Note. We have seen that any symmetric matrix is orthogonally diagonalizable. The converse statement is also true:

Proposition

If a matrix A is orthogonally diagonalizable then A is a symmetric matrix.

Recall:

1) An orthogonal matrix $Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{bmatrix}$ is a square matrix such that $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is an orthonormal set of vectors, i.e.:

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

2) If Q is an orthogonal matrix then $Q^{-1} = Q^T$

3) A square matrix A is orthogonally diagonalizable if there exist an orthogonal matrix Q and a diagonal matrix D such that

$$A = QDQ^{-1} = QDQ^T$$

4) A matrix A is orthogonally diagonalizable if and only if A is a symmetric matrix (i.e. $A^T = A$).

Yet another view of matrix multiplication

Note. If C is an $n \times 1$ matrix and D is an $1 \times n$ matrix then CD is an $n \times n$ matrix.

Propostion

Let A be an $n \times n$ matrix with columns $\mathbf{v}_1, \dots, \mathbf{v}_n$, and B be an $n \times n$ matrix with rows $\mathbf{w}_1, \dots, \mathbf{w}_n$:

$$A = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix} \qquad B = \begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_n \end{bmatrix}$$

Then

$$AB = \mathbf{v}_1\mathbf{w}_1 + \mathbf{v}_2\mathbf{w}_2 + \dots + \mathbf{v}_n\mathbf{w}_n$$

Example.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 1 \\ 7 & 2 \end{bmatrix}$$

Theorem

Let A be a symmetric matrix with orthogonal diagonalization

$$A = QDQ^T$$

If

$$Q = \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_n \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & 0 \\ 0 & \dots & \lambda_n \end{bmatrix}$$

then

$$A = \lambda_1(\mathbf{u}_1\mathbf{u}_1^T) + \lambda_2(\mathbf{u}_2\mathbf{u}_2^T) + \ldots + \lambda_n(\mathbf{u}_n\mathbf{u}_n^T)$$

Note. The above formula is called the *spectral decomposition* of the matrix A.

Example.

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$$