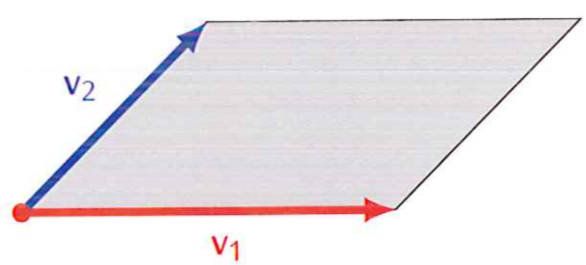
Recall:

$$\underset{\text{matrix}}{A} \longrightarrow \underset{\text{number}}{\det A} \begin{cases} = 0 & \text{if } A \text{ is not invertible} \\ \neq 0 & \text{if } A \text{ is invertible} \end{cases}$$

Note. Any two vectors in $\mathbb{R}^2$ define a parallelogram:



Notation

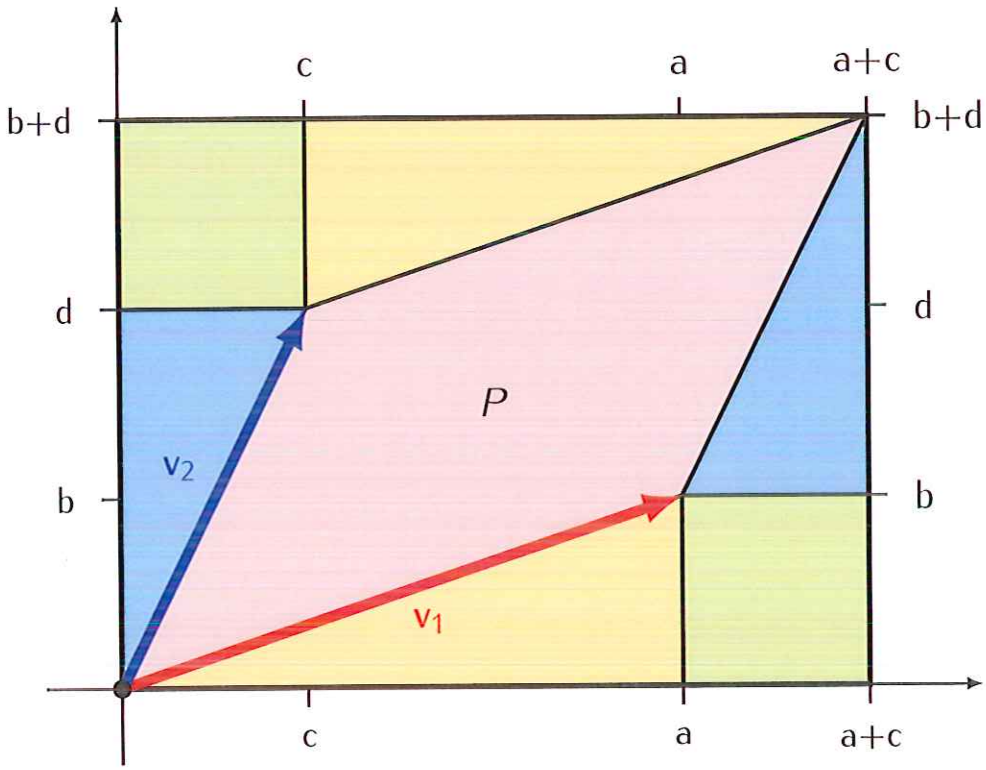
$$\text{area}(\mathbf{v}_1, \mathbf{v}_2) = \begin{pmatrix} \text{area of the parallelogram} \\ \text{defined by } \mathbf{v}_1 \text{ and } \mathbf{v}_2 \end{pmatrix}$$

Theorem

If $v_1, v_2 \in \mathbb{R}^2$ then
$$\operatorname{area}(v_1, v_2) = \left|\det\begin{bmatrix} v_1 & v_2 \end{bmatrix}\right|$$

Idea of the proof.

$$v_1 = \begin{bmatrix} a \\ b \end{bmatrix}, \quad v_2 = \begin{bmatrix} c \\ d \end{bmatrix} \qquad [v_1 \ v_2] = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$



