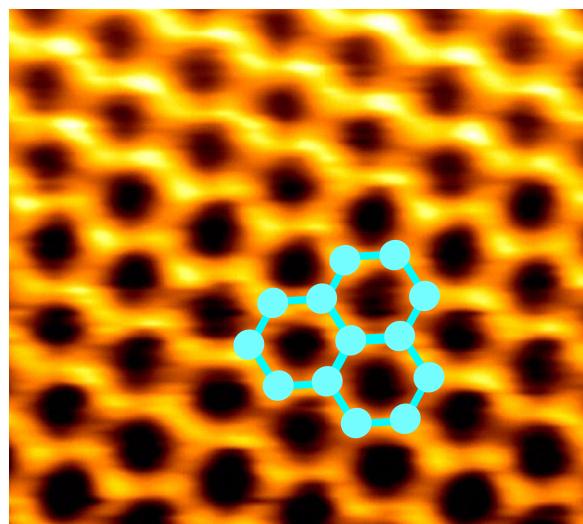
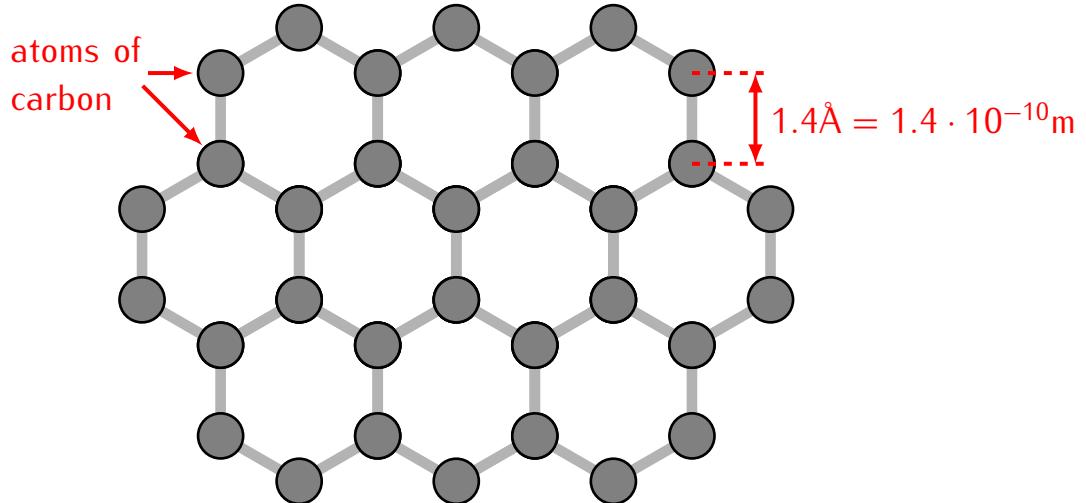


**Note.** Choosing a convenient basis can simplify computations.

**Example.** Graphene lattice.



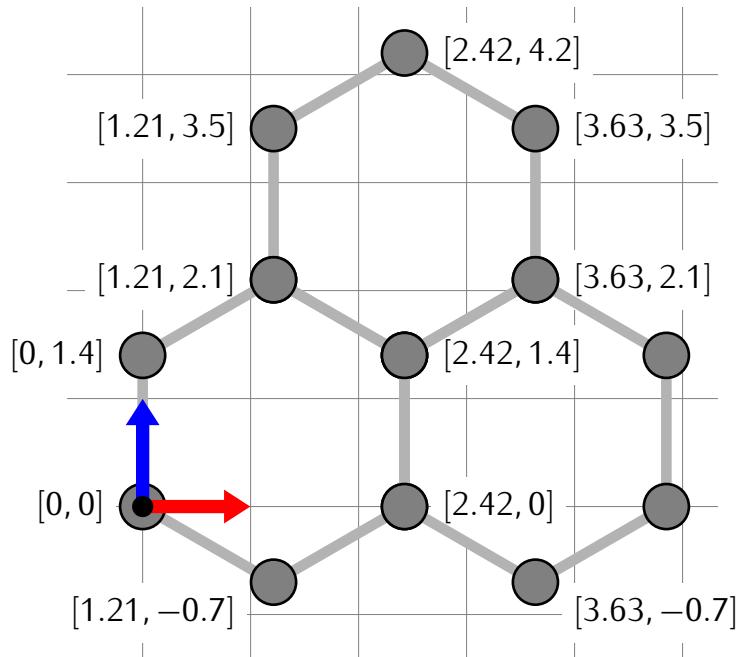
*Image of graphene taken with an atomic force microscope.  
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## Coordinates of atoms in the graphene lattice

In the standard basis  
 $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$ :

