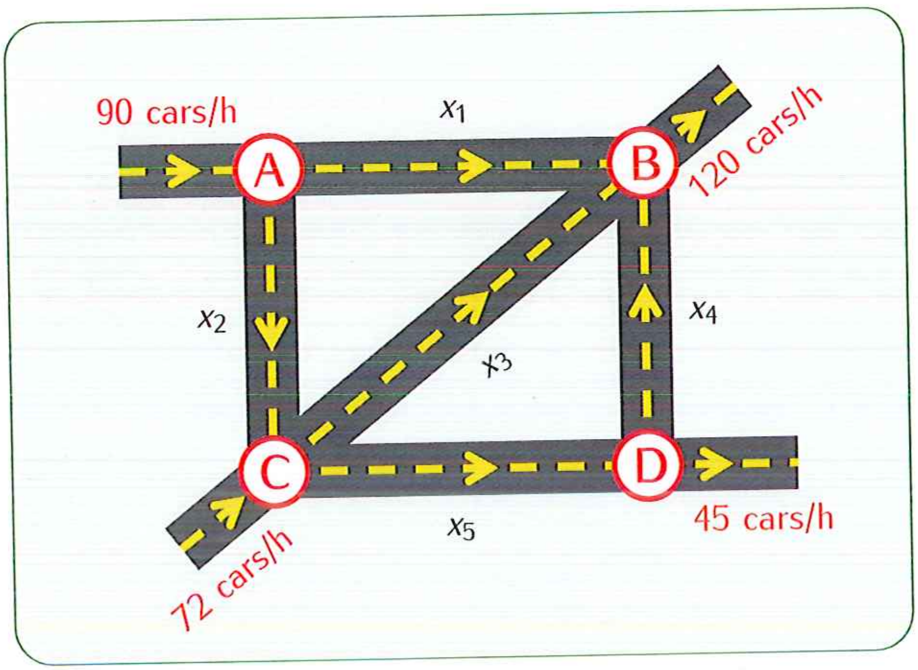
Problem. Find the flow rate of cars on each segment of streets:



Solution:

FLOW IN = FLOW OUT

@A: $90 = x_1 + x_2$

@B: $x_1 + x_3 + x_4 = 120$

@C: $72 + x_2 = x_3 + x_5$

@D: $x_5 = 45 + x_4$

$$\rightarrow \begin{cases} x_1 + x_2 = 90 \\ x_1 + x_3 + x_4 = 120 \\ -x_2 + x_3 + x_5 = 72 \\ -x_4 + x_5 = 45 \end{cases}$$

augmented matrix:

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 90 \\ 1 & 0 & 1 & 1 & 0 & 120 \\ 0 & -1 & 1 & 0 & 1 & 72 \\ 0 & 0 & 0 & -1 & 1 & 45 \end{array}\right] \end{array} \xrightarrow{\text{row red.}} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

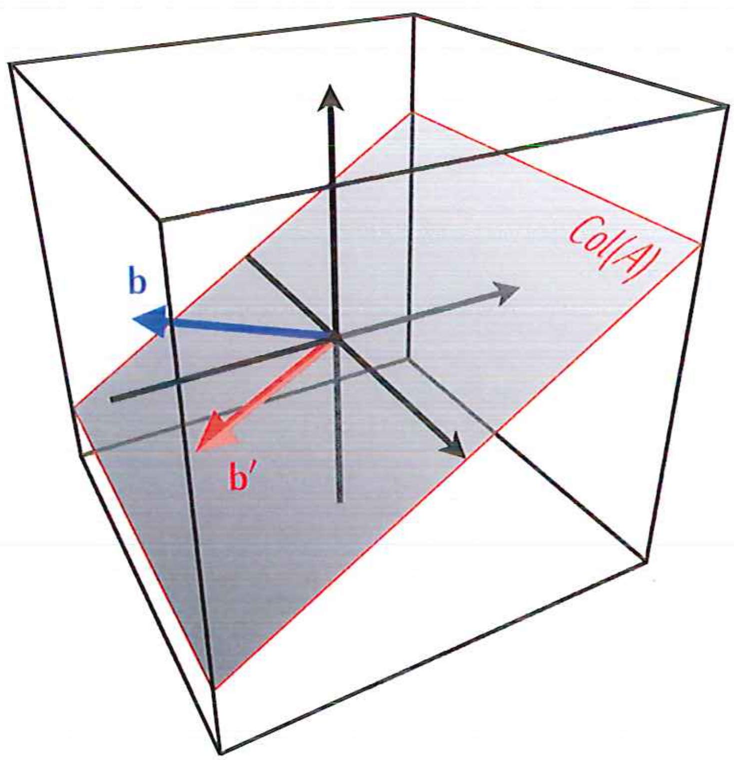
leading one in the last column, so no solutions

Upshot.

