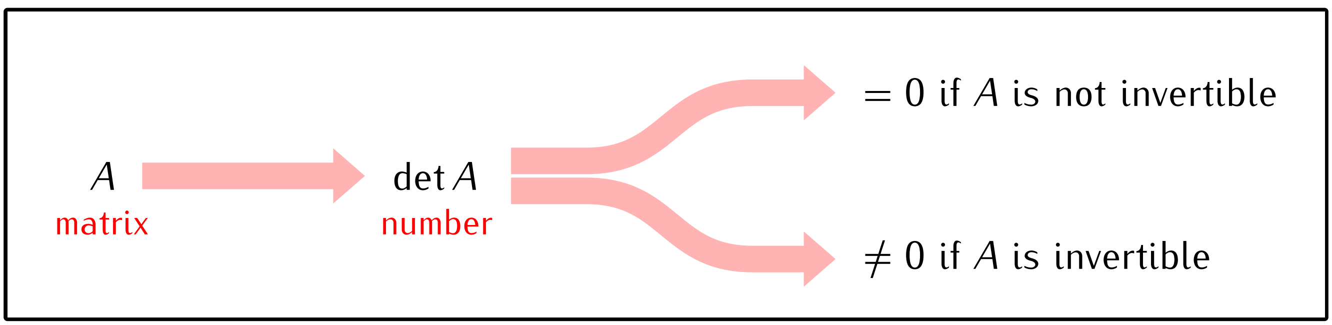
Recall: If an $n \times n$ matrix A is invertible then:

- the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b} \in \mathbb{R}^n$
- the linear transformation $T_A \colon \mathbb{R}^n \to \mathbb{R}^n$, $T_A(\mathbf{v}) = A\mathbf{v}$ has an inverse function.

Determinants recognize which matrices are invertible:



Example: Determinant for a 1×1 matrix.

$$A = [\ a\ ]$$

Note: $A^{-1} = [a^{-1}]$ since $A \cdot A^{-1} = [a] \cdot [a^{-1}] = \underline{[1]}$

1x1 identity matrix

This gives:

$A = [a]$ is invertible if and only if $a \neq 0$

Thus we can define:

If $A = [a]$ then:

$\det A = a$

Example: Determinant for a 2×2 matrix.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Recall: A is invertible if it has a pivot position in every row and column.

Case 1: $a \neq 0$

$$\begin{bmatrix} a & b \\ c & c \end{bmatrix} \cdot \left(\frac{1}{a}\right) \longrightarrow \begin{bmatrix} 1 & b/a \\ c & d \end{bmatrix} (-c) \longrightarrow \begin{bmatrix} 1 & b/a \\ 0 & \underbrace{d - \frac{cb}{a}} \end{bmatrix}$$

if not zero this is a pivot

Upshot: If $d - \frac{cb}{a} \neq 0$ then A is invertible.

$$\boxed{ad - bc \neq 0}$$

Case 2: $c \neq 0$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} c & d \\ a & b \end{bmatrix} \cdot \left(\frac{1}{c}\right) \longrightarrow \begin{bmatrix} 1 & d/c \\ a & b \end{bmatrix} (-a) \longrightarrow \begin{bmatrix} 1 & d/c \\ 0 & \underbrace{b - \frac{ad}{c}} \end{bmatrix}$$

if not zero this is a pivot

Upshot: If $b - \frac{ad}{c} \neq 0$ then A is invertible.

$$bc - ad \neq 0$$

$$\boxed{ad - bc \neq 0}$$

Case 3: $a = 0$, $c = 0$

Then $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is not invertible and $ad - bc = 0$,

This gives: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$

Thus we can define:

$$\det A = ad - bc$$

Definition

If A is an $n \times n$ matrix then for $1 \leq i, j \leq n$ the (i, j)-*minor* of A is the matrix A_{ij} obtained by deleting the i^{th} row and j^{th} column of A.

Example.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ \cancel{4} & 5 & 6 \\ \cancel{7} & 8 & 9 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} \cancel{1} & 2 & 3 \\ \cancel{4} & \cancel{5} & \cancel{6} \\ \cancel{7} & 8 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}$$

$$\vdots$$

Definition

Let A be an $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

1) If $n = 1$, i.e. $A = \begin{bmatrix} a_{11} \end{bmatrix}$, then $\det A = a_{11}$

2) If $n > 1$ then

$$\begin{aligned} \det A = & \quad (-1)^{1+1} a_{11} \cdot \det A_{11} \\ & + \ (-1)^{1+2} a_{12} \cdot \det A_{12} \\ & \dots \qquad \dots \quad \dots \quad \dots \\ & + \ (-1)^{1+n} a_{1n} \cdot \det A_{1n} \end{aligned}$$

