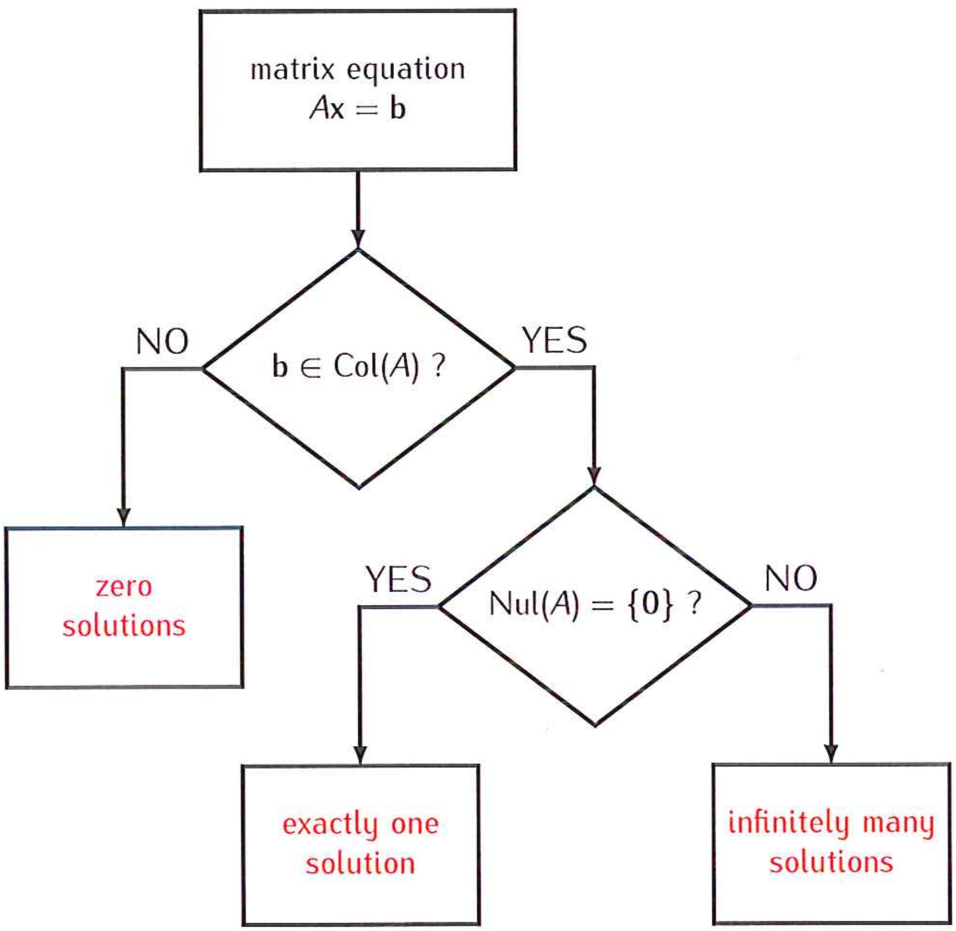
Recall:

1) We can multiply vectors by matrices.

2) Matrix equation: $A\mathbf{x} = \mathbf{b}$



$\mathrm{Col}(A) =$ (span of column vectors of A)

e.g.: $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ $\quad \mathrm{Col}(A) = \mathrm{Span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right)$

$\mathrm{Nul}(A) =$ (set of solutions of $A\mathbf{x} = \mathbf{0}$)

Recall: Nul(A) can be always described as a span of some vectors.

Example. Find the null space of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -2 & -2 & 1 & -5 \\ 1 & 1 & -1 & 3 \end{bmatrix}$$

Solution: We need to solve $Ax = 0$

aug. matrix:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ -2 & -2 & 1 & -5 & 0 \\ 1 & 1 & -1 & 3 & 0 \end{array}\right] \xrightarrow{\text{row red.}} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & \underbrace{0}_{\text{free}} & 0 & \underbrace{0}_{\text{free}} & 0 \end{array}\right]$$

Solutions:

$$\begin{cases} x_1 = -x_2 - 2x_4 \\ x_2 = \text{free} \\ x_3 = x_4 \\ x_4 = \text{free} \end{cases}$$

vector form:

$$x = \begin{bmatrix} -x_2 - 2x_4 \\ x_2 \\ x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2x_4 \\ 0 \\ x_4 \\ x_4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} x_4$$

This gives:

$$\text{Nul}(A) = \text{Span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right)$$

Example. Solve the matrix equation $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -2 & -2 & 1 & -5 \\ 1 & 1 & -1 & 3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Solution:

aug. matrix:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 1 \\ -2 & -2 & 1 & -5 & 0 \\ 1 & 1 & -1 & 3 & -1 \end{array}\right] \xrightarrow{\text{row red.}} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & \underbrace{0}_{\text{free}} & 0 & \underbrace{0}_{\text{free}} & 0 \end{array}\right] \end{array}$$

Solutions:

$$\begin{cases} x_1 = 1 - x_2 - 2x_4 \\ x_2 = \text{free} \\ x_3 = 2 + x_4 \\ x_4 = \text{free} \end{cases}$$

vector form:

$$\mathbf{x} = \begin{bmatrix} 1 - x_2 - 2x_4 \\ x_2 \\ 2 + x_4 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}}_{\substack{\text{a particular} \\ \text{solution of } A\mathbf{x}=\mathbf{b}}} + \underbrace{x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}}_{\substack{\text{a vector from} \\ \text{Nul}(A)}}$$

Proposition

Let $\mathbf{v}_0$ be some chosen solution of a matrix equation $A\mathbf{x} = \mathbf{b}$. Then any other solution $\mathbf{v}$ of this equation is of the form

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{n}$$

where $\mathbf{n} \in \mathrm{Nul}(A)$.