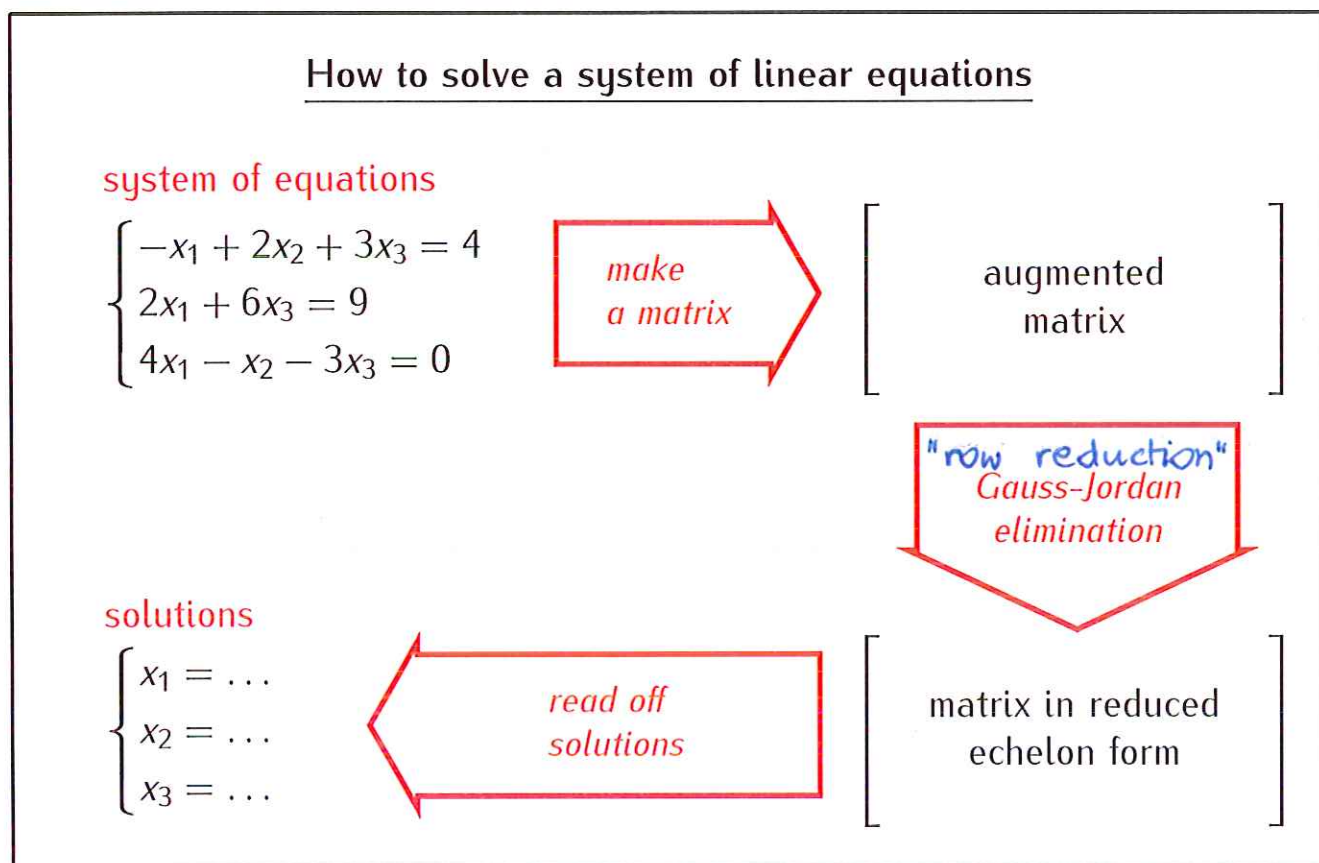


Recall:

- Every system of linear equations can be represented by a matrix
- Elementary row operations:
  - interchange of two rows
  - multiplication of a row by a non-zero number
  - addition of a multiple of one row to another row.
- Elementary row operations do not change solutions of systems of linear equations.

## Definition

A matrix is in the *row echelon form* if:

- 1) the first non-zero entry of each row is a 1 ("a leading one");
- 2) the leading one in each row is to the right of the leading one in the row above it.

A matrix is in the *reduced row echelon form* if in addition it satisfies:

- 3) all entries above each leading one are 0.

leading ones

$$\begin{bmatrix} \textcircled{1} & * & * & * & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & \textcircled{1} & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & \textcircled{1} & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(\* = any number)

## Example

$$\begin{bmatrix} \boxed{1} & \boxed{0} & 4 & \boxed{0} & 7 & \boxed{0} \\ 0 & \boxed{1} & 5 & \boxed{0} & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 3 & \boxed{0} \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

reduced row  
echelon form

$$\begin{bmatrix} \boxed{1} & 2 & 4 & 6 & 7 & 0 \\ 0 & \boxed{1} & 5 & 0 & 1 & 2 \\ 0 & 0 & 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

row echelon  
form

$$\begin{bmatrix} \boxed{1} & 0 & 4 & 0 & 7 & 0 \\ 0 & \boxed{0} & 1 & 0 & 1 & 0 \\ 0 & \boxed{1} & 3 & 6 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

NOT row echelon  
form

### Fact

If a system of linear equations is represented by a matrix in the reduced echelon form then it is easy to solve the system.

