

Recall:

- A vector space is a set V equipped with operations of addition and multiplication by scalars. These operations must satisfy some properties.
- Some examples of vector spaces:
 - 1) \mathbb{R}^n = the vector space of column vectors.
 - 2) $\mathcal{F}(\mathbb{R})$ = the vector space of all functions $f: \mathbb{R} \rightarrow \mathbb{R}$.
 - 3) $C(\mathbb{R})$ = the vector space of all continuous functions $f: \mathbb{R} \rightarrow \mathbb{R}$.
 - 4) $C^\infty(\mathbb{R})$ = the vector space of all smooth functions $f: \mathbb{R} \rightarrow \mathbb{R}$.
 - 5) $M_{m,n}(\mathbb{R})$ = the vector space of all $m \times n$ matrices.
 - 6) \mathbb{P} = the vector space of all polynomials.
 - 7) \mathbb{P}_n = the vector space of polynomials of degree $\leq n$.

- If V, W are vector spaces then a linear transformation is a function $T: V \rightarrow W$ such that

- 1) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

- 2) $T(c\mathbf{v}) = cT(\mathbf{v})$

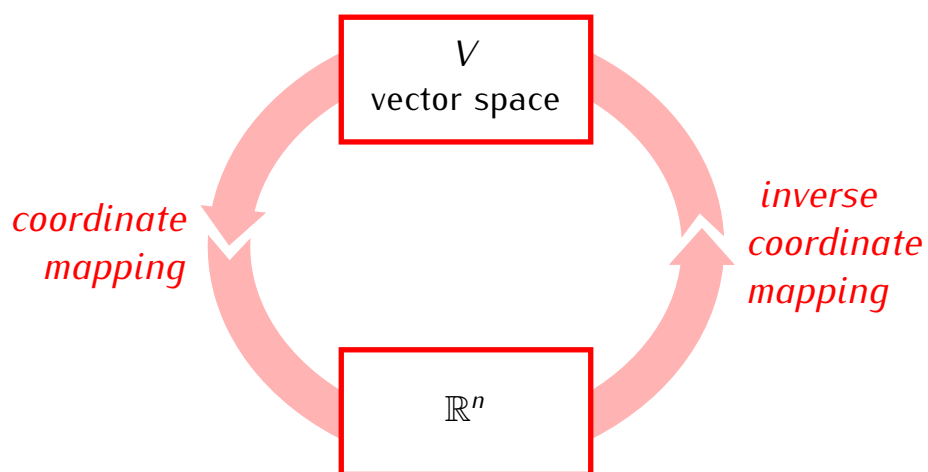
- Many problems involving \mathbb{R}^n can be easily solved using row reduction, matrix multiplication etc.
- The same types of problems involving other vector spaces can be difficult to solve.

Next goal:

