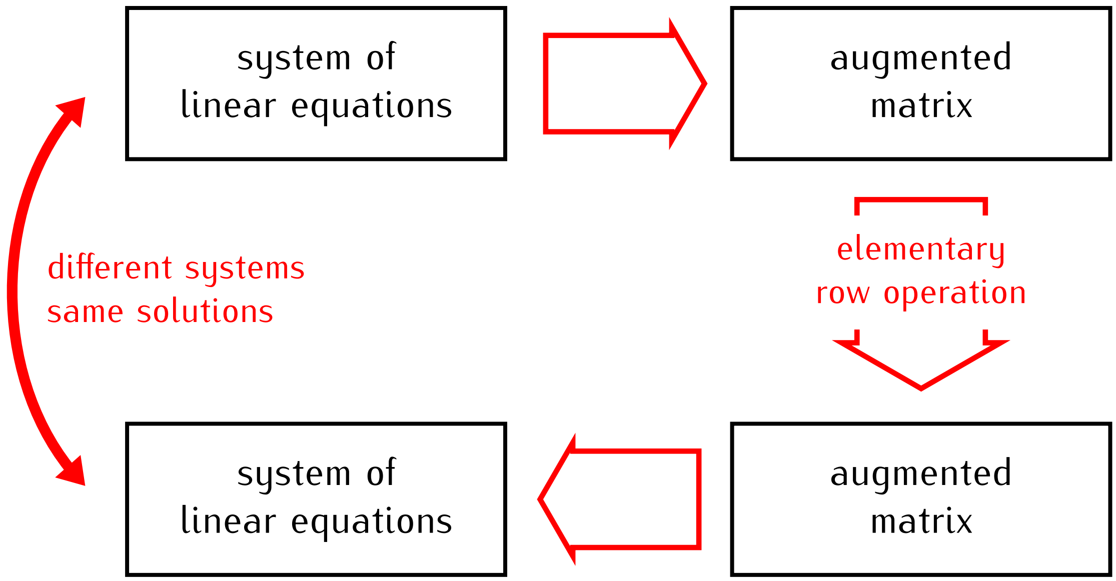
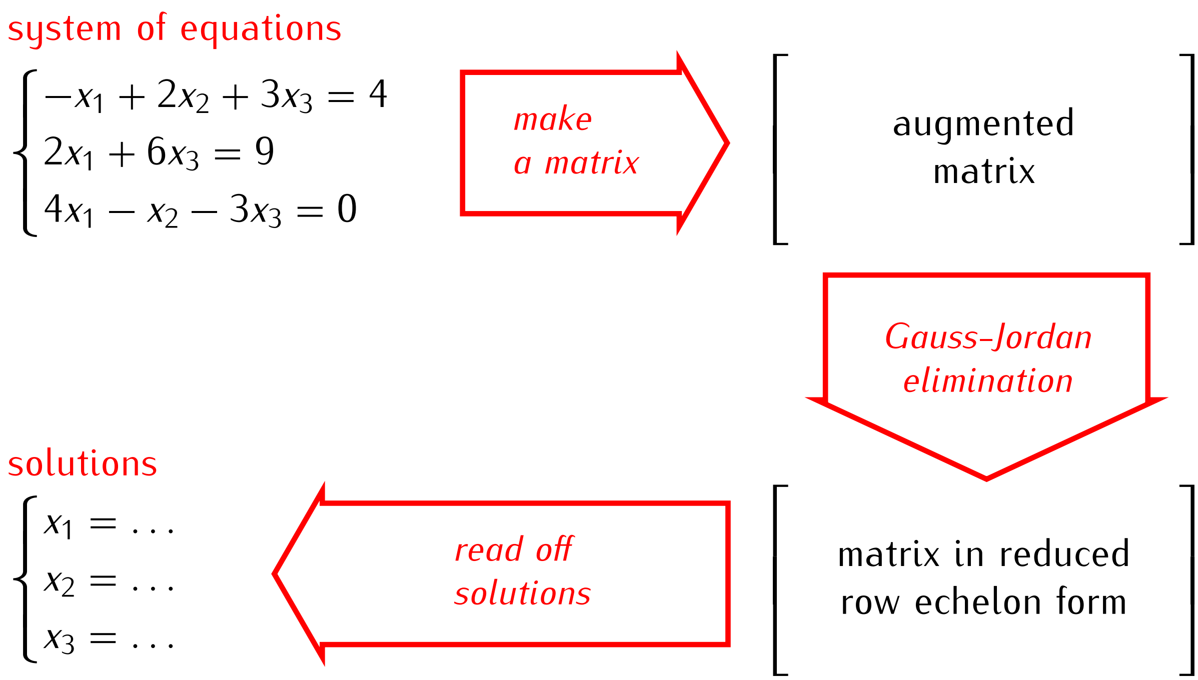
Proposition

Elementary row operations do not change solutions of the system of equations represented by a matrix.



Recall:

How to solve a system of linear equations



- Every system of linear equations can be represented by a matrix

- Elementary row operations:
  - interchange of two rows
  - multiplication of a row by a non-zero number
  - addition of a multiple of one row to another row.

- Elementary row operations do not change solutions of systems of linear equations.

Definition

A matrix is in the *row echelon form* if:

1) the first non-zero entry of each row is a 1 (“a leading one”);

2) the leading one in each row is to the right of the leading one in the row above it.

A matrix is in the *reduced row echelon form* if in addition it satisfies:

3) all entries above each leading one are 0.

$$\begin{bmatrix} 1 & * & * & * & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

($*$ = any number)

Fact

If a system of linear equations is represented by a matrix in the reduced row echelon form then it is easy to solve the system.

Example

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Proposition

A matrix in the reduced row echelon form represents an inconsistent system if and only if it contains a row of the form

$$\left[\begin{array}{cccccc} 0 & 0 & 0 \ldots & 0 & 1 \end{array}\right]$$

i.e. with the leading one in the last column.

Example

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Note

In an augmented matrix in the reduced row echelon form free variables correspond to columns of the coefficient matrix that do not contain leading ones.

Example

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 8 \end{array}\right]$$

Note

A matrix in the reduced row echelon represents a system of equations with exactly one solution if and only if it has a leading one in every column except for the last one.

Gauss-Jordan elimination process (= row reduction)

$$\begin{bmatrix} \quad \text{matrix} \quad \end{bmatrix} \xrightarrow{\substack{\textit{elementary} \\ \textit{row operations}}} \begin{bmatrix} \text{matrix} \\ \text{in reduced row} \\ \text{echelon form} \end{bmatrix}$$

1. Interchange rows, if necessary, to bring a non-zero element to the top of the first non-zero column of the matrix.

2. Multiply the first row so that its first non-zero entry becomes 1.

3. Add multiples of the first row to eliminate non-zero entries below the leading one.

4. Ignore the first row; apply steps 1-3 to the rest of the matrix.

5. Eliminate non-zero entries above all leading ones.

Example.

$$\begin{bmatrix} 0 & 4 & -8 & 0 & 4 \\ 2 & 6 & -6 & -2 & -4 \\ 2 & 7 & -8 & 0 & -1 \end{bmatrix}$$