Recall:

1) If

$$\mathbf{u} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

are vectors in $\mathbb{R}^n$ then:

- $\mathbf{u} \cdot \mathbf{v} = a_1 b_1 + \ldots + a_n b_n$
- $||\mathbf{u}|| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$
- $\text{dist}(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$

2) Vectors $\mathbf{u}, \mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.

3) Pythagorean theorem: $\mathbf{u}, \mathbf{v}$ are orthogonal if and only if

$$||\mathbf{u}||^2 + ||\mathbf{v}||^2 = ||\mathbf{u} + \mathbf{v}||^2$$

4) If $V \subseteq \mathbb{R}^n$ is a subspace then an orthogonal basis of V is a basis which consists of vectors that are orthogonal to one another.

5) If $\mathcal{B} = \{\mathbf{v}_1, \ldots \mathbf{v}_k\}$ is an orthogonal basis of V and $\mathbf{w} \in V$ then

$$[\ \mathbf{w}\ ]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

where $c_i = \dfrac{\mathbf{w} \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i}$.

6) Gram-Schmidt process:

$$\boxed{\begin{array}{c}\text{a basis}\\ \{\mathbf{v}_1, \dots, \mathbf{v}_k\}\\ \text{of } V \subseteq \mathbb{R}^n\end{array}} \xrightarrow{\textit{G-S process}} \boxed{\begin{array}{c}\text{an orthogonal basis}\\ \{\mathbf{w}_1, \dots, \mathbf{w}_k\}\\ \text{of } V\end{array}}$$

$$\mathbf{w}_1 = \mathbf{v}_1$$

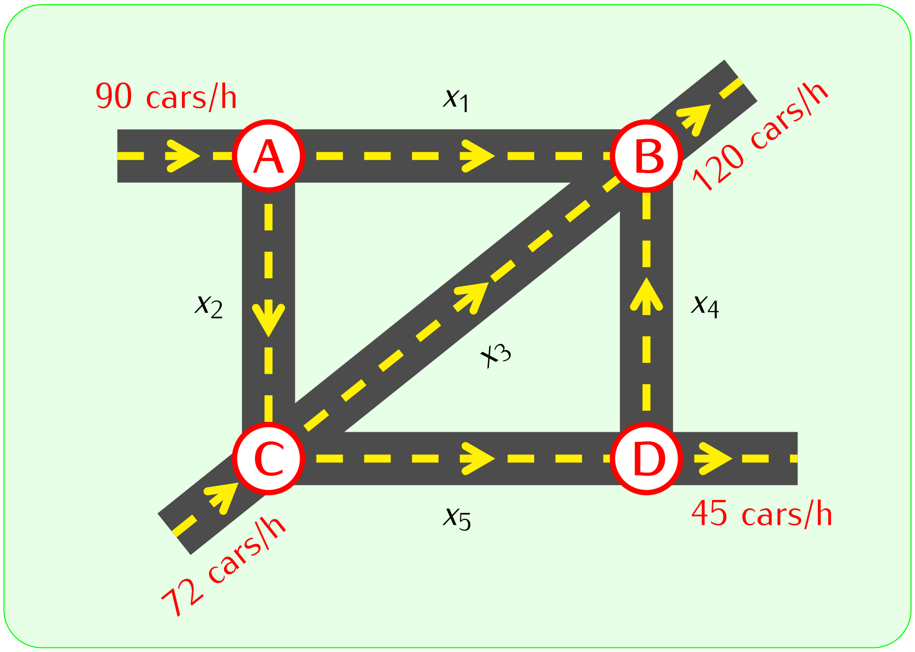
$$\mathbf{w}_2 = \mathbf{v}_2 - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_2}{\mathbf{w}_1 \cdot \mathbf{w}_1}\right) \mathbf{w}_1$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_3}{\mathbf{w}_1 \cdot \mathbf{w}_1}\right) \mathbf{w}_1 - \left(\frac{\mathbf{w}_2 \cdot \mathbf{v}_3}{\mathbf{w}_2 \cdot \mathbf{w}_2}\right) \mathbf{w}_2$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$\mathbf{w}_k = \mathbf{v}_k - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_k}{\mathbf{w}_1 \cdot \mathbf{w}_1}\right) \mathbf{w}_1 - \left(\frac{\mathbf{w}_2 \cdot \mathbf{v}_k}{\mathbf{w}_2 \cdot \mathbf{w}_2}\right) \mathbf{w}_2 - \dots - \left(\frac{\mathbf{w}_{k-1} \cdot \mathbf{v}_k}{\mathbf{w}_{k-1} \cdot \mathbf{w}_{k-1}}\right) \mathbf{w}_{k-1}$$

