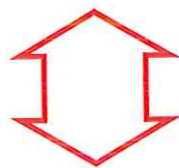


So far:

$$\begin{cases} 2x_1 + 4x_2 + 6x_3 + 3x_4 = 7 \\ 3x_1 + 2x_2 + 2x_3 + 9x_4 = 3 \\ 5x_1 + 8x_2 + 3x_3 + 3x_4 = 9 \end{cases}$$

system of
linear equations



$$x_1 \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 2 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 9 \end{bmatrix}$$

vector equation

Next:

$$A \cdot x = b$$

$$\begin{bmatrix} 2 & 4 & 6 & 3 \\ 3 & 2 & 2 & 9 \\ 5 & 8 & 3 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 9 \end{bmatrix}$$

matrix equation

Definition

Let A be an $m \times n$ matrix with columns v_1, v_2, \dots, v_n and let w be a vector in \mathbb{R}^n :

$$A = [v_1 \ v_2 \ \dots \ v_n] \quad w = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

The product Aw is a vector in \mathbb{R}^m given by

$$Aw = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

Example.

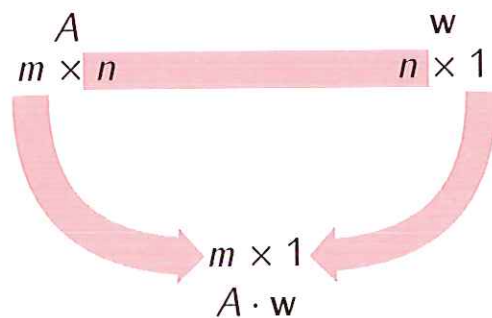
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad w = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{aligned} Aw &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix} + (-2) \cdot \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 1 \cdot \begin{bmatrix} 3 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 12 \end{bmatrix} + \begin{bmatrix} -4 \\ -10 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 8 \end{bmatrix} \end{aligned}$$

Properties of matrix-vector multiplication

1) The product Aw is defined only if

(number of columns of A) = (number of entries of w)



e.g.:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 7 \\ 0 \\ 5 \end{bmatrix}$$

2×3 4×1

2) $A(v + w) = Av + Aw$ no match, so this multiplication is not defined

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \left(\begin{bmatrix} 7 \\ -1 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ 3 \end{bmatrix}$$

3) If c is a scalar then $A(cw) = c(Aw)$.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \left(5 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5 \cdot \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$$

Example. Solve the matrix equation

$$\underbrace{\begin{bmatrix} 1 & 1 & -4 \\ 1 & -2 & 3 \\ 3 & -3 & 0 \end{bmatrix}}_{\text{A}} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\text{x}} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{\text{b}}$$

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

vector equation

augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & -4 & 1 \\ 1 & -2 & 3 & 2 \\ 3 & -3 & 0 & 3 \end{array} \right] \xrightarrow{\text{row red.}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$\underbrace{\hspace{10em}}_{\text{A}} \quad \underbrace{\hspace{1em}}_{\text{b}}$

solutions:

$$\begin{cases} x_1 = 3 \\ x_2 = 2 \\ x_3 = 1 \end{cases}$$

in vector form:

$$\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

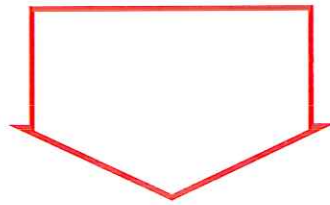
Check:

$$\begin{bmatrix} 1 & 1 & -4 \\ 1 & -2 & 3 \\ 3 & -3 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

How to solve a matrix equation

$$Ax = b$$

matrix equation



$$[A \mid b]$$

augmented matrix



$$[\text{reduced matrix}]$$



$$x = \dots$$

solutions