

**Recall:** An  $n \times n$  matrix  $A$  defines a linear transformation

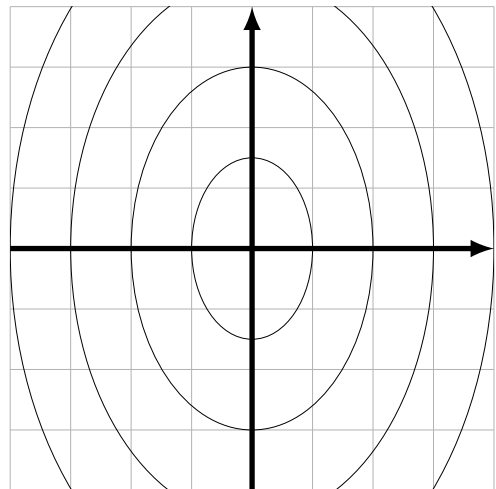
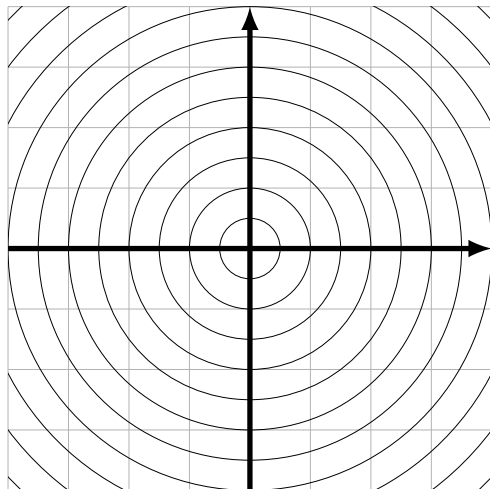
$$T_A: \mathbb{R}^n \rightarrow \mathbb{R}^n$$

given by  $T_A(\mathbf{v}) = A\mathbf{v}$ .

**Next goal:** Understand this linear transformation better.

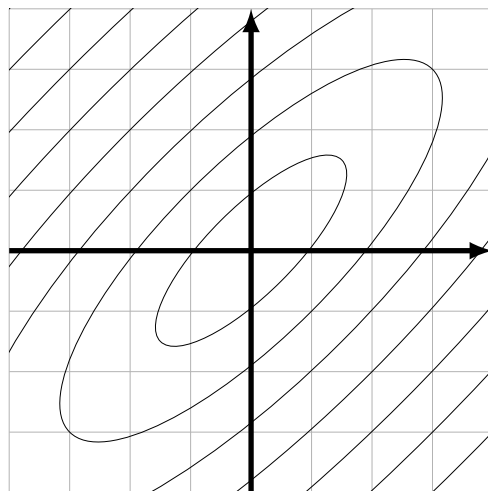
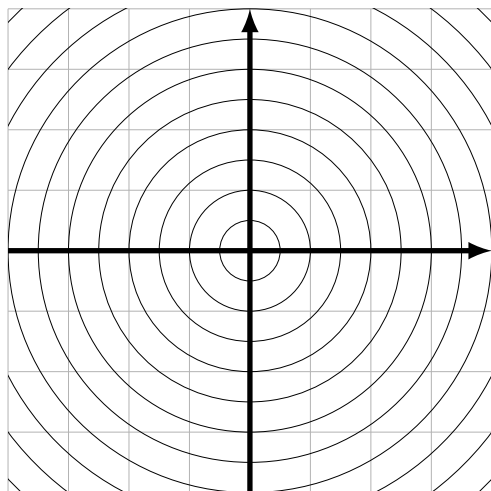
**Example.**

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$



**Example.**

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$



### Definition

Let  $A$  be an  $n \times n$  matrix. If  $\mathbf{v} \in \mathbb{R}^n$  is a non-zero vector and  $\lambda$  is a scalar such that

$$A\mathbf{v} = \lambda\mathbf{v}$$

then we say that

- $\lambda$  is an *eigenvalue* of  $A$
- $\mathbf{v}$  is an *eigenvector* of  $A$  corresponding to  $\lambda$ .

**Example.**

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

**Example.**

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

## Computation of eigenvalues

Recall:  $I_n = n \times n$  identity matrix:

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

### Proposition

If  $A$  be an  $n \times n$  matrix then  $\lambda \in \mathbb{R}$  is an eigenvalue of  $A$  if and only if the matrix equation

$$(A - \lambda I_n)\mathbf{x} = \mathbf{0}$$

has a non-trivial solution.

### Proposition

If  $B$  is an  $n \times n$  matrix then equation

$$Bx = 0$$

has a non-trivial solution if and only if the matrix  $B$  is not invertible.

### Proposition

If  $A$  be an  $n \times n$  matrix then  $\lambda \in \mathbb{R}$  is an eigenvalue of  $A$  if and only if

$$\det(A - \lambda I_n) = 0$$

**Example.** Find all eigenvalues of the following matrix:

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

### Definition

If  $A$  is an  $n \times n$  matrix then

$$P(\lambda) = \det(A - \lambda I_n)$$

is a polynomial of degree  $n$ .  $P(\lambda)$  is called the *characteristic polynomial* of the matrix  $A$ .

### Upshot

If  $A$  is a square matrix then

$$\text{eigenvalues of } A = \text{roots of } P(\lambda)$$

**Example.**

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

### Corollary

An  $n \times n$  matrix can have at most  $n$  distinct eigenvalues.

## Computation of eigenvectors

### Proposition

If  $\lambda$  is an eigenvalue of an  $n \times n$  matrix  $A$  then

$$\left\{ \begin{array}{l} \text{eigenvectors of } A \\ \text{corresponding to } \lambda \end{array} \right\} = \left\{ \begin{array}{l} \text{vectors in} \\ \text{Nul}(A - \lambda I_n) \end{array} \right\}$$



### Corollary/Definition

If  $A$  is an  $n \times n$  matrix and  $\lambda$  is an eigenvalue of  $A$  then the set of all eigenvectors corresponding to  $\lambda$  is a subspace of  $\mathbb{R}^n$ .

This subspace is called the *eigenspace* of  $A$  corresponding to  $\lambda$ .

### Proposition

If  $\lambda$  is an eigenvalue of an  $n \times n$  matrix  $A$  then

$$\left\{ \begin{array}{l} \text{eigenspace of } A \\ \text{corresponding to } \lambda \end{array} \right\} = \text{Nul}(A - \lambda I_n)$$

**Example.** Consider the following matrix:

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Recall that eigenvalues of  $A$  are  $\lambda_1 = 1$  and  $\lambda_2 = 5$ . Compute bases of eigenspaces of  $A$  corresponding to these eigenvalues.

**Solution.**

$$\underline{\lambda_1 = 1}$$

$$\underline{\lambda_2 = 5}$$