Recall: If A is an upper triangular matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

then $\det A = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$.

Note. If A is a square matrix then the reduced echelon form of A is always upper triangular.

$$A \xrightarrow{\text{row red.}} \begin{bmatrix} \text{reduced} \\ \text{matrix} \end{bmatrix}$$

arbitrary square matrix

upper triangular so easy to compute the determinant

Note: There is a direct way of computing the determinant of a 3x3 matrix:

$$\begin{array}{cccccc} + & + & + & - & - & - \end{array}$$

$$A = \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}_{\text{matrix}} \underbrace{\begin{matrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{matrix}}_{\text{copy the first two columns}}$$

$$\det A = (1\cdot 5\cdot 9) + (2\cdot 6\cdot 7) + (3\cdot 4\cdot 8) - (3\cdot 5\cdot 7) - (1\cdot 6\cdot 8) - (2\cdot 4\cdot 9)$$

Warning: This works for 3x3 matrices only.

Theorem

Let A and B be $n \times n$ matrices.

1) If B is obtained from A by interchanging two rows (or two columns) then

$$\det B = -\det A$$

2) If B is obtained from A by multiplying one row (or one column) of A by a scalar k then

$$\det B = k \cdot \det A$$

2) If B is obtained from A by adding a multiple of one row of A to another row (or adding a multiple of one column to another column) then

$$\det B = \det A$$

Example.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 7 \\ 2 & 5 & 1 \end{bmatrix}$$

$\det A = 1 \cdot C_{11} + 2 \cdot C_{12} + 3 \cdot C_{13}$

$C_{11} = (-1)^{1+1} \cdot \det \begin{bmatrix} 0 & 7 \\ 5 & 1 \end{bmatrix}, \; C_{12} = \ldots$

$$B = \begin{bmatrix} 4 \cdot 1 & 4 \cdot 2 & 4 \cdot 3 \\ 1 & 0 & 7 \\ 2 & 5 & 1 \end{bmatrix}$$

$\det B = 4 \cdot 1 \cdot C_{11} + 4 \cdot 2 \cdot C_{12} + 4 \cdot 3 \cdot C_{13}$

$C_{11} = (-1)^{1+1} \cdot \det \begin{bmatrix} 0 & 7 \\ 5 & 1 \end{bmatrix}, \; C_{12} = \ldots$

so:

$\det B = 4 \cdot (1 \cdot C_{11} + 2 \cdot C_{12} + 3 \cdot C_{13})$

$= 4 \cdot \det A$

Computation of determinants via row reduction

Idea. To compute $\det A$, row reduce A to the echelon form. Keep track how the determinant changes at each step of the row reduction process.

Example. Compute $\det A$ where

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 4 & 0 & 10 \\ 3 & 4 & 1 & 7 \\ -2 & 5 & 3 & 0 \end{bmatrix}$$

$$\det\begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 4 & 0 & 10 \\ 3 & 4 & 1 & 7 \\ -2 & 5 & 3 & 0 \end{bmatrix} = (-1)\cdot\det\begin{bmatrix} 2 & 4 & 0 & 10 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 1 & 7 \\ -2 & 5 & 3 & 0 \end{bmatrix}\begin{matrix}\cdot\left(\frac{1}{2}\right)\\ \\ \\ \\ \end{matrix}$$

$$= 2\cdot(-1)\cdot\det\begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 1 & 7 \\ -2 & 5 & 3 & 0 \end{bmatrix}\begin{matrix}(-3)\\ (2)\end{matrix} = 2\cdot(-1)\cdot\det\begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & -2 & 1 & -8 \\ 0 & 9 & 3 & 10 \end{bmatrix}\begin{matrix}2\\ (-9)\end{matrix}$$

reciprocal of $\frac{1}{2}$

$$= 2\cdot(-1)\cdot\det\begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -15 & -17 \end{bmatrix}\begin{matrix}\\ \\ \cdot 3\\ \end{matrix} = 2\cdot(-1)\cdot\det\begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & -23 \end{bmatrix}$$

upper triangular

$$= 2\cdot(-1)\cdot(1\cdot 1\cdot 5\cdot(-23))$$

$$= \underline{\underline{230}}$$

Theorem

If A is a square matrix then A is invertible if and only if $\det A \neq 0$

Recall: A is invertible if and only if its reduced echelon form is the identity matrix.

Proof:

$$\det A = (\text{some non-zero number}) \cdot \det \begin{bmatrix} \text{reduced} \\ \text{echelon} \\ \text{form} \\ \text{of } A \end{bmatrix}$$

1) if A is invertible then

$$\det A = (\text{some non-zero number}) \cdot \overbrace{\det I}^{1} \neq 0$$

the identity matrix

2) if A is not invertible then

$$\begin{bmatrix} \text{reduced} \\ \text{form} \\ \text{of } A \end{bmatrix} = \begin{array}{l} \text{upper triangular matrix} \\ \text{with a zero on the main diagonal} \end{array}$$

$$\left(\text{e.g.} \quad \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

This gives:

$$\det \left(\begin{bmatrix} \text{reduced} \\ \text{form} \\ \text{of } A \end{bmatrix} \right) = 0$$

so: $\det A = 0$.

Further properties of determinants

1) $\det(A^T) = \det A$
2) $\det(AB) = (\det A) \cdot (\det B)$
3) $\det(A^{-1}) = (\det A)^{-1}$

Note. In general $\det(A + B) \neq \det A + \det B$.

example:

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\det A = 0$ $\det B = 0$

$\det(A+B) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$