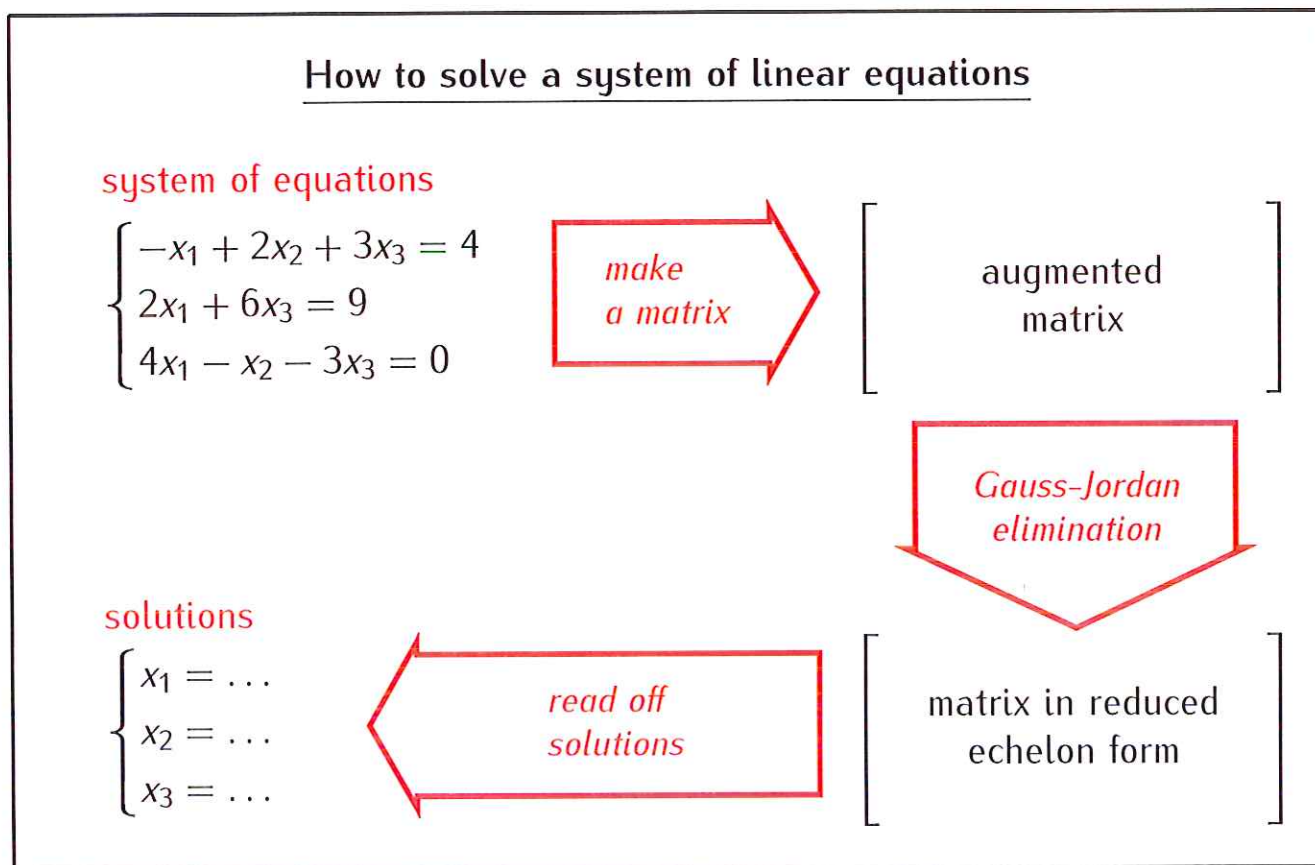
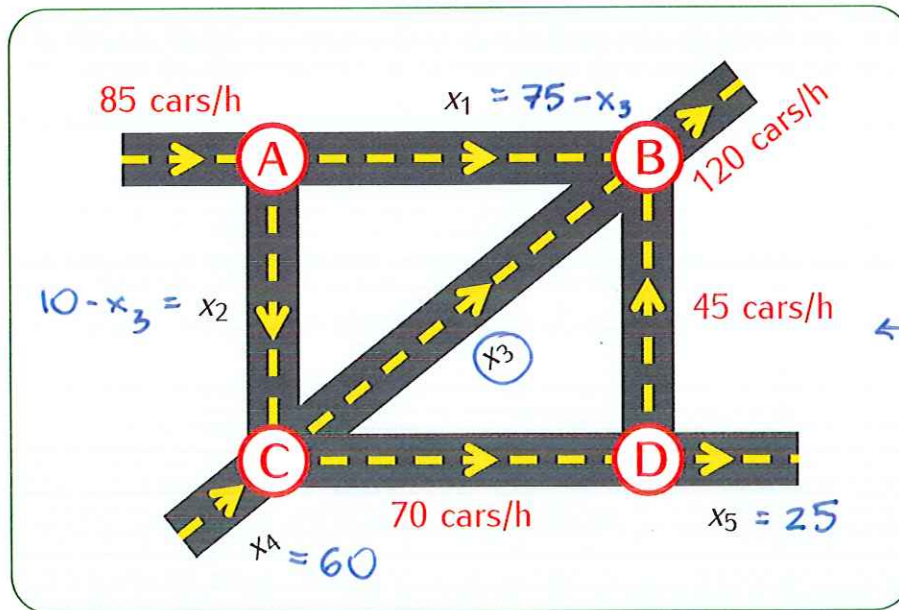


