

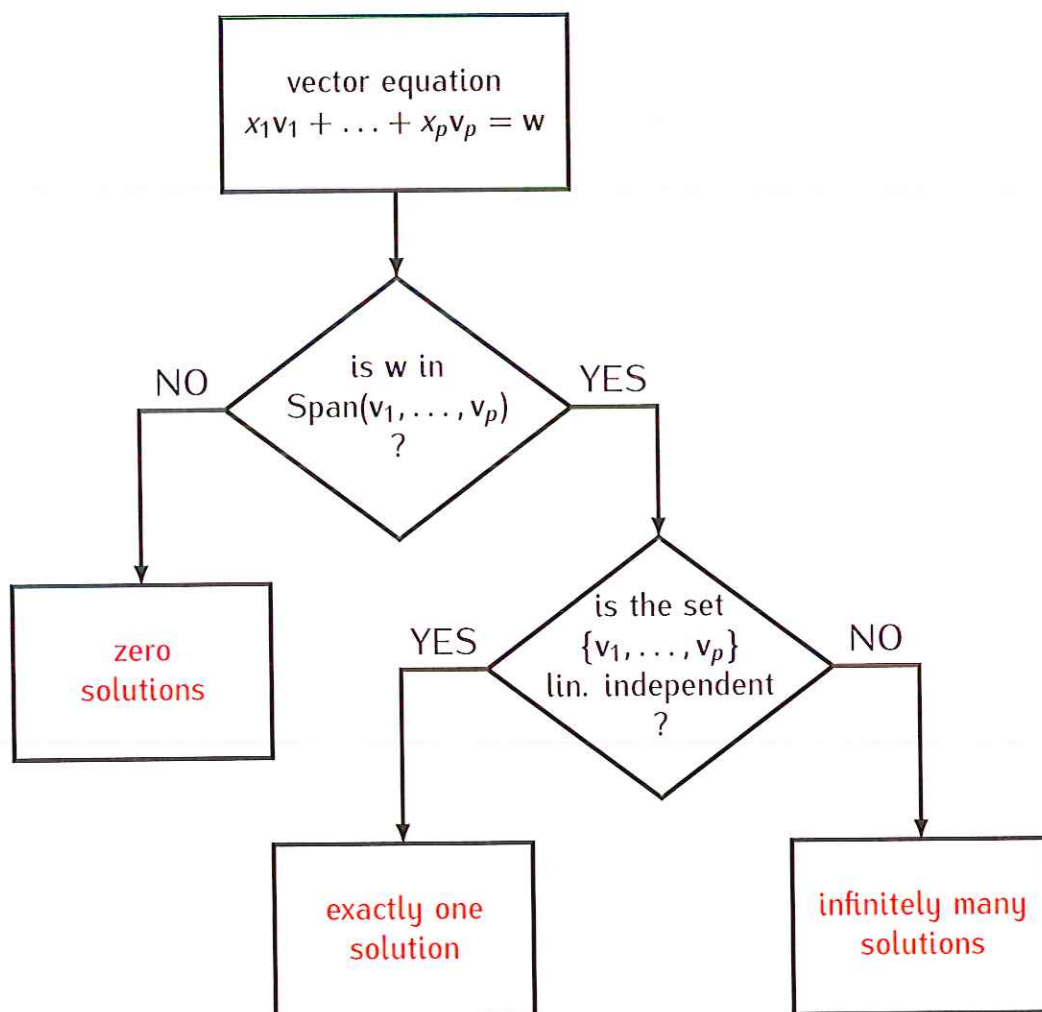
Recall:

$$1) \text{Span}(v_1, \dots, v_p) = \left\{ \begin{array}{l} \text{the set of all} \\ \text{linear combinations} \\ c_1 v_1 + \dots + c_p v_p \end{array} \right\}$$

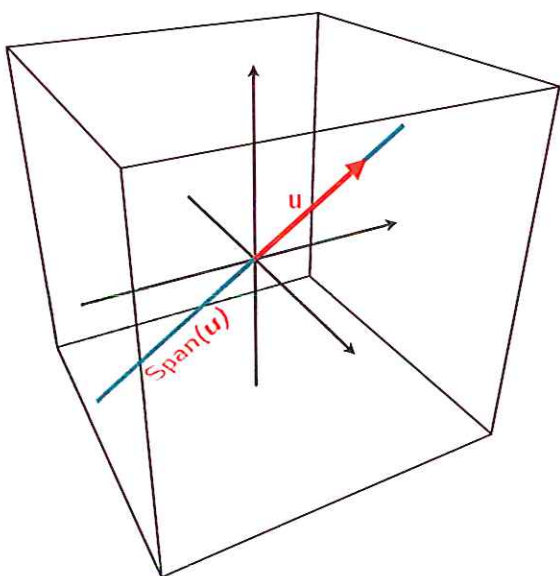
2) A set of vectors $\{v_1, \dots, v_p\}$ is linearly independent if the equation

