

Operations on matrices so far:

- addition/subtraction $A \pm B$
- scalar multiplication $c \cdot A$
- matrix multiplication $A \cdot B$
- matrix transpose A^T

Next: How to divide matrices?

Note: if a, b - numbers then:

$$1) \frac{a}{b} = a \cdot b^{-1}$$

2) b^{-1} is the number such that $b \cdot b^{-1} = 1$

Definition

A matrix A is *invertible* if there exists a matrix B such that

$$A \cdot B = B \cdot A = I$$

(where I = the identity matrix). In such case we say that B is the *inverse* of A and we write $B = A^{-1}$.

Example.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ is invertible, } A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Check:

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Note: Not every matrix is invertible.

Example:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Assume that $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a matrix

such that $AB = I$. Then:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}$$

The first column gives:

$$\left. \begin{array}{l} 1 = a+c \\ 0 = a+c \end{array} \right\} \leftarrow \text{impossible}$$

Thus $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not invertible.

Matrix inverses and matrix equations

Proposition

If A is an invertible matrix then for any vector \mathbf{b} the equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution.

Proof: If $A\mathbf{x} = \mathbf{b}$ then

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$

$$I\mathbf{x} = A^{-1}\mathbf{b}$$

$$\mathbf{x} = \underbrace{A^{-1}\mathbf{b}}$$

↑ the unique solution of $A\mathbf{x} = \mathbf{b}$

Example. Solve the following matrix equation:

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_A \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Recall: A is invertible, $A^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$

This gives:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} //$$

Recall: If A be is $m \times n$ matrix then:

- the matrix equation $Ax = b$ has a solution for every b if and only if A has a pivot position in every row;
- the matrix equation $Ax = b$ has exactly one solution for $b \in \text{Col}(A)$ if and only if A has a pivot position in every column.

$$\left[A \mid b \right] \xrightarrow{\text{row red.}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \square \\ 0 & 1 & 2 & \square \\ 0 & 0 & 0 & \square \end{array} \right]$$

we can get a leading 1 in the column of constants and so no solutions

$$\left[A \mid b \right] \xrightarrow{\text{row red.}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & \square \\ 0 & 0 & 1 & \square \end{array} \right]$$

this column gives a free variable, and so infinitely many solutions

Theorem

If a matrix A is invertible then it must be a square matrix.

For a square matrix A the following conditions are equivalent:

- 1) A is an invertible matrix.
- 2) The matrix A has a pivot position in every row and column.
- 3) The reduced row echelon form of A is the identity matrix I_n .

If A is invertible then:

$$\left(\begin{array}{c} \text{number of} \\ \text{rows of } A \end{array} \right) = \left(\begin{array}{c} \text{number of} \\ \text{pivot positions} \end{array} \right) = \left(\begin{array}{c} \text{number of} \\ \text{columns of } A \end{array} \right)$$

Proposition

If A is an $n \times n$ invertible matrix then

$$A^{-1} = [w_1 \ w_2 \ \dots \ w_n]$$

where w_i is the solution of $Ax = e_i$.

Proof: We have:

$$I_n = \begin{bmatrix} \overbrace{1}^{e_1} & \overbrace{0}^{e_2} & \dots & \overbrace{0}^{e_n} \\ 0 & 1 & & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \dots & 1 \end{bmatrix} = [e_1 \ e_2 \ \dots \ e_n]$$

This gives:

$$[e_1 \ e_2 \ \dots \ e_n] = I_n = AA^{-1} = A \cdot [w_1 \ w_2 \ \dots \ w_n] = [Aw_1 \ Aw_2 \ \dots \ Aw_n]$$

We obtain:

$$Aw_1 = e_1, \ Aw_2 = e_2, \ \dots, \ Aw_n = e_n$$

Example.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \xrightarrow{\text{row red.}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \text{ so } A \text{ is invertible}$$

$$A^{-1} = [w_1 \ w_2] \quad \text{where:} \quad \begin{aligned} w_1 &= (\text{solution of } Ax = e_1) \\ w_2 &= (\text{solution of } Ax = e_2) \end{aligned}$$

Solve $Ax = e_1$:

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ -1 & 1 & 0 \end{array} \right] \xrightarrow{\text{row red.}} \left[\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \end{array} \right] \quad \text{so: } w_1 = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$$

Solve $Ax = e_2$:

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 1 \end{array} \right] \xrightarrow{\text{row red.}} \left[\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{array} \right] \quad \text{so: } w_2 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

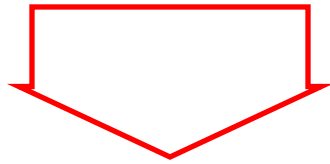
This gives:

$$A^{-1} = [w_1 \ w_2] = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

Simplification:
How to solve several matrix equations with the same
coefficient matrix at the same time

$$Ax = \mathbf{b}_1, Ax = \mathbf{b}_2, \dots, Ax = \mathbf{b}_n$$

matrix of equations



$$\left[A \mid \mathbf{b}_1 \ \mathbf{b}_2 \ \dots \ \mathbf{b}_n \right]$$

augmented matrix



$$\left[\begin{array}{c|c} & \end{array} \right]$$

reduced matrix



solutions

Example. Solve the vector equations $Ax = e_1$ and $Ax = e_2$ where

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solution:

augmented matrix:

$$[A \mid e_1 \ e_2] =$$

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array} \right]$$

row
red.

$$\left[\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & -1/2 & 1/2 \end{array} \right]$$

row reduced
form of A

solution
of $Ax = e_2$

solution
of $Ax = e_1$

Summary:
How to invert a matrix

Example: $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

1) Augment A by the identity matrix.

