6) Gram-Schmidt process:

$$\boxed{\begin{array}{c}\text{a basis}\\ \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}\\ \text{of } V \subseteq \mathbb{R}^n\end{array}} \xrightarrow{\textit{G-S process}} \boxed{\begin{array}{c}\text{an orthogonal basis}\\ \{\mathbf{w}_1, \ldots, \mathbf{w}_k\}\\ \text{of } V\end{array}}$$

$$\mathbf{w}_1 = \mathbf{v}_1$$

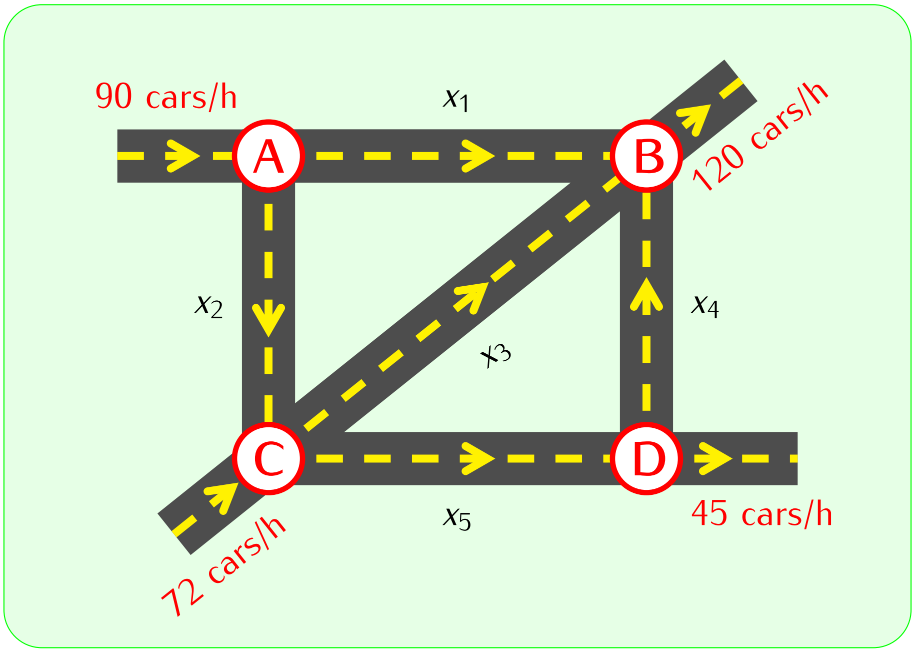
$$\mathbf{w}_2 = \mathbf{v}_2 - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_2}{\mathbf{w}_1 \cdot \mathbf{w}_1}\right)\mathbf{w}_1$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_3}{\mathbf{w}_1 \cdot \mathbf{w}_1}\right)\mathbf{w}_1 - \left(\frac{\mathbf{w}_2 \cdot \mathbf{v}_3}{\mathbf{w}_2 \cdot \mathbf{w}_2}\right)\mathbf{w}_2$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\mathbf{w}_k = \mathbf{v}_k - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_k}{\mathbf{w}_1 \cdot \mathbf{w}_1}\right)\mathbf{w}_1 - \left(\frac{\mathbf{w}_2 \cdot \mathbf{v}_k}{\mathbf{w}_2 \cdot \mathbf{w}_2}\right)\mathbf{w}_2 - \ldots - \left(\frac{\mathbf{w}_{k-1} \cdot \mathbf{v}_k}{\mathbf{w}_{k-1} \cdot \mathbf{w}_{k-1}}\right)\mathbf{w}_{k-1}$$

Problem. Find the flow rate of cars on each segment of streets:

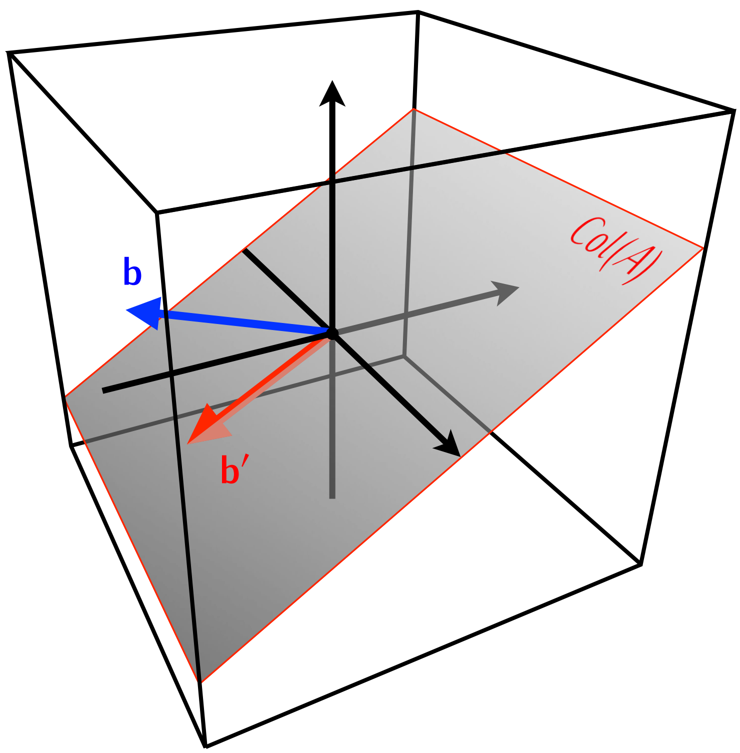


Upshot.

- Recall: a matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b} \in \text{Col}(A)$.

- In practical applications we may obtain a matrix equation that has no solutions, i.e. where $\mathbf{b} \notin \text{Col}(A)$.

- In such cases we may look for approximate solutions as follows:
  - replace $\mathbf{b}$ by a vector $\mathbf{b}'$ such that $\mathbf{b}' \in \text{Col}(A)$ and $\text{dist}(\mathbf{b}, \mathbf{b}')$ is a as small as possible.
  - then solve $A\mathbf{x} = \mathbf{b}'$



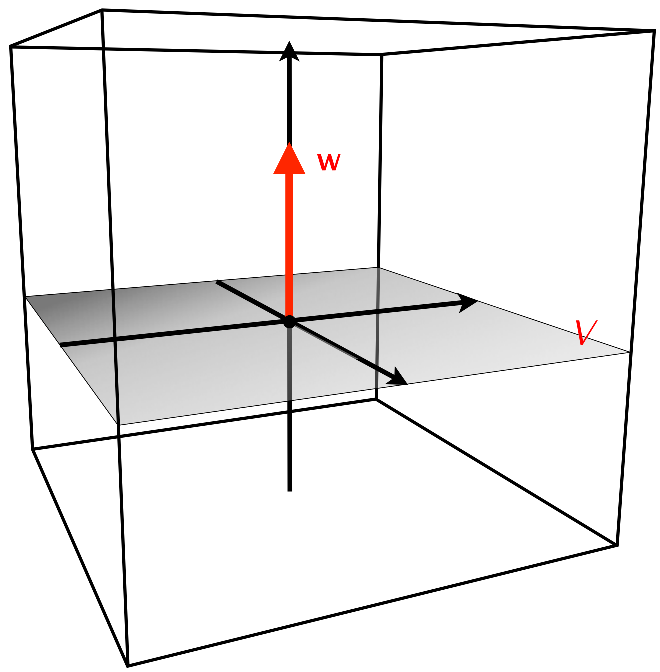
Definition

Given $\mathbf{b}' \in \text{Col}(A)$ as above we will say that a vector $\mathbf{v}$ is a *least square solution* of the equation $A\mathbf{x} = \mathbf{b}$ if $\mathbf{v}$ is a solution of the equation $A\mathbf{x} = \mathbf{b}'$.

Next: How to find the vector $\mathbf{b}'$?

Definition

Let V be a subspace of $\mathbb{R}^n$. A vector $\mathbf{w} \in \mathbb{R}^n$ is *orthogonal to* V if $\mathbf{w} \cdot \mathbf{v} = 0$ for all $\mathbf{v} \in V$.