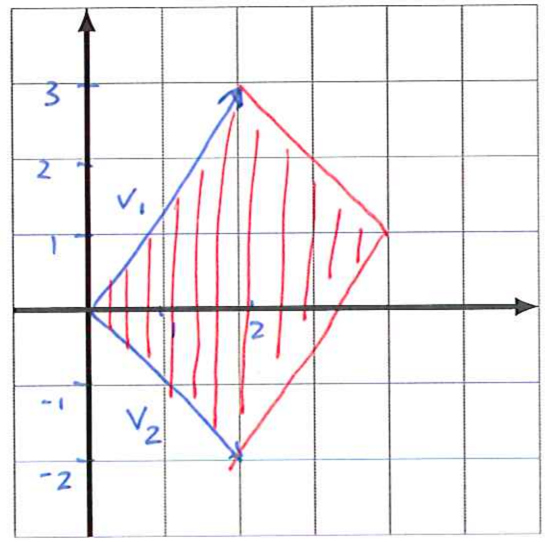
$$\begin{array}{r} \operatorname{area}(P) \\ +\ \tfrac{1}{2}ab \\ +\ \tfrac{1}{2}ab \\ +\ \tfrac{1}{2}cd \\ +\ \tfrac{1}{2}cd \\ +\ cb \\ +\ cb \\ \hline (a+c)(b+d) \end{array}$$

<u>We obtain:</u>

$$\begin{aligned} \operatorname{area}(P) &= (a+c)\cdot(b+d) - ab - cd - 2cb \\ &= (\cancel{ab} + ad + \cancel{cb} + \cancel{cd}) - \cancel{ab} - \cancel{cd} - \cancel{2}cb \\ &= ad - cb = \left|\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right| = \left|\det[v_1 \ v_2]\right| \end{aligned}$$

Example.

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$



$$\text{area}(v_1, v_2) = |\det[v_1 \ v_2]|$$

$$= \left| \det \begin{bmatrix} 2 & 2 \\ 3 & -2 \end{bmatrix} \right|$$

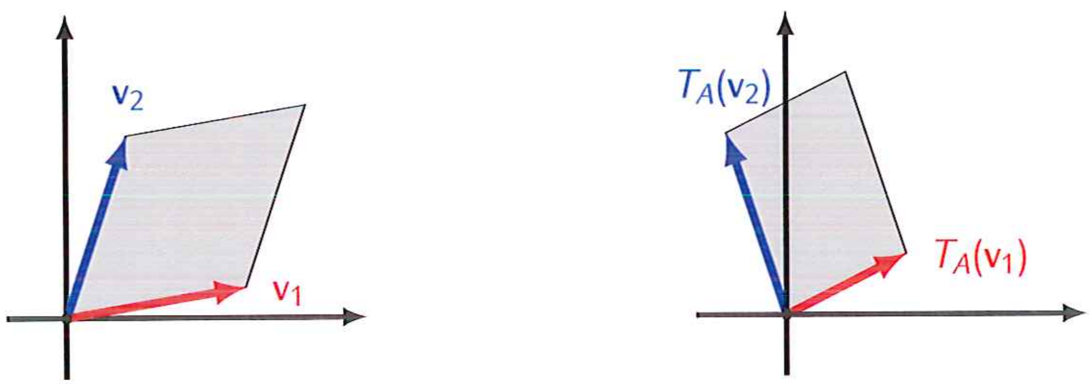
$$= |2\cdot(-2) - 2\cdot 3| = |-4-6| = 10$$

<u>Determinants and linear transformations</u>

Recall: If A is a 2×2 matrix then it defines a linear transformation

$$T_A\colon \mathbb{R}^2 \to \mathbb{R}^2 \qquad T_A(\mathbf{v}) = A\mathbf{v}$$

Note. T_A maps parallelograms to parallelograms:



Theorem

If A is a 2×2 matrix and $\mathbf{v}_1, \mathbf{v}_1 \in \mathbb{R}^2$ then

$$\operatorname{area}(T_A(\mathbf{v}_1), T_A(\mathbf{v}_2)) = |\det A| \cdot \operatorname{area}(\mathbf{v}_1, \mathbf{v}_2)$$

Proof:

$$\operatorname{area}(T_A(v_1), T_A(v_2)) = \operatorname{area}(Av_1, Av_2)$$
$$= |\det[Av_1 \ Av_2]|$$

Recall: $[Av_1 \ Av_2] = A\cdot[v_1 \ v_2]$

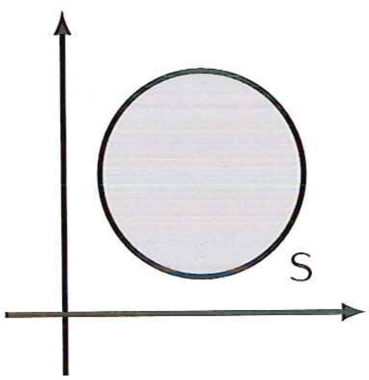
So: $\operatorname{area}(T_A(v_1), T_A(v_2)) = |\det(A\cdot[v_1, v_2])|$
$$= |(\det A)\cdot(\det[v_1 \ v_2])|$$
$$= |\det A|\cdot \operatorname{area}(v_1, v_2)$$

