & -8 & 0 & -1 \end{array}\right]$$

(2) row reduction

(see the previous page for computations)

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & \boxed{3} & 0 & -4 \\ 0 & 1 & \boxed{-2} & 0 & 1 \\ 0 & 0 & \boxed{0} & 1 & 1 \end{array}\right]$$

free variable

$\leftarrow$

$$\begin{cases} x_1 + 3x_3 = -4 \\ x_2 - 2x_3 = 1 \\ x_4 = 1 \end{cases}$$

simplify:

$$\begin{cases} x_1 = -3x_3 - 4 \\ x_2 = 2x_3 + 1 \\ x_3 = \text{free} \\ x_4 = 1 \end{cases}$$

(infinitely many solutions)