- Recall: a matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b} \in \mathrm{Col}(A)$.

- In practical applications we may obtain a matrix equation that has no solutions, i.e. where $\mathbf{b} \notin \mathrm{Col}(A)$.

- In such cases we may look for approximate solutions as follows:
  - replace $\mathbf{b}$ by a vector $\mathbf{b}'$ such that $\mathbf{b}' \in \mathrm{Col}(A)$ and $\mathrm{dist}(\mathbf{b}, \mathbf{b}')$ is a as small as possible.
  - then solve $A\mathbf{x} = \mathbf{b}'$



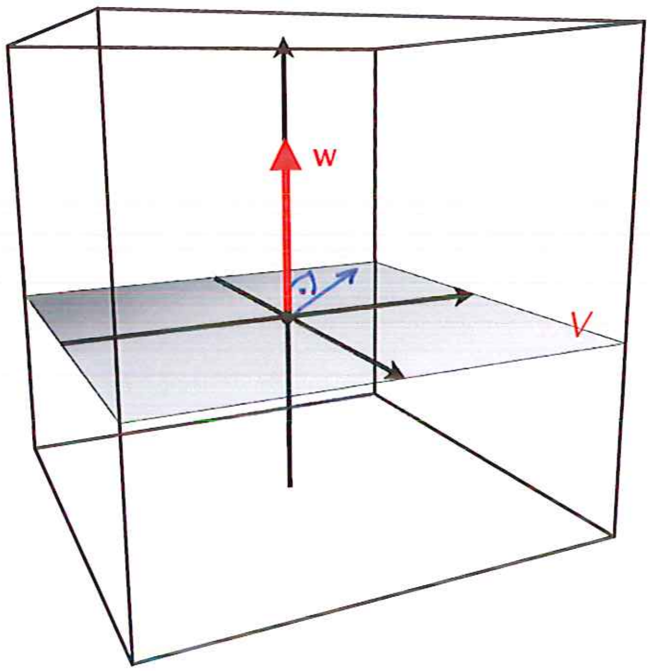
Definition

Given $\mathbf{b}' \in \mathrm{Col}(A)$ as above we will say that a vector $\mathbf{v}$ is a *least square solution* of the equation $A\mathbf{x} = \mathbf{b}$ if $\mathbf{v}$ is a solution of the equation $A\mathbf{x} = \mathbf{b}'$.

Next: How to find the vector $\mathbf{b}'$?

Definition

Let V be a subspace of $\mathbb{R}^n$. A vector $\mathbf{w} \in \mathbb{R}^n$ is *orthogonal to V* if $\mathbf{w} \cdot \mathbf{v} = 0$ for all $\mathbf{v} \in V$.



Proposition

If $V = \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ then a vector $\mathbf{w} \in \mathbb{R}^n$ is orthogonal to V if and only if $\mathbf{w} \cdot \mathbf{v}_i = 0$ for $i = 1, \ldots, k$.

Proof: Assume that w is orthogonal to $v_1, \ldots, v_k$.

If $v \in \mathrm{Span}(v_1, \ldots, v_k)$ then $v = c_1 v_1 + c_2 v_2 + \ldots + c_k v_k$

This gives:

