Definition

A *(real) vector space* is a set V together with two operations:

- addition

$$\begin{aligned} V \times V &\longrightarrow V \\ (\mathbf{u}, \ \ \mathbf{v}) &\longmapsto \ \mathbf{u} + \mathbf{v} \end{aligned}$$

- multiplication by scalars

$$\begin{aligned} \mathbb{R} \times V &\longrightarrow V \\ (c, \ \ \mathbf{v}) &\longmapsto \ c \cdot \mathbf{v} \end{aligned}$$

Moreover the following conditions must be satisfied:

1) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3) there is an element $\mathbf{0} \in V$ such that $\mathbf{0} + \mathbf{u} = \mathbf{u}$ for any $\mathbf{u} \in V$
4) for any $\mathbf{u} \in V$ there is an element $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7) $(cd)\mathbf{u} = c(d\mathbf{u})$
8) $1\mathbf{u} = \mathbf{u}$

Elements of V are called *vectors*.

Theorem

If V is a vectors space then:

1) $c \cdot \mathbf{0} = \mathbf{0}$ where $c \in \mathbb{R}$ and $\mathbf{0} \in V$ is the zero vector;

2) $0 \cdot \mathbf{u} = \mathbf{0}$ where $0 \in \mathbb{R}$, $\mathbf{u} \in V$ and $\mathbf{0}$ is the zero vector;

3) $(-1) \cdot \mathbf{u} = -\mathbf{u}$

Examples of vector spaces.

Defitnition

Let V be a vector space. A *subspace* of V is a subset $W \subseteq V$ such that

1) $\mathbf{0} \in W$
2) if $\mathbf{u}, \mathbf{v} \in W$ then $\mathbf{u} + \mathbf{v} \in W$
3) if $\mathbf{u} \in W$ and $c \in \mathbb{R}$ then $c\mathbf{u} \in W$.

Example.

Recall: $\mathbb{P}$ = the vector space of all polynomials.

Proposition

Let V be a vector space and $W \subseteq V$ is a subspace then W is itself a vector space.

Example.

Recall: $\mathcal{F}(\mathbb{R})$ = the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$

Some interesting subspaces of $\mathcal{F}(\mathbb{R})$:

1) $C(\mathbb{R})$ = the subspace of all continuous functions $f : \mathbb{R} \to \mathbb{R}$

2) $C^n(\mathbb{R})$ = the subspace of all functions $f : \mathbb{R} \to \mathbb{R}$ that are differentiable n or more times.

3) $C^\infty(\mathbb{R})$ = the subspace of all smooth functions $f : \mathbb{R} \to \mathbb{R}$ (i.e. functions that have derivatives of all orders: f', f'', f''', $\dots$).

Note. If V is a vector space then:

1) the biggest subspace of V is V itself;

2) the smallest subspace of V is the subspace $\{\mathbf{0}\}$ consisting of the zero vector only;

3) if a subspace of V contains a non-zero vector, then it contains infinitely many vectors.

Definition

Let V, W be vector spaces A *linear transformation* is a function

$$T\colon V \to W$$

which satisfies the following conditions:

1) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$

2) $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $\mathbf{v} \in V$ and any scalar c.

Proposition

If $T\colon V \to W$ is a linear transformation then $T(\mathbf{0}) = \mathbf{0}$.