Recall: If A is an upper triangular matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

then $\det A = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$.

Note. If A is a square matrix then the row echelon form of A is always upper triangular.

$$\begin{bmatrix} \text{square} \\ \text{matrix} \end{bmatrix} \xrightarrow{\text{row red}} \begin{bmatrix} \text{reduced} \\ \text{matrix} \end{bmatrix}$$

upper triangular
so easy to compute
the determinant

Theorem

Let A and B be $n \times n$ matrices.

1) If B is obtained from A by interchanging two rows (or two columns) then

$$\det B = -\det A$$

2) If B is obtained from A by multiplying one row (or one column) of A by a scalar k then

$$\det B = k \cdot \det A$$

2) If B is obtained from A by adding a multiple of one row of A to another row (or adding a multiple of one column to another column) then

$$\det B = \det A$$

Example.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 7 \\ 2 & 5 & 1 \end{bmatrix}$$

$$\det A = 1 \cdot C_{11} + 2 \cdot C_{12} + 3 \cdot C_{13}$$

$$C_{11} = (-1)^{1+1} \cdot \det \begin{bmatrix} 0 & 7 \\ 5 & 1 \end{bmatrix}$$

$$C_{12} = \dots$$

$$B = \begin{bmatrix} 4 \cdot 1 & 4 \cdot 2 & 4 \cdot 3 \\ 1 & 0 & 7 \\ 2 & 5 & 1 \end{bmatrix}$$

$$\det B = (4 \cdot 1) \cdot C_{11} + (4 \cdot 2) \cdot C_{12} + (4 \cdot 3) \cdot C_{13}$$

$$C_{11} = (-1)^{1+1} \cdot \det \begin{bmatrix} 0 & 7 \\ 5 & 1 \end{bmatrix}$$

$$C_{12} = \dots$$

So: $\det B = 4 \cdot (1 \cdot C_{11} + 2 \cdot C_{12} + 3 \cdot C_{13})$

$= 4 \cdot \det A$

Computation of determinants via row reduction

Idea. To compute $\det A$, row reduce A to the row echelon form. Keep track how the determinant changes at each step of the row reduction process.

Example. Compute $\det A$ where

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 4 & 0 & 10 \\ 3 & 4 & 1 & 7 \\ -2 & 5 & 3 & 0 \end{bmatrix}$$

$$\det \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 4 & 0 & 10 \\ 3 & 4 & 1 & 7 \\ -2 & 5 & 3 & 0 \end{bmatrix} = (-1)\cdot \det \begin{bmatrix} 2 & 4 & 0 & 10 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 1 & 7 \\ -2 & 5 & 3 & 0 \end{bmatrix} \cdot \left(\tfrac{1}{2}\right)$$

$$= 2\cdot(-1)\det \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 1 & 7 \\ -2 & 5 & 3 & 0 \end{bmatrix} \;(-3)\;(2) = 2\cdot(-1)\cdot\det \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & -2 & 1 & -8 \\ 0 & 9 & 3 & 10 \end{bmatrix} \;(2)\;(-9)$$

reciprocal of ½

$$= 2\cdot(-1)\cdot\det \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -15 & -17 \end{bmatrix} \;(3) = 2\cdot(-1)\cdot\det \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & -23 \end{bmatrix}$$

upper triangular

$$= 2\cdot(-1)\cdot 1\cdot 1\cdot 5\cdot(-23)$$

$$= 230$$

Theorem

If A is a square matrix then A is invertible if and only if $\det A \neq 0$

<u>**Recall:**</u> A is invertible if and only if its reduced row echelon form is the identity matrix.

<u>Proof:</u>

$$\det A = (\text{some non-zero number}) \cdot \det \begin{bmatrix} \text{reduced} \\ \text{echelon} \\ \text{form of } A \end{bmatrix}$$

1) If A is invertible then

$$\det A = (\text{some non-zero number}) \cdot \overbrace{\det \underbrace{I}_{\text{the identity matrix}}}^{\substack{1 \\ \shortparallel}} \neq 0$$

2) If A is non-invertible then

$$\begin{bmatrix} \text{reduced form} \\ \text{of } A \end{bmatrix} = \text{upper triangular matrix with a zero on the main diagonal}$$

$$\left(\text{e.g. } \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

This gives:

$$\det \begin{bmatrix} \text{reduced form} \\ \text{of } A \end{bmatrix} = 0$$

<u>So:</u> $\det A = 0$.

Further properties of determinants

1) $\det(A^T) = \det A$
2) $\det(AB) = (\det A) \cdot (\det B)$
3) $\det(A^{-1}) = (\det A)^{-1}$

Note. In general $\underbrace{\det(A + B) \neq \det A + \det B.}$

example:

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\det A = 0$ $\det B = 0$

$\det(A+B) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$