**Recall:**

**1)** A square matrix $A$ is diagonalizable if there exists an invertible matrix $P$ and a diagonal matrix $D$ such that

$$A = PDP^{-1}$$

**2)** If $A$ is diagonalizable then it is easy to compute powers of $A$:

$$A^k = PD^kP^{-1}$$

**3)** An $n \times n$ matrix $A$ is a diagonalizable if and only if it has $n$ linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. In such case we have:

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} \qquad \begin{array}{l} \lambda_1 = \text{eigenvalue corresponding to } \mathbf{v}_1 \\ \lambda_2 = \text{eigenvalue corresponding to } \mathbf{v}_2 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \lambda_n = \text{eigenvalue corresponding to } \mathbf{v}_n \end{array}$$

**4)** Not every square matrix is diagonalizable.

**Definition**

An *orthogonal matrix* is square matrix $Q$ such that $Q^TQ = I$ (i.e. $Q^T = Q^{-1}$).

**Example.**

$$Q = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}$$

$$Q^TQ = \begin{bmatrix} 2/3 & 1/3 & 1/3 \\ -2/3 & 2/3 & 2/3 \\ 1/3 & 1/3 & -2/3 \end{bmatrix} \cdot \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 1/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Proposition**

A square matrix $Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{bmatrix}$ is an orthogonal matrix if and only if $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is an orthonormal set of vectors, i.e.:

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

**Note.** If $\mathbf{v}$, $\mathbf{w}$ are vectors in $\mathbb{R}^n$ then

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T\mathbf{w}$$

**Example.**

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$v^T w = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = v \cdot w$$

Proof of Proposition:

$$Q^TQ = \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix} \cdot \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} = \begin{bmatrix} u_1^Tu_1 & u_1^Tu_2 & \cdots & u_1^Tu_n \\ \vdots & & & \vdots \\ u_n^Tu_1 & u_n^Tu_2 & \cdots & u_n^Tu_n \end{bmatrix} = \begin{bmatrix} u_1 \cdot u_1 & u_1 \cdot u_2 & \cdots & u_1 \cdot u_n \\ \vdots & & & \\ u_n \cdot u_1 & u_n \cdot u_2 & \cdots & u_n \cdot u_n \end{bmatrix}$$

It follows that

$$Q^TQ = I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix} \quad \text{if and only if} \quad u_i \cdot u_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

## Definition

A square matrix $A$ is *orthogonally diagonalizable* if there exists an orthogonal matrix $Q$ and a diagonal matrix $D$ such that

$$A = QDQ^{-1} = QDQ^T$$

## Proposition

An $n \times n$ matrix $A$ is orthogonally diagonalizable if and only if it has $n$ orthogonal eigenvectors.

<u>Proof:</u> Let $v_1, \ldots, v_n$ - orthogonal eigenvectors of $A$.

Take: $u_1 = \frac{v_1}{\|v_1\|}, \ldots, u_n = \frac{v_n}{\|v_n\|}$.

Then $u_1, \ldots, u_n$ are also eigenvectors of $A$ and we have

$$u_i \cdot u_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

This means that $Q = [u_1 \ u_2 \cdots u_n]$ is an orthogonal matrix.

We have:

$$A = QDQ^{-1} = QDQ^T$$

where:

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$\lambda_1 =$ eigenvalue of $A$ corresponding to $u_1$

$\vdots$

$\lambda_n =$ eigenvalue of $A$ corresponding to $u_n$

$\blacksquare$

**Definition**

A square matrix $A$ is *symmetric* if $A^T = A$

**Example.**

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 5 & 6 \\ 3 & 5 & 7 & 8 \\ 4 & 6 & 8 & 9 \end{bmatrix}$$

**Proposition**

If a matrix $A$ is orthogonally diagonalizable then $A$ is a symmetric matrix.

Proof: If $A = QDQ^T$ then

$$A^T = (QDQ^T)^T = (Q^T)^T D^T Q^T = QDQ^T = A$$

$\blacksquare$

**Spectral Theorem**

Every symmetric matrix is orthogonally diagonalizable.

**Theorem**

If $A$ is a symmetric matrix and $\lambda_1, \lambda_2$ are two different eigenvalues of $A$, then eigenvectors corresponding to $\lambda_1$ are orthogonal to eigenvectors corresponding to $\lambda_2$.

