Recall: An $n \times n$ matrix A defines a linear transformation

$$T_A \colon \mathbb{R}^n \to \mathbb{R}^n$$

given by $T_A(\mathbf{v}) = A\mathbf{v}$.

Next goal: Understand this linear transformation better.

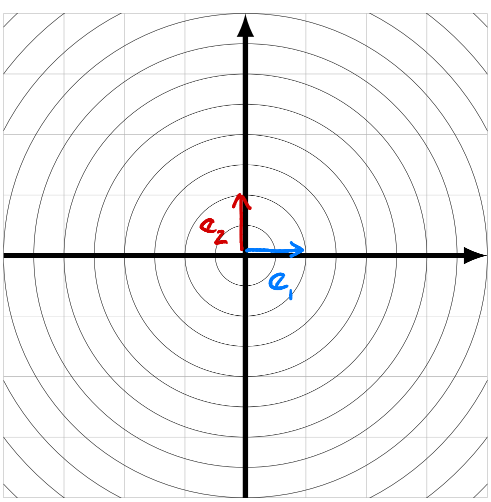
Example.

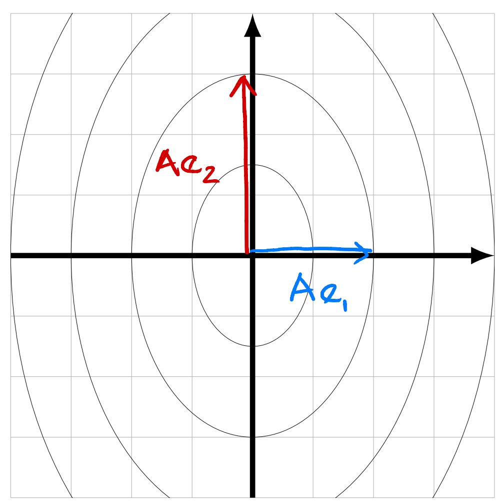
$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$v \longmapsto Av$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad Ae_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \cdot e_1$$

$$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad Ae_2 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} = 3 \cdot e_2$$





Example.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

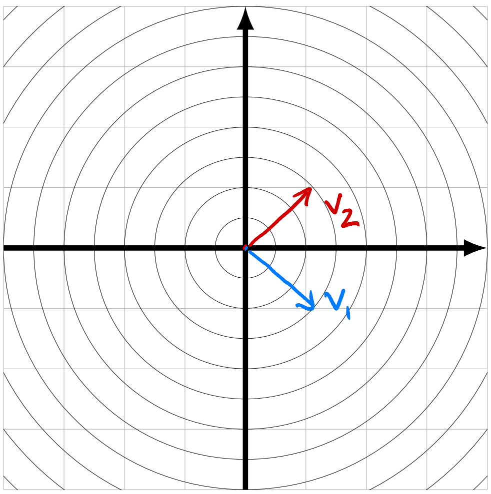
$$T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$v \longmapsto Av$$

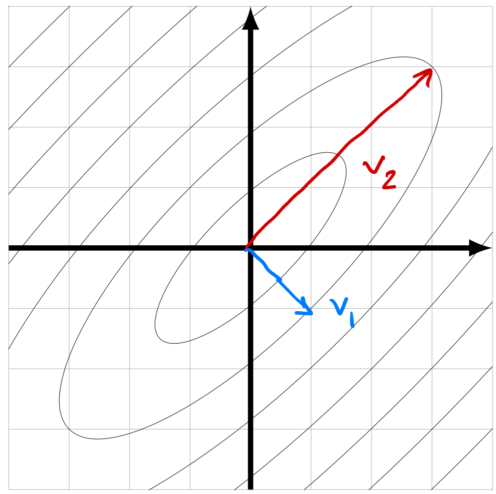
$$Ae_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad Ae_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Take $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$Av_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \cdot v_1$$

$$Av_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \cdot v_2$$





Definition

Let A be an $n \times n$ matrix. If $\mathbf{v} \in \mathbb{R}^n$ is a non-zero vector and λ is a scalar such that

$$A\mathbf{v} = \lambda\mathbf{v}$$

then we say that

- λ is an *eigenvalue* of A
- $\mathbf{v}$ is an *eigenvector* of A corresponding to λ.

Example.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

We had $A \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

So: $\lambda_1 = 2$, $\lambda_2 = 3$ are eigenvalues of A.

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 2$.

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 3$.

Example.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

We had $A \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $A \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So: $\lambda_1 = 1$, $\lambda_2 = 3$ are eigenvalues of A.

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 1$.

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 3$.

Computation of eigenvalues

Recall: $I_n = n \times n$ identity matrix:

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Propostiton

If A be an $n \times n$ matrix then $\lambda \in \mathbb{R}$ is an eigenvalue of A if and only if the matrix equation

$$(A - \lambda I_n)\mathbf{x} = \mathbf{0}$$

has a non-trivial solution.

Propostiton

If B is an $n \times n$ matrix then equation

$$B\mathbf{x} = \mathbf{0}$$

has a non-trivial solution if and only of the matrix B is not invertible.

Propostiton

If A be an $n \times n$ matrix then $\lambda \in \mathbb{R}$ is an eigenvalue of A if and only if

$$\det(A - \lambda I_n) = 0$$

Example. Find all eigenvalues of the following matrix:

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Definition

If A is an $n \times n$ matrix then

$$P(\lambda) = \det(A - \lambda I_n)$$

is a polynomial of degree n. $P(\lambda)$ is called the *characteristic polynomial* of the matrix A.

Upshot

If A is a square matrix then

$$\text{eigenvalues of } A = \text{roots of } P(\lambda)$$

Example.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Corollary

An $n \times n$ matrix can have at most n distinct eigenvalues.

Computation of eigenvectors

Proposition

If λ is an eigenvalue of an $n \times n$ matrix A then

$$\left\{ \begin{matrix} \text{eigenvectors of } A \\ \text{corresponding to } \lambda \end{matrix} \right\} = \left\{ \begin{matrix} \text{vectors in} \\ \mathrm{Nul}(A - \lambda I_n) \end{matrix} \right\}$$

Corollary/Definition

If A is an $n \times n$ matrix and λ is an eigenvalue of A then the set of all eigenvectors corresponding to λ is a subspace of $\mathbb{R}^n$.

This subspace is called the *eigenspace* of A corresponding to λ.

Proposition

If λ is an eigenvalue of an $n \times n$ matrix A then

$$\left\{ \begin{array}{c} \text{eigenspace of } A \\ \text{corresponding to } \lambda \end{array} \right\} = \mathrm{Nul}(A - \lambda I_n)$$

Example. Consider the following matrix:

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Recall that eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 5$. Compute bases of eigenspaces of A corresponding to these eigenvalues.

Solution.

$\underline{\lambda_1 = 1}$