Instructions.

In this problem you will be given a few statements. For each statement you need to decide if it is true or not, and justify your answer.

How to justify your answer.

• In order to show that a statement is false, it suffices to give a counterexample. For example, consider the the statement:

The last digit of every even number is either 2, 4, or 8.

To show that this statement is false, it is enough to point out that, for example, 10 is an even number, but its last digit is 0.

• In order to show that a statement is true, you need to provide a reasoning explaining why it is true in all instances. Giving one example when it is true will not suffice, since the statement may not work in some other cases. For example, consider the the statement:

If n is an even number then $n + 2$ is also an even number.

You can justify that this is true as follows. Even numbers are integers which are multiples of 2. If n is even then $n = 2m$ for some integer m. Then $n + 2 = 2m + 2 = 2(m + 1)$, which shows that $n + 2$ is even.

Note

This problem will not be collected or graded. However, problems of this type will appear on exams in this course. Sample solutions are provided at the end.

For each of the statements given below decide if it is true or false. If you decide that it is true, justify your answer. If you think it is false give a counterexample.

a) If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are vectors in $\mathbb{R}^3$ and $\mathbf{v}_1 \in \mathrm{Span}(\mathbf{v}_2, \mathbf{v}_3)$ then the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

b) If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are vectors in $\mathbb{R}^3$ and the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, then the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also linearly independent.

c) If A is a 2×3 matrix, and $\mathbf{v}$ is a vector in the column space $\mathrm{Col}(A)$ then $\mathbf{v} \in \mathbb{R}^3$.

d) If A is a matrix, $\mathrm{Nul}(A)$ is the null space of A, and $\mathbf{v}$ is a non-zero vector such that $\mathbf{v} \in \mathrm{Nul}(A)$, then $\mathrm{Nul}(A)$ must contain infinitely many vectors.

Here are solutions to the questions from the previous page. You should try to answer all questions by yourself before reading these solutions.

a) TRUE. If $\mathbf{v}_1 \in \text{Span}(\mathbf{v}_2, \mathbf{v}_3)$ then $\mathbf{v}_1 = c_2\mathbf{v}_2 + c_3\mathbf{v}_3$, for some scalars c_1, c_2, or equivalently

$$\mathbf{v}_1 - c_2\mathbf{v}_2 - c_3\mathbf{v}_3 = \mathbf{0}$$

This means that the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has a non-trivial solution $x_1 = 1$, $x_2 = -c_2$, $x_3 = -c_3$. Thus the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

b) FALSE. For example, take:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

The set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent (since the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{0}$ has only one, trivial solution). On the other hand the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent because the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has non-trivial solutions, e.g. $x_1 = 1$, $x_2 = 1$, $x_3 = -1$.

c) FALSE. The column space of a matrix A consists of linear combinations of columns of A. Since every column of a 2×3 matrix has 2 entries, thus every vector of $\text{Col}(A)$ is a vector in $\mathbb{R}^2$, and not in $\mathbb{R}^3$.

d) TRUE. The null space of A consists of all vectors which are solutions of the matrix equation $A\mathbf{x} = \mathbf{0}$. This equation always has the trivial solution $\mathbf{x} = \mathbf{0}$. If $\mathbf{v}$ is a non-zero vector in $\text{Nul}(A)$, then $\mathbf{x} = \mathbf{v}$ is another solution of this equation. This implies that $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, and so $\text{Nul}(A)$ consists of infinitely many vectors.