Problem. Find the flow rate of cars on each segment of streets:



Solution:

FLOW IN = FLOW OUT

@ A: $90 = x_1 + x_2$

@ B: $x_1 + x_3 + x_4 = 120$

@ C: $72 + x_2 = x_3 + x_5$

@ D: $x_5 = 45 + x_4$

$$\rightarrow \begin{cases} x_1 + x_2 = 90 \\ x_1 + x_3 + x_4 = 120 \\ -x_2 + x_3 + x_5 = 72 \\ -x_4 + x_5 = 45 \end{cases}$$

augmented matrix

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & \end{matrix} \\ \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 90 \\ 1 & 0 & 1 & 1 & 0 & 120 \\ 0 & -1 & 1 & 0 & 1 & 72 \\ 0 & 0 & 0 & -1 & 1 & 45 \end{array}\right] \end{array} \xrightarrow{\text{row red.}} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

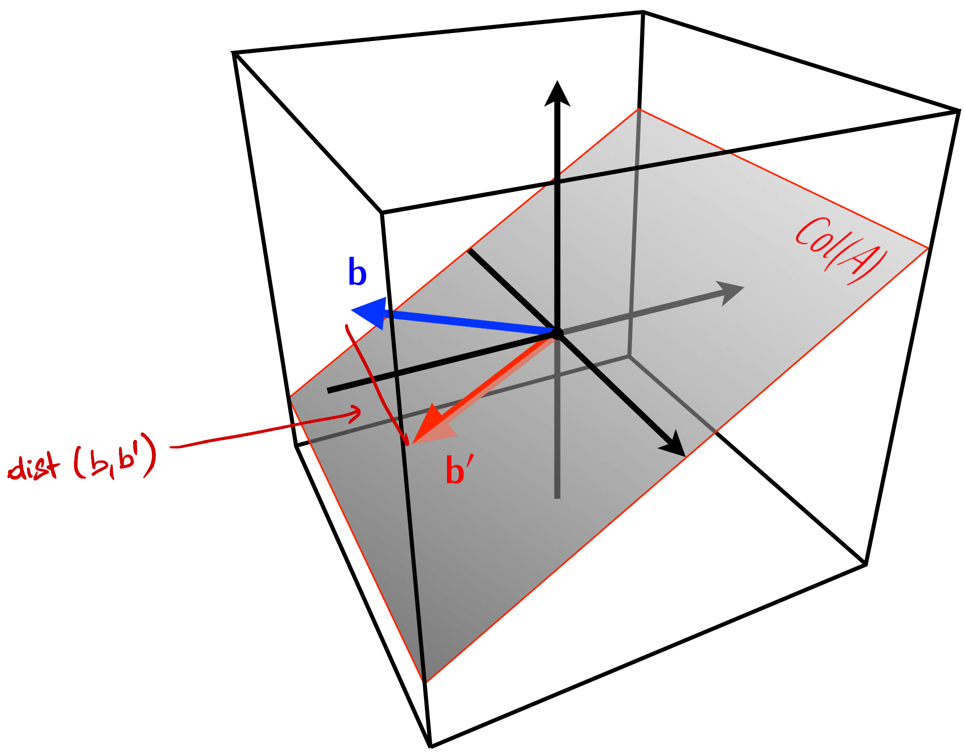
leading one in the last column, so: no solutions

Upshot.

- Recall: a matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b} \in \mathrm{Col}(A)$.

- In practical applications we may obtain a matrix equation that has no solutions, i.e. where $\mathbf{b} \notin \mathrm{Col}(A)$.

- In such cases we may look for approximate solutions as follows:
  - replace $\mathbf{b}$ by a vector $\mathbf{b}'$ such that $\mathbf{b}' \in \mathrm{Col}(A)$ and $\mathrm{dist}(\mathbf{b}, \mathbf{b}')$ is a as small as possible.
  - then solve $A\mathbf{x} = \mathbf{b}'$