Example. ($n = 2$)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\begin{aligned} \det A &= (-1)^{1+1} a_{11} \cdot \det A_{11} + (-1)^{1+2} a_{12} \cdot \det A_{12} \\ &= 1 \cdot 1 \cdot \det [4] + (-1) \cdot 2 \cdot \det [3] \\ &= 1 \cdot 4 - 2 \cdot 3 = -2 \end{aligned}$$

Note

If A is a 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

then $\det A = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$

Example. (n=3)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\det A = (-1)^{1+1} \cdot a_{11} \cdot \det A_{11} + (-1)^{1+2} \cdot a_{12} \cdot \det A_{12} + (-1)^{1+3} \cdot a_{13} \cdot \det A_{13}$$

$$= 1 \cdot 1 \cdot \det \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} + (-1) \cdot 2 \cdot \det \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} + 1 \cdot 3 \cdot \det \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

$$= 1 \cdot (5 \cdot 9 - 6 \cdot 8) - 2 \cdot (4 \cdot 9 - 6 \cdot 7) + 3 \cdot (4 \cdot 8 - 5 \cdot 7)$$

$$= 1 \cdot (-3) - 2 \cdot (-6) + 3 \cdot (-3)$$

$$= -3 + 12 - 9 = 0$$

A direct way of computing the determinant of a 3×3 matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\left[\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}\right] \begin{array}{cc} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{array}$$

$-\quad -\quad -\quad +\quad +\quad +$

$$\det A = (1\cdot 5\cdot 9) + (2\cdot 6\cdot 7) + (3\cdot 4\cdot 8) - (3\cdot 5\cdot 7) - (1\cdot 6\cdot 8) - (2\cdot 4\cdot 9)$$

$$= 45 + 84 + 96 - 105 - 48 - 72$$

$$= 225 - 225 = 0$$

Example (n=4)

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 4 & 0 & 1 \\ 2 & 1 & 6 & 1 \\ 3 & 5 & 7 & 0 \end{bmatrix}$$

$$\det A = (-1)^{1+1} \cdot a_{11} \cdot \det A_{11} + (-1)^{1+2} \cdot a_{12} \cdot \det A_{12} + (-1)^{1+3} a_{13} \cdot \det A_{13} + (-1)^{1+4} a_{14} \cdot \det A_{14}$$

$$= 1 \cdot 1 \cdot \det A_{11} + \underbrace{(-1) \cdot 0 \cdot \det A_{12}}_{= 0} + 1 \cdot 2 \cdot \det A_{13} + \underbrace{(-1) \cdot 0 \cdot \det A_{14}}_{= 0}$$

$$\det A_{11} = \det \begin{bmatrix} 4 & 0 & 1 \\ 1 & 6 & 1 \\ 5 & 7 & 0 \end{bmatrix} \begin{matrix} 4 & 0 \\ 1 & 6 \\ 5 & 7 \end{matrix}$$

$$= 4 \cdot 6 \cdot 0 + 0 \cdot 1 \cdot 5 + 1 \cdot 1 \cdot 7 - 1 \cdot 6 \cdot 5 - 4 \cdot 1 \cdot 7 - 0 \cdot 1 \cdot 0$$

$$= 0 + 0 + 7 - 30 - 28 - 0$$

$$= -51$$

$$\det A_{13} = \det \begin{bmatrix} 0 & 4 & 1 \\ 2 & 1 & 1 \\ 3 & 5 & 0 \end{bmatrix} \begin{matrix} 0 & 4 \\ 2 & 1 \\ 3 & 5 \end{matrix}$$

$$= 0 \cdot 1 \cdot 0 + 4 \cdot 1 \cdot 3 + 1 \cdot 2 \cdot 5 - 1 \cdot 1 \cdot 3 - 0 \cdot 1 \cdot 5 - 4 \cdot 2 \cdot 0$$

$$= 0 + 12 + 10 - 3 - 0 - 0$$

$$= 19$$

We obtain:

$$\det A = 1 \cdot 1 \cdot (-51) + 2 \cdot 19 = -51 + 38 = -13$$

Note. In order to compute the determinant of an $n \times n$ matrix in this way we need to compute:

$$\begin{array}{rl}
n & \text{determinants of } (n-1)\times(n-1) \text{ matrices} \\
n(n-1) & \text{determinants of } (n-2)\times(n-2) \text{ matrices} \\
n(n-1)(n-2) & \text{determinants of } (n-3)\times(n-3) \text{ matrices} \\
\dots\quad\dots\quad\dots\quad\dots & \quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots \\
n(n-1)(n-2)\cdot\ldots\cdot 3 & \text{determinants of } 2\times 2 \text{ matrices}
\end{array}$$

E.g. for a 25×25 matrix we would need to compute

$$25 \cdot 24 \cdot 23 \cdot \ldots \cdot 3 = 7,755,605,021,665,492,992,000,000$$

determinants of 2×2 matrices.

<u>**Next:**</u> How to compute determinants faster.