**Recall:** If $\mathbf{v}$, $\mathbf{w}$ are vectors in $\mathbb{R}^n$ then

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w}$$

Proof of Theorem:

Let $v$ = eigenvector of $A$ corresponding to $\lambda_1$
$w$ = eigenvector of $A$ corresponding to $\lambda_2$

We have:

$$\lambda_1 (v \cdot w) = (Av) \cdot w = (Av)^T w = (v^T A^T) w \underset{\substack{\uparrow \\ A^T = A}}{=} v^T A w$$

$$= v^T (\lambda_2 w) = \lambda_2 (v^T w) = \lambda_2 (v \cdot w)$$

This gives:

$$\lambda_1 (v \cdot w) = \lambda_2 (v \cdot w)$$

$$(\lambda_1 - \lambda_2)(v \cdot w) = 0$$

Since $\lambda_1 \neq \lambda_2$ we have $\lambda_1 - \lambda_2 \neq 0$, so $v \cdot w = 0$.

**Example.**
Find three orthogonal eigenvectors of the following symmetric matrix:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Solution:

1) Find eigenvalues of A:

$$P(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{bmatrix}$$

$$= -\lambda^3 + 6\lambda^2 - 9\lambda + 4$$

$$(\text{eigenvalues of } A) = (\text{roots of } P(\lambda))$$

$$= (\lambda_1 = 4,\ \lambda_2 = 1)$$

2) Find a basis of the eigenspace for each eigenvalue:

$$\begin{pmatrix} \text{eigenspace} \\ \text{for } \lambda_1 = 4 \end{pmatrix} = \mathrm{Nul}(A - 4I) \qquad \text{basis: } \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\begin{pmatrix} \text{eigenspace} \\ \text{for } \lambda_2 = 1 \end{pmatrix} = \mathrm{Nul}(A - 1I) \qquad \text{basis: } \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Upshot: We have 3 lin. independent eigenvectors:

$$\underbrace{\overset{v_1}{\begin{bmatrix}1\\1\\1\end{bmatrix}}}_{\lambda_1=4},\ \underbrace{\overset{v_2}{\begin{bmatrix}-1\\0\\1\end{bmatrix}},\ \overset{v_3}{\begin{bmatrix}-1\\1\\0\end{bmatrix}}}_{\lambda_2=1}$$

Note: 1) $v_1$ is orthogonal to $v_2$ and $v_3$ (since it corresponds to a different eigenvalue).

2) $v_2, v_3$ are not orthogonal to each other: $v_1 \cdot v_2 = 1 \neq 0$.

To fix this, we need to use the find an orthogonal basis of the eigenspace of $\lambda_1 = 1$

$$w_2 = v_2 = \begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$$w_3 = w_3 - \left(\frac{w_2 \cdot v_3}{w_2 \cdot w_2}\right) w_2 = \begin{bmatrix}-1/2\\1\\-1/2\end{bmatrix}$$

We obtain 3 orthogonal eigenvectors:

$$\underbrace{\overset{v_1}{\begin{bmatrix}1\\1\\1\end{bmatrix}}}_{\lambda_1=4},\ \underbrace{\overset{w_2}{\begin{bmatrix}-1\\0\\1\end{bmatrix}},\ \overset{w_3}{\begin{bmatrix}-1/2\\1\\-1/2\end{bmatrix}}}_{\lambda_2=1}$$

**Upshot.** How to find $n$ orthogonal eigenvectors for a symmetric $n \times n$ matrix $A$:

**1)** Find eigenvalues of $A$.

**2)** Find a basis of the eigenspace for each eigenvalue.

**3)** Use the Gram-Schmidt process to find an orthogonal basis of each eigenspace.

**Example.** Find an orthogonal diagonalization of the following symmetric matrix:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

<u>Solution:</u>

The previous example gives a diagonalization of $A$:

$$A = PDP^{-1} \qquad P = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & -1 & -1/2 \\ 1 & 0 & 1 \\ 1 & 1 & -1/2 \end{bmatrix} \end{array} \qquad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is not an orthogonal diagonalization since $P$ is not an orthogonal matrix:

$$P^TP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3/2 \end{bmatrix}$$

To get an orthogonal matrix take $Q = \left[ \frac{v_1}{\|v_1\|} \ \frac{v_2}{\|v_2\|} \ \frac{v_3}{\|v_3\|} \right]$

$\|v_1\| = \sqrt{3}$ , $\|v_2\| = \sqrt{2}$ $\|v_3\| = \sqrt{3/2} = \frac{\sqrt{6}}{2}$

$$Q = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$$

We get:

$$A = QDQ^{-1} = QDQ^T$$

where is the same as before: $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.