

Recall:

1) The dot product in  $\mathbb{R}^n$ :

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \dots a_n b_n$$

2) Properties of the dot product:

- a)  $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- b)  $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
- c)  $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$
- d)  $\mathbf{u} \cdot \mathbf{u} \geq 0$  and  $\mathbf{u} \cdot \mathbf{u} = 0$  if and only if  $\mathbf{u} = \mathbf{0}$ .

2) Using the dot product we can define:

- length of vectors
- distance between vectors
- orthogonality of vectors
- orthogonal and orthonormal bases
- orthogonal projection of a vector onto a subspace of  $\mathbb{R}^n$
- ...

Next: Generalization to arbitrary vector spaces.

### Definition

Let  $V$  be a vector space. An *inner product* on  $V$  is a function

$$\begin{aligned} V \times V &\longrightarrow \mathbb{R} \\ \mathbf{u}, \mathbf{v} &\longmapsto \langle \mathbf{u}, \mathbf{v} \rangle \end{aligned}$$

such that:

- a)  $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
- b)  $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
- c)  $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$
- d)  $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$  and  $\langle \mathbf{u}, \mathbf{u} \rangle = 0$  if and only if  $\mathbf{u} = \mathbf{0}$ .

### Definition

Let  $V$  be a vector space with an inner product  $\langle \cdot, \cdot \rangle$ .

- 1) The *length* (or *norm*) of a vector  $\mathbf{v}$  is the number

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

- 2) The *distance* between vectors  $\mathbf{u}, \mathbf{v} \in V$  is the number

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

- 3) Vectors  $\mathbf{u}, \mathbf{v} \in V$  are *orthogonal* if  $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ .

**Example.** The dot product is an inner product in  $\mathbb{R}^n$ .

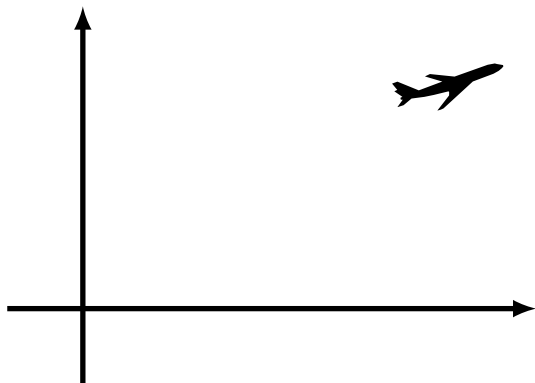
**Example.** Let  $p_1, \dots, p_n$  be any positive numbers. For vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$

$$\mathbf{u} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

define:

$$\langle \mathbf{u}, \mathbf{v} \rangle = p_1(a_1 b_1) + p_2(a_2 b_2) + \dots + p_n(a_n b_n)$$

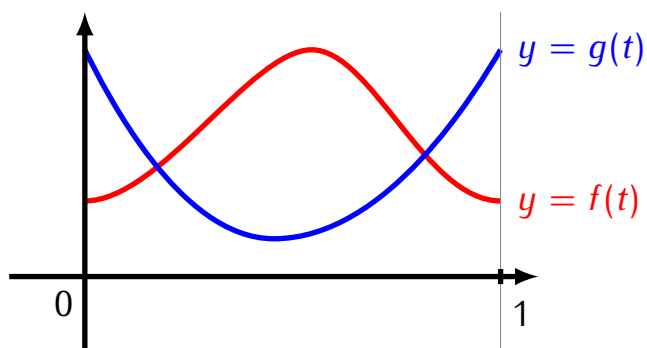
This gives an inner product in  $\mathbb{R}^n$ .



**Example.** Let  $C[0, 1]$  be the vector space of continuous functions  $f: [0, 1] \rightarrow \mathbb{R}$ . Define:

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt$$

This is an inner product on  $C[0, 1]$ .



**Example.** Compute the length of the function

$$f(t) = 1 + t^2$$

in  $C[0, 1]$ .

### Definition

Let  $V$  be a vector space with an inner product  $\langle \cdot, \cdot \rangle$ , and let  $W$  be a subspace of  $V$ . A vector  $\mathbf{v} \in V$  is *orthogonal to  $W$*  if  $\langle \mathbf{v}, \mathbf{w} \rangle = 0$  for all  $\mathbf{w} \in W$ .

