Recall: If A is an $m \times n$ matrix then

$$A \cdot \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$$

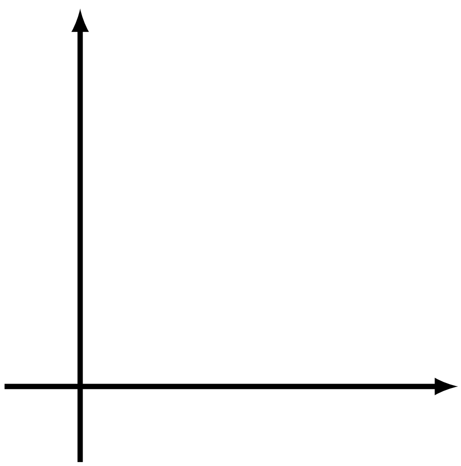
Definition

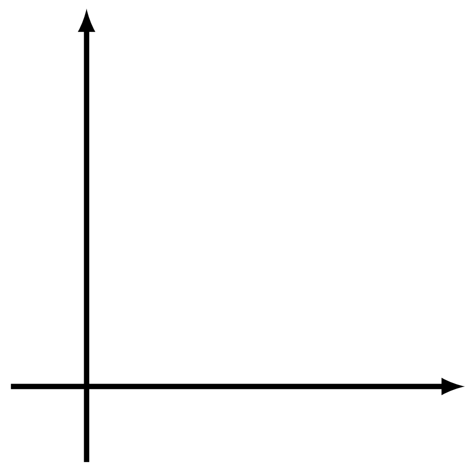
If A is an $m \times n$ matrix then the function

$$T_A \colon \mathbb{R}^n \to \mathbb{R}^m$$

given by $T_A(\mathbf{v}) = A\mathbf{v}$ is called the *matrix transformation associated to* A.

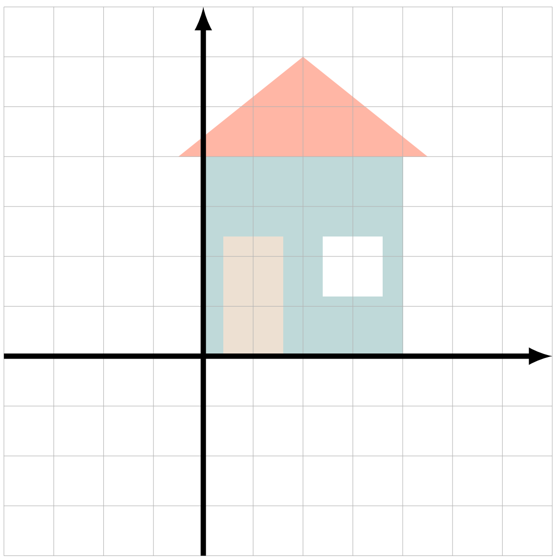
Geometric interpretation of matrix transformations $\mathbb{R}^2 \to \mathbb{R}^2$

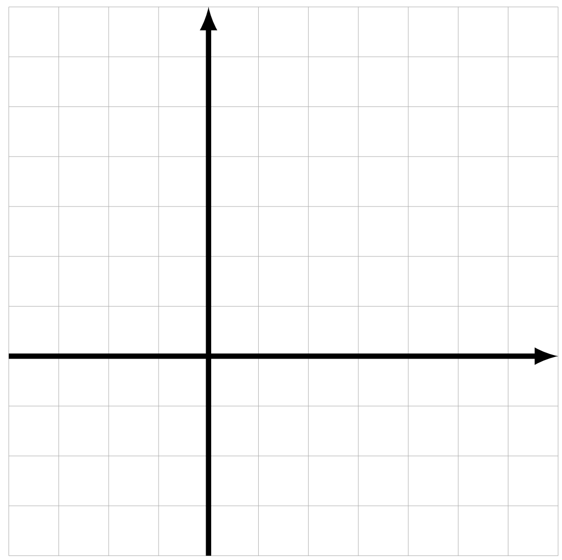




Example.

