Recall:

1) A square matrix A is diagonalizable if there exists an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}$$

2) If A is diagonalizable then it is easy to compute powers of A:

$$A^k = PD^kP^{-1}$$

3) An $n \times n$ matrix A is a diagonalizable if and only if it has n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. In such case we have:

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} \qquad \begin{array}{l} \lambda_1 = \text{eigenvalue corresponding to } \mathbf{v}_1 \\ \lambda_2 = \text{eigenvalue corresponding to } \mathbf{v}_2 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \lambda_n = \text{eigenvalue corresponding to } \mathbf{v}_n \end{array}$$

4) Not every square matrix is diagonalizable.

Definition

A square matrix A is *symmetric* if $A^T = A$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 5 & 6 \\ 3 & 5 & 7 & 8 \\ 4 & 6 & 8 & 9 \end{bmatrix}$$

Note: A matrix is symmetric if it entries above the main diagonal are the same as the corresponding entries below the main diagonal.

Theorem

Every symmetric matrix is diagonalizable.

Theorem

If A is a symmetric matrix and λ_1, λ_2 are two different eigenvalues of A, then eigenvectors corresponding to λ_1 are orthogonal to eigenvectors corresponding to λ_2.

Note. If $\mathbf{v}$, $\mathbf{w}$ are vectors in $\mathbb{R}^n$ then

$$\underbrace{\mathbf{v} \cdot \mathbf{w}}_{\text{dot product}} = \underbrace{\mathbf{v}^T \mathbf{w}}_{\text{matrix multiplication}}$$

Example.

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$v^T \cdot w = \underbrace{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}}_{1\times 3 \text{ matrix}} \cdot \underbrace{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}}_{3\times 1 \text{ matrix}} = \begin{bmatrix} 1\cdot 4 + 2\cdot 5 + 3\cdot 6 \end{bmatrix} = \underbrace{v \cdot w}_{\text{dot product}}$$

Proof of Theorem:

Let v = eigenvector corresponding to λ_1

w = eigenvector corresponding to λ_2

We have:

$$\lambda_1 (v \cdot w) = (\lambda_1 v) \cdot w = (Av) \cdot w = (Av)^T w$$

$$= (v^T A^T) w \underset{A = A^T}{=} (v^T A) w = v^T (Aw) = v^T (\lambda_2 w) = \lambda_2 (v^T w) = \lambda_2 (v \cdot w)$$

This gives:

$$\lambda_1 (v \cdot w) = \lambda_2 (v \cdot w)$$

$$(\lambda_1 - \lambda_2)(v \cdot w) = 0$$

Since $\lambda_1 \neq \lambda_2$ we obtain $\lambda_1 - \lambda_2 \neq 0$, so $v \cdot w = 0$.

Theorem

If A is an $n \times n$ symmetric matrix then A has n orthogonal eigenvectors.

Example.

a) Find three orthogonal eigenvectors of the following symmetric matrix:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

b) Use these eigenvectors to diagonalize this matrix.

Solution:

1) Find eigenvalues of A:

$$P(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{bmatrix} = -\lambda^3 + 6\lambda^2 - 9\lambda + 4$$

$$(\text{eigenvalues of } A) = (\text{roots of } P(\lambda)) = (\lambda_1 = 4,\ \lambda_2 = 1)$$

2) Find a basis of eigenspace for each eigenvalue.

$$\begin{pmatrix} \text{eigenspace} \\ \text{for } \lambda_1 = 4 \end{pmatrix} = \mathrm{Nul}(A - 4I) \qquad \text{basis: } \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\begin{pmatrix} \text{eigenspace} \\ \text{for } \lambda_2 = 1 \end{pmatrix} = \mathrm{Nul}(A - 1 \cdot I) \qquad \text{basis: } \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Upshot: We have 3 linearly independent eigenvectors:

$$\underbrace{\overset{v_1}{\begin{bmatrix}1\\1\\1\end{bmatrix}}}_{\lambda_1=4},\ \underbrace{\overset{v_2}{\begin{bmatrix}-1\\0\\1\end{bmatrix}},\ \overset{v_3}{\begin{bmatrix}-1\\1\\0\end{bmatrix}}}_{\lambda_2=1}$$

Note: 1) v_1 is orthogonal to v_2 and v_3 (since it corresponds to a different eigenvalue.

2) v_2, v_3 are not orthogonal to each other: $v_1 \cdot v_2 = 1 \neq 0$

To fix this, we need to use the Gram-Schmidt process to find an orthogonal basis of the eigenspace of $\lambda_1 = 1$:

$$w_2 = v_2 = \begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$$w_3 = v_3 - \left(\frac{w_2 \cdot v_3}{w_2 \cdot w_2}\right) w_2 = \begin{bmatrix}-1/2\\1\\-1/2\end{bmatrix}$$

We obtain 3 orthogonal eigenvectors

$$\underbrace{\overset{v_1}{\begin{bmatrix}1\\1\\1\end{bmatrix}}}_{\lambda_1=4},\ \underbrace{\overset{w_2}{\begin{bmatrix}-1\\0\\1\end{bmatrix}},\ \overset{w_3}{\begin{bmatrix}-1/2\\1\\-1/2\end{bmatrix}}}_{\lambda_2=1}$$

This gives a diagonalization of A:

$$A = P \cdot D \cdot P^{-1} \qquad P = \begin{bmatrix}1 & -1 & -1/2\\1 & 0 & 1\\1 & 1 & -1/2\end{bmatrix} \qquad D = \begin{bmatrix}4 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Upshot. How to find n orthogonal eigenvectors for a symmetric $n \times n$ matrix A:

1) Find eigenvalues of A.