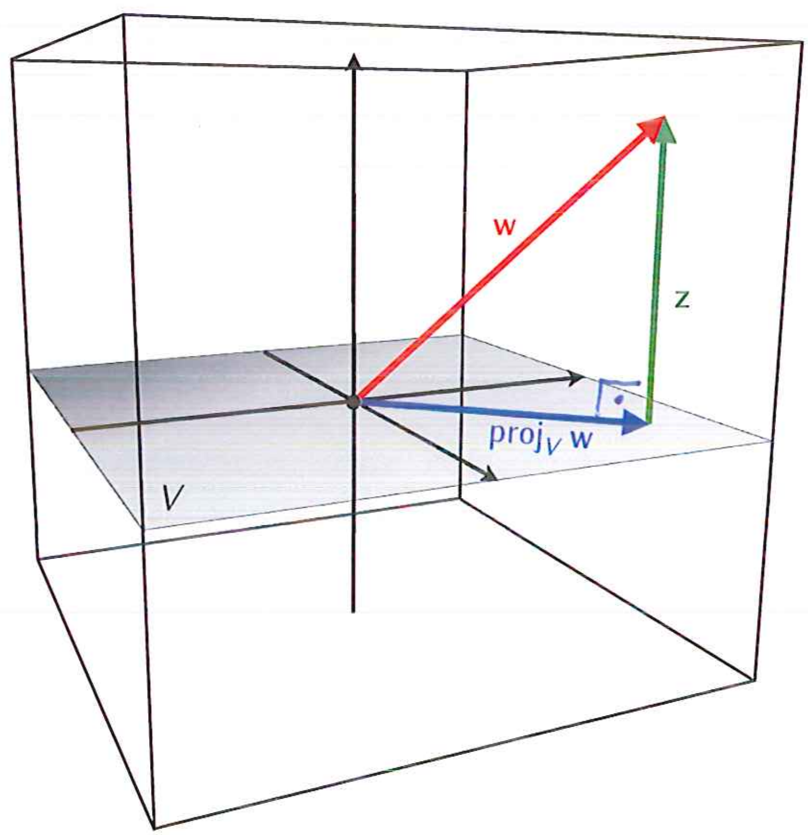
$$w \cdot v = w \cdot (c_1 v_1 + c_2 v_2 + \ldots + c_k v_k)$$
$$= c_1 (w \cdot v_1) + c_2 (w \cdot v_2) + \ldots + c_k (w \cdot v_k) = 0$$

So: w is orthogonal to every vector $v \in \mathrm{Span}(v_1, \ldots, v_k)$.

Definition

Let V be a subspace of $\mathbb{R}^n$ and let $\mathbf{w} \in \mathbb{R}^n$ the *orthogonal projection of* $\mathbf{w}$ *onto* V is a vector $\text{proj}_V\mathbf{w}$ such that

1) $\text{proj}_V\mathbf{w} \in V$
2) the vector $\mathbf{z} = \mathbf{w} - \text{proj}_V\mathbf{w}$ is orthogonal to V.



The Best Approximation Theorem

If V is a subspace of $\mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^n$ then $\text{proj}_V\mathbf{w}$ is a vector in V which is closest to $\mathbf{w}$:

$$\text{dist}(\mathbf{w}, \text{proj}_V\mathbf{w}) \leq \text{dist}(\mathbf{w}, \mathbf{v})$$

for all $\mathbf{v} \in V$.

Proof:

Let $v \in V$. We want to show:

$$\underbrace{\text{dist}(w, \text{proj}_V w)}_{\| w - \text{proj}_V w \|} \leq \underbrace{\text{dist}(w, v)}_{\| w - v \|}$$

Note: 1) $w - \text{proj}_V w$ is a vector orthogonal to V (by definition of $\text{proj}_V w$)

2) $\text{proj}_V w - v \in V$ (since v and $\text{proj}_V w$ are vectors in V)

This gives: $(w - \text{proj}_V w) \cdot (\text{proj}_V w - v) = 0$

By the Pythagorean Theorem we obtain:

$$\| w - \text{proj}_V w \|^2 + \underbrace{\| \text{proj}_V w - v \|^2}_{\geq 0} = \underbrace{\| (w - \cancel{\text{proj}_V w}) + (\cancel{\text{proj}_V w} - v) \|^2}_{\| w - v \|^2}$$

