Recall:

If $A = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$ is an $m \times n$ matrix then:

1) $\mathrm{Col}(A) = \mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_n)$

2) $\mathrm{Nul}(A) = \{\mathbf{v} \in \mathbb{R}^m \mid A\mathbf{v} = \mathbf{0}\}$

Col(A) is a subspace of $\mathbb{R}^m$.

e.g. $A = \underbrace{\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}}\} m=2$ $\quad \mathrm{Col}(A) = \mathrm{Span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix}\right) \subseteq \mathbb{R}^2$

Nul(A) is a subspace of $\mathbb{R}^n$.

e.g.: $\mathrm{Nul}\left(\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}\right) = \left\{ \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \;\middle|\; \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$

Construction of a basis of Col(A)

Example:

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \qquad \text{Col}(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix}\right) \subseteq \mathbb{R}^2.$$

Lemma

Let V be a vector space, and let $\mathbf{v}_1, \ldots, \mathbf{v}_p \in V$. If a vector $\mathbf{v}_i$ is a linear combination of the other vectors then

$$\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p) = \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_p)$$

Upshot. One can construct a basis of a vector space V as follows:

- Start with a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ such that $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p) = V$.
- Keep removing vectors without changing the span, until you get a linearly independent set.

Example. Find a basis of Col(A) where A is the following matrix:

$$A = \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

matrix in the reduced row echelon form.

Solution

$\mathrm{Col}(A) = \mathrm{Span}(v_1, v_2, \cancel{v_3}, v_4, \cancel{v_5}, v_6) \subseteq \mathbb{R}^5$

Note: $v_3 = 2v_1 + 3v_2$

$v_5 = v_1 - v_2 + 3v_4$

This gives:

$\mathrm{Col}(A) = \mathrm{Span}(v_1, v_2, v_4, v_6)$

Note: The set $\{v_1, v_2, v_4, v_6\}$ is linearly independent, so it is a basis of $\mathrm{Col}(A)$.

In general: If A is a matrix in the reduced echelon form then the set of all columns of A which contain leading ones is a basis of $\mathrm{Col}(A)$.

Example. Find a basis of Col(A) where A is the following matrix:

$$A = \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \end{array}$$

Solution:

$\mathrm{Col}(A) = \mathrm{Span}(v_1, v_2, v_3, v_4, v_5)$

$$\begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \end{array} \xrightarrow{\text{row red}} \begin{array}{c} \begin{array}{ccccc} w_1 & w_2 & w_3 & w_4 & w_5 \end{array} \\ \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{cases} v_2 = -2v_1 \\ v_4 = -v_1 + 2v_3 \\ v_5 = 3v_1 - 2v_3 \end{cases} \longleftarrow \begin{cases} w_2 = -2w_1 \\ w_4 = -w_1 + 2w_3 \\ w_5 = 3w_1 - 2w_3 \end{cases}$$

This gives: $\mathrm{Col}(A) = \mathrm{Span}(v_1, v_3)$.

Check: the set $\{v_1, v_3\}$ is linearly independent, so it is a basis of $\mathrm{Col}(A)$.

In general: If A is a matrix then the set of pivot columns of A is a basis of $\mathrm{Col}(A)$.

Construction of a basis of Nul(A)

Example. Find a basis of Nul(A) where A is the following matrix:

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Solution:

Recall: We know how to find a spanning set of Nul(A):

$$\left[\begin{array}{ccccc|c} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{array}\right] \xrightarrow{\text{row red}} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & \underbrace{0}_{\text{free}} & 0 & \underbrace{0}_{\text{free}} & \underbrace{0}_{\text{free}} & 0 \end{array}\right]$$

We get: $v \in \text{Nul}(A)$ if

$$v = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \cdot \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{w_1} + x_4 \cdot \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{w_2} + x_5 \cdot \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{w_3}$$

This gives: $\text{Nul}(A) = \text{Span}(w_1, w_2, w_3)$

So: the set $\{w_1, w_2, w_3\}$ is a spanning set of Nul(A).

Note: The set obtained in this way is always linearly independent, so it is a basis of Nul (A).

Upshot. If A is matrix then:

$$\dim \mathrm{Col}(A) = \text{the number of pivot columns of } A$$
$$\dim \mathrm{Nul}(A) = \text{the number of non-pivot columns of } A$$

Definition

If A is a matrix then:

- the dimension of $\mathrm{Col}(A)$ is called the *rank* of A and it is denoted $\mathrm{rank}(A)$
- the dimension of $\mathrm{Nul}(A)$ is called the *nullity* of A.

E.g: If A is the matrix from the last example then

$\dim \mathrm{Col}(A) = 2$ so $\mathrm{rank}\, A = 2$

$\dim \mathrm{Nul}(A) = 3$ so nullity of $A = 3$

The Rank Theorem

If A is an $m \times n$ matrix then

$$\mathrm{rank}(A) + \dim \mathrm{Nul}(A) = n$$

Proof:

$\mathrm{rank}(A) + \dim \mathrm{Nul}(A) = $ (the number of pivot columns) + (the number of non-pivot columns)

= (the number of all columns)

$= n.$

Example. Let A be a 100×101 matrix such that $\dim \mathrm{Nul}(A) = 1$. Show that the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbb{R}^{100}$.

Solution:

Recall: $Ax = b$ has a solution if $b \in \mathrm{Col}(A)$

So: $Ax = b$ has solution for each $b \in \mathbb{R}^{100}$ if $\mathrm{Col}(A) = \mathbb{R}^{100}$.

Thus we need to show: $\mathrm{Col}(A) = \mathbb{R}^{100}$.

We have:

$$\dim \mathrm{Col}(A) = \mathrm{rank}\, A = \underbrace{101}_{\text{number of columns of } A} - \underbrace{1}_{\text{the nullity of } A} = 100$$

We obtain:

1) $\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^{100}$.

2) $\dim \mathrm{Col}(A) = 100 = \dim \mathbb{R}^{100}$

This gives: $\mathrm{Col}(A) = \mathbb{R}^{100}$.

Example. Let A be a 5×9. Can the null space of A have dimension 3?

Solution:

We have: $\dim \mathrm{Nul}(A) = \underbrace{9}_{\text{the number of columns of } A} - \dim \mathrm{Col}(A)$

$\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^5$, so $\dim \mathrm{Col}(A) \leqslant \dim \mathbb{R}^5 = 5$

We obtain: $\dim \mathrm{Nul}(A) \geqslant 4$

In particular there is no 5×9 matrix A with $\dim \mathrm{Nul}(A) = 3$.