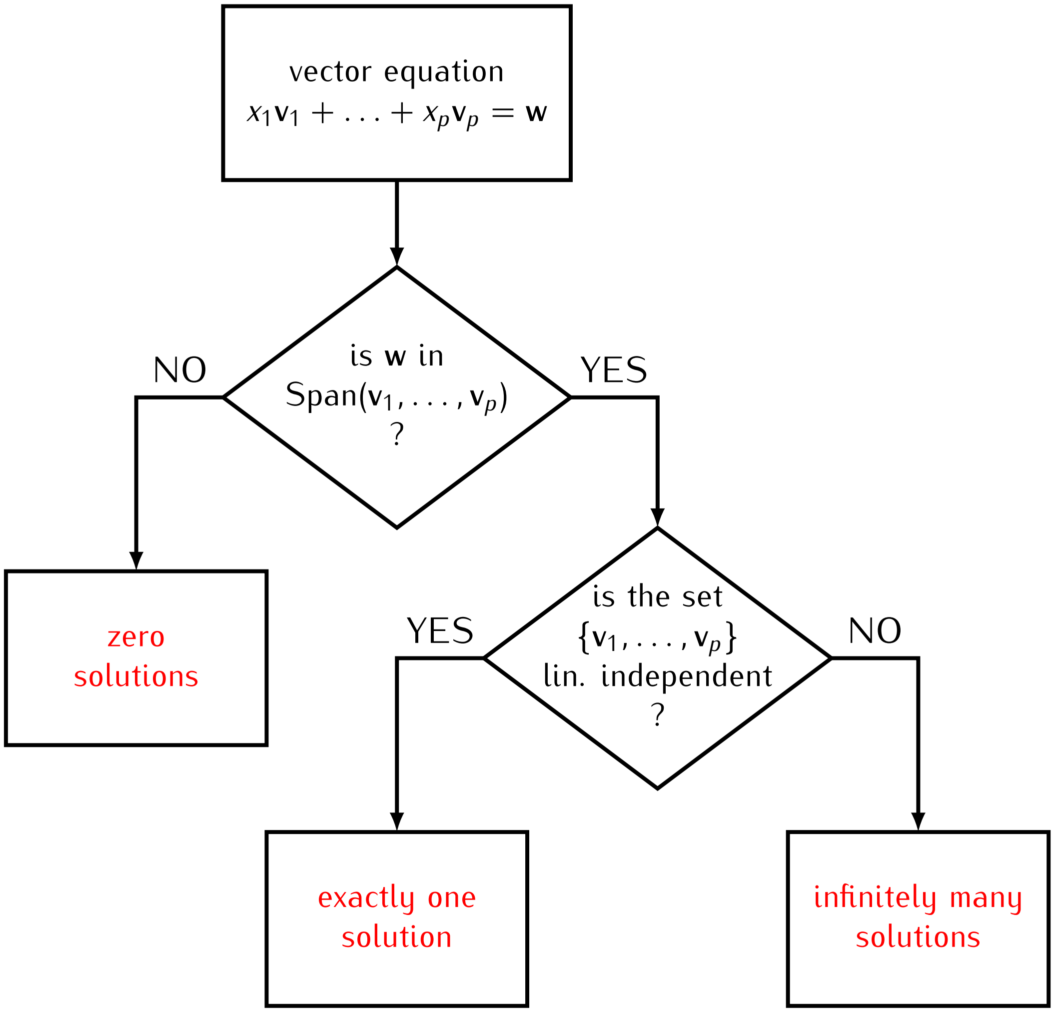
Recall:

1) $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p) = \left\{ \begin{array}{c} \text{the set of all} \\ \text{linear combinations} \\ c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p \end{array} \right\}$

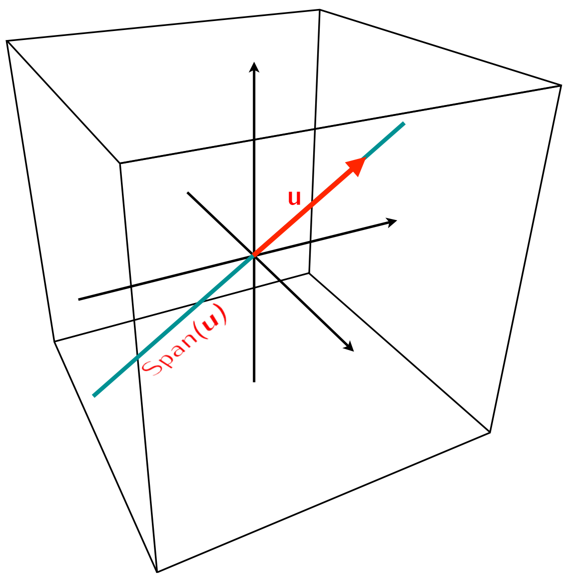
2) A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent if the equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