$$x_1 v_1 + \dots + x_p v_p = 0$$

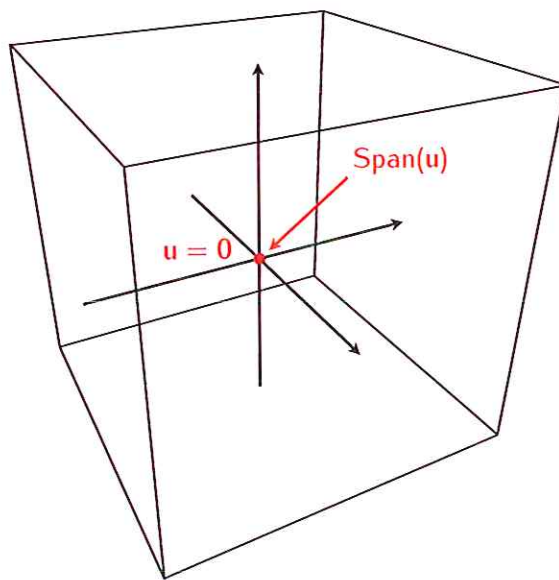
has only one, trivial solution $x_1 = 0, \dots, x_p = 0$.



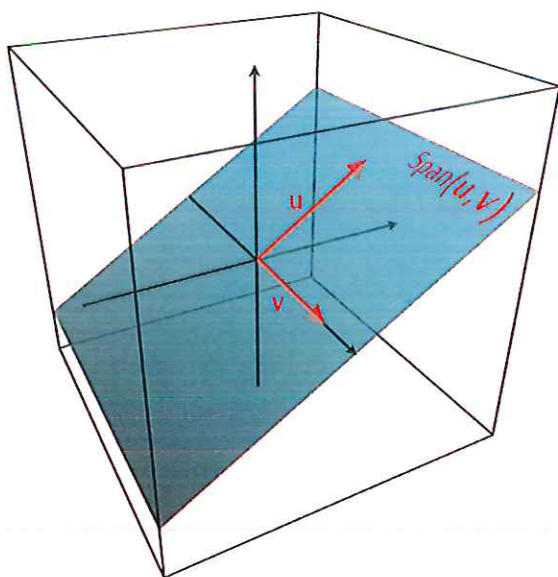
Linear independence vs. Span



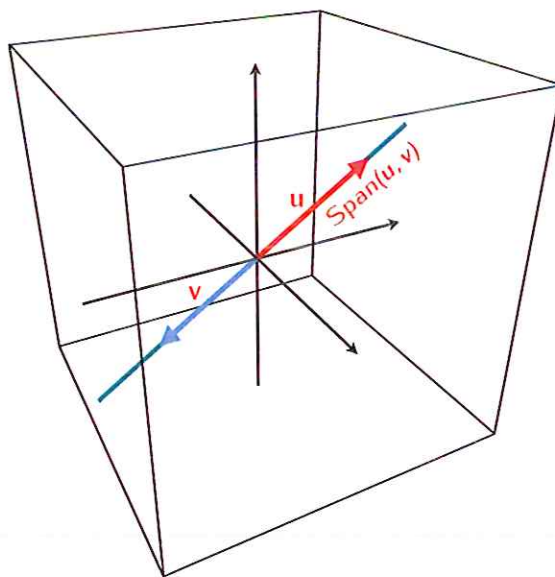
$\{u\}$ linearly independent



$\{u\}$ linearly dependent



$\{u, v\}$ linearly independent



$\{u, v\}$ linearly dependent

Theorem

Let $\{v_1, \dots, v_p\}$ be a set of vectors in \mathbb{R}^n . The following conditions are equivalent:

- 1) The set $\{v_1, \dots, v_p\}$ is linearly dependent.
- 2) For some v_i we have $v_i \in \text{Span}(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_p)$.
- 3) For some v_i we have

$$\text{Span}(v_1, \dots, v_p) = \text{Span}(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_p)$$

Example.

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The set $\{v_1, v_2, v_3\}$ is linearly dependent e.g.

$$2v_1 - v_2 + 0v_3 = 0$$

This gives:

$$(*) \quad v_2 = 2v_1 + 0v_3$$

so: $v_2 \in \text{Span}(v_1, v_3)$

Also, assume that $w \in \text{Span}(v_1, v_2, v_3)$

Then:

$$w = c_1 v_1 + c_2 v_2 + c_3 v_3$$

but using $(*)$ we obtain:

$$\begin{aligned} w &= c_1 v_1 + c_2 (2v_1 + 0v_3) + c_3 v_3 \\ &= (c_1 + 2c_2) v_1 + c_3 v_3 \end{aligned}$$

So: $w \in \text{Span}(v_1, v_3)$.