Recall:

- A basis of a vector space V is a set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ such that

  1) $\mathrm{Span}(\mathbf{b}_1, \ldots, \mathbf{b}_n) = V$

  2) The set $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is linearly independent.
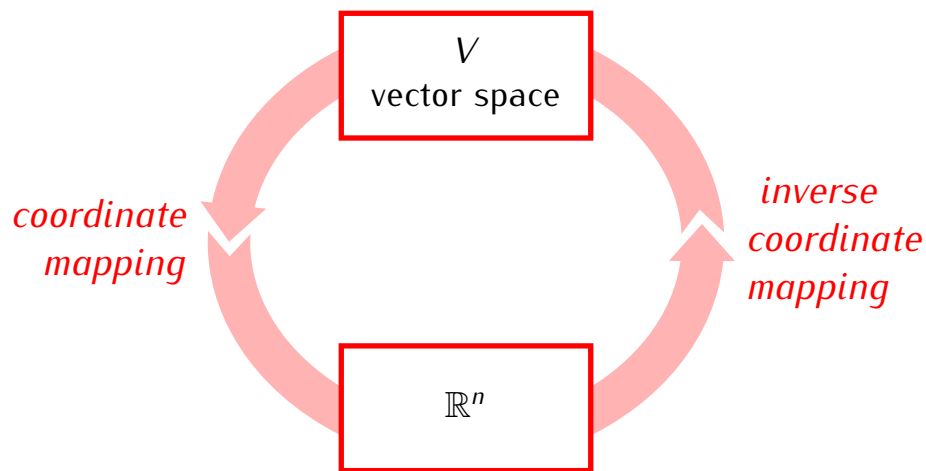
- For $\mathbf{v} \in V$ let $c_1, \ldots, c_n$ be the unique numbers such that

$$c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n = \mathbf{v}$$

The vector

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$$

is called the *coordinate vector of* $\mathbf{v}$ *relative to the basis* $\mathcal{B}$.

V
vector space

coordinate mapping

inverse coordinate mapping

$\mathbb{R}^n$

Theorem

Let $\mathcal{B}$ be a basis of a vector space V. If $\mathbf{v}_1, \ldots \mathbf{v}_p, \mathbf{w} \in V$ then:

1) Solutions of the equation $x_1 \mathbf{v}_1 + \ldots + x_p \mathbf{v}_p = \mathbf{w}$ are the same as solutions of the equation $x_1 \left[\mathbf{v}_1 \right]_{\mathcal{B}} + \ldots + x_p \left[\mathbf{v}_p \right]_{\mathcal{B}} = \left[\mathbf{w} \right]_{\mathcal{B}}$.

2) The set of vectors $\{\mathbf{v}_1, \ldots \mathbf{v}_p\}$ is linearly independent if and only if the set $\left\{ \left[\mathbf{v}_1 \right]_{\mathcal{B}}, \ldots, \left[\mathbf{v}_p \right]_{\mathcal{B}} \right\}$ is linearly independent.

3) $\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p) = V$ if any only if $\mathrm{Span}\left(\left[\mathbf{v}_1 \right]_{\mathcal{B}}, \ldots, \left[\mathbf{v}_p \right]_{\mathcal{B}} \right) = \mathbb{R}^n$.

4) $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a basis of V if and only if $\left\{ \left[\mathbf{v}_1 \right]_{\mathcal{B}}, \ldots, \left[\mathbf{v}_p \right]_{\mathcal{B}} \right\}$ is a basis of $\mathbb{R}^n$.

Example. Recall that $\mathbb{P}_2$ is the vector space of polynomials of degree ≤ 2. Consider the following polynomials in $\mathbb{P}_2$:

$$p_1(t) = 1 + 2t + t^2$$
$$p_2(t) = 3 + t + 2t^2$$
$$p_3(t) = 1 - 8t - t^2$$

Check if the set $\{p_1, p_2, p_3\}$ is linearly independent.

> **Theorem**
>
> Let $\{v_1, \ldots, v_p\}$ be vectors in $\mathbb{R}^n$. The set $\{v_1, \ldots, v_p\}$ is a basis of $\mathbb{R}^n$ if and only if the matrix
> $$A = \begin{bmatrix} v_1 & \ldots & v_p \end{bmatrix}$$
> has a pivot position in every row and in every column (i.e. if A is an invertible matrix).

> **Corollary**
>
> Every basis of $\mathbb{R}^n$ consists of n vectors.

> **Theorem**
>
> Let V be a vector space. If V has a basis consisting of n vectors then every basis of V consists of n vectors.

> **Definition**
>
> A vector space has *dimension n* if V has a basis consisting of n vectors. Then we write $\dim V = n$.

Example.

Theorem

Let V be a vector space such that $\dim V = n$, and let $\mathbf{v}_1, \ldots \mathbf{v}_p \in V$.

1) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a spanning set of V then $p \geq n$.

2) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a linearly independent set then $p \leq n$.

Corollary

Let V be a vector space such that $\dim V = n$. If W be a subspace of V then $\dim W \leq n$. Moreover, if $\dim W = n$ then $W = V$.

Note.

1) One can show that every vector space has a basis.

2) Some vector spaces have bases consisting of infinitely many vectors. If V is such vector space then we write $\dim V = \infty$.

Example.