Recall: If A is an $m \times n$ matrix then

$$A \cdot \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}}_{\substack{\text{vector in} \\ \mathbb{R}^n}} = \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}}_{\substack{\text{vector in} \\ \mathbb{R}^m}}$$

Example:

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}_{2\times 3 \text{ matrix}} \cdot \underbrace{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}}_{\text{vector in } \mathbb{R}^3} = 2 \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 1 \cdot \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 9 \end{bmatrix}}_{\text{vector in } \mathbb{R}^2}$$

$$\mathbb{R}^3 \xrightarrow{\cdot A} \mathbb{R}^2$$
$$v \longmapsto Av$$

Definition

If A is an $m \times n$ matrix then the function

$$T_A \colon \mathbb{R}^n \to \mathbb{R}^m$$

given by $T_A(\mathrm{v}) = A\mathrm{v}$ is called the *matrix transformation associated to* A.

Example.

Let $T_A\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the matrix transformation defined by the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{bmatrix}$$

1) Compute $T_A(\mathrm{v})$ where $\mathrm{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

$$T_A(v) = Av = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

2) Find a vector v such that $T_A(\mathrm{v}) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

We need to find a vector $v = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that

$$\underline{Av = \begin{bmatrix} 5 \\ 6 \end{bmatrix}}$$

↑ vector equation

<u>aug. matrix:</u>

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 1 & 3 & 3 & 6 \end{array}\right] \xrightarrow{\text{row red}} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & 0 & 1 \end{array}\right] \end{array} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3-3x_3 \\ 1 \\ x_3 \end{bmatrix}$$

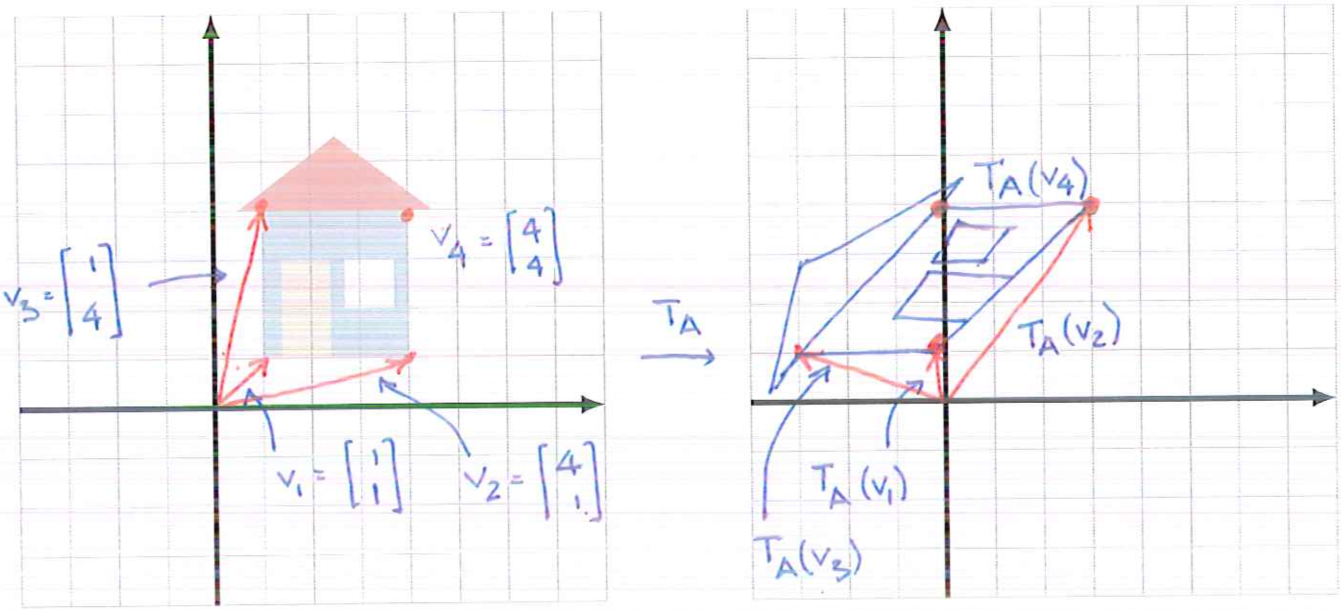
e.g. $x_3 = 0 \Rightarrow v = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

