Recall:

1) The dot product in $\mathbb{R}^n$:

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \ldots a_n b_n$$

2) Properties of the dot product:

a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

c) $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$

d) $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

2) Using the dot product we can define:

- length of vectors
- distance between vectors
- orthogonality of vectors
- orthogonal and orthonormal bases
- orthogonal projection of a vector onto a subspace of $\mathbb{R}^n$
- ...

Next: Generalization to arbitrary vector spaces.

Definition

Let V be a vector space. An *inner product* on V is a function

$$\begin{array}{rcl} V \times V & \longrightarrow & \mathbb{R} \\ \mathbf{u}, \ \ \mathbf{v} & \longmapsto & \langle \mathbf{u}, \mathbf{v} \rangle \end{array}$$

such that:

a) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$

b) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

c) $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$

d) $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Definition

Let V be a vector space with an inner product $\langle \ , \ \rangle$.

1) The *length* (or *norm*) of a vector $\mathbf{v}$ is the number

$$||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

2) The *distance* between vectors $\mathbf{u}, \mathbf{v} \in V$ is the number

$$\mathrm{dist}(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$$

3) Vectors $\mathbf{u}, \mathbf{v} \in V$ are *orthogonal* if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Example. The dot product is an inner product in $\mathbb{R}^n$.

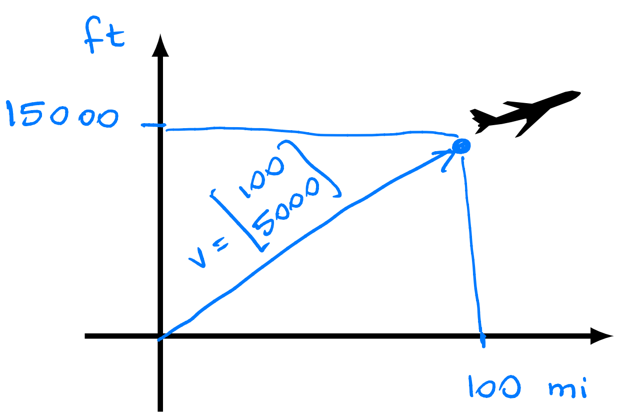
Example. Let $p_1, \dots, p_n$ be any positive numbers. For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$

$$\mathbf{u} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

define:

$$\langle \mathbf{u}, \mathbf{v} \rangle = p_1(a_1 b_1) + p_2(a_2, b_2) + \dots + p_n(a_n b_n)$$

This gives an inner product in $\mathbb{R}^n$.



$1 \text{ ft} = 0.0002 \text{ mi}$

(denote : $\delta = 0.0002$)

length of $v = \sqrt{(100)^2 + (\delta \cdot 15000)^2}$

$= \sqrt{(100)^2 + \delta^2 \cdot (15000)^2}$ miles

<u>In $\mathbb{R}^2$ define</u>

$$\left\langle \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right\rangle = (a_1 b_1) + \delta^2 \cdot (a_2 \cdot b_2)$$

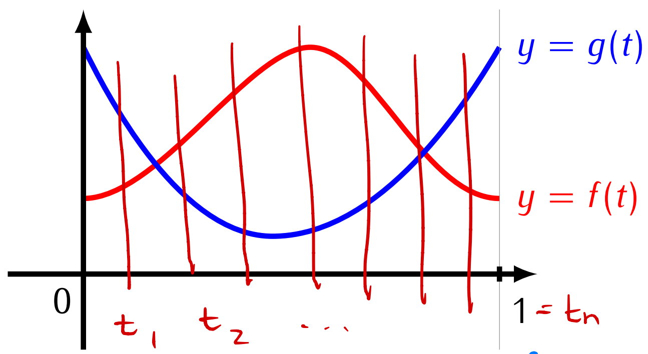
Then:

