

Recall:

1) The dot product in \mathbb{R}^n :

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \dots a_n b_n$$

2) Properties of the dot product:

- a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
- c) $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$
- d) $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

2) Using the dot product we can define:

- length of vectors
- distance between vectors
- orthogonality of vectors
- orthogonal and orthonormal bases
- orthogonal projection of a vector onto a subspace of \mathbb{R}^n
- ...

Next: Generalization to arbitrary vector spaces.

Definition

Let V be a vector space. An *inner product* on V is a function

$$\begin{aligned} V \times V &\longrightarrow \mathbb{R} \\ \mathbf{u}, \mathbf{v} &\longmapsto \langle \mathbf{u}, \mathbf{v} \rangle \end{aligned}$$

such that:

- a) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
- b) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
- c) $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$
- d) $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Definition

Let V be a vector space with an inner product $\langle \cdot, \cdot \rangle$.

- 1) The *length* (or *norm*) of a vector \mathbf{v} is the number

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

- 2) The *distance* between vectors $\mathbf{u}, \mathbf{v} \in V$ is the number

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

- 3) Vectors $\mathbf{u}, \mathbf{v} \in V$ are *orthogonal* if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Example. The dot product is an inner product in \mathbb{R}^n .

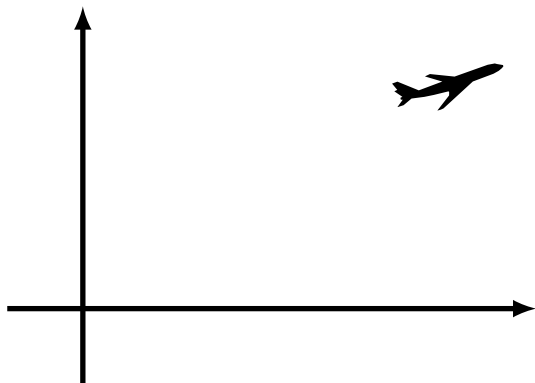
Example. Let p_1, \dots, p_n be any positive numbers. For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$

$$\mathbf{u} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

define:

$$\langle \mathbf{u}, \mathbf{v} \rangle = p_1(a_1 b_1) + p_2(a_2 b_2) + \dots + p_n(a_n b_n)$$

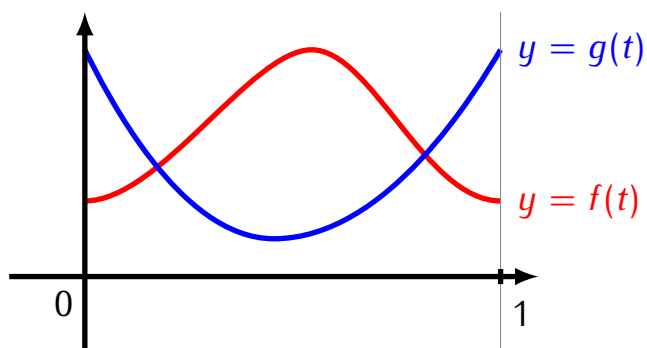
This gives an inner product in \mathbb{R}^n .



Example. Let $C[0, 1]$ be the vector space of continuous functions $f: [0, 1] \rightarrow \mathbb{R}$. Define:

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt$$

This is an inner product on $C[0, 1]$.



Example. Compute the length of the function

$$f(t) = 1 + t^2$$

in $C[0, 1]$.

Definition

Let V be a vector space with an inner product $\langle \cdot, \cdot \rangle$, and let W be a subspace of V . A vector $\mathbf{v} \in V$ is *orthogonal to W* if $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in W$.

Definition

Let V be a vector space with an inner product $\langle \cdot, \cdot \rangle$, and let W be a subspace of V . The *orthogonal projection of a vector $\mathbf{v} \in V$ onto W* is a vector $\text{proj}_W \mathbf{v}$ such that

- 1) $\text{proj}_W \mathbf{v} \in W$
- 2) the vector $\mathbf{z} = \mathbf{v} - \text{proj}_W \mathbf{v}$ is orthogonal to W .

