

Recall:

- A basis of a vector space V is a set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ such that

- 1) $\text{Span}(\mathbf{b}_1, \dots, \mathbf{b}_n) = V$
- 2) The set $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is linearly independent.

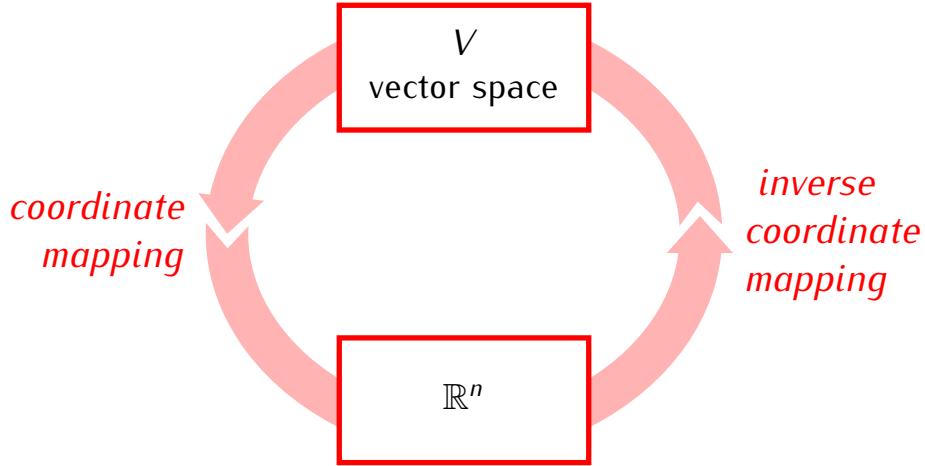
- For $\mathbf{v} \in V$ let c_1, \dots, c_n be the unique numbers such that

$$c_1\mathbf{b}_1 + \dots + c_n\mathbf{b}_n = \mathbf{v}$$

The vector

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$$

is called the *coordinate vector of \mathbf{v} relative to the basis \mathcal{B}* .



Theorem

Let \mathcal{B} be a basis of a vector space V . If $v_1, \dots, v_p, w \in V$ then:

- 1) Solutions of the equation $x_1v_1 + \dots + x_pv_p = w$ are the same as solutions of the equation $x_1[v_1]_{\mathcal{B}} + \dots + x_p[v_p]_{\mathcal{B}} = [w]_{\mathcal{B}}$.
- 2) The set of vectors $\{v_1, \dots, v_p\}$ is linearly independent if and only if the set $\{[v_1]_{\mathcal{B}}, \dots, [v_p]_{\mathcal{B}}\}$ is linearly independent.
- 3) $\text{Span}(v_1, \dots, v_p) = V$ if and only if $\text{Span}([v_1]_{\mathcal{B}}, \dots, [v_p]_{\mathcal{B}}) = \mathbb{R}^n$.
- 4) $\{v_1, \dots, v_p\}$ is a basis of V if and only if $\{[v_1]_{\mathcal{B}}, \dots, [v_p]_{\mathcal{B}}\}$ is a basis of \mathbb{R}^n .

Example. Recall that \mathbb{P}_2 is the vector space of polynomials of degree ≤ 2 . Consider the following polynomials in \mathbb{P}_2 :

$$p_1(t) = 1 + 2t + t^2$$

$$p_2(t) = 3 + t + 2t^2$$

$$p_3(t) = 1 - 8t - t^2$$

Check if the set $\{p_1, p_2, p_3\}$ is linearly independent.

Recall:

In \mathbb{P}_2 we have the standard base $\mathcal{E} = \{1, t, t^2\}$

We have:

$$[p_1]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad [p_2]_{\mathcal{E}} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad [p_3]_{\mathcal{E}} = \begin{bmatrix} 1 \\ -8 \\ -1 \end{bmatrix}$$

It will suffice to check if the set $\{[p_1]_{\mathcal{E}}, [p_2]_{\mathcal{E}}, [p_3]_{\mathcal{E}}\}$ is linearly independent.

Augmented matrix:

$$\left[\begin{array}{ccc} 1 & 3 & 1 \\ 2 & 1 & -8 \\ 1 & 2 & -1 \end{array} \right] \xrightarrow{\text{row red.}} \left[\begin{array}{ccc|c} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right]$$

↑
no leading one
so $\{[p_1]_{\mathcal{E}}, [p_2]_{\mathcal{E}}, [p_3]_{\mathcal{E}}\}$
is not linearly indep.

This shows that the set $\{p_1, p_2, p_3\}$ is not linearly independent.

Theorem

Let $\{v_1, \dots, v_p\}$ be vectors in \mathbb{R}^n . The set $\{v_1, \dots, v_p\}$ is a basis of \mathbb{R}^n if and only if the matrix

$$A = [v_1 \ \dots \ v_p]$$

has a pivot position in every row and in every column (i.e. if A is an invertible matrix).

