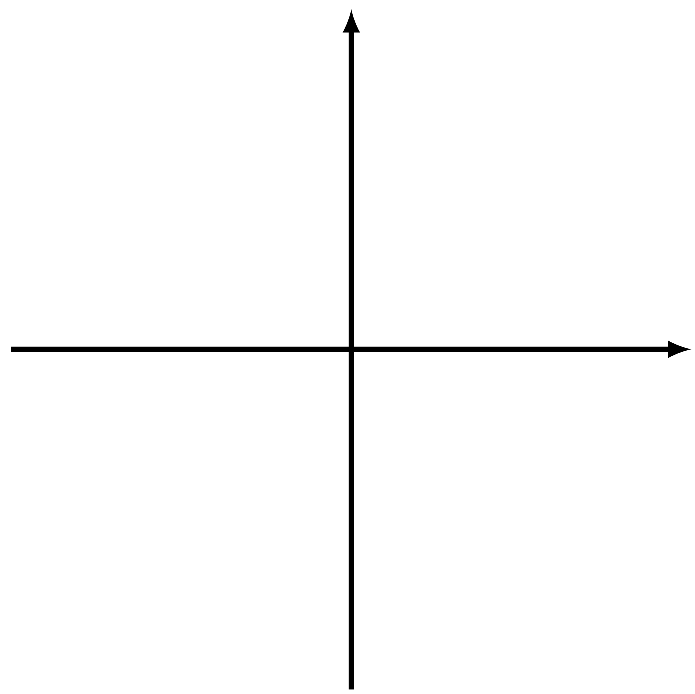
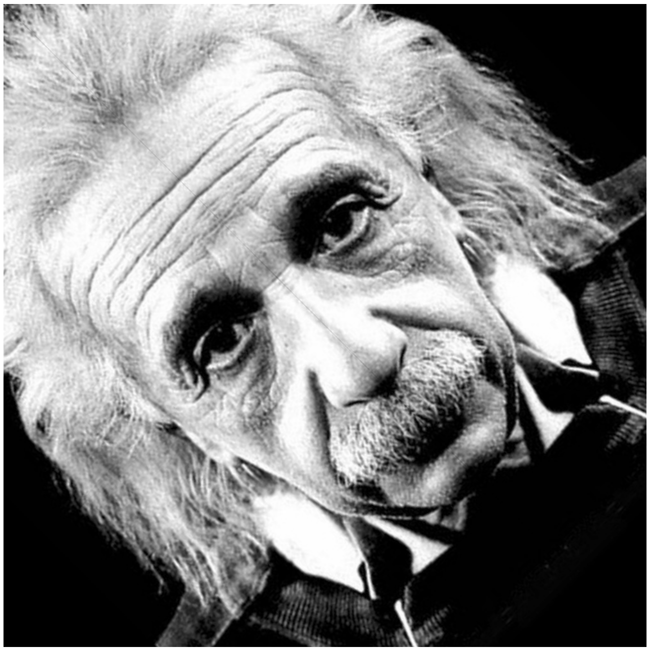
Spectral decomposition and linear transformations

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$$





- The size of this image is 1000×1000 pixels.
- The color of each pixel is represented by an integer between 0 (black) and 255 (white).
- The whole image is described by a (symmetric) matrix A consisting of $1000 \times 1000 = 1,000,000$ numbers
- Each number is stored in 1 byte, so the image file size is $1,000,000$ bytes ($\approx$ 1 MB).

How to make the image file smaller:

1) Find the spectral decomposition of the matrix A:

$$A = \lambda_1(\mathbf{u}_1\mathbf{u}_1^T) + \lambda_2(\mathbf{u}_2\mathbf{u}_2^T) + \ldots + \lambda_{1000}(\mathbf{u}_{1000}\mathbf{u}_{1000}^T)$$