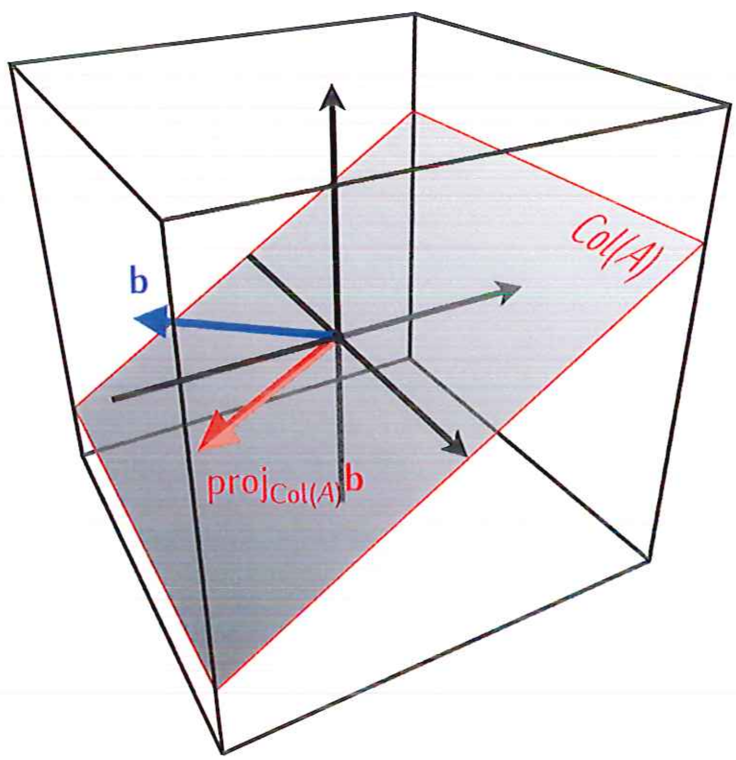
We obtain: $\| w - \text{proj}_V w \|^2 \leq \| w - v \|^2$

so: $\| w - \text{proj}_V w \| \leq \| w - v \|$

Corollary

The least square solutions of a matrix equation $A\mathbf{x} = \mathbf{b}$ are solutions of the equation

$$A\mathbf{x} = \text{proj}_{\text{Col}(A)}\mathbf{b}$$



<u>Next:</u> If V is a subspace of $\mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^n$ how to compute $\text{proj}_V\mathbf{w}$?

Theorem

If V is a subspace of $\mathbb{R}^n$ with an orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and $\mathbf{w} \in \mathbb{R}^n$ then

$$\mathrm{proj}_V \mathbf{w} = \left(\frac{\mathbf{w}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1 + \ldots + \left(\frac{\mathbf{w}\cdot\mathbf{v}_k}{\mathbf{v}_k\cdot\mathbf{v}_k}\right)\mathbf{v}_k$$

Proof: We need to check:

1) $\mathrm{proj}_V w \in V$
2) $w - \mathrm{proj}_V w$ is orthogonal to V

1) By the formula in the theorem $\mathrm{proj}_V w \in \mathrm{Span}(v_1, \ldots, v_k)$, and since $\{v_1, \ldots, v_k\}$ is a basis of V we have $\mathrm{Span}(v_1, \ldots, v_k) = V$. Thus $\mathrm{proj}_V w \in V$

2) Since $V = \mathrm{Span}(v_1, \ldots, v_k)$ it is enough to check that $w - \mathrm{proj}_V w$ is orthogonal to $v_1, \ldots, v_k$.

We have:

$$(w - \mathrm{proj}_V w)\cdot v_1 = w\cdot v_1 - (\mathrm{proj}_V w)\cdot v_1$$

$$= w\cdot v_1 - \Bigg[\underbrace{\left(\frac{w\cdot v_1}{\cancel{v_1\cdot v_1}}\right)\cancel{v_1\cdot v_1}}_{w\cdot v_1} + \left(\frac{w\cdot v_2}{v_2\cdot v_2}\right)\underbrace{v_2\cdot v_1}_{=0} + \ldots + \left(\frac{w\cdot v_k}{v_k\cdot v_k}\right)\underbrace{v_k\cdot v_1}_{=0}\Bigg]$$

$$= 0$$

By the same argument $(w - \mathrm{proj}_V w)\cdot v_i = 0$ for $i = 1, 2, \ldots, k$.

Corollary

If V is a subspace of $\mathbb{R}^n$ with an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and $\mathbf{w} \in \mathbb{R}^n$ then

$$\mathrm{proj}_V \mathbf{w} = (\mathbf{w}\cdot\mathbf{v}_1)\,\mathbf{v}_1 + \ldots + (\mathbf{w}\cdot\mathbf{v}_k)\,\mathbf{v}_k$$

Example. Let

$$\mathcal{B} = \left\{ \begin{array}{ccc} v_1 & v_2 & v_3 \\ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, & \begin{bmatrix} 2 \\ -4 \\ 5 \\ 2 \end{bmatrix}, & \begin{bmatrix} 4 \\ 1 \\ 0 \\ -2 \end{bmatrix} \end{array} \right\}, \qquad \mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

The set $\mathcal{B}$ is an orthogonal basis of some subspace V of $\mathbb{R}^4$. Compute $\text{proj}_V \mathbf{w}$.

