

Next: From systems of linear equations to vector equations.

$$\begin{cases} x_1 + 2x_2 = 4 \\ 2x_1 + 7x_2 = 9 \\ 4x_1 + x_2 = 0 \end{cases} \quad \Rightarrow \quad x_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 0 \end{bmatrix}$$

Why vectors and vector equations are useful:

- They show up in many applications (velocity vectors, force vectors etc.)
- They give a better geometric picture of systems of linear equations.

### Definition

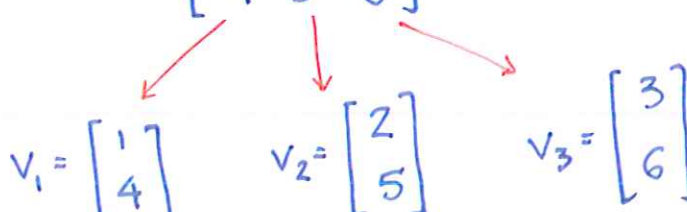
A *column vector* is a matrix with one column.

### Example

$$u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad v = \begin{bmatrix} 5 \\ -1 \\ 0 \\ 7 \end{bmatrix}$$

**Note.** Columns of a matrix are column vectors.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = [v_1 \ v_2 \ v_3]$$



$$v_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad v_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \quad v_3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

### Notation

$\mathbb{R}^n$  is the set of all column vectors with  $n$  entries.

$$\mathbb{R}^2 = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \mid a_1, a_2 \in \mathbb{R} \right\} = \text{the set of all vectors with 2 entries}$$

$$\mathbb{R}^3 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \mid a_1, a_2, a_3 \in \mathbb{R} \right\} = \text{the set of all vectors with 3 entries}$$

## Operations on vectors in $\mathbb{R}^n$

### 1) Addition of vectors:

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

e.g.:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 5 \\ -1 \\ 7 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 10 \end{bmatrix}$$

Note: We can add two vectors only if they have the same number of entries.