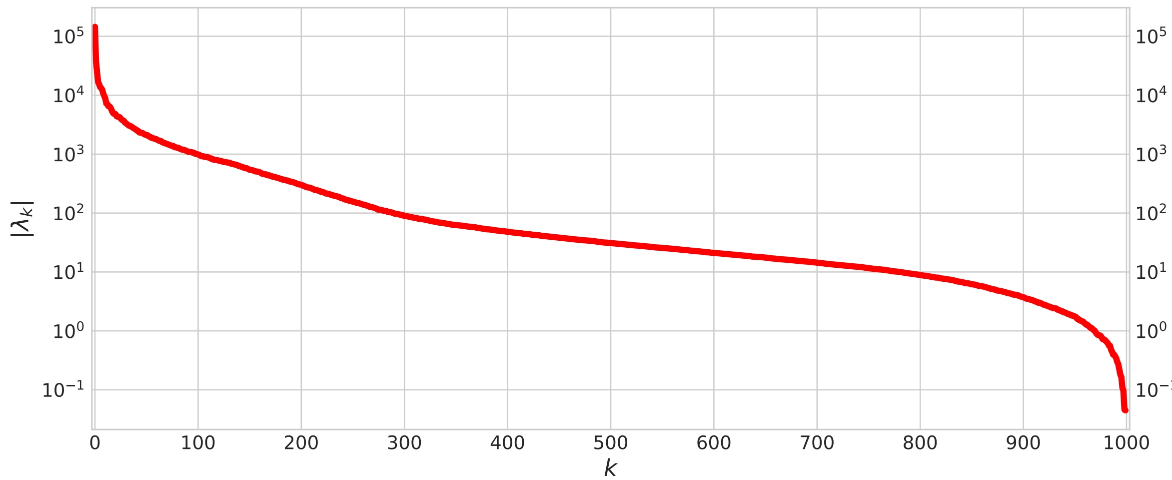
where $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_{1000}|$.

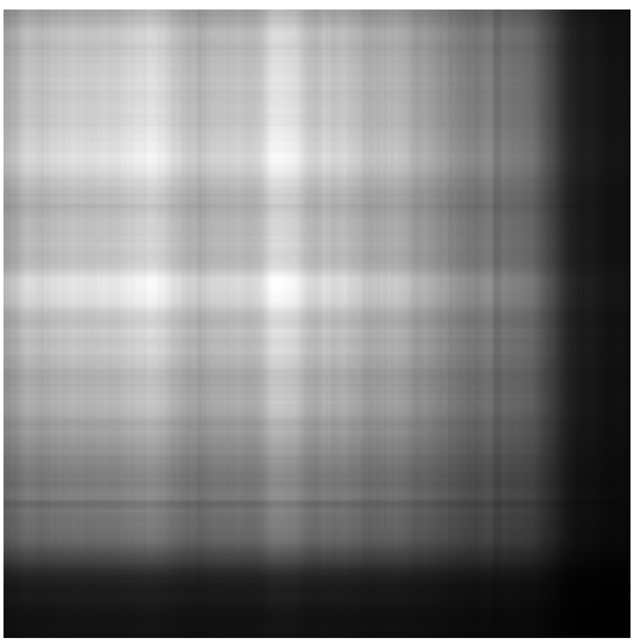
2) For $k = 1, \ldots, 1000$ define:

$$B_k = \lambda_1(\mathbf{u}_1\mathbf{u}_1^T) + \lambda_2(\mathbf{u}_2\mathbf{u}_2^T) + \ldots + \lambda_k(\mathbf{u}_k\mathbf{u}_k^T)$$

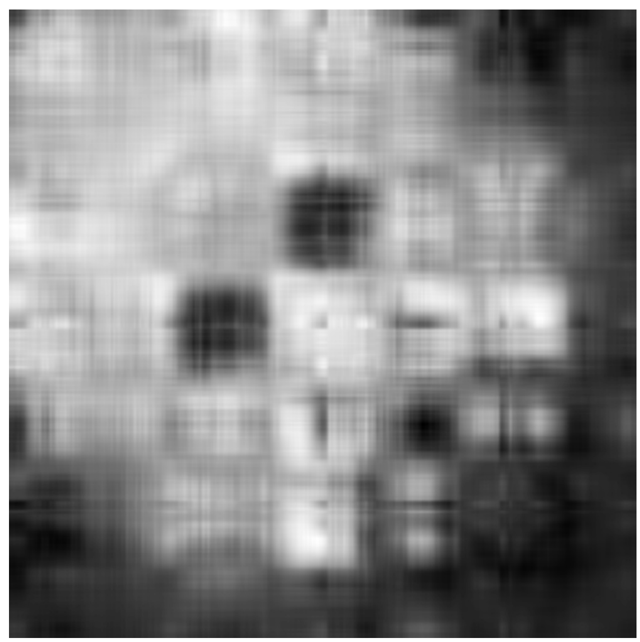
This matrix approximates the matrix A and can be stored using $k \cdot (1000 + 1)$ numbers (i.e. $k \cdot (1000 + 1)$ bytes).

Eigenvalues of the matrix A





matrix B_1
1001 bytes
compression 1000:1



matrix B_5
5005 bytes
compression 200:1