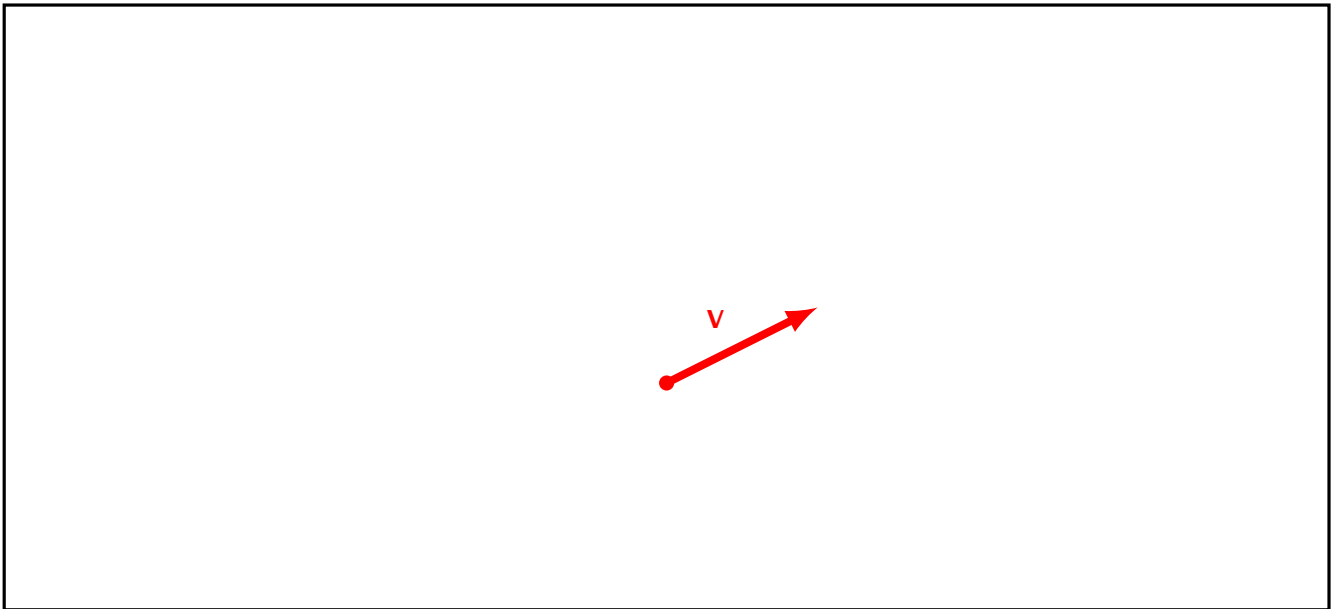
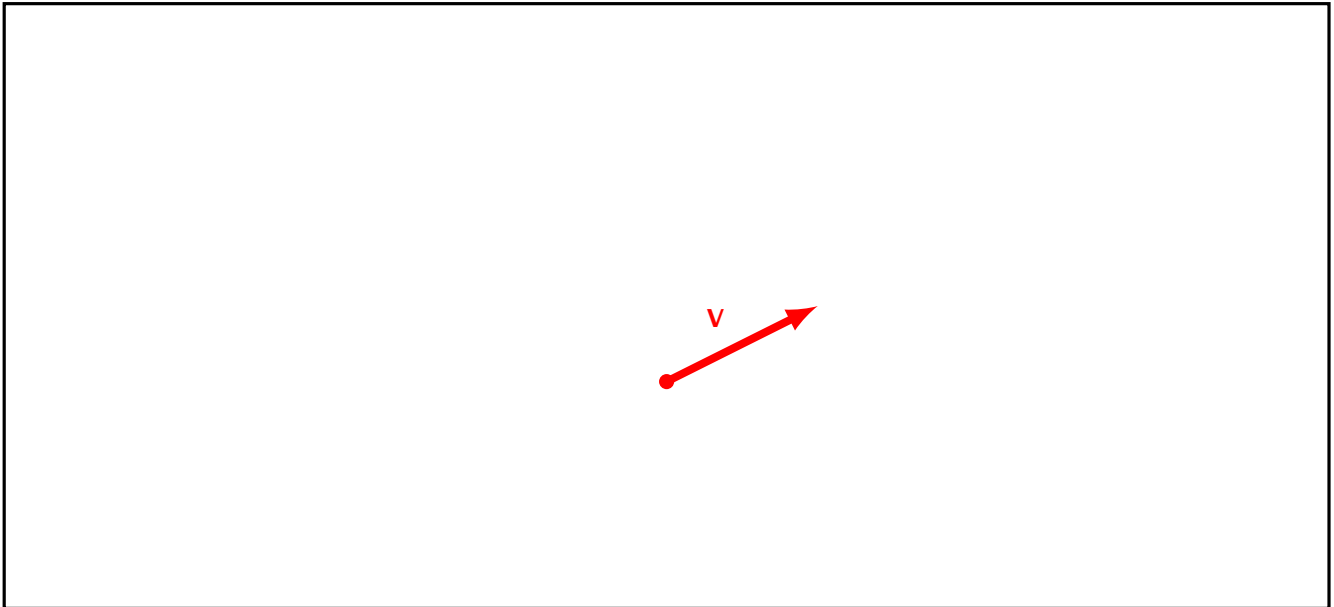
If V is a *finite dimensional* vector space then we can construct a *coordinate mapping*

$$V \rightarrow \mathbb{R}^n$$

which lets us turn computations in V into computations in \mathbb{R}^n .



