

**Example:** Determinant for a  $2 \times 2$  matrix.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

### Definition

If  $A$  is an  $n \times n$  matrix then for  $1 \leq i, j \leq n$  the  $(i, j)$ -minor of  $A$  is the matrix  $A_{ij}$  obtained by deleting the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column of  $A$ .

Example.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

### Definition

Let  $A$  be an  $n \times n$  matrix

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

1) If  $n = 1$ , i.e.  $A = [a_{11}]$ , then  $\det A = a_{11}$

2) If  $n > 1$  then

$$\begin{aligned} \det A = & (-1)^{1+1} a_{11} \cdot \det A_{11} \\ & + (-1)^{1+2} a_{12} \cdot \det A_{12} \\ & \dots \quad \dots \quad \dots \quad \dots \\ & + (-1)^{1+n} a_{1n} \cdot \det A_{1n} \end{aligned}$$

**Example.** ( $n = 2$ )

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

### Note

If  $A$  is a  $2 \times 2$  matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

then  $\det A = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$

**Example.** (n=3)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

### A direct way of computing the determinant of a $3 \times 3$ matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

**Example** (n=4)

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 4 & 0 & 1 \\ 2 & 1 & 6 & 1 \\ 3 & 5 & 7 & 0 \end{bmatrix}$$

**Note.** In order to compute the determinant of an  $n \times n$  matrix in this way we need to compute:

$$\begin{array}{rcl}
 & n & \text{determinants of } (n-1) \times (n-1) \text{ matrices} \\
 & n(n-1) & \text{determinants of } (n-2) \times (n-2) \text{ matrices} \\
 & n(n-1)(n-2) & \text{determinants of } (n-3) \times (n-3) \text{ matrices} \\
 & \dots & \dots \\
 & n(n-1)(n-2) \cdot \dots \cdot 3 & \text{determinants of } 2 \times 2 \text{ matrices}
 \end{array}$$

E.g. for a  $25 \times 25$  matrix we would need to compute

$$25 \cdot 24 \cdot 23 \cdot \dots \cdot 3 = 7,755,605,021,665,492,992,000,000$$

determinants of  $2 \times 2$  matrices.

Next: How to compute determinants faster.

**Definition**

If  $A$  is an  $n \times n$  matrix and  $1 \leq i, j \leq n$  then the  $ij$ -cofactor of  $A$  is the number

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

**Example.**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

**Note.** By the definition of the determinant we have:

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$$