$x_3 = 1 \Rightarrow v = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\vdots$

Geometric interpretation of matrix transformations $\mathbb{R}^2 \to \mathbb{R}^2$

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \qquad T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad v \longmapsto Av$$



$$T_A(v_1) = A \cdot v_1 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T_A(v_2) = A \cdot v_2 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$T_A(v_3) = A v_3 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

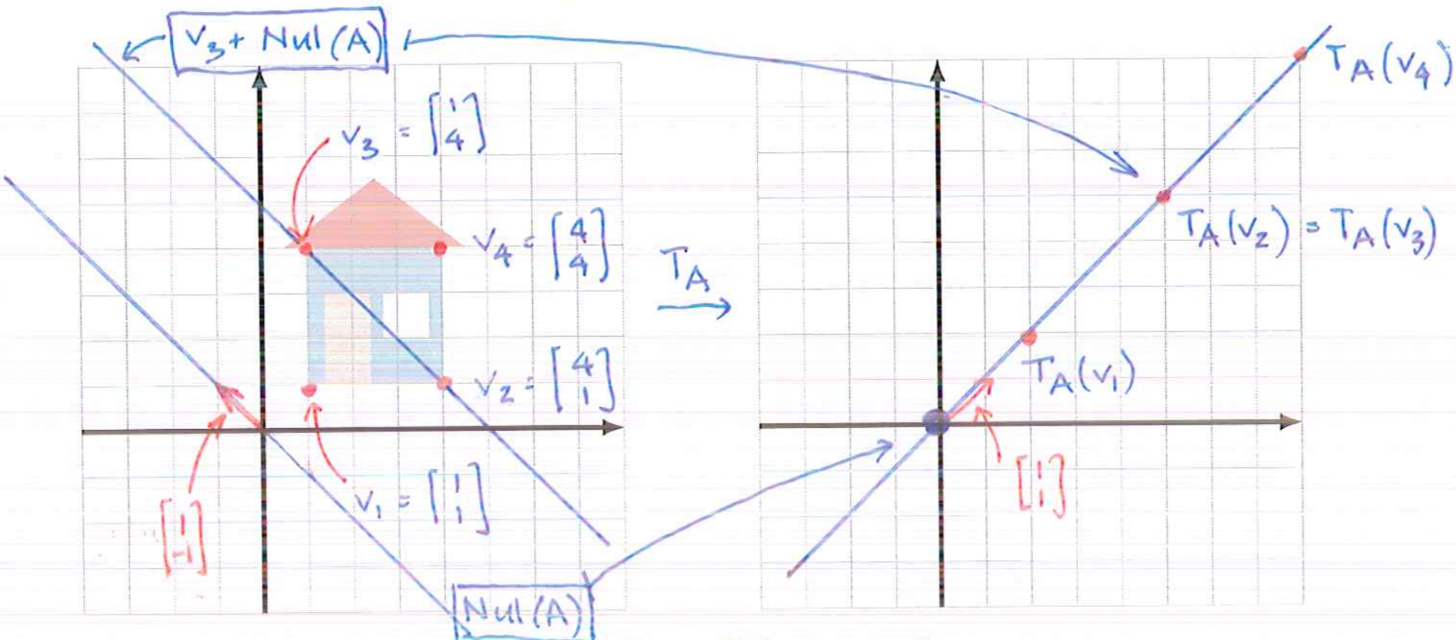
$$T_A(v_4) = A v_4 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

Null spaces, column spaces and matrix transformations

Example.

