

Definition

A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ in \mathbb{R}^n is an *orthogonal set* if each pair each pair of vectors in this set is orthogonal, i.e.

$$\mathbf{v}_i \cdot \mathbf{v}_j = 0$$

for all $i \neq j$.

Example.

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is an orthogonal set in \mathbb{R}^3 .

Example.

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix} \right\}$ is another orthogonal set in \mathbb{R}^3 .

Proposition

If $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an orthogonal set of non-zero vectors in \mathbb{R}^n then this set is linearly independent.

Recall: Any linearly independent set of n vectors in \mathbb{R}^n is a basis of \mathbb{R}^n .

Corollary

If $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is an orthogonal set of n non-zero vectors in \mathbb{R}^n then this set is a basis of \mathbb{R}^n .

Definition

If V is a subspace of \mathbb{R}^n then we say that a set $\{v_1, \dots, v_k\}$ is an *orthogonal basis* of V if

- 1) $\{v_1, \dots, v_k\}$ is a basis of V and
- 2) $\{v_1, \dots, v_k\}$ is an orthogonal set.

Recall. If $\mathcal{B} = \{v_1, \dots, v_k\}$ is a basis of a vector space V and $w \in V$ then the coordinate vector of w relative to \mathcal{B} is the vector

$$[w]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

where c_1, \dots, c_k are scalars such that $c_1 v_1 + \dots + c_k v_k = w$.

Proposition

If $\mathcal{B} = \{v_1, \dots, v_k\}$ is an orthogonal basis of V and $w \in V$ then

$$[w]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

$$\text{where } c_i = \frac{w \cdot v_i}{v_i \cdot v_i} = \frac{w \cdot v_i}{\|v_i\|^2}$$

Example. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix} \right\}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

The set \mathcal{B} is an orthogonal basis of \mathbb{R}^3 . Compute $[\mathbf{w}]_{\mathcal{B}}$.

Theorem (Gram-Schmidt Process)

Let $\{v_1, \dots, v_k\}$ be a basis of V . Define vectors $\{w_1, \dots, w_k\}$ as follows:

$$w_1 = v_1$$

$$w_2 = v_2 - \left(\frac{w_1 \cdot v_2}{w_1 \cdot w_1} \right) w_1$$

$$w_3 = v_3 - \left(\frac{w_1 \cdot v_3}{w_1 \cdot w_1} \right) w_1 - \left(\frac{w_2 \cdot v_3}{w_2 \cdot w_2} \right) w_2$$

... ..

$$w_k = v_k - \left(\frac{w_1 \cdot v_k}{w_1 \cdot w_1} \right) w_1 - \left(\frac{w_2 \cdot v_k}{w_2 \cdot w_2} \right) w_2 - \dots - \left(\frac{w_{k-1} \cdot v_k}{w_{k-1} \cdot w_{k-1}} \right) w_{k-1}$$

Then the set $\{w_1, \dots, w_k\}$ is an orthogonal basis of V .

Example. In \mathbb{R}^4 take

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 7 \\ 4 \\ 3 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 7 \\ 7 \\ 8 \end{bmatrix}$$

The set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of some subspace $V \subseteq \mathbb{R}^4$. Find an orthogonal basis of V .

Definition

An orthogonal basis $\mathcal{B} = \{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ of V is called an *orthonormal basis* if $\|\mathbf{w}_i\| = 1$ for $i = 1, \dots, k$.

Proposition

If $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an orthonormal basis of V and $\mathbf{w} \in V$ then

$$[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

where $c_i = \mathbf{w} \cdot \mathbf{v}_i$.

Note. If $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an orthogonal basis of V then

$$\mathcal{C} = \left\{ \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \dots, \frac{\mathbf{v}_k}{\|\mathbf{v}_k\|} \right\}$$

is an orthonormal basis of V .