Recall: An $n \times n$ matrix A defines a linear transformation

$$T_A\colon \mathbb{R}^n \to \mathbb{R}^n$$

given by $T_A(\mathbf{v}) = A\mathbf{v}$.

Next goal: Understand this linear transformation better.

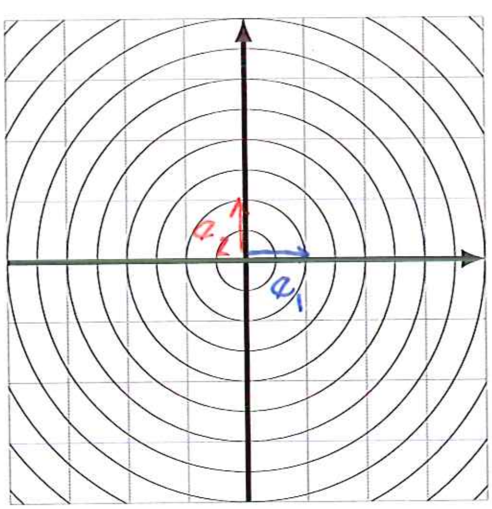
Example.

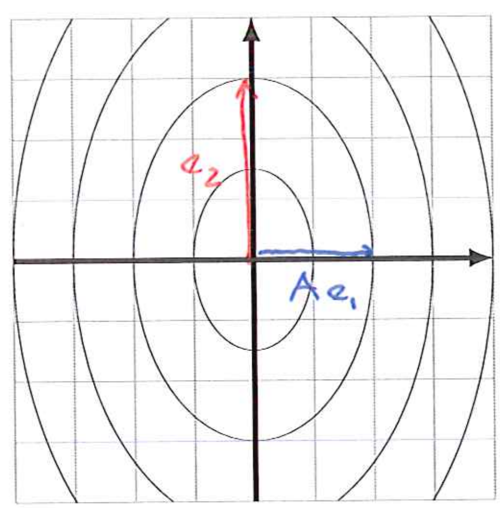
$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$$\begin{aligned} T_A : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ v &\longmapsto Av \end{aligned}$$

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad Ae_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \cdot e_1$

$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad Ae_2 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} = 3 \cdot e_2$





Example.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

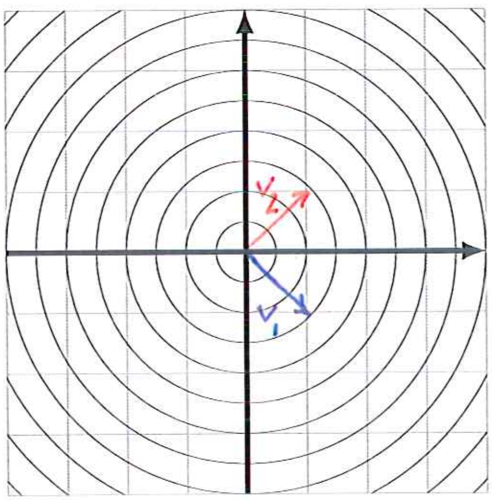
$$T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$v \longmapsto Av$$

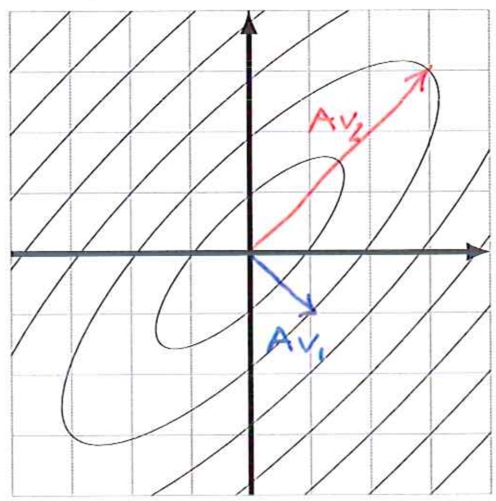
$$Ae_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad Ae_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Take: $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$Av_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \cdot v_1$$

$$Av_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \cdot v_2$$





Definition

Let A be an $n \times n$ matrix. If $\mathrm{v} \in \mathbb{R}^n$ is a non-zero vector and λ is a scalar such that

$$A\mathrm{v} = \lambda \mathrm{v}$$

then we say that

- λ is an *eigenvalue* of A
- v is an *eigenvector* of A corresponding to λ.

Example.

$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ We had: $A \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

So: $\lambda_1 = 2$ and $\lambda_2 = 3$ are eigenvalues of A.

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 2$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_2 = 3$

Example.

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ We had: $A \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $A \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So: $\lambda_1 = 1$ and $\lambda_2 = 3$ are eigenvalues of A.

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 1$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_2 = 3$

Computation of eigenvalues

Recall: $I_n = n \times n$ identity matrix:

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Note:

$$\lambda I_n = \begin{bmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \lambda \end{bmatrix}$$

For any $v \in \mathbb{R}^n$ we have:

$$(\lambda I_n)v = \lambda(I_n v) = \lambda v$$

Propostiton

If A be an $n \times n$ matrix then $\lambda \in \mathbb{R}$ is an eigenvalue of A if and only if the matrix equation

$$(A - \lambda I_n)\mathrm{x} = \mathbf{0}$$

has a non-trivial solution.

Proof: λ is an eigenvalue of A

$\Updownarrow$

there is a vector $v \neq 0$ such that $Av = \lambda v = (\lambda I_n)v$

$\Updownarrow$

there is a vector $v \neq 0$ such that $(A - \lambda I_n)v = 0$

$\Updownarrow$

the equation $(A - \lambda I_n)x = 0$ has a non-trivial solution.

Propostiton

If B is an $n \times n$ matrix then equation

$$Bx = \mathbf{0}$$

has ~~only~~ a non-trivial solution if and only of the matrix B is not invertible.

Proof: $Bx = 0$ has a non-trivial solution

$\Updownarrow$

not every column of B is a pivot column

$\Updownarrow$

B is not invertible

Propostiton

If A be an $n \times n$ matrix then $\lambda \in \mathbb{R}$ is an eigenvalue of A if and only if

$$\det(A - \lambda I_n) = 0$$

Proof: λ is an eigenvalue of A

$\Updownarrow$

$(A - \lambda I_n)x = 0$ has a non-trivial solution

$\Updownarrow$

$A - \lambda I_n$ is not invertible

$\Updownarrow$

$\det(A - \lambda I_n) = 0$

Example. Find all eigenvalues of the following matrix:

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Solution: We need to find $\lambda \in \mathbb{R}$ such that $\det(A - \lambda I) = 0$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right)$$

$$= \det \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$$

$$= (2-\lambda)(3-\lambda)(2-\lambda) + 2\cdot 1\cdot 1 + 1\cdot 1\cdot 2$$
$$- 1\cdot(3-\lambda)\cdot 1 - (2-\lambda)\cdot 1\cdot 2 - 1\cdot 2\cdot(2-\lambda)$$
$$= -\lambda^3 + 7\lambda^2 - 11\lambda + 5$$

We obtain: λ is an eigenvalue of A if and only if

$$-\lambda^3 + 7\lambda - 11\lambda + 5 = 0$$

Check: the only solutions of this equation are $\lambda_1 = 1$ and $\lambda_2 = 5$

We obtain: The matrix A has two eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = 5$.

Definition

If A is an $n \times n$ matrix then

$$P(\lambda) = \det(A - \lambda I_n)$$

is a polynomial of degree n. $P(\lambda)$ is called the *characteristic polynomial* of the matrix A.

Upshot

If A is a square matrix then

$$\text{eigenvalues of } A = \text{roots of } P(\lambda)$$

Example.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

We had:

$$P(\lambda) = \det(A - \lambda I) = \underbrace{-\lambda^3 + 7\lambda^2 - 11\lambda + 5}_{\text{the characteristic polynomial of } A}$$

Corollary

An $n \times n$ matrix can have at most n distinct eigenvalues.

Proof: The characteristic polynomial $P(\lambda)$ of A is a polynomial of degree n, so it can have at most n distinct roots.

Computation of eigenvectors

Recall: If B is a matrix then

$$\mathrm{Nul}(B) = \{v \mid Bv = 0\}$$

Proposition

If λ is an eigenvalue of an $n \times n$ matrix A then

$$\left\{\begin{matrix}\text{eigenvectors of } A \\ \text{corresponding to } \lambda\end{matrix}\right\} = \left\{\begin{matrix}\text{vectors in} \\ \mathrm{Nul}(A - \lambda I_n)\end{matrix}\right\}$$

Proof:

$$v \in \mathrm{Nul}(A - \lambda I)$$

$$\Updownarrow$$

$$(A - \lambda I)v = 0$$

$$\Updownarrow$$

$$Av = (\lambda I)v$$

$$\Updownarrow$$

$$Av = \lambda v$$

$$\Updownarrow$$

v is an eigenvector of A corresponding to λ

Recall: If B is an $m \times n$ matrix then $\text{Nul}(B)$ is a subspace of $\mathbb{R}^n$.

Corollary/Definition

If A is an $n \times n$ matrix and λ is an eigenvalue of A then the set of all eigenvectors corresponding to λ is a subspace of $\mathbb{R}^n$.

This subspace is called the *eigenspace* of A corresponding to λ.

Proposition

If λ is an eigenvalue of an $n \times n$ matrix A then

$$\left\{ \begin{array}{c} \text{eigenspace of } A \\ \text{corresponding to } \lambda \end{array} \right\} = \text{Nul}(A - \lambda I_n)$$

Example. Consider the following matrix:

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Recall that eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 5$. Compute bases of eigenspaces of A corresponding to these eigenvalues.

Solution.

$\underline{\lambda_1 = 1}$

$$\begin{pmatrix}\text{eigenspace of} \\ \lambda_1 = 1\end{pmatrix} = \mathrm{Nul}(A - 1 \cdot I)$$

$$= \mathrm{Nul}\left(\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \mathrm{Nul}\left(\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\right)$$

We need to solve:

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right] \xrightarrow[\text{red.}]{\text{row}} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \underbrace{0}_{\text{free}} & \underbrace{0}_{\text{free}} & 0 \end{array}\right] \end{array} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} x_3$$

<u>We obtain:</u>

$$\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\} = \begin{pmatrix}\text{a basis} \\ \text{of } \mathrm{Nul}(A - 1 \cdot I)\end{pmatrix} = \begin{pmatrix}\text{a basis of the} \\ \text{eigenspace of } A \\ \text{corresponding to } \lambda_1 = 1\end{pmatrix}$$

$\underline{\lambda_2 = 5}$

$$\begin{pmatrix} \text{eigenspace of} \\ \lambda_2 = 5 \end{pmatrix} = \text{Nul}\,(A - 5I)$$

$$= \text{Nul}\left(\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}\right)$$

$$= \text{Nul}\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$$

We need to solve:

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -3 & 2 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 2 & -3 & 0 \end{array}\right] \xrightarrow{\text{row red.}} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & \underbrace{0}_{\text{free}} & 0 \end{array}\right] \end{array} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x_3$$

<u>We obtain:</u>

$$\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\} = \begin{pmatrix} \text{a basis} \\ \text{of Nul}\,(A-5I) \end{pmatrix} = \begin{pmatrix} \text{a basis of the eigenspace} \\ \text{corresponding to } \lambda_2 = 5 \end{pmatrix}$$