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array} \right]$$

2) Reduce the augmented matrix.

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array} \right] \xrightarrow{\text{row red.}} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array} \right]$$

2) If after the row reduction the matrix on the left is the identity matrix, then A is invertible and

$$A^{-1} = \text{the matrix on the right}$$

Otherwise A is not invertible.

In our example A is invertible
and $A^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

Properties of matrix inverses

1) If A is invertible then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

2) If A, B are invertible then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

3) If A is invertible then A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

Matrix inverses and matrix transformations

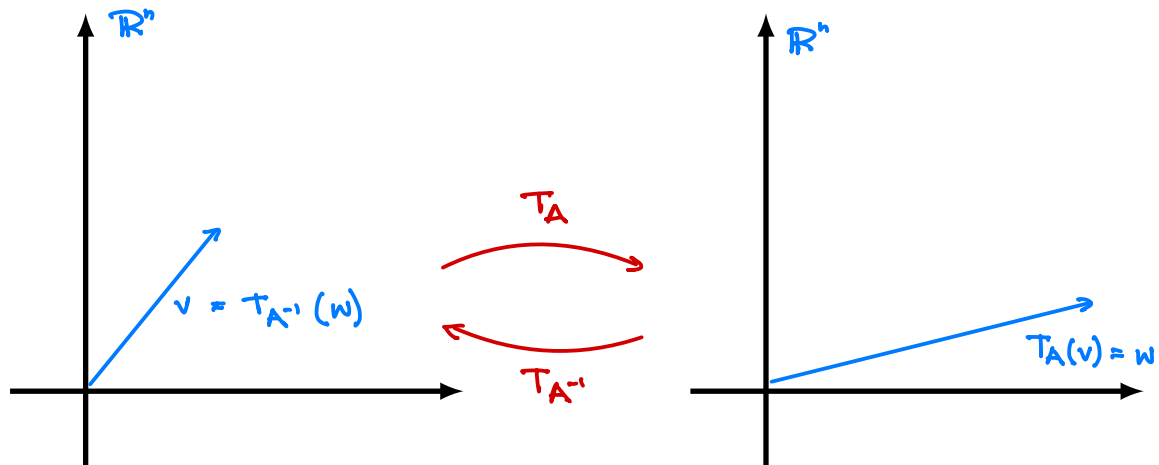
A - invertible $n \times n$ matrix

$$T_A: \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$v \mapsto Av$$

$$T_{A^{-1}}: \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$w \mapsto A^{-1}w$$

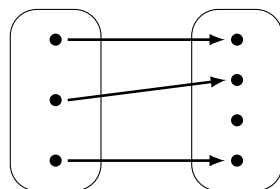


$$T_{A^{-1}}(T_A(v)) = T_{A^{-1}}(Av) = A^{-1}(Av) = (A^{-1}A)v = Iv = v$$

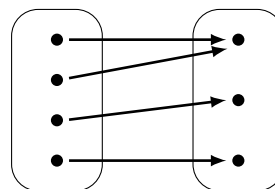
$$T_A(T_{A^{-1}}(w)) = \dots = w$$

Note

- If A is an $n \times n$ invertible matrix then the matrix transformation $T_A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ has the inverse function $T_{A^{-1}}: \mathbb{R}^n \rightarrow \mathbb{R}^n$.
- As a consequence the function T_A is both onto and one-to-one.



not onto

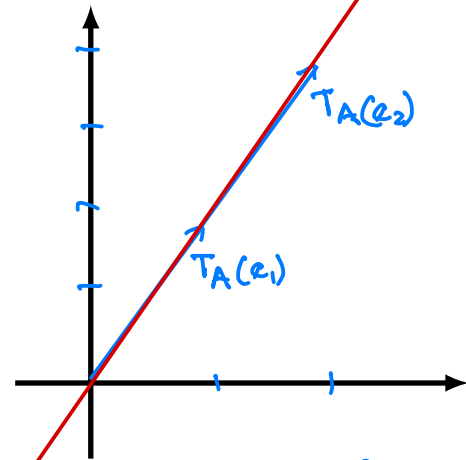
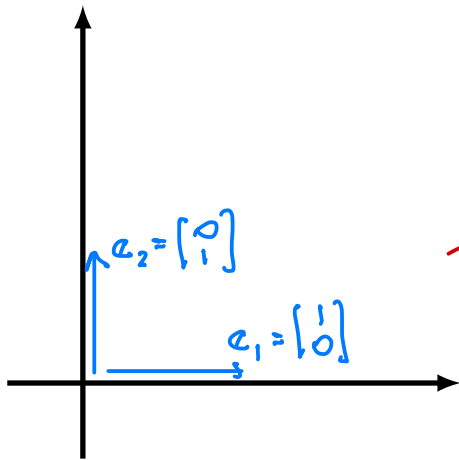


not one-to-one

Example.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$T_A: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$
$$v \mapsto Av$$



$\text{Col}(A)$ = the space
of all possible values
of T_A

$$T_A(e_1) = Ae_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$T_A(e_2) = Ae_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

- T_A is not onto, so A is not invertible
- T_A is also not one-to-one since e.g.
$$T_A\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = T_A\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$