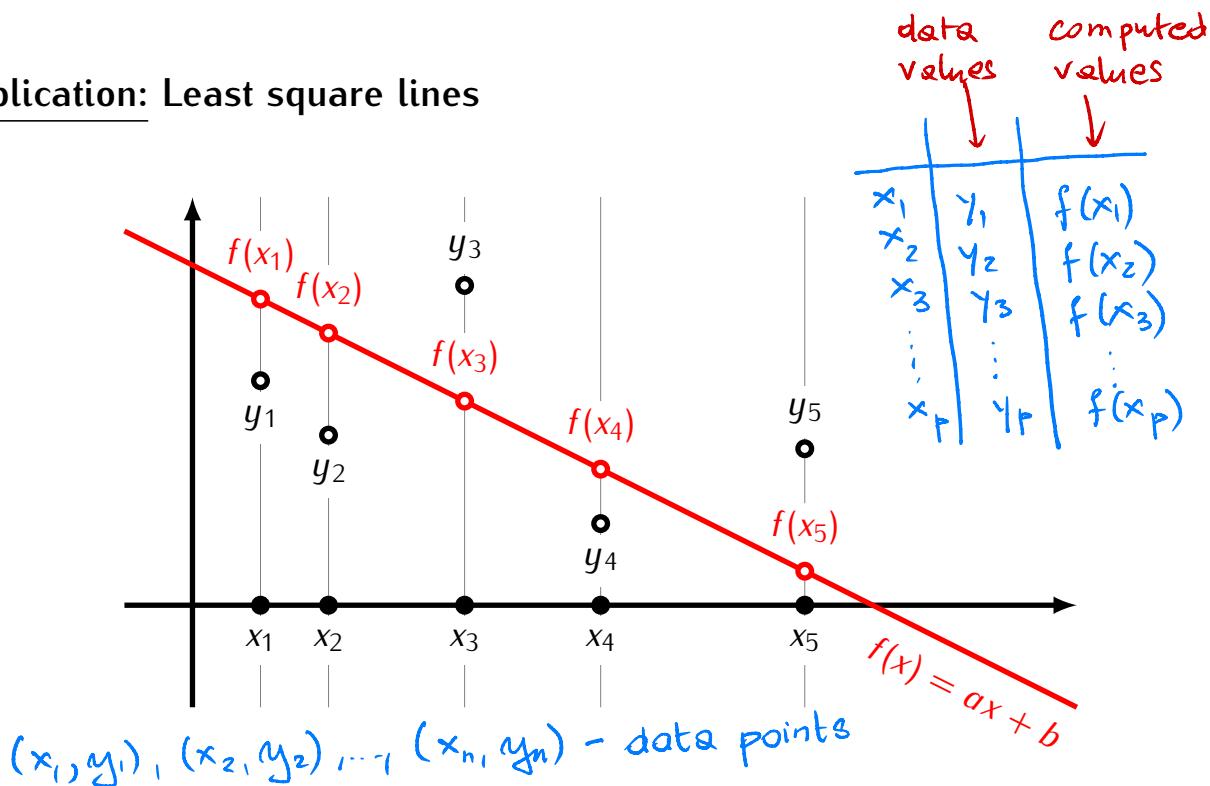


Application: Least square lines**Definition**

If $(x_1, y_1), \dots, (x_p, y_p)$ are points on the plane then the *least square line* for these points is the line given by an equation $f(x) = ax + b$ such that the number

$$\text{dist} \left(\begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}, \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_p) \end{bmatrix} \right) = \sqrt{(y_1 - f(x_1))^2 + \dots + (y_p - f(x_p))^2}$$

is the smallest possible.

