**Recall:** If A is an  $m \times n$  matrix then

$$A \cdot \left[ \begin{array}{c} b_1 \\ \vdots \\ b_n \end{array} \right] = \left[ \begin{array}{c} c_1 \\ \vdots \\ c_m \end{array} \right]$$

#### **Definition**

If A is an  $m \times n$  matrix then the function

$$T_A \colon \mathbb{R}^n \to \mathbb{R}^m$$

given by  $T_A(\mathbf{v}) = A\mathbf{v}$  is called the matrix transformation associated to A.

### Example.

Let  $T_A \colon \mathbb{R}^3 \to \mathbb{R}^2$  be the matrix transformation defined by the matrix

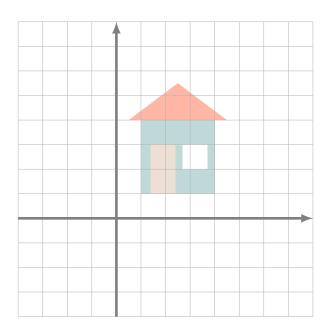
$$A = \left[ \begin{array}{ccc} 1 & 2 & 3 \\ 1 & 3 & 3 \end{array} \right]$$

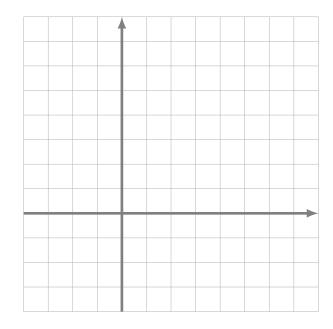
**1)** Compute 
$$T_A(\mathbf{v})$$
 where  $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ .

2) Find a vector **v** such that 
$$T_A(\mathbf{v}) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$
.

# Geometric interpretation of matrix transformations $\mathbb{R}^2 \to \mathbb{R}^2$

$$A = \left[ \begin{array}{cc} 1 & -1 \\ 1 & 0 \end{array} \right]$$

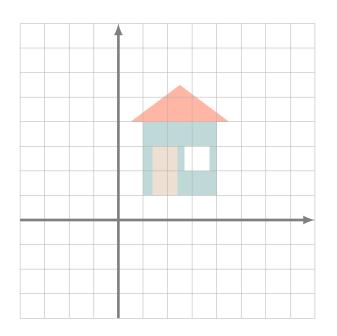


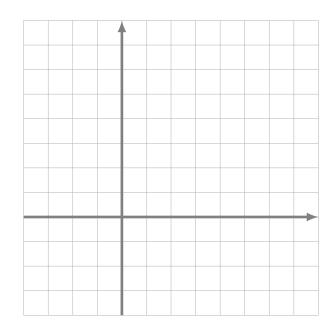


## Null spaces, column spaces and matrix transformations

## Example.

$$A = \left[ \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right]$$





#### Note

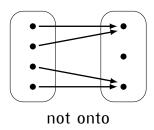
If  $T_A : \mathbb{R}^n \to \mathbb{R}^m$  is a linear transformation associated to a matrix A then:

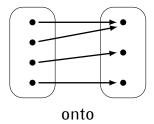
- Col(A) = the set of values of  $T_A$ .
- Nul(A) = the set of vectors v such that  $T_A(v) = 0$ .
- $T_A(v) = T_A(w)$  if and only if w = v + n for some  $n \in Nul(A)$ .

#### Recall:

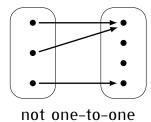
A function  $F: \mathbb{R}^n \to \mathbb{R}^m$  is:

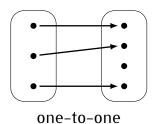
• *onto* if for each  $\mathbf{b} \in \mathbb{R}^m$  there is  $\mathbf{v} \in \mathbb{R}^n$  such that  $F(\mathbf{v}) = \mathbf{b}$ ;





• one-to-one if for any  $v_1, v_2$  such that  $v_1 \neq v_2$  we have  $F(v_2) \neq F(v_2)$ .





#### **Proposition**

Let A be an  $m \times n$  matrix. The following conditions are equivalent:

- **1)** The matrix transformation  $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$  is onto.
- 2)  $Col(A) = \mathbb{R}^m$ .
- 3) The matrix A has a pivot position in every row.

#### **Proposition**

Let A be an  $m \times n$  matrix. The following conditions are equivalent:

- 1) The matrix transformation  $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$  is one-to-one.
- 2)  $Nul(A) = \{0\}.$
- 3) The matrix A has a pivot position in every column.

**Example.** For the following  $2 \times 2$  matrix A check if the matrix transformation  $T_A \colon \mathbb{R}^2 \to \mathbb{R}^2$  is onto and if it is one-to-one.

$$A = \left[ \begin{array}{cc} 1 & -1 \\ 1 & 0 \end{array} \right]$$

**Example.** For the following  $3 \times 4$  matrix A check if the matrix transformation  $T_A \colon \mathbb{R}^4 \to \mathbb{R}^3$  is onto and if it is one-to-one.

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -2 & -2 & 1 & -5 \\ 1 & 1 & -1 & 4 \end{bmatrix}$$

### Proposition

Let A be an  $m \times n$  matrix. If the matrix transformation  $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$  is both onto and one-to-one then we must have m = n (i.e. A must be a square matrix).