Recall:

- A basis of a vector space $V$ is a set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ such that
  1) $\mathrm{Span}(\mathbf{b}_1, \ldots, \mathbf{b}_n) = V$
  2) The set $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is linearly independent.

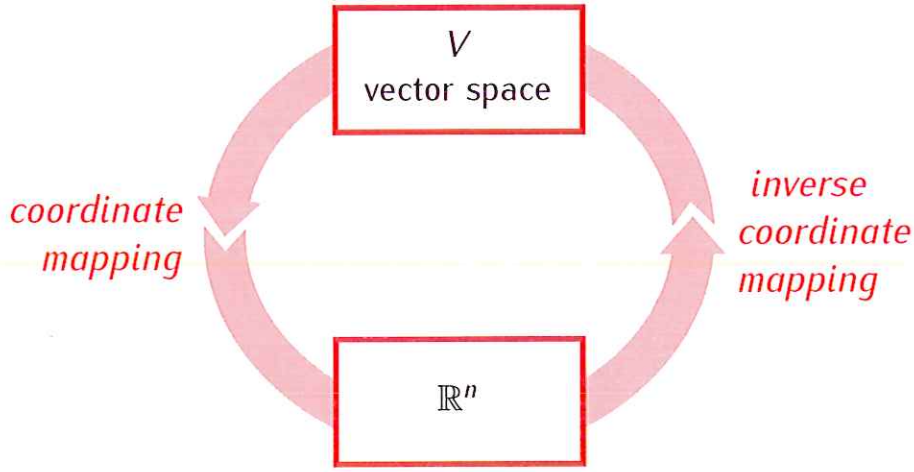
- For $\mathbf{v} \in V$ let $c_1, \ldots, c_n$ be the unique numbers such that

$$c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n = \mathbf{v}$$

The vector

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$$

is called the *coordinate vector of* $\mathbf{v}$ *relative to the basis* $\mathcal{B}$.




**Theorem**

Let $\mathcal{B}$ be a basis of a vector space $V$. If $v_1, \dots v_p, w \in V$ then:

1) Solutions of the equation $x_1 v_1 + \dots + x_p v_p = w$ are the same as solutions of the equation $x_1 [v_1]_{\mathcal{B}} + \dots + x_p [v_p]_{\mathcal{B}} = [w]_{\mathcal{B}}$.

2) The set of vectors $\{v_1, \dots v_p\}$ is linearly independent if and only if the set $\{[v_1]_{\mathcal{B}}, \dots, [v_p]_{\mathcal{B}}\}$ is linearly independent.

3) $\text{Span}(v_1, \dots, v_p) = V$ if any only if $\text{Span}([v_1]_{\mathcal{B}}, \dots, [v_p]_{\mathcal{B}}) = \mathbb{R}^n$.

4) $\{v_1, \dots, v_p\}$ is a basis of $V$ if and only if $\{[v_1]_{\mathcal{B}}, \dots, [v_p]_{\mathcal{B}}\}$ is a basis of $\mathbb{R}^n$.

**Example.** Recall that $\mathbb{P}_2$ is the vector space of polynomials of degree $\leq 2$. Consider the following polynomials in $\mathbb{P}_2$:

$$\begin{aligned} p_1(t) &= 1 + 2t + t^2 \\ p_2(t) &= 3 + t + 2t^2 \\ p_3(t) &= 1 - 8t - t^2 \end{aligned}$$

Check if the set $\{p_1, p_2, p_3\}$ is linearly independent.

Recall:

In $\mathbb{P}_2$ we have the standard basis $\mathcal{E} = \{1, t, t^2\}$.

We have:

$$[p_1]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad [p_2]_{\mathcal{E}} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \quad [p_3]_{\mathcal{E}} = \begin{bmatrix} 1 \\ -8 \\ -1 \end{bmatrix}$$

It will suffice to check if the set $\{[p_1]_{\mathcal{E}}, [p_2]_{\mathcal{E}}, [p_3]_{\mathcal{E}}\}$ is linearly independent.

We need to solve: $x_1 [p_1]_{\mathcal{E}} + x_2 [p_2]_{\mathcal{E}} + x_3 [p_3]_{\mathcal{E}} = 0$

Augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 2 & 1 & -8 & 0 \\ 1 & 2 & -1 & 0 \end{array}\right] \xrightarrow{\text{row red.}} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \quad \begin{cases} x_1 = 5x_3 \\ x_2 = -2x_3 \\ x_3 = \text{free} \end{cases}$$

Infinitely many solutions, so $\{[p_1]_{\mathcal{E}}, [p_2]_{\mathcal{E}}, [p_3]_{\mathcal{E}}\}$ is not lin. indep., and so $\{p_1, p_2, p_3\}$ is also not lin. indep.

**Theorem**

Let $\{v_1, \dots, v_p\}$ be vectors in $\mathbb{R}^n$. The set $\{v_1, \dots, v_p\}$ is a basis of $\mathbb{R}^n$ if and only if the matrix

$$A = \begin{bmatrix} v_1 & \dots & v_p \end{bmatrix}$$

has a pivot position in every row and in every column (i.e. if $A$ is an invertible matrix).

Proof:

By definition $\{v_1, \dots, v_p\}$ is a basis of $\mathbb{R}^n$ if and only if

1) $\{v_1, \dots, v_p\}$ is lin. independent ($\Leftrightarrow$ $[v_1 \dots v_p]$ has a pivot position in each column)

2) $\mathrm{Span}(v_1, \dots, v_p) = \mathbb{R}^n$ ($\Leftrightarrow$ $[v_1 \dots v_p]$ has a pivot position in each row)

Example: $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $v_3 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

$$[v_1 \ v_2 \ v_3] = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \xrightarrow{\text{row red}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \leftarrow$$

$\mathrm{Span}(v_1, v_2, v_3) = \mathbb{R}^2$ but $\{v_1, v_2, v_3\}$ is not lin. indep. so $\{v_1, v_2, v_3\}$ is not a basis of $\mathbb{R}^2$.