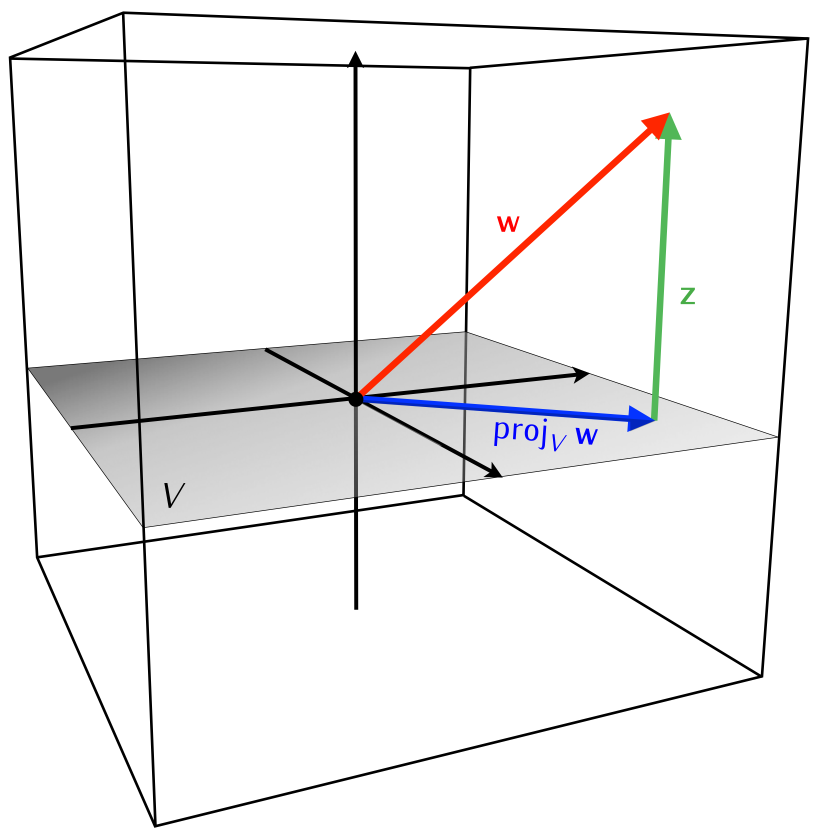
Proposition

If $V = \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ then a vector $\mathbf{w} \in \mathbb{R}^n$ is orthogonal to V if and only if $\mathbf{w} \cdot \mathbf{v}_i = 0$ for $i = 1, \ldots, k$.

Definition

Let V be a subspace of $\mathbb{R}^n$ and let $\mathbf{w} \in \mathbb{R}^n$ the *orthogonal projection of* $\mathbf{w}$ *onto* V is a vector $\mathrm{proj}_V\mathbf{w}$ such that

1) $\mathrm{proj}_V\mathbf{w} \in V$
2) the vector $\mathbf{z} = \mathbf{w} - \mathrm{proj}_V\mathbf{w}$ is orthogonal to V.



The Best Approximation Theorem

If V is a subspace of $\mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^n$ then $\mathrm{proj}_V \mathbf{w}$ is a vector in V which is closest to $\mathbf{w}$:

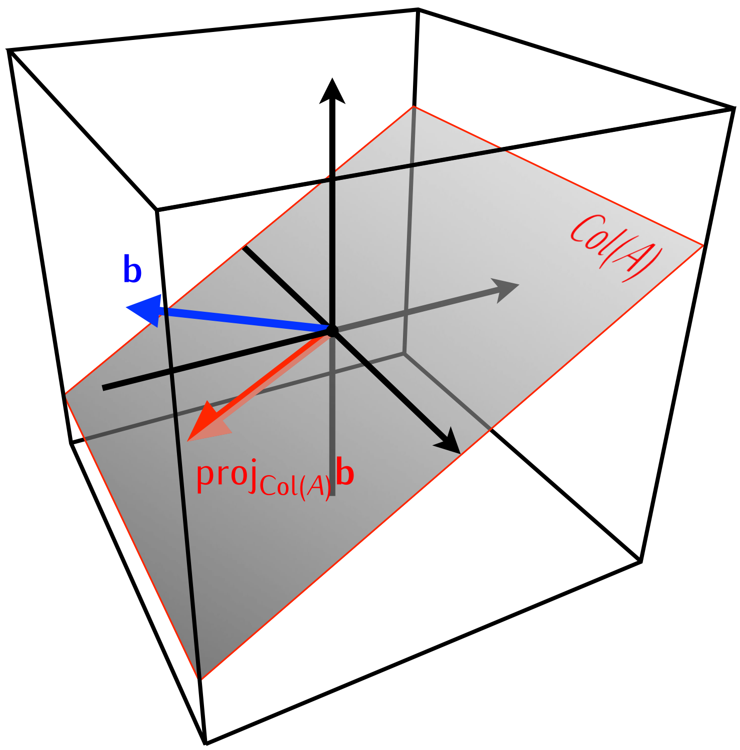
$$\mathrm{dist}(\mathbf{w}, \mathrm{proj}_V \mathbf{w}) \leq \mathrm{dist}(\mathbf{w}, \mathbf{v})$$