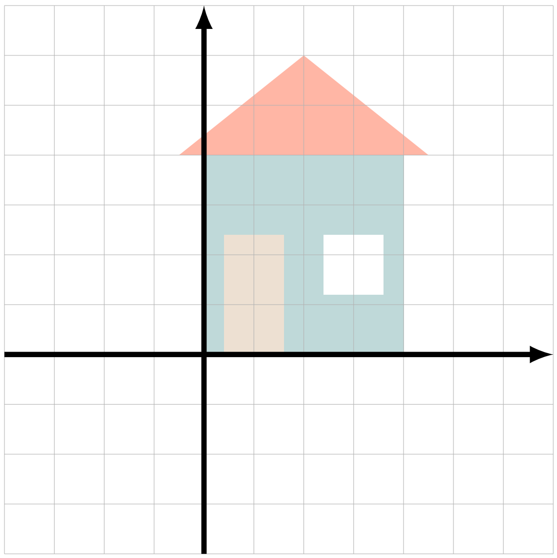
$$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$$

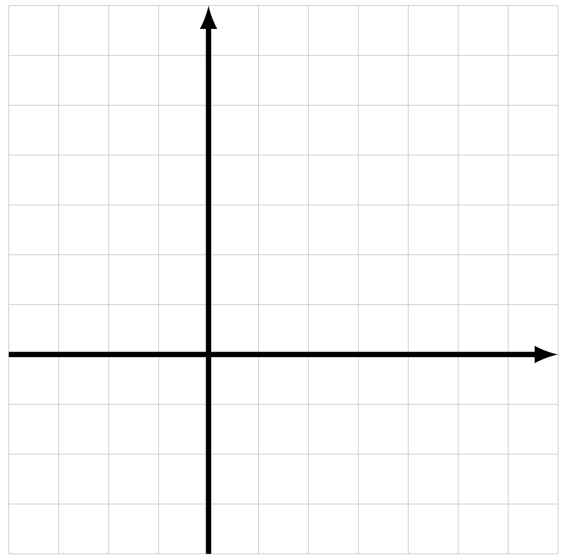




Example.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$





Recall:

A function $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is:

- *onto* if for each $\mathbf{b} \in \mathbb{R}^m$ there is $\mathbf{v} \in \mathbb{R}^m$ such that $F(\mathbf{v}) = \mathbf{b}$;

- *one-to-one* if for any $\mathbf{v}_1, \mathbf{v}_2$ such that $\mathbf{v}_1 \neq \mathbf{v}_2$ we have $F(\mathbf{v}_2) \neq F(\mathbf{v}_2)$.

Proposition

Let A be an $m \times n$ matrix. The following conditions are equivalent:

1) The matrix transformation $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ is onto.
2) $\mathrm{Col}(A) = \mathbb{R}^m$.
3) The matrix A has a pivot position in every row.

Proposition

Let A be an $m \times n$ matrix. The following conditions are equivalent:

1) The matrix transformation $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one.
2) $\mathrm{Nul}(A) = \{0\}$.
3) The matrix A has a pivot position in every column.

Example. For the following 3×4 matrix A check if the matrix transformation $T_A \colon \mathbb{R}^4 \to \mathbb{R}^3$ is onto and if it is one-to-one.

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -2 & -2 & 1 & -5 \\ 1 & 1 & -1 & 4 \end{bmatrix}$$

Example. For the following 3×3 matrix A check if the matrix transformation $T_A \colon \mathbb{R}^3 \to \mathbb{R}^3$ is onto and if it is one-to-one.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 2 & 5 \end{bmatrix}$$

Proposition

Let A be an $m \times n$ matrix. If the matrix transformation $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$ is both onto and one-to-one then we must have $m = n$ (i.e. A must be a square matrix).