Linear Algebra

$$\mathbb{R}^n = \begin{pmatrix} \text{set of all column vectors} \\ \text{with } n \text{ entries} \end{pmatrix}$$

Column vectors can be added and multiplied by real numbers.

Linear transformation is a function

$$T: \mathbb{R}^n \to \mathbb{R}^m$$
, $T(\mathbf{v}) = A\mathbf{v}$

It satisfies:

- $\bullet \ T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
- $T(c\mathbf{v}) = cT(\mathbf{v})$

Tupical problem: given a vector **b** find all vectors \mathbf{x} such that

$$T(\mathbf{x}) = \mathbf{b}$$

(i.e solve the equation $A\mathbf{x} = \mathbf{b}$).

Fact: Such vectors x are of the form

$$x = v_0 + n$$

where:

- \mathbf{v}_0 is some distinguished solution of Ax = b;
- $n \in Nul(A)$ (i.e. n is a solution of $A\mathbf{x} = \mathbf{0}$).

Calculus

$$C^{\infty}(\mathbb{R}) = \begin{pmatrix} \text{set of all smooth} \\ \text{functions } f \colon \mathbb{R} \to \mathbb{R} \end{pmatrix}$$

Functions can be added and multiplied by real numbers.

Differentiation is a function

$$D \colon C^{\infty}(\mathbb{R}) \to C^{\infty}(\mathbb{R}), \quad D(f) = f'$$

It satisfies:

- D(f+g) = D(f) + D(g)• D(cf) = cD(f)

Typical problem: given a function qfind all functions f such that

$$D(f) = g$$

(i.e find antiderivatives of q).

Fact: Such functions *f* are of the form

$$f = F + C$$

where:

- F is some distinguished antiderivative of q;
- C is a constant function (i.e. C is a solution of D(f) = 0).

Definition

A (real) vector space is a set V together with two operations:

addition

$$\begin{array}{ccc}
V \times V \longrightarrow V \\
(\mathbf{u}, & \mathbf{v}) \longmapsto & \mathbf{u} + \mathbf{v}
\end{array}$$

• multiplication by scalars

$$\mathbb{R} \times V \longrightarrow V$$

$$(c, \mathbf{v}) \longmapsto c \cdot \mathbf{v}$$

Moreover the following conditions must be satisfied:

- 1) u + v = v + u
- 2) (u + v) + w = u + (v + w)
- 3) there is an element $\mathbf{0} \in V$ such that $\mathbf{0} + \mathbf{u} = \mathbf{u}$ for any $\mathbf{u} \in V$
- 4) for any $\mathbf{u} \in V$ there is an element $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
- 5) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- $6) \quad (c+d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
- 7) (cd)u = c(du)
- 8) 1u = u

Elements of V are called *vectors*.

Theorem

If V is a vectors space then:

- 1) $c \cdot \mathbf{0} = \mathbf{0}$ where $c \in \mathbb{R}$ and $\mathbf{0} \in V$ is the zero vector;
- 2) $0 \cdot \mathbf{u} = \mathbf{0}$ where $0 \in \mathbb{R}$, $\mathbf{u} \in V$ and $\mathbf{0}$ is the zero vector;
- 3) $(-1) \cdot u = -u$

Examples of vector spaces.

Defitnition

Let V be a vector space. A subspace of V is a subset $W\subseteq V$ such that

- 1) $0 \in W$
- 2) if $\mathbf{u}, \mathbf{v} \in W$ then $\mathbf{u} + \mathbf{v} \in W$
- 3) if $\mathbf{u} \in W$ and $c \in \mathbb{R}$ then $c\mathbf{u} \in W$.

Example.

Recall: \mathbb{P} = the vector space of all polynomials.

Proposition

Let V be a vector space and $W\subseteq V$ is a subspace then W is itself a vector space.

Example.

Recall: $\mathcal{F}(\mathbb{R})$ = the vector space of all functions $f: \mathbb{R} \to \mathbb{R}$

Some interesting subspaces of $\mathcal{F}(\mathbb{R})$:

- 1) $C(\mathbb{R}) = \text{the subspace of all continuous functions } f : \mathbb{R} \to \mathbb{R}$
- 2) $C^n(\mathbb{R}) = \text{the subspace of all functions } f \colon \mathbb{R} \to \mathbb{R} \text{ that are differentiable } n \text{ or more times.}$
- 3) $C^{\infty}(\mathbb{R}) = \text{the subspace of all smooth functions } f : \mathbb{R} \to \mathbb{R}$ (i.e. functions that have derivatives of all orders: f', f'', f''', . . .).

Note. If V is a vector space then:

- 1) the biggest subspace of V is V itself;
- 2) the smallest subspace of V is the subspace $\{\mathbf{0}\}$ consisting of the zero vector only;
- 3) if a subspace of V contains a non-zero vector, then it contains infinitely many vectors.