Definition

If A is an $n \times n$ matrix and $1 \leq i, j \leq n$ then the *ij-cofactor of A* is the number

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Example.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$C_{23} = (-1)^{2+3} \cdot \det A_{23} = (-1)^{2+3} \cdot \det \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix} = (-1) \cdot (1 \cdot 8 - 2 \cdot 7) = \underline{\underline{6}}$$

Note. By the definition of the determinant we have:

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}$$

Theorem

Let A be an $n \times n$ matrix.

1) For any $1 \leq i \leq n$ we have

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

(cofactor expansion across the i^{th} row).

2) For any $1 \leq j \leq n$ we have

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$

(cofactor expansion down the j^{th} column).

Example.

$$A = \begin{bmatrix} 1 & 3 & 0 & 4 \\ 0 & 4 & 6 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 5 & 0 & 0 \end{bmatrix}$$

Cofactor expansion down the 3rd column:

$$\det A = 0 \cdot C_{13} + 6 \cdot C_{23} + 0 \cdot C_{33} + 0 \cdot C_{43}$$
$$= 6 \cdot C_{23}$$

$$C_{23} = (-1)^{2+3} \cdot \det \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 1 \\ 3 & 5 & 0 \end{bmatrix} = (-1) \cdot 32 = -32$$

$$\det \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 1 \\ 3 & 5 & 0 \end{bmatrix} = 4 \cdot C_{13} + 1 \cdot C_{23} + 0 \cdot C_{33} = 4 \cdot 7 + 1 \cdot 4 = 32$$

cofactor exp. down the 3rd column

$$C_{13} = (-1)^{1+3} \det \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} = 1 \cdot (2 \cdot 5 - 1 \cdot 3) = 7$$

$$C_{23} = (-1)^{2+3} \cdot \det \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix} = (-1) \cdot (1 \cdot 5 - 3 \cdot 3) = 4$$

<u>We obtain</u>: $\det A = 6 \cdot (-32) = \underline{\underline{-192}}$

Example. Compute the determinant of the following matrix:

$$A=\begin{bmatrix}
1 & 0 & 0 & 3 & 0 & 0 & 2 & 0 & 3 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & \pi & 0 & 0 & 0 & 6 & 0 & 0 & 5 & 6 & 0 & 2 & 0 & 7 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 11 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 2 & 0 & 4 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 2 & 1 & 2 & 3 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 2 & 7 & 0 & -4 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 1 & 0 & 4 & 3 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 8 & 7 & 7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 2 & 8 & 9 & 0 & 3 & 3 & 2 & 5 & 6 & 3 & 8 & 9 & 2 & 6 & 2 & 2 & 1
\end{bmatrix}$$

$\det A = (-1)^{1+1}\cdot 1 \cdot (-1)^{1+1}\cdot 2 \cdot (-1)^{1+1}\cdot 1 \cdot (-1)^{1+1}\cdot\left(-\frac{1}{2}\right)\cdot(-1)^{17+1}\cdot 2\cdot \ldots$

Definition

An square matrix is *upper triangular* is all its entries below the main diagonal are 0.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

Proposition

If A is an upper triangular matrix as above then

$$\det A = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$$

Example:

$$A = \begin{bmatrix} 4 & 2 & 3 & 4 \\ 0 & 3 & 6 & 7 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$\det A \underset{\uparrow}{=} 4 \cdot (-1)^{1+1} \cdot \det \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

cof. exp. down 1^{st} col.

$$\underset{\uparrow}{=} 4 \cdot \left(3 \cdot (-1)^{1+1} \cdot \det \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} \right)$$

cof. exp. down 1^{st} col

$$\underset{\uparrow}{=} 4 \cdot 3 \cdot \left(2 \cdot (-1)^{1+1} \cdot \det [5] \right) = 4 \cdot 3 \cdot 2 \cdot 5 = 120$$

cof. exp. down 1^{st} col.