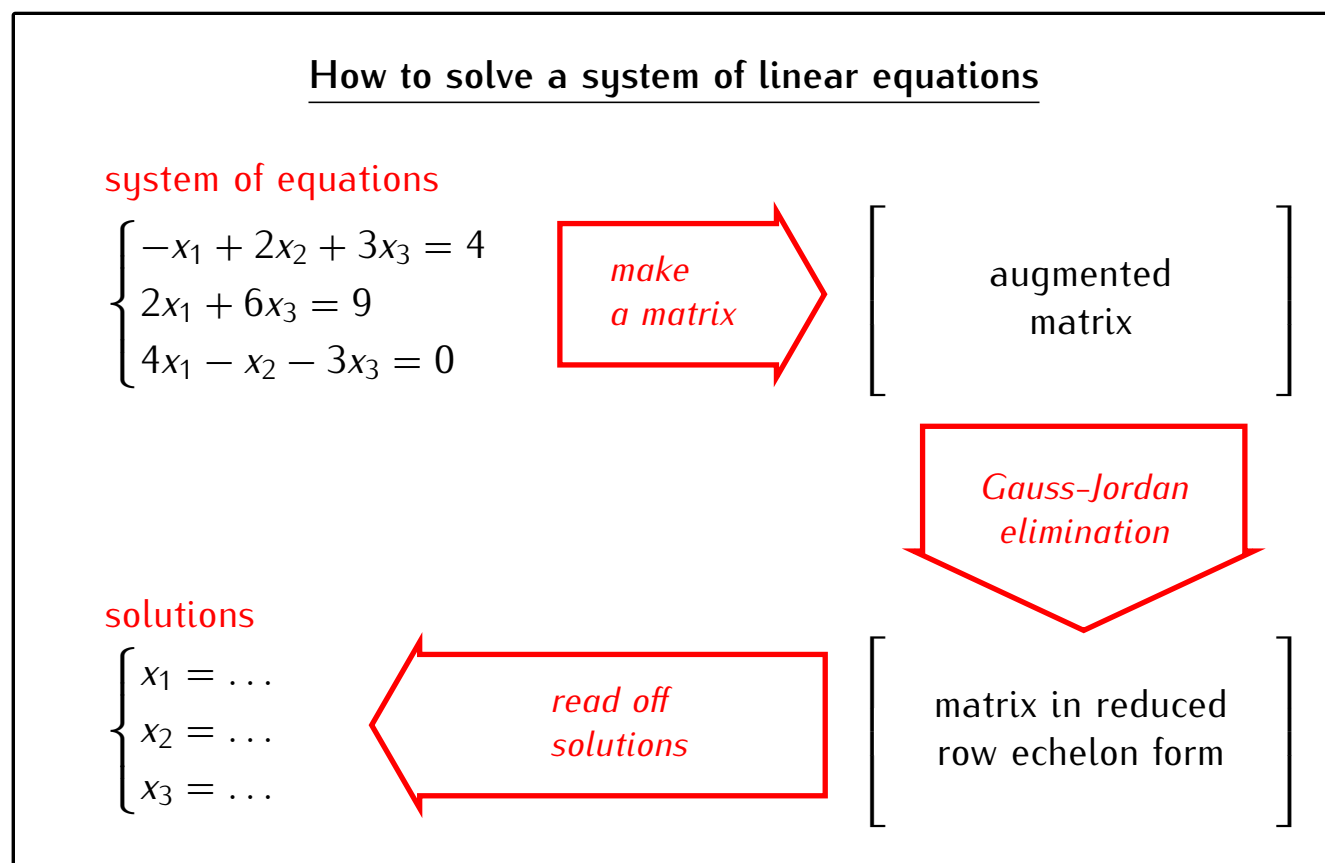


Recall:

- Every system of linear equations can be represented by a matrix
- Elementary row operations:
  - interchange of two rows
  - multiplication of a row by a non-zero number
  - addition of a multiple of one row to another row.
- Elementary row operations do not change solutions of systems of linear equations.

## Definition

A matrix is in the *row echelon form* if:

- 1) the first non-zero entry of each row is a 1 (“a leading one”);
- 2) the leading one in each row is to the right of the leading one in the row above it.

A matrix is in the *reduced row echelon form* if in addition it satisfies:

- 3) all entries above each leading one are 0.

$$\begin{bmatrix} 1 & * & * & * & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(\* = any number)

## Example

$$\begin{bmatrix} 1 & 0 & 4 & 0 & 7 & 0 \\ 0 & 1 & 5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 & 6 & 7 & 0 \\ 0 & 1 & 5 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 4 & 0 & 7 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 6 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

### Fact

If a system of linear equations is represented by a matrix in the reduced row echelon form then it is easy to solve the system.

### Example

$$\left[ \begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

### Proposition

A matrix in the reduced row echelon form represents an inconsistent system if and only if it contains a row of the form

$$[0 \quad 0 \quad 0 \quad \dots \quad 0 \quad 1]$$

i.e. with the leading one in the last column.

### Example

$$\left[ \begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

#### Note

In an augmented matrix in the reduced row echelon form free variables correspond to columns of the coefficient matrix that do not contain leading ones.

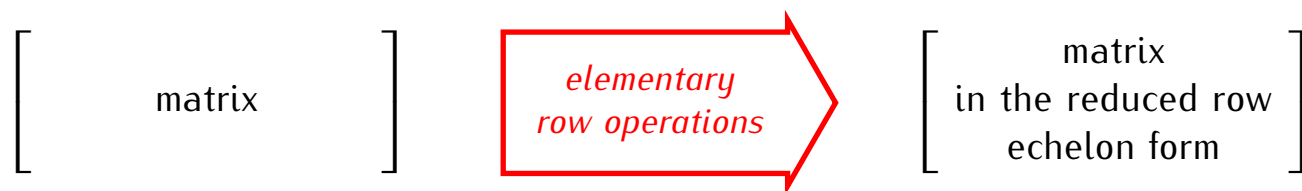
### Example

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 8 \end{array} \right]$$

### Note

A matrix in the reduced row echelon form represents a system of equations with exactly one solution if and only if it has a leading one in every column except for the last one.

## Gauss-Jordan elimination process (= row reduction)



- ① Interchange rows, if necessary, to bring a non-zero element to the top of the first non-zero column of the matrix.
- ② Multiply the first row so that its first non-zero entry becomes 1.
- ③ Add multiples of the first row to eliminate non-zero entries below the leading one.
- ④ Ignore the first row; apply steps 1–3 to the rest of the matrix.
- ⑤ Eliminate non-zero entries above all leading ones.

**Example.**

$$\begin{bmatrix} 0 & 4 & -8 & 0 & 4 \\ 2 & 6 & -6 & -2 & -4 \\ 2 & 7 & -8 & 0 & -1 \end{bmatrix}$$

### How to solve systems of linear equations: example

$$\begin{cases} 4x_2 - 8x_3 = 4 \\ 2x_1 + 6x_2 - 6x_3 - 2x_4 = -4 \\ 2x_1 + 7x_2 - 8x_3 = -1 \end{cases}$$