Operations on matrices so far:

- addition/subtraction $A \pm B$
- scalar multiplication $c \cdot A$
- matrix multiplication $A \cdot B$
- matrix transpose A^T

Next: How to divide matrices?

Note: if a, b - numbers then:

1) $a/b = a \cdot b^{-1}$

2) b^{-1} is a number such that $b \cdot b^{-1} = 1$

Definition

A matrix A is *invertible* if there exists a matrix B such that

$$A \cdot B = B \cdot A = I$$

(where $I =$ the identity matrix). In such case we say that B is the *inverse* of A and we write $B = A^{-1}$.

Example.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Check:

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Note: Not every matrix is invertible.

Example:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Assume that $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a matrix such that $AB = I$

Then:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}$$

The first column gives:

$$1 = a + c$$
$$0 = a + c$$

— impossible

Thus $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not invertible.

Matrix inverses and matrix equations

Proposition

If A is an invertible matrix then for any vector $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution.

Proof: If $Ax = b$ then

$$A^{-1}Ax = A^{-1}b$$
$$Ix = A^{-1}b$$
$$x = \underbrace{A^{-1}b}_{\uparrow \text{ the unique solution of } Ax=b}$$

Example. Solve the following matrix equation:

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{A} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Recall: A is invertible, $A^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$

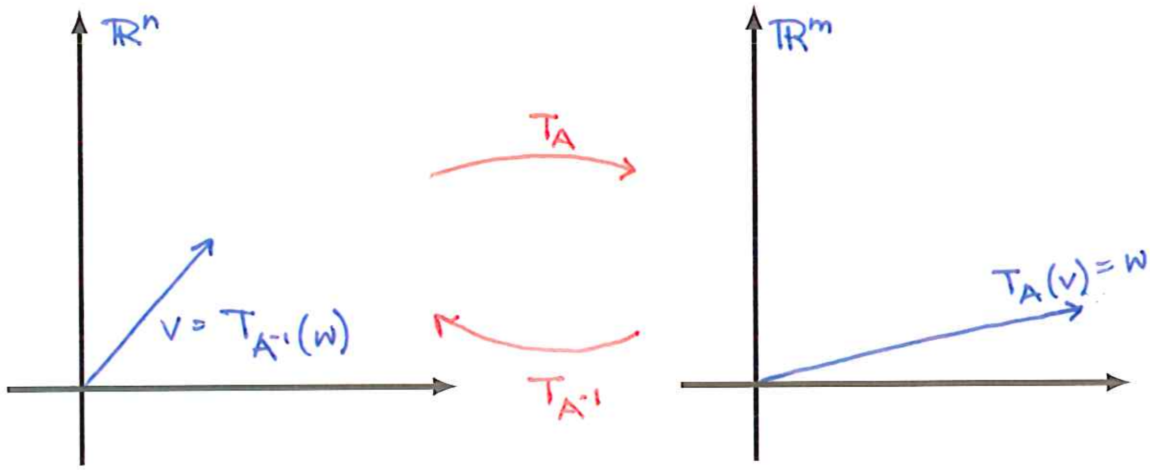
This gives:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} \quad //$$

Matrix inverses and matrix transformations

A - $m \times n$ matrix

$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$v \longmapsto Av$$

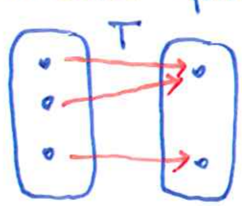


$$T_{A^{-1}}(T_A(v)) = T_{A^{-1}}(Av) = A^{-1}(Av) = (A^{-1}A)v = Iv = v$$

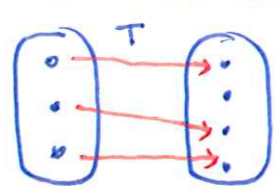
$$T_A(T_{A^{-1}}(w)) = \quad \ldots \quad\quad\quad \ldots \quad = w$$

If A is an invertible matrix then $T_{A^{-1}}$ is the inverse function of T_A.

