Recall:

<div style="border:1px solid black">

How to solve a system of linear equations

system of equations

$$\begin{cases} -x_1 + 2x_2 + 3x_3 = 4 \\ 2x_1 + 6x_3 = 9 \\ 4x_1 - x_2 - 3x_3 = 0 \end{cases}$$

make a matrix ⟶

$$\begin{bmatrix} \text{augmented} \\ \text{matrix} \end{bmatrix}$$

Gauss-Jordan elimination ⟱

solutions

$$\begin{cases} x_1 = \dots \\ x_2 = \dots \\ x_3 = \dots \end{cases}$$

⟵ *read off solutions*

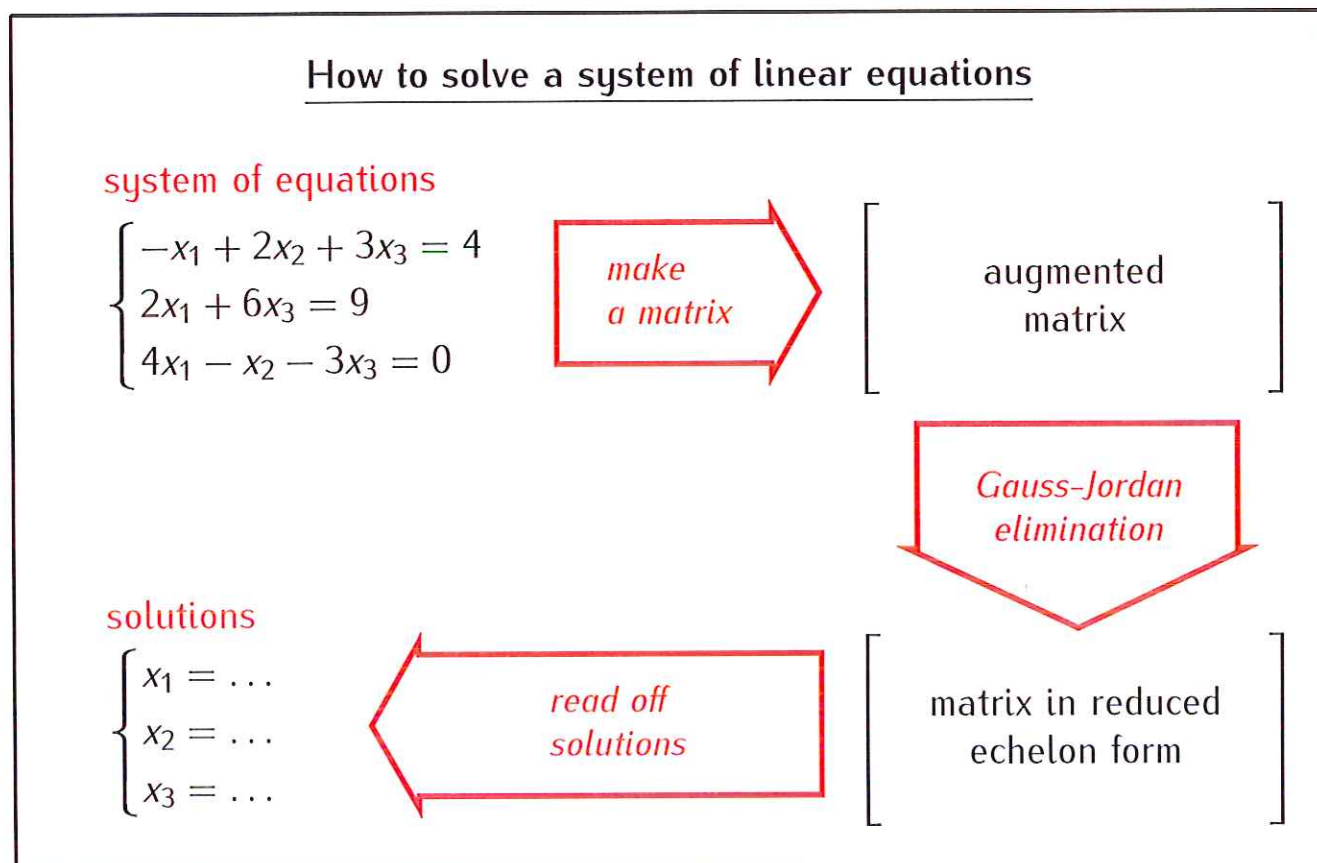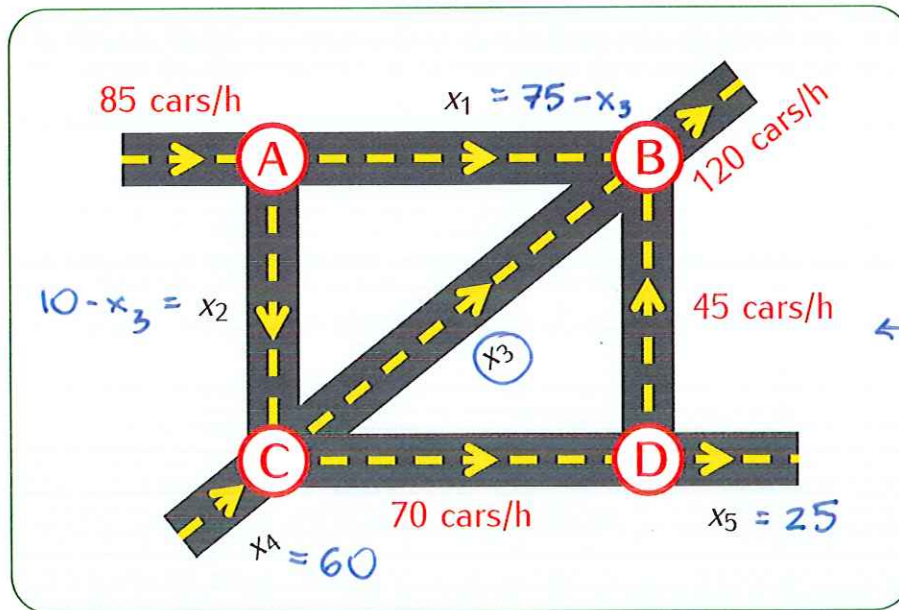$$\begin{bmatrix} \text{matrix in reduced} \\ \text{echelon form} \end{bmatrix}$$

