

Recall: An $n \times n$ matrix A defines a linear transformation

$$T_A: \mathbb{R}^n \rightarrow \mathbb{R}^n$$

given by $T_A(\mathbf{v}) = A\mathbf{v}$.

Next goal: Understand this linear transformation better.

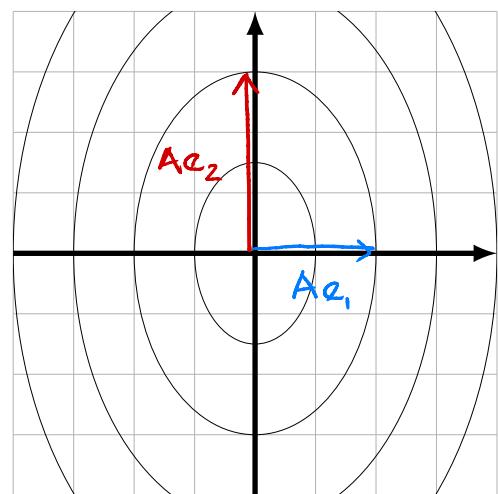
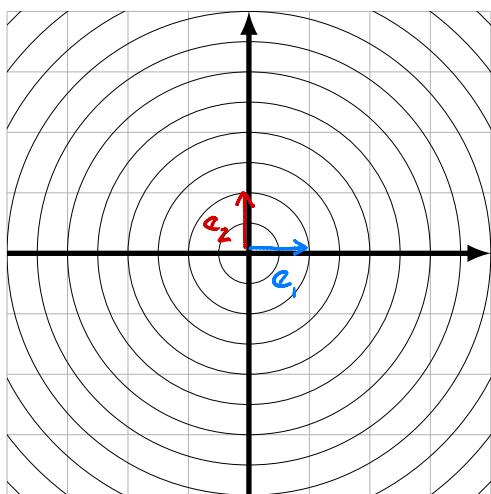
Example.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad T_A : \mathbb{R}^2 \xrightarrow{\sim} \mathbb{R}^2$$

$\downarrow \longmapsto Av$

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad Ae_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \cdot e_1$$

$$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad Ae_2 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} = 3 \cdot e_2$$



Example.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$T_A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$v \mapsto Av$$

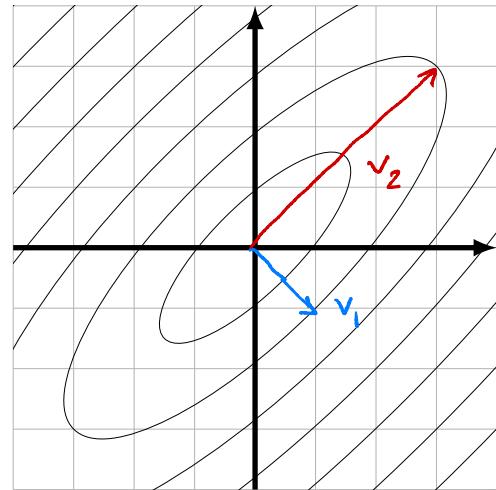
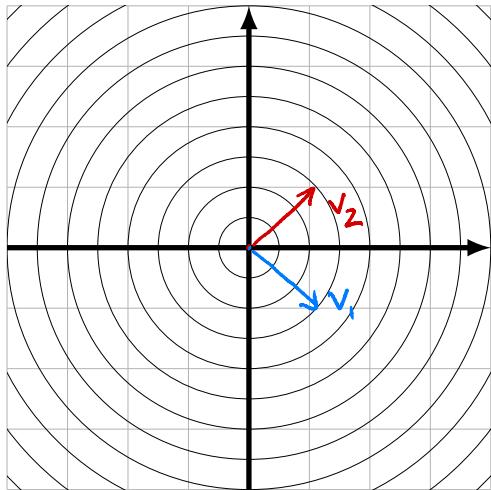
$$Av_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$Av_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Take $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$Av_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \cdot v_1$$

$$Av_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \cdot v_2$$



Definition

Let A be an $n \times n$ matrix. If $\mathbf{v} \in \mathbb{R}^n$ is a non-zero vector and λ is a scalar such that

$$A\mathbf{v} = \lambda\mathbf{v}$$

then we say that

- λ is an *eigenvalue* of A
- \mathbf{v} is an *eigenvector* of A corresponding to λ .

Example.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad \text{We had } A \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So: $\lambda_1 = 2, \lambda_2 = 3$ are eigenvalues of A .

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 2$.

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_2 = 3$.

Example.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{We had } A \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad A \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So: $\lambda_1 = 1, \lambda_2 = 3$ are eigenvalues of A .

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 1$.

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_2 = 3$.

Computation of eigenvalues

Recall: $I_n = n \times n$ identity matrix:

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Note:

$$\lambda I_n = \begin{bmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda \end{bmatrix}$$

For any $v \in \mathbb{R}^n$ we have:

$$(\lambda I_n)v = \lambda \cdot (I_n v) = \lambda v$$

Propositon

If A be an $n \times n$ matrix then $\lambda \in \mathbb{R}$ is an eigenvalue of A if and only if the matrix equation

$$(A - \lambda I_n)x = 0$$

has a non-trivial solution.

Proof: λ is an eigenvalue of A

\Updownarrow

there is a vector $v \neq 0$ such that $Av = \lambda v = (\lambda I_n)v$

\Updownarrow

there is a vector $v \neq 0$ such that $(A - \lambda I_n)v = 0$

Propostiton

If B is an $n \times n$ matrix then equation

$$Bx = 0$$

has a non-trivial solution if and only of the matrix B is not invertible.

