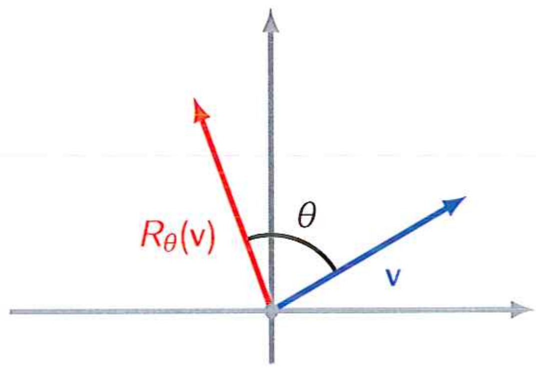
Problem: How to recognize if a function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation?

Example. Rotation by an angle θ:



$$R_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$$

Question:

1) Is R_θ a matrix transformation?
That is, is there a matrix A such that
$$R_\theta(v) = Av$$
for all $v \in \mathbb{R}^2$?

2) If so, what is this matrix A?

Definition

A function $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a *linear transformation* if it satisfies the following conditions:

1) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$
2) $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $\mathbf{v} \in \mathbb{R}^n$ and scalar c.

Proposition

Every matrix transformation is a linear transformation.

Proof: Let A be an $m \times n$ matrix:

$$\begin{aligned} T_A : \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ v &\longmapsto Av \end{aligned}$$

We have:

1) $T_A(u+v) = A(u+v) = Au + Av = T_A(u) + T_A(v)$

2) $T_A(cu) = A(cu) = c(Au) = cT_A(u)$

Theorem

Every linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation:

$$T = T_A$$

for some matrix A.

Proof:

Let

$$\underbrace{e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \ldots, \ e_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}_{\text{"standard basis vectors"}} \in \mathbb{R}^n$$

Take $A = [T(e_1) \ T(e_2) \ \cdots \ T(e_n)]$

We will show that $T(u) = A \cdot u$ for any $u \in \mathbb{R}^n$

Indeed:

If $u = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ then $u = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \ldots \ c_n \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

so: $u = c_1 e_1 + c_2 e_2 + \ldots + c_n e_n$

This gives:

$$T(u) = T(c_1 e_1 + c_2 e_2 + \ldots \ c_n e_n) = T(c_1 e_1) + T(c_2 e_2) + \ldots + T(c_n e_n)$$

$$= c_1 T(e_1) + c_2 T(e_2) + \ldots + c_n T(e_n)$$

$$= [T(e_1) \ T(e_2) \ \ldots \ T(e_n)] \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = A \cdot u$$

Corollary

If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then $T = T_A$ where A is the matrix given by

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \end{bmatrix}$$

This matrix is called the *standard matrix* of T.

Example. Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be the function given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 0 \\ 2x_1 \end{bmatrix}$$

Check if T is a linear transformation. If it is, find its standard matrix.

Solution:

1) Check if $T(u+v) = T(u) + T(v)$

Let $u = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $v = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$\left.\begin{aligned} T(u) &= T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} a_1 + a_2 \\ 0 \\ 2a_1 \end{bmatrix} \\ T(v) &= T\left(\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\right) = \begin{bmatrix} b_1 + b_2 \\ 0 \\ 2b_1 \end{bmatrix} \end{aligned}\right\} T(u) + T(v) = \begin{bmatrix} a_1 + a_2 + b_1 + b_2 \\ 0 \\ 2a_1 + 2b_1 \end{bmatrix}$$

$$T(u+v) = T\left(\begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}\right) = \begin{bmatrix} (a_1+b_1) + (a_2+b_2) \\ 0 \\ 2(a_1+b_1) \end{bmatrix}$$

This gives: $T(u) + T(v) = T(u+v)$

2) Similarly we can check that $T(cu) = cT(u)$.

This shows that T is a linear transformation

The standard matrix of T:

$$A = [\,T(e_1)\;\; T(e_2)\,]$$

where $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We get:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 2 & 0 \end{bmatrix}$$

Check:

$$A \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} x_1 + x_2 \\ 0 \\ 2x_1 \end{bmatrix}}_{= \, T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)}$$

Example. Let $S\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the function given by

$$S\cancel{T}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 + x_2 \\ x_2 \\ 3x_1 \end{bmatrix}$$

Check if S is a linear transformation. If it is, find its standard matrix.