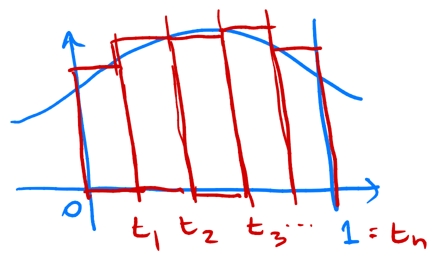
$$\|v\| = \sqrt{\langle v, v \rangle} = \sqrt{a_1^2 + \delta^2 a_2^2}$$

Example. Let $C[0,1]$ be the vector space of continuous functions $f\colon [0,1] \to \mathbb{R}$. Define:

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt$$

This is an inner product on $C[0,1]$.





$$\int_0^1 f(t)\,dt \approx \frac{1}{n}\cdot f(t_1) + \dots + \frac{1}{n} f(t_n)$$
$$= \frac{1}{n}\left(f(t_1) + \dots + f(t_n)\right)$$

<u>So:</u>

$$\langle f, g\rangle \approx \frac{1}{n}\left(f(t_1)g(t_1) + f(t_2)g(t_2) + \dots + f(t_n)g(t_n)\right)$$

Example. Compute the length of the function

$$f(t) = 1 + t^2$$

in $C[0,1]$.

<u>Solution</u>: $\|f\| = \sqrt{\langle f, f\rangle}$

$$\langle f, f\rangle = \int_0^1 f(t)\cdot f(t)\,dt = \int_0^1 (1+t^2)^2 dt = \int_0^1 1 + 2t^2 + t^4\,dt$$
$$= \left(t + \frac{2}{3}t^2 + \frac{1}{5}t^5\right)\Big|_{t=0}^{t=1} = \frac{28}{15}$$

<u>So:</u> $\|f\| = \sqrt{28/15}$

Definition

Let V be a vector space with an inner product $\langle\ ,\ \rangle$, and let W be a subspace of V. A vector $\mathbf{v} \in V$ is *orthogonal to* W if $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in W$.

Definition

Let V be a vector space with an inner product $\langle\ ,\ \rangle$, and let W be a subspace of V. The *orthogonal projection of a vector* $\mathbf{v} \in V$ *onto* W is a vector $\mathrm{proj}_W \mathbf{v}$ such that

1) $\mathrm{proj}_W \mathbf{v} \in W$
2) the vector $\mathbf{z} = \mathbf{v} - \mathrm{proj}_W \mathbf{v}$ is orthogonal to W.

Best Approximation Theorem

If V is a vector space with an inner product $\langle\ ,\ \rangle$, W is a subspace of V, and $\mathbf{v} \in V$, then $\mathrm{proj}_W \mathbf{v}$ is the vector of V which is the closest to $\mathbf{v}$:

$$\mathrm{dist}(\mathbf{v}, \mathrm{proj}_W \mathbf{v}) \leq \mathrm{dist}(\mathbf{v}, \mathbf{w})$$

for all $\mathbf{w} \in W$.

