<u>Recall:</u>

1) A square matrix A is diagonalizable if there exists an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}$$

2) If A is diagonalizable then it is easy to compute powers of A:

$$A^k = PD^kP^{-1}$$

3) An $n \times n$ matrix A is a diagonalizable if and only if it has n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$. In such case we have:

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \qquad \begin{array}{l} \lambda_1 = \text{eigenvalue corresponding to } \mathbf{v}_1 \\ \lambda_2 = \text{eigenvalue corresponding to } \mathbf{v}_2 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \lambda_n = \text{eigenvalue corresponding to } \mathbf{v}_n \end{array}$$

4) Not every square matrix is diagonalizable.

Definition

A square matrix A is *symmetric* if $A^T = A$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 5 & 6 \\ 3 & 5 & 7 & 8 \\ 4 & 6 & 8 & 9 \end{bmatrix}$$

Note: A square matrix is symmetric if its entries above the main diagonal are the same as the corresponding entries below the main diagonal.

Theorem

Every symmetric matrix is diagonalizable.

Theorem

If A is a symmetric matrix and λ_1, λ_2 are two different eigenvalues of A, then eigenvectors corresponding to λ_1 are orthogonal to eigenvectors corresponding to λ_2.

Note. If v, w are vectors in $\mathbb{R}^n$ then

$$\underbrace{\mathbf{v} \cdot \mathbf{w}}_{\text{dot product}} = \underbrace{\mathbf{v}^T \mathbf{w}}_{\text{matrix multiplication}}$$

Example.

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$v^T w = \underbrace{[1\ 2\ 3]}_{\substack{1\times 3\\ \text{matrix}}} \cdot \underbrace{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}}_{\substack{3\times 1\\ \text{matrix}}} = \underbrace{[4\cdot 1 + 5\cdot 2 + 6\cdot 3]}_{\substack{1\times 1\\ \text{matrix}}} = [32] = \underbrace{v \cdot w}_{\substack{\text{dot}\\ \text{product}}}$$

Proof of theorem:

Let v = eigenvector corresponding to λ_1
w = —— " —— " —— " —— λ_2

We have:

$$\lambda_1(v\cdot w) = (\lambda_1 v)\cdot w = (Av)\cdot w = (Av)^T w$$
$$= (v^T A^T) w \underset{A^T = A}{=} (v^T A) w = v^T (Aw) = v^T(\lambda_2 w)$$
$$= \lambda_2 (v^T w)$$
$$= \lambda_2 (v \cdot w)$$

This gives: $\lambda_1 (v\cdot w) = \lambda_2 (v\cdot w)$
$(\lambda_1 - \lambda_2)(v \cdot w) = 0$

Since $\lambda_1 \neq \lambda_2$ we get $(\lambda_1 - \lambda_2) \neq 0$, so $v \cdot w = 0$.

Theorem

If A is an $n \times n$ symmetric matrix then A has n othogonal eigenvectors.

Example.

a) Find three orthogonal eigenvectors of the following symmetric matrix:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

b) Use these eigenvectors to diagonalize this matrix.

Solution:

1) Find eigenvalues of A:

$$P(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{bmatrix} = -\lambda^3 + 6\lambda^2 - 9\lambda + 4$$

$$\left(\text{eigenvalues of } A\right) = \left(\text{roots of } P(\lambda)\right) = \left(\lambda_1 = 4,\ \lambda_2 = 1\right)$$

2) Find a basis of eigenspace for each eigenvalue:

$$\begin{pmatrix} \text{eigenspace for} \\ \lambda_1 = 4 \end{pmatrix} = \text{Nul}(A - 4I) = \text{Nul}\left(\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\right)$$

basis : $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$$\begin{pmatrix} \text{eigenspace for} \\ \lambda_2 = 1 \end{pmatrix} = \text{Nul}(A - 1I) = \text{Nul}\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right)$$

basis: $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

Upshot. We have 3 linearly independent eigenvectors of A:

$$\underbrace{\overset{v_1}{\begin{bmatrix}1\\1\\1\end{bmatrix}}}_{\lambda_1=4}, \underbrace{\overset{v_2}{\begin{bmatrix}-1\\0\\1\end{bmatrix}}, \overset{v_3}{\begin{bmatrix}-1\\1\\0\end{bmatrix}}}_{\lambda_2=1}$$

Note: 1) v_1 is orthogonal to v_2 and v_3 (since it corresponds to a different eigenvalue

2) v_2 and v_3 are not orthogonal to each other: $v_2 \cdot v_3 = 1 \neq 0$

To fix this we need to use the Gram-Schmidt process to find an orthogonal basis of the eigenspace of $\lambda_1 = 1$:

$$w_2 = v_2 = \begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$$w_3 = v_3 - \left(\frac{w_2 \cdot v_3}{w_2 \cdot w_2}\right) w_2 = \begin{bmatrix}-1\\1\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}-1/2\\1\\-1/2\end{bmatrix}$$

We obtain 3 orthogonal eigenvectors:

$$\underbrace{\overset{v_1}{\begin{bmatrix}1\\1\\1\end{bmatrix}}}_{\lambda_1=4}, \underbrace{\overset{w_2}{\begin{bmatrix}-1\\0\\1\end{bmatrix}}, \overset{w_3}{\begin{bmatrix}-1/2\\1\\-1/2\end{bmatrix}}}_{\lambda_2=1}$$

This gives a diagonalization of A:

$$A = PDP^{-1} \quad \text{where} \quad P = \begin{bmatrix}1 & -1 & -1/2\\1 & 0 & 1\\1 & 1 & -1/2\end{bmatrix} \quad D = \begin{bmatrix}4 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Upshot. How to find n orthogonal eigenvectors for a symmetric $n \times n$ matrix A:

1) Find eigenvalues of A.

