

Recall:

1) Let A be an $n \times n$ matrix. If $\mathbf{v} \in \mathbb{R}^n$ is a non-zero vector and λ is a scalar such that

$$A\mathbf{v} = \lambda\mathbf{v}$$

then

- λ is an eigenvalue of A
- \mathbf{v} is an eigenvector of A corresponding to λ .

2) The characteristic polynomial of an $n \times n$ matrix A is the polynomial given by the formula

$$P(\lambda) = \det(A - \lambda I_n)$$

where I_n is the $n \times n$ identity matrix.

3) If A is a square matrix then

$$\text{eigenvalues of } A = \text{roots of } P(\lambda)$$

4) If λ is an eigenvalue of an $n \times n$ matrix A then

$$\left\{ \begin{array}{l} \text{eigenvectors of } A \\ \text{corresponding to } \lambda \end{array} \right\} = \left\{ \begin{array}{l} \text{vectors in} \\ \text{Nul}(A - \lambda I_n) \end{array} \right\}$$

Motivating example: Fibonacci numbers

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, ...

Problem. Find a formula for the n -th Fibonacci number F_n .

General Problem. If A is a square matrix how to compute A^k quickly?

Easy case:

Definition

A square matrix D is *diagonal matrix* if all its entries outside the main diagonal are zeros:

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

Proposition

If D is a diagonal matrix as above then

$$D^k = \begin{bmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^k \end{bmatrix}$$

Definition

A square matrix A is a *diagonalizable* if A is of the form

$$A = PDP^{-1}$$

where D is a diagonal matrix and P is an invertible matrix.

Example.

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ is a diagonalizable matrix:

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix}^{-1}$$

Proposition

If A is a diagonalizable matrix, $A = PDP^{-1}$, then

$$A^k = PD^kP^{-1}$$

Example.

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Compute A^{10} .

Diagonalization Theorem

1) An $n \times n$ matrix A is a diagonalizable if and only if it has n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

2) In such case $A = PDP^{-1}$ where :

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \quad \begin{array}{l} \lambda_1 = \text{eigenvalue corresponding to } \mathbf{v}_1 \\ \lambda_2 = \text{eigenvalue corresponding to } \mathbf{v}_2 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \lambda_n = \text{eigenvalue corresponding to } \mathbf{v}_n \end{array}$$

Example. Diagonalize the following matrix if possible:

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Note. Not every matrix is diagonalizable.

Example. Check if the following matrix is diagonalizable:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

Proposition

If A is an $n \times n$ matrix with n distinct eigenvalues then A is diagonalizable.

Back to Fibonacci numbers:

$$\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{n-1} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$