Theorem

If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a linearly dependent set of vectors in then:

1) for some $\mathbf{v}_i$ we have $\mathbf{v}_i \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_p)$.

2) for some $\mathbf{v}_i$ we have

$$\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p) = \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_p)$$

Example.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So far:

$$\begin{cases} 2x_1 + 4x_2 + 6x_3 + 3x_4 = 7 \\ 3x_1 + 2x_2 + 2x_3 + 9x_4 = 3 \\ 5x_1 + 8x_2 + 3x_3 + 3x_4 = 9 \end{cases}$$

system of
linear equations

$$\Updownarrow$$

$$x_1 \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 2 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 9 \end{bmatrix}$$

vector equation

Next:

$$\begin{bmatrix} 2 & 4 & 6 & 3 \\ 3 & 2 & 2 & 9 \\ 5 & 8 & 3 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 9 \end{bmatrix}$$

matrix equation

Definition

Let A be an $m \times n$ matrix with columns $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and let $\mathbf{w}$ be a vector in $\mathbb{R}^n$:

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

The product $A\mathbf{w}$ is a vector in $\mathbb{R}^m$ given by

$$A\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

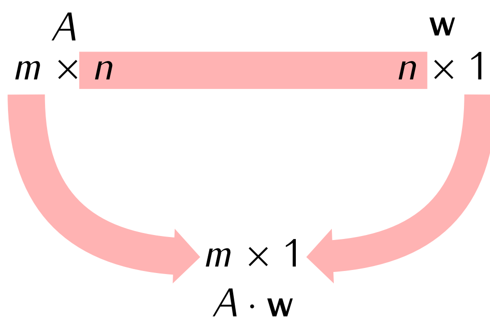
Example.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

Properties of matrix-vector multiplication

1) The product $A\mathbf{w}$ is defined only if

$$(\text{number of columns of } A) = (\text{number of entries of } \mathbf{w})$$



2) $A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w}$

3) If c is a scalar then $A(c\mathbf{w}) = c(A\mathbf{w})$.

Example. Solve the matrix equation

$$\begin{bmatrix} 1 & 1 & -4 \\ 1 & -2 & 3 \\ 3 & -3 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$