Definition

A *homogenous vector equation* is a vector equation of the form

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

(i.e. with the zero vector as the vector of constants).

Definition

Let $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is *linearly independent* if the homogenous equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

has only one, trivial solution $x_1 = 0, \ldots, x_p = 0$. Otherwise the set is *linearly dependent*.

Theorem

Let $v_1, \ldots, v_p \in \mathbb{R}^n$. If the set $\{v_1, \ldots, v_p\}$ is linearly independent then the equation

$$x_1 v_1 + \ldots + x_p v_p = w$$

has exactly one solution for any vector $w \in \text{Span}(v_1, \ldots, v_p)$.

If the set is linearly dependent then this equation has infinitely many solutions for any $w \in \text{Span}(v_1, \ldots, v_p)$.

Example. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ -12 \end{bmatrix}$$

Check is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

> **Note**
>
> A set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly independent if and only if every column of the matrix
> $$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_p \end{bmatrix}$$
> is a pivot column.

Some properties of linearly (in)dependent sets

1) A set consisting of one vector $\{v_1\}$ is linearly dependent if and only if $v_1 = 0$.

2) A set consisting of two vectors $\{v_1, v_2\}$ is linearly dependent if and only if one vector is a scalar multiple of the other.

3) If $\{v_1, \ldots, v_p\}$ is a set of p vectors in $\mathbb{R}^n$ and $p > n$ then this set is linearly dependent.

Upshot: how to find the number of solutions of a vector equation

vector equation
$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{w}$$

is $\mathbf{w}$ in
$\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p)$
?

NO

YES

zero
solutions

is the set
$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$
lin. independent
?

YES

NO

exactly one
solution

infinitely many
solutions