Definition

Let V, W be vector spaces A *linear transformation* is a function

$$T\colon V \to W$$

which satisfies the following conditions:

1) $T(\mathrm{u} + \mathrm{v}) = T(\mathrm{u}) + T(\mathrm{v})$ for all $\mathrm{u}, \mathrm{v} \in V$
2) $T(c\mathrm{v}) = cT(\mathrm{v})$ for any $\mathrm{v} \in V$ and any scalar c.

Example: If A is an $m \times n$ matrix then it defines a linear transformation:

$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$v \longmapsto Av$$

Example:

Recall: $C^{\infty}(\mathbb{R}) = \left\{ \begin{array}{l} \text{the vector space of all} \\ \text{smooth functions } f: \mathbb{R} \to \mathbb{R} \end{array} \right\}$

Take: $D: C^{\infty}(\mathbb{R}) \longrightarrow C^{\infty}(\mathbb{R})$

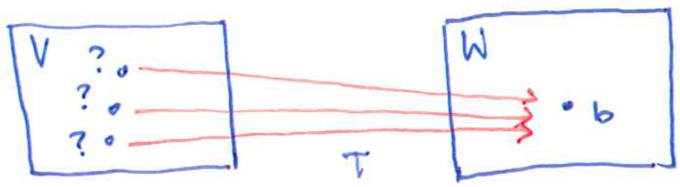
$D(f) = f'$ ← the derivative of f

D is a linear transformation:

1) $D(f+g) = (f+g)' = f' + g' = D(f) + D(g)$
2) $D(cf) = (cf)' = c \cdot f' = c \cdot D(f)$

Note. If $T\colon V \to W$ is a linear transformation then for any vector $\mathbf{b} \in W$ we can consider the equation

$$T(\mathrm{x}) = \mathbf{b}$$



Example:

If $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ - a (matrix) linear transformation
$v \longmapsto Av$

then the equation $T_A(x) = b$ is the same as the matrix equation $Ax = b$.

Example:

Take $D\colon C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$
$f \longmapsto f'$

For $g \in C^\infty(\mathbb{R})$ the equation $D(x) = g$ is the same as the differential equation

$$\frac{dx}{dt} = g$$

This equation is solved by integration:

$$x(t) = \int g(t)\,dt$$

Definition

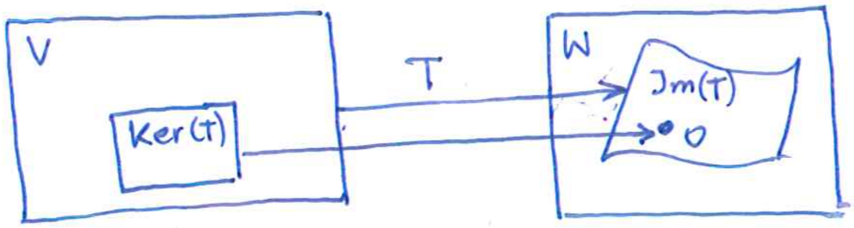
If $T\colon V \to W$ is a linear transformation then:

1) The *kernel* of T is the set

$$\mathrm{Ker}(T) = \{\mathrm{v} \in V \mid T(\mathrm{v}) = \mathbf{0}\}$$

2) The *image* of T is the set

$$\mathrm{Im}(T) = \{\mathrm{w} \in W \mid \mathrm{w} = T(\mathrm{v}) \text{ for some } \mathrm{v} \in V\}$$



Example:

A - $m \times n$ matrix

$$T_A\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$v \longmapsto Av$$

$$\mathrm{Ker}(T_A) = \{v \in \mathbb{R}^n \mid T_A(v) = 0\} = \{v \in \mathbb{R}^n \mid Av = 0\} = \mathrm{Nul}(A) \leftarrow \text{the null space of } A$$

$$\mathrm{Im}(T_A) = \{b \in \mathbb{R}^m \mid T_A(v) = b \text{ for some } v \in \mathbb{R}^n\} = \{b \in \mathbb{R}^m \mid Av = b \text{ ——''—— }\} = \underbrace{\mathrm{Col}(A)}_{\text{the column space of } A}$$

Example:

$$D\colon C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$$
$$f \longmapsto f'$$

$$\mathrm{Ker}(D) = \{f \in C^\infty(\mathbb{R}) \mid f' = 0\} = \{\text{the set of all constant functions}\}$$

$$\mathrm{Im}(D) = \{g \in C^\infty(\mathbb{R}) \mid g = f' \text{ for some } f \in C^\infty(\mathbb{R})\} = C^\infty(\mathbb{R})$$

Proposition

If $T: V \to W$ is a linear transformation then:

1) $\mathrm{Ker}(T)$ is a subspace of V

2) $\mathrm{Im}(T)$ is a subspace of W

Proof of 1)

We need to check:

(i) $0 \in \mathrm{Ker}(T)$

(ii) if $u, v \in \mathrm{Ker}(T)$ then $u+v \in \mathrm{Ker}(T)$

(iii) if $u \in \mathrm{Ker}(T)$ then $cu \in \mathrm{Ker}(T)$

(i) Check: if T is a linear transformation then $T(0)=0$, so $0 \in \mathrm{Ker}(T)$.

(ii) If $u, v \in \mathrm{Ker}(T)$ then $T(u)=0$, $T(v)=0$
so $T(u+v) = T(u)+T(v) = 0+0 = 0$.
Thus $u+v \in \mathrm{Ker}(T)$

(iii) If $u \in \mathrm{Ker}(T)$ then $T(u)=0$
so: $T(cu) = cT(u) = c \cdot 0 = 0$
and so $cu \in \mathrm{Ker}(T)$

Proof of 2) is similar

Theorem

If $T: V \to W$ is a linear transformation and v_0 is a solution of the equation

$$T(\mathrm{x}) = \mathbf{b}$$

then all other solutions of this equation are vectors of the form

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{n}$$

where $\mathbf{n} \in \mathrm{Ker}(T)$.

Proof:

If v_0 is a solution of $T(x) = b$ and $n \in \mathrm{Ker}(T)$ then

$$T(v_0 + n) = T(v_0) + T(n) = b + 0 = b$$

So: $v_0 + n$ is also a solution of $T(x) = b$

Proof of the converse is similar.

Example.

$$D\colon C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$$
$$f \longmapsto f'$$

Recall:

$\mathrm{Ker}(D) = \{$all constant functions$\}$

Let $g(t) = t^2$

Solutions of $D(x) = g$ are functions f such that $f'(t) = g(t) = t^2$

This gives: solutions of $D(x) = g$ are functions

$$f(t) = \int t^2\,dt = \underbrace{\tfrac{1}{3}t^3}_{\substack{\text{a particular}\\ \text{solution of}\\ D(x)=g}} + \underbrace{C}_{\substack{\text{a constant function}\\ \text{i.e. a function from}\\ \mathrm{Ker}(D)}}$$

Proposition

If $T \colon V \to W$ is a linear transformation then

1) T is onto if and only if $\operatorname{Im}(T) = W$
2) T is one-to-one if and only if $\operatorname{Ker}(T) = \{\mathbf{0}\}$.