Definition

Let V be a vector space. An *inner product* on V is a function

$$\begin{aligned} V \times V &\longrightarrow \mathbb{R} \\ \mathbf{u}, \ \mathbf{v} &\longmapsto \langle \mathbf{u}, \mathbf{v} \rangle \end{aligned}$$

such that:

a) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$

b) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

c) $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$

d) $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Definition

Let V be a vector space with an inner product $\langle \ , \ \rangle$.

1) The *length* (or *norm*) of a vector $\mathbf{v}$ is the number

$$||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

2) The *distance* between vectors $\mathbf{u}, \mathbf{v} \in V$ is the number

$$\text{dist}(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$$

3) Vectors $\mathbf{u}, \mathbf{v} \in V$ are *orthogonal* if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Example. The dot product is an inner product in $\mathbb{R}^n$.

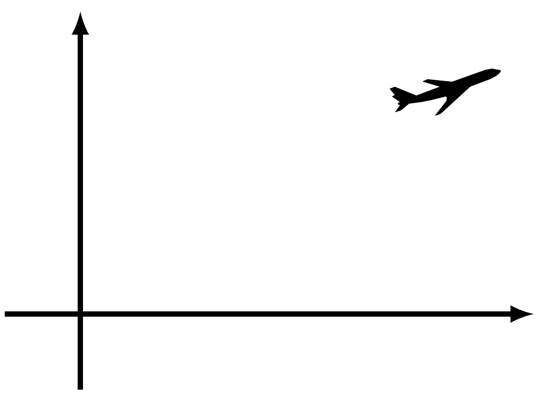
Example. Let $p_1, \dots, p_n$ be any positive numbers. For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$

$$\mathbf{u} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

define:

$$\langle \mathbf{u}, \mathbf{v} \rangle = p_1(a_1 b_1) + p_2(a_2, b_2) + \ldots + p_n(a_n b_n)$$

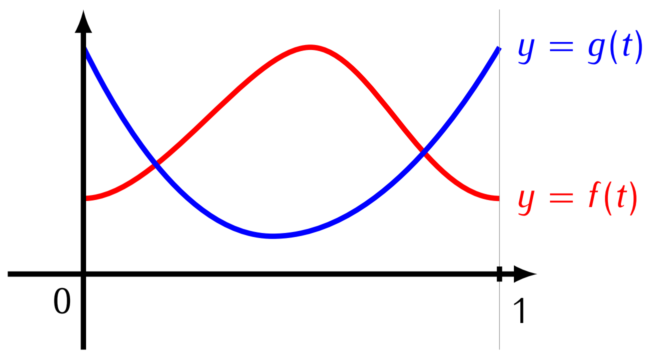
This gives an inner product in $\mathbb{R}^n$.



Example. Let $C[0,1]$ be the vector space of continuous functions $f\colon [0,1] \to \mathbb{R}$. Define:

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt$$

This is an inner product on $C[0,1]$.



Example. Compute the length of the function

$$f(t) = 1 + t^2$$

in $C[0,1]$.

Definition

Let V be a vector space with an inner product $\langle\ ,\ \rangle$, and let W be a subspace of V. A vector $\mathbf{v} \in V$ is *orthogonal to* W if $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in W$.

Definition

Let V be a vector space with an inner product $\langle\ ,\ \rangle$, and let W be a subspace of V. The *orthogonal projection of a vector* $\mathbf{v} \in V$ *onto* W is a vector $\mathrm{proj}_W\mathbf{v}$ such that

1) $\mathrm{proj}_W\mathbf{v} \in W$

2) the vector $\mathbf{z} = \mathbf{v} - \mathrm{proj}_W\mathbf{v}$ is orthogonal to W.

