MTH 309H Practice Exam 3

1. Consider the following vectors in $\mathbb{R}^4$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of some subspace V of $\mathbb{R}^4$.

a) Find an orthogonal basis $\mathcal{D} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of the subspace V.

b) Compute the vector $\mathrm{proj}_V \mathbf{u}$, the orthogonal projection of $\mathbf{u}$ onto V.

2. Find the equation $f(x) = ax + b$ of the least square line for the points $(1, 0)$, $(-1, 2)$, $(2, 1)$.

3. Find a basis of the eigenspace for the eigenvalue $\lambda = 3$ of the following matrix:

$$A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$$

4. Consider the following matrix A and vector $\mathbf{v}$:

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Knowing that A has an eigenvalue $\lambda_1 = -2$ and that $\mathbf{v}$ is an eigenvector of A, diagonalize this matrix. That is, find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$. Note: you do not need to compute P^{-1}.

5. Consider the following vectors:

$$\mathbf{w}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \qquad \mathbf{w}_2 = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

Assume that B is a symmetric matrix with eigenvalues $\lambda_1 = 9$ and $\lambda_2 = -9$ such that $\mathbf{w}_1$ and $\mathbf{w}_2$ are eigenvectors of B corresponding to the eigenvalue $\lambda_1 = 9$.

a) Find a vector $\mathbf{v}$, which is eigenvector of B corresponding to the eigenvalue $\lambda_2 = -9$.

b) Compute the matrix B.

6. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^2$ which satisfy $||\mathbf{u} + \mathbf{v}|| = ||\mathbf{u} - \mathbf{v}||$ then $\mathbf{u}$ must be orthogonal to $\mathbf{v}$.

b) If A is a 3×3 matrix which is both symmetric and orthogonal then $A^3 = A$.

c) If A is an $n \times n$ matrix and $\mathbf{v}$ is eigenvector of A, then $\mathbf{v}$ is also an eigenvector of A^2.

d) If A is an $n \times n$ matrix and λ is eigenvalue of A, then λ is also an eigenvalue of A^2.