Recall:

Next: Some applications of systems of linear equations:

- Computations of traffic flow.
- Balancing chemical equations.
- Google PageRank.

Computations of traffic flow



In order to get full information about the flow of traffic we would need to measure the flow x_3 .

Problem. Find the flow rate of cars on each segment of streets.

Note:

- flow into an intersection = flow out of that intersection
- total flow in = total flow out

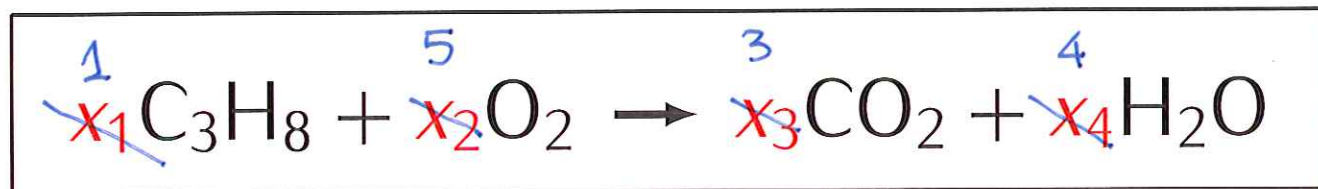
$$\begin{array}{lcl}
 \text{total :} & \text{IN} & = \text{OUT} \\
 \text{③ A :} & 85 + x_4 & = 120 + x_5 \\
 \text{② B :} & 85 & = x_1 + x_2 \\
 \text{① B :} & x_1 + x_3 + 45 & = 120 \\
 \text{② C :} & x_2 + x_4 & = x_3 + 70 \\
 \text{① D :} & 70 & = x_5 + 45
 \end{array}
 \Rightarrow
 \begin{cases}
 x_4 - x_5 = 35 \\
 x_1 + x_2 = 85 \\
 x_1 + x_3 = 75 \\
 x_2 - x_3 + x_4 = 70 \\
 x_5 = 25
 \end{cases}$$

augmented matrix:

$$\begin{array}{c}
 \begin{array}{ccccc|c}
 x_1 & x_2 & x_3 & x_4 & x_5 & \\
 \hline
 0 & 0 & 0 & 1 & -1 & 35 \\
 1 & 1 & 0 & 0 & 0 & 85 \\
 1 & 0 & 1 & 0 & 0 & 75 \\
 0 & 1 & -1 & 1 & 0 & 70 \\
 0 & 0 & 0 & 0 & 1 & 25
 \end{array}
 \xrightarrow{\text{now reduction}}
 \begin{array}{ccccc|c}
 x_1 & x_2 & x_3 & x_4 & x_5 & \\
 \hline
 1 & 0 & 1 & 0 & 0 & 75 \\
 0 & 1 & -1 & 0 & 0 & 10 \\
 0 & 0 & 0 & 1 & 0 & 60 \\
 0 & 0 & 0 & 0 & 1 & 25 \\
 0 & 0 & 0 & 0 & 0 & 0
 \end{array}
 \rightarrow
 \begin{cases}
 x_1 = 75 - x_3 \\
 x_2 = 10 + x_3 \\
 x_3 = \text{free} \\
 x_4 = 60 \\
 x_5 = 25
 \end{cases}
 \end{array}$$

Balancing chemical equations

Burning propane:



Note:

- The numbers x_1, x_2, x_3, x_4 are integers.
- The number of atoms of each element on the left side is the same as the number of atoms of that element on the right side.