Best Approximation Theorem

If V is a vector space with an inner product $\langle\ ,\ \rangle$, W is a subspace of V, and $\mathbf{v} \in V$, then $\mathrm{proj}_W\mathbf{v}$ is the vector of V which is the closest to $\mathbf{v}$:

$$\mathrm{dist}(\mathbf{v}, \mathrm{proj}_W\mathbf{v}) \leq \mathrm{dist}(\mathbf{v}, \mathbf{w})$$

for all $\mathbf{w} \in W$.

Theorem

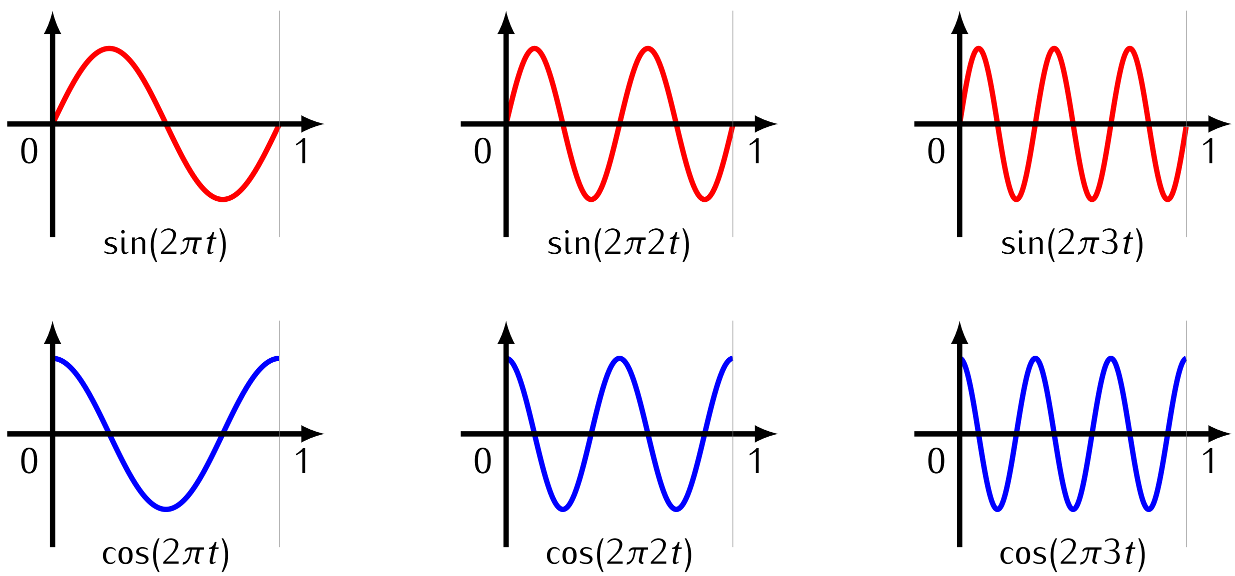
Let V is a vector space with an inner product $\langle\ ,\ \rangle$, and let W be a subspace of V. If $\mathcal{B} = \{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is an orthogonal basis of W (i.e. a basis such that $\langle \mathbf{w}_i, \mathbf{w}_j \rangle = 0$ for all $i \neq j$) then for $\mathbf{v} \in V$ we have:

$$\mathrm{proj}_W\mathbf{v} = \frac{\langle \mathbf{v}, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 + \ldots + \frac{\langle \mathbf{v}, \mathbf{w}_k \rangle}{\langle \mathbf{w}_k, \mathbf{w}_k \rangle}\mathbf{w}_k$$

Application: Fourier approximations.

Goal: Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. Find the best possible approximation of f of the form

$$
\begin{aligned}
P(t) = \quad & a_0 \\
& + a_1 \sin(2\pi t) \quad + \; b_1 \cos(2\pi t) \\
& + a_2 \sin(2\pi 2t) \; + \; b_2 \cos(2\pi 2t) \\
& \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
& + a_n \sin(2\pi n t) \; + \; b_n \cos(2\pi n t)
\end{aligned}
$$



Note: Let W_n be a subspace of $C[0,1]$ given by:

$$W_n = \mathrm{Span}(1, \sin(2\pi t), \cos(2\pi t), \dots, \sin(2\pi n t), \cos(2\pi n t))$$

By the Best Approximation Theorem, the best approximation of f is obtained if we take $P(t) = \mathrm{proj}_{W_n} f(t)$.

