**Recall:** If $A$ is an $m \times n$ matrix then

$$A \cdot \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$$

**Definition**

If $A$ is an $m \times n$ matrix then the function

$$T_A\colon \mathbb{R}^n \to \mathbb{R}^m$$

given by $T_A(\mathbf{v}) = A\mathbf{v}$ is called the *matrix transformation associated to* $A$.

**Example.**

Let $T_A\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the matrix transformation defined by the matrix

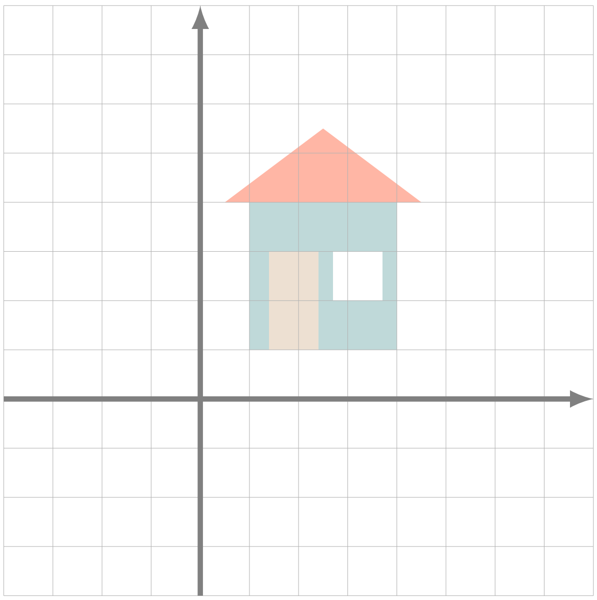
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{bmatrix}$$

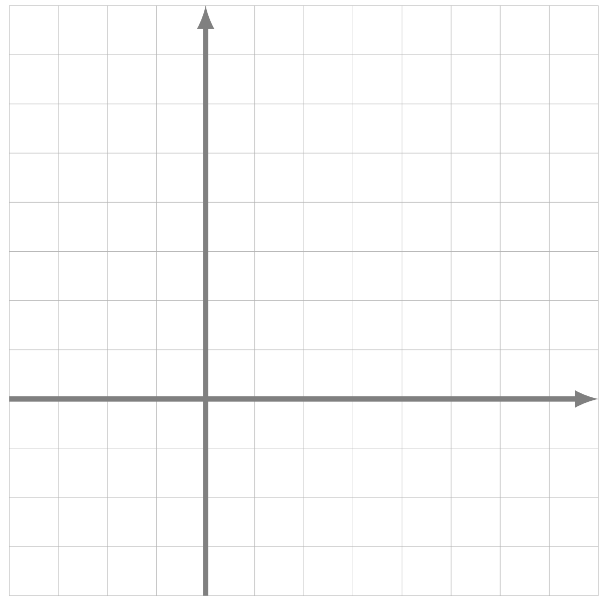
**1)** Compute $T_A(\mathbf{v})$ where $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

**2)** Find a vector $\mathbf{v}$ such that $T_A(\mathbf{v}) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

## Geometric interpretation of matrix transformations $\mathbb{R}^2 \to \mathbb{R}^2$

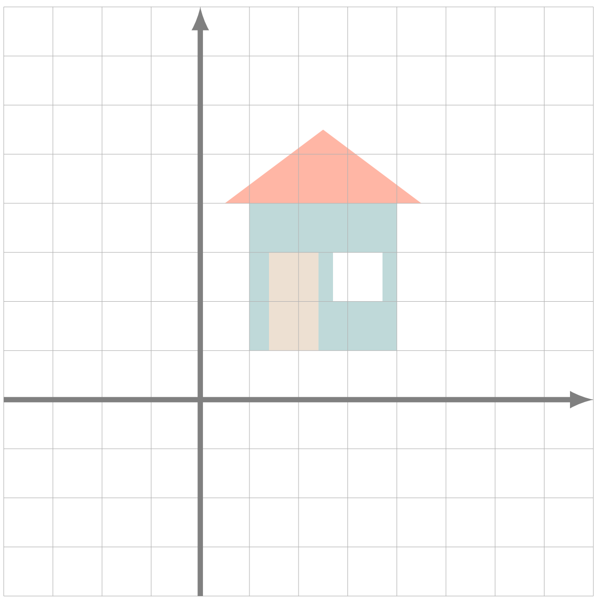
$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