### Definition

Let  $V$  be a vector space with an inner product  $\langle \cdot, \cdot \rangle$ , and let  $W$  be a subspace of  $V$ . The *orthogonal projection of a vector  $\mathbf{v} \in V$  onto  $W$*  is a vector  $\text{proj}_W \mathbf{v}$  such that

- 1)  $\text{proj}_W \mathbf{v} \in W$
- 2) the vector  $\mathbf{z} = \mathbf{v} - \text{proj}_W \mathbf{v}$  is orthogonal to  $W$ .

### Best Approximation Theorem

If  $V$  is a vector space with an inner product  $\langle \cdot, \cdot \rangle$ ,  $W$  is a subspace of  $V$ , and  $\mathbf{v} \in V$ , then  $\text{proj}_W \mathbf{v}$  is the vector of  $W$  which is the closest to  $\mathbf{v}$ :

$$\text{dist}(\mathbf{v}, \text{proj}_W \mathbf{v}) \leq \text{dist}(\mathbf{v}, \mathbf{w})$$

for all  $\mathbf{w} \in W$ .

### Theorem

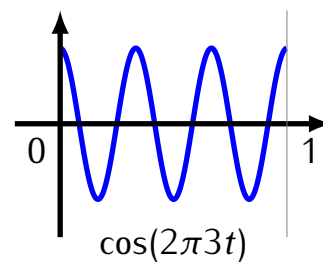
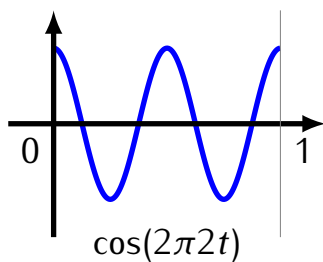
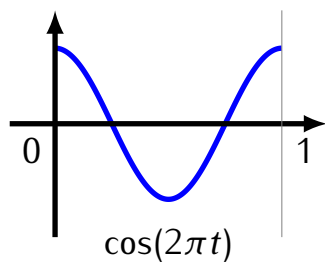
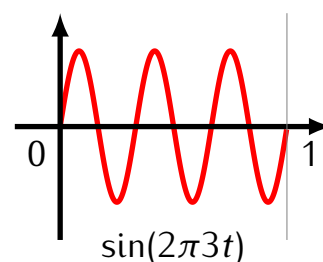
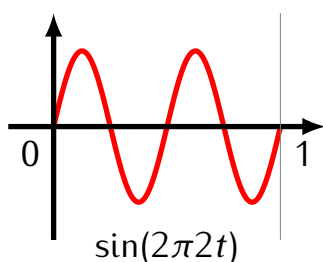
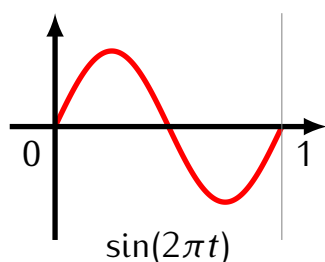
Let  $V$  is a vector space with an inner product  $\langle \cdot, \cdot \rangle$ , and let  $W$  be a subspace of  $V$ . If  $\mathcal{B} = \{\mathbf{w}_1, \dots, \mathbf{w}_k\}$  is an orthogonal basis of  $W$  (i.e. a basis such that  $\langle \mathbf{w}_i, \mathbf{w}_j \rangle = 0$  for all  $i \neq j$ ) then for  $\mathbf{v} \in V$  we have:

$$\text{proj}_W \mathbf{v} = \frac{\langle \mathbf{v}, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} \mathbf{w}_1 + \dots + \frac{\langle \mathbf{v}, \mathbf{w}_k \rangle}{\langle \mathbf{w}_k, \mathbf{w}_k \rangle} \mathbf{w}_k$$

**Application:** Fourier approximations.

**Goal:** Let  $f: [0, 1] \rightarrow \mathbb{R}$  be a continuous function. Find the best possible approximation of  $f$  of the form

$$\begin{aligned} P(t) = & a_0 \\ & + a_1 \sin(2\pi t) + b_1 \cos(2\pi t) \\ & + a_2 \sin(2\pi 2t) + b_2 \cos(2\pi 2t) \\ & \dots \dots \dots \dots \dots \dots \dots \dots \\ & + a_n \sin(2\pi nt) + b_n \cos(2\pi nt) \end{aligned}$$



**Note:** Let  $W_n$  be a subspace of  $C[0, 1]$  given by:

$$W_n = \text{Span}(1, \sin(2\pi t), \cos(2\pi t), \dots, \sin(2\pi nt), \cos(2\pi nt))$$

By the Best Approximation Theorem, the best approximation of  $f$  is obtained if we take  $P(t) = \text{proj}_{W_n} f(t)$ .