for all $\mathbf{v} \in V$.

Corollary

The least square solutions of a matrix equation $A\mathbf{x} = \mathbf{b}$ are solutions of the equation

$$A\mathbf{x} = \mathrm{proj}_{\mathrm{Col}(A)}\mathbf{b}$$



<u>**Next:**</u> If V is a subspace of $\mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^n$ how to compute $\mathrm{proj}_V\mathbf{w}$?

Theorem

If V is a subspace of $\mathbb{R}^n$ with an orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and $\mathbf{w} \in \mathbb{R}^n$ then

$$\text{proj}_V \mathbf{w} = \left(\frac{\mathbf{w} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1 + \ldots + \left(\frac{\mathbf{w} \cdot \mathbf{v}_k}{\mathbf{v}_k \cdot \mathbf{v}_k}\right) \mathbf{v}_k$$

Corollary

If V is a subspace of $\mathbb{R}^n$ with an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and $\mathbf{w} \in \mathbb{R}^n$ then

$$\text{proj}_V \mathbf{w} = (\mathbf{w} \cdot \mathbf{v}_1) \mathbf{v}_1 + \ldots + (\mathbf{w} \cdot \mathbf{v}_k) \mathbf{v}_k$$

Example. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 0 \\ -2 \end{bmatrix} \right\}, \qquad \mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

The set $\mathcal{B}$ is an orthogonal basis of some subspace V of $\mathbb{R}^4$. Compute $\text{proj}_V \mathbf{w}$.

Note. In general if V is a subspace of $\mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^n$ then in order to find $\text{proj}_V \mathbf{w}$ we need to do the following:

1) find a basis of V.

2) use the Gram-Schmidt process to get an orthogonal basis of V

3) use the orthogonal basis to compute $\text{proj}_V \mathbf{w}$.

Example. Consider the following matrix A and vector $\mathbf{u}$:

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 3 & 4 & 2 \\ 2 & 6 & 3 & -1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

Compute $\text{proj}_{\text{Col}(A)} \mathbf{u}$.

Example. Find least square solutions of the matrix equation $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 90 \\ 120 \\ 72 \\ 45 \end{bmatrix}$$

Recall:

1) The least square solutions of a matrix equation $A\mathbf{x} = \mathbf{b}$ are the solutions of the equation

$$A\mathbf{x} = \text{proj}_{\text{Col}(A)}\mathbf{b}$$

2) If $A\mathbf{x} = \mathbf{b}$ is a consistent equation, then $\mathbf{b} \in \text{Col}(A)$, and $\text{proj}_{\text{Col}(A)}\mathbf{b} = \mathbf{b}$. In such case the least square solutions of $A\mathbf{x} = \mathbf{b}$ are just the ordinary solutions.

3) If $A\mathbf{x} = \mathbf{b}$ is inconsistent, then the least square solutions are the best substitute for the (nonexistent) ordinary solutions.

4) If $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an orthogonal basis of a subspace V of $\mathbb{R}^n$ then

$$\text{proj}_V\mathbf{w} = \left(\frac{\mathbf{w}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1 + \dots + \left(\frac{\mathbf{w}\cdot\mathbf{v}_k}{\mathbf{v}_k\cdot\mathbf{v}_k}\right)\mathbf{v}_k$$

5) If $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an arbitrary basis of V then we can use the Gram-Schmidt process to obtain an orthogonal basis of V.