Motivation: How to assign coordinates to vectors



Definition

If V is a vector space then vector $\mathbf{w} \in V$ is a *linear combination* of vectors $\mathbf{v}_1, \dots, \mathbf{v}_p \in V$ if there exist scalars c_1, \dots, c_p such that

$$\mathbf{w} = c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p$$

Definition

If V is a vector space and $\mathbf{v}_1, \dots, \mathbf{v}_p$ are vectors in V then

$$\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_p) = \left\{ \begin{array}{l} \text{the set of all} \\ \text{linear combinations} \\ c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p \end{array} \right\}$$

Definition

If V is a vector space and $\mathbf{v}_1, \dots, \mathbf{v}_p$ are vectors in V such that

$$V = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_p)$$

the the set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is called the *spanning set* of V .

Example.

Definition

If V is a vector space and $v_1, \dots, v_p \in V$ then the set $\{v_1, \dots, v_p\}$ is *linearly independent* if the homogenous equation

$$x_1 v_1 + \dots + x_p v_p = \mathbf{0}$$

has only one, trivial solution $x_1 = 0, \dots, x_p = 0$. Otherwise the set is *linearly dependent*.

Theorem

Let V be a vector space, and let $v_1, \dots, v_p \in V$. If the set $\{v_1, \dots, v_p\}$ is linearly independent then the equation

$$x_1 v_1 + \dots + x_p v_p = \mathbf{w}$$

has exactly one solution for any vector $\mathbf{w} \in \text{Span}(v_1, \dots, v_p)$.

Example.

Recall: $\mathcal{F}(\mathbb{R})$ = the vector space of all functions $f: \mathbb{R} \rightarrow \mathbb{R}$. Let $f, g, h \in \mathcal{F}(\mathbb{R})$ be the following functions:

$$f(t) = \sin(t), \quad g(t) = \cos(t), \quad h(t) = \cos^2(t)$$

Check if the set $\{f, g, h\}$ is linearly independent.

Example.

Let $f, g, h \in \mathcal{F}(\mathbb{R})$ be the following functions:

$$f(t) = \sin^2(t), \quad g(t) = \cos^2(t), \quad h(t) = \cos 2t$$

Check if the set $\{f, g, h\}$ is linearly independent.

Definition

A *basis* of a vector space V is an ordered set of vectors

$$\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$$

such that

- 1) $\text{Span}(\mathbf{b}_1, \dots, \mathbf{b}_n) = V$
- 2) The set $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is linearly independent.

Theorem

A set $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis of a vector space V if and only if for each $\mathbf{v} \in V$ the vector equation

$$x_1 \mathbf{b}_1 + \dots + x_n \mathbf{b}_n = \mathbf{v}$$

has a unique solution.

Definition

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis of a vector space V . For $\mathbf{v} \in V$ let c_1, \dots, c_n be the unique numbers such that

$$c_1 \mathbf{b}_1 + \dots + c_n \mathbf{b}_n = \mathbf{v}$$

Then the vector

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$$

is called the *coordinate vector of \mathbf{v} relative to the basis \mathcal{B}* and it is denoted by $[\mathbf{v}]_{\mathcal{B}}$.

Example. Let $\mathcal{E} = \{1, t, t^2\}$ be the standard basis of \mathbb{P}_2 , and let

$$p(t) = 3 + 2t - 4t^2$$

Find the coordinate vector $[p]_{\mathcal{E}}$.

Example. Let $\mathcal{B} = \{1, 1 + t, 1 + t + t^2\}$. One can check that \mathcal{B} is a basis of \mathbb{P}_2 .
Let

$$p(t) = 3 + 2t - 4t^2$$

Find the coordinate vector $[p]_{\mathcal{B}}$.