### 2) Multiplication by scalars:

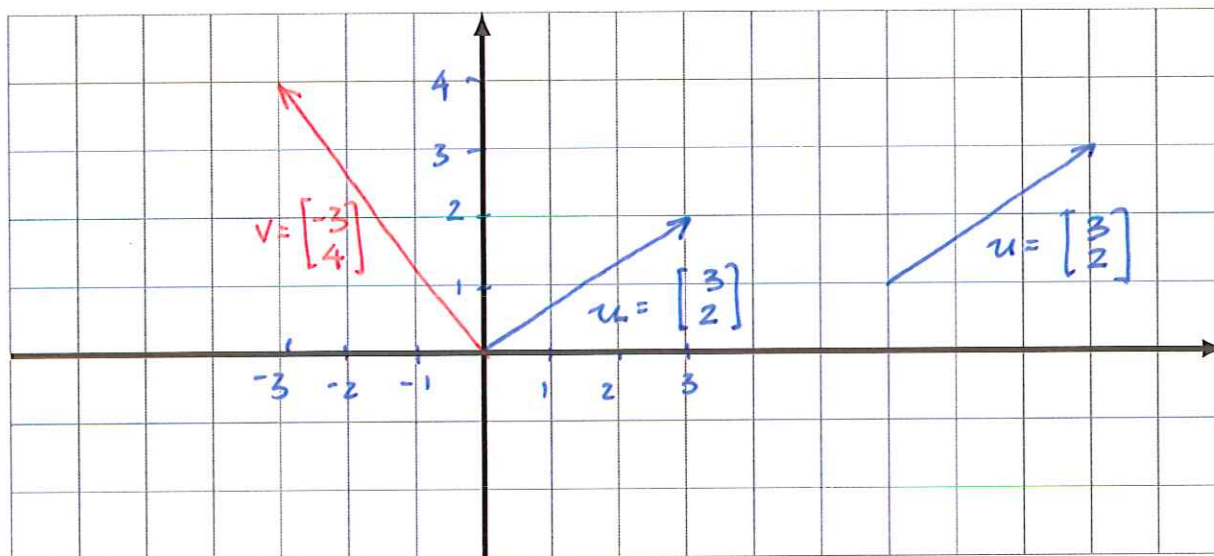
scalars = real numbers

$$c \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}$$

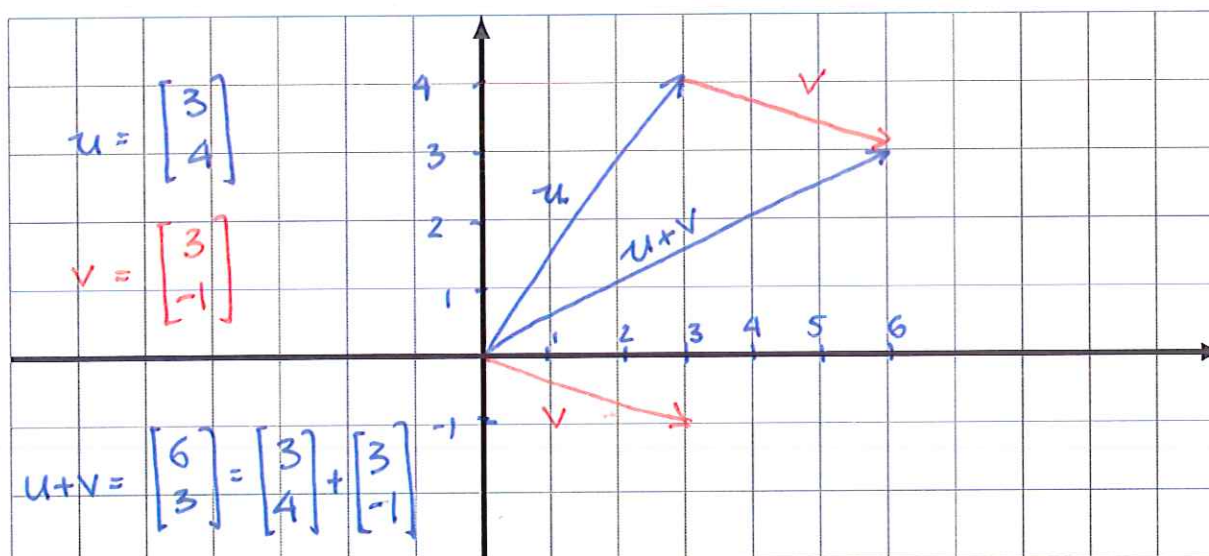
e.g.:  $3 \cdot \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 15 \end{bmatrix}$

