Proposition

If $T: V \to W$ is a linear transformation then

1) T is onto if and only if $\mathrm{Im}(T) = W$

2) T is one-to-one if and only if $\mathrm{Ker}(T) = \{\mathbf{0}\}$.

Recall:

• A vector space is a set V equipped with operations of addition and multiplication by scalars. These operations must satisfy some properties.

• Some examples of vector spaces:

1) $\mathbb{R}^n$ = the vector space of column vectors.

2) $\mathcal{F}(\mathbb{R})$ = the vector space of all functions $f\colon \mathbb{R} \to \mathbb{R}$.

3) $C(\mathbb{R})$ = the vector space of all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$.

4) $C^\infty(\mathbb{R})$ = the vector space of all smooth functions $f\colon \mathbb{R} \to \mathbb{R}$.

5) $M_{m,n}(\mathbb{R})$ = the vector space of all $m \times n$ matrices.

6) $\mathbb{P}$ = the vector space of all polynomials.

7) $\mathbb{P}_n$ = the vector space of polynomials of degree $\leq n$.

• If V, W are vector spaces then a linear transformation is a function $T\colon V \to W$ such that

1) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

2) $T(c\mathbf{v}) = cT(\mathbf{v})$

• Many problems involving $\mathbb{R}^n$ can be easily solved using row reduction, matrix multiplication etc.

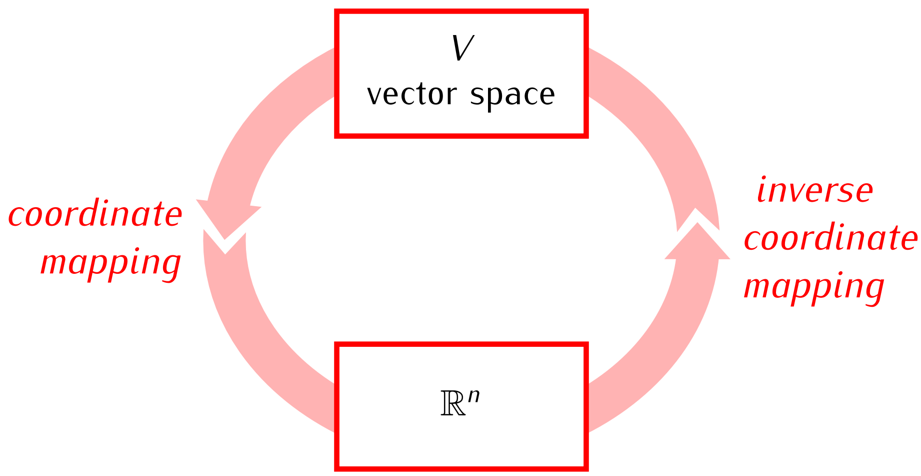
• The same types of problems involving other vector spaces can be difficult to solve.

Next goal:

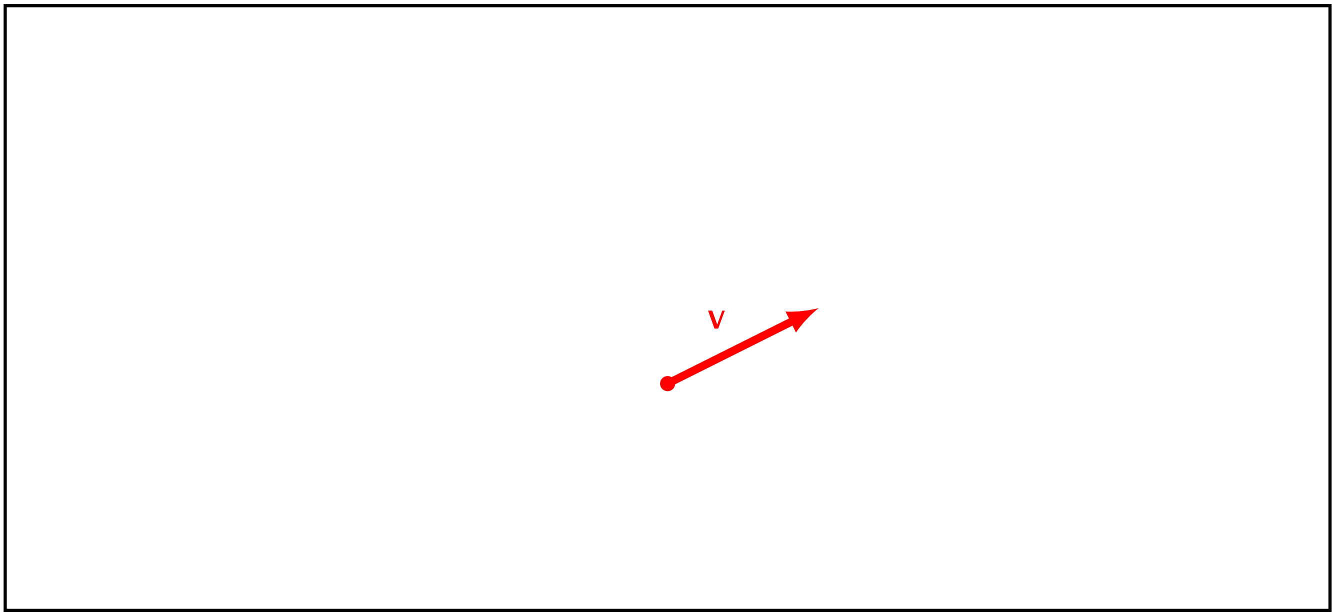
If V is a *finite dimensional* vector space then we can construct a *coordinate mapping*

