

**Definition**

A *homogenous vector equation* is a vector equation of the form

$$x_1 \mathbf{v}_1 + \dots + x_p \mathbf{v}_p = \mathbf{0}$$

(i.e. with the zero vector as the vector of constants).

**Definition**

Let  $\mathbf{v}_1, \dots, \mathbf{v}_p \in \mathbb{R}^n$ . The set  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is *linearly independent* if the homogenous equation

$$x_1 \mathbf{v}_1 + \dots + x_p \mathbf{v}_p = \mathbf{0}$$

has only one, trivial solution  $x_1 = 0, \dots, x_p = 0$ . Otherwise the set is *linearly dependent*.

### Theorem

Let  $\mathbf{v}_1, \dots, \mathbf{v}_p \in \mathbb{R}^n$ . If the set  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is linearly independent then the equation

$$x_1\mathbf{v}_1 + \dots + x_p\mathbf{v}_p = \mathbf{w}$$

has exactly one solution for any vector  $\mathbf{w} \in \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_p)$ .

If the set is linearly dependent then this equation has infinitely many solutions for any  $\mathbf{w} \in \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_p)$ .

**Example.** Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ -12 \end{bmatrix}$$

Check if the set  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is linearly independent.

### Note

A set  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is linearly independent if and only if every column of the matrix

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_p \end{bmatrix}$$

is a pivot column.

### Some properties of linearly (in)dependent sets

1) A set consisting of one vector  $\{v_1\}$  is linearly dependent if and only if  $v_1 = 0$ .

2) A set consisting of two vectors  $\{v_1, v_2\}$  is linearly dependent if and only if one vector is a scalar multiple of the other.

3) If  $\{v_1, \dots, v_p\}$  is a set of  $p$  vectors in  $\mathbb{R}^n$  and  $p > n$  then this set is linearly dependent.

## Upshot: how to find the number of solutions of a vector equation