Theorem

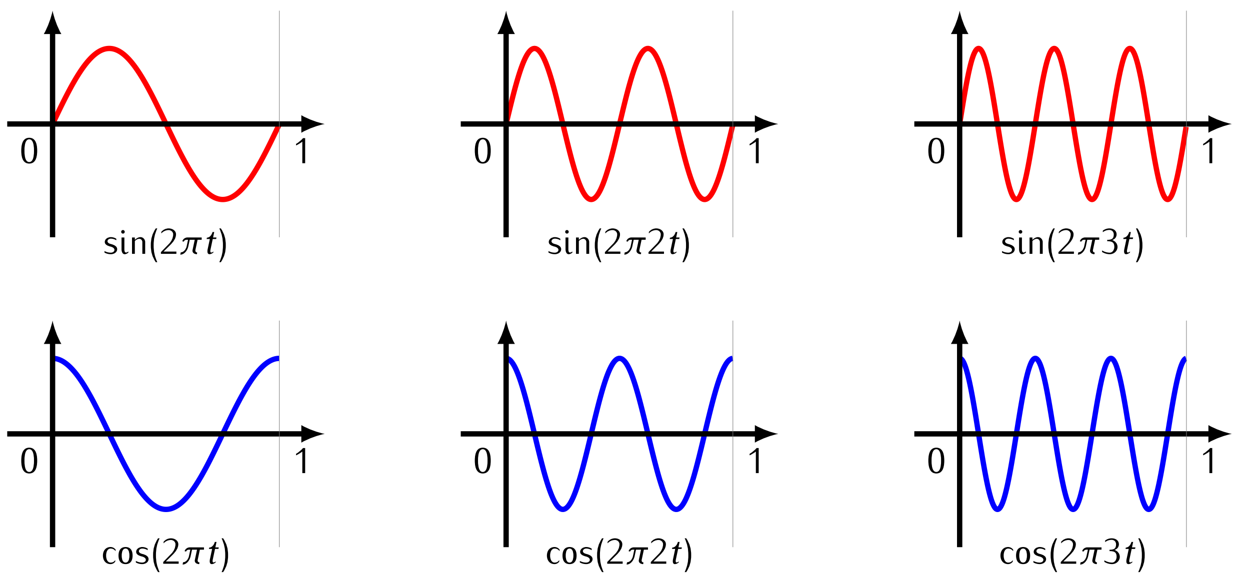
Let V is a vector space with an inner product $\langle\ ,\ \rangle$, and let W be a subspace of V. If $\mathcal{B} = \{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is an orthogonal basis of W (i.e. a basis such that $\langle \mathbf{w}_i, \mathbf{w}_j \rangle = 0$ for all $i \neq j$) then for $\mathbf{v} \in V$ we have:

$$\mathrm{proj}_W \mathbf{v} = \frac{\langle \mathbf{v}, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} \mathbf{w}_1 + \ldots + \frac{\langle \mathbf{v}, \mathbf{w}_k \rangle}{\langle \mathbf{w}_k, \mathbf{w}_k \rangle} \mathbf{w}_k$$

<u>**Application:**</u> Fourier approximations.

Goal: Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. Find the best possible approximation of f of the form

$$
\begin{aligned}
P(t) = \quad & a_0 \\
& + a_1 \sin(2\pi t) \quad + \ b_1 \cos(2\pi t) \\
& + a_2 \sin(2\pi 2t) \ + \ b_2 \cos(2\pi 2t) \\
& \dots \ \dots \ \dots \quad \dots \ \dots \ \dots \ \dots \\
& + a_n \sin(2\pi n t) \ + \ b_n \cos(2\pi n t)
\end{aligned}
$$



Note: Let W_n be a subspace of $C[0,1]$ given by:

$$W_n = \operatorname{Span}(1, \sin(2\pi t), \cos(2\pi t), \dots, \sin(2\pi n t), \cos(2\pi n t))$$

By the Best Approximation Theorem, the best approximation of f is obtained if we take $P(t) = \operatorname{proj}_{W_n} f(t)$.

Theorem

The set

$$\{1, \sin(2\pi t), \cos(2\pi t), \dots, \sin(2\pi n t), \cos(2\pi n t)\}$$

is an orthogonal basis of W_n.

Corollary

If $f \in C[0,1]$ then

$$\begin{aligned} \text{proj}_{W_n} f(t) = \quad & a_0 \\ & + a_1 \sin(2\pi t) \quad + \quad b_1 \cos(2\pi t) \\ & + a_2 \sin(2\pi 2t) \; + \; b_2 \cos(2\pi 2t) \\ & \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ & + a_n \sin(2\pi n t) \; + \; b_n \cos(2\pi n t) \end{aligned}$$

where:

$$a_0 = \frac{\langle f, 1\rangle}{\langle 1, 1\rangle} = \int_0^1 f(t)dt$$

and for $k > 0$:

$$a_k = \frac{\langle f, \sin(2\pi k t)\rangle}{\langle \sin(2\pi k t), \sin(2\pi k t)\rangle} = 2\int_0^1 f(t) \cdot \sin(2\pi k t)dt$$

$$b_k = \frac{\langle f, \cos(2\pi k t)\rangle}{\langle \cos(2\pi k t), \cos(2\pi k t)\rangle} = 2\int_0^1 f(t) \cdot \cos(2\pi k t)dt$$

Example. Compute $\mathrm{proj}_{W_n} f(t)$ for the function $f(t) = t$.

