Recall: If A is square matrix then the *ij-cofactor of A* is the number

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Definition

If A is an $n \times n$ matrix then the *adjoint* (or *adjugate*) of A is the matrix

$$\mathrm{adj}A = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Theorem

If A is an invertible matrix then

$$A^{-1} = \frac{1}{\det A} \cdot \mathrm{adj}A$$

Idea of proof: We need to show: $A \cdot \left(\frac{1}{\det A} \mathrm{adj} A\right) = I$

or equivalently: $A \cdot \mathrm{adj} A = \det A \cdot I = \begin{bmatrix} \det A & & 0 \\ & \det A \ddots & \\ 0 & & \det A \end{bmatrix}$

$$A \cdot \mathrm{adj} A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ C_{12} & & C_{n2} \\ \vdots & & \\ C_{1n} & & C_{nn} \end{bmatrix}$$

e.g: the entry in the 1^{st} row, 1^{st} column of $A \cdot \mathrm{adj} A$ is:

$$[a_{11}\ a_{12} \cdots a_{1n}] \cdot \begin{bmatrix} C_{11} \\ C_{12} \\ \vdots \\ C_{1n} \end{bmatrix} = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n} = \det A$$

cofactor expansion across the 1^{st} row

Example. Compute A^{-1} for

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \cdot \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

cofactor exp. across 2^{nd} row

$$\det A = \det \begin{bmatrix} 1 & 1 & 2 \\ 4 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} = (-1)^{2+1} \cdot 4 \cdot \det \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= (-1) \cdot 4 \cdot (1 \cdot 1 - 2 \cdot 1) = 4 /\!/$$

$$C_{11} = (-1)^{1+1} \cdot \det \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = 0$$

$$C_{12} = (-1)^{1+2} \cdot \det \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix} = -4$$

$$C_{13} = (-1)^{1+3} \cdot \det \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix} = 4$$

$$C_{21} = (-1)^{2+1} \cdot \det \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = 1$$

$$\vdots$$

This gives:

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 0 & 1 & \cdots \\ -4 & \cdots & \cdots \\ 4 & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} 0 & 1/4 & \cdots \\ -1 & \cdots & \cdots \\ 1 & \cdots & \cdots \end{bmatrix}$$

Recall: If A is an invertible matrix then the equation $A\mathbf{x} = \mathbf{b}$ has only one solution: $\mathbf{x} = A^{-1}\mathbf{b}$.

Definition

If A is an $n \times n$ matrix and $\mathbf{b} \in \mathbb{R}^n$ then $A_i(\mathbf{b})$ is the matrix obtained by replacing the i^{th} column of A with $\mathbf{b}$.

Example.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$$

$$A_1(b) = \begin{bmatrix} 10 & 2 & 3 \\ 20 & 5 & 6 \\ 30 & 8 & 9 \end{bmatrix} \qquad A_2(b) = \begin{bmatrix} 1 & 10 & 3 \\ 4 & 20 & 6 \\ 7 & 30 & 9 \end{bmatrix}$$

Theorem (Cramer's Rule)

If A is an $n \times n$ invertible matrix and $\mathbf{b} \in \mathbb{R}^n$ then the unique solution of the equation

$$A\mathbf{x} = \mathbf{b}$$

is given by

$$\mathbf{x} = \frac{1}{\det A} \begin{bmatrix} \det A_1(\mathbf{b}) \\ \vdots \\ \det A_n(\mathbf{b}) \end{bmatrix}$$

Proof: We have: $x = A^{-1}b$

Then use the determinant formula for A^{-1}.

Example. Solve the equation

$$\begin{bmatrix} 1 & 1 & 2 \\ 4 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

We have:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{\det A} \cdot \begin{bmatrix} \det A_1(b) \\ \det A_2(b) \\ \det A_3(b) \end{bmatrix}$$

$$\det A = 4$$

$$\det A_1(b) = \det \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix} = 2$$

$$\det A_2(b) = \det \begin{bmatrix} 1 & 1 & 2 \\ 4 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix} = 18$$

$$\det A_3(b) = \det \begin{bmatrix} 1 & 1 & 1 \\ 4 & 0 & 2 \\ 1 & 1 & 3 \end{bmatrix} = -8$$

We obtain

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 \\ 18 \\ -8 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/9 \\ -2 \end{bmatrix}$$