### Theorem

The set

$$\{1, \sin(2\pi t), \cos(2\pi t), \dots, \sin(2\pi n t), \cos(2\pi n t)\}$$

is an orthogonal basis of  $W_n$ .

### Corollary

If  $f \in C[0, 1]$  then

$$\begin{aligned} \text{proj}_{W_n} f(t) = & a_0 \\ & + a_1 \sin(2\pi t) + b_1 \cos(2\pi t) \\ & + a_2 \sin(2\pi 2t) + b_2 \cos(2\pi 2t) \\ & \dots \dots \dots \dots \dots \dots \dots \dots \\ & + a_n \sin(2\pi n t) + b_n \cos(2\pi n t) \end{aligned}$$

where:

$$a_0 = \frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} = \int_0^1 f(t) dt$$

and for  $k > 0$ :

$$a_k = \frac{\langle f, \sin(2\pi k t) \rangle}{\langle \sin(2\pi k t), \sin(2\pi k t) \rangle} = 2 \int_0^1 f(t) \cdot \sin(2\pi k t) dt$$

$$b_k = \frac{\langle f, \cos(2\pi k t) \rangle}{\langle \cos(2\pi k t), \cos(2\pi k t) \rangle} = 2 \int_0^1 f(t) \cdot \cos(2\pi k t) dt$$

**Example.** Compute  $\text{proj}_{W_n} f(t)$  for the function  $f(t) = t$ .



**Application:** Polynomial approximations.

**Goal:** Let  $f: [0, 1] \rightarrow \mathbb{R}$  be a continuous function. Find the best possible approximation of  $f$  given by a polynomial  $P(t)$  of degree  $\leq n$ :

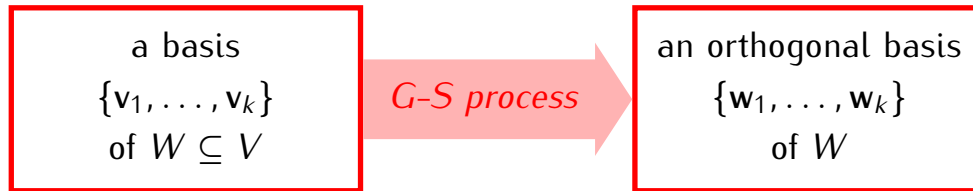
$$P(t) = a_0 + a_1 t + \dots + a_n t^n$$

**Note:** Let  $\mathbb{P}_n$  be the subspace of  $C[0, 1]$  consisting of all polynomials of degree  $\leq n$ :

$$\mathbb{P}_n = \{a_0 + a_1 t + \dots + a_n t^n \mid a_k \in \mathbb{R}\}$$

By the Best Approximation Theorem, the best approximation of  $f$  is obtained if we take  $P(t) = \text{proj}_{\mathbb{P}_n} f(t)$ .

## Gram-Schmidt process:



### Theorem (Gram-Schmidt Process)

Let  $V$  be a vector space with an inner product  $\langle \cdot, \cdot \rangle$ , and let  $W$  be a subspace of  $V$ . Let  $\{v_1, \dots, v_k\}$  be a basis of  $W$ . Define vectors  $\{w_1, \dots, w_k\}$  as follows:

$$w_1 = v_1$$

$$w_2 = v_2 - \frac{\langle w_1, v_2 \rangle}{\langle w_1, w_1 \rangle} w_1$$

$$w_3 = v_3 - \frac{\langle w_1, v_3 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle w_2, v_3 \rangle}{\langle w_2, w_2 \rangle} w_2$$

... ..

$$w_k = v_k - \frac{\langle w_1, v_k \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle w_2, v_k \rangle}{\langle w_2, w_2 \rangle} w_2 - \dots - \frac{\langle w_{k-1}, v_k \rangle}{\langle w_{k-1}, w_{k-1} \rangle} w_{k-1}$$

Then the set  $\{w_1, \dots, w_k\}$  is an orthogonal basis of  $W$ .

**Example.** Find an orthogonal basis of the subspace  $\mathbb{P}_2$  of the vector space  $C[0, 1]$ .

**Example.** Compute  $\text{proj}_{\mathbb{P}_2} f(t)$  for  $f(t) = \sqrt{t}$ .