Proof: $Bx = 0$ has a non-trivial solution

\Updownarrow

not every column of B is a pivot column

\Updownarrow

B is not invertible

Propostiton

If A be an $n \times n$ matrix then $\lambda \in \mathbb{R}$ is an eigenvalue of A if and only if

$$\det(A - \lambda I_n) = 0$$

Proof: λ is an eigenvalue of A

\Updownarrow

$(A - \lambda I_n)x = 0$ has a non-trivial solution

\Updownarrow

$(A - \lambda I_n)$ is not invertible

\Updownarrow

$$\det(A - \lambda I_n) = 0$$

Example. Find all eigenvalues of the following matrix:

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Solution We need to find $\lambda \in \mathbb{R}$ such that $\det(A - \lambda I) = 0$

$$\begin{aligned}\det(A - \lambda I) &= \det \left(\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right) \\ &= \det \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix} \\ &= (2-\lambda)(3-\lambda)(2-\lambda) + 2 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 2 \\ &\quad - 1 \cdot (3-\lambda) \cdot 1 - (2-\lambda) \cdot 1 \cdot 2 - 2 \cdot 1 \cdot (2-\lambda) \\ &= -\lambda^3 + 7\lambda^2 - 11\lambda + 5\end{aligned}$$

We obtain: λ is an eigenvalue of A if and only if

$$-\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$

Check: The only solutions of this equation are $\lambda_1 = 1, \lambda_2 = 5$

We obtain: The matrix A has two eigenvalues: $\lambda_1 = 1, \lambda_2 = 5$.

Definition

If A is an $n \times n$ matrix then

$$P(\lambda) = \det(A - \lambda I_n)$$

is a polynomial of degree n . $P(\lambda)$ is called the *characteristic polynomial* of the matrix A .

Upshot

If A is a square matrix then

$$\text{eigenvalues of } A = \text{roots of } P(\lambda)$$

Example.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

We had : $P(\lambda) = \det(A - \lambda I) = -\underline{\lambda^3 + 7\lambda^2 - 11\lambda + 5}$
the characteristic polynomial of A

Corollary

An $n \times n$ matrix can have at most n distinct eigenvalues.

Proof: The characteristic polynomial $P(\lambda)$ of A is a polynomial of degree n , so it can have at most n distinct roots.

Computation of eigenvectors

Recall: If B is a matrix then

$$\text{Nul}(B) = \{v \mid Bv = 0\}$$

Proposition

If λ is an eigenvalue of an $n \times n$ matrix A then

$$\left\{ \begin{array}{l} \text{eigenvectors of } A \\ \text{corresponding to } \lambda \end{array} \right\} = \left\{ \begin{array}{l} \text{vectors in} \\ \text{Nul}(A - \lambda I_n) \end{array} \right\}$$

Proof: $v \in \text{Nul}(A - \lambda I)$

$$\Updownarrow (A - \lambda I)v = 0$$

$$\Updownarrow Av = (\lambda I)v$$

$$\Updownarrow Av = \lambda v$$

Recall: If B is an $m \times n$ matrix then $\text{Nul}(B)$ is a subspace of \mathbb{R}^n .

Corollary/Definition

If A is an $n \times n$ matrix and λ is an eigenvalue of A then the set of all eigenvectors corresponding to λ is a subspace of \mathbb{R}^n .

This subspace is called the *eigenspace* of A corresponding to λ .

Proposition

If λ is an eigenvalue of an $n \times n$ matrix A then

$$\left\{ \begin{array}{l} \text{eigenspace of } A \\ \text{corresponding to } \lambda \end{array} \right\} = \text{Nul}(A - \lambda I_n)$$

Example. Consider the following matrix:

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Recall that eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 5$. Compute bases of eigenspaces of A corresponding to these eigenvalues.

Solution.

$$\underline{\lambda_1 = 1}$$

$$\begin{aligned} (\text{eigenspace} \\ \text{of } \lambda_1=1) &= \text{Nul}(A - 1 \cdot I) \\ &= \text{Nul}\left(\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \text{Nul}\left(\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\right) \end{aligned}$$

We need to solve:

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & | & 0 \\ 1 & 2 & 1 & | & 0 \\ 1 & 2 & 1 & | & 0 \end{bmatrix} \xrightarrow{\text{row red}} \begin{bmatrix} x_1 & x_2 & x_3 \\ 1 & 2 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

We obtain:

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} = \left(\begin{array}{l} \text{a basis of} \\ \text{Nul}(A - 1 \cdot I) \end{array} \right) = \left(\begin{array}{l} \text{a basis of the} \\ \text{eigenspace of } A \\ \text{corresponding to } \lambda_1=1 \end{array} \right)$$

$$\lambda_2 = 5$$

$$\begin{pmatrix} \text{eigenspace} \\ \text{of } \lambda_1 = 5 \end{pmatrix} = \text{Nul}(A - 5 \cdot I)$$

$$= \text{Nul} \left(\begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \right) = \text{Nul} \left(\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \right)$$

We need to solve:

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 2 & 1 & | & 0 \\ 1 & -2 & 1 & | & 0 \\ 1 & 2 & -3 & | & 0 \end{bmatrix} \xrightarrow{\text{row red}} \begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We obtain:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} = \left(\begin{array}{c} \text{a basis of} \\ \text{Nul}(A - 5I) \end{array} \right) = \left(\begin{array}{c} \text{a basis of the} \\ \text{eigenspace of } A \\ \text{corresponding to } \lambda_1 = 5 \end{array} \right)$$