LEFT = RIGHT

$$\begin{array}{lcl} \text{C:} & 3x_1 & = x_3 \\ \text{H:} & 8x_1 & = 2x_4 \\ \text{O:} & 2x_2 & = 2x_3 + x_4 \end{array}$$

$$\left\{ \begin{array}{l} 3x_1 - x_3 = 0 \\ 8x_1 - x_4 = 0 \\ 2x_2 - 2x_3 - x_4 = 0 \end{array} \right.$$

homogenous system

(i.e. zeros only on the right hand side)

$$\left[\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ 3 & 0 & -1 & 0 & 0 \\ 8 & 0 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array} \right] \xrightarrow{\text{now red}} \left[\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ 1 & 0 & 0 & -1/4 & 0 \\ 0 & 1 & 0 & -5/4 & 0 \\ 0 & 0 & 1 & -3/4 & 0 \end{array} \right]$$

free

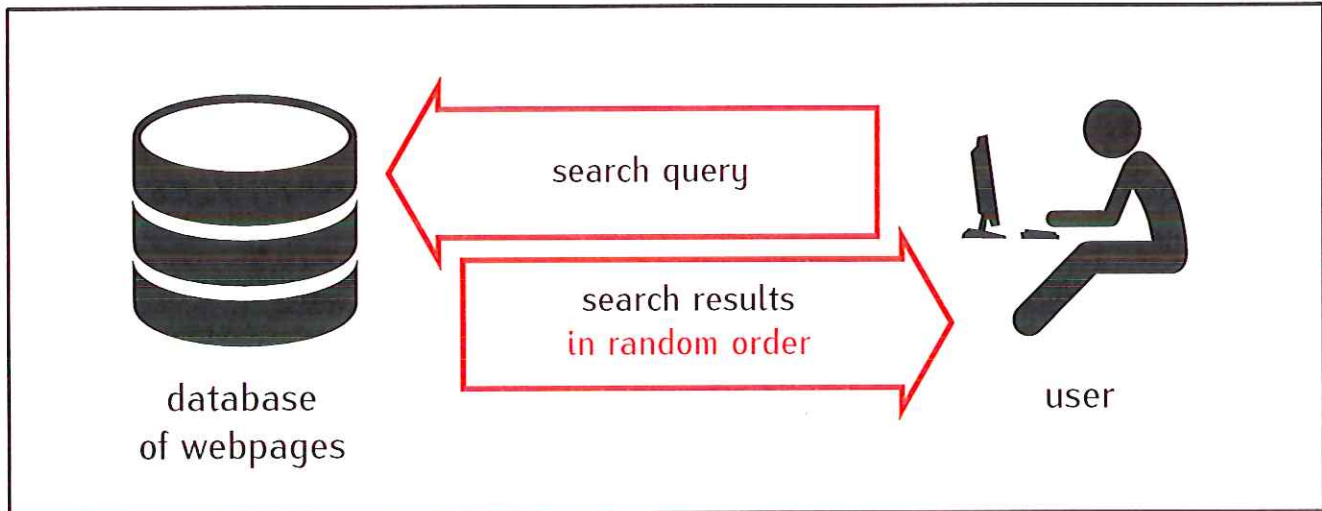
$$\left\{ \begin{array}{l} x_1 = \frac{1}{4}x_4 \\ x_2 = \frac{5}{4}x_4 \\ x_3 = \frac{3}{4}x_4 \\ x_4 = \text{free} \end{array} \right.$$

set
 $x_4 = 4$

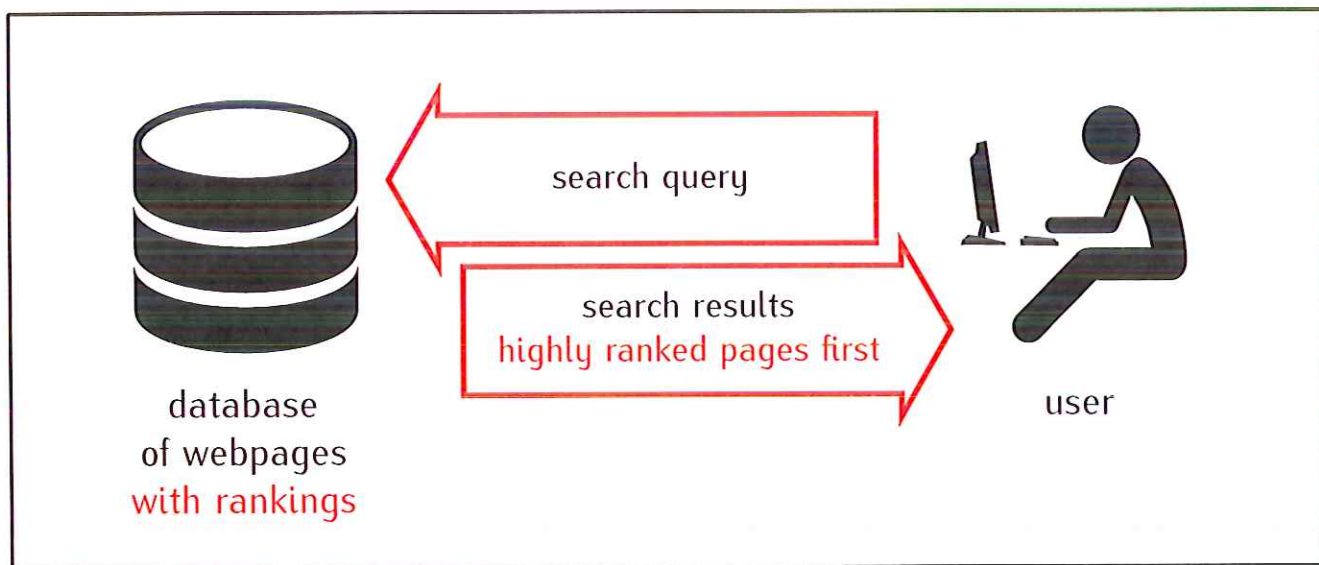
$$\left\{ \begin{array}{l} x_1 = 1 \\ x_2 = 5 \\ x_3 = 3 \\ x_4 = 4 \end{array} \right.$$

