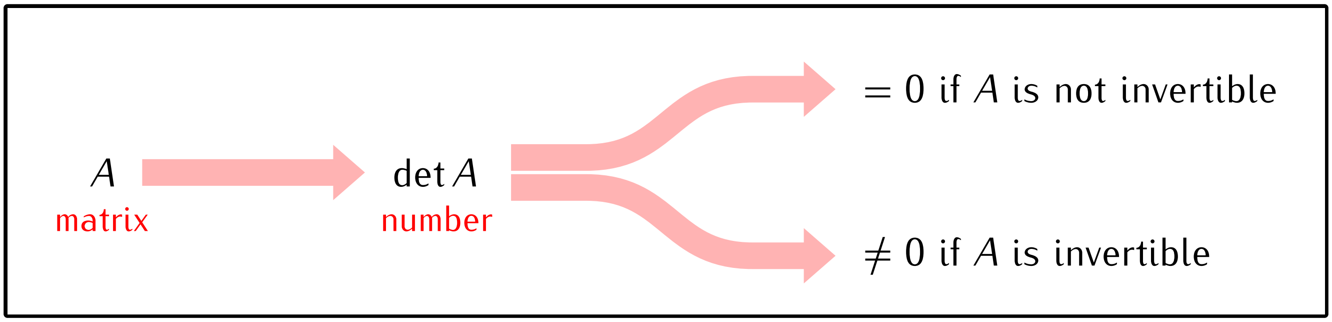
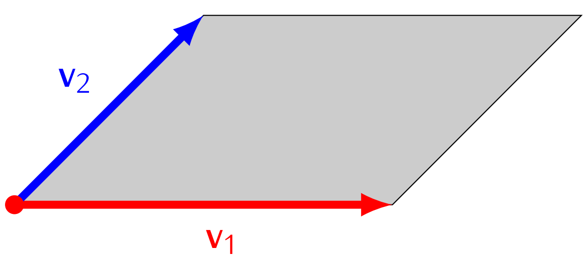
Example. Solve the equation

$$\begin{bmatrix} 1 & 1 & 2 \\ 4 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Recall:



Note. Any two vectors in $\mathbb{R}^2$ define a parallelogram:



Notation

$$\operatorname{area}(\mathbf{v}_1, \mathbf{v}_2) = \left(\begin{array}{c}\text{area of the parallelogram}\\ \text{defined by } \mathbf{v}_1 \text{ and } \mathbf{v}_2\end{array}\right)$$

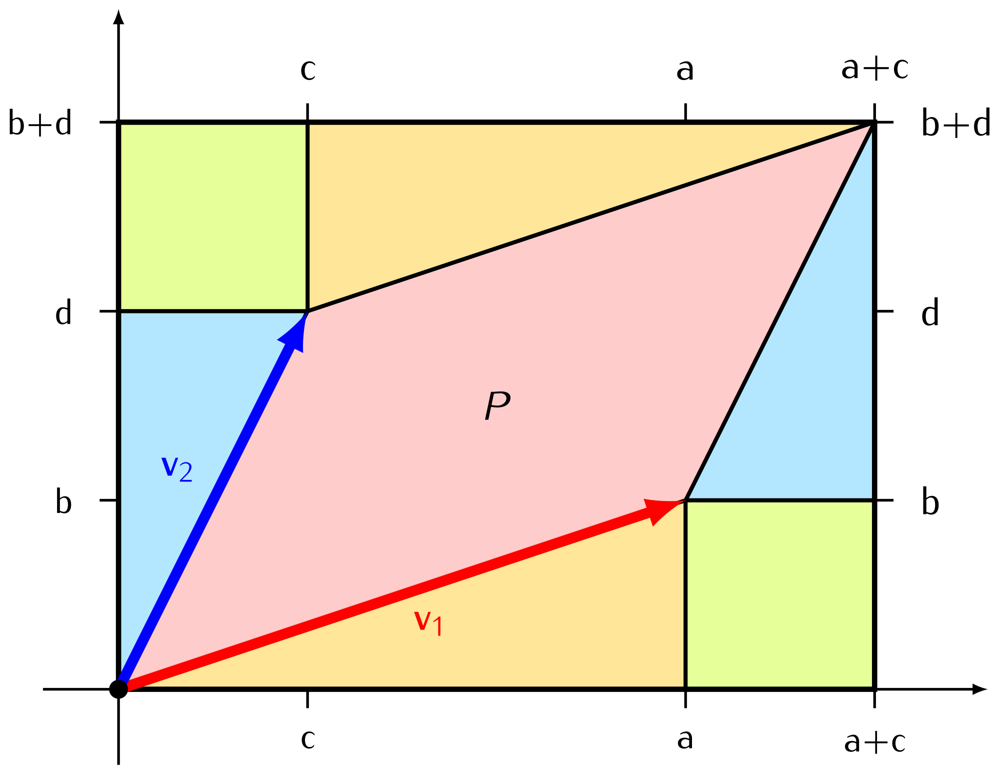
Theorem

If $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ then

$$\text{area}(\mathbf{v}_1, \mathbf{v}_2) = \left|\det\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\right|$$

Idea of the proof.

