

Linear Algebra	Calculus
$\mathbb{R}^n = \left( \begin{array}{c} \text{set of all column vectors} \\ \text{with } n \text{ entries} \end{array} \right)$	$C^\infty(\mathbb{R}) = \left( \begin{array}{c} \text{set of all smooth} \\ \text{functions } f: \mathbb{R} \rightarrow \mathbb{R} \end{array} \right)$
<p><b>Column vectors</b> can be added and multiplied by real numbers.</p>	<p><b>Functions</b> can be added and multiplied by real numbers.</p>
<p><b>Linear transformation</b> is a function</p> $T: \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad T(\mathbf{v}) = A\mathbf{v}$ <p>It satisfies:</p> <ul style="list-style-type: none"> <li>• <math>T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})</math></li> <li>• <math>T(c\mathbf{v}) = cT(\mathbf{v})</math></li> </ul>	<p><b>Differentiation</b> is a function</p> $D: C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}), \quad D(f) = f'$ <p>It satisfies:</p> <ul style="list-style-type: none"> <li>• <math>D(f + g) = D(f) + D(g)</math></li> <li>• <math>D(cf) = cD(f)</math></li> </ul>
<p><b>Typical problem:</b> given a vector <math>\mathbf{b}</math> find all vectors <math>\mathbf{x}</math> such that</p> $T(\mathbf{x}) = \mathbf{b}$ <p>(i.e solve the equation <math>A\mathbf{x} = \mathbf{b}</math>).</p>	<p><b>Typical problem:</b> given a function <math>g</math> find all functions <math>f</math> such that</p> $D(f) = g$ <p>(i.e find antiderivatives of <math>g</math>).</p>
<p><b>Fact:</b> Such vectors <math>\mathbf{x}</math> are of the form</p> $\mathbf{x} = \mathbf{v}_0 + \mathbf{n}$ <p>where:</p> <ul style="list-style-type: none"> <li>• <math>\mathbf{v}_0</math> is some distinguished solution of <math>A\mathbf{x} = \mathbf{b}</math>;</li> <li>• <math>\mathbf{n} \in \text{Nul}(A)</math> (i.e. <math>\mathbf{n}</math> is a solution of <math>A\mathbf{x} = \mathbf{0}</math>).</li> </ul>	<p><b>Fact:</b> Such functions <math>f</math> are of the form</p> $f = F + C$ <p>where:</p> <ul style="list-style-type: none"> <li>• <math>F</math> is some distinguished antiderivative of <math>g</math>;</li> <li>• <math>C</math> is a constant function (i.e. <math>C</math> is a solution of <math>D(f) = 0</math>).</li> </ul>

## Definition

A (real) vector space is a set  $V$  together with two operations:

- addition

$$\begin{aligned} V \times V &\longrightarrow V \\ (\mathbf{u}, \mathbf{v}) &\longmapsto \mathbf{u} + \mathbf{v} \end{aligned}$$

- multiplication by scalars

$$\begin{aligned} \mathbb{R} \times V &\longrightarrow V \\ (c, \mathbf{v}) &\longmapsto c \cdot \mathbf{v} \end{aligned}$$

Moreover the following conditions must be satisfied:

- 1)  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- 2)  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- 3) there is an element  $\mathbf{0} \in V$  such that  $\mathbf{0} + \mathbf{u} = \mathbf{u}$  for any  $\mathbf{u} \in V$
- 4) for any  $\mathbf{u} \in V$  there is an element  $-\mathbf{u} \in V$  such that  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
- 5)  $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- 6)  $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
- 7)  $(cd)\mathbf{u} = c(d\mathbf{u})$
- 8)  $1\mathbf{u} = \mathbf{u}$

Elements of  $V$  are called *vectors*.

### Theorem

If  $V$  is a vectors space then:

- 1)  $c \cdot \mathbf{0} = \mathbf{0}$  where  $c \in \mathbb{R}$  and  $\mathbf{0} \in V$  is the zero vector;
- 2)  $0 \cdot \mathbf{u} = \mathbf{0}$  where  $0 \in \mathbb{R}$ ,  $\mathbf{u} \in V$  and  $\mathbf{0}$  is the zero vector;
- 3)  $(-1) \cdot \mathbf{u} = -\mathbf{u}$

**Examples of vector spaces.**