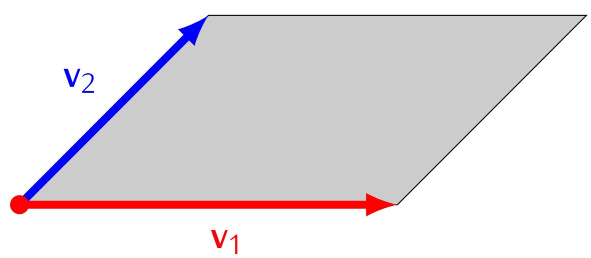
Recall:

A (matrix) $\longrightarrow$ $\det A$ (number) $\longrightarrow$

- $= 0$ if A is not invertible
- $\neq 0$ if A is invertible

Note. Any two vectors in $\mathbb{R}^2$ define a parallelogram:



Notation

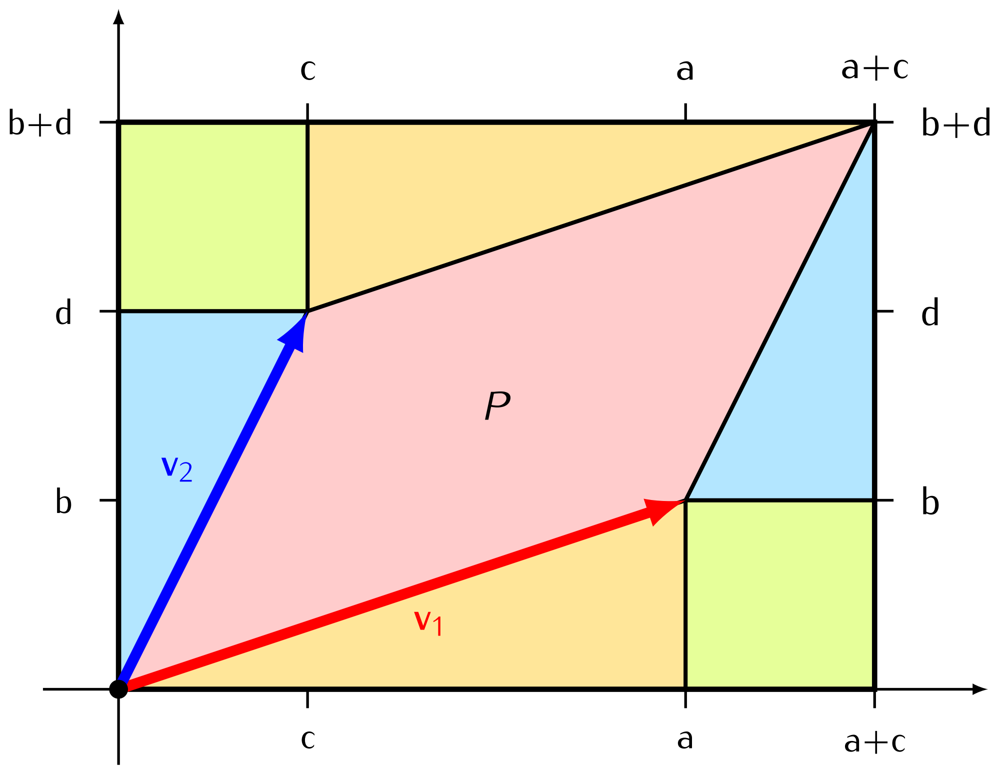
$$\text{area}(\mathbf{v}_1, \mathbf{v}_2) = \begin{pmatrix}\text{area of the parallelogram} \\ \text{defined by } \mathbf{v}_1 \text{ and } \mathbf{v}_2\end{pmatrix}$$

Theorem

If $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ then
$$\operatorname{area}(\mathbf{v}_1, \mathbf{v}_2) = \left|\det \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\right|$$

Idea of the proof.

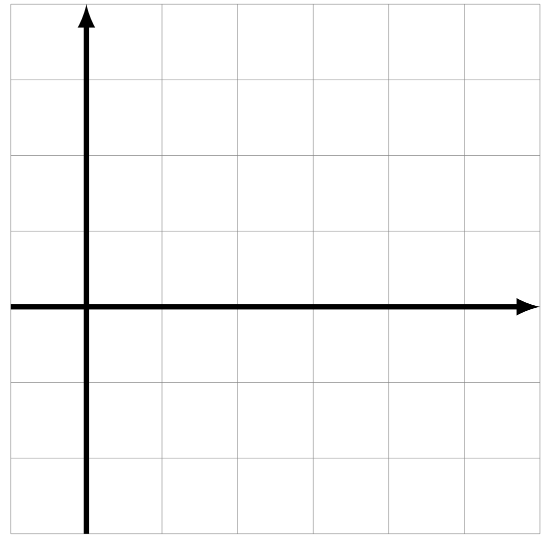
$$\mathbf{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} c \\ d \end{bmatrix}$$



$$\begin{array}{r} \operatorname{area}(P) \\ +\ \frac{1}{2}ab \\ +\ \frac{1}{2}ab \\ +\ \frac{1}{2}cd \\ +\ \frac{1}{2}cd \\ +\ cb \\ +\ cb \\ \hline (a+c)(b+d) \end{array}$$

Example.

