Instructions.

In this problem you will be given a few statements. For each statement you need to decide if it is true or not, and justify your answer.

How to justify your answer.

• In order to show that a statement is false, it suffices to give a counterexample. For example, consider the the statement:

The last digit of every even number is either 2, 4, or 8.

To show that this statement is false, it is enough to point out that, for example, 10 is an even number, but its last digit is 0.

• In order to show that a statement is true, you need to provide a reasoning explaining why it is true in all instances. Giving one example when it is true will not suffice, since the statement may not work in some other cases. For example, consider the the statement:

If n is an even number then $n + 2$ is also an even number.

You can justify that this is true as follows. Even numbers are integers which are multiples of 2. If n is even then $n = 2m$ for some integer m. Then $n + 2 = 2m + 2 = 2(m + 1)$, which shows that $n + 2$ is even.

Note

This problem will not be collected or graded. However, problems of this type will appear on exams in this course. Sample solutions are provided at the end.

For each of the statements given below decide if it is true or false. If you decide that it is true, justify your answer. If you think it is false give a counterexample.

a) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 matrix which is invertible, then the matrix $B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$ is also invertible.

b) If V is a vector space and $T\colon V \to V$ is a linear transformation then $T(\mathbf{0}) = \mathbf{0}$ (where $\mathbf{0}$ denotes the zero vector in V).

c) If W be a vector space and V_1, V_2 are subspaces of W, then $V_1 \cup V_2$ is also a subspace of W. Here $V_1 \cup V_2$ denotes the union of V_1 and V_2, i.e. the set of vectors which belong to either V_1 or V_2 (or to both).

d) If W be a vector space and V_1, V_2 are subspaces of W, then $V_1 \cap V_2$ is also a subspace of W. Here $V_1 \cap V_2$ denotes the intersection of V_1 and V_2, i.e. the set of vectors which belong both V_1 and V_2.

Here are solutions to the questions from the previous page. You should try to answer all questions by yourself before reading these solutions.

a) TRUE. The matrix B is obtained from A by interchanging rows, so $\det B = -\det A$. Since A is invertible we have $\det A \neq 0$. Therefore $\det B \neq 0$, which shows that B is invertible.

b) TRUE. We have:

$$T(\mathbf{0}) = T(\mathbf{0} + \mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0})$$

Subtracting $T(\mathbf{0})$ from both sides we obtain $\mathbf{0} = T(\mathbf{0})$.

c) FALSE. Take for example $W = \mathbb{R}^2$. Let V_1 be the subspace consisting of all vectors of the form $\begin{bmatrix} a \\ 0 \end{bmatrix}$, where a is an arbitrary number. Let V_2 be the subspace consisting of all vectors of the form $\begin{bmatrix} 0 \\ b \end{bmatrix}$, where again b is an arbitrary number. We have $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in V_1$ and $\mathbf{e}_2 \in \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in V_2$. This gives $\mathbf{e}_1, \mathbf{e}_2 \in V_1 \cup V_2$. However $\mathbf{e}_1 + \mathbf{e}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in $V_1 \cup V_2$, since this vector is neither in V_1 nor in V_2. This shows that $V_1 \cup V_2$ is not a subspace of $\mathbb{R}^2$.

d) TRUE. To show this we need to check that

(1) $\mathbf{0} \in V_1 \cap V_2$
(2) if $\mathbf{u}, \mathbf{v} \in V_1 \cap V_2$ then $\mathbf{u} + \mathbf{v} \in V_1 \cap V_2$
(3) if $\mathbf{u} \in V_1 \cap V_2$ and $c \in \mathbb{R}$ then $c\mathbf{u} \in V_1 \cap V_2$.

Checking (1): Since V_1, V_2 are subspaces, we have $\mathbf{0} \in V_1$ and $\mathbf{0} \in V_2$, so $\mathbf{0} \in V_1 \cap V_2$.

Checking (2): Since $\mathbf{u}, \mathbf{v} \in V_1 \cap V_2$, it means that $\mathbf{u}, \mathbf{v} \in V_1$ and $\mathbf{u}, \mathbf{v} \in V_2$. Since V_1 and V_2 are subspaces this gives that $\mathbf{u} + \mathbf{v} \in V_1$ and $\mathbf{u} + \mathbf{v} \in V_2$. This means that $\mathbf{u} + \mathbf{v} \in V_1 \cap V_2$.

Checking (3): Since $\mathbf{u} \in V_1 \cap V_2$, it means that $\mathbf{u} \in V_1$ and $\mathbf{u} \in V_2$. Since V_1 and V_2 are subspaces this gives that $c\mathbf{u} \in V_1$ and $c\mathbf{u} \in V_2$ for any real number $\mathbb{R}$. This means that $c\mathbf{u} \in V_1 \cap V_2$.