Definition

A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ in $\mathbb{R}^n$ is an *orthogonal set* if each pair each pair of vectors in this set is orthogonal, i.e.

$$\mathbf{v}_i \cdot \mathbf{v}_j = 0$$

for all $i \neq j$.

Example.

$$\begin{array}{ccc} e_1 & e_2 & e_3 \\ \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\right. & \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, & \left.\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\} \end{array}$$

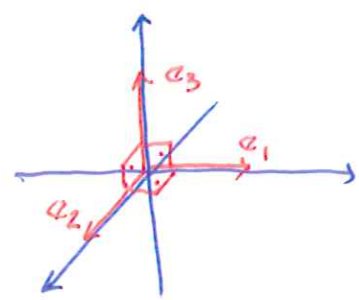
is an orthogonal set in $\mathbb{R}^3$.

Check:

$e_1 \cdot e_2 = 0$

$e_1 \cdot e_3 = 0$

$e_2 \cdot e_3 = 0$



Example.

$$\begin{array}{ccc} v_1 & v_2 & v_3 \\ \left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},\right. & \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, & \left.\begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix}\right\} \end{array}$$

is another orthogonal set in $\mathbb{R}^3$.

Check: $v_1 \cdot v_2 = 1 \cdot (-3) + 2 \cdot 0 + 3 \cdot 1 = -3 + 3 = 0$

$v_1 \cdot v_3 = \ \ldots \ = 0$

$v_2 \cdot v_3 = \ \ldots \ = 0$

Proposition

If $\{v_1, \dots, v_k\}$ is an orthogonal set of non-zero vectors in $\mathbb{R}^n$ then this set is linearly independent.

<u>Proof:</u> Assume that

$$c_1 v_1 + c_2 v_2 + \dots + c_k v_k = 0$$

We need to show that $c_1 = c_2 = \dots = c_k = 0$

We have:

$$\underbrace{v_1 \cdot (c_1 v_1 + c_2 v_2 + \dots + c_k v_k)} = v_1 \cdot 0 = 0$$

$$\|$$

$$\underbrace{v_1 \cdot (c_1 v_1) + v_1 \cdot (c_2 v_2) + \dots + v_1 \cdot (c_k v_k)} = 0$$

$$\|$$

$$c_1 (v_1 \cdot v_1) + c_2 \underbrace{(v_1 \cdot v_2)}_{\substack{\| \\ 0}} + \dots + c_k \underbrace{(v_1 \cdot v_k)}_{\substack{\| \\ 0}} = 0$$

This gives: $c_1 (v_1 \cdot v_1) = 0$

Since $v_1 \neq 0$ we have $v_1 \cdot v_1 \neq 0$, so $c_1 = 0$.

In the same way we get $c_2 = 0, \dots, c_k = 0$

Recall: Any linearly independent set of n vectors in $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$.

Corollary

If $\{v_1, \dots, v_n\}$ is an orthogonal set of n non-zero vectors in $\mathbb{R}^n$ then this set is a basis of $\mathbb{R}^n$.

Definition

If V is a subspace of $\mathbb{R}^n$ then we say that a set $\{\mathbf{v}_1, \dots \mathbf{v}_k\}$ is an *orthogonal basis* of V if

1) $\{\mathbf{v}_1, \dots \mathbf{v}_k\}$ is a basis of V and
2) $\{\mathbf{v}_1, \dots \mathbf{v}_k\}$ is an orthogonal set.

Recall. If $\mathcal{B} = \{\mathbf{v}_1, \dots \mathbf{v}_k\}$ is a basis of a vector space V and $\mathbf{w} \in V$ then the coordinate vector of $\mathbf{w}$ relative to $\mathcal{B}$ is the vector

$$[\,\mathbf{w}\,]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

where $c_1, \dots, c_k$ are scalars such that $c_1\mathbf{v}_1 + \dots + c_k\mathbf{v}_k = \mathbf{w}$.