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$v \longmapsto Av$$



$$T_A(v_1) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$T_A(v_2) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$T_A(v_2) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$T_A(v_4) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$$

$$(\text{values of } T_A) = (\text{vectors in } \mathrm{Col}(A)) = \mathrm{Span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

$$(\text{vectors } v \text{ s.t. } T_A(v) = 0) = (\text{vectors in } \mathrm{Nul}(A)) = \mathrm{Span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$$

Note

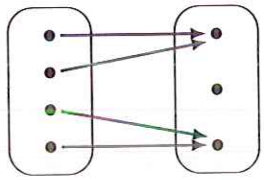
If $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation associated to a matrix A then:

- $\mathrm{Col}(A)$ = the set of values of T_A.
- $\mathrm{Nul}(A)$ = the set of vectors $\mathbf{v}$ such that $T_A(\mathbf{v}) = \mathbf{0}$.
- $T_A(\mathbf{v}) = T_A(\mathbf{w})$ if and only if $\mathbf{w} = \mathbf{v} + \mathbf{n}$ for some $\mathbf{n} \in \mathrm{Nul}(A)$.

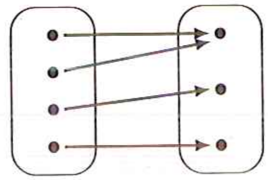
Recall:

A function $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is:

- *onto* if for each $\mathbf{b} \in \mathbb{R}^m$ there is $v \in \mathbb{R}^m$ such that $F(v) = \mathbf{b}$;

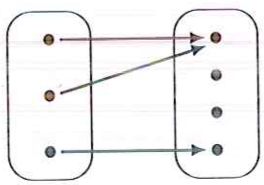


not onto

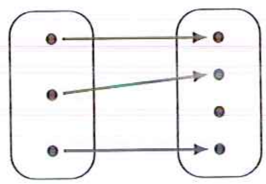


onto

- *one-to-one* if for any v_1, v_2 such that $v_1 \neq v_2$ we have $F(v_2) \neq F(v_2)$.



not one-to-one



one-to-one

Proposition

Let A be an $m \times n$ matrix. The following conditions are equivalent:

1) The matrix transformation $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ is onto.

2) $\mathrm{Col}(A) = \mathbb{R}^m$.

3) The matrix A has a pivot position in every row.

Proposition

Let A be an $m \times n$ matrix. The following conditions are equivalent:

1) The matrix transformation $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one.

2) $\mathrm{Nul}(A) = \{0\}$.

3) The matrix A has a pivot position in every column.

Example. For the following 3×3 matrix A check if the matrix transformation $T_A \colon \mathbb{R}^3 \to \mathbb{R}^3$ is onto and if it is one-to-one.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\Big\downarrow \text{row red.}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

pivot pos. in every row $\Rightarrow$ $\text{Col}(A) = \mathbb{R}^2$

<u>so</u>: T_A is onto

pivot pos. in every column $\Rightarrow$ $\text{Nul}(A) = \{0\}$

<u>so</u>: T_A is one-to one

Example. For the following 3×4 matrix A check if the matrix transformation $T_A \colon \mathbb{R}^4 \to \mathbb{R}^3$ is onto and if it is one-to-one.

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -2 & -2 & 1 & -5 \\ 1 & 1 & -1 & 4 \end{bmatrix}$$

$$\Big\downarrow \text{row red}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

pivot pos. in every row $\Rightarrow$ $\text{Col}(A) = \mathbb{R}^3$

<u>so</u>: T_A is onto

no pivot position in the second column $\Rightarrow$ $\text{Nul}(A) \neq \{0\}$

<u>so</u> T_A is not one-to one

Proposition

Let A be an $m \times n$ matrix. If the matrix transformation $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$ is both onto and one-to-one then we must have $m = n$ (i.e. A must be a square matrix).