Recall:

1) The dot product in \mathbb{R}^n :

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = a_1b_1 + a_2b_2 + \dots + a_nb_n$$

2) Properties of the dot product:

- a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- b) $(u + v) \cdot w = u \cdot w + v \cdot w$
- c) $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$
- d) $\mathbf{u} \cdot \mathbf{u} \ge 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

2) Using the dot product we can define:

- length of vectors
- distance between vectors
- orthogonality of vectors
- orthogonal and orthonormal bases
- ullet orthogonal projection of a vector onto a subspace of \mathbb{R}^n
- ...

Next: Generalization to arbitrary vector spaces.

Definition

Let V be a vector space. An *inner product* on V is a function

$$V \times V \longrightarrow \mathbb{R}$$

$$u, v \longmapsto \langle u, v \rangle$$

such that:

- a) $\langle u, v \rangle = \langle v, u \rangle$
- b) $\langle u+v,w \rangle = \langle u,w \rangle + \langle v,w \rangle$
- c) $\langle c\mathbf{u}, \mathbf{v} \rangle = c \langle \mathbf{u}, \mathbf{v} \rangle$
- d) $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Definition

Let V be a vector space with an inner product $\langle \ , \ \rangle$.

1) The *length* (or *norm*) of a vector **v** is the number

$$||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

2) The *distance* between vectors $\mathbf{u}, \mathbf{v} \in V$ is the number

$$dist(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$$

3) Vectors $\mathbf{u}, \mathbf{v} \in V$ are orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Example. The dot product is an inner product in \mathbb{R}^n .

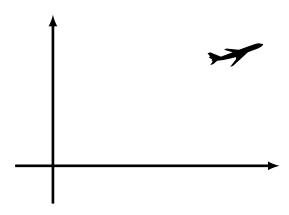
Example. Let p_1, \ldots, p_n be any positive numbers. For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$

$$\mathbf{u} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

define:

$$\langle \mathbf{u}, \mathbf{v} \rangle = p_1(a_1b_1) + p_2(a_2, b_2) + \ldots + p_n(a_nb_n)$$

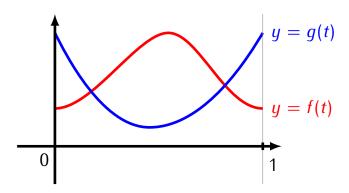
This gives an inner product in \mathbb{R}^n .



Example. Let C[0,1] be the vector space of continuous functions $f:[0,1] \to \mathbb{R}$. Define:

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt$$

This is an inner product on C[0, 1].



Example. Compute the length of the function

$$f(t) = 1 + t^2$$

in C[0, 1].

Definition

Let V be a vector space with an inner product \langle , \rangle , and let W be a subspace of V. A vector $\mathbf{v} \in V$ is *orthogonal to* W if $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in W$.

Definition

Let V be a vector space with an inner product $\langle \ , \ \rangle$, and let W be a subspace of V. The *orthogonal projection of a vector* $\mathbf{v} \in V$ *onto* W is a vector $\mathsf{proj}_W \mathbf{v}$ such that

- 1) $\operatorname{proj}_{W} \mathbf{v} \in W$
- 2) the vector $\mathbf{z} = \mathbf{v} \operatorname{proj}_{W} \mathbf{v}$ is orthogonal to W.

Best Approximation Theorem

If V is a vector space with an inner product \langle , \rangle , W is a subspace of V, and $\mathbf{v} \in V$, then $\text{proj}_W \mathbf{v}$ is the vector of V which is the closest to \mathbf{v} :

$$dist(v, proj_{W}v) \leq dist(v, w)$$

for all $\mathbf{w} \in W$.

Theorem

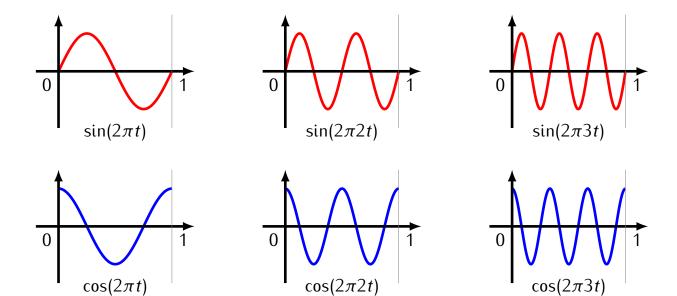
Let V is a vector space with an inner product \langle , \rangle , and let W be a subspace of V. If $\mathcal{B} = \{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ is an orthogonal basis of W (i.e. a basis such that $\langle \mathbf{w}_i, \mathbf{w}_j \rangle = 0$ for all $i \neq j$) then for $\mathbf{v} \in V$ we have:

$$\operatorname{proj}_{W} \mathbf{v} = \frac{\langle \mathbf{v}, \mathbf{w}_{1} \rangle}{\langle \mathbf{w}_{1}, \mathbf{w}_{1} \rangle} \mathbf{w}_{1} + \ldots + \frac{\langle \mathbf{v}, \mathbf{w}_{k} \rangle}{\langle \mathbf{w}_{k}, \mathbf{w}_{k} \rangle} \mathbf{w}_{k}$$

Application: Fourier approximations.