Theorem

The set
$$\{1, \sin(2\pi t), \cos(2\pi t), \dots, \sin(2\pi n t), \cos(2\pi n t)\}$$
is an orthogonal basis of W_n.

Corollary

If $f \in C[0,1]$ then

$$\begin{aligned}\operatorname{proj}_{W_n} f(t) = \quad & a_0 \\ & + a_1 \sin(2\pi t) \quad + \quad b_1 \cos(2\pi t) \\ & + a_2 \sin(2\pi 2t) \; + \; b_2 \cos(2\pi 2t) \\ & \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ & + a_n \sin(2\pi n t) \; + \; b_n \cos(2\pi n t)\end{aligned}$$

where:

$$a_0 = \frac{\langle f, 1\rangle}{\langle 1, 1\rangle} = \int_0^1 f(t)dt$$

and for $k > 0$:

$$a_k = \frac{\langle f, \sin(2\pi k t)\rangle}{\langle \sin(2\pi k t), \sin(2\pi k t)\rangle} = 2\int_0^1 f(t)\cdot\sin(2\pi k t)dt$$

$$b_k = \frac{\langle f, \cos(2\pi k t)\rangle}{\langle \cos(2\pi k t), \cos(2\pi k t)\rangle} = 2\int_0^1 f(t)\cdot\cos(2\pi k t)dt$$

Example. Compute $\text{proj}_{W_n} f(t)$ for the function $f(t) = t$.

Application: Polynomial approximations.

Goal: Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. Find the best possible approximation of f given by a polynomial $P(t)$ of degree $\leq n$:

$$P(t) = a_0 + a_1 t + \ldots + a_n t^n$$

Note: Let $\mathbb{P}_n$ be the subspace of $C[0,1]$ consisting of all polynomials of degree $\leq n$:

$$\mathbb{P}_n = \{a_0 + a_1 t + \ldots + a_n t^n \mid a_k \in \mathbb{R}\}$$

By the Best Approximation Theorem, the best approximation of f is obtained if we take $P(t) = \mathrm{proj}_{\mathbb{P}_n} f(t)$.

Gram-Schmidt process:

a basis		an orthogonal basis
$\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$	*G-S process* $\Longrightarrow$	$\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$
of $W \subseteq V$		of W

Theorem (Gram-Schmidt Process)

Let V be a vector space with an inner product $\langle\ ,\ \rangle$, and let W be a subspace of V. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a basis of W. Define vectors $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ as follows:

$$\mathbf{w}_1 = \mathbf{v}_1$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{w}_1, \mathbf{v}_2 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{w}_1, \mathbf{v}_3 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \frac{\langle \mathbf{w}_2, \mathbf{v}_3 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\mathbf{w}_k = \mathbf{v}_k - \frac{\langle \mathbf{w}_1, \mathbf{v}_k \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \frac{\langle \mathbf{w}_2, \mathbf{v}_k \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2 - \ldots - \frac{\langle \mathbf{w}_{k-1}, \mathbf{v}_k \rangle}{\langle \mathbf{w}_{k-1}, \mathbf{w}_{k-1} \rangle}\mathbf{w}_{k-1}$$

Then the set $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is an orthogonal basis of W.

Example. Find an orthogonal basis of the subspace $\mathbb{P}_2$ of the vector space $C[0,1]$.

Example. Compute $\operatorname{proj}_{\mathbb{P}_2} f(t)$ for $f(t) = \sqrt{t}$.

Recall: An $n \times n$ matrix A defines a linear transformation

$$T_A \colon \mathbb{R}^n \to \mathbb{R}^n$$

given by $T_A(\mathbf{v}) = A\mathbf{v}$.

Next goal: Understand this linear transformation better.

Example.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$