A simple error correcting code: triple repeat.

message: 1011

Problem: The encoded message is 3 times longer than the original message.

Better error correction: Hamming (7,4) code.

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

<div align="center" style="color:red">decoding matrix</div>

<div align="center" style="color:red">encoding matrix</div>

message: 10111101

Encoding.

1) Split the message into vectors with 4 entries, and multiply each vector by the encoding matrix E.

2) Reduce all numbers obtained in step 1 modulo 2. That is, add or subtract from each number a multiple of 2 to get either 0 or 1.

Encoded message:

Received message:

Decoding. Split the received message into vectors with 7 entries, multiply each vector by the decoding matrix D, and reduce modulo 2.
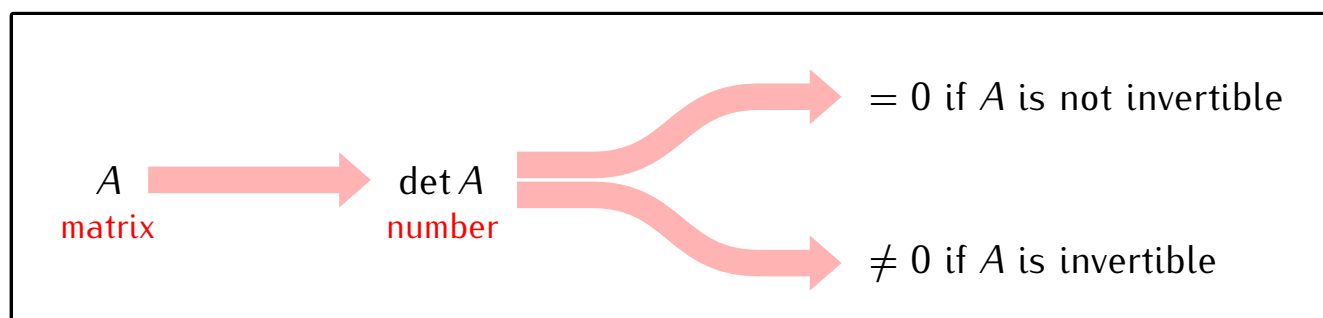
Decoded message:

How the Hamming code works:

Recall: If an $n \times n$ matrix A is invertible then:

- the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b} \in \mathbb{R}^n$
- the linear transformation $T_A \colon \mathbb{R}^n \to \mathbb{R}^n$, $T_A(\mathbf{v}) = A\mathbf{v}$ has an inverse function.

Determinants recognize which matrices are invertible:

$\begin{array}{c} A \\ \text{matrix} \end{array} \longrightarrow \begin{array}{c} \det A \\ \text{number} \end{array} \begin{cases} = 0 \text{ if } A \text{ is not invertible} \\ \neq 0 \text{ if } A \text{ is invertible} \end{cases}$

Example: Determinant for a 1×1 matrix.

$$A = \begin{bmatrix} a \end{bmatrix}$$

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