- **Linear equations**

1) Three forms of equations:

   - system of linear equations

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad\quad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n = b_m \end{cases}$$

   - vector equation

$$x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

   - matrix equation

$$\begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

2) How to solve matrix equations:
   - row reduction (= Gauss-Jordan elimination)

3) Related notions:
   - elementary row operations
   - reduced echelon form of a matrix
   - leading ones
   - pivot positions and pivot columns

- **Vectors**

1) $\mathbb{R}^n$ = the set of all vectors with n entries

2) Operations on vectors in $\mathbb{R}^n$:

- addition
- multiplication by scalars

3) Geometric interpretation of vectors and vector operations.

4) Linear combinations of vectors.

5) Span of a set of vectors.

6) Linear independence of vectors.

- **Matrices**

1) Operations on Matrices:

- addition $A + B$
- multiplication by scalars cA
- matrix multiplication AB
- matrix Transpose A^T
- matrix inverse A^{-1}

2) Properties of the matrix algebra:

❶ $AB \neq BA$
❷ $(AB)^T = B^T A^T$
❸ $(A^T)^T = A$
❹ $(A + B)^T = A^T + B^T$
❺ $(A^T)^{-1} = (A^{-1})^T$
❻ $(AB)^{-1} = B^{-1}A^{-1}$
❼ $(A^{-1})^{-1} = A$
❽ $(A + B)^{-1} \neq A^{-1} + B^{-1}$

3) Column space of a matrix $\mathrm{Col}(A)$.

4) Null space of a matrix $\mathrm{Nul}(A)$, and its representation as a span of vectors.

- **Matrix transformations and linear transformations**

1) An $m \times n$ matrix defines a matrix transformation

$$T_A\colon \mathbb{R}^n \to \mathbb{R}^m, \qquad T_A(\mathbf{v}) = A\mathbf{v}$$

2) Composition of matrix transformations = matrix multiplication:

$$T_A \circ T_B = T_{AB}$$

3) Linear transformation is a function $T\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
(ii) $T(c\mathbf{v}) = cT(\mathbf{v})$

4) Every matrix transformation is a linear transformation.

5) Every linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation:

$$T = T_A$$

where $A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \quad \dots \quad & T(\mathbf{e}_n) \end{bmatrix}$ is the standard matrix of A.

6) A matrix transformation $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ is onto if any only if:

– $\mathrm{Col}(A) = \mathbb{R}^m$
– A has a pivot position in every row

7) A matrix transformation $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if any only if:

– $\mathrm{Nul}(A) = \{\mathbf{0}\}$
– A has a pivot position in every column

8) If A is an invertible matrix then $T_{A^{-1}}$ is the inverse function of T_A.

- **Sample TRUE/FALSE questions and answers**

Here is a sample of true/false questions. Questons of this type will be a part of the exam. In order to answer these questions you need to provide reasoning. Simply writing TRUE or FALSE as an answer will give you very little or no credit. In order to show that a statement is false, it suffices to give one example illustrating that it is false. In order to show that a statement is true, you need to provide a reasoning explaining why it is true in all instances - giving one example in this case will not suffice, since the statement may not work for some other examples.

For each of the statements given below decide if it is true or false. If you decide that it is true justify your answer. If you think it is false give a counterexample.

a) If a system of linear equations has more variables than equations then it must have infinitely many solutions.

b) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors such that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then the set $\{\mathbf{u}, \mathbf{v}\}$ is also linearly independent.

c) If A and B are invertible matrices then $A + B$ is also an invertible matrix.

d) If $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then $T(\mathbf{0}) = \mathbf{0}$ where $\mathbf{0}$ is the zero vector.

e) If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are vectors in $\mathbb{R}^2$ such that the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent then $\mathbf{v}_1 \in \text{Span}(\mathbf{v}_2, \mathbf{v}_3)$.

f) If A if a 4×5 matrix then the equation $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

g) If $\mathbf{v}, \mathbf{w}$ are solutions of the matrix equation $A\mathbf{x} = \mathbf{b}$ then the vector $\mathbf{v} + \mathbf{w}$ is in the null space $\text{Nul}(A)$.

Here are solutions to the sample TRUE/FALSE questions from the previous page. You should try to answer all questions by yourself before reading these solutions.

a) FALSE. A system which has more variables than equations may have zero solutions. Example: the system

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_1 + x_2 + x_3 = 2 \end{cases}$$

has no solutions.

b) TRUE. If the set $\{\mathbf{u}, \mathbf{v}\}$ were linearly dependent then the equation

$$x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{0}$$

would have a non-trivial solution $x_1 = c_1$, $x_2 = c_2$. This would mean that the equation

$$x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{0}$$

also has a non-trivial solution $x_1 = c_1$, $x_2 = c_2$, $x_3 = 0$, so the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ would be linearly dependent.

c) FALSE. For example $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ are invertible matrices, but $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible.

d) TRUE. Since T is a linear transformation we have

$$T(\mathbf{0}) = T(\mathbf{0} + \mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0})$$

subtracting $T(\mathbf{0})$ from both sides we obtain $\mathbf{0} = T(\mathbf{0})$.

e) FALSE. Take e.g.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Then the set $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linearly dependent (since e.g. $0\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$), but $\mathbf{v}_1 \notin \text{Span}(\mathbf{v}_2, \mathbf{v}_3)$.

f) TRUE. The equation $A\mathbf{x} = \mathbf{0}$ is consistent since it has at least one trivial solution $\mathbf{x} = \mathbf{0}$. Moreover, A has more columns than rows, which means that one of the columns does not have a pivot position. This column will correspond to a free variable, which will give infinitely many solutions.

g) FALSE. Take e.g.:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Both $\mathbf{v}$ and $\mathbf{w}$ are solutions of the equation $A\mathbf{x} = \mathbf{b}$, but

$$A(\mathbf{v} + \mathbf{w}) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Therefore $A(\mathbf{v} + \mathbf{w})$ is not the zero vector, which means that $\mathbf{v} + \mathbf{w}$ is not in $\mathrm{Nul}(A)$.