Recall: A vector equation

$$x_1\mathbf{v}_1 + \ldots + x_n\mathbf{v}_n = \mathbf{b}$$

has a solution if and only if $\mathbf{b} \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$.

Equivalently: If $A = [v_1 \ \ldots \ v_p]$ then the matrix equation

$$Ax = b$$

has a solution if $b \in \mathrm{Span}(v_1, \ldots, v_p)$

Definition

If A is a matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$:

$$A = \begin{bmatrix} \mathbf{v}_1 & \ldots & \mathbf{v}_n \end{bmatrix}$$

then the set $\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is called the *column space* of A and it is denoted $\mathrm{Col}(A)$.

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \mathrm{Col}(A) = \mathrm{Span}\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}\right)$$

Upshot. A matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b} \in \text{Col}(A)$.

Question: What conditions on the matrix A guarantee that the equation $A\mathbf{x} = \mathbf{b}$ has a solution for an arbitrary vector $\mathbf{b}$?

Example.

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

augmented matrix of $Ax = b$:

$$[A \mid b] \xrightarrow{\text{row red.}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \square \\ 0 & 1 & 0 & -1 & \square \\ 0 & 0 & 1 & 2 & \square \end{array}\right]$$

no place for a leading one here, so $Ax = b$ will always have a solution

Example.

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 7 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[A \mid b] \xrightarrow{\text{row red.}} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & \square \\ 0 & 1 & 1 & 1 & \square \\ 0 & 0 & 0 & 0 & \square \end{array}\right]$$

by choosing an appropriate vector b we will get a leading 1 here. Thus $Ax = b$ will not have a solution for some b.

Proposition

A matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b}$ if and only if A has a pivot position in every row.

In such case $\mathrm{Col}(A) = \mathbb{R}^m$, where m is the number of rows of A.

Recall: A vector equation

$$x_1\mathbf{v}_1 + \ldots + x_n\mathbf{v}_n = \mathbf{b}$$

has only one solution for each $\mathbf{b} \in \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ if and only if the homogenous equation

$$x_1\mathbf{v}_1 + \ldots + x_n\mathbf{v}_n = \mathbf{0}$$

has only the trivial solution $x_1 = 0, \ldots, x_n = 0$.

Reformulation for matrix equations:

A matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has only one solution for each $\mathbf{b} \in \text{Col}(A)$ if and only if the homogenous equation

$$A\mathbf{x} = \mathbf{0}$$

has only the trivial solution $\mathbf{x} = \mathbf{0}$.

the zero vector

Definition

If A is a matrix then the set of solution of the homogenous equation

$$A\mathbf{x} = \mathbf{0}$$

is called the *null space* of A and it is denoted $\text{Nul}(A)$.

Upshot. A matrix equation $A\mathbf{x} = \mathbf{b}$ has only one solution for each $\mathbf{b} \in \text{Col}(A)$ if and only if $\text{Nul}(A) = \{\mathbf{0}\}$.

Example. Find the null space of the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Solution. We need to solve:

$$Ax = 0$$

augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{array}\right] \xrightarrow{\text{row red.}} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

solutions:

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$$

in vector form

$$x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So: $\mathrm{Nul}(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} = \mathrm{Span}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} \right)$

Proposition

$\mathrm{Nul}(A) = \{0\}$ if and only if the matrix A has a pivot position in every column.

Example. Find the null space of the matrix

$$A = \begin{bmatrix} 3 & 1 & -2 & 1 & 5 \\ 1 & 0 & 1 & 0 & 1 \\ 5 & 2 & -5 & 5 & 3 \end{bmatrix}$$

Solution: We need to solve $Ax = 0$

augmented matrix

$$\left[\begin{array}{ccccc|c} 3 & 1 & -2 & 1 & 5 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 5 & 2 & -5 & 5 & 3 & 0 \end{array}\right] \xrightarrow{\text{row red.}} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -5 & 0 & 4 & 0 \\ 0 & 0 & \underbrace{0}_{\text{free}} & 1 & \underbrace{-2}_{\text{free}} & 0 \end{array}\right]$$

solutions:

$$\begin{cases} x_1 = -x_3 - x_5 \\ x_2 = 5x_3 - 4x_5 \\ x_3 = x_3 \\ x_4 = 2x_5 \\ x_5 = x_5 \end{cases}$$

in vector form:

$$x = \begin{bmatrix} -x_3 - x_5 \\ 5x_3 + 4x_5 \\ x_3 \\ 2x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 5x_3 \\ x_3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_5 \\ -4x_5 \\ 0 \\ 2x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -4 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\mathrm{Nul}(A) = \left\{ x_3 \begin{bmatrix} -1 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -4 \\ 0 \\ 2 \\ 1 \end{bmatrix} \;\middle|\; x_1, x_2 \in \mathbb{R} \right\} = \mathrm{Span}\left(\begin{bmatrix} -1 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right)$$

Note

If A is an $m \times n$ matrix then $\mathrm{Nul}(A)$ can be always described as a span of some vectors in $\mathbb{R}^n$.

Upshot: how to find the number of solutions of a matrix equation