### Example

$$\begin{array}{c} x_1 \quad x_2 \quad x_3 \quad x_4 \\ \left[ \begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right] \end{array}$$

$$\rightarrow \begin{cases} x_1 + 3x_3 = 0 \\ x_1 + 7x_3 = 0 \\ x_4 = 0 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 = 1 \end{cases}$$

$$0 = 1$$

can't happen!

so: NO SOLUTIONS

### Proposition

A matrix in the reduced echelon form represents an inconsistent system if and only if it contains a row of the form

$$[0 \quad 0 \quad 0 \quad \dots \quad 0 \quad 1]$$

i.e. with the leading one in the last column.

### Example

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \hline 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

no leading one  
in the column of  $x_3$

$$\begin{cases} x_1 + 3x_3 = 0 \\ x_2 + 7x_3 = 0 \\ x_4 = 0 \\ 0 = 0 \end{cases}$$

↓

$$\begin{cases} x_1 = -3x_3 \\ x_2 = -7x_3 \\ x_3 = \text{any number} \\ x_4 = 0 \end{cases}$$

$x_3$  is a free variable

$$\begin{cases} x_1 = -3x_3 \\ x_2 = -7x_3 \\ x_3 = x_3 \\ x_4 = 0 \end{cases}$$

INFINITELY MANY SOLUTIONS

### Note

In an augmented matrix in the reduced echelon form free variables correspond to columns of the coefficient matrix that do not contain leading ones.

### Example

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \hline 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 8 \end{array}$$

$$\begin{cases} x_1 = 5 \\ x_2 = 6 \\ x_3 = 7 \\ x_4 = 8 \end{cases}$$

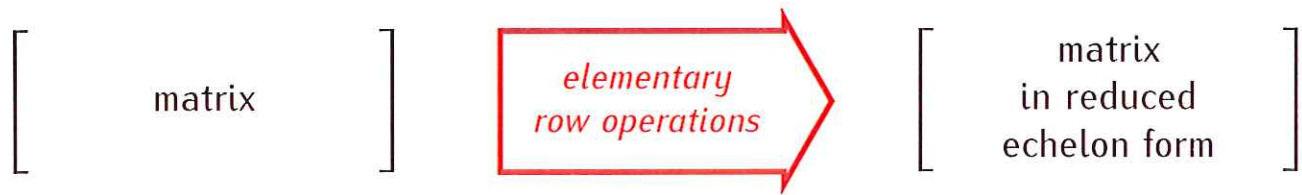
EXACTLY ONE SOLUTION

### Note

A matrix in the reduced row echelon form represents a system of equations with exactly one solution if and only if it has a leading one in every column except for the last one.



## Gauss-Jordan elimination process (= row reduction)



- ① Interchange rows, if necessary, to bring a non-zero element to the top of the first non-zero column of the matrix.
- ② Multiply the first row so that its first non-zero entry becomes 1.
- ③ Add multiples of the first row to eliminate non-zero entries below the leading one.
- ④ Ignore the first row; apply steps 1-3 to the rest of the matrix.
- ⑤ Eliminate non-zero entries above all leading ones.

Example.

$$\begin{bmatrix} 0 & 4 & -8 & 0 & 4 \\ 2 & 6 & -6 & -2 & -4 \\ 2 & 7 & -8 & 0 & -1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 2 & 6 & -6 & -2 & -4 \\ 0 & 4 & -8 & 0 & 4 \\ 2 & 7 & -8 & 0 & -1 \end{bmatrix} \cdot \frac{1}{2}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 3 & -3 & -1 & -2 \\ 0 & 4 & -8 & 0 & 4 \\ 2 & 7 & -8 & 0 & -1 \end{bmatrix} \cdot (-2)$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 3 & -3 & -1 & -2 \\ 0 & 4 & -8 & 0 & 4 \\ 0 & 1 & -2 & 2 & 3 \end{bmatrix} \cdot \frac{1}{4}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 3 & -3 & -1 & -2 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 1 & -2 & 2 & 3 \end{bmatrix} \cdot (-1)$$

$$\begin{bmatrix} 1 & 3 & -3 & -1 & -2 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix} \cdot \frac{1}{2}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 3 & -3 & -1 & -2 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \cdot 1$$

matrix in the echelon form  
(not reduced yet!)

$$\downarrow$$

$$\begin{bmatrix} 1 & 3 & -3 & 0 & -1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \cdot (-3)$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & -4 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

matrix in the reduced  
echelon form

(THE END)

## How to solve systems of linear equations: example

$$\begin{cases} 4x_2 - 8x_3 = 4 \\ 2x_1 + 6x_2 - 6x_3 - 2x_4 = -4 \\ 2x_1 + 7x_2 - 8x_3 = -1 \end{cases}$$

①  
→

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \hline 0 & 4 & -8 & 0 & 4 \\ 2 & 6 & -6 & -2 & -4 \\ 2 & 7 & -8 & 0 & -1 \end{array}$$

↓ ② row reduction

(see the previous page for computations)

$$\begin{cases} x_1 + 3x_3 = -4 \\ x_2 - 2x_3 = 1 \\ x_4 = 1 \end{cases}$$

←

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \hline 1 & 0 & 3 & 0 & -4 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}$$

↑ free variable

Simplify:

$$\begin{cases} x_1 = -3x_3 - 4 \\ x_2 = 2x_3 + 1 \\ x_3 = \text{free} \\ x_4 = 1 \end{cases}$$

(infinitely many solutions)