Other operations on matrices

1) Addition.

If $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \dots & b_{mn} \end{bmatrix}$ are $m \times n$ matrices then

$$A + B = \begin{bmatrix} a_{11} + b_{11} & \dots & a_{1n} + b_{1n} \\ \vdots & & \vdots \\ a_{m1} + b_{m1} & \dots & a_{mn} + b_{mn} \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 1 & 1 & 5 \\ 8 & 6 & 9 \end{bmatrix}$$

Note. The sum $A + B$ is defined only if A and B have the same dimensions.

1) Scalar multiplication.

If $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$, and c is a scalar then

$$cA = \begin{bmatrix} ca_{11} & \dots & ca_{1n} \\ \vdots & & \vdots \\ ca_{m1} & \dots & ca_{mn} \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$3 \cdot A = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}$$

Properties of matrix algebra

1) $(AB)C = A(BC)$

2) $(A+B)C = AC + BC$
$A(B+C) = AB + AC$

3) $I_n =$ the $n \times n$ identity matrix:

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

If A is an $m \times n$ matrix then

$$A \cdot I_n = A$$

$$I_m \cdot A = A$$

Note:

1) If $v = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ then

$$I_n v = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = c_1 \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \dots + c_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}}_{= v}$$

So $I_n v = v$ for any vector v.

$$T_{I^n} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$v \longmapsto I_n v = v$$

2) For any $m \times n$ matrix A we have: $A \cdot I_n = A$
$I_m \cdot A = A$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "T_{I_n}"] & \mathbb{R}^n \arrow[r, "T_A"] & \mathbb{R}^n \\ v \arrow[r] \arrow[rr, bend right, "T_{A \cdot I_n} = T_A"'] & I_n v = v \arrow[r] & A(I_n v) = Av \end{tikzcd}$$

Non-commutativity of matrix multiplication

1) If AB is defined then BA need not be defined.

e.g.:

$$A = \underset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \qquad B = \underset{2\times 3}{\begin{bmatrix} 0 & -1 & 7 \\ 5 & 4 & 2 \end{bmatrix}}$$

$\underset{(2\times 2)\ (2\times 3)}{A\cdot B}$ - defined $\qquad$ $\underset{(2\times 3)\ (2\times 2)}{B\cdot A}$ - not defined

2) Even if both AB and BA are both defined then usually

$$AB \neq BA$$

Example:

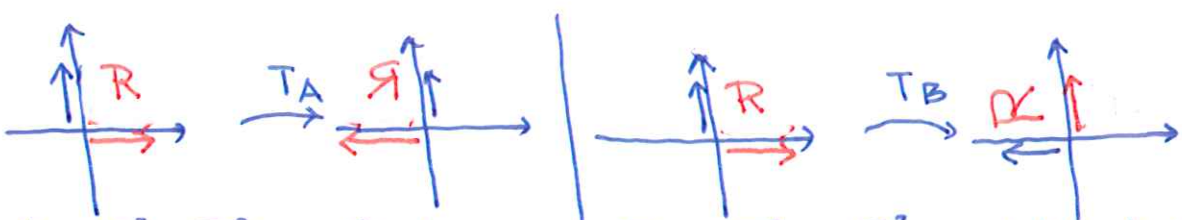
$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

not equal !

$$BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

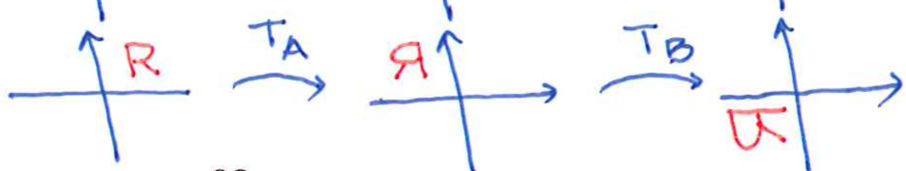
Note:



$T_A : \mathbb{R}^2 \to \mathbb{R}^2$ - reflection about y-axis $\qquad$ $T_B : \mathbb{R}^2 \to \mathbb{R}^2$ - rotation by 90°

$T_{AB} = T_A \circ T_B$

$T_{BA} = T_B \circ T_A$



One more operation on matrices: matrix transpose

Definition

The transpose of a matrix A is the matrix A^T such that

$$(\text{rows of } A^T) = (\text{columns of } A)$$

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

2×3 $\qquad$ 3×2

Properties of transpose

1) $(A^T)^T = A$
2) $(A + B)^T = (A^T + B^T)$
3) $(AB)^T = B^T A^T$