Theorem

Let A and B be $n \times n$ matrices.

1) If B is obtained from A by interchanging two rows (or two columns) then

$$\det B = -\det A$$

2) If B is obtained from A by multiplying one row (or one column) of A by a scalar k then

$$\det B = k \cdot \det A$$

2) If B is obtained from A by adding a multiple of one row of A to another row (or adding a multiple of one column to another column) then

$$\det B = \det A$$

Example.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 7 \\ 2 & 5 & 1 \end{bmatrix}$$

Computation of determinants via row reduction

Idea. To compute $\det A$, row reduce A to the row echelon form. Keep track how the determinant changes at each step of the row reduction process.

Example. Compute $\det A$ where

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 4 & 0 & 10 \\ 3 & 4 & 1 & 7 \\ -2 & 5 & 3 & 0 \end{bmatrix}$$

Theorem

If A is a square matrix then A is invertible if and only if $\det A \neq 0$

Recall: A is invertible if and only if its reduced row echelon form is the identity matrix.

Further properties of determinants

1) $\det(A^T) = \det A$

2) $\det(AB) = (\det A) \cdot (\det B)$

3) $\det(A^{-1}) = (\det A)^{-1}$

Note. In general $\det(A + B) \neq \det A + \det B$.

Recall: If A is square matrix then the *ij-cofactor of A* is the number

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Definition

If A is an $n \times n$ matrix then the *adjoint* (or *adjugate*) of A is the matrix

$$\mathrm{adj}A = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Theorem

If A is an invertible matrix then

$$A^{-1} = \frac{1}{\det A} \cdot \mathrm{adj}A$$