Recall:

Vector equations are equivalent to systems of linear equations:

$$x_{1}\begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_{2}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

$$\begin{cases} 2x_{1} + 4x_{2} = 7 \\ 3x_{1} + 2x_{2} = 3 \end{cases}$$
vector
equation
$$\begin{cases} \text{system of linear equations} \end{cases}$$

Upshot. A vector equation can have either:

- no solutions
- exactly one solution
- infinitely many solutions

Next:

- When does a vector equation have a solution?
- When does it have exactly one solution?

Definition

A vector $\mathbf{w} \in \mathbb{R}^n$ is a *linear combination* of vectors $\mathbf{v}_1, \dots \mathbf{v}_p \in \mathbb{R}^n$ if there exists scalars c_1, \dots, c_p such that

$$\mathbf{w} = c_1 \mathbf{v}_1 + \ldots + c_p \mathbf{v}_p$$

Equivalently: A vector \mathbf{w} is a linear combination of vectors $\mathbf{v}_1, \dots \mathbf{v}_p$ is the vector equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{w}$$

has a solution.

Example.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix}$$

Example. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 9 \\ 3 \\ 6 \end{bmatrix}$$

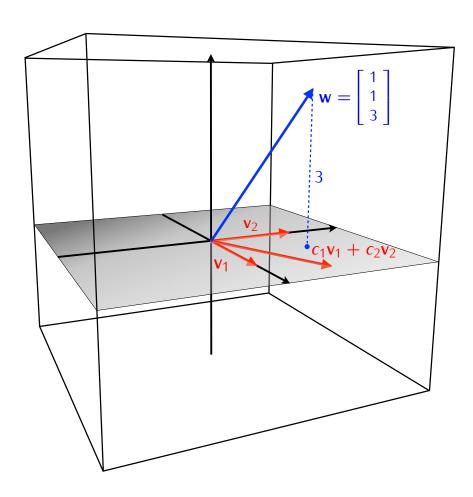
Express ${\bf w}$ as a linear combination of ${\bf v}_1, {\bf v}_2, {\bf v}_3$ or show that this is not possible.

Example. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

Express ${\bf w}$ as a linear combination of ${\bf v}_1, {\bf v}_2$ or show that this is not possible.

Geometric picture of the last example



Definition

If $\mathbf{v}_1,\dots,\mathbf{v}_p$ are vectors in \mathbb{R}^n then

$$Span(v_1, ..., v_p) = \begin{cases} the set of all \\ linear combinations \\ c_1v_1 + ... + c_pv_p \end{cases}$$

Example.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

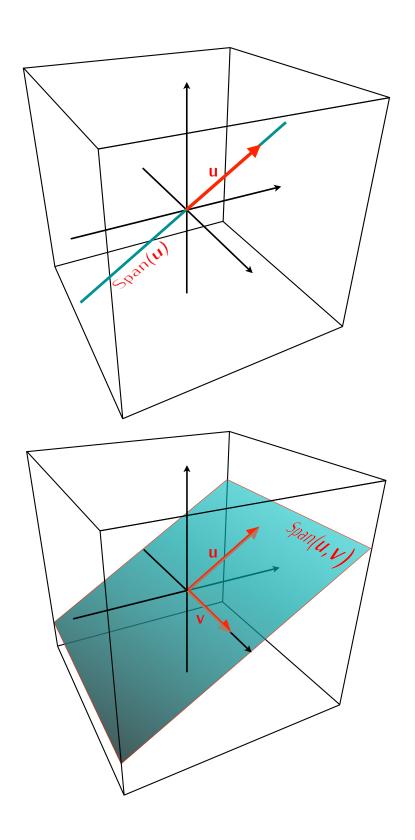
Proposition

A vector \mathbf{w} is in $\mathrm{Span}(\mathbf{v}_1,\ldots,\mathbf{v}_p)$ if and only if the vector equation

$$x_1\mathbf{v}_1+\ldots+x_p\mathbf{v}_p=\mathbf{w}$$

has a solution.

Geometric interpretation of Span



Proposition

For arbitrary vectors $\mathbf{v}_1, \dots, \mathbf{v}_p \in \mathbb{R}^n$ the zero vector $\mathbf{0} \in \mathbb{R}^n$ is in Span $(\mathbf{v}_1, \dots, \mathbf{v}_p)$.