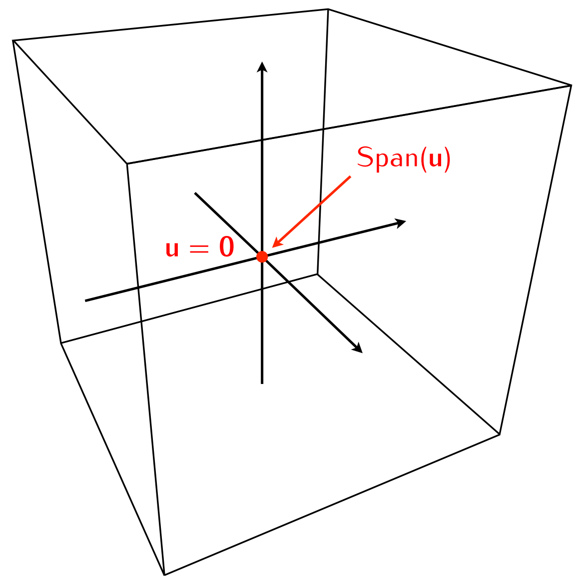
has only one, trivial solution $x_1 = 0, \ldots, x_p = 0$.


vector equation
$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{w}$
is $\mathbf{w}$ in Span($\mathbf{v}_1, \ldots, \mathbf{v}_p$) ?
NO
YES
zero solutions
is the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ lin. independent ?
YES
NO
exactly one solution
infinitely many solutions

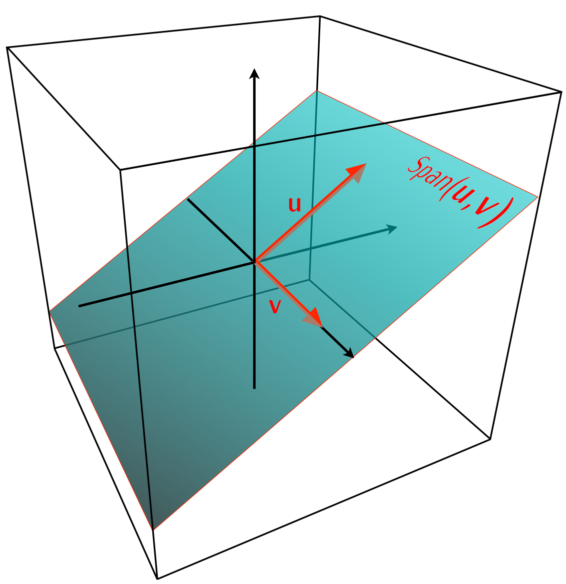
Linear independence vs. Span



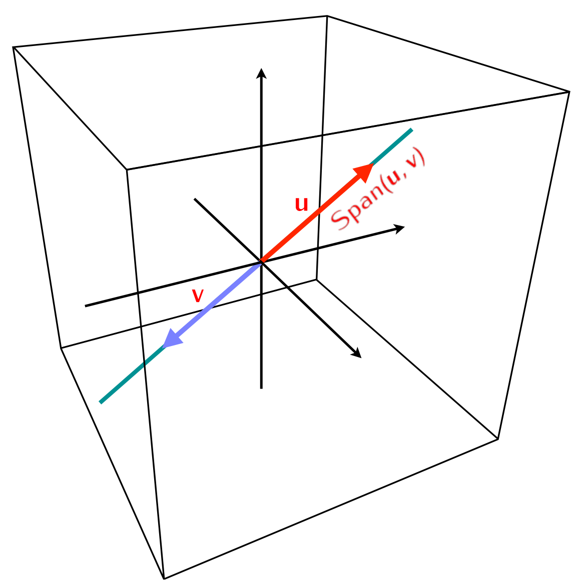
{u} linearly independent



{u} linearly dependent



{u, v} linearly independent



{u, v} linearly dependent

Theorem

If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a linearly dependent set of vectors in then:

1) for some $\mathbf{v}_i$ we have $\mathbf{v}_i \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_p)$.

2) for some $\mathbf{v}_i$ we have

$$\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p) = \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_p)$$

Example.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$