- **Inner product**

1) Inner product and orthogonality in $\mathbb{R}^n$:

- definitions of $\mathbf{u} \cdot \mathbf{v}$, $||\mathbf{v}||$, $\text{dist}(\mathbf{u}, \mathbf{v})$
- orthogonality of vectors
- Pythagorean theorem
- orthogonal sets of vectors
- orthogonal bases and coordinate systems
- Gram-Schmidt process:

a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of $V \subseteq \mathbb{R}^n$	*G-S process* $\Rightarrow$	an orthogonal basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ of V

$$\mathbf{w}_1 = \mathbf{v}_1$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_2}{\mathbf{w}_1 \cdot \mathbf{w}_1}\right)\mathbf{w}_1$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_3}{\mathbf{w}_1 \cdot \mathbf{w}_1}\right)\mathbf{w}_1 - \left(\frac{\mathbf{w}_2 \cdot \mathbf{v}_3}{\mathbf{w}_2 \cdot \mathbf{w}_2}\right)\mathbf{w}_2$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\mathbf{w}_k = \mathbf{v}_k - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_k}{\mathbf{w}_1 \cdot \mathbf{w}_1}\right)\mathbf{w}_1 - \left(\frac{\mathbf{w}_2 \cdot \mathbf{v}_k}{\mathbf{w}_2 \cdot \mathbf{w}_2}\right)\mathbf{w}_2 - \ldots - \left(\frac{\mathbf{w}_{k-1} \cdot \mathbf{v}_k}{\mathbf{w}_{k-1} \cdot \mathbf{w}_{k-1}}\right)\mathbf{w}_{k-1}$$

2) Orthogonal projections:

– If V is a subspace of $\mathbb{R}^n$ with an orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and $\mathbf{w} \in \mathbb{R}^n$ then

$$\mathrm{proj}_V \mathbf{w} = \left(\frac{\mathbf{w} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1 + \ldots + \left(\frac{\mathbf{w} \cdot \mathbf{v}_k}{\mathbf{v}_k \cdot \mathbf{v}_k}\right) \mathbf{v}_k$$

– main property: $\mathrm{proj}_V \mathbf{w} \in V$, $\mathbf{w} - \mathrm{proj}_V \mathbf{w} \in V^\perp$

– Best Approximation Theorem:

$$\mathrm{dist}(\mathbf{w}, \mathrm{proj}_V \mathbf{w}) \leq \mathrm{dist}(\mathbf{w}, \mathbf{v})$$

for all $\mathbf{v} \in V$.

3) Least square solutions.

– computation:

$$\begin{pmatrix} \text{least square solutions of} \\ A\mathbf{x} = \mathbf{b} \end{pmatrix} = \begin{pmatrix} \text{solutions of} \\ A\mathbf{x} = \mathrm{proj}_{\mathrm{Col}(A)} \mathbf{b} \end{pmatrix}$$

– much faster computation:

$$\begin{pmatrix} \text{least square solutions of} \\ A\mathbf{x} = \mathbf{b} \end{pmatrix} = \begin{pmatrix} \text{solutions of} \\ A^T A\mathbf{x} = A^T \mathbf{b} \end{pmatrix}$$

– application: least square fitting of lines and curves.

4) General inner product spaces.

• **Eigenvalues and eigenvectors**

1) Definition.

2) Computation:

– if A is an $n \times n$ matrix then

$$\text{eigenvalues of } A = \begin{pmatrix} \text{roots of the characteristic polynomial} \\ P(\lambda) = \det(A - \lambda I) \end{pmatrix}$$

– if λ is an eigenvalue of A then

$$(\text{the eigenspace of } A \text{ corresponding to } \lambda) = \mathrm{Nul}(A - \lambda I)$$

3) Diagonalization of matrices:

– A square matrix A is diagonalizable if there exists an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}$$

– An $n \times n$ matrix A is a diagonalizable if and only if it has n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. In such case we have:

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} \qquad \begin{array}{l} \lambda_1 = \text{eigenvalue corresponding to } \mathbf{v}_1 \\ \lambda_2 = \text{eigenvalue corresponding to } \mathbf{v}_2 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \lambda_n = \text{eigenvalue corresponding to } \mathbf{v}_n \end{array}$$

– Not every matrix is diagonalizable.