Solution:

$$a_0 = \int_0^1 1 \cdot f(t)\,dt = \int_0^1 t\,dt = \frac{1}{2}t^2 \Big|_{t=0}^{t=1} = 1/2$$

$$a_k = 2\int_0^1 f(t)\sin(2\pi k t)\,dt = 2\int_0^1 t\sin(2\pi k t)\,dt = -\frac{1}{\pi k}$$

$$b_k = 2\int_0^1 f(t)\cdot\cos(2\pi k t)\,dt = 2\int_0^1 t\cos(2\pi k t)\,dt = 0$$

This gives

$$\begin{aligned}\mathrm{proj}_{W_n} f(t) = \frac{1}{2} &- \frac{1}{\pi}\sin(2\pi t) \\ &- \frac{1}{2\pi}\sin(2\pi 2t) \\ &- \frac{1}{3\pi}\sin(2\pi 3t) \\ &\;\vdots \\ &- \frac{1}{n\pi}\sin(2\pi n t)\end{aligned}$$

Application: Polynomial approximations.

Goal: Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. Find the best possible approximation of f given by a polynomial $P(t)$ of degree $\leq n$:

$$P(t) = a_0 + a_1 t + \ldots + a_n t^n$$

Note: Let $\mathbb{P}_n$ be the subspace of $C[0,1]$ consisting of all polynomials of degree $\leq n$:

$$\mathbb{P}_n = \{a_0 + a_1 t + \ldots + a_n t^n \mid a_k \in \mathbb{R}\}$$

By the Best Approximation Theorem, the best approximation of f is obtained if we take $P(t) = \mathrm{proj}_{\mathbb{P}_n} f(t)$.

Note: In order to compute $\mathrm{proj}_{\mathbb{P}_n} f(t)$ we need an orthogonal basis of $\mathbb{P}_n$.

$\mathbb{P}_n$ has the standard basis $\mathcal{E} = \{1, t, \ldots, t^n\}$ but this basis is not orthogonal:

e.g.:

$$\langle t, t^2 \rangle = \int_0^1 t \cdot t^2 \, dt = \int_0^1 t^3 \, dt = \frac{1}{4} t^4 \Big|_{t=0}^{t=1} = \frac{1}{4} \neq 0$$

Gram-Schmidt process:

a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of $W \subseteq V$ $\xrightarrow{\text{G-S process}}$ an orthogonal basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ of W

Theorem (Gram-Schmidt Process)

Let V be a vector space with an inner product $\langle\ ,\ \rangle$, and let W be a subspace of V. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a basis of W. Define vectors $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ as follows:

$$\mathbf{w}_1 = \mathbf{v}_1$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{w}_1, \mathbf{v}_2 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{w}_1, \mathbf{v}_3 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \frac{\langle \mathbf{w}_2, \mathbf{v}_3 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\mathbf{w}_k = \mathbf{v}_k - \frac{\langle \mathbf{w}_1, \mathbf{v}_k \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \frac{\langle \mathbf{w}_2, \mathbf{v}_k \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2 - \ldots - \frac{\langle \mathbf{w}_{k-1}, \mathbf{v}_k \rangle}{\langle \mathbf{w}_{k-1}, \mathbf{w}_{k-1} \rangle}\mathbf{w}_{k-1}$$

Then the set $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is an orthogonal basis of W.

Example. Find an orthogonal basis of the subspace $\mathbb{P}_2$ of the vector space $C[0,1]$.