</div>

Next: Some applications of systems of linear equations:

- Computations of traffic flow.
- Balancing chemical equations.
- Google PageRank.

Computations of traffic flow

Inside the diagram:
- 85 cars/h (top left, into A)
- $x_1 = 75 - x_3$ (top, A to B)
- 120 cars/h (top right, out of B)
- $10 - x_3 = x_2$ (left, A to C)
- 45 cars/h (right, into B)
- x_3 (diagonal)
- 70 cars/h (bottom, C to D)
- $x_5 = 25$ (bottom right, out of D)
- $x_4 = 60$ (bottom left, into C)

In order to get full information about the flow of traffic we would need to measure the flow x_3.

Problem. Find the flow rate of cars on each segment of streets.

Note:

- flow into an intersection = flowout of that intersection
- total flow in = total flow out

$$
\begin{array}{lccc}
 & \text{IN} & = & \text{OUT} \\
\text{total}: & 85 + x_4 & = & 120 + x_5 \\
@A: & 85 & = & x_1 + x_2 \\
@B: & x_1 + x_3 + 45 & = & 120 \\
@C: & x_2 + x_4 & = & x_3 + 70 \\
@D: & 70 & = & x_5 + 45
\end{array}
$$

$$
\longrightarrow
\begin{cases}
x_4 - x_5 = 35 \\
x_1 + x_2 = 85 \\
x_1 + x_3 = 75 \\
x_2 - x_3 + x_4 = 70 \\
x_5 = 25
\end{cases}
$$

<u>augmented metrix:</u>

$$
\begin{array}{ccccc}
x_1 & x_2 & x_3 & x_4 & x_5
\end{array}
$$

$$
\left[\begin{array}{ccccc|c}
0 & 0 & 0 & 1 & -1 & 35 \\
1 & 1 & 0 & 0 & 0 & 85 \\
1 & 0 & 1 & 0 & 0 & 75 \\
0 & 1 & -1 & 1 & 0 & 70 \\
0 & 0 & 0 & 0 & 1 & 25
\end{array}\right]
\xrightarrow[\text{reduction}]{\text{now}}
$$

$$
\begin{array}{ccccc}
x_1 & x_2 & x_3 & x_4 & x_5
\end{array}
$$

$$
\left[\begin{array}{ccccc|c}
① & 0 & 1 & 0 & 0 & 75 \\
0 & ① & -1 & 0 & 0 & 10 \\
0 & 0 & 0 & ① & 0 & 60 \\
0 & 0 & 0 & 0 & ① & 25 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\longrightarrow
\begin{cases}
x_1 = 75 - x_3 \\
x_2 = 10 + x_3 \\
x_3 = \text{free} \\
x_4 = 60 \\
x_5 = 25
\end{cases}
$$

