

Recall:

- A basis of a vector space V is a set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ such that
 - 1) $\text{Span}(\mathbf{b}_1, \dots, \mathbf{b}_n) = V$
 - 2) The set $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is linearly independent.

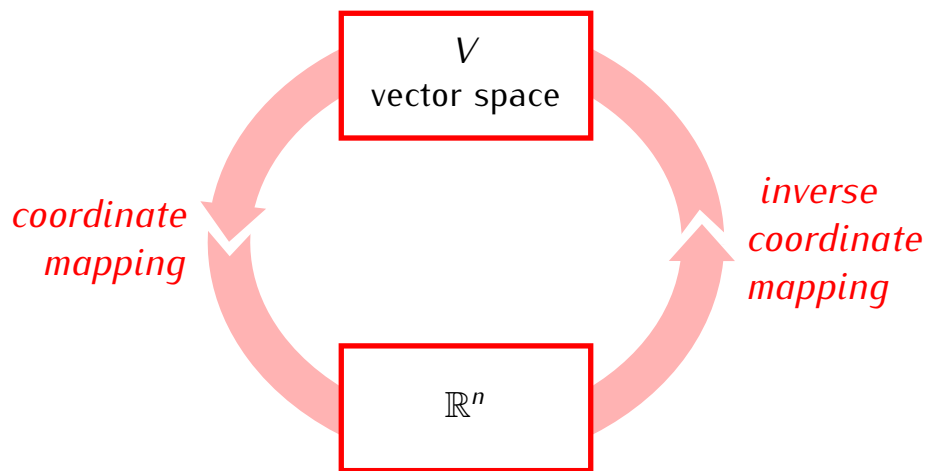
- For $\mathbf{v} \in V$ let c_1, \dots, c_n be the unique numbers such that

$$c_1 \mathbf{b}_1 + \dots + c_n \mathbf{b}_n = \mathbf{v}$$

The vector

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$$

is called the *coordinate vector of \mathbf{v} relative to the basis \mathcal{B}* .



Theorem

Let \mathcal{B} be a basis of a vector space V . If $\mathbf{v}_1, \dots, \mathbf{v}_p, \mathbf{w} \in V$ then:

- 1) Solutions of the equation $x_1 \mathbf{v}_1 + \dots + x_p \mathbf{v}_p = \mathbf{w}$ are the same as solutions of the equation $x_1 [\mathbf{v}_1]_{\mathcal{B}} + \dots + x_p [\mathbf{v}_p]_{\mathcal{B}} = [\mathbf{w}]_{\mathcal{B}}$.
- 2) The set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly independent if and only if the set $\{[\mathbf{v}_1]_{\mathcal{B}}, \dots, [\mathbf{v}_p]_{\mathcal{B}}\}$ is linearly independent.
- 3) $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_p) = V$ if and only if $\text{Span}([\mathbf{v}_1]_{\mathcal{B}}, \dots, [\mathbf{v}_p]_{\mathcal{B}}) = \mathbb{R}^n$.
- 4) $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a basis of V if and only if $\{[\mathbf{v}_1]_{\mathcal{B}}, \dots, [\mathbf{v}_p]_{\mathcal{B}}\}$ is a basis of \mathbb{R}^n .

Example. Recall that \mathbb{P}_2 is the vector space of polynomials of degree ≤ 2 . Consider the following polynomials in \mathbb{P}_2 :

$$p_1(t) = 1 + 2t + t^2$$

$$p_2(t) = 3 + t + 2t^2$$

$$p_3(t) = 1 - 8t - t^2$$

Check if the set $\{p_1, p_2, p_3\}$ is linearly independent.

Theorem

Let $\{v_1, \dots, v_p\}$ be vectors in \mathbb{R}^n . The set $\{v_1, \dots, v_p\}$ is a basis of \mathbb{R}^n if and only if the matrix

$$A = \begin{bmatrix} v_1 & \dots & v_p \end{bmatrix}$$

has a pivot position in every row and in every column (i.e. if A is an invertible matrix).

Corollary

Every basis of \mathbb{R}^n consists of n vectors.

Theorem

Let V be a vector space. If V has a basis consisting of n vectors then every basis of V consists of n vectors.

Definition

A vector space has *dimension* n if V has a basis consisting of n vectors. Then we write $\dim V = n$.

Example.

Theorem

Let V be a vector space such that $\dim V = n$, and let $v_1, \dots, v_p \in V$.

- 1) If $\{v_1, \dots, v_p\}$ is a spanning set of V then $p \geq n$.
- 2) If $\{v_1, \dots, v_p\}$ is a linearly independent set then $p \leq n$.

Corollary

Let V be a vector space such that $\dim V = n$. If W be a subspace of V then $\dim W \leq n$. Moreover, if $\dim W = n$ then $W = V$.

Note.

- 1) One can show that every vector space has a basis.
- 2) Some vector spaces have bases consisting of infinitely many vectors. If V is such vector space then we write $\dim V = \infty$.

Example.