#### **Definition**

A matrix is in the row echelon form if:

- 1) the first non-zero entry of each row is a 1 ("a leading one");
- 2) the leading one in each row is to the right of the leading one in the row above it.

A matrix is in the reduced row echelon form if in addition it satisfies:

3) all entries above each leading one are 0.

$$\begin{bmatrix} 1 & * & * & * & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(\* = any number)

## Example

$$\begin{bmatrix} 1 & 0 & 4 & 0 & 7 & 0 \\ 0 & 1 & 5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 4 & 6 & 7 & 0 \\ 0 & 1 & 5 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 4 & 0 & 7 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 6 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

#### **Fact**

If a system of linear equations is represented by a matrix in the reduced row echelon form then it is easy to solve the system.

## Example

$$\left[\begin{array}{ccc|cccc}
1 & 0 & 3 & 0 & 0 \\
0 & 1 & 7 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{array}\right]$$

# Proposition

A matrix in the reduced row echelon form represents an inconsistent system if and only if it contains a row of the form

i.e. with the leading one in the last column.

# Example

$$\left[\begin{array}{ccc|ccc|c}
1 & 0 & 3 & 0 & 0 \\
0 & 1 & 7 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]$$

#### Note

In an augmented matrix in the reduced row echelon form free variables correspond to columns of the coefficient matrix that do not contain leading ones.

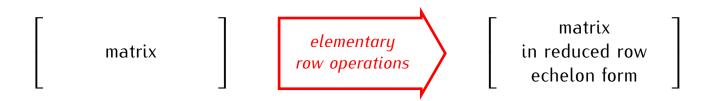
# Example

$$\left[\begin{array}{ccc|ccc|c}
1 & 0 & 0 & 0 & 5 \\
0 & 1 & 0 & 0 & 6 \\
0 & 0 & 1 & 0 & 7 \\
0 & 0 & 0 & 1 & 8
\end{array}\right]$$

#### Note

A matrix in the reduced row echelon form represents a system of equations with exactly one solution if and only if it has a leading one in every column except for the last one.

## Gauss-Jordan elimination process (= row reduction)



- 1 Interchange rows, if necessary, to bring a non-zero element to the top of the first non-zero column of the matrix.
- $\bigcirc$  Multiply the first row so that its first non-zero entry becomes 1.
- Add multiples of the first row to eliminate non-zero entries below the leading one.
- (4) Ignore the first row; apply steps 1-3 to the rest of the matrix.
- (5) Eliminate non-zero entries above all leading ones.

# Example.

$$\begin{bmatrix}
0 & 4 & -8 & 0 & 4 \\
2 & 6 & -6 & -2 & -4 \\
2 & 7 & -8 & 0 & -1
\end{bmatrix}$$

# How to solve systems of linear equations: example

$$\begin{cases} 4x_2 - 8x_3 = 4 \\ 2x_1 + 6x_2 - 6x_3 - 2x_4 = -4 \\ 2x_1 + 7x_2 - 8x_3 = -1 \end{cases}$$

## 4. Pivot positions and pivot columns

$$\begin{bmatrix} 0 & 4 & -8 & 0 & | & 4 \\ 2 & 6 & -6 & -2 & | & -4 \\ 2 & 7 & -8 & 0 & | & -1 \end{bmatrix} \xrightarrow{row} \begin{bmatrix} 1 & 0 & 3 & 0 & | & -4 \\ 0 & 1 & -2 & 0 & | & 1 \\ 0 & 0 & 0 & 1 & | & 1 \end{bmatrix}$$

#### **Definition**

A *pivot position* in a matrix is a position that after the row reduction contains a leading one.

A pivot column of a matrix is a column that contains a pivot position.

#### Theorem

- 1) A system of linear equations is inconsistent if and only if the last column of its augmented matrix is a pivot column.
- 2) Free variables of the system correspond to non-pivot columns of the coefficient matrix.
- 3) The system has only one solution if and only if every column of its augmented matrix is a pivot column, except for the last column.