**Corollary**

Every basis of $\mathbb{R}^n$ consists of $n$ vectors.

**Theorem**

Let $V$ be a vector space. If $V$ has a basis consisting of $n$ vectors then every basis of $V$ consists of $n$ vectors.

Proof:

Let $\mathcal{B} = \{b_1, \dots, b_n\}$ and $\mathcal{D} = \{d_1, \dots, d_m\}$ be two bases of V. We have:

1) For each $v \in V$ the coordinate vector $[v]_{\mathcal{B}}$ is a vector in $\mathbb{R}^n$.

2) Since $\{d_1, \dots, d_m\}$ is a basis of V, the set $\{[d_1]_{\mathcal{B}}, \dots, [d_m]_{\mathcal{B}}\}$ is a basis of $\mathbb{R}^n$.

Since every basis of $\mathbb{R}^n$ consists of n vectors we obtain $m = n$.

**Definition**

A vector space has *dimension* $n$ if $V$ has a basis consisting of $n$ vectors. Then we write $\dim V = n$.

**Example.**

1) In $\mathbb{R}^n$ take

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

This gives the standard basis of $\mathbb{R}^n$:

$$\mathcal{E} = \{e_1, \ldots, e_n\}$$

Since $\mathcal{E}$ consists of $n$ vectors we obtain that $\dim \mathbb{R}^n = n$.

2) Recall: $\mathbb{P}_n = \{$the vector space of polynomials of degree $\leq n\}$

$$= \{a_0 + a_1 t + \ldots + a_n t^n \mid a_i \in \mathbb{R}\}$$

The standard basis of $\mathbb{P}_n$: $\mathcal{E} = \{1, t, t^2, \ldots, t^n\}$

Since $\mathcal{E}$ consists of $n+1$ vectors we get: $\dim \mathbb{P}_n = n+1$

3) Recall: $M_{m,n}(\mathbb{R}) = \{$the vector space of $m \times n$ matrices$\}$

$$= \left\{ \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \;\middle|\; a_{ij} \in \mathbb{R} \right\}$$

Let $B_{kl} = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \\ b_{m1} & \cdots & b_{mn} \end{bmatrix}$ - matrix s.t. $b_{ij} = \begin{cases} 1 & \text{if } i=k, j=l \\ 0 & \text{otherwise} \end{cases}$

(e.g: $B_{11} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$)

Check: $\mathcal{B} = \{B_{11}, B_{12}, \ldots, B_{mn}\}$ is a basis of $M_{m,n}(\mathbb{R})$.

Since $\mathcal{B}$ consists of $m \cdot n$ elements we get that

$$\dim M_{m,n}(\mathbb{R}) = m \cdot n$$

**Theorem**

Let $V$ be a vector space such that $\dim V = n$, and let $\mathbf{v}_1, \dots \mathbf{v}_p \in V$.

1) If $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a spanning set of $V$ then $p \geq n$.

2) If $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a linearly independent set then $p \leq n$.

__Proof:__ It is enough to check it if $V = \mathbb{R}^n$.

1) If $v_1, \dots, v_p$ are vectors in $\mathbb{R}^n$ and $p < n$ then the matrix $[v_1 \dots v_p]$ cannot have a pivot position in each row, so $\mathrm{Span}(v_1, \dots, v_p) \neq \mathbb{R}^n$ e.i. $\{v_1, \dots, v_p\}$ is not a spanning set of $\mathbb{R}^n$

2) If $v_1, \dots, v_p$ are vectors in $\mathbb{R}^n$ and $p > n$ then the matrix $[v_1 \dots v_p]$ cannot have a pivot position in each column, so the set $\{v_1, \dots, v_p\}$ is not linearly independent.

**Corollary**

Let $V$ be a vector space such that $\dim V = n$. If $W$ be a subspace of $V$ then $\dim W \leq n$. Moreover, if $\dim W = n$ then $W = V$.

__Proof:__ If $\dim W = m$ then $W$ has a basis consisting of $m$ vectors $\{w_1, \dots, w_m\}$. Since this set is linearly independent and $w_1, \dots, w_m \in V$ by part 2) of the Theorem above we obtain that

$$\dim W = m \leqslant n = \dim V$$

**Note.**

1) One can show that every vector space has a basis.

2) Some vector spaces have bases consisting of infinitely many vectors. If $V$ is such vector space then we write $\dim V = \infty$.

**Example.**

1) $\mathbb{P} = \{$the vector space of all polynomials$\}$
$= \{a_0 + a_1 t + \ldots + a_n t^n \mid a_i \in \mathbb{R},\ n \geq 0\}$

The set $\mathcal{E} = \{1, t, t^2, t^3, \ldots\}$ is a basis of $\mathbb{P}$. Since $\mathcal{E}$ consists of infinitely many elements we get that $\dim \mathbb{P} = \infty$

2) $C^\infty(\mathbb{R}) = \{$ the vector space of all smooth functions $f: \mathbb{R} \to \mathbb{R}\}$.

Since $\mathbb{P}$ is a subspace of $C^\infty(\mathbb{R})$ and $\dim \mathbb{P} = \infty$ we get that $\dim C^\infty(\mathbb{R}) = \infty$.

It is not possible to write explicitly a basis of $C^\infty(\mathbb{R})$.