Propostion

If $\mathcal{B} = \{\mathbf{v}_1, \dots \mathbf{v}_k\}$ is an orthogonal basis of V and $\mathbf{w} \in V$ then

$$[\,\mathbf{w}\,]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

where $c_i = \dfrac{\mathbf{w} \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i} = \dfrac{\mathbf{w} \cdot \mathbf{v}_i}{||\mathbf{v}_i||^2}$

Proof: If $[w]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$ then $w = c_1v_1 + \dots + c_kv_k$

We have: $w \cdot v_1 = (c_1v_1 + c_2v_2 + \dots + c_kv_k) \cdot v_1$

$= c_1(v_1 \cdot v_1) + c_2\underbrace{(v_2 \cdot v_1)}_{=0} + \dots + c_k\underbrace{(v_k \cdot v_1)}_{=0}$

So: $w \cdot v_1 = c_1(v_1 \cdot v_1)$

and so: $\boxed{c_1 = \dfrac{w \cdot v_1}{v_1 \cdot v_1}}$

In the same way we get:

$c_i = \dfrac{w \cdot v_i}{v_i \cdot v_i}$ for $i = 1, 2, \dots, k$

Example. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix} \right\}, \qquad \mathbf{w} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

The set $\mathcal{B}$ is an orthogonal basis of $\mathbb{R}^3$. Compute $[\,\mathbf{w}\,]_{\mathcal{B}}$.

Solution:

$$[w]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$c_1 = \frac{w \cdot v_1}{v_1 \cdot v_1} = \frac{3\cdot 1 + 2\cdot 2 + 1\cdot 3}{1^2 + 2^2 + 3^2} = \frac{10}{14} = \frac{5}{7}$$

$$c_2 = \frac{w \cdot v_2}{v_2 \cdot v_2} = \frac{3\cdot(-3) + 2\cdot 0 + 1\cdot 1}{(-3)^2 + 0^2 + 1^2} = \frac{-8}{10} = -\frac{4}{5}$$

$$c_3 = \frac{w \cdot v_3}{v_3 \cdot v_3} = \frac{3\cdot 1 + 2\cdot(-5) + 1\cdot 3}{1^2 + (-5)^2 + 3^2} = \frac{-4}{35}$$

We get:

$$[w]_{\mathcal{B}} = \begin{bmatrix} 5/7 \\ -4/5 \\ -4/35 \end{bmatrix}$$

Check: $w = 5/7\, v_1 - 4/5\, v_2 - 4/35\, v_3$

Theorem (Gram-Schmidt Process)

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of V. Define vectors $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ as follows:

$$\mathbf{w}_1 = \mathbf{v}_1$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_2}{\mathbf{w}_1 \cdot \mathbf{w}_1}\right)\mathbf{w}_1$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_2}{\mathbf{w}_1 \cdot \mathbf{w}_1}\right)\mathbf{w}_1 - \left(\frac{\mathbf{w}_2 \cdot \mathbf{v}_3}{\mathbf{w}_2 \cdot \mathbf{w}_2}\right)\mathbf{w}_2$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\mathbf{w}_k = \mathbf{v}_k - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_k}{\mathbf{w}_1 \cdot \mathbf{w}_1}\right)\mathbf{w}_1 - \left(\frac{\mathbf{w}_2 \cdot \mathbf{v}_k}{\mathbf{w}_2 \cdot \mathbf{w}_2}\right)\mathbf{w}_2 - \ldots - \left(\frac{\mathbf{w}_{k-1} \cdot \mathbf{v}_k}{\mathbf{w}_{k-1} \cdot \mathbf{w}_{k-1}}\right)\mathbf{w}_{k-1}$$

Then the set $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is an orthogonal basis of V.

e.g.: check $w_1 \cdot w_2$

$$\begin{aligned} w_1 \cdot w_2 &= w_1 \cdot \left(v_2 - \left(\frac{w_1 \cdot v_2}{w_1 \cdot w_1}\right) w_1\right) \\ &= v_1 \cdot \left(v_2 - \left(\frac{v_1 \cdot v_2}{v_1 \cdot v_1}\right) v_1\right) \\ &= v_1 \cdot v_2 - \left(\frac{v_1 \cdot v_2}{\cancel{v_1 \cdot v_1}}\right)(\cancel{v_1 \cdot v_1}) \\ &= v_1 \cdot v_2 - v_1 \cdot v_2 = 0 \end{aligned}$$

Example. In $\mathbb{R}^4$ take

$$\mathrm{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}, \quad \mathrm{v}_2 = \begin{bmatrix} 7 \\ 4 \\ 3 \\ -3 \end{bmatrix}, \quad \mathrm{v}_3 = \begin{bmatrix} 5 \\ 7 \\ 7 \\ 8 \end{bmatrix}$$

The set $\mathcal{B} = \{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\}$ is a basis of some subspace $V \subseteq \mathbb{R}^4$. Find an orthogonal basis of V.