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

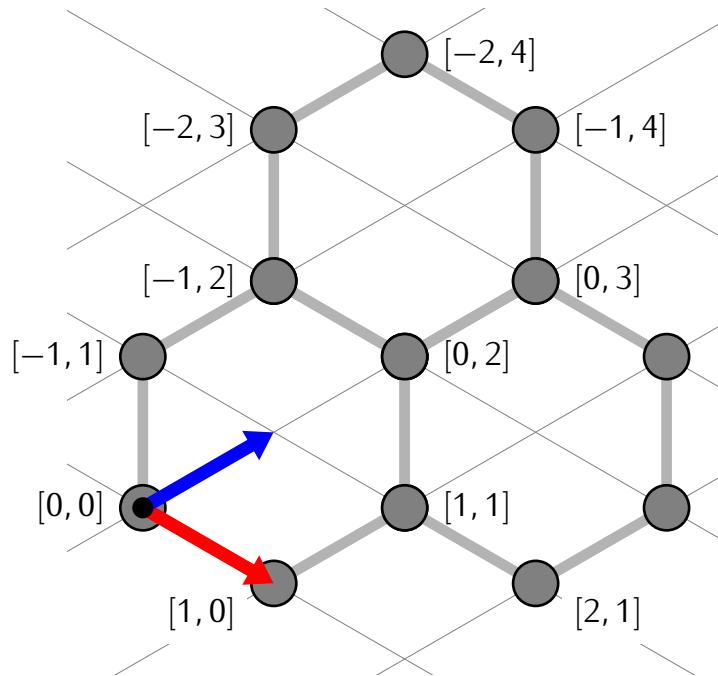
$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



In a more convenient basis  $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ :

$$\mathbf{b}_1 = \begin{bmatrix} 1.21 \\ -0.7 \end{bmatrix}$$

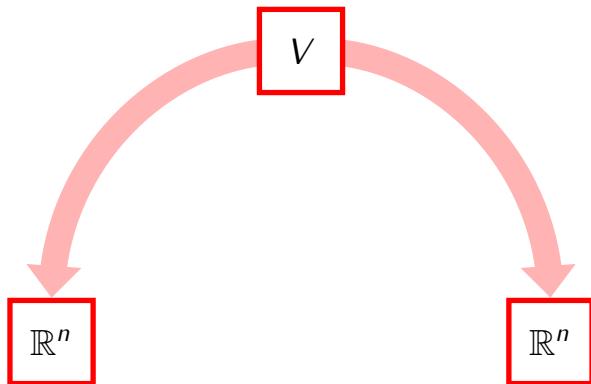
$$\mathbf{b}_2 = \begin{bmatrix} 1.21 \\ 0.7 \end{bmatrix}$$



**Problem** Let

$$\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}, \quad \mathcal{D} = \{\mathbf{d}_1, \dots, \mathbf{d}_1\}$$

be two bases of a vector space  $V$ , and let  $\mathbf{v} \in V$ . Assume that we know  $[\mathbf{v}]_{\mathcal{B}}$ . What is  $[\mathbf{v}]_{\mathcal{D}}$ ?



## Definition

Let  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  and  $\mathcal{D} = \{\mathbf{d}_1, \dots, \mathbf{d}_1\}$  be two bases of a vector space  $V$ . The matrix

$$P_{\mathcal{D} \leftarrow \mathcal{B}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{D}} & [\mathbf{b}_2]_{\mathcal{D}} & \cdots & [\mathbf{b}_n]_{\mathcal{D}} \end{bmatrix}$$

is called the *change of coordinates matrix* from the basis  $\mathcal{B}$  to the basis  $\mathcal{D}$ .

## Propostion

Let  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  and  $\mathcal{D} = \{\mathbf{d}_1, \dots, \mathbf{d}_1\}$  be two bases of a vector space  $V$ . For any vector  $\mathbf{v} \in V$  we have

$$[\mathbf{v}]_{\mathcal{D}} = P_{\mathcal{D} \leftarrow \mathcal{B}} \cdot [\mathbf{v}]_{\mathcal{B}}$$

**Example.** Let  $\mathbb{P}_2$  be the vector space of polynomials of degree  $\leq 2$ . Consider two bases of  $\mathbb{P}_2$ :

$$\mathcal{B} = \{1, 1 + t, 1 + t + t^2\}$$
$$\mathcal{D} = \{1 + t, 1 - 5t, 2 + t^2\}$$

1) Compute the change of coordinates matrix  $P_{\mathcal{D} \leftarrow \mathcal{B}}$ .

2) Let  $p \in \mathbb{P}_2$  be a polynomial such that

$$[p]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

Compute  $[p]_{\mathcal{D}}$ .

## Proposition

If  $\mathcal{B}, \mathcal{D}, \mathcal{E}$  are three bases of a vector space  $V$  then:

- 1)  $P_{\mathcal{B} \leftarrow \mathcal{D}} = (P_{\mathcal{D} \leftarrow \mathcal{B}})^{-1}$
- 2)  $P_{\mathcal{E} \leftarrow \mathcal{B}} = P_{\mathcal{E} \leftarrow \mathcal{D}} \cdot P_{\mathcal{D} \leftarrow \mathcal{B}}$

What we want:



What we have:

