Definition

A matrix is in the *row echelon form* if:

1) the first non-zero entry of each row is a 1 (“a leading one”);
2) the leading one in each row is to the right of the leading one in the row above it.

A matrix is in the *reduced row echelon form* if in addition it satisfies:

3) all entries above each leading one are 0.

$$\begin{bmatrix} 1 & * & * & * & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

($* =$ any number)

Example

$$\begin{bmatrix} 1 & 0 & 4 & 0 & 7 & 0 \\ 0 & 1 & 5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 4 & 6 & 7 & 0 \\ 0 & 1 & 5 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 4 & 0 & 7 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 6 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Fact

If a system of linear equations is represented by a matrix in the reduced row echelon form then it is easy to solve the system.

Example

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Proposition

A matrix in the reduced row echelon form represents an inconsistent system if and only if it contains a row of the form

$$\left[\begin{array}{cccccc} 0 & 0 & 0 & \dots & 0 & 1 \end{array}\right]$$

i.e. with the leading one in the last column.

Example

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Note

In an augmented matrix in the reduced row echelon form free variables correspond to columns of the coefficient matrix that do not contain leading ones.

Example

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 8 \end{array}\right]$$

Note

A matrix in the reduced row echelon form represents a system of equations with exactly one solution if and only if it has a leading one in every column except for the last one.

Gauss-Jordan elimination process (= row reduction)

$$\left[\quad \text{matrix} \quad\right] \xrightarrow{\textit{elementary row operations}} \left[\begin{array}{c}\text{matrix}\\ \text{in reduced row}\\ \text{echelon form}\end{array}\right]$$

1. Interchange rows, if necessary, to bring a non-zero element to the top of the first non-zero column of the matrix.

2. Multiply the first row so that its first non-zero entry becomes 1.

3. Add multiples of the first row to eliminate non-zero entries below the leading one.

4. Ignore the first row; apply steps 1-3 to the rest of the matrix.

5. Eliminate non-zero entries above all leading ones.

Example.

$$\begin{bmatrix} 0 & 4 & -8 & 0 & 4 \\ 2 & 6 & -6 & -2 & -4 \\ 2 & 7 & -8 & 0 & -1 \end{bmatrix}$$

How to solve systems of linear equations: example

$$\begin{cases} 4x_2 - 8x_3 = 4 \\ 2x_1 + 6x_2 - 6x_3 - 2x_4 = -4 \\ 2x_1 + 7x_2 - 8x_3 = -1 \end{cases}$$

$$\left[\begin{array}{cccc|c} 0 & 4 & -8 & 0 & 4 \\ 2 & 6 & -6 & -2 & -4 \\ 2 & 7 & -8 & 0 & -1 \end{array}\right] \xrightarrow{\textit{row reduction}} \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & -4 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Definition

A *pivot position* in a matrix is a position that after the row reduction contains a leading one.

A *pivot column* of a matrix is a column that contains a pivot position.

Theorem

1) A system of linear equations is inconsistent if and only if the last column of its augmented matrix is a pivot column.

2) Free variables of the system correspond to non-pivot columns of the coefficient matrix.

3) The system has only one solution if and only if every column of its augmented matrix is a pivot column, except for the last column.