Operations on matrices so far:

- addition/subtraction $A \pm B$
- scalar multiplication $c \cdot A$
- matrix multiplication $A \cdot B$
- matrix transpose A^T

Next: How to divide matrices?

Definition

A matrix A is *invertible* if there exists a matrix B such that

$$A \cdot B = B \cdot A = I$$

(where I = the identity matrix). In such case we say that B is the *inverse* of A and we write $B = A^{-1}$.

Example.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ is invertible,} \quad A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Matrix inverses and matrix equations

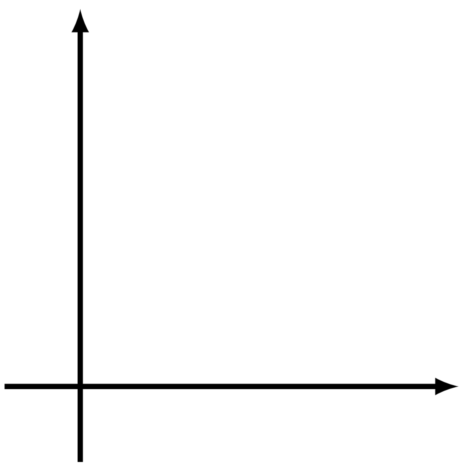
Proposition

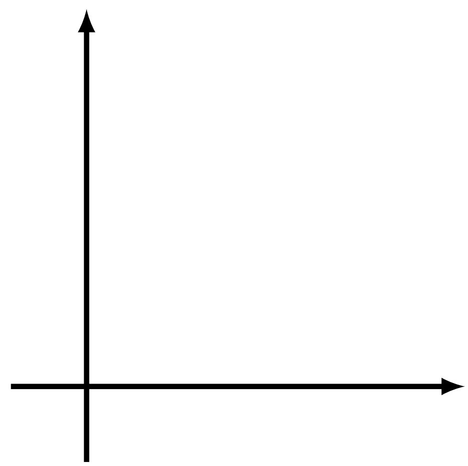
If A is an invertible matrix then for any vector $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution.

Example. Solve the following matrix equation:

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

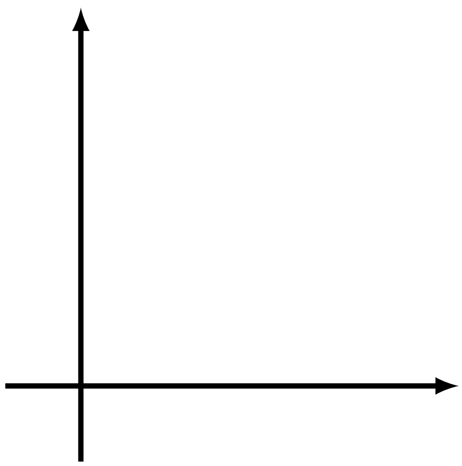
Matrix inverses and matrix transformations

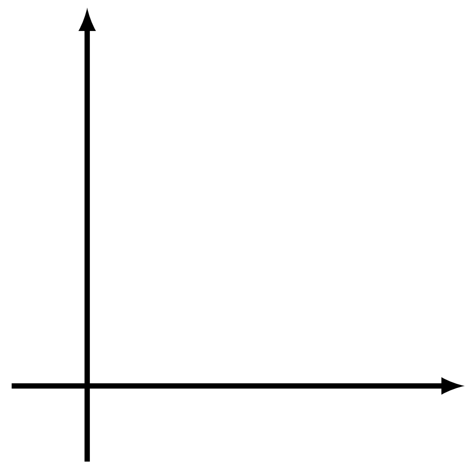




Example.

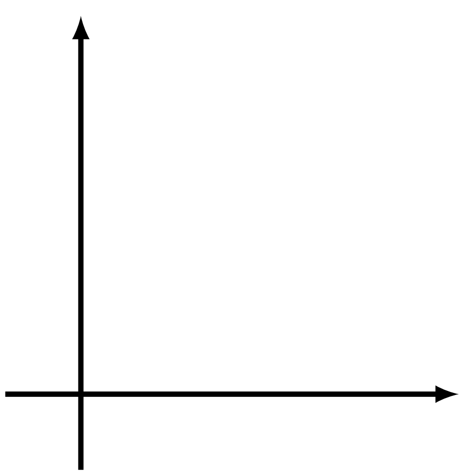
$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

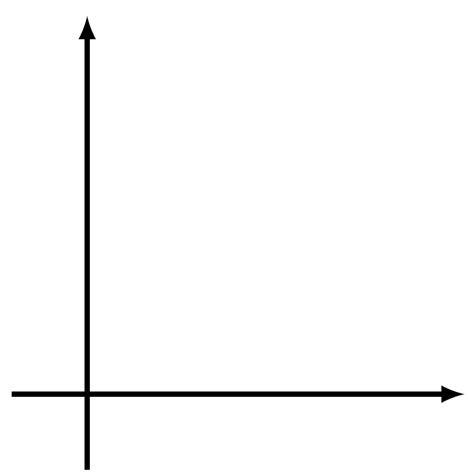




Example.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$





Upshot. If an $m \times n$ matrix A is invertible then the matrix transformation $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$ must be one-to-one and onto.

Recall: If A be is $m \times n$ matrix then the matrix transformation $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$ is:

- onto if and only if A has a pivot position in every row
- one-to-one if and only if A has a pivot position in every column.

Theorem

If A is not a square matrix then it is not invertible.

If A is a square matrix then the following conditions are equivalent:

1) A is an invertible matrix.

2) The matrix A has a pivot position in every row and column.

3) The reduced row echelon form of A is the identity matrix I_n.

Proposition

If A is an $n \times n$ invertible matrix then

$$A^{-1} = \left[\begin{array}{cccc} \mathbf{w}_1 & \mathbf{w}_2 & \dots & \mathbf{w}_n \end{array}\right]$$

where $\mathbf{w}_i$ is the solution of $A\mathbf{x} = \mathbf{e}_i$.

Example.

$A = \left[\begin{array}{rr} 1 & -1 \\ -1 & 1 \end{array}\right]$

Simplification:
How to solve several matrix equations with the same coefficient matrix at the same time

$$A\mathbf{x} = \mathbf{b}_1, \ A\mathbf{x} = \mathbf{b}_2, \ \ldots, \ A\mathbf{x} = \mathbf{b}_n$$

matrix of equations

$$\left[\begin{array}{c|cccc} A & \mathbf{b}_1 & \mathbf{b}_2 & \ldots & \mathbf{b}_n \end{array}\right]$$

augmented matrix

row reduction

$$\left[\begin{array}{c|c} & \phantom{\mathbf{b}_1 \ \mathbf{b}_2 \ \ldots \ \mathbf{b}_n} \end{array}\right]$$

reduced matrix

read off solutions

solutions

Example. Solve the vector equations $A\mathbf{x} = \mathbf{e}_1$ and $A\mathbf{x} = \mathbf{e}_2$ where

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Summary:
How to invert a matrix

Example: $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

1) Augment A by the identity matrix.

2) Reduce the augmented matrix.

2) If after the row reduction the matrix on the left is the identity matrix, then A is invertible and

$$A^{-1} = \text{the matrix on the right}$$

Otherwise A is not invertible.

Properties of matrix inverses

1) If A is invertible then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

2) If A, B are invertible then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

3) If A is invertible then A^{T} is invertible and

$$(A^{T})^{-1} = (A^{-1})^{T}$$