Best Approximation Theorem

If V is a vector space with an inner product $\langle \cdot, \cdot \rangle$, W is a subspace of V , and $\mathbf{v} \in V$, then $\text{proj}_W \mathbf{v}$ is the vector of W which is the closest to \mathbf{v} :

$$\text{dist}(\mathbf{v}, \text{proj}_W \mathbf{v}) \leq \text{dist}(\mathbf{v}, \mathbf{w})$$

for all $\mathbf{w} \in W$.

Theorem

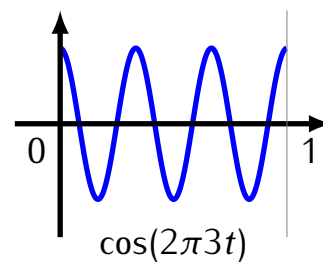
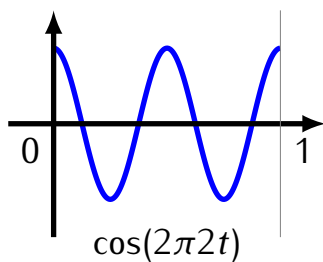
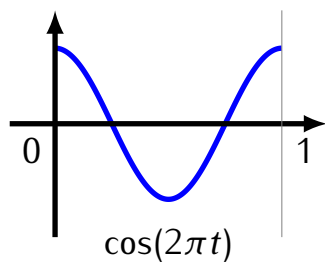
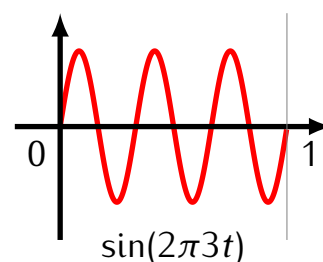
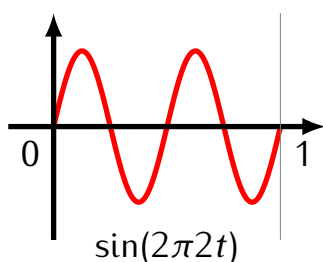
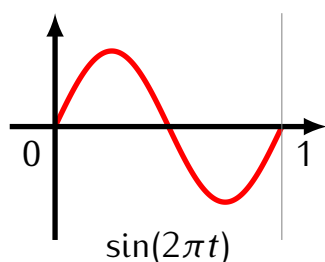
Let V is a vector space with an inner product $\langle \cdot, \cdot \rangle$, and let W be a subspace of V . If $\mathcal{B} = \{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ is an orthogonal basis of W (i.e. a basis such that $\langle \mathbf{w}_i, \mathbf{w}_j \rangle = 0$ for all $i \neq j$) then for $\mathbf{v} \in V$ we have:

$$\text{proj}_W \mathbf{v} = \frac{\langle \mathbf{v}, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} \mathbf{w}_1 + \dots + \frac{\langle \mathbf{v}, \mathbf{w}_k \rangle}{\langle \mathbf{w}_k, \mathbf{w}_k \rangle} \mathbf{w}_k$$

Application: Fourier approximations.

Goal: Let $f: [0, 1] \rightarrow \mathbb{R}$ be a continuous function. Find the best possible approximation of f of the form

$$\begin{aligned}
 P(t) = & a_0 \\
 & + a_1 \sin(2\pi t) + b_1 \cos(2\pi t) \\
 & + a_2 \sin(2\pi 2t) + b_2 \cos(2\pi 2t) \\
 & \dots \dots \dots \dots \dots \dots \dots \dots \\
 & + a_n \sin(2\pi nt) + b_n \cos(2\pi nt)
 \end{aligned}$$



Note: Let W_n be a subspace of $C[0, 1]$ given by:

$$W_n = \text{Span}(1, \sin(2\pi t), \cos(2\pi t), \dots, \sin(2\pi nt), \cos(2\pi nt))$$

By the Best Approximation Theorem, the best approximation of f is obtained if we take $P(t) = \text{proj}_{W_n} f(t)$.