<u>Solution</u>

1) Check if $S(u) + S(v) = S(u+v)$

$$u = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad v = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$S(u) = S\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} 1 + a_2 \\ a_2 \\ 3a_1 \end{bmatrix}$$

$$S(v) = S\left(\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\right) = \begin{bmatrix} 1 + b_2 \\ b_2 \\ 3b_2 \end{bmatrix}$$

$$S(u) + S(v) = \begin{bmatrix} 2 + a_2 + b_2 \\ a_2 + b_2 \\ 3a_1 + 3b_1 \end{bmatrix}$$

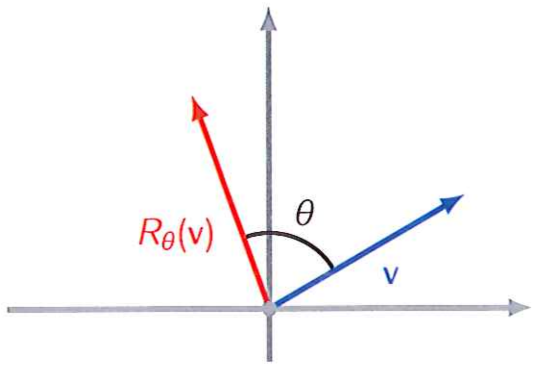
$$S(u+v) = S\left(\begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}\right) = \begin{bmatrix} 1 + (a_2 + b_2) \\ a_2 + b_2 \\ 3(a_1 + b_1) \end{bmatrix}$$

not equal

<u>We get</u>: $S(u) + S(v) \neq S(u+v)$

This shows that S is <u>not</u> a linear transformation and thus it can't be represented by a matrix.

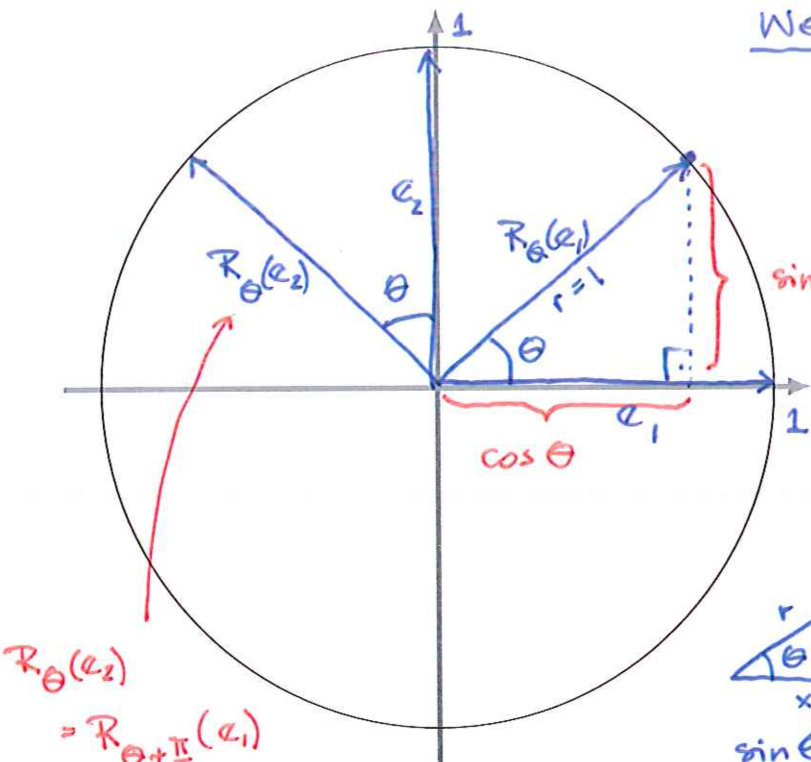
Back to rotations:



$R_\theta \colon \mathbb{R}^2 \to \mathbb{R}^2$

- One can check that R_θ is a linear transformation
- The standard matrix of R_θ:

$$A = [R_\theta(e_1) \quad R_\theta(e_2)]$$



We get:

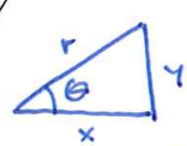
$$R_\theta(e_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$R_\theta(e_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

We get:

$$\boxed{A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$$

the standard matrix of the rotation by an angle θ



$\sin\theta = \frac{y}{r}$

$\cos\theta = \frac{x}{r}$