Theorem

Any A an $m \times n$ matrix can be written as a product

$$A = U\Sigma V^T$$

where:

- $U = \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_m \end{bmatrix}$ is an $m \times m$ orthogonal matrix.
- $V = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$ is an $n \times n$ orthogonal matrix.
- Σ is an $m \times n$ matrix of the following form:

$$\begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \sigma_n \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \sigma_m & 0 & \cdots & 0 \end{bmatrix}$$

(if $n \leq m$) (if $n \geq m$)

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$.

Note.

- The numbers $\sigma_1, \sigma_2, \ldots$ are called *singular values* of A.
- The vectors $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are called *left singular vectors* of A.
- Then the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are called *right singular vectors* of A.
- The formula $A = U\Sigma V^T$ is called a *singular value decomposition (SVD)* of A.
- The matrix Σ is uniquely determined, but U and V depend on some choices.

Theorem

Let A be a matrix with a singular value decomposition

$$A = U\Sigma V^T$$

If

$$U = \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_m \end{bmatrix} \qquad V = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$$

and $\sigma_1, \dots, \sigma_r$ are singular values of A then then

$$A = \sigma_1(\mathbf{u}_1\mathbf{v}_1^T) + \sigma_2(\mathbf{u}_2\mathbf{v}_2^T) + \dots + \sigma_r(\mathbf{u}_r\mathbf{v}_r^T)$$

e.g.:

$$\underset{3\times 2}{A} = \underset{3\times 3}{U} \cdot \underset{3\times 2}{\Sigma} \cdot \underset{2\times 2}{V^T}$$

$$U = [u_1 \; u_2 \; u_3]$$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}$$

$$V = [v_1 \; v_2]$$

$$A = [u_1 \; u_2 \; u_3] \cdot \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}$$

$$= [\sigma_1 u_1 \;\; \sigma_2 u_2] \cdot \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}$$

$$= \sigma_1(u_1 v_1^T) + \sigma_2(u_2 v_2^T)$$

Application: Image compression

- The size of this image is 800×700 pixels.
- The color of each pixel is represented by an integer between 0 (black) and 255 (white).
- The whole image is described by a matrix A consisting of $800 \times 700 = 560,000$ numbers.
- Each number is stored in 1 byte, so the image file size is $560,000$ bytes (≈ 0.53 MB).

How to make the image file smaller:

1) Compute SVD of the matrix A:

$$A = U\Sigma V^T$$

where

$$U = \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_m \end{bmatrix} \qquad V = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$$

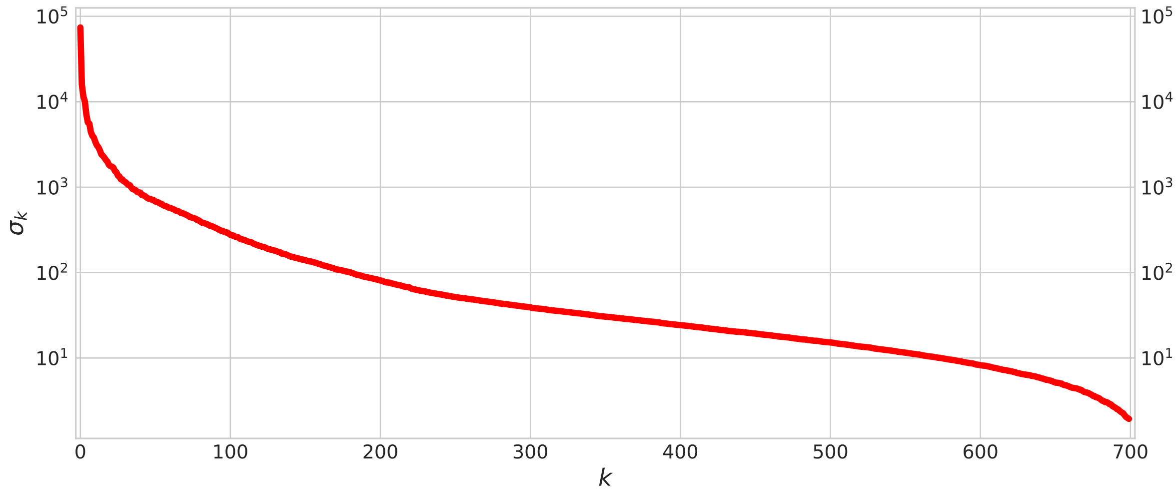
and $\sigma_1, \dots, \sigma_r$ are singular values of A.

