Note.

1) One can show that every vector space has a basis.

2) Some vector spaces have bases consisting of infinitely many vectors. If V is such vector space then we write $\dim V = \infty$.

Example.

Recall:

If $A = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$ is an $m \times n$ matrix then:

1) $\mathrm{Col}(A) = \mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_n)$

2) $\mathrm{Nul}(A) = \{\mathbf{v} \in \mathbb{R}^m \mid A\mathbf{v} = \mathbf{0}\}$

Construction of a basis of Col(A)

Lemma

Let V be a vector space, and let $\mathbf{v}_1, \ldots, \mathbf{v}_p \in V$. If a vector $\mathbf{v}_i$ is a linear combination of the other vectors then

$$\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p) = \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_p)$$

Upshot. One can construct a basis of a vector space V as follows:

- Start with a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ such that $\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p) = V$.
- Keep removing vectors without changing the span, until you get a linearly independent set.

Example. Find a basis of $\mathrm{Col}(A)$ where A is the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example. Find a basis of Col(A) where A is the following matrix:

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Construction of a basis of Nul(A)

Example. Find a basis of Nul(A) where A is the following matrix:

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Upshot. If A is matrix then:

$$\dim \operatorname{Col}(A) = \text{the number of pivot columns of } A$$

$$\dim \operatorname{Nul}(A) = \text{the number of non-pivot columns of } A$$

Definition

If A is a matrix then:

- the dimension of $\operatorname{Col}(A)$ is called the *rank* of A and it is denoted $\operatorname{rank}(A)$
- the dimension of $\operatorname{Nul}(A)$ is called the *nullity* of A.

The Rank Theorem

If A is an $m \times n$ matrix then

$$\operatorname{rank}(A) + \dim \operatorname{Nul}(A) = n$$

Example. Let A be a 100×101 matrix such that $\dim \mathrm{Nul}(A) = 1$. Show that the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbb{R}^{100}$.