Definition

If

$$\mathbf{u} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

are vectors in $\mathbb{R}^n$ then the *inner product* (or *dot product*) of $\mathbf{u}$ and $\mathbf{v}$ is the number

$$\mathbf{u} \cdot \mathbf{v} = a_1 b_1 + \ldots + a_n b_n$$

Example :

$$u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad v = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$u \cdot v = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 4 + 10 + 18 = 32$$

Properties of the dot product:

1) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
3) $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$
4) $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Definition

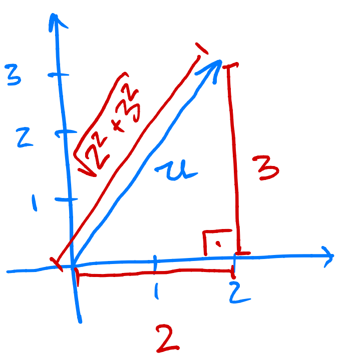
If $\mathbf{u} \in \mathbb{R}^n$ then the *length* (or the *norm*) of $\mathbf{u}$ is the number

$$||\mathbf{u}|| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$$

Note. If $\mathbf{u} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ then $||\mathbf{u}|| = \sqrt{a_1^2 + \ldots + a_n^2}$.

Example:

$u = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \in \mathbb{R}^2$ $\qquad ||u|| = \sqrt{2^2 + 3^2} = \sqrt{4+9} = \sqrt{13}$



Properties of the norm:

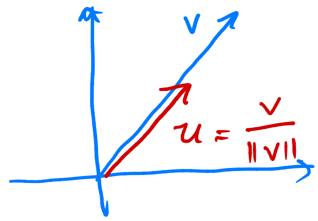
1) $||u|| \geq 0$ and $||u|| = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

2) $||cu|| = |c| \cdot ||u||$

Definition

A vector $\mathbf{u} \in \mathbb{R}^n$ is an *unit vector* if $||\mathbf{u}|| = 1$.

Note: If $v \in \mathbb{R}^n$, $v \neq 0$ then $u = \frac{1}{||v||} \cdot v$ is the unit vector pointing in the same direction as v

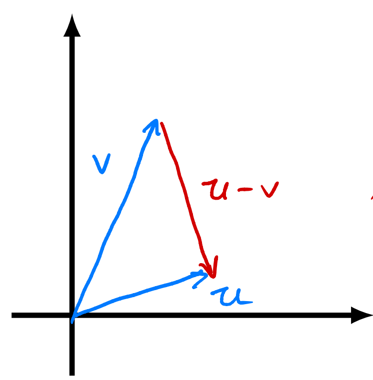


$||u|| = \frac{1}{||v||} \cdot ||v|| = 1$

Definition

If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ then the *distance* between $\mathbf{u}$ and $\mathbf{v}$ is the number

$$\text{dist}(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$$



$\text{dist}(u, v) = ||u - v||$

Note. If $\mathbf{u} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ then

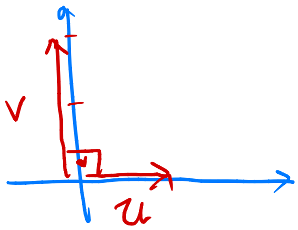
$$\text{dist}(\mathbf{u}, \mathbf{v}) = \sqrt{(a_1 - b_1)^2 + \ldots + (a_n - b_n)^2}$$

Definition

Vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are *orthogonal* if $\mathbf{u} \cdot \mathbf{v} = 0$.

Example:

$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$



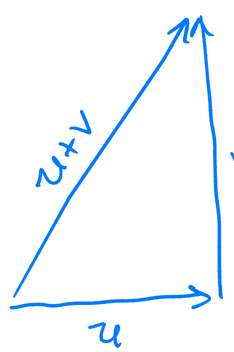
$u \cdot v = 1 \cdot 0 + 0 \cdot 2 = 0$

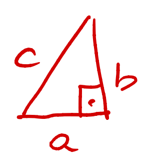
so u, v are orthogonal.

Pythagorean Theorem

Vectors $\mathbf{u}, \mathbf{v}$ are orthogonal if and only if

$$||\mathbf{u}||^2 + ||\mathbf{v}||^2 = ||\mathbf{u} + \mathbf{v}||^2$$





$a^2 + b^2 = c^2$

Proof:

$$\begin{aligned} \|u+v\|^2 &= (u+v) \cdot (u+v) \\ &= u \cdot u + 2(u \cdot v) + v \cdot v \\ &= \|u\|^2 + 2(u \cdot v) + \|v\|^2 \end{aligned}$$

This gives: $\|u+v\|^2 = \|u\|^2 + \|v\|^2$

if and only if $u \cdot v = 0$.