Note: If $f(x) = ax + b$ then

$$\begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_p) \end{bmatrix} = \begin{bmatrix} ax_1 + b \\ ax_2 + b \\ \vdots \\ ax_p + b \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_p & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix}$$

This gives: $f(x) = ax + b$ is the least square line if the distance

$$\text{dist} \left(\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_p & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} \right)$$

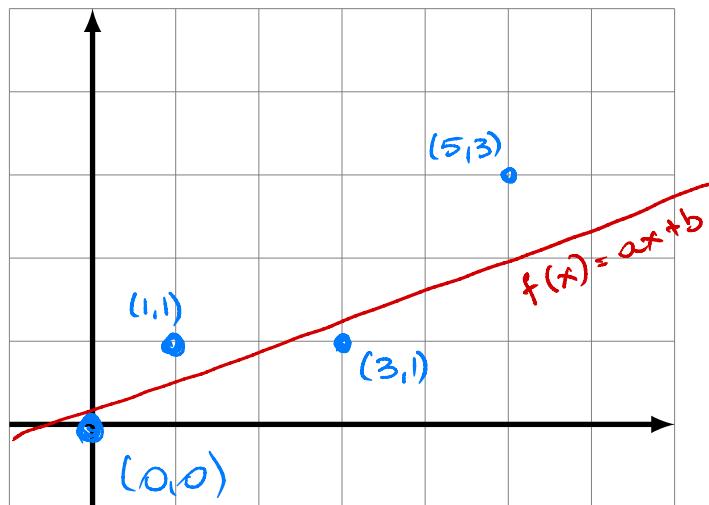
is as small as possible.

Proposition

The line $f(x) = ax + b$ is the least square line for points $(x_1, y_1), \dots, (x_p, y_p)$ if the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ is the least square solution of the equation

$$\begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_p & 1 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}$$

Example. Find the equation of the least square line for the points $(0, 0)$, $(1, 1)$, $(3, 1)$, $(5, 3)$.



Solution: The least square line is given by

$$f(x) = ax + b$$

where $\begin{bmatrix} a \\ b \end{bmatrix}$ is a least square solution of

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 5 & 1 \end{bmatrix}}_A \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix}}_b$$

$$A^T A = \begin{bmatrix} 35 & 9 \\ 9 & 4 \end{bmatrix} \quad A^T b = \begin{bmatrix} 19 \\ 5 \end{bmatrix}$$

Normal equation:

$$\begin{bmatrix} 35 & 9 \\ 9 & 4 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 19 \\ 5 \end{bmatrix}$$

aug. matrix:

$$\left[\begin{array}{cc|c} 35 & 9 & 19 \\ 9 & 4 & 5 \end{array} \right] \xrightarrow{\text{row red}} \left[\begin{array}{cc|c} 1 & 0 & 31/59 \\ 0 & 1 & 4/59 \end{array} \right]$$

$$\begin{cases} a = 31/59 \\ b = 4/59 \end{cases}$$

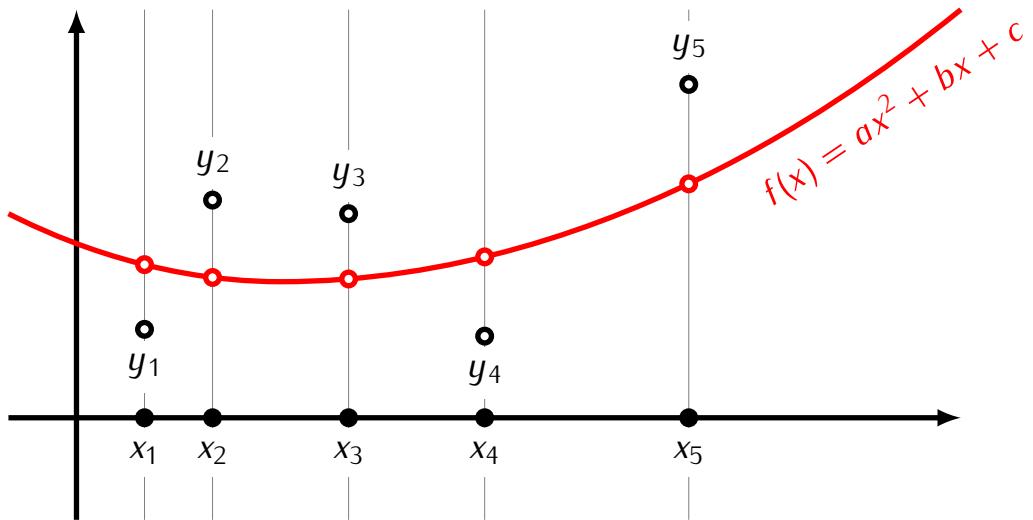
The least square line:

$$f(x) = \frac{31}{59}x + \frac{4}{59}$$

Application: Least square curves

The above procedure can be used to determine curves other than lines that fit a set of points in the least square sense.

