Recall: If A is an $m \times n$ matrix then

$$A \cdot \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$$

Definition

If A is an $m \times n$ matrix then the function

$$T_A \colon \mathbb{R}^n \to \mathbb{R}^m$$

given by $T_A(\mathbf{v}) = A\mathbf{v}$ is called the *m*atrix transformation associated to A.

Example.

Let $T_A\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the matrix transformation defined by the matrix

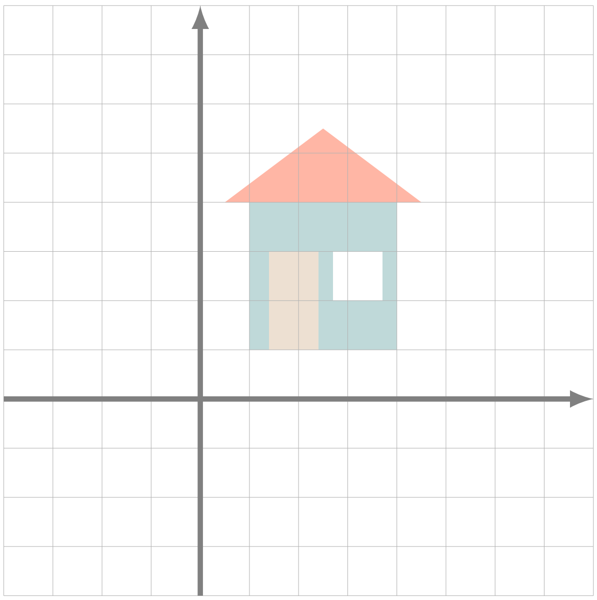
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{bmatrix}$$

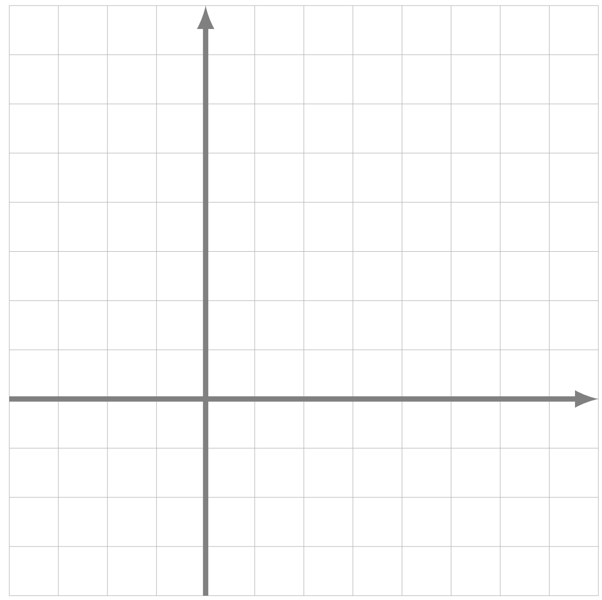
1) Compute $T_A(\mathbf{v})$ where $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

2) Find a vector $\mathbf{v}$ such that $T_A(\mathbf{v}) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

Geometric interpretation of matrix transformations $\mathbb{R}^2 \to \mathbb{R}^2$

