Definition

Let V, W be vector spaces A $linear\ transformation$ is a function

$$T\colon V\to W$$

which satisfies the following conditions:

- 1) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$
- 2) $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $\mathbf{v} \in V$ and any scalar c.

Proposition

If $T: V \to W$ is a linear transformation then $T(\mathbf{0}) = \mathbf{0}$.

Note. If $T\colon V\to W$ is a linear transformation then for any vector $\mathbf{b}\in W$ we can consider the equation

$$T(\mathbf{x}) = \mathbf{b}$$

Definition

If $T: V \to W$ is a linear transformation then:

1) The kernel of T is the set

$$\mathsf{Ker}(T) = \{ \mathsf{v} \in V \mid T(\mathsf{v}) = \mathbf{0} \}$$

2) The image of T is the set

$$Im(T) = \{ w \in W \mid w = T(v) \text{ for some } v \in V \}$$

Proposition

If $T: V \to W$ is a linear transformation then:

- 1) Ker(T) is a subspace of V
- 2) $\operatorname{Im}(T)$ is a subspace of W

Theorem

If $T: V \to W$ is a linear transformation and v_0 is a solution of the equation

$$T(\mathbf{x}) = \mathbf{b}$$

then all other solutions of this equation are vectors of the form

$$\mathbf{v}=\mathbf{v}_0+\mathbf{n}$$

where $n \in Ker(T)$.

Example.

$$D\colon C^{\infty}(\mathbb{R}) \longrightarrow C^{\infty}(\mathbb{R})$$
$$f \longmapsto f'$$

Proposition

If $T: V \to W$ is a linear transformation then

- 1) T is onto if and only if $\operatorname{Im}(T) = W$
- 2) T is one-to-one if and only if $Ker(T) = \{0\}$.