Solution: apply the Gram - Schmidt process

$$w_1 = v_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}$$

$$w_2 = v_2 - \left(\frac{w_1 \cdot v_2}{w_1 \cdot w_1}\right) w_1$$

$$w_1 \cdot v_1 = 2\cdot 7 + 1\cdot 4 + 3\cdot 3 + (-1)\cdot(-3) = 30$$

$$w_1 \cdot w_1 = 2^2 + 1^2 + 3^2 + (-1)^2 = 15$$

so: $$w_2 = \begin{bmatrix} 7 \\ 4 \\ 3 \\ -3 \end{bmatrix} - \underset{2}{\cancel{\frac{30}{15}}} \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -3 \\ -1 \end{bmatrix}$$

$$w_3 = v_3 - \left(\frac{w_1 \cdot v_3}{w_1 \cdot w_1}\right) w_1 - \left(\frac{w_2 \cdot v_3}{w_2 \cdot w_2}\right) w_2$$

$$w_1 \cdot w_1 = 15$$

$$w_1 \cdot v_3 = 2\cdot 5 + 1\cdot 7 + 3\cdot 7 + (-1)\cdot 8 = 30$$

$$w_2 \cdot v_3 = 3\cdot 5 + 2\cdot 7 + (-3)\cdot 7 + (-1)\cdot 8 = 0$$

$$w_2 \cdot w_2 = 3^2 + 2^2 + (-3)^2 + (-1)^2 = 23$$

so: $$w_3 = \begin{bmatrix} 5 \\ 7 \\ 7 \\ 8 \end{bmatrix} - \underset{2}{\cancel{\frac{30}{15}}} \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix} + \underset{0}{\cancel{\frac{0}{23}}} \begin{bmatrix} 3 \\ 2 \\ -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 1 \\ 10 \end{bmatrix}$$

We obtain an orthogonal basis of V:

$$\{w_1, w_2, w_3\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 1 \\ 10 \end{bmatrix} \right\}$$

Definition

An orthogonal basis $\mathcal{B} = \{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ of V is called an *orthonormal basis* if $||\mathbf{w}_i|| = 1$ for $i = 1, \ldots, k$.

Propostion

If $\mathcal{B} = \{\mathbf{v}_1, \ldots \mathbf{v}_k\}$ is an orthonormal basis of V and $\mathbf{w} \in V$ then

$$\left[\, \mathbf{w} \,\right]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

where $c_i = \mathbf{w} \cdot \mathbf{v}_i$.

Note. If $\mathcal{B} = \{\mathbf{v}_1, \ldots \mathbf{v}_k\}$ is an orthogonal basis of V then

$$\mathcal{C} = \left\{ \frac{\mathbf{v}_1}{||\mathbf{v}_1||}, \ldots, \frac{\mathbf{v}_k}{||\mathbf{v}_k||} \right\}$$

is an orthonormal basis of V.

Example: In the last example we had:

$$\{w_1, w_2, w_3\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 1 \\ 10 \end{bmatrix} \right\}$$ – orthogonal basis of some subspace $V \subseteq \mathbb{R}^4$

orthonormal basis of V:

$$\left\{ \frac{w_1}{||w_1||}, \frac{w_2}{||w_2||}, \frac{w_3}{||w_3||} \right\} = \left\{ \frac{1}{\sqrt{15}} \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{23}} \begin{bmatrix} 3 \\ 2 \\ -3 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{127}} \begin{bmatrix} 1 \\ 5 \\ 1 \\ 10 \end{bmatrix} \right\}$$

$$||w_1|| = \sqrt{2^2 + 1^2 + 3^2 + (-1)^2} = \sqrt{15}$$

$$||w_2|| = \sqrt{3^2 + 2^2 + (-3)^2 + (-1)^2} = \sqrt{23}$$

$$||w_3|| = \sqrt{1^2 + 5^2 + 1^2 + 10^2} = \sqrt{127}$$