Recall:

1) The dot product in $\mathbb{R}^n$:

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = a_1b_1 + a_2b_2 + \ldots a_nb_n$$

2) Properties of the dot product:

a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

c) $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$

d) $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

2) Using the dot product we can define:

- length of vectors
- distance between vectors
- orthogonality of vectors
- orthogonal and orthonormal bases
- orthogonal projection of a vector onto a subspace of $\mathbb{R}^n$
- ...

Next: Generalization to arbitrary vector spaces.

Definition

Let V be a vector space. An *inner product* on V is a function

$$\begin{aligned} V \times V &\longrightarrow \mathbb{R} \\ \mathbf{u},\ \mathbf{v} &\longmapsto \langle \mathbf{u}, \mathbf{v} \rangle \end{aligned}$$

such that:

a) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$

b) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

c) $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$

d) $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Definition

Let V be a vector space with an inner product $\langle\ ,\ \rangle$.

1) The *length* (or *norm*) of a vector $\mathbf{v}$ is the number $||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.

2) The *distance* between vectors $\mathbf{u}, \mathbf{v} \in V$ is the number $\text{dist}(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$.

3) Vectors $\mathbf{u}, \mathbf{v} \in V$ are *orthogonal* if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Example. The dot product is an inner product in $\mathbb{R}^n$.

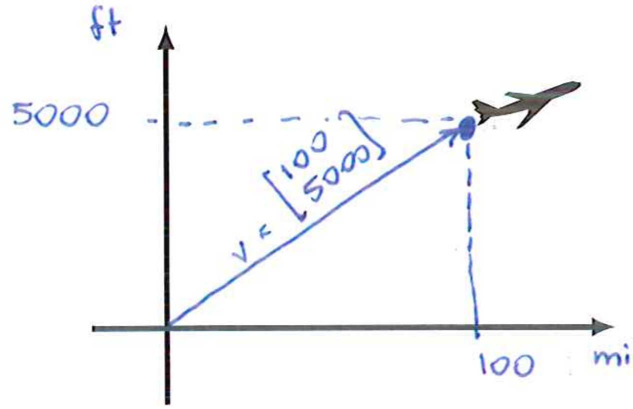
Example. Let $p_1, \ldots, p_n$ be any positive numbers. For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$

$$\mathbf{u} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

define:

$$\langle \mathbf{u}, \mathbf{v} \rangle = p_1(a_1 b_1) + p_2(a_2, b_2) + \ldots + p_n(a_n b_n)$$

This gives an inner product in $\mathbb{R}^n$.



$1\text{ft} = 0.0002 \text{ mi}$
(denote: $\delta = 0.0002$)

length of v $= \sqrt{100^2 + (\delta \cdot 5000)^2}$

$= \sqrt{100^2 + \delta^2 \cdot 5000^2}$ miles

In $\mathbb{R}^2$ define:

$$\left\langle \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \right\rangle = a_1 b_1 + \delta^2 a_2 b_2$$

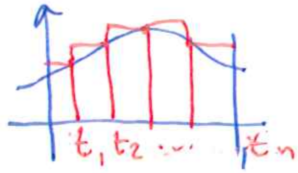
Then:

$$\|v\| = \sqrt{\langle v, v \rangle} = \sqrt{a_1^2 + \delta^2 a_2^2}$$

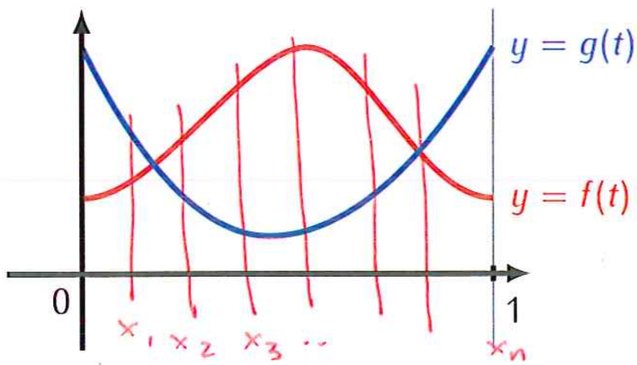
Example. Let $C[0,1]$ be the vector space of continuous functions $f\colon [0,1] \to \mathbb{R}$. Define:

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt$$

This is an inner product on $C[0,1]$.



$$\int_0^1 f(t)\,dt \approx \frac{1}{n} f(t_1) + \dots \frac{1}{n} f(t_n) = \frac{1}{n}\left(f(t_1) + \dots + f(t_n)\right)$$



$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt$$

$$\approx \frac{1}{n}\left(f(t_1)g(t_1) + f(t_2)g(t_2) + \dots + f(t_n)g(t_n)\right)$$

Example. Compute the length of the function

$$f(t) = 1 + t^2$$

in $C[0,1]$.

Solution:

$$\|f\| = \sqrt{\langle f, f\rangle}$$

$$\langle f, f\rangle = \int_0^1 f(t)\cdot f(t)\,dt = \int_0^1 (1+t^2)^2 dt = \int_0^1 1 + 2t^2 + t^4\, dt$$

$$= \left(t + \frac{2}{3}t^3 + \frac{1}{5}t^5\right)\Bigg|_{t=0}^{t=1} = \frac{28}{15}$$

So: $\|f\| = \sqrt{\frac{28}{15}}$

Definition

Let V be a vector space with an inner product $\langle\ ,\ \rangle$, and let W be a subspace of V. A vector $\mathrm{v} \in V$ is *orthogonal to* W if $\langle \mathrm{v}, \mathrm{w} \rangle = 0$ for all $\mathrm{w} \in W$.

Definition

Let V be a vector space with an inner product $\langle\ ,\ \rangle$, and let W be a subspace of V. The *orthogonal projection of a vector* $\mathrm{v} \in V$ *onto* W is a vector $\mathrm{proj}_W \mathrm{v}$ such that

1) $\mathrm{proj}_W \mathrm{v} \in W$
2) the vector $\mathrm{z} = \mathrm{v} - \mathrm{proj}_W \mathrm{v}$ is orthogonal to W.

Best Approximation Theorem

If V is a vector space with an inner product $\langle\ ,\ \rangle$, W is a subspace of V, and $\mathrm{v} \in V$, then $\mathrm{proj}_W \mathrm{v}$ is the vector of V which is the closest to v:

$$\mathrm{dist}(\mathrm{v}, \mathrm{proj}_W \mathrm{v}) \leq \mathrm{dist}(\mathrm{v}, \mathrm{w})$$

for all $\mathrm{w} \in W$.

Theorem

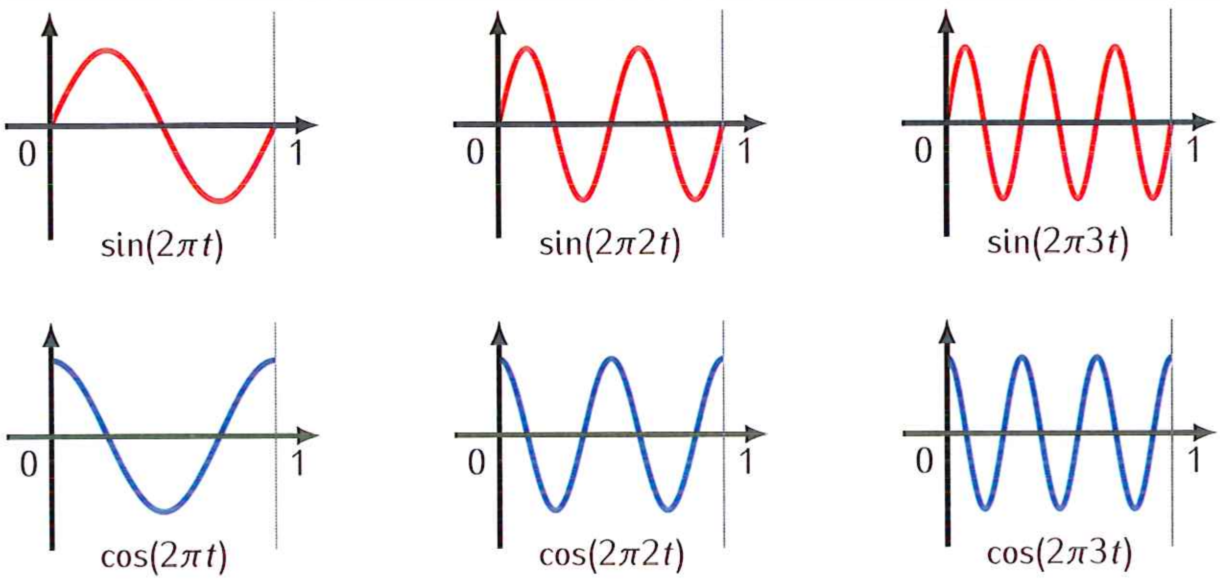
Let V is a vector space with an inner product $\langle\ ,\ \rangle$, and let W be a subspace of V. If $\mathcal{B} = \{\mathrm{w}_1, \ldots, \mathrm{w}_k\}$ is an orthogonal basis of W (i.e. a basis such that $\langle \mathrm{w}_i, \mathrm{w}_j \rangle = 0$ for all $i \neq j$) then for $\mathrm{v} \in V$ we have:

$$\mathrm{proj}_W \mathrm{v} = \frac{\langle \mathrm{v}, \mathrm{w}_1 \rangle}{\langle \mathrm{w}_1, \mathrm{w}_1 \rangle} \mathrm{w}_1 + \ldots + \frac{\langle \mathrm{v}, \mathrm{w}_k \rangle}{\langle \mathrm{w}_k, \mathrm{w}_k \rangle} \mathrm{w}_k$$

Application: Fourier approximations.

Goal: Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. Find the best possible approximation of f of the form

$$\begin{aligned} P(t) = \quad & a_0 \\ & + a_1 \sin(2\pi t) \;\; + \;\; b_1 \cos(2\pi t) \\ & + a_2 \sin(2\pi 2t) \; + \;\; b_2 \cos(2\pi 2t) \\ & \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ & + a_n \sin(2\pi n t) + \;\; b_n \cos(2\pi n t) \end{aligned}$$



Note: Let W_n be a subspace of $C[0,1]$ given by:

$$W_n = \mathrm{Span}(1, \sin(2\pi t), \cos(2\pi t), \ldots, \sin(2\pi n t), \cos(2\pi n t))$$

By the Best Approximation Theorem, the best approximation of f is obtained if we take $P(t) = \mathrm{proj}_{W_n} f(t)$.

Theorem

The set
$$\{1, \sin(2\pi t), \cos(2\pi t), \ldots, \sin(2\pi n t), \cos(2\pi n t)\}$$
is an orthogonal basis of W_n.

Corollary

If $f \in C[0,1]$ then

$$\begin{aligned}
\mathrm{proj}_{W_n} f(t) = \quad & a_0 \\
& + a_1 \sin(2\pi t) \quad + \quad b_1 \cos(2\pi t) \\
& + a_2 \sin(2\pi 2t) \ + \quad b_2 \cos(2\pi 2t) \\
& \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
& + a_n \sin(2\pi n t) \ + \quad b_n \cos(2\pi n t)
\end{aligned}$$

where:

$$a_0 = \frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} = \int_0^1 f(t)dt$$

and for $k > 0$:

$$a_k = \frac{\langle f, \sin(2\pi k t) \rangle}{\langle \sin(2\pi k t), \sin(2\pi k t) \rangle} = 2\int_0^1 f(t) \cdot \sin(2\pi k t) dt$$

$$b_k = \frac{\langle f, \cos(2\pi k t) \rangle}{\langle \cos(2\pi k t), \cos(2\pi k t) \rangle} = 2\int_0^1 f(t) \cdot \cos(2\pi k t) dt$$

Example. Compute $\mathrm{proj}_{W_n} f(t)$ for the function $f(t) = t$.

