Recall:

Let A be a matrix with a singular value decomposition

$$A = U\Sigma V^T$$

If

$$U = \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_m \end{bmatrix} \qquad V = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$$

and $\sigma_1, \dots, \sigma_r$ are singular values of A then then

$$A = \sigma_1(\mathbf{u}_1\mathbf{v}_1^T) + \sigma_2(\mathbf{u}_2\mathbf{v}_2^T) + \dots + \sigma_r(\mathbf{u}_r\mathbf{v}_r^T)$$

Example: Movie ratings:

	Matrix	Amelie	Alien	Casablanca	Interstellar
user_1	5	0	5	0	4
user_2	5	0	3	0	5
user_3	0	5	0	5	1
user_4	1	5	0	4	0
user_5	4	0	4	0	3
user_6	0	5	0	4	0
user_7	3	0	3	0	2

Singular value decomposition of the matrix of movie ratings:

$$U = \begin{bmatrix} -0.6 & 0.1 & -0.3 & -0.2 & 0.2 & -0.7 & -0.2 \\ -0.5 & 0.1 & 0.8 & 0.2 & 0.1 & 0.1 & 0.1 \\ -0.1 & -0.6 & 0.2 & -0.7 & -0.4 & 0.0 & 0.0 \\ -0.1 & -0.5 & -0.1 & 0.7 & -0.4 & -0.1 & -0.2 \\ -0.5 & 0.1 & -0.3 & -0.1 & -0.1 & 0.7 & -0.4 \\ -0.1 & -0.6 & -0.1 & 0.0 & 0.8 & 0.1 & 0.2 \\ -0.3 & 0.1 & -0.3 & 0.0 & -0.3 & 0.1 & 0.8 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 13.6 & 0 & 0 & 0 & 0 \\ 0 & 11.4 & 0 & 0 & 0 \\ 0 & 0 & 1.9 & 0 & 0 \\ 0 & 0 & 0 & 1.1 & 0 \\ 0 & 0 & 0 & 0 & 0.3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} -0.6 & 0.1 & 0.0 & 0.7 & -0.4 \\ -0.1 & -0.7 & -0.1 & 0.3 & 0.6 \\ -0.5 & 0.1 & -0.7 & -0.4 & 0.2 \\ -0.1 & -0.6 & 0.0 & -0.4 & -0.7 \\ -0.5 & 0.1 & 0.7 & -0.4 & 0.3 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & 5 & 0 & 4 \\ 5 & 0 & 3 & 0 & 5 \\ 0 & 5 & 0 & 5 & 1 \\ 1 & 5 & 0 & 4 & 0 \\ 4 & 0 & 4 & 0 & 3 \\ 0 & 5 & 0 & 4 & 0 \\ 3 & 0 & 3 & 0 & 2 \end{bmatrix} \approx \begin{array}{cc} & \begin{array}{cc} \text{scifi} & \text{romance} \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{array} & \begin{bmatrix} -0.6 & 0.1 \\ -0.5 & 0.1 \\ -0.1 & -0.6 \\ -0.1 & -0.5 \\ -0.5 & 0.1 \\ -0.1 & -0.6 \\ -0.3 & 0.1 \end{bmatrix} \end{array} \cdot \begin{bmatrix} 13.6 & 0 \\ 0 & 11.4 \end{bmatrix} \cdot \begin{array}{cc} \begin{array}{ccccc} \text{Matrix} & \text{Amelie} & \text{Alien} & \text{Casablanca} & \text{Interstellar} \end{array} & \\ \begin{bmatrix} -0.6 & -0.1 & -0.5 & -0.1 & -0.5 \\ 0.1 & -0.7 & 0.1 & -0.6 & 0.1 \end{bmatrix} & \begin{array}{l} \leftarrow \text{scifi} \\ \leftarrow \text{romance} \end{array} \end{array}$$

$$= \bar{U} \qquad = \bar{\Sigma} \qquad = \bar{V}^T$$

Problem. A new movie "Captive State" was rated by the seven users as follows: 4, 4, 0, 1, 4, 0, 0. What kind of movie it is?

Question: How to get from a movie ratings vector to a movie classification vector?

$$\underbrace{[\,r_1 \;\; r_2 \;\; \cdots \;\; r_n\,]}_{=A} \rightsquigarrow \underbrace{[\,v_1 \;\; v_2 \;\; \cdots \;\; v_n\,]}_{=V^T}$$

columns = movie rating vectors

columns = movie classification vectors

We have: $A \approx \bar{U}^T \cdot \bar{\Sigma} \cdot \bar{V}^T$

$$\bar{U}^T A \approx \underbrace{\bar{U}^T \bar{U}}_{= I} \cdot \bar{\Sigma} \cdot \bar{V}^T$$

$$\bar{U}^T A \approx \bar{\Sigma} \bar{V}^T$$

Note:

$$\Sigma = \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_r \end{bmatrix}, \quad \sigma_i \neq 0$$

so: $\bar{\Sigma}$ is invertible

$$\bar{\Sigma}^{-1} = \begin{bmatrix} \sigma_1^{-1} & & & 0 \\ & \sigma_2^{-1} & & \\ & & \ddots & \\ 0 & & & \sigma_r^{-1} \end{bmatrix}$$

We get: $\bar{\Sigma}^{-1} \bar{U}^T A \approx V^T$

This gives: if r is a column vector of movie ratings then its classification vector is $\bar{\Sigma}^{-1} \bar{U}^T \cdot r$

In our example:

$$\bar{\Sigma}^{-1} \bar{U}^T \cdot \begin{bmatrix} 4 \\ 4 \\ 0 \\ 1 \\ 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.48 \\ 0.05 \end{bmatrix}$$