Example: Least square parabolas



Definition

If $(x_1, y_1), \dots, (x_p, y_p)$ are points on the plane then the *least square parabola* for these points is the parabola given by an equation $f(x) = ax^2 + bx + c$ such that the number

$$\text{dist} \left(\begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}, \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_p) \end{bmatrix} \right) = \sqrt{(y_1 - f(x_1))^2 + \dots + (y_p - f(x_p))^2}$$

is the smallest possible.

Note: If $f(x) = ax^2 + bx + c$ then

$$\begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_p) \end{bmatrix} = \begin{bmatrix} ax_1^2 + bx_1 + c \\ ax_2^2 + bx_2 + c \\ \vdots \\ ax_p^2 + bx_p + c \end{bmatrix} = \begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ \vdots & \vdots & \vdots \\ x_p^2 & x_p & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

This gives: $f(x) = ax^2 + bx + c$ is the least square parabola if the distance

$$\text{dist} \left(\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ \vdots & \vdots & \vdots \\ x_p^2 & x_p & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} \right)$$

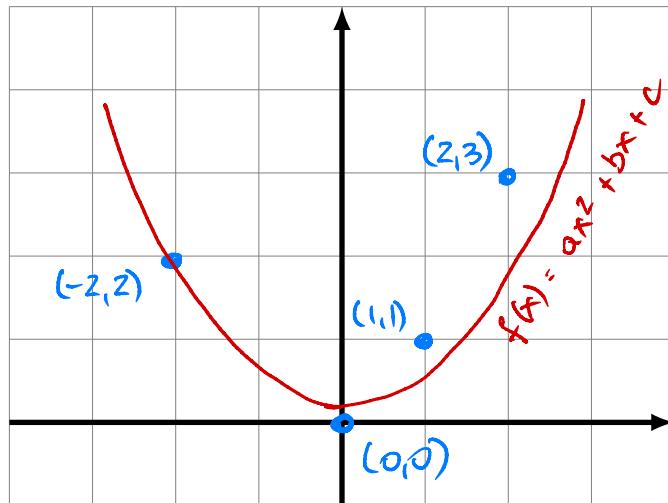
is as small as possible.

Proposition

The parabola $f(x) = ax^2 + bx + c$ is the least square parabola for points $(x_1, y_1), \dots, (x_p, y_p)$ if the vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is the least square solution of the equation

$$\begin{bmatrix} x_1^2 & x_1 & 1 \\ \vdots & \vdots & \vdots \\ x_p^2 & x_p & 1 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}$$

Example. Find the equation of the least square parabola for the points $(-2, 2)$, $(0, 0)$, $(1, 1)$, $(2, 3)$.



Solution: We need to find a least square solution of the equation

$$\begin{bmatrix} x^2 & x & 1 \\ (-2)^2 & -2 & 1 \\ 0^2 & 0 & 1 \\ 1^2 & 1 & 1 \\ 2^2 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y \\ 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}$$

$\underbrace{\quad\quad\quad}_{A}$

Normal equation:

$$\begin{bmatrix} 33 & 1 & 9 \\ 1 & 9 & 1 \\ 9 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 21 \\ 3 \\ 6 \end{bmatrix}$$

$\underbrace{\quad\quad\quad}_{A^T A}$

aug. matrix.

$$\left[\begin{array}{ccc|c} 33 & 1 & 9 & 21 \\ 1 & 9 & 1 & 3 \\ 9 & 1 & 4 & 6 \end{array} \right] \xrightarrow{\text{row red}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 27/37 \\ 0 & 1 & 0 & 51/185 \\ 0 & 0 & 1 & -39/185 \end{array} \right]$$

$$\begin{cases} a = 27/37 \\ b = 51/185 \\ c = -39/185 \end{cases}$$

The least square parabola:

$$f(x) = \frac{27}{37}x^2 + \frac{51}{185}x - \frac{39}{185}$$