Recall:

1) Let A be an $n \times n$ matrix. If $\mathbf{v} \in \mathbb{R}^n$ is a non-zero vector and λ is a scalar such that

$$A\mathbf{v} = \lambda\mathbf{v}$$

then

- λ is an eigenvalue of A
- $\mathbf{v}$ is an eigenvector of A corresponding to λ.

2) The characteristic polynomial of an $n \times n$ matrix A is the polynomial given by the formula

$$P(\lambda) = \det(A - \lambda I_n)$$

where I_n is the $n \times n$ identity matrix.

3) If A is a square matrix then

$$\text{eigenvalues of } A = \text{roots of } P(\lambda)$$

4) If λ is an eigenvalue of an $n \times n$ matrix A then

$$\left\{ \begin{matrix} \text{eigenvectors of } A \\ \text{corresponding to } \lambda \end{matrix} \right\} = \left\{ \begin{matrix} \text{vectors in} \\ \text{Nul}(A - \lambda I_n) \end{matrix} \right\}$$

Motivating example: Fibonacci numbers

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233,\ 377,\ \ldots$$

$F_1\ F_2\ F_3\ F_4\ \ldots$

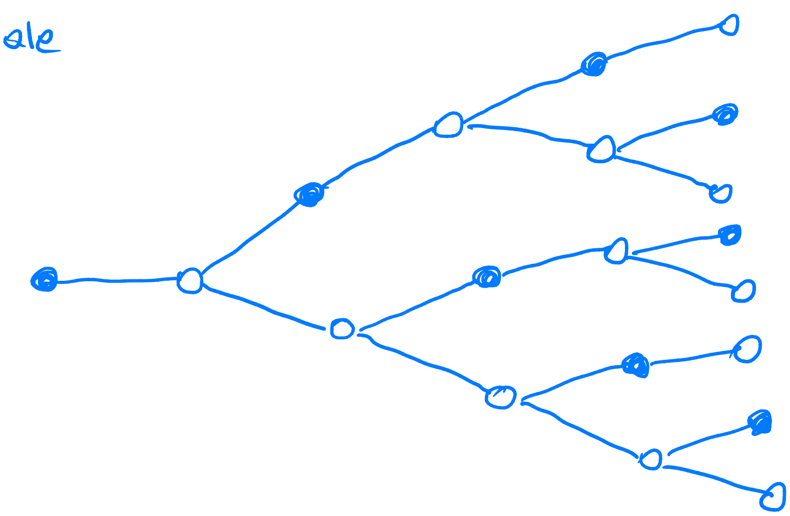
Recursive formula:

$$\begin{cases} F_1 = 1, \quad F_2 = 1 \\ F_{n+1} = F_n + F_{n-1} \quad \text{for } n \geq 2 \end{cases}$$

Fibonacci numbers and the honeybee family tree

● male

○ female



1 1 2 3 5 8 ...

Problem. Find a formula for the n-th Fibonacci number F_n.

Note:

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} F_{n-1} \\ F_n \end{bmatrix} = \begin{bmatrix} 0 \cdot F_{n-1} + 1 \cdot F_n \\ 1 \cdot F_{n-1} + 1 \cdot F_n \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$$

This gives:

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{matrix} F_1 \\ F_2 \end{matrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{matrix} F_2 \\ F_3 \end{matrix}}_{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \quad \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{matrix} F_2 \\ F_3 \end{matrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{matrix} F_3 \\ F_4 \end{matrix}}_{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \quad \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{matrix} F_3 \\ F_4 \end{matrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \begin{matrix} F_4 \\ F_5 \end{matrix}}_{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^3 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

In general:

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{n-1} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$$

Problem:

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{n-1} = ?$$

General Problem. If A is a square matrix how to compute A^k quickly?

Easy case:

Definition

A square matrix D is *diagonal matrix* if all its entries outside the main diagonal are zeros:

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

e.g.:

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Proposition

If D is a diagonal matrix as above then

$$D^k = \begin{bmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n^k \end{bmatrix}$$

e.g.:

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad D^2 = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \qquad D^3 = \begin{bmatrix} 8 & 0 \\ 0 & 27 \end{bmatrix} \qquad \dots$$

Definition

A square matrix A is a *diagonalizable* if A is of the form

$$A = PDP^{-1}$$

where D is a diagonal matrix and P is an invertible matrix.

Example.

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ is a diagonalizable matrix:

$$A = \underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix}}_{P} \cdot \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} \cdot \underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix}^{-1}}_{P^{-1}}$$

Proposition

If A is a diagonalizable matrix, $A = PDP^{-1}$, then

$$A^k = PD^kP^{-1}$$

Proof:

$$A^k = \underbrace{A \cdot A \cdot A \cdot \ldots \cdot A}_{k \text{ times}}$$

$$= (P \cdot D \cdot \cancel{P^{-1}}) \cdot (\cancel{P} \cdot D \cdot \cancel{P^{-1}}) \cdot (\cancel{P} \cdot D \cdot \cancel{P^{-1}}) \cdot \ldots \cdot (\cancel{P} \cdot D \cdot P^{-1})$$

$$= P \cdot \underbrace{D \cdot D \cdot D \cdot \ldots \cdot D}_{k \text{ times}} \cdot P^{-1} = P \cdot D^k \cdot P^{-1}$$

Example.

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Compute A^{10}.

Solution:

We had:

$$A = \underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix}}_{P} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix}^{-1}}_{P^{-1}}$$

$$A^{10} = P \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{10} \cdot P^{-1}$$

$$= P \cdot \begin{bmatrix} 2^{10} & 0 & 0 \\ 0 & (-1)^{10} & 0 \\ 0 & 0 & 1^{10} \end{bmatrix} \cdot P^{-1}$$

$$= P \cdot \begin{bmatrix} 1024 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P^{-1}$$

$$= \begin{bmatrix} 342 & 341 & 341 \\ 341 & 342 & 341 \\ 341 & 341 & 342 \end{bmatrix}$$

Diagonalization Theorem

1) An $n \times n$ matrix A is a diagonalizable if and only if it has n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

2) In such case $A = PDP^{-1}$ where :

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

λ_1 = eigenvalue corresponding to $\mathbf{v}_1$
λ_2 = eigenvalue corresponding to $\mathbf{v}_2$
... ...
λ_n = eigenvalue corresponding to $\mathbf{v}_n$

Proof: Assume that A is diagonalizable :

$$A = P \cdot D \cdot P^{-1}$$

$$P = [v_1 \; v_2 \cdots v_n] \;,\quad D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

1) Columns of P are linearly independent since P is invertible.

2) $A = PDP^{-1} \Rightarrow AP = P \cdot D$

We have:

$$AP = A[v_1 \; v_2 \cdots v_n] = [Av_1 \; Av_2 \cdots Av_n]$$

$$P \cdot D = [v_1 \; v_2 \cdots v_n] \cdot \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = [\lambda_1 v_1 \; \lambda_2 v_2 \cdots \lambda_n v_n]$$

Thus the equation $AP = DP$ gives:

$$[Av_1 \; Av_2 \cdots Av_n] = [\lambda v_1 \; \lambda_2 v_2 \cdots \lambda_n v_n]$$

So:

$Av_1 = \lambda_1 v_1$ (e.i. v_1 is an eigenvector corresponding to λ_1)

$Av_2 = \lambda_2 v_2$ (e.i. v_2 is an eigenvector corresponding to λ_2)

$\vdots$

$Av_n = \lambda_n v_n$ (e.i. v_n is an eigenvector corresponding to λ_n)

Example. Diagonalize the following matrix if possible:

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Solution: We want to find an invertible matrix P and a diagonal matrix D such that $A = P \cdot D \cdot P^{-1}$.

① Find eigenvalues of A

Characteristic polynomial of A:

$$P(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 4-\lambda & 0 & 0 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} = -\lambda^3 + 10\cdot\lambda^2 - 32\lambda + 32$$

(eigenvalues of A) = (roots of $P(\lambda)$) = $(\lambda_1 = 2,\ \lambda_2 = 4)$

② Calculate bases of eigenspaces:

$\lambda_1 = 2$

(basis of eigenspace for $\lambda_1 = 2$) = (basis of $\mathrm{Nul}(A - 2\cdot I)$) = $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

$\lambda_1 = 4$

(basis of eigenspace for $\lambda_1 = 4$) = (basis of $\mathrm{Nul}(A - 4\cdot I)$) = $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

Fact: eigenvectors corresponding to distinct eigenvalues are linearly independent.

Upshot: A has 3 linearly independent eigenvectors:

$$\underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{\substack{\text{eigenv.} \\ \lambda_1 = 2}},\ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\substack{\text{eigenvalue} \\ \lambda_2 = 4}}$$

This gives: A is diagonalizable:

$$A = P \cdot D \cdot P^{-1} \qquad P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Note. Not every matrix is diagonalizable.

Example. Check if the following matrix is diagonalizable:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

Solution:

1. Find eigenvalues of A:

$$P(\lambda) = \det \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2$$

$P(\lambda)$ has only one root $\lambda = 2$, so this is the only eigenvalue of A.

2. Calculate bases of eigenspaces:

$$(\text{basis of eigenspace for } \lambda = 2) = (\text{basis of } \mathrm{Nul}(A - 2I))$$

$$= \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

This shows that A does not have 2 linearly independent eigenvectors, so it is not diagonalizable.

Proposition

If A is an $n \times n$ matrix with n distinct eigenvalues then A is diagonallizable.

Proof:

Let $\lambda_1, \dots, \lambda_n$ - eigenvalues of A.

Take v_i - an eigenvector corresponding to λ_i.

Since eigenvectors corresponding to different eigenvalues are linearly independent, we get that $v_1, v_2, \dots, v_n$ are linearly independent eigenvectors of A.

Back to Fibonacci numbers:

$$\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{n-1} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

In order to compute $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{n-1}$ diagonalize the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

① Eigenvalues of A:

$$P(\lambda) = \det \begin{bmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = \lambda^2 - \lambda + 1$$

"golden ratio"

$$(\text{eigenvalues of } A) = (\text{roots of } P(\lambda)) = \left(\lambda_1 = \overbrace{\frac{1+\sqrt{5}}{2}}^{\text{"golden ratio"}},\ \lambda_2 = \frac{1-\sqrt{5}}{2}\right)$$

Note: $\lambda_1 + \lambda_2 = 1$
$\lambda_1 \cdot \lambda_2 = -1$

② Bases of eigenspaces of A:

$$(\text{basis of eigenspace for } \lambda_1) = \left\{ \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \right\}$$

$$(\text{basis of eigenspace for } \lambda_2) = \left\{ \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} \right\}$$

We obtain: A is diagonalizable:

$$A = P \cdot D \cdot P^{-1} \quad \text{where} \quad P = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

This gives:

$$\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = A^{n-1} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = P \cdot D^{n-1} \cdot P^{-1} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = P \cdot \begin{bmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{bmatrix} \cdot P^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^n - \lambda_2^n \\ \lambda_1^{n+1} - \lambda_2^{n+1} \end{bmatrix}$$

We obtain:

$$\boxed{F_n = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)}$$

Binet's formula