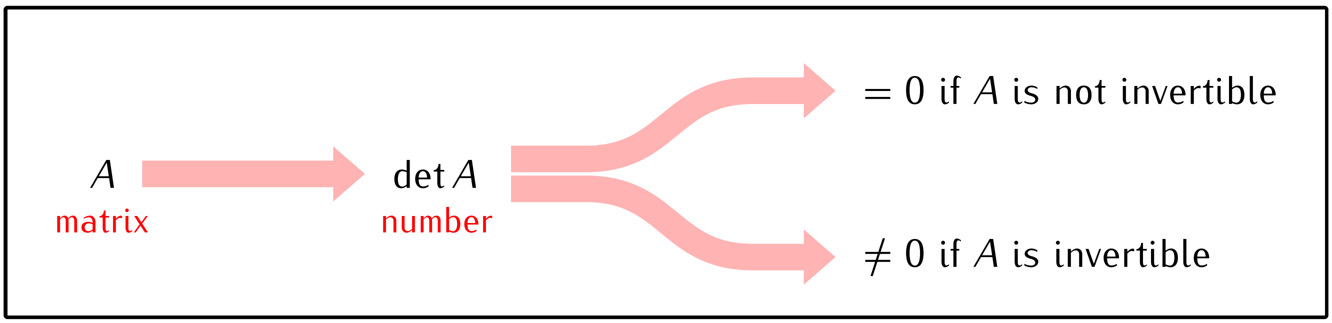
Recall: If an $n \times n$ matrix A is invertible then:

- the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b} \in \mathbb{R}^n$
- the linear transformation $T_A \colon \mathbb{R}^n \to \mathbb{R}^n$, $T_A(\mathbf{v}) = A\mathbf{v}$ has an inverse function.

Determinants recognize which matrices are invertible:



Example: Determinant for a 1×1 matrix.

$$A = \begin{bmatrix} a \end{bmatrix}$$

Example: Determinant for a 2×2 matrix.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Definition

If A is an $n \times n$ matrix then for $1 \leq i, j \leq n$ the (i, j)-*minor* of A is the matrix A_{ij} obtained by deleting the i^{th} row and j^{th} column of A.

Example.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Definition

Let A be an $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

1) If $n = 1$, i.e. $A = \begin{bmatrix} a_{11} \end{bmatrix}$, then $\det A = a_{11}$

2) If $n > 1$ then

$$\begin{aligned} \det A = & \quad (-1)^{1+1} a_{11} \cdot \det A_{11} \\ & + \ (-1)^{1+2} a_{12} \cdot \det A_{12} \\ & \dots \qquad \dots \quad \dots \quad \dots \\ & + \ (-1)^{1+n} a_{1n} \cdot \det A_{1n} \end{aligned}$$

Example. $(n = 2)$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Note

If A is a 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

then $\det A = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$

Example. (n=3)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

A direct way of computing the determinant of a 3×3 matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Example (n=4)

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 4 & 0 & 1 \\ 2 & 1 & 6 & 1 \\ 3 & 5 & 7 & 0 \end{bmatrix}$$

Note. In order to compute the determinant of an $n \times n$ matrix in this way we need to compute:

$$\begin{array}{rl} n & \text{determinants of } (n-1)\times(n-1) \text{ matrices} \\ n(n-1) & \text{determinants of } (n-2)\times(n-2) \text{ matrices} \\ n(n-1)(n-2) & \text{determinants of } (n-3)\times(n-3) \text{ matrices} \\ \dots\ \dots\ \dots\ \dots & \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots \\ n(n-1)(n-2)\cdot\ldots\cdot 3 & \text{determinants of } 2\times 2 \text{ matrices} \end{array}$$

E.g. for a 25×25 matrix we would need to compute

$$25 \cdot 24 \cdot 23 \cdot \ldots \cdot 3 = 7,755,605,021,665,492,992,000,000$$

determinants of 2×2 matrices.

Next: How to compute determinants faster.