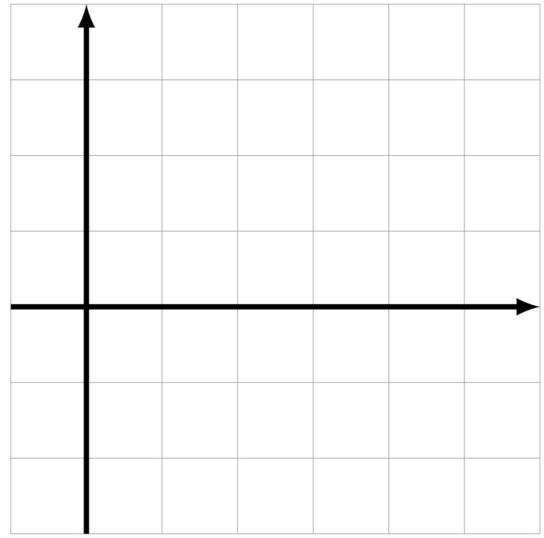
$$\mathbf{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} c \\ d \end{bmatrix}$$



$$\begin{array}{r} \text{area}(P) \\ +\ \frac{1}{2}ab \\ +\ \frac{1}{2}ab \\ +\ \frac{1}{2}cd \\ +\ \frac{1}{2}cd \\ +\ cb \\ +\ cb \\ \hline (a+c)(b+d) \end{array}$$

Example.

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

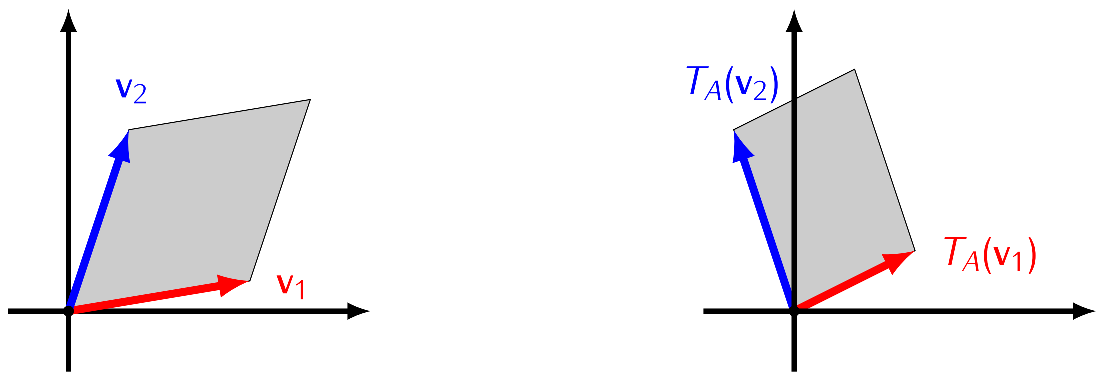


Determinants and linear transformations

Recall: If A is a 2×2 matrix then it defines a linear transformation

$$T_A \colon \mathbb{R}^2 \to \mathbb{R}^2 \qquad T_A(\mathbf{v}) = A\mathbf{v}$$

Note. T_A maps parallelograms to parallelograms:



Theorem

If A is a 2×2 matrix and $\mathbf{v}_1, \mathbf{v}_1 \in \mathbb{R}^2$ then

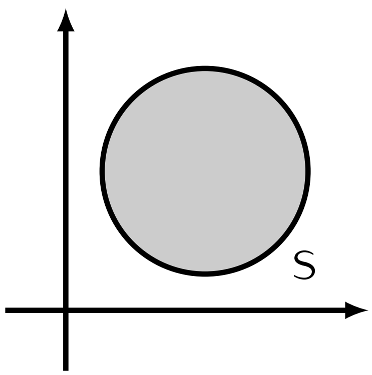
$$\mathrm{area}(T_A(\mathbf{v}_1), T_A(\mathbf{v}_2)) = |\det A| \cdot \mathrm{area}(\mathbf{v}_1, \mathbf{v}_2)$$

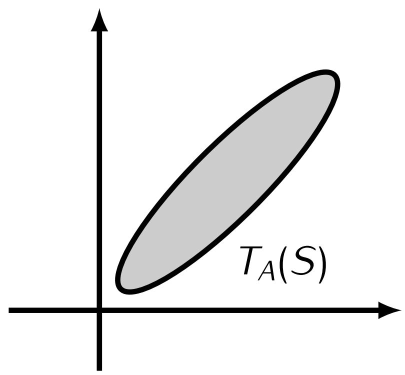
Generalization:

Theorem

If A is a 2×2 matrix then for any region S of $\mathbb{R}^2$ we have:

$$\text{area}(T_A(S)) = |\det A| \cdot \text{area}(S)$$





Idea of the proof.

The area of S can be approximated by the sum of small squares covering S.