Goal: Let $f:[0,1] \to \mathbb{R}$ be a continuous function. Find the best possible approximation of f of the form

$$P(t) = a_0 + a_1 \sin(2\pi t) + b_1 \cos(2\pi t) + a_2 \sin(2\pi 2t) + b_2 \cos(2\pi 2t) + a_n \sin(2\pi nt) + b_n \cos(2\pi nt)$$



Note: Let W_n be a subspace of C[0,1] given by:

$$W_n = \operatorname{Span}(1, \sin(2\pi t), \cos(2\pi t), \dots, \sin(2\pi n t), \cos(2\pi n t))$$

By the Best Approximation Theorem, the best approximation of f is obtained if we take $P(t) = \text{proj}_{W_n} f(t)$.

Theorem

The set

$$\{1, \sin(2\pi t), \cos(2\pi t), \dots, \sin(2\pi n t), \cos(2\pi n t)\}$$

is an orthogonal basis of W_n .

Corollary

If $f \in C[0, 1]$ then

$$proj_{W_n} f(t) = a_0$$

$$+ a_1 \sin(2\pi t) + b_1 \cos(2\pi t)$$

$$+ a_2 \sin(2\pi 2t) + b_2 \cos(2\pi 2t)$$

$$\cdots \cdots \cdots$$

$$+ a_n \sin(2\pi nt) + b_n \cos(2\pi nt)$$

where:

$$a_0 = \frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} = \int_0^1 f(t) dt$$

and for k > 0:

$$a_k = \frac{\langle f, \sin(2\pi kt) \rangle}{\langle \sin(2\pi kt), \sin(2\pi kt) \rangle} = 2 \int_0^1 f(t) \cdot \sin(2\pi kt) dt$$

$$b_k = \frac{\langle f, \cos(2\pi kt) \rangle}{\langle \cos(2\pi kt), \cos(2\pi kt) \rangle} = 2 \int_0^1 f(t) \cdot \cos(2\pi kt) dt$$

Example. Compute $\operatorname{proj}_{W_n} f(t)$ for the function f(t) = t.

Application: Polynomial approximations.

Goal: Let $f:[0,1] \to \mathbb{R}$ be a continuous function. Find the best possible approximation of f given by a polynomial P(t) of degree $\leq n$:

$$P(t) = a_0 + a_1 t + \ldots + a_n t^n$$

Note: Let \mathbb{P}_n be the subspace of C[0,1] consisting of all polynomials of degree $\leq n$:

$$\mathbb{P}_n = \{a_0 + a_1t + \ldots + a_nt^n \mid a_k \in \mathbb{R}\}\$$

By the Best Approximation Theorem, the best approximation of f is obtained if we take $P(t) = \text{proj}_{\mathbb{P}_n} f(t)$.

Gram-Schmidt process:

a basis
$$\{v_1, \ldots, v_k\}$$
 of $W \subseteq V$ an orthogonal basis $\{w_1, \ldots, w_k\}$ of W

Theorem (Gram-Schmidt Process)

Let V be a vector space with an inner product \langle , \rangle , and let W be a subspace of V. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a basis of W. Define vectors $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ as follows:

$$\mathbf{w}_1 = \mathbf{v}_1$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{w}_1, \mathbf{v}_2 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} \mathbf{w}_1$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{w}_1, \mathbf{v}_3 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} \mathbf{w}_1 - \frac{\langle \mathbf{w}_2, \mathbf{v}_3 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle} \mathbf{w}_2$$

...

$$\mathbf{w}_{k} = \mathbf{v}_{k} - \frac{\langle \mathbf{w}_{1}, \mathbf{v}_{k} \rangle}{\langle \mathbf{w}_{1}, \mathbf{w}_{1} \rangle} \mathbf{w}_{1} - \frac{\langle \mathbf{w}_{2}, \mathbf{v}_{k} \rangle}{\langle \mathbf{w}_{2}, \mathbf{w}_{2} \rangle} \mathbf{w}_{2} - \ldots - \frac{\langle \mathbf{w}_{k-1}, \mathbf{v}_{k} \rangle}{\langle \mathbf{w}_{k-1}, \mathbf{w}_{k-1} \rangle} \mathbf{w}_{k-1}$$

Then the set $\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ is an orthogonal basis of W.

Example. Find an orthogonal basis of the subspace \mathbb{P}_2 of the vector space C[0,1].

Example. Compute $\operatorname{proj}_{\mathbb{P}_2} f(t)$ for $f(t) = \sqrt{t}$.