Linear Algebra	Calculus
$\mathbb{R}^n = \begin{pmatrix}\text{set of all column vectors} \\ \text{with } n \text{ entries}\end{pmatrix}$	$C^\infty(\mathbb{R}) = \begin{pmatrix}\text{set of all smooth} \\ \text{functions } f\colon \mathbb{R} \to \mathbb{R}\end{pmatrix}$
Column vectors can be added and multiplied by real numbers.	**Functions** can be added and multiplied by real numbers.
Linear transformation is a function $$T\colon \mathbb{R}^n \to \mathbb{R}^m, \quad T(\mathbf{v}) = A\mathbf{v}$$ It satisfies: • $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ • $T(c\mathbf{v}) = cT(\mathbf{v})$	**Differentiation** is a function $$D\colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \quad D(f) = f'$$ It satisfies: • $D(f + g) = D(f) + D(g)$ • $D(cf) = cD(f)$
Typical problem: given a vector $\mathbf{b}$ find all vectors $\mathbf{x}$ such that $$T(\mathbf{x}) = \mathbf{b}$$ (i.e solve the equation $A\mathbf{x} = \mathbf{b}$).	**Typical problem:** given a function g find all functions f such that $$D(f) = g$$ (i.e find antiderivatives of g).
Fact: Such vectors x are of the form $$\mathbf{x} = \mathbf{v}_0 + \mathbf{n}$$ where: • $\mathbf{v}_0$ is some distinguished solution of $A\mathbf{x} = \mathbf{b}$; • $\mathbf{n} \in \mathrm{Nul}(A)$ (i.e. $\mathbf{n}$ is a solution of $A\mathbf{x} = \mathbf{0}$).	**Fact:** Such functions f are of the form $$f = F + C$$ where: • F is some distinguished antiderivative of g; • C is a constant function (i.e. C is a solution of $D(f) = 0$).

Definition

A *(real) vector space* is a set V together with two operations:

- addition

$$\begin{aligned} V \times V &\longrightarrow V \\ (\mathbf{u}, \quad \mathbf{v}) &\longmapsto \quad \mathbf{u} + \mathbf{v} \end{aligned}$$

- multiplication by scalars

$$\begin{aligned} \mathbb{R} \times V &\longrightarrow V \\ (c, \quad \mathbf{v}) &\longmapsto \quad c \cdot \mathbf{v} \end{aligned}$$

Moreover the following conditions must be satisfied:

1) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3) there is an element $\mathbf{0} \in V$ such that $\mathbf{0} + \mathbf{u} = \mathbf{u}$ for any $\mathbf{u} \in V$
4) for any $\mathbf{u} \in V$ there is an element $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7) $(cd)\mathbf{u} = c(d\mathbf{u})$
8) $1\mathbf{u} = \mathbf{u}$

Elements of V are called *vectors*.

Theorem

If V is a vectors space then:

1) $c \cdot \mathbf{0} = \mathbf{0}$ where $c \in \mathbb{R}$ and $\mathbf{0} \in V$ is the zero vector;

2) $0 \cdot \mathbf{u} = \mathbf{0}$ where $0 \in \mathbb{R}$, $\mathbf{u} \in V$ and $\mathbf{0}$ is the zero vector;

3) $(-1) \cdot \mathbf{u} = -\mathbf{u}$

Proof of 2):

We have:

$$0 \cdot u = (0+0) \cdot u \overset{\text{by (6)}}{=} 0 \cdot u + 0 \cdot u$$

This gives:

$$\underset{\substack{\| \leftarrow \text{by (4)} \\ \mathbf{0}}}{(0 \cdot u) + (-(0 \cdot u))} = \underbrace{(0 \cdot u + 0 \cdot u) + (-(0 \cdot u))}$$

$$\| \leftarrow \text{by (2)}$$

$$(0 \cdot u) + (0 \cdot u + (-(0 \cdot u)))$$

$$\| \leftarrow \text{by (4)}$$

$$0 \cdot u + \mathbf{0}$$

$$\| \leftarrow \text{by (3) and (1)}$$

$$0 \cdot u$$

So: $\mathbf{0} = 0 \cdot u$.

Examples of vector spaces.

1) $\mathbb{R}^n = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \middle| \; a_1, \dots, a_n \in \mathbb{R} \right\}$

2) $\mathcal{F}(\mathbb{R}) = \{$the set of all functions $f: \mathbb{R} \to \mathbb{R}\}$

3) $\mathbb{P} = \{$the set of all polynomials of variable $t\}$

$= \{ a_0 + a_1 t + \dots + a_m t^m \mid a_i \in \mathbb{R}, \; m \geqslant 0 \}$

4) $M_{m,n}(\mathbb{R}) = \{$ the set of all $m \times n$ matrices $\}$

$= \left\{ \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \middle| \; a_{ij} \in \mathbb{R} \right\}$