Solution :

$$a_0 = \int_0^1 1 \cdot f(t)\,dt = \int_0^1 t\,dt = \frac{1}{2}t^2 \Big|_{t=0}^{t=1} = 1/2$$

$$a_k = 2\int_0^1 f(t)\cdot \sin(2\pi k t)\,dt = 2\int_0^1 t \sin(2\pi k t)\,dt = -\frac{1}{\pi k}$$

$$b_k = 2\int_0^1 f(t)\cdot \cos(2\pi k t)\,dt = 2\int_0^1 t\cos(2\pi k t)\,dt = 0$$

This gives:

$$\begin{aligned}\mathrm{proj}_{W_n} f(t) = \frac{1}{2} &- \frac{1}{\pi}\sin(2\pi t) \\ &- \frac{1}{2\pi}\sin(2\pi 2t) \\ &- \frac{1}{3\pi}\sin(2\pi 3t) \\ &\vdots \\ &- \frac{1}{n\pi}\sin(2\pi n t)\end{aligned}$$

Application: Polynomial approximations.

Goal: Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. Find the best possible approximation of f given by a polynomial $P(t)$ of degree $\leq n$:

$$P(t) = a_0 + a_1 t + \ldots + a_n t^n$$

Note: Let $\mathbb{P}_n$ be the subspace of $C[0,1]$ consisting of all polynomials of degree $\leq n$:

$$\mathbb{P}_n = \{a_0 + a_1 t + \ldots + a_n t^n \mid a_k \in \mathbb{R}\}$$

By the Best Approximation Theorem, the best approximation of f is obtained if we take $P(t) = \mathrm{proj}_{\mathbb{P}_n} f(t)$.

Note: In order to compute $\mathrm{proj}_{\mathbb{P}_n} f(t)$ we need an orthogonal basis of $\mathbb{P}_n$.

$\mathbb{P}_n$ has the standard basis: $\mathcal{E} = \{1, t, t^2, \ldots, t^n\}$ but this basis is not orthogonal

e.g.

$$\langle t, t^2 \rangle = \int_0^1 t \cdot t^2 \, dt = \int_0^1 t^3 \, dt = \frac{1}{4} t^4 \Big|_{t=0}^{t=1} = \frac{1}{4} \neq 0$$

Gram-Schmidt process:

a basis $\{v_1, \ldots, v_k\}$ of $W \subseteq V$	$\xrightarrow{\text{G-S process}}$	an orthogonal basis $\{w_1, \ldots, w_k\}$ of W

Theorem (Gram-Schmidt Process)

Let V be a vector space with an inner product $\langle\ ,\ \rangle$, and let W be a subspace of V. Let $\{v_1, \ldots, v_k\}$ be a basis of W. Define vectors $\{w_1, \ldots, w_k\}$ as follows:

$$
\begin{aligned}
w_1 &= v_1 \\
w_2 &= v_2 - \frac{\langle w_1, v_2 \rangle}{\langle w_1, w_1 \rangle} w_1 \\
w_3 &= v_3 - \frac{\langle w_1, v_3 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle w_2, v_3 \rangle}{\langle w_2, w_2 \rangle} w_2 \\
\cdots &\quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
w_k &= v_k - \frac{\langle w_1, v_k \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle w_2, v_k \rangle}{\langle w_2, w_2 \rangle} w_2 - \ldots - \frac{\langle w_{k-1}, v_k \rangle}{\langle w_{k-1}, w_{k-1} \rangle} w_{k-1}
\end{aligned}
$$

Then the set $\{w_1, \ldots, w_k\}$ is an orthogonal basis of W.

Example. Find an orthogonal basis of the subspace $\mathbb{P}_2$ of the vector space $C[0,1]$.