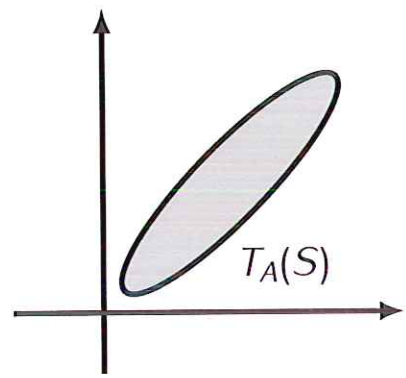
Generalization:

Theorem

If A is a 2×2 matrix then for any region S of $\mathbb{R}^2$ we have:

$$\text{area}(T_A(S)) = |\det A| \cdot \text{area}(S)$$

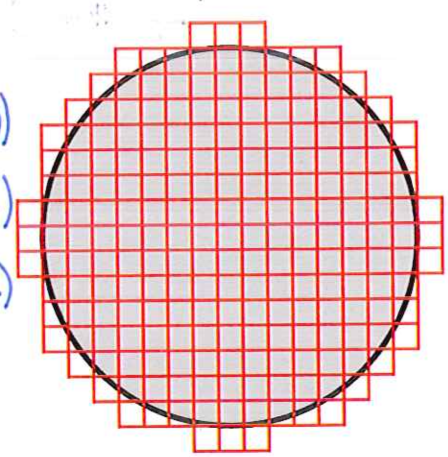




Idea of the proof.

The area of S can be approximated by the sum of small squares covering S.

$$\begin{aligned}
\text{area}(S) &\approx \sum \text{area (small square)} \\
\text{area}(T_A(S)) &\approx \sum \text{area}(T_A(\text{small square})) \\
&= \sum |\det A| \cdot \text{area (small sq.)} \\
&= |\det A| \cdot \sum \text{area (small sq.)} \\
&\approx |\det A| \cdot \text{area}(S)
\end{aligned}$$



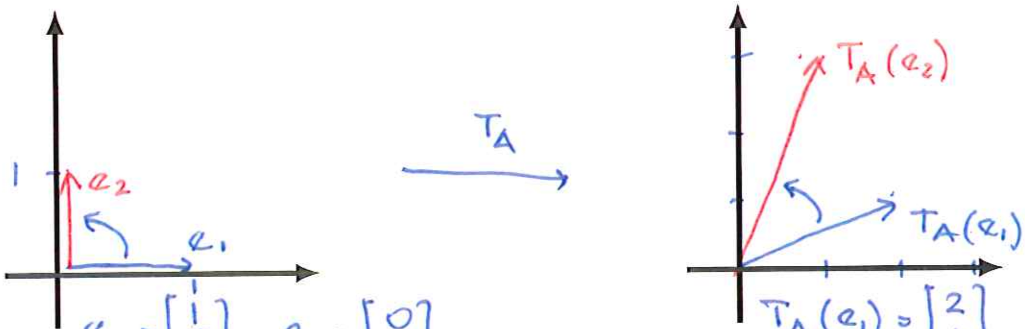
Sign of the determinant

Example.

$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ $\underbrace{\det A = 5 > 0}$

$T_A\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$v \longmapsto Av$



$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $T_A(e_1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $T_A(e_2) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

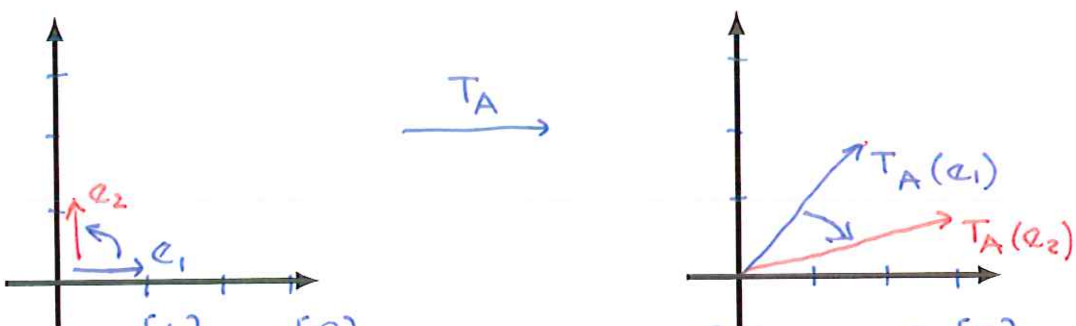
T_A preserves direction of angles between vectors
(We say that T_A <u>preserves orientation</u>)

Example.

$A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$ $\underbrace{\det A = -4}$

$T_A\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$v \longmapsto Av$



$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $T_A(e_1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $T_A(e_2) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

T_A reverses direction of anges between vectors
(We say that T_A <u>reverses orientation</u>)

Theorem

If A is a 2×2 matrix then the linear transformation $T_A\colon \mathbb{R}^2 \to \mathbb{R}^2$ preserves orientation if $\det A > 0$ and reverses orientation if $\det A < 0$.