

Linear Algebra	Calculus
$\mathbb{R}^n = \left(\begin{array}{l} \text{set of all column vectors} \\ \text{with } n \text{ entries} \end{array} \right)$	$C^\infty(\mathbb{R}) = \left(\begin{array}{l} \text{set of all smooth} \\ \text{functions } f: \mathbb{R} \rightarrow \mathbb{R} \end{array} \right)$
Column vectors can be added and multiplied by real numbers.	Functions can be added and multiplied by real numbers.
Linear transformation is a function $T: \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad T(\mathbf{v}) = A\mathbf{v}$	Differentiation is a function $D: C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}), \quad D(f) = f'$
It satisfies: <ul style="list-style-type: none"> • $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ • $T(c\mathbf{v}) = cT(\mathbf{v})$ 	It satisfies: <ul style="list-style-type: none"> • $D(f + g) = D(f) + D(g)$ • $D(cf) = cD(f)$
Typical problem: given a vector \mathbf{b} find all vectors \mathbf{x} such that $T(\mathbf{x}) = \mathbf{b}$ (i.e solve the equation $A\mathbf{x} = \mathbf{b}$).	Typical problem: given a function g find all functions f such that $D(f) = g$ (i.e find antiderivatives of g).
Fact: Such vectors \mathbf{x} are of the form $\mathbf{x} = \mathbf{v}_0 + \mathbf{n}$ where: <ul style="list-style-type: none"> • \mathbf{v}_0 is some distinguished solution of $A\mathbf{x} = \mathbf{b}$; • $\mathbf{n} \in \text{Nul}(A)$ (i.e. \mathbf{n} is a solution of $A\mathbf{x} = \mathbf{0}$). 	Fact: Such functions f are of the form $f = F + C$ where: <ul style="list-style-type: none"> • F is some distinguished antiderivative of g; • C is a constant function (i.e. C is a solution of $D(f) = 0$).

Definition

A (real) vector space is a set V together with two operations:

- addition

$$\begin{aligned} V \times V &\longrightarrow V \\ (\mathbf{u}, \mathbf{v}) &\longmapsto \mathbf{u} + \mathbf{v} \end{aligned}$$

- multiplication by scalars

$$\begin{aligned} \mathbb{R} \times V &\longrightarrow V \\ (c, \mathbf{v}) &\longmapsto c \cdot \mathbf{v} \end{aligned}$$

Moreover the following conditions must be satisfied:

- 1) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- 2) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- 3) there is an element $\mathbf{0} \in V$ such that $\mathbf{0} + \mathbf{u} = \mathbf{u}$ for any $\mathbf{u} \in V$
- 4) for any $\mathbf{u} \in V$ there is an element $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
- 5) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- 6) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
- 7) $(cd)\mathbf{u} = c(d\mathbf{u})$
- 8) $1\mathbf{u} = \mathbf{u}$

Elements of V are called *vectors*.

Theorem

If V is a vectors space then:

- 1) $c \cdot \mathbf{0} = \mathbf{0}$ where $c \in \mathbb{R}$ and $\mathbf{0} \in V$ is the zero vector;
- 2) $0 \cdot \mathbf{u} = \mathbf{0}$ where $0 \in \mathbb{R}$, $\mathbf{u} \in V$ and $\mathbf{0}$ is the zero vector;
- 3) $(-1) \cdot \mathbf{u} = -\mathbf{u}$

Examples of vector spaces.