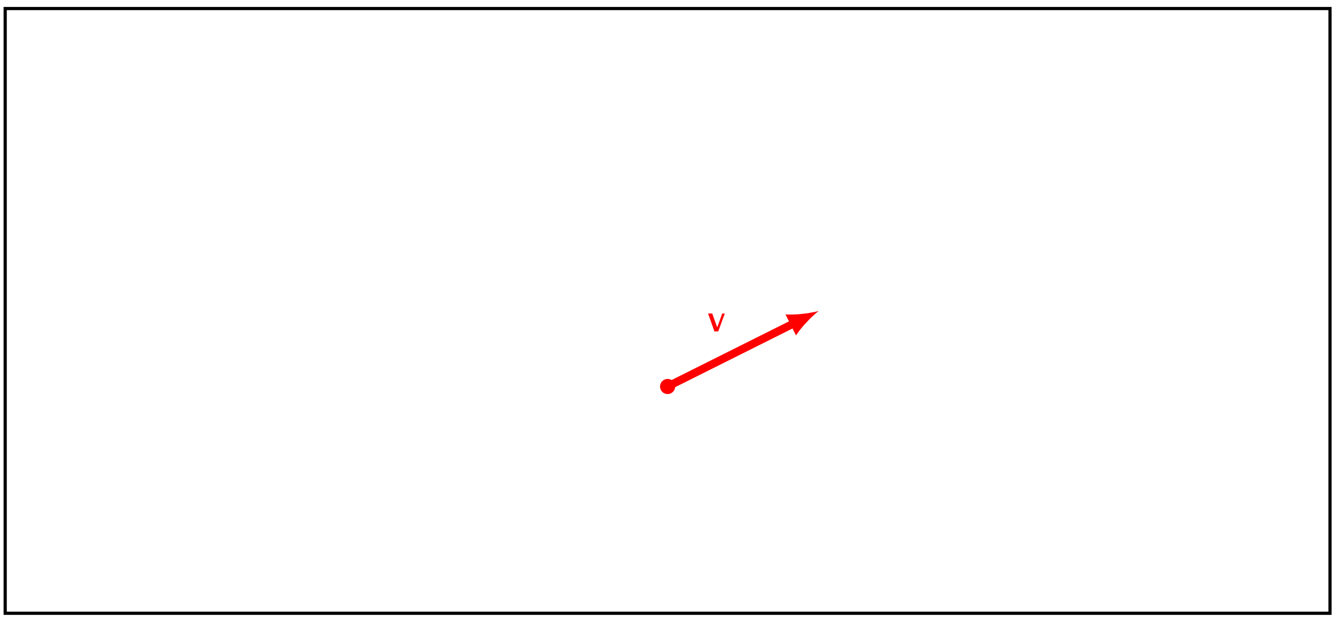
$$V \to \mathbb{R}^n$$

which lets us turn computations in V into computations in $\mathbb{R}^n$.



Motivation: How to assign coordinates to vectors





Definition

If V is a vector space then vector $\mathbf{w} \in V$ is a *linear combination* of vectors $\mathbf{v}_1, \ldots \mathbf{v}_p \in V$ if there exist scalars $c_1, \ldots, c_p$ such that

$$\mathbf{w} = c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p$$

Definition

If V is a vector space and $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are vectors in V then

$$\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p) = \left\{ \begin{array}{c} \text{the set of all} \\ \text{linear combinations} \\ c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p \end{array} \right\}$$

Definition

If V is a vector space and $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are vectors in V such that

$$V = \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p)$$

the the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is called the *spanning set* of V.

Example.

Definition

If V is a vector space and $\mathbf{v}_1, \ldots, \mathbf{v}_p \in V$ then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is *linearly independent* if the homogenous equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

has only one, trivial solution $x_1 = 0, \ldots, x_p = 0$. Otherwise the set is *linearly dependent*.

Theorem

Let V be a vector space, and let $\mathbf{v}_1, \ldots, \mathbf{v}_p \in V$. If the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent then the equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{w}$$

has exactly one solution for any vector $\mathbf{w} \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p)$.

Example.

Recall: $\mathcal{F}(\mathbb{R})$ = the vector space of all functions $f\colon \mathbb{R} \to \mathbb{R}$. Let $f, g, h \in \mathcal{F}(\mathbb{R})$ be the following functions:

$$f(t) = \sin(t), \quad g(t) = \cos(t), \quad h(t) = \cos^2(t)$$

Check if the set $\{f, g, h\}$ is linearly independent.

Example.

Let $f, g, h \in \mathcal{F}(\mathbb{R})$ be the following functions:

$$f(t) = \sin^2(t), \quad g(t) = \cos^2(t), \quad h(t) = \cos 2t$$

Check if the set $\{f, g, h\}$ is linearly independent.

Definition

A *basis* of a vector space V is an ordered set of vectors

$$\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$$

such that

1) $\mathrm{Span}(\mathbf{b}_1, \ldots, \mathbf{b}_n) = V$

2) The set $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is linearly independent.

Theorem

A set $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis of a vector space V if any only if for each $\mathbf{v} \in V$ the vector equation

$$x_1\mathbf{b}_1 + \ldots + x_n\mathbf{b}_n = \mathbf{v}$$

has a unique solution.

Definition

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis of a vector space V. For $\mathbf{v} \in V$ let $c_1, \ldots, c_n$ be the unique numbers such that

$$c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n = \mathbf{v}$$

Then the vector

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$$

is called the *coordinate vector of* $\mathbf{v}$ *relative to the basis* $\mathcal{B}$ and it is denoted by $[\mathbf{v}]_{\mathcal{B}}$.