Solution:

$$\text{proj}_V w = \left(\frac{w \cdot v_1}{v_1 \cdot v_1}\right) v_1 + \left(\frac{w \cdot v_2}{v_2 \cdot v_2}\right) v_2 + \left(\frac{w \cdot v_3}{v_3 \cdot v_3}\right) v_3$$

$$w \cdot v_1 = 1 \cdot 1 + 2 \cdot 2 + 2 \cdot 0 + 1 \cdot 3 = 8$$

$$v_1 \cdot v_1 = 1^2 + 2^2 + 0^2 + 3^2 = 14$$

$$w \cdot v_2 = 1 \cdot 2 + 2 \cdot (-4) + 2 \cdot 5 + 1 \cdot 2 = 6$$

$$v_2 \cdot v_2 = 2^2 + (-4)^2 + 5^2 + 2^2 = 49$$

$$w \cdot v_3 = 1 \cdot 4 + 2 \cdot 1 + 2 \cdot 0 + 1 \cdot (-2) = 4$$

$$v_3 \cdot v_3 = 4^2 + 1^2 + 0^2 + (-2)^2 = 21$$

This gives:

$$\text{proj}_V w = \frac{8}{14} v_1 + \frac{6}{49} v_2 + \frac{4}{21} v_3 = \frac{8}{14} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} + \frac{6}{49} \begin{bmatrix} 2 \\ -4 \\ 5 \\ 2 \end{bmatrix} + \frac{4}{21} \begin{bmatrix} 4 \\ 1 \\ 0 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 232/147 \\ 124/147 \\ 90/147 \\ 232/147 \end{bmatrix}$$

Note. In general if V is a subspace of $\mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^n$ then in order to find $\mathrm{proj}_V \mathbf{w}$ we need to do the following:

1) find a basis of V.
2) use the Gram-Schmidt process to get an orthogonal basis of V
3) use the orthogonal basis to compute $\mathrm{proj}_V \mathbf{w}$.

Example. Consider the following matrix A and vector $\mathbf{u}$:

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 3 & 4 & 2 \\ 2 & 6 & 3 & -1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

Compute $\mathrm{proj}_{\mathrm{Col}(A)} \mathbf{u}$.

Solution:

① Find a basis of Col (A):

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 3 & 4 & 2 \\ 2 & 6 & 3 & -1 \end{bmatrix} \xrightarrow{\text{row red}} \begin{bmatrix} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{basis of Col}(A) = \left\{ \overset{v_1}{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}} \right\}$$

② Use the G-S process to get an orthogonal basis of Col (A):

$$w_1 = v_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \qquad w_2 = v_2 - \left(\frac{w_1 \cdot v_2}{w_1 \cdot w_1}\right) w_1 = v_2 - \frac{10}{5} w_1 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

orthogonal basis:

$$\{w_1, w_2\} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$$

③ Calculate $\mathrm{proj}_V u$ using the orthogonal basis:

$$\mathrm{proj}_V u = \left(\frac{u \cdot w_1}{w_1 \cdot w_1}\right) w_1 + \left(\frac{u \cdot w_2}{w_2 \cdot w_2}\right) w_2 = \frac{3}{5} w_1 + \frac{6}{6} w_2$$

$$= \frac{3}{5} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 13/5 \\ 1/5 \end{bmatrix}$$

Example. Find least square solutions of the matrix equation $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 90 \\ 120 \\ 72 \\ 45 \end{bmatrix}$$

Solution:

exercise:

1. Find a basis of Col(A)
2. Use the Gram-Schmidt process to get an orthogonal basis of Col(A)
3. Use the orthogonal basis to compute $\text{proj}_{\text{Col}(A)} b$
4. Solve the equation

$$Ax = \text{proj}_{\text{Col}(A)} b$$

Solutions of this equation are the least square solutions of $Ax = b$.

Note: Next time we will simplify this procedure.