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

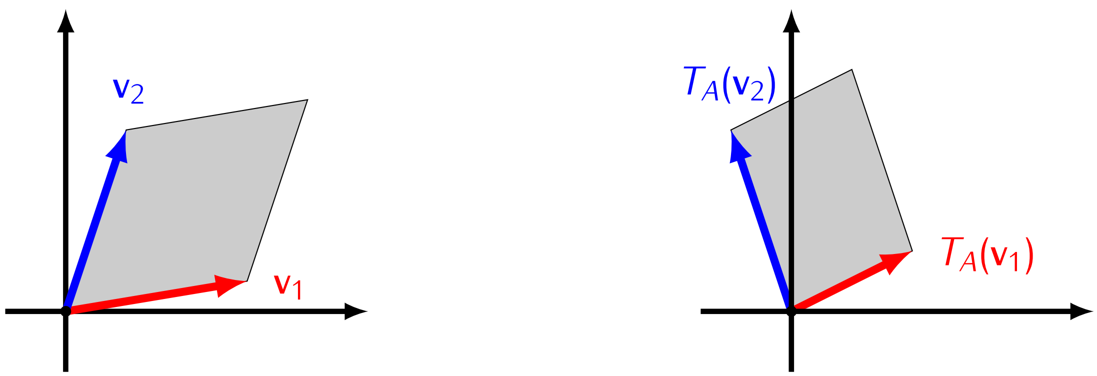


Determinants and linear transformations

Recall: If A is a 2×2 matrix then it defines a linear transformation

$$T_A\colon \mathbb{R}^2 \to \mathbb{R}^2 \qquad T_A(\mathbf{v}) = A\mathbf{v}$$

Note. T_A maps parallelograms to parallelograms:



Theorem

If A is a 2×2 matrix and $\mathbf{v}_1, \mathbf{v}_1 \in \mathbb{R}^2$ then

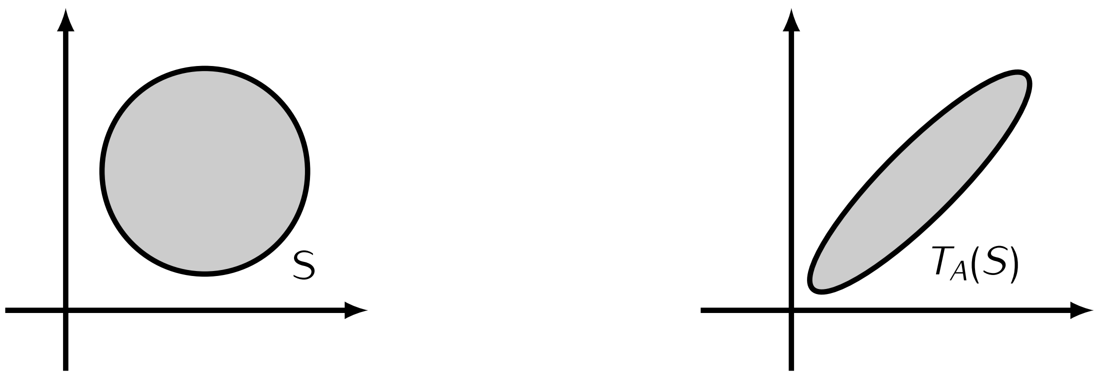
$$\text{area}(T_A(\mathbf{v}_1), T_A(\mathbf{v}_2)) = |\det A| \cdot \text{area}(\mathbf{v}_1, \mathbf{v}_2)$$

Generalization:

Theorem

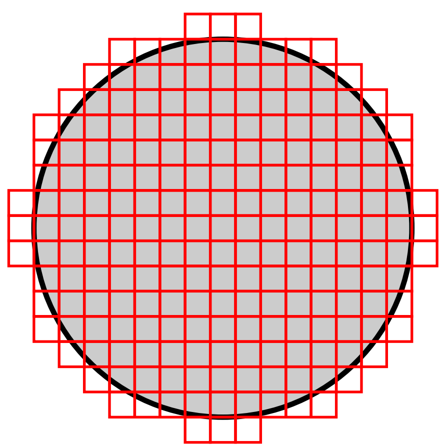
If A is a 2×2 matrix then for any region S of $\mathbb{R}^2$ we have:

$$\text{area}(T_A(S)) = |\det A| \cdot \text{area}(S)$$



Idea of the proof.

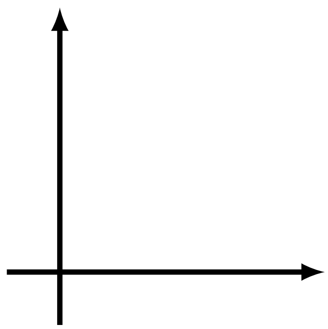
The area of S can be approximated by the sum of small squares covering S.

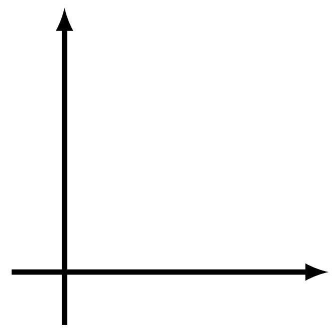


Sign of the determinant

Example.

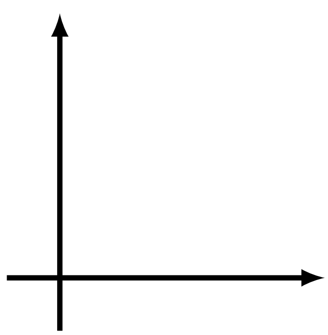
$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

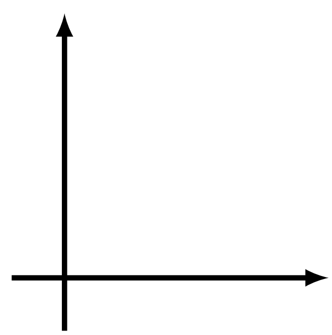




Example.

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$$





Theorem

If A is a 2×2 matrix then the linear transformation $T_A\colon \mathbb{R}^2 \to \mathbb{R}^2$ preserves orientation if $\det A > 0$ and reverses orientation if $\det A < 0$.