20

Balancing chemical equations

Burning propane:

$$\overset{1}{\cancel{x_1}}C_3H_8 + \overset{5}{x_2}O_2 \;\longrightarrow\; \overset{3}{x_3}CO_2 + \overset{4}{x_4}\!$$
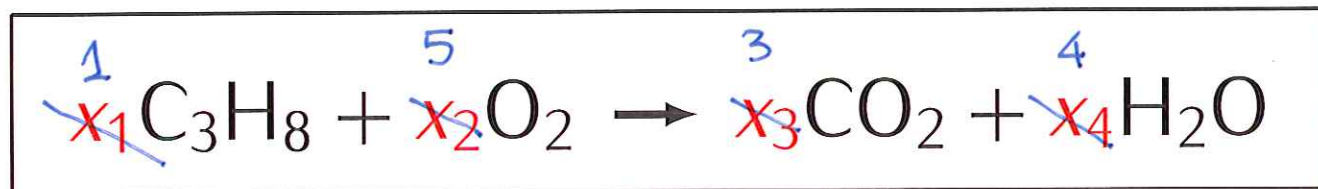
Note:

- The numbers x_1, x_2, x_3, x_4 are integers.

- The number of atoms of each element on the left side is the s
  number of atoms of that element on the right side.

$$\mathsf{LEFT} \;=\; \mathsf{RIGHT}$$

$$
\begin{array}{llll}
C; & 3x_1 & = & x_3 \\
H: & 8x_1 & = & 2x_4 \\
O: & 2x_2 & = & 2x_3 + x_4
\end{array}
$$

$$\Biggr\} \quad \Rightarrow \quad \begin{cases} 3x_1 - x_3 = 0 \\ 8x_1 - x_4 = 0 \\ 2x_2 - 2x_3 - x_4 \end{cases}$$

homogenous s
(i.e. zeros only
right hand sid

$$
\left[\begin{array}{cccc|c}
x_1 & x_2 & x_3 & x_4 & \\
3 & 0 & -1 & 0 & 0 \\
8 & 0 & 0 & -1 & 0 \\
0 & 2 & -2 & -1 & 0
\end{array}\right]
\xrightarrow[\text{red}]{\text{now}}
\left[\begin{array}{cccc|c}
x_1 & x_2 & x_3 & x_4 & \\
1 & 0 & 0 & -1/4 & 0 \\
0 & 1 & 0 & -5/4 & 0 \\
0 & 0 & 1 & -3/4 & 0
\end{array}\right]
$$

$\underbrace{\qquad}_{\text{free}}$

$$\begin{cases} x_1 = \tfrac{1}{4}x_4 \\ x_2 = \tfrac{5}{4}x_4 \\ x_3 = \tfrac{3}{4}x_4 \\ x_4 = \text{free} \end{cases}$$

$$\boxed{\begin{array}{c}\text{set} \\ x_4 = 4\end{array}} \;\longrightarrow\; \begin{cases} x_1 = 1 \\ x_2 = 5 \\ x_3 = 3 \\ x_4 = 1 \end{cases}$$

Google PageRank

Early search engines:

database
of webpages

search query

search results
in random order

user

Google search engine:

database
of webpages
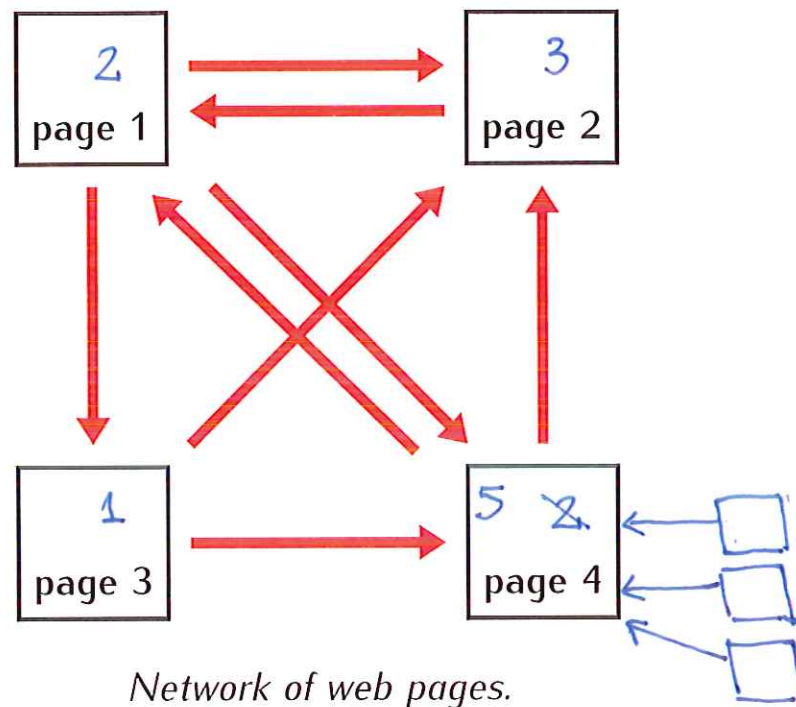with rankings

search query

search results
highly ranked pages first

user
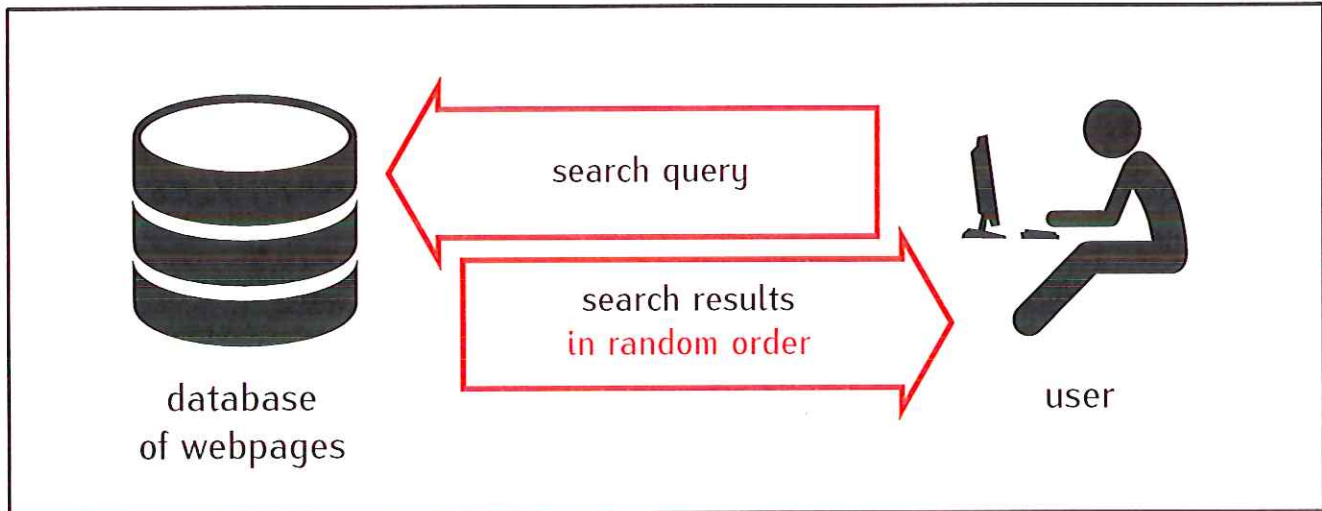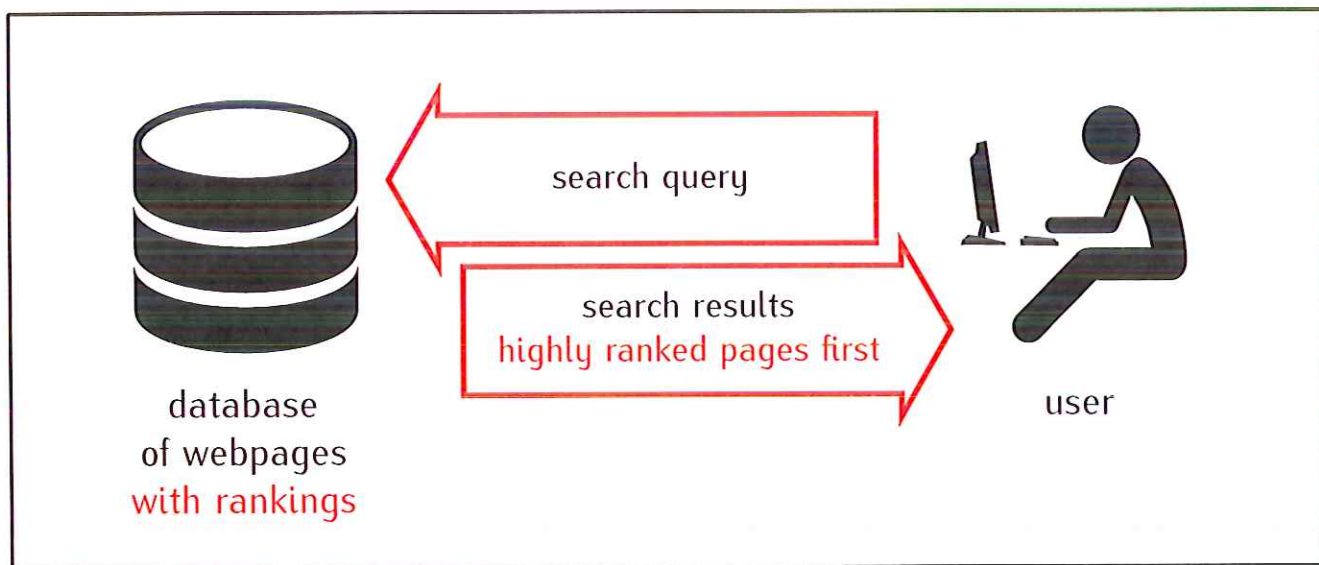
How to rank webpages?