## Geometric interpretation of vectors in $\mathbb{R}^2$

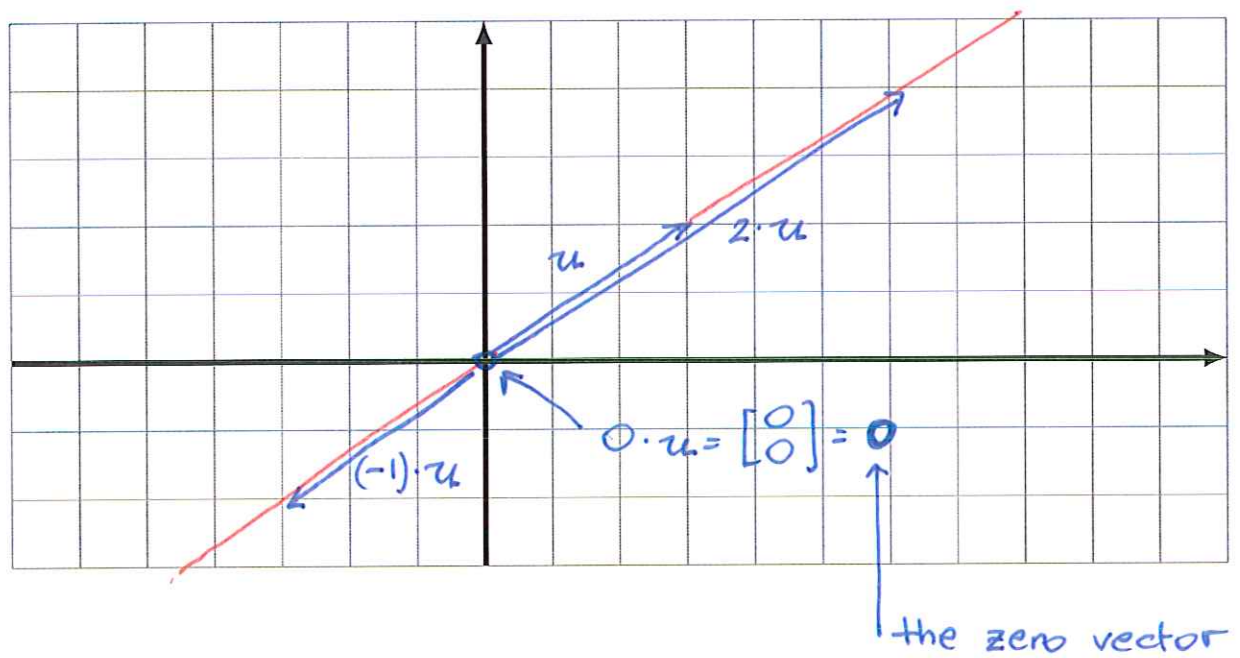
Vector coordinates:



Vector addition:



## Scalar multiplication:



## Vector equations

$$x_1 \underbrace{v_1}_{\uparrow} + x_2 \underbrace{v_2}_{\uparrow} + \dots + x_p \underbrace{v_p}_{\uparrow} = \underbrace{w}_{\uparrow}$$

vectors in  $\mathbb{R}^n$

Example. Solve the following vector equation:

$$x_1 \overset{v_1}{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + x_2 \overset{v_2}{\begin{bmatrix} 4 \\ -2 \end{bmatrix}} = \overset{w_2}{\begin{bmatrix} 10 \\ 3 \end{bmatrix}}$$

$$\parallel$$
$$\begin{bmatrix} 2x_1 \\ 3x_1 \end{bmatrix} + \begin{bmatrix} 4x_2 \\ -2x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$$

$$\parallel$$
$$\begin{bmatrix} 2x_1 + 4x_2 \\ 3x_1 - 2x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$$

$$\Downarrow$$
$$\begin{cases} 2x_1 + 4x_2 = 10 \\ 3x_1 - 2x_2 = 3 \end{cases}$$

system of linear equations

augmented matrix :

$$\begin{array}{ccc|c} 2 & 4 & 10 \\ 3 & -2 & 3 \end{array} \xrightarrow{\text{row red.}} \begin{array}{cc|c} \overset{x_1}{1} & \overset{x_2}{0} & 2 \\ 0 & 1 & 3/2 \end{array} \xrightarrow{\text{solutions}} \begin{cases} x_1 = 2 \\ x_2 = 3/2 \end{cases}$$

