

Recall:

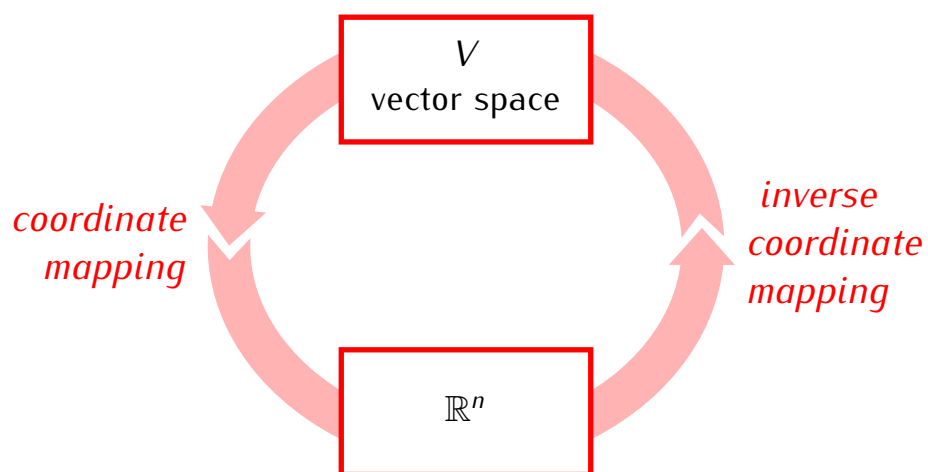
- A vector space is a set  $V$  equipped with operations of addition and multiplication by scalars. These operations must satisfy some properties.
- Some examples of vector spaces:
  - 1)  $\mathbb{R}^n$  = the vector space of column vectors.
  - 2)  $\mathcal{F}(\mathbb{R})$  = the vector space of all functions  $f: \mathbb{R} \rightarrow \mathbb{R}$ .
  - 3)  $C(\mathbb{R})$  = the vector space of all continuous functions  $f: \mathbb{R} \rightarrow \mathbb{R}$ .
  - 4)  $C^\infty(\mathbb{R})$  = the vector space of all smooth functions  $f: \mathbb{R} \rightarrow \mathbb{R}$ .
  - 5)  $M_{m,n}(\mathbb{R})$  = the vector space of all  $m \times n$  matrices.
  - 6)  $\mathbb{P}$  = the vector space of all polynomials.
  - 7)  $\mathbb{P}_n$  = the vector space of polynomials of degree  $\leq n$ .
- If  $V, W$  are vector spaces then a linear transformation is a function  $T: V \rightarrow W$  such that
  - 1)  $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
  - 2)  $T(c\mathbf{v}) = cT(\mathbf{v})$
- Many problems involving  $\mathbb{R}^n$  can be easily solved using row reduction, matrix multiplication etc.
- The same types of problems involving other vector spaces can be difficult to solve.

Next goal:

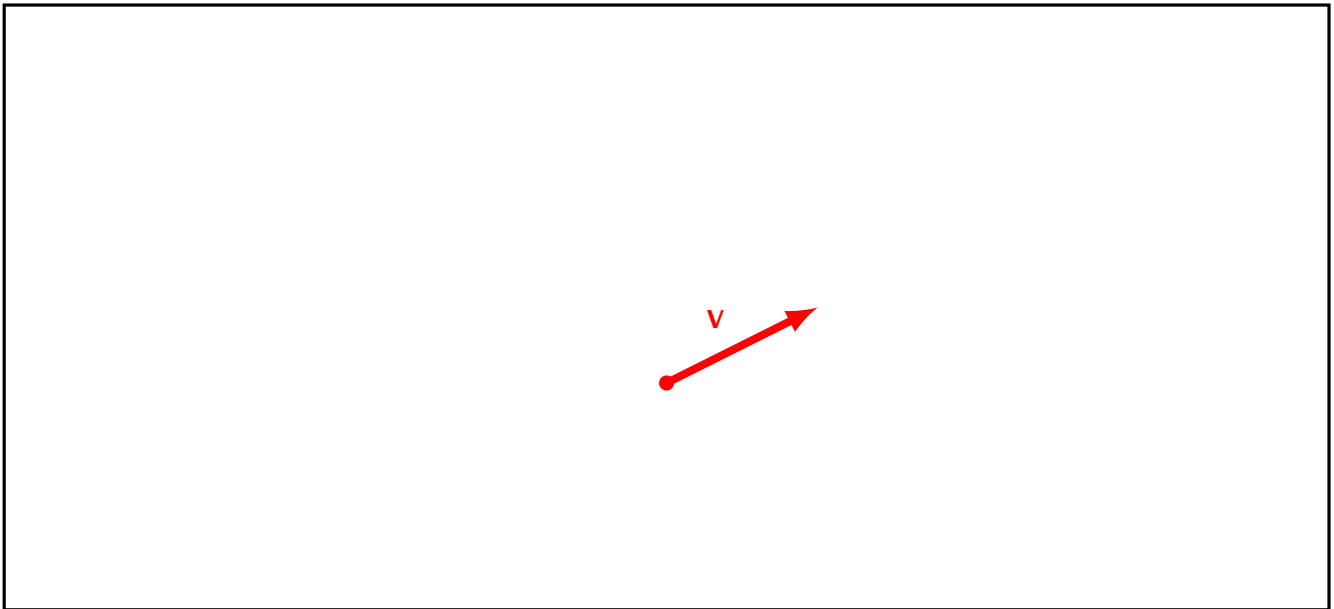
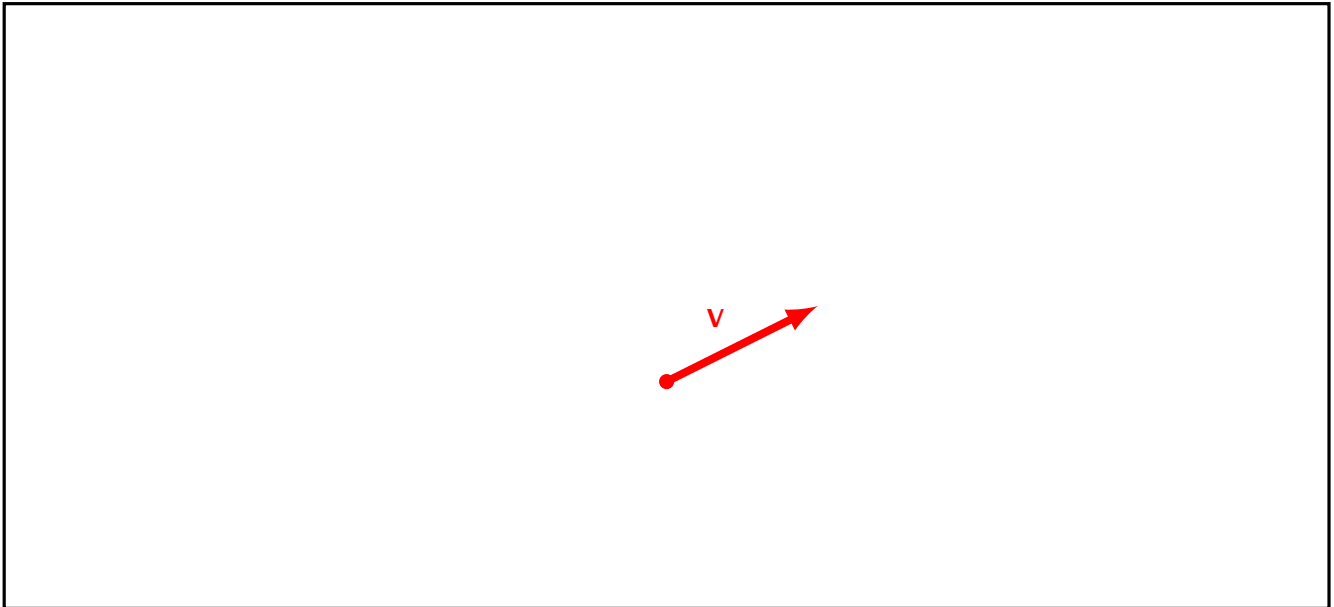
If  $V$  is a *finite dimensional* vector space then we can construct a *coordinate mapping*

$$V \rightarrow \mathbb{R}^n$$

which lets us turn computations in  $V$  into computations in  $\mathbb{R}^n$ .