Proof:

By definition $\{v_1, \dots, v_p\}$ is a basis of \mathbb{R}^n if and only if

1) $\{v_1, v_2, \dots, v_p\}$ is linearly independent

(e.g. $[v_1 \ v_2 \ \dots \ v_p]$ has a pivot position in every column).

2) $\text{Span}(v_1, v_2, \dots, v_p) = \mathbb{R}^n$

(e.g. $[v_1 \ v_2 \ \dots \ v_p]$ has a pivot position in every row).

Example:

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad v_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad v_3 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$[v_1 \ v_2 \ v_3] = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \xrightarrow{\text{row red}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \leftarrow \begin{array}{l} \text{Span}(v_1, v_2, v_3) = \mathbb{R}^2 \\ \text{but } \{v_1, v_2, v_3\} \text{ is} \\ \text{not linearly indep.,} \\ \text{so it is not a basis} \\ \text{of } \mathbb{R}^2. \end{array}$$

Corollary

Every basis of \mathbb{R}^n consists of n vectors.

Theorem

Let V be a vector space. If V has a basis consisting of n vectors then every basis of V consists of n vectors.

Proof: Let $B = \{b_1, b_2, \dots, b_n\}$, $D = \{d_1, d_2, \dots, d_m\}$ be two bases of V .

We have:

1) For each $v \in V$ the coordinate vector $[v]_B$ is a vector in \mathbb{R}^n .

2) Since $\{d_1, \dots, d_m\}$ is a basis of V , the set $\{[d_1]_B, \dots, [d_m]_B\}$ is a basis of \mathbb{R}^n .

Since every basis of \mathbb{R}^n consists of n vectors we obtain $m=n$.

Definition

A vector space has *dimension n* if V has a basis consisting of n vectors. Then we write $\dim V = n$.

Example.

1) In \mathbb{R}^n take the standard basis

$$\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$$

Since this basis consists of n vectors we obtain: $\dim \mathbb{R}^n = n$.

2) Recall: $P_n = \{\text{the vector space of polynomials of degree } \leq n\}$

The standard basis of P_n :

$$\mathcal{E} = \{1, t, t^2, \dots, t^n\}$$

Since \mathcal{E} consists of $n+1$ vectors we have: $\dim P_n = n+1$.

Theorem

Let V be a vector space such that $\dim V = n$, and let $v_1, \dots, v_p \in V$.

- 1) If $\{v_1, \dots, v_p\}$ is a spanning set of V then $p \geq n$.
- 2) If $\{v_1, \dots, v_p\}$ is a linearly independent set then $p \leq n$.

Proof: It is enough to check it for $V = \mathbb{R}^n$.

- 1) If v_1, \dots, v_p are vectors in \mathbb{R}^n and $p < n$ then the matrix $[v_1 \ v_2 \ \dots \ v_p]$ cannot have a pivot position in every row, so $\text{Span}(v_1, \dots, v_p) \neq \mathbb{R}^n$.
- 2) If v_1, \dots, v_p are vectors in \mathbb{R}^n and $p > n$ then the matrix $[v_1 \ v_2 \ \dots \ v_p]$ cannot have a pivot position in every column so the set $\{v_1, v_2, \dots, v_p\}$ is not linearly independent.

Corollary

Let V be a vector space such that $\dim V = n$. If W be a subspace of V then $\dim W \leq n$. Moreover, if $\dim W = n$ then $W = V$.

Proof: If $\dim W = m$ then W has a basis consisting of m vectors: $\{w_1, \dots, w_m\}$. Since this set is linearly independent, and $w_1, \dots, w_m \in V$ by the Theorem above we obtain:

$$\dim W = m \leq n = \dim V$$

Note.

- 1) One can show that every vector space has a basis.
- 2) Some vector spaces have bases consisting of infinitely many vectors. If V is such vector space then we write $\dim V = \infty$.

Example.

$$\begin{aligned} 1) \mathbb{P} &= \{\text{the vector space of all polynomials}\} \\ &= \{a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n \mid a_i \in \mathbb{R}, n \geq 0\} \end{aligned}$$

The set $E = \{1, t, t^2, \dots\}$ is a basis of \mathbb{P} .

Since E consists of infinitely many elements,
we get that $\dim E = \infty$.

$$2) C^\infty(\mathbb{R}) = \{\text{the vector space of all smooth functions } f: \mathbb{R} \rightarrow \mathbb{R}\}$$

Since \mathbb{P} is a subspace of $C^\infty(\mathbb{R})$ and $\dim \mathbb{P} = \infty$
we get that $\dim C^\infty(\mathbb{R}) = \infty$.

It is not possible to write explicitly a basis of $C^\infty(\mathbb{R})$.