$v_1 \quad v_2 \quad w$

## How to solve a vector equation

$$x_1 v_1 + \dots + x_p v_p = w$$

vector of equation

*make  
a matrix*

$$\left[ v_1 \quad \dots \quad v_p \mid w \right]$$

augmented matrix

*row  
reduction*

$$\left[ \text{reduced matrix} \right]$$

*read off  
solutions*

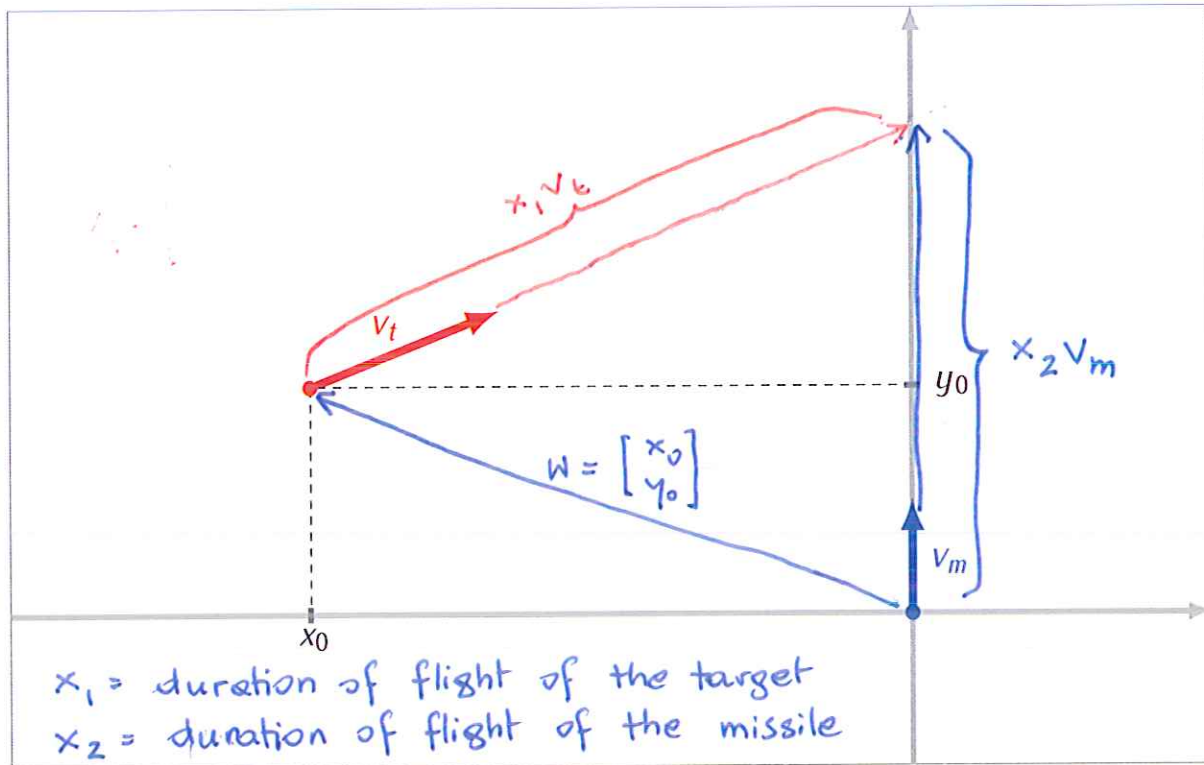
$$\begin{cases} x_1 = \dots \\ \dots \quad \dots \\ x_p = \dots \end{cases}$$

solutions



### Example: Target shooting.

At time  $t = 0$  a target is observed at the position  $(x_0, y_0)$  moving in the direction of the vector  $v_t$ . The target is moving at such speed, that it travels the length of  $v_t$  in one second. Find  $t_0$  such that a missile positioned at the point  $(0, 0)$  will intercept the target if it is fired at the time  $t_0$ . The missile travels the length of the vector  $v_m$  in one second.



$$t_0 = x_1 - x_2$$

so, it is enough to find  $x_1, x_2$

$$W + x_1 v_t = x_2 v_m$$

$$x_1 v_t - x_2 v_m = -W \quad \leftarrow \text{solve for } x_1, x_2$$

e.g.:

$$v_t = \begin{bmatrix} 11 \\ 1 \end{bmatrix}, v_m = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, W = \begin{bmatrix} -165 \\ 25 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 11 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 165 \\ -25 \end{bmatrix}$$

aug. matrix

$$\left[ \begin{array}{cc|c} 11 & 0 & 165 \\ 1 & -4 & -25 \end{array} \right] \xrightarrow{\text{row red}} \left[ \begin{array}{cc|c} 1 & 0 & 15 \\ 0 & 1 & 10 \end{array} \right]$$

$$\begin{cases} x_1 = 15 \\ x_2 = 10 \end{cases} \Rightarrow t_0 = 15 - 10 = 5 //$$