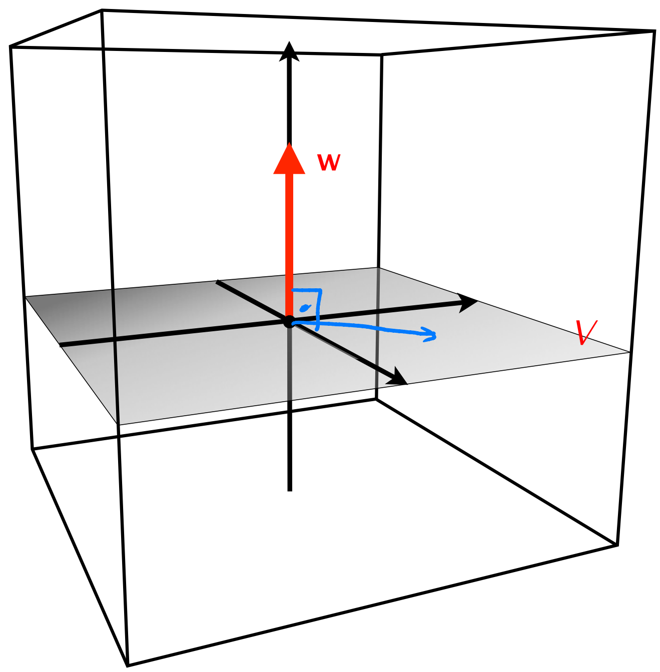
Definition

Given $\mathbf{b}' \in \mathrm{Col}(A)$ as above we will say that a vector $\mathbf{v}$ is a *least square solution* of the equation $A\mathbf{x} = \mathbf{b}$ if $\mathbf{v}$ is a solution of the equation $A\mathbf{x} = \mathbf{b}'$.

Next: How to find the vector $\mathbf{b}'$?

Definition

Let V be a subspace of $\mathbb{R}^n$. A vector $\mathbf{w} \in \mathbb{R}^n$ is *orthogonal to* V if $\mathbf{w} \cdot \mathbf{v} = 0$ for all $\mathbf{v} \in V$.



Proposition

If $V = \mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ then a vector $\mathbf{w} \in \mathbb{R}^n$ is orthogonal to V if and only if $\mathbf{w} \cdot \mathbf{v}_i = 0$ for $i = 1, \dots, k$.

Proof: Assume that w is orthogonal to $v_1, \dots, v_k$.
If $v \in \mathrm{Span}(v_1, \dots, v_k)$ then $v = c_1 v_1 + \dots + c_k v_k$.
This gives:

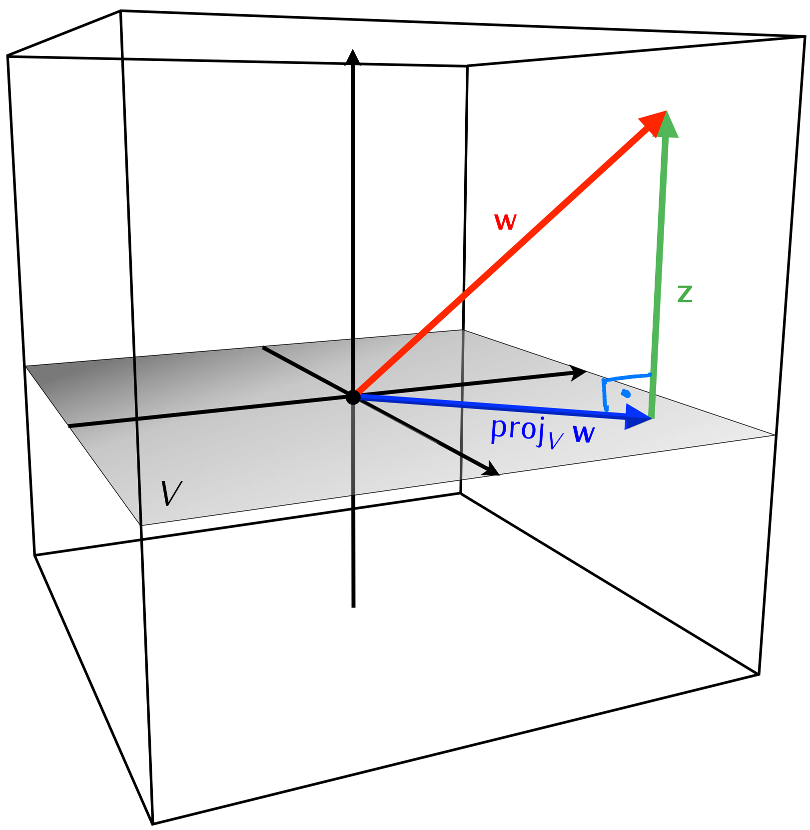
$$w \cdot v = w \cdot (c_1 v_1 + \dots + c_k v_k)$$
$$= c_1 (w \cdot v_1) + \dots + c_k (w \cdot v_k) = 0$$

So: w is orthogonal to every vector $v \in \mathrm{Span}(v_1, \dots, v_k)$.

