

Definition

Let V, W be vector spaces A *linear transformation* is a function

$$T: V \rightarrow W$$

which satisfies the following conditions:

- 1) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$
- 2) $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $\mathbf{v} \in V$ and any scalar c .

Note. If $T: V \rightarrow W$ is a linear transformation then for any vector $\mathbf{b} \in W$ we can consider the equation

$$T(\mathbf{x}) = \mathbf{b}$$

Definition

If $T: V \rightarrow W$ is a linear transformation then:

1) The *kernel* of T is the set

$$\text{Ker}(T) = \{v \in V \mid T(v) = 0\}$$

2) The *image* of T is the set

$$\text{Im}(T) = \{w \in W \mid w = T(v) \text{ for some } v \in V\}$$

Proposition

If $T: V \rightarrow W$ is a linear transformation then:

- 1) $\text{Ker}(T)$ is a subspace of V
- 2) $\text{Im}(T)$ is a subspace of W

Theorem

If $T: V \rightarrow W$ is a linear transformation and v_0 is a solution of the equation

$$T(\mathbf{x}) = \mathbf{b}$$

then all other solutions of this equation are vectors of the form

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{n}$$

where $\mathbf{n} \in \text{Ker}(T)$.

Example.

$$\begin{aligned} D: C^\infty(\mathbb{R}) &\longrightarrow C^\infty(\mathbb{R}) \\ f &\longmapsto f' \end{aligned}$$