Note: A function $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ can have an inverse function only if T is one-to-one and onto:



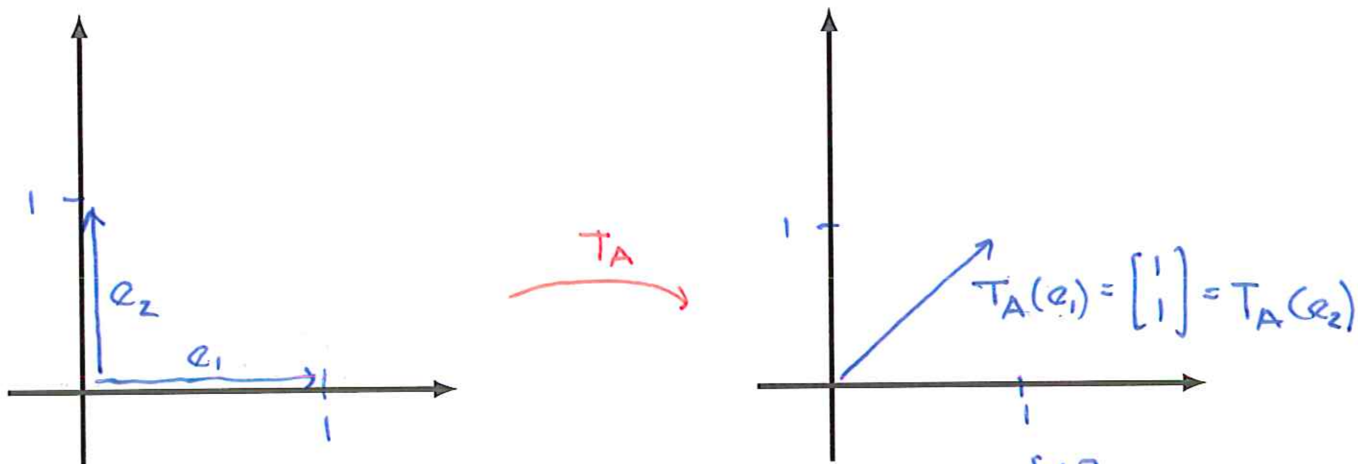
not one to one



not onto

Example.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad v \longmapsto Av$$



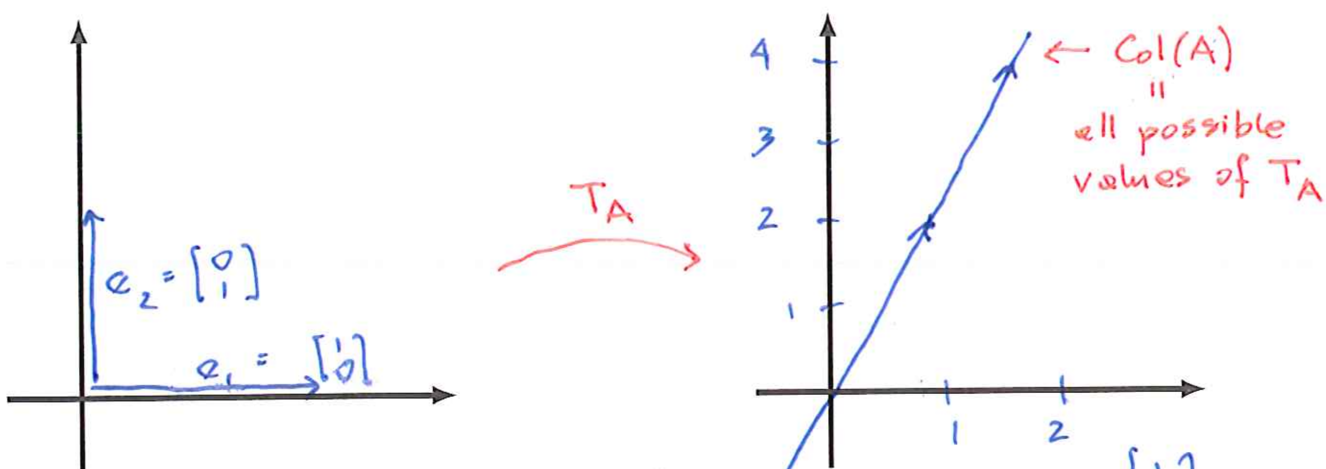
$T_A(e_1) = Ae_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$T_A(e_2) = Ae_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

T_A is not one-to-one, so it does not have an inverse function. Thus $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not invertible.

