

## Construction of a basis of $\text{Col}(A)$

### Lemma

Let  $V$  be a vector space, and let  $\mathbf{v}_1, \dots, \mathbf{v}_p \in V$ . If a vector  $\mathbf{v}_i$  is a linear combination of the other vectors then

$$\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_p) = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_p)$$

**Upshot.** One can construct a basis of a vector space  $V$  as follows:

- Start with a set of vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  such that  $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_p) = V$ .
- Keep removing vectors without changing the span, until you get a linearly independent set.

**Example.** Find a basis of  $\text{Col}(A)$  where  $A$  is the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**Example.** Find a basis of  $\text{Col}(A)$  where  $A$  is the following matrix:

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

### Construction of a basis of $\text{Nul}(A)$

**Example.** Find a basis of  $\text{Nul}(A)$  where  $A$  is the following matrix:

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

**Upshot.** If  $A$  is matrix then:

$\dim \text{Col}(A) =$  the number of pivot columns of  $A$

$\dim \text{Nul}(A) =$  the number of non-pivot columns of  $A$

### Definition

If  $A$  is a matrix then:

- the dimension of  $\text{Col}(A)$  is called the *rank* of  $A$  and it is denoted  $\text{rank}(A)$
- the dimension of  $\text{Nul}(A)$  is called the *nullity* of  $A$ .

### The Rank Theorem

If  $A$  is an  $m \times n$  matrix then

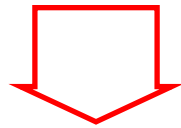
$$\text{rank}(A) + \dim \text{Nul}(A) = n$$

**Example.** Let  $A$  be a  $100 \times 101$  matrix such that  $\dim \text{Nul}(A) = 1$ . Show that the equation  $A\mathbf{x} = \mathbf{b}$  has a solution for each  $\mathbf{b} \in \mathbb{R}^{100}$ .

**Example.** Let  $A$  be a  $5 \times 9$ . Can the null space of  $A$  have dimension 3?

**Recall:** Any basis  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  of a vector space  $V$  defines a coordinate system:

$$\mathbf{v} = c_1 \mathbf{b}_1 + \dots + c_n \mathbf{b}_n = \mathbf{v}$$



$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