– If A is diagonalizable, $A = PDP^{-1}$ then

$$A^k = PD^kP^{-1}$$

4) Symmetric matrices and orthogonal diagonalization.

– An orthogonal matrix $Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{bmatrix}$ is a square matrix such that $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is an orthonormal set of vectors, i.e.:

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

– If Q is an orthogonal matrix then $Q^{-1} = Q^T$

– A square matrix A is orthogonally diagonalizable if there exist an orthogonal matrix Q and a diagonal matrix D such that

$$A = QDQ^{-1} = QDQ^T$$

– A matrix A is orthogonally diagonalizable if and only if A is a symmetric matrix (i.e. $A^T = A$).

– Spectral decomposition of a symmetric matrix:

$$A = \lambda_1(\mathbf{u}_1\mathbf{u}_1^T) + \lambda_2(\mathbf{u}_2\mathbf{u}_2^T) + \dots + \lambda_n(\mathbf{u}_n\mathbf{u}_n^T)$$

5) The singular value decomposition of a matrix:

$$A = U\Sigma V^T$$

- **Sample TRUE/FALSE questions and answers**

Here is a sample of true/false questions. Questions of this type will be a part of the exam. In order to answer these questions you need to provide reasoning. Simply writing TRUE or FALSE as an answer will give you very little or no credit. In order to show that a statement is false, it suffices to give one example illustrating that it is false. In order to show that a statement is true, you need to provide a reasoning explaining why it is true in all instances - giving one example in this case will not suffice, since the statement may not work for some other examples.

For each of the statements given below decide if it is true or false. If you decide that it is true justify your answer. If you think it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ which satisfy $||\mathbf{u} + \mathbf{v}|| = ||\mathbf{u} - \mathbf{v}||$ then $\mathbf{u}$ must be orthogonal to $\mathbf{v}$.

b) If $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are vectors in $\mathbb{R}^3$ such that $\mathbf{u}_1$ is orthogonal to $\mathbf{u}_2$, and $\mathbf{u}_2$ is orthogonal to $\mathbf{u}_3$, then $\mathbf{u}_1$ must be orthogonal to $\mathbf{u}_3$.

c) If A is an $n \times n$ matrix and $\mathbf{v}$ is eigenvector of A, then $\mathbf{v}$ is also an eigenvector of A^2.

d) If A is an $n \times n$ matrix and λ is eigenvalue of A, then λ is also an eigenvalue of A^2.

Here are solutions to the sample TRUE/FALSE questions from the previous page. You should try to answer all questions by yourself before reading these solutions.

a) TRUE. If $||\mathbf{u}+\mathbf{v}|| = ||\mathbf{u}-\mathbf{v}||$, then

$$(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = ||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}-\mathbf{v}||^2 = (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})$$

Since

$$(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v}$$

and

$$(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} - 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v}$$

this gives $2(\mathbf{u}\cdot\mathbf{v}) = -2(\mathbf{u}\cdot\mathbf{v})$, which holds only if $\mathbf{u}\cdot\mathbf{v} = 0$.

b) FALSE. Take e.g

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Then $\mathbf{u}_1\cdot\mathbf{u}_2 = 0$, $\mathbf{u}_2\cdot\mathbf{u}_3 = 0$, but $\mathbf{u}_1\cdot\mathbf{u}_3 \neq 0$.

c) TRUE. If $\mathbf{v}$ is an eigenvector of A corresponding to an eigenvalue λ, then

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda\cdot(A\mathbf{v}) = \lambda^2\mathbf{v}$$

This means that $\mathbf{v}$ is an eigenvector of A^2 corresponding to an eigenvalue λ^2.

d) FALSE. Take e.g.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Since the characteristic polynomial of this matrix is $P(\lambda) = (2-\lambda)(3-\lambda)$, the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 3$. On the other hand

$$A^2 = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$$

and since the characteristic polynomial of A^2 is $P(\lambda) = (4-\lambda)(9-\lambda)$, the eigenvalues of A^2 are $\lambda_1 = 4$ and $\lambda_2 = 9$. Thus e.g. 2 is an eigenvalue of A, but it is not an eigenvalue of A^2.