### **Recall:** A vector equation

$$x_1\mathbf{v}_1 + \ldots + x_n\mathbf{v}_n = \mathbf{b}$$

has a solution if and only if  $b \in Span(v_1, ..., v_n)$ .

#### **Definition**

If A is a matrix with columns  $v_1, \ldots, v_n$ :

$$A = [\mathbf{v}_1 \ldots \mathbf{v}_n]$$

then the set  $Span(v_1, ..., v_n)$  is called the *column space* of A and it is denoted Col(A).

**Upshot.** A matrix equation  $A\mathbf{x} = \mathbf{b}$  has a solution if and only if  $\mathbf{b} \in \text{Col}(A)$ .

**Question:** What conditions on the matrix A guarantee that the equation  $A\mathbf{x} = \mathbf{b}$  has a solution for an arbitrary vector  $\mathbf{b}$ ?

Example.

Example.

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 7 \end{bmatrix} \qquad \begin{array}{c} row \\ reduction \end{array} \qquad \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

## Proposition

A matrix equation  $A\mathbf{x} = \mathbf{b}$  has a solution for any  $\mathbf{b}$  if and only if A has a pivot position in every row.

In such case  $Col(A) = \mathbb{R}^m$ , where m is the number of rows of A.

### **Recall:** A vector equation

$$x_1\mathbf{v}_1 + \ldots + x_n\mathbf{v}_n = \mathbf{b}$$

has only one solution for each  $\mathbf{b} \in \text{Span}(v_1, \dots, v_n)$  if and only if the homogenous equation

$$x_1\mathbf{v}_1+\ldots+x_n\mathbf{v}_n=\mathbf{0}$$

has only the trivial solution  $x_1 = 0, ..., x_n = 0$ .

#### **Definition**

If A is a matrix then the set of solution of the homogenous equation

$$A\mathbf{x} = \mathbf{0}$$

is called the *null space* of A and it is denoted Nul(A).

**Upshot.** A matrix equation  $A\mathbf{x} = \mathbf{b}$  has only one solution for each  $\mathbf{b} \in \operatorname{Col}(A)$  if and only if  $\operatorname{Nul}(A) = \{\mathbf{0}\}$ .

**Example.** Find the null space of the matrix

$$A = \left[ \begin{array}{cc} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{array} \right]$$

# **Proposition**

 $Nul(A) = \{0\}$  if and only if the matrix A has a pivot position in every column.

**Example.** Find the null space of the matrix

$$A = \begin{bmatrix} 3 & 1 & -2 & 1 & 5 \\ 1 & 0 & 1 & 0 & 1 \\ 5 & 2 & -5 & 5 & 3 \end{bmatrix}$$

## Note

If A is an  $m \times n$  matrix then Nul(A) can be always described as a span of some vectors in  $\mathbb{R}^n$ .