Very simple ranking:

$$\text{ranking of a page} = \left(\begin{array}{c} \text{number of links} \\ \text{pointing to that page} \end{array} \right)$$

Network of web pages.

Problem. This is very easy to manipulate.

How to rank webpages?

Google PageRank: Links from highly ranked pages are worth more than links from lower ranked pages.

If:

- the rank of a page is x
- the page has n links to other pages

then each link from that page is worth x/n.

$$\begin{cases} x_1 = x_2 + \frac{1}{2}x_4 \\ x_2 = \frac{1}{3}x_1 + \frac{1}{2}x_3 + \frac{1}{2}x_4 \\ x_3 = \frac{1}{3}x_1 \\ x_4 = \frac{1}{3}x_1 + \frac{1}{2}x_3 \end{cases}$$

$12/31$ — page 1: x_1, $\frac{1}{3}x_1$ → page 2: x_2 $9/31$, x_2 ←

$\frac{1}{3}x_1$, $\frac{1}{3}x_1$, $\frac{1}{2}x_4$, $\frac{1}{2}x_3$, $\frac{1}{2}x_4$

page 3: x_3 $4/31$, $\frac{1}{2}x_3$ → page 4: x_4 $6/31$

↓ simplify

$$\begin{cases} x_1 - x_2 - \frac{1}{2}x_4 = 0 \\ -\frac{1}{3}x_1 + x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 = 0 \\ -\frac{1}{3}x_1 + x_3 = 0 \\ -\frac{1}{3}x_1 - \frac{1}{2}x_3 + x_4 = 0 \\ x_1 + x_2 + x_3 + x_4 = 1 \end{cases}$$

This system has a trivial solution
$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 0$

Adding this equation eliminates the trivial solution

aug. matrix

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ 1 & -1 & 0 & -\frac{1}{2} & 0 \\ -\frac{1}{3} & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{3} & 0 & 1 & 0 & 0 \\ -\frac{1}{3} & 0 & -\frac{1}{2} & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{array}$$

row red. →

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ 1 & 0 & 0 & 0 & 12/31 \\ 0 & 1 & 0 & 0 & 9/31 \\ 0 & 0 & 1 & 0 & 4/31 \\ 0 & 0 & 0 & 1 & 6/31 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

solutions →

$$\begin{cases} x_1 = 12/31 \\ x_2 = 9/31 \\ x_3 = 4/31 \\ x_4 = 6/31 \end{cases}$$

24