## Motivation: How to assign coordinates to vectors



### Definition

If  $V$  is a vector space then vector  $\mathbf{w} \in V$  is a *linear combination* of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_p \in V$  if there exist scalars  $c_1, \dots, c_p$  such that

$$\mathbf{w} = c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p$$

### Definition

If  $V$  is a vector space and  $\mathbf{v}_1, \dots, \mathbf{v}_p$  are vectors in  $V$  then

$$\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_p) = \left\{ \begin{array}{l} \text{the set of all} \\ \text{linear combinations} \\ c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p \end{array} \right\}$$

### Definition

If  $V$  is a vector space and  $\mathbf{v}_1, \dots, \mathbf{v}_p$  are vectors in  $V$  such that

$$V = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_p)$$

the the set  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is called the *spanning set* of  $V$ .

**Example.**

### Definition

If  $V$  is a vector space and  $\mathbf{v}_1, \dots, \mathbf{v}_p \in V$  then the set  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is *linearly independent* if the homogenous equation

$$x_1\mathbf{v}_1 + \dots + x_p\mathbf{v}_p = \mathbf{0}$$

has only one, trivial solution  $x_1 = 0, \dots, x_p = 0$ . Otherwise the set is *linearly dependent*.

### Theorem

Let  $V$  be a vector space, and let  $\mathbf{v}_1, \dots, \mathbf{v}_p \in V$ . If the set  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is linearly independent then the equation

$$x_1\mathbf{v}_1 + \dots + x_p\mathbf{v}_p = \mathbf{w}$$

has exactly one solution for any vector  $\mathbf{w} \in \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_p)$ .

### Definition

A *basis* of a vector space  $V$  is an ordered set of vectors

$$\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$$

such that

- 1)  $\text{Span}(\mathbf{b}_1, \dots, \mathbf{b}_n) = V$
- 2) The set  $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  is linearly independent.

### Theorem

A set  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  is a basis of a vector space  $V$  if and only if for each  $\mathbf{v} \in V$  the vector equation

$$x_1 \mathbf{b}_1 + \dots + x_n \mathbf{b}_n = \mathbf{v}$$

has a unique solution.

### Definition

Let  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  be a basis of a vector space  $V$ . For  $\mathbf{v} \in V$  let  $c_1, \dots, c_n$  be the unique numbers such that

$$c_1 \mathbf{b}_1 + \dots + c_n \mathbf{b}_n = \mathbf{v}$$

Then the vector

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$$

is called the *coordinate vector of  $\mathbf{v}$  relative to the basis  $\mathcal{B}$*  and it is denoted by  $[\mathbf{v}]_{\mathcal{B}}$ .



**Example.** Let  $\mathcal{E} = \{1, t, t^2\}$  be the standard basis of  $\mathbb{P}_2$ , and let

$$p(t) = 3 + 2t - 4t^2$$

Find the coordinate vector  $[p]_{\mathcal{E}}$ .

**Example.** Let  $\mathcal{B} = \{1, 1 + t, 1 + t + t^2\}$ . One can check that  $\mathcal{B}$  is a basis of  $\mathbb{P}_2$ .  
Let

$$p(t) = 3 + 2t - 4t^2$$

Find the coordinate vector  $[p]_{\mathcal{B}}$ .

**Example.** Consider the following vectors in  $\mathbb{R}^2$ :

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \mathbf{b}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

One can check that  $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$  is a basis of  $\mathbb{R}^2$ . Find the coordinate vector  $[\mathbf{v}]_{\mathcal{B}}$ .