

- Linear equations

1) Three forms of equations:

– system of linear equations

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{cases}$$

– vector equation

$$x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

– matrix equation

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

2) How to solve matrix equations:

– row reduction (= Gauss-Jordan elimination)

3) Related notions:

- elementary row operations
- reduced echelon form of a matrix
- leading ones
- pivot positions and pivot columns

• Vectors

- 1) \mathbb{R}^n = the set of all vectors with n entries
- 2) Operations on vectors in \mathbb{R}^n :
 - addition
 - multiplication by scalars
- 3) Geometric interpretation of vectors and vector operations.
- 4) Linear combinations of vectors.
- 5) Span of a set of vectors.
- 6) Linear independence of vectors.

• Matrices

- 1) Operations on Matrices:
 - addition $A + B$
 - multiplication by scalars cA
 - matrix multiplication AB
 - matrix Transpose A^T
 - matrix inverse A^{-1}
- 2) Properties of the matrix algebra:

❶ $AB \neq BA$

❷ $(AB)^T = B^T A^T$

❸ $(A^T)^T = A$

❹ $(A + B)^T = A^T + B^T$

❺ $(A^T)^{-1} = (A^{-1})^T$

❻ $(AB)^{-1} = B^{-1}A^{-1}$

❼ $(A^{-1})^{-1} = A$

❽ $(A + B)^{-1} \neq A^{-1} + B^{-1}$

- 3) Column space of a matrix $\text{Col}(A)$.
- 4) Null space of a matrix $\text{Nul}(A)$, and its representation as a span of vectors.

• **Matrix transformations and linear transformations**

1) An $m \times n$ matrix defines a matrix transformation

$$T_A: \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad T_A(\mathbf{v}) = A\mathbf{v}$$

2) Composition of matrix transformations = matrix multiplication:

$$T_A \circ T_B = T_{AB}$$

3) Linear transformation is a function $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

(ii) $T(c\mathbf{v}) = cT(\mathbf{v})$

4) Every matrix transformation is a linear transformation.

5) Every linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a matrix transformation:

$$T = T_A$$

where $A = [T(\mathbf{e}_1) \ T(\mathbf{e}_2) \ \dots \ T(\mathbf{e}_n)]$ is the standard matrix of A .

6) If $T_A: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a matrix transformation then

– $\text{Col}(A)$ = the set of all values of T_A

– $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$ if and only if $\mathbf{v}_1 - \mathbf{v}_2 \in \text{Nul}(A)$

7) A matrix transformation $T_A: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is onto if and only if:

– $\text{Col}(A) = \mathbb{R}^m$

– A has a pivot position in every row

8) A matrix transformation $T_A: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is one-to-one if and only if:

– $\text{Nul}(A) = \{\mathbf{0}\}$

– A has a pivot position in every column

9) If A is an invertible matrix then $T_{A^{-1}}$ is the inverse function of T_A .