Example.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad v \longmapsto Av$$



$T_A(e_1) = A \cdot e_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$T_A(e_2) = A \cdot e_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

T_A is not onto, so it does not have an inverse function. Thus $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is not invertible.

Upshot. If an $m \times n$ matrix A is invertible then the matrix transformation $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ must be one-to-one and onto.

Recall: If A be is $m \times n$ matrix then the matrix transformation $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ is:

- onto if and only if A has a pivot position in every row
- one-to-one if and only if A has a pivot position in every column.

Upshot: If A - invertible then A has a pivot position in every row and every column

$$\begin{bmatrix} & A & \end{bmatrix} \xrightarrow[\text{red}]{\text{row}} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Theorem

If A is an $n \times n$ matrix then the following conditions are equivalent:

1) A is an invertible matrix.
2) The matrix A has a pivot position in every row and column.
3) The reduced echelon form of A is the identity matrix I_n.

Proposition

If A is an $n \times n$ invertible matrix then

$$A^{-1} = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \dots & \mathbf{w}_n \end{bmatrix}$$

where $\mathbf{w}_i$ is the solution of $A\mathbf{x} = \mathbf{e}_i$.

Proof: $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, e_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

Note: $I_n = [e_1 \ e_2 \ \dots \ e_n]$

We have:

$$[e_1 \ e_2 \dots e_n] = I_n = AA^{-1} = A \cdot [w_1 \ w_2 \cdots w_n]$$
$$= [Aw_1 \ Aw_2 \ \cdots \ Aw_n]$$

This gives: $Aw_1 = e_1$, $Aw_2 = e_2$, $\vdots$, $Aw_n = e_n$

Example.

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \xrightarrow[\text{red.}]{\text{row}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \text{ so } A \text{ is invertible}$$

$A^{-1} = [w_1 \ w_2]$ where $w_1 = (\text{solution of } Ax = e_1)$
$w_2 = (\text{solution of } Ax = e_2)$

Solve $Ax = e_1$:

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ -1 & 1 & 0 \end{array}\right] \xrightarrow{\text{row red.}} \left[\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \end{array}\right] \quad \text{so: } w_1 = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$$

Solve $Ax = e_2$:

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 1 \end{array}\right] \xrightarrow{\text{row red.}} \left[\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{array}\right] \quad \text{so: } w_2 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

This gives:

$$A^{-1} = [w_1 \ w_2] = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

Simplification:
How to solve several matrix equations with the same coefficient matrix at the same time

$A\mathrm{x} = \mathbf{b}_1,\ A\mathrm{x} = \mathbf{b}_2,\ \ldots,\ A\mathrm{x} = \mathbf{b}_n$

matrix of equations

$\left[\, A \,\middle|\, \mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \ldots \;\; \mathbf{b}_n \,\right]$

augmented matrix

row reduction

$\left[\quad \middle| \qquad\qquad\qquad \right]$

reduced matrix

read off solutions

solutions

Example. Solve the vector equations $A\mathbf{x} = \mathbf{e}_1$ and $A\mathbf{x} = \mathbf{e}_2$ where

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solution:

augmented matrix:

$$[A \mid e_1 \; e_2] = \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

$$\Big\downarrow \text{ row red.}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & -1/2 & 1/2 \end{array}\right]$$

row reduced form of A

solution of $Ax = e_1$

solution of $Ax = e_2$

Summary:
How to invert a matrix

Example: $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

1) Augment A by the identity matrix.

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]$$

2) Reduce the augmented matrix.

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{\text{row red.}} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

2) If after the row reduction the matrix on the left is the identity matrix, then A is invertible and

$$A^{-1} = \text{the matrix on the right}$$

Otherwise A is not invertible.

In our example A is invertible, $A^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

Properties of matrix inverses

1) If A is invertible then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

2) If A, B are invertible then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= A \cdot I \cdot A^{-1} \\ &= A \cdot A^{-1} \\ &= I\end{aligned}$$

3) If A is invertible then A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T$$