

Recall:

1) Let  $A$  be an  $n \times n$  matrix. If  $\mathbf{v} \in \mathbb{R}^n$  is a non-zero vector and  $\lambda$  is a scalar such that

$$A\mathbf{v} = \lambda\mathbf{v}$$

then

- $\lambda$  is an eigenvalue of  $A$
- $\mathbf{v}$  is an eigenvector of  $A$  corresponding to  $\lambda$ .

2) The characteristic polynomial of an  $n \times n$  matrix  $A$  is the polynomial given by the formula

$$P(\lambda) = \det(A - \lambda I_n)$$

where  $I_n$  is the  $n \times n$  identity matrix.

3) If  $A$  is a square matrix then

$$\text{eigenvalues of } A = \text{roots of } P(\lambda)$$

4) If  $\lambda$  is an eigenvalue of an  $n \times n$  matrix  $A$  then

$$\left\{ \begin{array}{l} \text{eigenvectors of } A \\ \text{corresponding to } \lambda \end{array} \right\} = \left\{ \begin{array}{l} \text{vectors in} \\ \text{Nul}(A - \lambda I_n) \end{array} \right\}$$

Motivating example: Fibonacci numbers

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, ...

**Problem.** Find a formula for the  $n$ -th Fibonacci number  $F_n$ .

**General Problem.** If  $A$  is a square matrix how to compute  $A^k$  quickly?

Easy case:

### Definition

A square matrix  $D$  is *diagonal matrix* if all its entries outside the main diagonal are zeros:

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

### Proposition

If  $D$  is a diagonal matrix as above then

$$D^k = \begin{bmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^k \end{bmatrix}$$

### Definition

A square matrix  $A$  is a *diagonalizable* if  $A$  is of the form

$$A = PDP^{-1}$$

where  $D$  is a diagonal matrix and  $P$  is an invertible matrix.

**Example.**

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$  is a diagonalizable matrix:

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix}^{-1}$$

### Proposition

If  $A$  is a diagonalizable matrix,  $A = PDP^{-1}$ , then

$$A^k = PD^kP^{-1}$$

**Example.**

Let  $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ . Compute  $A^{10}$ .

## Diagonalization Theorem

1) An  $n \times n$  matrix  $A$  is a diagonalizable if and only if it has  $n$  linearly independent eigenvectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ .

2) In such case  $A = PDP^{-1}$  where :

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \quad \begin{array}{l} \lambda_1 = \text{eigenvalue corresponding to } \mathbf{v}_1 \\ \lambda_2 = \text{eigenvalue corresponding to } \mathbf{v}_2 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \lambda_n = \text{eigenvalue corresponding to } \mathbf{v}_n \end{array}$$

**Example.** Diagonalize the following matrix if possible:

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$



**Note.** Not every matrix is diagonalizable.

**Example.** Check if the following matrix is diagonalizable:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

### Proposition

If  $A$  is an  $n \times n$  matrix with  $n$  distinct eigenvalues then  $A$  is diagonalizable.

Back to Fibonacci numbers:

$$\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{n-1} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$