

**Instructions.**

In this problem you will be given a few statements. For each statement you need to decide if it is true or not, and justify your answer.

**How to justify your answer.**

- In order to show that a statement is false, it suffices to give a counterexample. For example, consider the the statement:

*The last digit of every even number is either 2, 4, or 8.*

To show that this statement is false, it is enough to point out that, for example, 10 is an even number, but its last digit is 0.

- In order to show that a statement is true, you need to provide a reasoning explaining why it is true in all instances. Giving one example when it is true will not suffice, since the statement may not work in some other cases. For example, consider the the statement:

*If  $n$  is an even number then  $n + 2$  is also an even number.*

You can justify that this is true as follows. Even numbers are integers which are multiples of 2. If  $n$  is even then  $n = 2m$  for some integer  $m$ . Then  $n + 2 = 2m + 2 = 2(m + 1)$ , which shows that  $n + 2$  is even.

**Note**

This problem will not be collected or graded. However, problems of this type will appear on exams in this course. Sample solutions are provided at the end.

For each of the statements given below decide if it is true or false. If you decide that it is true, justify your answer. If you think it is false give a counterexample.

- a) If a set  $\{v_1, v_2\}$  is a basis of  $\mathbb{R}^2$  then the set  $\{v_1, v_1 + v_2\}$  is also a basis of  $\mathbb{R}^2$ .
- b) If  $S = \{v_1, v_2, v_3\}$  is set of vectors in a vector space  $V$  such that  $v_1$  is the zero vector, then  $S$  cannot be a basis of  $V$ .
- c) If  $V$  and  $W$  are subspaces of  $\mathbb{R}^2$  and  $\dim V = \dim W$  then  $V = W$ .
- d) If  $V$  and  $W$  are subspaces of  $\mathbb{R}^2$  and  $\dim V = \dim W = 2$  then  $V = W$ .

Here are solutions to the questions from the previous page. You should try to answer all questions by yourself before reading these solutions.

- a) TRUE. If a set  $\{v_1, v_2\}$  is a basis of  $\mathbb{R}^2$  then the matrix  $\begin{bmatrix} v_1 & v_2 \end{bmatrix}$  is invertible, i.e. if  $\det \begin{bmatrix} v_1 & v_2 \end{bmatrix} \neq 0$ . Since

$$\det \begin{bmatrix} v_1 & v_1 + v_2 \end{bmatrix} = \det \begin{bmatrix} v_1 & v_2 \end{bmatrix} \neq 0$$

we obtain that  $\begin{bmatrix} v_1 & v_1 + v_2 \end{bmatrix}$  is an invertible matrix, which shows that the set  $\{v_1, v_1 + v_2\}$  is a basis of  $\mathbb{R}^2$ .

- b) TRUE. In order to be a basis the set  $\{v_1, v_2, v_3\}$  must be linearly independent, i.e. the equation

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = \mathbf{0}$$

must have only one, trivial solution. However, if  $v_1 = \mathbf{0}$  this equation has infinitely many solutions, since we can take  $x_1$  to be any number, and  $x_2 = 0, x_3 = 0$ .

- c) FALSE. Take e.g.

$$V = \text{Span} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \quad W = \text{Span} \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

Then  $\dim V = \dim W = 1$ , but  $V \neq W$ .

- d) TRUE. Recall that if  $W$  is a subspace of a vector space  $V$  and  $\dim V = \dim W$  then  $V = W$ . In our case  $\dim V = \dim W = \dim \mathbb{R}^2$  which gives  $V = W = \mathbb{R}^2$ .