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

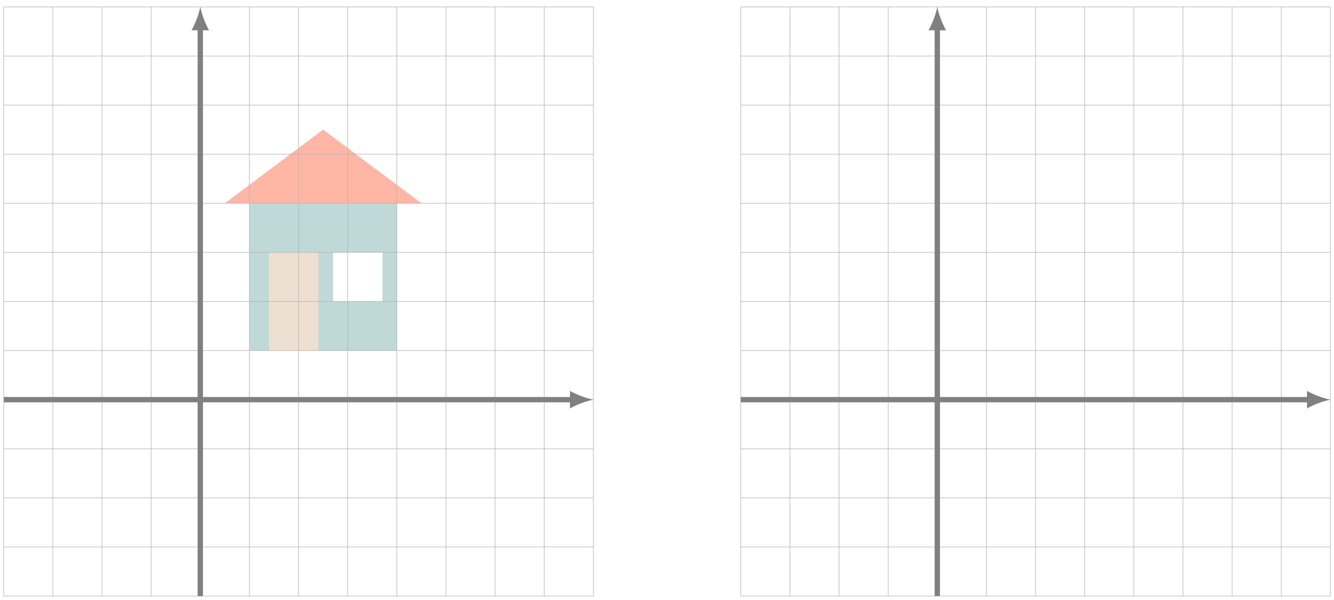




Null spaces, column spaces and matrix transformations

Example.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$



Note

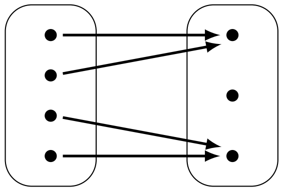
If $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation associated to a matrix A then:

- $\mathrm{Col}(A) =$ the set of values of T_A.
- $\mathrm{Nul}(A) =$ the set of vectors $\mathbf{v}$ such that $T_A(\mathbf{v}) = \mathbf{0}$.
- $T_A(\mathbf{v}) = T_A(\mathbf{w})$ if and only if $\mathbf{w} = \mathbf{v} + \mathbf{n}$ for some $\mathbf{n} \in \mathrm{Nul}(A)$.

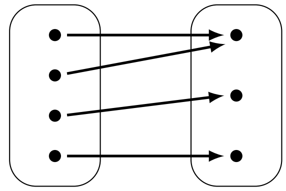
Recall:

A function $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is:

- *onto* if for each $\mathbf{b} \in \mathbb{R}^m$ there is $\mathbf{v} \in \mathbb{R}^n$ such that $F(\mathbf{v}) = \mathbf{b}$;

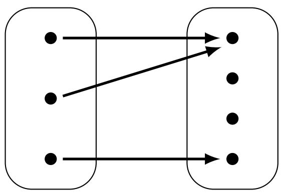


not onto

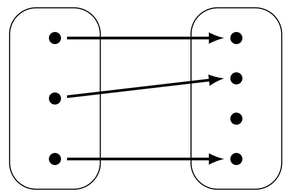


onto

- *one-to-one* if for any $\mathbf{v}_1, \mathbf{v}_2$ such that $\mathbf{v}_1 \neq \mathbf{v}_2$ we have $F(\mathbf{v}_2) \neq F(\mathbf{v}_2)$.



not one-to-one



one-to-one

Proposition

Let A be an $m \times n$ matrix. The following conditions are equivalent:

1) The matrix transformation $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ is onto.

2) $\mathrm{Col}(A) = \mathbb{R}^m$.

3) The matrix A has a pivot position in every row.

Proposition

Let A be an $m \times n$ matrix. The following conditions are equivalent:

1) The matrix transformation $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one.

2) $\mathrm{Nul}(A) = \{0\}$.

3) The matrix A has a pivot position in every column.

Example. For the following 2×2 matrix A check if the matrix transformation $T_A \colon \mathbb{R}^2 \to \mathbb{R}^2$ is onto and if it is one-to-one.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

Example. For the following 3×4 matrix A check if the matrix transformation $T_A \colon \mathbb{R}^4 \to \mathbb{R}^3$ is onto and if it is one-to-one.

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -2 & -2 & 1 & -5 \\ 1 & 1 & -1 & 4 \end{bmatrix}$$

Proposition

Let A be an $m \times n$ matrix. If the matrix transformation $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$ is both onto and one-to-one then we must have $m = n$ (i.e. A must be a square matrix).