2) Find a basis of the eigenspace for each eigenvalue.

3) Use the Gram-Schmidt process to find an orthogonal basis of each eigenspace.

<u>Note:</u> Take the matrix P from the last example:

$$P = \begin{bmatrix} 1 & -1 & -1/2 \\ 1 & 0 & 1 \\ 1 & 1 & -1/2 \end{bmatrix}$$

<u>We have:</u>

$$P^T P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ -1/2 & 1 & -1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -1/2 \\ 1 & 0 & 1 \\ 1 & 1 & -1/2 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3/2 \end{bmatrix}}$$

this is almost the identity matrix, so P^T is almost the inverse of P

<u>Why it works this way:</u>

$$P = [w_1 \; w_2 \; w_3] \qquad P^T = \begin{bmatrix} w_1^T \\ w_2^T \\ w_3^T \end{bmatrix}$$

equal to 0 since w_1, w_2, w_3 are orthogonal

$$P^T P = \begin{bmatrix} w_1^T \\ w_2^T \\ w_3^T \end{bmatrix} \cdot [w_1 \; w_2 \; w_3] = \begin{bmatrix} w_1 \cdot w_1 & \boxed{w_1 \cdot w_2} & \boxed{w_1 \cdot w_3} \\ \boxed{w_2 \cdot w_1} & w_2 \cdot w_2 & \boxed{w_2 \cdot w_3} \\ \boxed{w_3 \cdot w_1} & \boxed{w_3 \cdot w_2} & w_3 \cdot w_3 \end{bmatrix}$$

Definition

A square matrix $Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{bmatrix}$ is an *orthogonal matrix* if $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is an orthonormal set of vectors, i.e.:

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Theorem

If Q is an orthogonal matrix then Q is invertible and $Q^{-1} = Q^T$.

Note. If $P = [\ \mathbf{v}_1 \quad \mathbf{v}_2 \ \ \dots \quad \mathbf{v}_n\ ]$ is a matrix with orthogonal columns, then

$$Q = \left[\ \frac{\mathbf{v}_1}{||\mathbf{v}_1||} \quad \frac{\mathbf{v}_2}{||\mathbf{v}_2||} \quad \dots \quad \frac{\mathbf{v}_n}{||\mathbf{v}_n||}\ \right]$$

is an orthogonal matrix.

Indeed:

$$\frac{v_i}{\|v_i\|} \cdot \frac{v_j}{\|v_j\|} = \frac{v_i \cdot v_j}{\|v_i\| \cdot \|v_j\|} = \begin{cases} 0 & \text{if } i \neq j \text{ since } v_i \cdot v_j = 0 \\ \frac{v_i \cdot v_i}{\|v_i\|^2} = \frac{\|v_i\|^2}{\|v_i\|^2} = 1 & \text{if } i = j \end{cases}$$

Theorem

If A is a symmetric matrix then A is *orthogonally diagonalizable*. That is, there exists an orthogonal matrix Q and a diagonal matrix D such that

$$A = QDQ^{-1} = QDQ^T$$

Proof: We had: if A - symmetric matrix then A has n orthogonal eigenvectors $v_1, v_2, \dots, v_n$.

Take:

$$Q = \left[\frac{v_1}{\|v_1\|} \quad \frac{v_2}{\|v_2\|} \quad \dots \quad \frac{v_n}{\|v_n\|} \right]$$

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & & \lambda_n \end{bmatrix}$$

where:
$\lambda_1 =$ eigenvalue for v_1
$\lambda_2 =$ eigenvalue for v_2
$\vdots$
$\lambda_n =$ eigenvalue for v_n

Then Q is an orthogonal matrix and $A = QDQ^T$.

Example. Find an orthogonal diagonalization of the following symmetric matrix:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Solution: We have already seen that A has 2 eigenvalues:

$$\lambda_1 = 4, \quad \lambda_2 = 1$$

and it has 3 orthogonal eigenvectors:

$$\underbrace{\overset{v_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}}_{\lambda_1 = 4}, \quad \underbrace{\overset{v_2}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}, \quad \overset{v_3}{\begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix}}}_{\lambda_2 = 1}$$

Take:

$$Q = \left[\frac{v_1}{\|v_1\|} \quad \frac{v_2}{\|v_2\|} \quad \frac{v_3}{\|v_3\|} \right]$$

$$\|v_1\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$
$$\|v_2\| = \sqrt{(-1)^2+0^2+1^2} = \sqrt{2}$$
$$\|v_3\| = \sqrt{(-\tfrac{1}{2})^2+1^2+(-\tfrac{1}{2})^2} = \frac{\sqrt{6}}{2}$$

So:

$$Q = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$$

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We have: $A = QDQ^{-1} = QDQ^T$

Note. We have seen that any symmetric matrix is orthogonally diagonalizable. The converse statement is also true:

Proposition

If a matrix A is orthogonally diagonalizable then A is a symmetric matrix.

<u>Proof:</u> If A is orthogonally diagonalizable then:

$$A = Q \cdot D \cdot Q^T$$

where Q - orthogonal matrix

D - diagonal matrix

<u>Note:</u> $D^T = D$

This gives:

$$\begin{aligned} A^T &= (QDQ^T)^T \\ &= (Q^T)^T \cdot D^T \cdot Q^T \\ &= Q \cdot D \cdot Q^T \\ &= A \end{aligned}$$

So A is a symmetric matrix