Solution: The standard basis of $\mathbb{P}_2$: $\mathcal{E} = \{\overset{v_1}{1}, \overset{v_2}{t}, \overset{v_3}{t^2}\}$

Use the Gram-Schmidt process to get an orthogonal basis $\{w_1, w_2, w_3\}$:

$$\boxed{w_1 = v_1 = 1}$$

$$w_2 = v_2 - \frac{\langle w_1, v_2\rangle}{\langle w_1, w_2\rangle} w_1$$

$$\langle w_1, v_2\rangle = \int_0^1 1\cdot t\,dt = \frac{1}{2}t^2\Big|_{t=0}^{t=1} = \frac{1}{2}$$

$$\langle w_1, w_1\rangle = \int_0^1 1\cdot 1\,dt = t\Big|_{t=0}^{t=1} = 1$$

So:

$$\boxed{w_2 = t - \frac{1/2}{1}\cdot 1 = t - \frac{1}{2}}$$

$$w_3 = v_3 - \frac{\langle w_1, v_3\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle w_2, v_3\rangle}{\langle w_2, w_2\rangle} w_2$$

$$\langle w_1, v_3\rangle = \int_0^1 1\cdot t^2\,dt = \frac{1}{3}t^3\Big|_{t=0}^{t=1} = \frac{1}{3}$$

$$\langle w_1, w_1\rangle = 1$$

$$\langle w_2, v_3\rangle = \int_0^1 \left(t - \frac{1}{2}\right)\cdot t^2\,dt = \int_0^1 t^3 - \frac{1}{2}t^2\,dt = \frac{1}{4}t^4 - \frac{1}{6}t^3\Big|_{t=0}^{t=1} = \frac{1}{12}$$

$$\langle w_2, w_2\rangle = \int_0^1 \left(t - \frac{1}{2}\right)^2 dt = \int_0^1 t^2 - t + \frac{1}{4}\,dt = \frac{1}{3}t^3 - \frac{1}{2}t^2 + \frac{1}{4}t\Big|_{t=0}^{t=1} = \frac{1}{12}$$

$$\boxed{w_3 = t^2 - \frac{1/3}{1}\cdot 1 - \frac{1/12}{1/12}\left(t - \frac{1}{2}\right) = t^2 - t + \frac{1}{6}}$$

An orthogonal basis of $\mathbb{P}_2$: $\left\{1, \left(t - \frac{1}{2}\right), \left(t^2 - t + \frac{1}{6}\right)\right\}$

Example. Compute $\text{proj}_{\mathbb{P}_2} f(t)$ for $f(t) = \sqrt{t}$.

Solution

We had: $\{\overset{w_1}{1}, \overset{w_2}{(t-\frac{1}{2})}, \overset{w_3}{(t^2-t+\frac{1}{6})}\}$ - an orthogonal basis of $\mathbb{P}_2$.

This gives:

$$\text{proj}_{\mathbb{P}_2} f(t) = \frac{\langle w_1, f\rangle}{\langle w_1, w_1\rangle}\cdot w_1 + \frac{\langle w_2, f\rangle}{\langle w_2, w_2\rangle}\cdot w_2 + \frac{\langle w_3, f\rangle}{\langle w_3, w_3\rangle}\cdot w_3$$

We had:

$$\langle w_1, w_1\rangle = 1$$

$$\langle w_2, w_2\rangle = 1/12$$

$$\langle w_3, w_3\rangle = \int_0^1 (t^2 - t + \tfrac{1}{6})^2 dt = \frac{1}{180}$$

$$\langle w_1, f\rangle = \int_0^1 1\cdot\sqrt{t}\, dt = \frac{2}{3}$$

$$\langle w_2, f\rangle = \int_0^1 (t-1/2)\cdot\sqrt{t}\, dt = \frac{1}{15}$$

$$\langle w_3, f\rangle = \int_0^1 (t^2-t+1/6)\sqrt{t}\, dt = -\frac{1}{315}$$

We obtain:

$$\text{proj}_{\mathbb{P}_2} f(t) = \frac{2/3}{1}\cdot 1 + \frac{1/15}{1/12}(t-\tfrac{1}{2}) + \frac{(-1/315)}{1/180}(t^2-t+\tfrac{1}{6})$$

$$= -\frac{4}{7}t^2 + \frac{48}{35}t + \frac{6}{35}$$