

Recall:

1) A square matrix  $A$  is diagonalizable if there exists an invertible matrix  $P$  and a diagonal matrix  $D$  such that

$$A = PDP^{-1}$$

2) If  $A$  is diagonalizable then it is easy to compute powers of  $A$ :

$$A^k = PD^kP^{-1}$$

3) An  $n \times n$  matrix  $A$  is a diagonalizable if and only if it has  $n$  linearly independent eigenvectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ . In such case we have:

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \quad \begin{array}{l} \lambda_1 = \text{eigenvalue corresponding to } \mathbf{v}_1 \\ \lambda_2 = \text{eigenvalue corresponding to } \mathbf{v}_2 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \lambda_n = \text{eigenvalue corresponding to } \mathbf{v}_n \end{array}$$

4) Not every square matrix is diagonalizable.

### Definition

A square matrix  $A$  is *symmetric* if  $A^T = A$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 5 & 6 \\ 3 & 5 & 7 & 8 \\ 4 & 6 & 8 & 9 \end{bmatrix}$$

### Theorem

Every symmetric matrix is diagonalizable.

### Theorem

If  $A$  is a symmetric matrix and  $\lambda_1, \lambda_2$  are two different eigenvalues of  $A$ , then eigenvectors corresponding to  $\lambda_1$  are orthogonal to eigenvectors corresponding to  $\lambda_2$ .

**Note.** If  $\mathbf{v}, \mathbf{w}$  are vectors in  $\mathbb{R}^n$  then

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w}$$

**Example.**

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

### Theorem

If  $A$  is an  $n \times n$  symmetric matrix then  $A$  has  $n$  orthogonal eigenvectors.

**Example.**

a) Find three orthogonal eigenvectors of the following symmetric matrix:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

b) Use these eigenvectors to diagonalize this matrix.

Upshot. How to find  $n$  orthogonal eigenvectors for a symmetric  $n \times n$  matrix  $A$ :

- 1) Find eigenvalues of  $A$ .
- 2) Find a basis of the eigenspace for each eigenvalue.
- 3) Use the Gram-Schmidt process to find an orthogonal basis of each eigenspace.

### Definition

A square matrix  $Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{bmatrix}$  is an *orthogonal matrix* if  $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$  is an orthonormal set of vectors, i.e.:

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

### Theorem

If  $Q$  is an orthogonal matrix then  $Q$  is invertible and  $Q^{-1} = Q^T$ .

**Note.** If  $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}$  is a matrix with orthogonal columns, then

$$Q = \begin{bmatrix} \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} & \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} & \dots & \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|} \end{bmatrix}$$

is an orthogonal matrix.

### Theorem

If  $A$  is a symmetric matrix then  $A$  is *orthogonally diagonalizable*. That is, there exists an orthogonal matrix  $Q$  and a diagonal matrix  $D$  such that

$$A = QDQ^{-1} = QDQ^T$$

**Example.** Find an orthogonal diagonalization of the following symmetric matrix:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$



**Note.** We have seen that any symmetric matrix is orthogonally diagonalizable. The converse statement is also true:

**Proposition**

If a matrix  $A$  is orthogonally diagonalizable then  $A$  is a symmetric matrix.