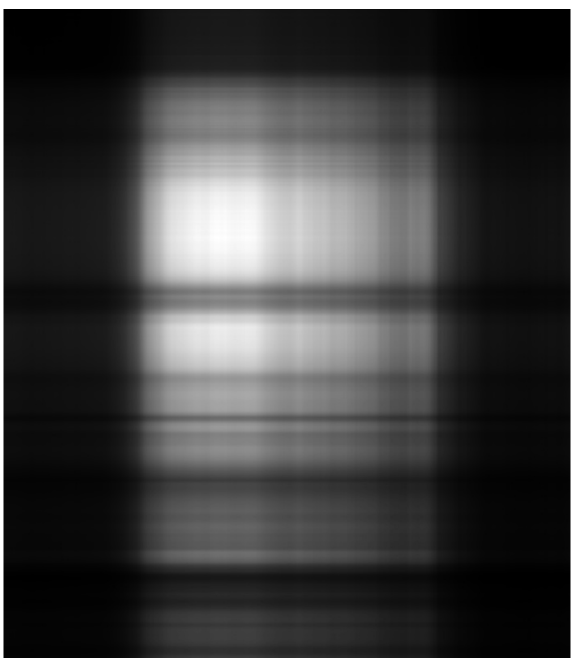
2) Replace A by the matrix

$$B_k = \sigma_1(\mathbf{u}_1\mathbf{v}_1^T) + \ldots + \sigma_k(\mathbf{u}_k\mathbf{v}_k^T)$$

for some $1 \leq k \leq 700$. This matrix can be stored using $k \cdot (800 + 700 + 1)$ numbers.

Singular values of the matrix A





matrix B_1
1501 bytes
compression 374:1

matrix B_5
7905 bytes
compression 75:1

matrix B_{10}
15,010 bytes
compression 37:1

matrix B_{20}
30,020 bytes
compression 18:1

matrix B_{50}
75,050 bytes
compression 7:1

matrix B_{100}
150,100 bytes
compression 4:1

How to compute SVD of a matrix A

Assume: $A = \underset{\text{orthogonal}}{\mathcal{U}} \cdot \underset{\text{diagonal}}{\Sigma} \cdot \underset{\text{orthogonal}}{V}^T$

$U^T = U^{-1}$

Then: $A^TA = (U\Sigma V^T)^T \cdot (U\Sigma V^T) = (\cancel{V^T})^{\cancel{T}} \cdot \Sigma^T \cancel{U^T U}\, \Sigma V^T = V(\Sigma^T\Sigma)V^T$

Note: $\Sigma^T\Sigma$ is a diagonal matrix with squares of singular values on the diagonal.

e.g.:

if $\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_1 & 0 \end{bmatrix}$ then $\Sigma^T\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

We obtain: $\underset{\text{symmetric}}{A^TA} = \underset{\text{orthogonal}}{V} \underset{\text{diagonal}}{(\Sigma^T\Sigma)} V^T$ ← This is an orthogonal diagonalization A^TA. We know how to compute it.

This gives matrices V and Σ:

(columns of V) = (orthogonal eigenvectors of A^TA)

(diagonal entries of Σ i.e. singular values of A) = $\sqrt{\text{eigenvalues of } A^TA}$

It remains to compute the matrix $\mathcal{U}$:

$A = \mathcal{U}\Sigma V^T$ gives $AV = \mathcal{U}\Sigma$

Note: If: $\mathcal{U} = [u_1 \dots u_m]$, $V = [v_1, \dots, v_n]$,

$\sigma_1, \dots, \sigma_r$ - non-zero singular values of A

then:

$AV = [Av_1 \dots Av_n]$

$U\Sigma = [\sigma_1 u_1 \dots \sigma_r u_r \; 0 \dots 0]$

so: $u_1 = \frac{1}{\sigma_1} Av_1, \dots, u_r = \frac{1}{\sigma_r} Av_r$

Vectors $u_{r+1}, \dots, u_m$ can be chosen in an arbitrary way so that $\{u_1, \dots, u_r, u_{r+1}, \dots, u_m\}$ is an orthonormal basis of $\mathbb{R}^m$.

