Recall:

1) A square matrix A is diagonalizable if there exists an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}$$

2) If A is diagonalizable then it is easy to compute powers of A:

$$A^k = PD^kP^{-1}$$

3) An $n \times n$ matrix A is a diagonalizable if and only if it has n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$. In such case we have:

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \qquad \begin{array}{l} \lambda_1 = \text{eigenvalue corresponding to } \mathbf{v}_1 \\ \lambda_2 = \text{eigenvalue corresponding to } \mathbf{v}_2 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \lambda_n = \text{eigenvalue corresponding to } \mathbf{v}_n \end{array}$$

4) Not every square matrix is diagonalizable.

Definition

A square matrix A is *symmetric* if $A^T = A$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 5 & 6 \\ 3 & 5 & 7 & 8 \\ 4 & 6 & 8 & 9 \end{bmatrix}$$

Theorem

Every symmetric matrix is diagonalizable.

Theorem

If A is a symmetric matrix and λ_1, λ_2 are two different eigenvalues of A, then eigenvectors corresponding to λ_1 are orthogonal to eigenvectors corresponding to λ_2.

Note. If $\mathbf{v}$, $\mathbf{w}$ are vectors in $\mathbb{R}^n$ then

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w}$$

Example.

$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

Theorem

If A is an $n \times n$ symmetric matrix then A has n orthogonal eigenvectors.

Example.

a) Find three orthogonal eigenvectors of the following symmetric matrix:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

b) Use these eigenvectors to diagonalize this matrix.

Upshot. How to find n orthogonal eigenvectors for a symmetric $n \times n$ matrix A:

1) Find eigenvalues of A.

2) Find a basis of the eigenspace for each eigenvalue.

3) Use the Gram-Schmidt process to find an orthogonal basis of each eigenspace.

Definition

A square matrix $Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{bmatrix}$ is an *orthogonal matrix* if $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is an orthonormal set of vectors, i.e.:

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Theorem

If Q is an orthogonal matrix then Q is invertible and $Q^{-1} = Q^T$.

Note. If $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}$ is a matrix with orthogonal columns, then

$$Q = \begin{bmatrix} \frac{\mathbf{v}_1}{||\mathbf{v}_1||} & \frac{\mathbf{v}_2}{||\mathbf{v}_2||} & \dots & \frac{\mathbf{v}_n}{||\mathbf{v}_n||} \end{bmatrix}$$

is an orthogonal matrix.

Theorem

If A is a symmetric matrix then A is *orthogonally diagonalizable*. That is, there exists an orthogonal matrix Q and a diagonal matrix D such that

$$A = QDQ^{-1} = QDQ^T$$

Example. Find an orthogonal diagonalization of the following symmetric matrix:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Note. We have seen that any symmetric matrix is orthogonally diagonalizable. The converse statement is also true:

Proposition

If a matrix A is orthogonally diagonalizable then A is a symmetric matrix.