Google PageRank

Early search engines:



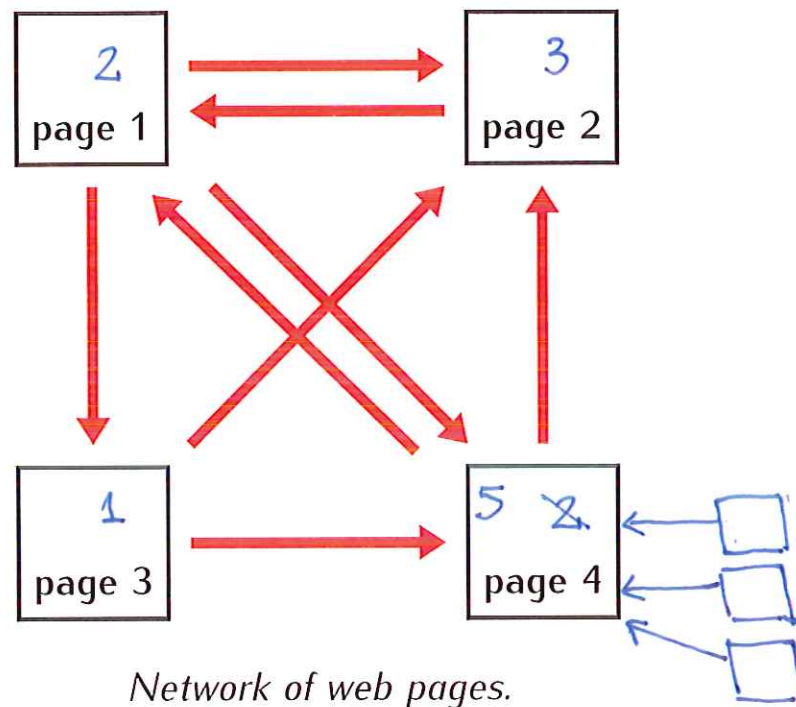
Google search engine:



How to rank webpages?

Very simple ranking:

$$\text{ranking of a page} = \left(\begin{array}{c} \text{number of links} \\ \text{pointing to that page} \end{array} \right)$$



Problem. This is very easy to manipulate.

How to rank webpages?

Google PageRank: Links from highly ranked pages are worth more than links from lower ranked pages.

If:

- the rank of a page is x
- the page has n links to other pages

then each link from that page is worth x/n .

$$\begin{cases} x_1 = x_2 + \frac{1}{2}x_4 \\ x_2 = \frac{1}{3}x_1 + \frac{1}{2}x_3 + \frac{1}{2}x_4 \\ x_3 = \frac{1}{3}x_1 \\ x_4 = \frac{1}{3}x_1 + \frac{1}{2}x_3 \end{cases}$$

↓ simplify

$$\begin{cases} x_1 - x_2 - \frac{1}{2}x_4 = 0 \\ -\frac{1}{3}x_1 + x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 = 0 \\ -\frac{1}{3}x_1 + x_3 = 0 \\ -\frac{1}{3}x_1 - \frac{1}{2}x_3 + x_4 = 0 \\ x_1 + x_2 + x_3 + x_4 = 1 \end{cases}$$

This system has a trivial solution

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0$$

Adding this equation eliminates the trivial solution

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & \\ 1 & -1 & 0 & -\frac{1}{2} & 0 \\ -\frac{1}{3} & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{3} & 0 & 1 & 0 & 0 \\ -\frac{1}{3} & 0 & -\frac{1}{2} & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

now red. →

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & \\ 1 & 0 & 0 & 0 & \frac{12}{31} \\ 0 & 1 & 0 & 0 & \frac{9}{31} \\ 0 & 0 & 1 & 0 & \frac{4}{31} \\ 0 & 0 & 0 & 1 & \frac{6}{31} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

solution:

$$\begin{cases} x_1 = \frac{12}{31} \\ x_2 = \frac{9}{31} \\ x_3 = \frac{4}{31} \\ x_4 = \frac{6}{31} \end{cases}$$