Theorem

The set

$$\{1, \sin(2\pi t), \cos(2\pi t), \dots, \sin(2\pi n t), \cos(2\pi n t)\}$$

is an orthogonal basis of W_n .

Corollary

If $f \in C[0, 1]$ then

$$\begin{aligned} \text{proj}_{W_n} f(t) = & a_0 \\ & + a_1 \sin(2\pi t) + b_1 \cos(2\pi t) \\ & + a_2 \sin(2\pi 2t) + b_2 \cos(2\pi 2t) \\ & \dots \dots \dots \dots \dots \dots \dots \dots \\ & + a_n \sin(2\pi n t) + b_n \cos(2\pi n t) \end{aligned}$$

where:

$$a_0 = \frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} = \int_0^1 f(t) dt$$

and for $k > 0$:

$$a_k = \frac{\langle f, \sin(2\pi k t) \rangle}{\langle \sin(2\pi k t), \sin(2\pi k t) \rangle} = 2 \int_0^1 f(t) \cdot \sin(2\pi k t) dt$$

$$b_k = \frac{\langle f, \cos(2\pi k t) \rangle}{\langle \cos(2\pi k t), \cos(2\pi k t) \rangle} = 2 \int_0^1 f(t) \cdot \cos(2\pi k t) dt$$

Example. Compute $\text{proj}_{W_n} f(t)$ for the function $f(t) = t$.

Application: Polynomial approximations.

Goal: Let $f: [0, 1] \rightarrow \mathbb{R}$ be a continuous function. Find the best possible approximation of f given by a polynomial $P(t)$ of degree $\leq n$:

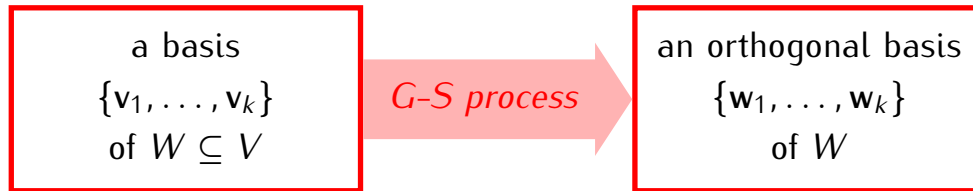
$$P(t) = a_0 + a_1 t + \dots + a_n t^n$$

Note: Let \mathbb{P}_n be the subspace of $C[0, 1]$ consisting of all polynomials of degree $\leq n$:

$$\mathbb{P}_n = \{a_0 + a_1 t + \dots + a_n t^n \mid a_k \in \mathbb{R}\}$$

By the Best Approximation Theorem, the best approximation of f is obtained if we take $P(t) = \text{proj}_{\mathbb{P}_n} f(t)$.

Gram-Schmidt process:



Theorem (Gram-Schmidt Process)

Let V be a vector space with an inner product $\langle \cdot, \cdot \rangle$, and let W be a subspace of V . Let $\{v_1, \dots, v_k\}$ be a basis of W . Define vectors $\{w_1, \dots, w_k\}$ as follows:

$$w_1 = v_1$$

$$w_2 = v_2 - \frac{\langle w_1, v_2 \rangle}{\langle w_1, w_1 \rangle} w_1$$

$$w_3 = v_3 - \frac{\langle w_1, v_3 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle w_2, v_3 \rangle}{\langle w_2, w_2 \rangle} w_2$$

... ..

$$w_k = v_k - \frac{\langle w_1, v_k \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle w_2, v_k \rangle}{\langle w_2, w_2 \rangle} w_2 - \dots - \frac{\langle w_{k-1}, v_k \rangle}{\langle w_{k-1}, w_{k-1} \rangle} w_{k-1}$$

Then the set $\{w_1, \dots, w_k\}$ is an orthogonal basis of W .

Example. Find an orthogonal basis of the subspace \mathbb{P}_2 of the vector space $C[0, 1]$.

Example. Compute $\text{proj}_{\mathbb{P}_2} f(t)$ for $f(t) = \sqrt{t}$.