Instructions.

In this problem you will be given a few statements. For each statement you need to decide if it is true or not, and justify your answer.

How to justify your answer.

• In order to show that a statement is false, it suffices to give a counterexample. For example, consider the the statement:

The last digit of every even number is either 2, 4, or 8.

To show that this statement is false, it is enough to point out that, for example, 10 is an even number, but its last digit is 0.

• In order to show that a statement is true, you need to provide a reasoning explaining why it is true in all instances. Giving one example when it is true will not suffice, since the statement may not work in some other cases. For example, consider the the statement:

If n is an even number then $n + 2$ is also an even number.

You can justify that this is true as follows. Even numbers are integers which are multiples of 2. If n is even then $n = 2m$ for some integer m. Then $n + 2 = 2m + 2 = 2(m + 1)$, which shows that $n + 2$ is even.

Note

This problem will not be collected or graded. However, problems of this type will appear on exams in this course. Sample solutions are provided at the end.

For each of the statements given below decide if it is true or false. If you decide that it is true, justify your answer. If you think it is false give a counterexample.

a) If A is a 2×4 matrix then $\text{rank}(A) \leq 2$.

b) If $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is an orthogonal basis of $\mathbb{R}^3$, and $\mathbf{v} \in \mathbb{R}^3$ is a vector orthogonal to both $\mathbf{b}_1$ and $\mathbf{b}_2$, then $\mathbf{v} = c\mathbf{b}_3$ for some scalar c.

c) If V is a subspace of $\mathbb{R}^3$ and $\mathbf{w} \in \mathbb{R}^3$ is a vector such that $\text{proj}_V \mathbf{w} = \frac{1}{2}\mathbf{w}$ then $\mathbf{w}$ must be the zero vector.

d) If $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are vectors in $\mathbb{R}^3$ such that $\mathbf{u}_1$ is orthogonal to $\mathbf{u}_2$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{u}_3$, then $\mathbf{u}_1$ must be othogonal to $\mathbf{u}_3$

Here are solutions to the questions from the previous page. You should try to answer all questions by yourself before reading these solutions.

a) TRUE. Recall that $\text{rank}(A) = \dim \text{Col}(A)$, and that $\dim \text{Col}(A)$ is equal to the number of pivot columns of A. Since the matrix A has two rows, it can have at most two pivot positions (one in each row), and so it has at most two pivot columns.

b) TRUE. Let

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

be the coordinate vector of $\mathbf{v}$ relative to the basis $\mathcal{B}$. Then

$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3$$

Also, since the basis $\mathcal{B}$ is orthogonal, we have $c_i = \dfrac{\mathbf{b}_i \cdot \mathbf{v}}{\mathbf{b}_i \cdot \mathbf{b}_i}$ for $i = 1, 2, 3$. By assumption $\mathbf{b}_1 \cdot \mathbf{v} = \mathbf{b}_2 \cdot \mathbf{v} = 0$, so $c_1 = c_2 = 0$. This gives $\mathbf{v} = c_3\mathbf{b}_3$.

c) TRUE. Recall that the vector $\mathbf{w} - \text{proj}_V\mathbf{w}$ is orthogonal to the subspace V. Since $\text{proj}_V\mathbf{w} \in V$ this means in particular that $\mathbf{w} - \text{proj}_V\mathbf{w}$ is orthogonal to $\text{proj}_V\mathbf{w}$. If $\text{proj}_V\mathbf{w} = \frac{1}{2}\mathbf{w}$ then $\mathbf{w} - \text{proj}_V\mathbf{w} = \frac{1}{2}\mathbf{w}$, so we obtain that $\frac{1}{2}\mathbf{w}$ is orthogonal to itself. However, the only vector orthogonal to itself is the zero vector. This gives $\frac{1}{2}\mathbf{w} = \mathbf{0}$, and so $\mathbf{w} = \mathbf{0}$.

d) FALSE. Take e.g.

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$