2) Find a basis of the eigenspace for each eigenvalue.

3) Use the Gram-Schmidt process to find an orthogonal basis of each eigenspace.

Note: Take the matrix P from the last example:

$$P = \begin{bmatrix} 1 & -1 & -1/2 \\ 1 & 0 & 1 \\ 1 & 1 & -1/2 \end{bmatrix}$$

We have:

$$P^T P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ -1/2 & 1 & -1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -1/2 \\ 1 & 0 & 1 \\ 1 & 1 & -1/2 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3/2 \end{bmatrix}}_{\substack{\text{this is almost} \\ \text{the identity matrix,} \\ \text{so } P^T \text{ is almost} \\ \text{the inverse of } P}}$$

Why this works this way:

$$P = [w_1 \;\; w_2 \;\; w_2] \qquad P^T = \begin{bmatrix} w_1^T \\ w_2^T \\ w_3^T \end{bmatrix}$$

$$P^T P = \begin{bmatrix} w_1^T \\ w_2^T \\ w_3^T \end{bmatrix} \cdot [w_1 \;\; w_2 \;\; w_3] = \begin{bmatrix} w_1 \cdot w_1 & \overset{0}{\overset{\shortparallel}{\boxed{w_1 \cdot w_2}}} & \overset{0}{\overset{\shortparallel}{\boxed{w_1 \cdot w_3}}} \\ w_2 \cdot w_1 & w_2 \cdot w_2 & w_2 \cdot w_3 \\ w_3 \cdot w_1 & w_3 \cdot w_2 & w_3 \cdot w_3 \end{bmatrix}$$

since w_1, w_2, w_3 are orthogonal

Definition

A square matrix $Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{bmatrix}$ is an *orthogonal matrix* if $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is an orthonormal set of vectors, i.e.:

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Theorem

If Q is an orthogonal matrix then Q is invertible and $Q^{-1} = Q^T$.

Note. If $P = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$ is a matrix with orthogonal columns, then

$$Q = \begin{bmatrix} \frac{v_1}{||v_1||} & \frac{v_2}{||v_2||} & \dots & \frac{v_n}{||v_n||} \end{bmatrix}$$

is an orthogonal matrix.

Indeed: $\frac{v_i}{||v_i||} \cdot \frac{v_j}{||v_j||} = \frac{v_i \cdot v_j}{||v_i|| \cdot ||v_j||} = \begin{cases} 0 \text{ if } i \neq j & \text{since } v_i \cdot v_j = 0 \\ \frac{v_i \cdot v_i}{||v_i||^2} = \frac{||v_i||^2}{||v_i||^2} = 1 & \text{if } i = j \end{cases}$

Theorem

If A is a symmetric matrix then A is *orthogonally diagonalizable*. That is, there exists an orthogonal matrix Q and a diagonal matrix D such that

$$A = QDQ^{-1} = QDQ^T$$

Proof:

We had: if A is a symmetric $n \times n$ matrix then A has n orthogonal eigenvectors $v_1, v_2, \dots, v_n$

Take:

$$Q = \begin{bmatrix} \frac{v_1}{||v_1||} & \frac{v_2}{||v_2||} & \dots & \frac{v_n}{||v_n||} \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

where:
λ_1 = eigenvalue of v_1
λ_2 = —— " —— v_2
$\vdots$
λ_n = —— " —— v_n

Then Q is an orthogonal matrix and $A = QDQ^T$

Example. Find an orthogonal diagonalization of the following symmetric matrix:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

<u>Solution:</u> We have already seen that A has 2 eigenvalues:

$$\lambda_1 = 4 \,, \quad \lambda_2 = 1$$

and it has 3 orthogonal eigenvectors:

$$\underbrace{\overset{w_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}}_{\lambda_1 = 4} \quad \underbrace{\overset{w_2}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}} \quad \overset{w_3}{\begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix}}}_{\lambda_2 = 1}$$

<u>Take:</u> $Q = \left[\frac{w_1}{\|w_1\|} \quad \frac{w_2}{\|w_2\|} \quad \frac{w_3}{\|w_3\|} \right]$

$\|w_1\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

$\|w_2\| = \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2}$

$\|w_3\| = \sqrt{(-\frac{1}{2})^2 + 1^2 + (-\frac{1}{2})^2} = \frac{\sqrt{6}}{2}$

<u>so:</u>

$$Q = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$$

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>We have:</u>

$$A = QDQ^{-1} = QDQ^T$$

Note. We have seen that any symmetric matrix is orthogonally diagonalizable. The converse statement is also true:

Proposition

If a matrix A is orthogonally diagonalizable then A is a symmetric matrix.

Proof: If A is orthogonally diagonalizable then:

$$A = QDQ^T$$

for some orthogonal matrix Q and diagonal matrix D.

Note: $D^T = D$

This gives:

$$\begin{aligned} A^T &= (QDQ^T)^T \\ &= (Q^T)^T D^T Q^T \\ &= QDQ^T \\ &= A \end{aligned}$$

So A is a symmetric matrix.