The standard basis of $\mathbb{P}_2$: $\mathcal{E} = \{\overset{v_1}{1}, \overset{v_2}{t}, \overset{v_3}{t^2}\}$

Use G-S process to get an orthogonal basis:

$$\boxed{w_1 = v_1 = 1}$$

$$w_2 = v_2 - \frac{\langle w_1, v_2\rangle}{\langle w_1, w_1\rangle} w_1$$

$$\langle w_1, v_2\rangle = \int_0^1 1\cdot t\, dt = \frac{1}{2}t^2\Big|_{t=0}^{t=1} = \frac{1}{2}$$

$$\langle w_1, w_1\rangle = \int_0^1 1\cdot 1\, dt = t\Big|_{t=0}^{t=1} = 1$$

So: $$\boxed{w_2 = t - \frac{1/2}{1}\cdot 1 = \left(t - \frac{1}{2}\right)}$$

$$w_3 = v_3 - \frac{\langle w_1, v_3\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle w_2, v_3\rangle}{\langle w_2, w_2\rangle} w_2$$

$$\langle w_1, v_3\rangle = \int_0^1 1\cdot t^2\, dt = \frac{1}{3}t^3\Big|_{t=0}^{t=1} = 1/3$$

$$\langle w_1, w_1\rangle = 1$$

$$\langle w_2, v_3\rangle = \int_0^1 \left(t-\frac{1}{2}\right)t^2\, dt = \int_0^1 t^3 - \frac{1}{2}t^2 dt = \frac{1}{4}t^4 - \frac{1}{6}t^3\Big|_{t=0}^{t=1} = \frac{1}{12}$$

$$\langle w_2, w_2\rangle = \int_0^1 \left(t-\frac{1}{2}\right)^2 = \int_0^1 t^2 - t + \frac{1}{4}\, dt = \frac{1}{3}t^3 - \frac{1}{2}t^2 + \frac{1}{4}t\Big|_{t=1}^{t=0} = \frac{1}{12}$$

$$\boxed{w_3 = t^2 - \frac{1}{3}\cdot 1 - \cancel{\frac{1/12}{1/12}}\left(t - \frac{1}{2}\right) = t^2 - t + \frac{1}{6}}$$

Orthogonal basis of $\mathbb{P}_2$: $\left\{1, \left(t-\frac{1}{2}\right), \left(t^2 - t + \frac{1}{6}\right)\right\}$

Example. Compute $\text{proj}_{\mathbb{P}_2} f(t)$ for $f(t) = \sqrt{t}$.

Solution:

We had: $\{\overset{w_1}{1}, \overset{w_2}{(t-\frac{1}{2})}, \overset{w_3}{(t^2-t+\frac{1}{6})}\}$ - orthogonal basis of $\mathbb{P}_2$

This gives:

$$\text{proj}_{\mathbb{P}_2} f(t) = \frac{\langle w_1, f\rangle}{\langle w_1, w_1\rangle} \cdot w_1 + \frac{\langle w_2, f\rangle}{\langle w_2, w_2\rangle} \cdot w_2 + \frac{\langle w_3, f\rangle}{\langle w_3, w_3\rangle} \cdot w_3$$

$$\langle w_1, w_1\rangle = 1$$

$$\langle w_2, w_2\rangle = \frac{1}{12}$$

$$\langle w_3, w_3\rangle = \int_0^1 (t^2-t+\tfrac{1}{6})^2 dt = \frac{1}{180}$$

$$\langle w_1, f\rangle = \int_0^1 \sqrt{t}\, dt = \frac{2}{3} t^{3/2} \Big|_{t=0}^{t=1} = \frac{2}{3}$$

$$\langle w_2, f\rangle = \int_0^1 \sqrt{t}\,(t-\tfrac{1}{2}) dt = \frac{1}{15}$$

$$\langle w_3, f\rangle = \int_0^1 \sqrt{t}\,(t^2-t+\tfrac{1}{6}) dt = -\frac{1}{315}$$

$$\text{proj}_{\mathbb{P}_2} f(t) = \frac{2}{3}\cdot 1 + \frac{1/15}{1/12}(t-\tfrac{1}{2}) + \frac{(-1/315)}{(1/180)}(t^2-t+\tfrac{1}{6})$$

$$\boxed{= -\frac{4}{7}t^2 + \frac{48}{35} + \frac{6}{35}}$$