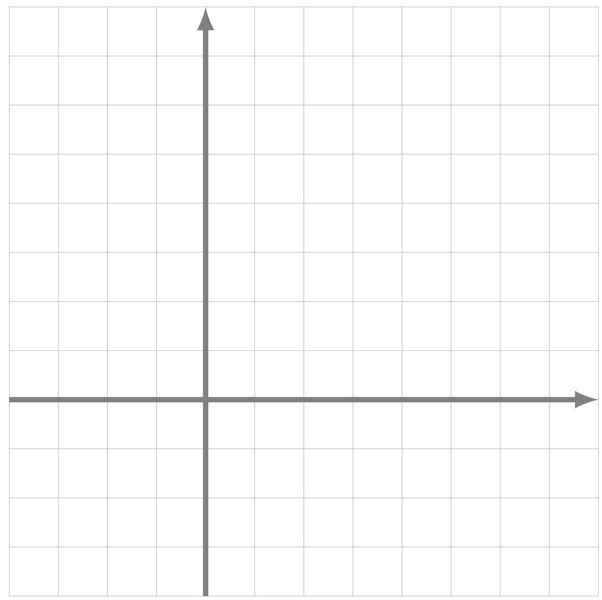




**Example.**

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$





**Note**

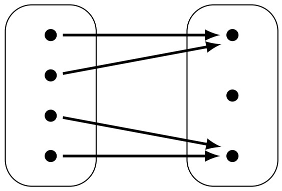
If $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation associated to a matrix $A$ then:

- $\mathrm{Col}(A)$ = the set of values of $T_A$.
- $\mathrm{Nul}(A)$ = the set of vectors $\mathbf{v}$ such that $T_A(\mathbf{v}) = \mathbf{0}$.
- $T_A(\mathbf{v}) = T_A(\mathbf{w})$ if and only if $\mathbf{w} = \mathbf{v} + \mathbf{n}$ for some $\mathbf{n} \in \mathrm{Nul}(A)$.

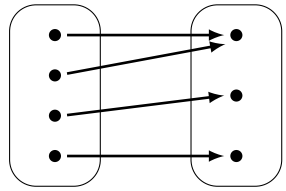
**Recall:**

A function $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is:

- *onto* if for each $\mathbf{b} \in \mathbb{R}^m$ there is $\mathbf{v} \in \mathbb{R}^n$ such that $F(\mathbf{v}) = \mathbf{b}$;

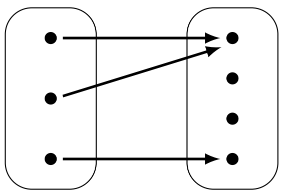


not onto

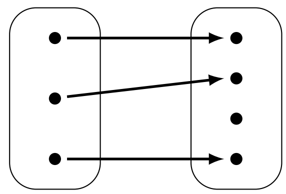


onto

- *one-to-one* if for any $\mathbf{v}_1, \mathbf{v}_2$ such that $\mathbf{v}_1 \neq \mathbf{v}_2$ we have $F(\mathbf{v}_2) \neq F(\mathbf{v}_2)$.



not one-to-one



one-to-one

**Proposition**

Let $A$ be an $m \times n$ matrix. The following conditions are equivalent:

**1)** The matrix transformation $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ is onto.

**2)** $\mathrm{Col}(A) = \mathbb{R}^m$.

**3)** The matrix $A$ has a pivot position in every row.

**Proposition**

Let $A$ be an $m \times n$ matrix. The following conditions are equivalent:

1) The matrix transformation $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one.

2) $\mathrm{Nul}(A) = \{0\}$.

3) The matrix $A$ has a pivot position in every column.

**Example.** For the following $2 \times 2$ matrix $A$ check if the matrix transformation $T_A\colon \mathbb{R}^2 \to \mathbb{R}^2$ is onto and if it is one-to-one.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

**Example.** For the following $3 \times 4$ matrix $A$ check if the matrix transformation $T_A\colon \mathbb{R}^4 \to \mathbb{R}^3$ is onto and if it is one-to-one.

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -2 & -2 & 1 & -5 \\ 1 & 1 & -1 & 4 \end{bmatrix}$$

**Proposition**

Let $A$ be an $m \times n$ matrix. If the matrix transformation $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ is both onto and one-to-one then we must have $m = n$ (i.e. $A$ must be a square matrix).