Definition

If A is an $n \times n$ matrix and $1 \le i, j \le n$ then the ij-cofactor of A is the number

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Example.

$$A = \left[\begin{array}{rrr} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array} \right]$$

Note. By the definition of the determinant we have:

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}$$

Theorem

Let A be an $n \times n$ matrix.

1) For any $1 \le i \le n$ we have

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

(cofactor expansion across the i^{th} row).

2) For any $1 \le j \le n$ we have

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$

(cofactor expansion down the j^{th} column).

Example.

$$A = \left[\begin{array}{cccc} 1 & 3 & 0 & 4 \\ 0 & 4 & 6 & 1 \\ 2 & 1 & 0 & 3 \\ 0 & 5 & 0 & 0 \end{array} \right]$$

Example. Compute the determinant of the following matrix:

Γ	1	0	0	3	0	0	2	0	3	0	0	0	0	e	0	0	0	3	0	0	0]
	0	2	0	0	π	0	0	0	6	0	0	5	6	0	2	0	7	0	0	0	0
	0	0	1	0	0	0	0	0	11	0	0	0	0	0	7	0	0	0	0	0	0
İ	0	0	0	$-\frac{1}{2}$	0	0	0	0	4	0	0	2	0	4	0	2	0	0	0	0	0
	0	0	0	Ō	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
	0	0	0	0	0	-1	0	0	0	0	9	0	0	0	2	1	2	3	4	0	0
l	0	0	0	0	0	0	3	1	0	0	-1	0	0	0	0	0	5	0	0	0	0
	0	0	0	0	0	0	2	1	0	0	0	0	0	0	12	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	-1	0	0	4	0	0
1	0	0	0	0	0	0	0	0	0	3	0	0	2	7	0	-4	0	0	3	0	0
	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	3	0	0	2	0	0
	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0	6	0
	0	0	0	0	0	0	0	0	0	0	0	0	$\frac{1}{4}$	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	<u>1</u>	0	1	0	4	3	2	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	8	7	7
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	-1	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0
L	0	0	0	0	2	8	9	0	3	3	2	5	6	3	8	9	2	6	2	2	1]

Definition

An square matrix is *upper triangular* is all its entries below the main diagonal are 0.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

Proposition

If A is an upper triangular matrix as above then

$$\det A = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$$