Instructions.

In this problem you will be given a few statements. For each statement you need to decide if it is true or not, and justify your answer.

How to justify your answer.

• In order to show that a statement is false, it suffices to give a counterexample. For example, consider the the statement:

The last digit of every even number is either 2, 4, or 8.

To show that this statement is false, it is enough to point out that, for example, 10 is an even number, but its last digit is 0.

• In order to show that a statement is true, you need to provide a reasoning explaining why it is true in all instances. Giving one example when it is true will not suffice, since the statement may not work in some other cases. For example, consider the the statement:

If n is an even number then $n + 2$ is also an even number.

You can justify that this is true as follows. Even numbers are integers which are multiples of 2. If n is even then $n = 2m$ for some integer m. Then $n + 2 = 2m + 2 = 2(m + 1)$, which shows that $n + 2$ is even.

Note

This problem will not be collected or graded. However, problems of this type will appear on exams in this course. Sample solutions are provided at the end.

For each of the statements given below decide if it is true or false. If you decide that it is true, justify your answer. If you think it is false give a counterexample.

a) If A, B, C are non-zero 2×2 matrices such that $AB = AC$ then $B = C$.

b) If A is 2×2 matrix such that the matrix transformation $T_A \colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $T_A(\mathbf{v}) = A\mathbf{v}$ is one-to-one, then T_A must be onto.

c) If A, B are invertible 2×2 matrices, and $B \neq -A$ then $A + B$ is also invertible.

d) If A is a 2×2 matrix then $(AA^T)^T = AA^T$ (where A^T denotes the transpose of the matrix A).

Here are solutions to the questions from the previous page. You should try to answer all questions by yourself before reading these solutions.

a) FALSE. Take for example

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$$

Then $AB = AC = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ but $B \neq C$.

b) TRUE. Recall that T_A is one-to-one if and only of the matrix A has a pivot position in every column, and that T_A is onto if and only of the matrix A has a pivot position in every row.
Since by assumption T_A is one-to-one and since A has 2 columns it follows that A has 2 pivot positions. Thus, since A has two rows, there will be a pivot position in every row of A. This means that T_A is onto.

c) FALSE. Take for example

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

Both A and B are invertible, but the matrix $A + B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not.

d) TRUE. Recall that for any matrices A, B we have $(AB)^T = B^T A^T$. Taking $B = A^T$ we obtain:

$$(AA^T)^T = (A^T)^T A^T$$

Since $(A^T)^T = A$ this gives $(AA^T)^T = AA^T$.