How to compute SVD of a matrix A

1) Compute an orthogonal diagonalization of the symmetric $n \times n$ matrix $A^T A$:

$$A^T A = QDQ^T$$

such that eigenvalues on the diagonal of the matrix D are arranged from the largest to the smallest. We set $V = Q$.

2) If

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

then $\sigma_i = \sqrt{\lambda_i}$. This gives the matrix Σ.

Note: if $n > m$ then we use only $\lambda_1, \dots, \lambda_m$. The remaining eigenvalues $\lambda_{m+1}, \dots, \lambda_n$ of D will be equal to 0 in this case.

3) Let $V = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$, and let $\sigma_1, \dots, \sigma_r$ be non-zero singular values of A. The first r columns of the matrix $U = \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_m \end{bmatrix}$ are given by

$$\mathbf{u}_i = \frac{1}{\sigma_i} A\mathbf{v}_i$$

The remaining columns $\mathbf{u}_{r+1}, \dots, \mathbf{u}_m$ can be added arbitrarily so that U is an orthogonal matrix (i.e.$\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$) is an orthonormal basis of $\mathbb{R}^m$.

Example. Find SVD of the following matrix:

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$A = \underset{3\times 3}{\mathcal{U}} \cdot \underset{3\times 2}{\Sigma} \cdot \underset{2\times 2}{V^T}$$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}$$

① Compute orthogonal diagonalization of A^TA:

$$A^TA = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$P(\lambda) = \det \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 3$$

eigenvalues of A^TA: $\lambda_1 = 3$, $\lambda_2 = 1$

$$\left(\text{basis of eigenspace for } \lambda_1 = 3\right) = \left\{ \overset{w_1}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}} \right\} \qquad \|w_1\| = \sqrt{2}$$

$$\left(\text{basis of eigenspace for } \lambda_2 = 1\right) = \left\{ \overset{w_2}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \right\} \qquad \|w_2\| = \sqrt{2}$$

We get: $A^TA = VDV^T$ where $V = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \qquad \sigma_1^2, \ \sigma_2^2$$

② We obtain:

$$V = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

③ Compute $\mathcal{U}$:

Let $\mathcal{U} = [u_1 \ u_2 \ u_3]$

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -2/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \overset{A}{\begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}} \cdot \overset{V}{\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}} = \overset{\mathcal{U}}{[u_1 \ u_2 \ u_3]} \cdot \overset{\Sigma}{\begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}} = [\sqrt{3}u_1 \ \ u_2]$$

So: $u_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1/\sqrt{2} \\ -2/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$ $\qquad u_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

$\underline{u_3 = ?}$

- Start with a vector linearly independent of u_1, u_2.
  In this example we can use e.g. $z_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

- $\{u_1, u_2, z_3\}$ is a basis of $\mathbb{R}^3$. Use Gram-Schmidt process to make it into an orthogonal basis.
  Since u_1, u_2 are already orthogonal it suffices to modify z_3:

$$w_3 = z_3 - \left(\frac{z_3 \cdot u_1}{u_1 \cdot u_1}\right) u_1 - \left(\frac{z_3 \cdot u_2}{u_2 \cdot u_2}\right) u_2 = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

Take $u_3 = \dfrac{w_3}{\|w_3\|} = \dfrac{1}{1/\sqrt{3}} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$

__We obtain:__

$$A = \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_2 \end{array} \\ \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \end{array} \cdot \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\mathcal{U} \quad \cdot \quad \Sigma \quad \cdot \quad V^T$$