Proposition

If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal set of non-zero vectors in $\mathbb{R}^n$ then this set is linearly independent.

Recall: Any linearly independent set of n vectors in $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$.

Corollary

If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthogonal set of n non-zero vectors in $\mathbb{R}^n$ then this set is a basis of $\mathbb{R}^n$.

Definition

If V is a subspace of $\mathbb{R}^n$ then we say that a set $\{\mathbf{v}_1, \ldots \mathbf{v}_k\}$ is an *orthogonal basis* of V if

1) $\{\mathbf{v}_1, \ldots \mathbf{v}_k\}$ is a basis of V and
2) $\{\mathbf{v}_1, \ldots \mathbf{v}_k\}$ is an orthogonal set.

Recall. If $\mathcal{B} = \{\mathbf{v}_1, \ldots \mathbf{v}_k\}$ is a basis of a vector space V and $\mathbf{w} \in V$ then the coordinate vector of $\mathbf{w}$ relative to $\mathcal{B}$ is the vector

$$[\ \mathbf{w}\ ]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

where $c_1, \ldots, c_k$ are scalars such that $c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k = \mathbf{w}$.

Propostion

If $\mathcal{B} = \{\mathbf{v}_1, \ldots \mathbf{v}_k\}$ is an orthogonal basis of V and $\mathbf{w} \in V$ then

$$[\ \mathbf{w}\ ]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

where $c_i = \dfrac{\mathbf{w} \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i} = \dfrac{\mathbf{w} \cdot \mathbf{v}_i}{||\mathbf{v}_i||^2}$

Example. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix} \right\}, \qquad \mathbf{w} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

The set $\mathcal{B}$ is an orthogonal basis of $\mathbb{R}^3$. Compute $\left[\ \mathbf{w}\ \right]_{\mathcal{B}}$.

Theorem (Gram-Schmidt Process)

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a basis of V. Define vectors $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ as follows:

$$\mathbf{w}_1 = \mathbf{v}_1$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_2}{\mathbf{w}_1 \cdot \mathbf{w}_1}\right)\mathbf{w}_1$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_3}{\mathbf{w}_1 \cdot \mathbf{w}_1}\right)\mathbf{w}_1 - \left(\frac{\mathbf{w}_2 \cdot \mathbf{v}_3}{\mathbf{w}_2 \cdot \mathbf{w}_2}\right)\mathbf{w}_2$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\mathbf{w}_k = \mathbf{v}_k - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_k}{\mathbf{w}_1 \cdot \mathbf{w}_1}\right)\mathbf{w}_1 - \left(\frac{\mathbf{w}_2 \cdot \mathbf{v}_k}{\mathbf{w}_2 \cdot \mathbf{w}_2}\right)\mathbf{w}_2 - \ldots - \left(\frac{\mathbf{w}_{k-1} \cdot \mathbf{v}_k}{\mathbf{w}_{k-1} \cdot \mathbf{w}_{k-1}}\right)\mathbf{w}_{k-1}$$

Then the set $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is an orthogonal basis of V.

Example. In $\mathbb{R}^4$ take

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 7 \\ 4 \\ 3 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 7 \\ 7 \\ 8 \end{bmatrix}$$

The set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of some subspace $V \subseteq \mathbb{R}^4$. Find an orthogonal basis of V.

Definition

An orthogonal basis $\mathcal{B} = \{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ of V is called an *orthonormal basis* if $||\mathbf{w}_i|| = 1$ for $i = 1, \dots, k$.

Propostion

If $\mathcal{B} = \{\mathbf{v}_1, \dots \mathbf{v}_k\}$ is an orthonormal basis of V and $\mathbf{w} \in V$ then

$$[\ \mathbf{w}\ ]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

where $c_i = \mathbf{w} \cdot \mathbf{v}_i$.

Note. If $\mathcal{B} = \{\mathbf{v}_1, \dots \mathbf{v}_k\}$ is an orthogonal basis of V then

$$\mathcal{C} = \left\{ \frac{\mathbf{v}_1}{||\mathbf{v}_1||}, \dots, \frac{\mathbf{v}_k}{||\mathbf{v}_k||} \right\}$$

is an orthonormal basis of V.

Recall:

1) If

$$\mathbf{u} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

are vectors in $\mathbb{R}^n$ then:

- $\mathbf{u} \cdot \mathbf{v} = a_1 b_1 + \ldots + a_n b_n$
- $||\mathbf{u}|| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$
- $\mathrm{dist}(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$

2) Vectors $\mathbf{u}, \mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.

3) Pythagorean theorem: $\mathbf{u}, \mathbf{v}$ are orthogonal if and only if

$$||\mathbf{u}||^2 + ||\mathbf{v}||^2 = ||\mathbf{u} + \mathbf{v}||^2$$

4) If $V \subseteq \mathbb{R}^n$ is a subspace then an orthogonal basis of V is a basis which consists of vectors that are orthogonal to one another.

5) If $\mathcal{B} = \{\mathbf{v}_1, \ldots \mathbf{v}_k\}$ is an orthogonal basis of V and $\mathbf{w} \in V$ then

$$\left[\ \mathbf{w}\ \right]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

where $c_i = \dfrac{\mathbf{w} \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i}$.