**Definition**

Let $V, W$ be vector spaces A *linear transformation* is a function

$$T\colon V \to W$$

which satisfies the following conditions:

1) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$
2) $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $\mathbf{v} \in V$ and any scalar $c$.

Example: If $A$ is an $m \times n$ matrix then it defines a linear transformation:

$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$v \longmapsto Av$$

Example:

Recall: $C^\infty(\mathbb{R}) = \left\{ \begin{array}{l} \text{the vector space of all} \\ \text{smooth functions } f:\mathbb{R}\to\mathbb{R} \end{array} \right\}$

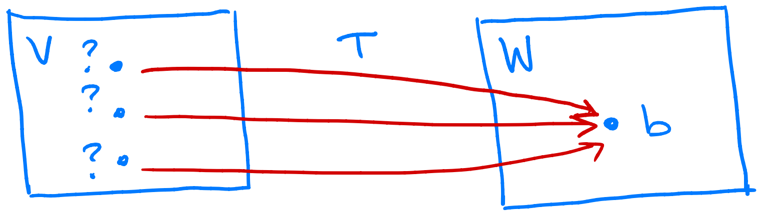
Take $D: C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$, $D(f) = f'$ ← the derivative of $f$.

D is a linear transformation:

1) $D(f+g) = (f+g)' = f' + g' = D(f) + D(g)$
2) $D(cf) = (cf)' = c \cdot f' = c \cdot D(f)$.

**Note.** If $T: V \to W$ is a linear transformation then for any vector $\mathbf{b} \in W$ we can consider the equation

$$T(\mathbf{x}) = \mathbf{b}$$




Example:

If $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $v \longmapsto Av$ - a (matrix) linear transformation

then the equation $T_A(x) = b$ is the same as the matrix equation $Ax = b$.

Example:

Take $D: C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$, $f(t) \longmapsto f'(t)$

For $g \in C^\infty(\mathbb{R})$ the equation $D(x) = g$ is the same as the differential equation

$$\frac{dx}{dt} = g$$

This equation is solved by integration:

$$x(t) = \int g(t)\,dt$$

## Definition

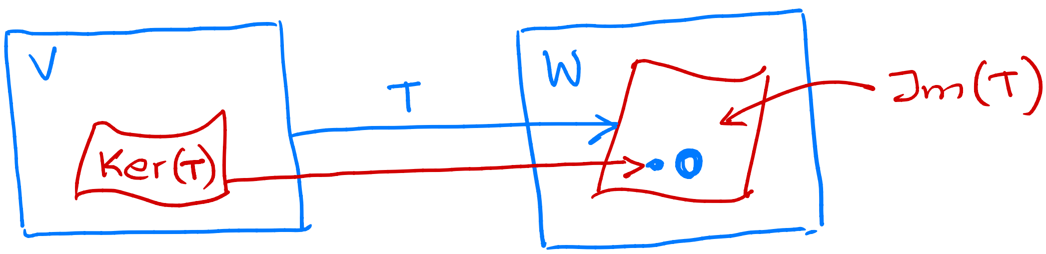
If $T\colon V \to W$ is a linear transformation then:

**1)** The *kernel* of $T$ is the set

$$\mathrm{Ker}(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}$$

**2)** The *image* of $T$ is the set

$$\mathrm{Im}(T) = \{\mathbf{w} \in W \mid \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V\}$$




Example:

$A$ - $m\times n$ matrix $\qquad T_A\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $\quad v \longmapsto Av$

$$\mathrm{Ker}(T_A) = \{v \in \mathbb{R}^n \mid T_A(v) = 0\}$$
$$= \{v \in \mathbb{R}^n \mid Av = 0\} = \mathrm{Nul}(A)$$

the null space of A

$$\mathrm{Im}(T_A) = \{b \in \mathbb{R}^m \mid T_A(v) = b \text{ for some } v \in \mathbb{R}^n\}$$
$$= \{b \in \mathbb{R}^m \mid Av = b \text{ —''— ''— }\}$$
$$= \mathrm{Col}(A)$$

← the column space of A.

Example:

$$D\colon C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R}), \quad f \longmapsto f'$$

$$\mathrm{Ker}(D) = \{f \in C^\infty(\mathbb{R}) \mid f' = 0\} = \{\text{the set of all constant functions}\}$$

$$\mathrm{Im}(D) = \{g \in C^\infty(\mathbb{R}) \mid g = f' \text{ for some } f \in C^\infty(\mathbb{R})\} = C^\infty(\mathbb{R})$$

**Proposition**

If $T: V \to W$ is a linear transformation then:

**1)** $\mathrm{Ker}(T)$ is a subspace of $V$

**2)** $\mathrm{Im}(T)$ is a subspace of $W$

**Theorem**

If $T: V \to W$ is a linear transformation and $v_0$ is a solution of the equation

$$T(\mathbf{x}) = \mathbf{b}$$

then all other solutions of this equation are vectors of the form

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{n}$$

where $\mathbf{n} \in \mathrm{Ker}(T)$.

Proof:

If $v_0$ is a solution of $T(x) = b$ and $n \in \mathrm{Ker}(T)$ then

$$T(v_0 + n) = T(v_0) + T(n) = b + \mathbf{0} = b$$

so: $v_0 + n$ is also a solution of $T(x) = b$.

Proof of the converse is similar.

**Example.**

$$D\colon C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$$
$$f \longmapsto f'$$

Recall:

$\operatorname{Ker}(D) = \{$ all constant functions $\}$

Let $g(t) = t^2$

Solutions of $D(x) = g$ are functions $f$ such that $f'(t) = g(t) = t^2$

This gives: solutions of $D(x) = g$ are functions

$$f(t) = \int t^2 dt = \underbrace{\tfrac{1}{3}t^3}_{\substack{\text{a particular}\\ \text{solution of}\\ D(x) = g}} + \underbrace{C}_{\substack{\text{a constant function}\\ \text{i.e a function}\\ \text{from } \operatorname{Ker}(D).}}$$