Definition

Let V be a subspace of $\mathbb{R}^n$ and let $\mathbf{w} \in \mathbb{R}^n$ the *orthogonal projection of* $\mathbf{w}$ *onto* V is a vector $\text{proj}_V\mathbf{w}$ such that

1) $\text{proj}_V\mathbf{w} \in V$
2) the vector $\mathbf{z} = \mathbf{w} - \text{proj}_V\mathbf{w}$ is orthogonal to V.



The Best Approximation Theorem

If V is a subspace of $\mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^n$ then $\text{proj}_V \mathbf{w}$ is a vector in V which is closest to $\mathbf{w}$:

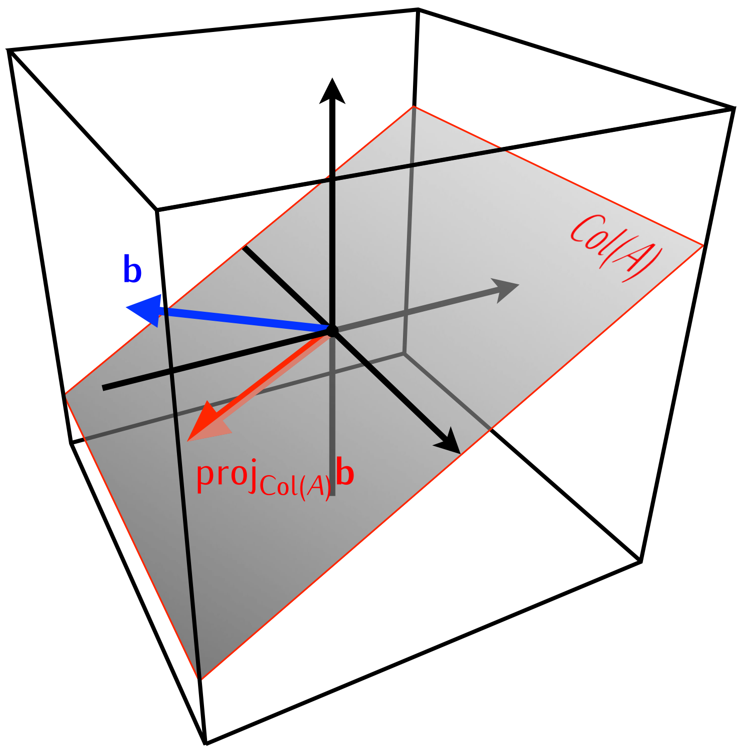
$$\text{dist}(\mathbf{w}, \text{proj}_V \mathbf{w}) \leq \text{dist}(\mathbf{w}, \mathbf{v})$$

for all $\mathbf{v} \in V$.

Corollary

The least square solutions of a matrix equation $A\mathbf{x} = \mathbf{b}$ are solutions of the equation

$$A\mathbf{x} = \mathrm{proj}_{\mathrm{Col}(A)}\mathbf{b}$$



<u>**Next:**</u> If V is a subspace of $\mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^n$ how to compute $\mathrm{proj}_V\mathbf{w}$?

Theorem

If V is a subspace of $\mathbb{R}^n$ with an orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and $\mathbf{w} \in \mathbb{R}^n$ then

$$\text{proj}_V \mathbf{w} = \left(\frac{\mathbf{w} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1 + \ldots + \left(\frac{\mathbf{w} \cdot \mathbf{v}_k}{\mathbf{v}_k \cdot \mathbf{v}_k}\right) \mathbf{v}_k$$

Corollary

If V is a subspace of $\mathbb{R}^n$ with an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and $\mathbf{w} \in \mathbb{R}^n$ then

$$\text{proj}_V \mathbf{w} = (\mathbf{w} \cdot \mathbf{v}_1)\, \mathbf{v}_1 + \ldots + (\mathbf{w} \cdot \mathbf{v}_k)\, \mathbf{v}_k$$

Example. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 0 \\ -2 \end{bmatrix} \right\}, \qquad \mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

The set $\mathcal{B}$ is an orthogonal basis of some subspace V of $\mathbb{R}^4$. Compute $\mathrm{proj}_V \mathbf{w}$.

Note. In general if V is a subspace of $\mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^n$ then in order to find $\mathrm{proj}_V \mathbf{w}$ we need to do the following:

1) find a basis of V.

2) use the Gram-Schmidt process to get an orthogonal basis of V

3) use the orthogonal basis to compute $\mathrm{proj}_V \mathbf{w}$.

Example. Consider the following matrix A and vector $\mathbf{u}$:

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 3 & 4 & 2 \\ 2 & 6 & 3 & -1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

Compute $\mathrm{proj}_{\mathrm{Col}(A)} \mathbf{u}$.

Example. Find least square solutions of the matrix equation $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 90 \\ 120 \\ 72 \\ 45 \end{bmatrix}$$