Further properties of determinants

1) $\det(A^T) = \det A$

2) $\det(AB) = (\det A) \cdot (\det B)$

3) $\det(A^{-1}) = (\det A)^{-1}$

Note. In general $\det(A + B) \neq \det A + \det B$.

Recall: If A is square matrix then the *ij-cofactor of A* is the number

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Definition

If A is an $n \times n$ matrix then the *adjoint* (or *adjugate*) of A is the matrix

$$\mathrm{adj} A = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Theorem

If A is an invertible matrix then

$$A^{-1} = \frac{1}{\det A} \cdot \mathrm{adj} A$$

Example. Compute A^{-1} for

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Recall: If A is an invertible matrix then the equation $A\mathbf{x} = \mathbf{b}$ has only one solution: $\mathbf{x} = A^{-1}\mathbf{b}$.

Definition

If A is an $n \times n$ matrix and $\mathbf{b} \in \mathbb{R}^n$ then $A_i(\mathbf{b})$ is the matrix obtained by replacing the i^{th} column of A with $\mathbf{b}$.

Example.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$$

Theorem (Cramer's Rule)

If A is an $n \times n$ invertible matrix and $\mathbf{b} \in \mathbb{R}^n$ then the unique solution of the equation

$$A\mathbf{x} = \mathbf{b}$$

is given by

$$\mathbf{x} = \frac{1}{\det A} \begin{bmatrix} \det A_1(\mathbf{b}) \\ \vdots \\ \det A_n(\mathbf{b}) \end{bmatrix}$$