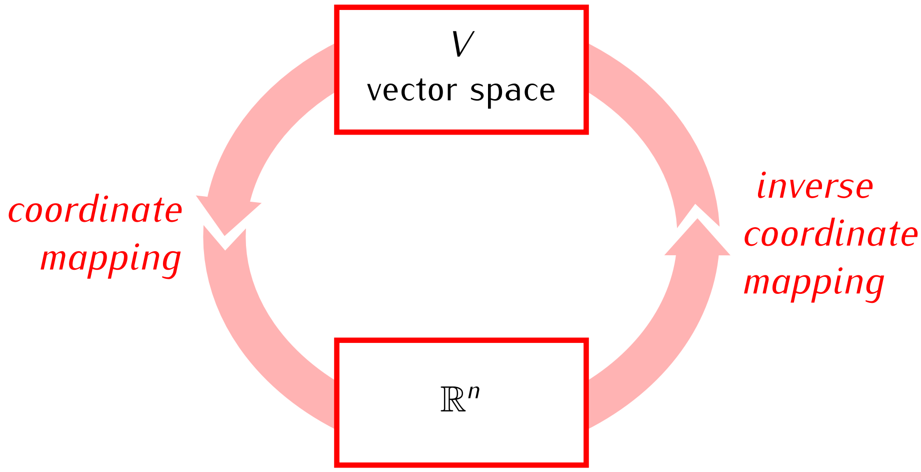

Theorem

Let $\mathcal{B}$ be a basis of a vector space V. If $\mathbf{v}_1, \dots \mathbf{v}_p, \mathbf{w} \in V$ then:

1) Solutions of the equation $x_1\mathbf{v}_1 + \dots + x_p\mathbf{v}_p = \mathbf{w}$ are the same as solutions of the equation $x_1 \left[\mathbf{v}_1\right]_{\mathcal{B}} + \dots + x_p \left[\mathbf{v}_p\right]_{\mathcal{B}} = \left[\mathbf{w}\right]_{\mathcal{B}}$.

2) The set of vectors $\{\mathbf{v}_1, \dots \mathbf{v}_p\}$ is linearly independent if and only if the set $\left\{\left[\mathbf{v}_1\right]_{\mathcal{B}}, \dots, \left[\mathbf{v}_p\right]_{\mathcal{B}}\right\}$ is linearly independent.

3) $\mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_p) = V$ if any only if $\mathrm{Span}\left(\left[\mathbf{v}_1\right]_{\mathcal{B}}, \dots, \left[\mathbf{v}_p\right]_{\mathcal{B}}\right) = \mathbb{R}^n$.

4) $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a basis of V if and only if $\left\{\left[\mathbf{v}_1\right]_{\mathcal{B}}, \dots, \left[\mathbf{v}_p\right]_{\mathcal{B}}\right\}$ is a basis of $\mathbb{R}^n$.

Example. Recall that $\mathbb{P}_2$ is the vector space of polynomials of degree ≤ 2. Consider the following polynomials in $\mathbb{P}_2$:

$$\begin{aligned} p_1(t) &= 1 + 2t + t^2 \\ p_2(t) &= 3 + t + 2t^2 \\ p_3(t) &= 1 - 8t - t^2 \end{aligned}$$

Check if the set $\{p_1, p_2, p_3\}$ is linearly independent.

Theorem

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be vectors in $\mathbb{R}^n$. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a basis of $\mathbb{R}^n$ if and only if the matrix

$$A = \begin{bmatrix} \mathbf{v}_1 & \ldots & \mathbf{v}_p \end{bmatrix}$$

has a pivot position in every row and in every column (i.e. if A is an invertible matrix).

Corollary

Every basis of $\mathbb{R}^n$ consists of n vectors.

Theorem

Let V be a vector space. If V has a basis consisting of n vectors then every basis of V consists of n vectors.

Definition

A vector space has *dimension* n if V has a basis consisting of n vectors. Then we write $\dim V = n$.

Example.

Theorem

Let V be a vector space such that $\dim V = n$, and let $\mathbf{v}_1, \dots \mathbf{v}_p \in V$.

1) If $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a spanning set of V then $p \geq n$.

2) If $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a linearly independent set then $p \leq n$.

Corollary

Let V be a vector space such that $\dim V = n$. If W be a subspace of V then $\dim W \leq n$. Moreover, if $\dim W = n$ then $W = V$.

Note.

1) One can show that every vector space has a basis.

2) Some vector spaces have bases consisting of infinitely many vectors. If V is such vector space then we write $\dim V = \infty$.

Example.

Recall:

If $A = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$ is an $m \times n$ matrix then:

1) $\mathrm{Col}(A) = \mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_n)$

2) $\mathrm{Nul}(A) = \{\mathbf{v} \in \mathbb{R}^m \mid A\mathbf{v} = \mathbf{0}\}$