<u>Recall:</u>

<div style="border:1px solid">

<u>How to solve a system of linear equations</u>

system of equations

$$\begin{cases} -x_1 + 2x_2 + 3x_3 = 4 \\ 2x_1 + 6x_3 = 9 \\ 4x_1 - x_2 - 3x_3 = 0 \end{cases}$$

*make
a matrix* $\Longrightarrow$

$$\begin{bmatrix} & \text{augmented} \\ & \text{matrix} & \end{bmatrix}$$

*Gauss–Jordan
elimination* $\Downarrow$

solutions

$$\begin{cases} x_1 = \ldots \\ x_2 = \ldots \\ x_3 = \ldots \end{cases}$$

$\Longleftarrow$ *read off
solutions*

$$\begin{bmatrix} & \text{matrix in reduced} \\ & \text{row echelon form} \end{bmatrix}$$

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- Every system of linear equations can be represented by a matrix

- Elementary row operations:
  - interchange of two rows
  - multiplication of a row by a non–zero number
  - addition of a multiple of one row to another row.

- Elementary row operations do not change solutions of systems of linear equations.

Definition

A matrix is in the *row echelon form* if:

1) the first non-zero entry of each row is a 1 ("a leading one");
2) the leading one in each row is to the right of the leading one in the row above it.

A matrix is in the *reduced row echelon form* if in addition it satisfies:

3) all entries above each leading one are 0.

$$\begin{bmatrix} 1 & * & * & * & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

($* = $ any number)

Example

$$\begin{bmatrix} 1 & 0 & 4 & 0 & 7 & 0 \\ 0 & 1 & 5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 4 & 6 & 7 & 0 \\ 0 & 1 & 5 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 4 & 0 & 7 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 6 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

$$
\left[\begin{array}{cccc|c}
1 & 0 & 3 & 0 & 0 \\
0 & 1 & 7 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{array}\right]
$$

Example

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Note

In an augmented matrix in the reduced row echelon form free variables correspond to columns of the coefficient matrix that do not contain leading ones.
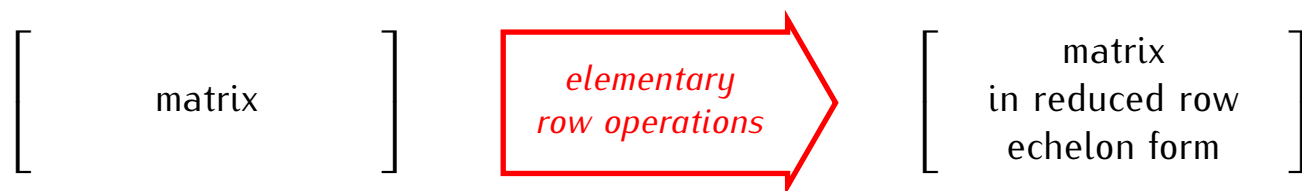
Example

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 8 \end{array}\right]$$

Note

A matrix in the reduced row echelon form represents a system of equations with exactly one solution if and only if it has a leading one in every column except for the last one.

Gauss-Jordan elimination process (= row reduction)

$$\begin{bmatrix} \text{matrix} \end{bmatrix} \xrightarrow{\;\text{\textit{elementary row operations}}\;} \begin{bmatrix} \text{matrix in reduced row echelon form} \end{bmatrix}$$

① Interchange rows, if necessary, to bring a non-zero element to the top of the first non-zero column of the matrix.

② Multiply the first row so that its first non-zero entry becomes 1.

③ Add multiples of the first row to eliminate non-zero entries below the leading one.

④ Ignore the first row; apply steps 1–3 to the rest of the matrix.

⑤ Eliminate non-zero entries above all leading ones.

Example.

$$\begin{bmatrix} 0 & 4 & -8 & 0 & 4 \\ 2 & 6 & -6 & -2 & -4 \\ 2 & 7 & -8 & 0 & -1 \end{bmatrix}$$

How to solve systems of linear equations: example

$$\begin{cases} 4x_2 - 8x_3 = 4 \\ 2x_1 + 6x_2 - 6x_3 - 2x_4 = -4 \\ 2x_1 + 7x_2 - 8x_3 = -1 \end{cases}$$