Theorem

Let $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$. If the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent then the equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{w}$$

has exactly one solution for any vector $\mathbf{w} \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p)$.

If the set is linearly dependent then this equation has infinitely many solutions for any $\mathbf{w} \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p)$.

Example. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ -12 \end{bmatrix}$$

Check is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

Note

A set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly independent if and only if every column of the matrix

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_p \end{bmatrix}$$

is a pivot column.

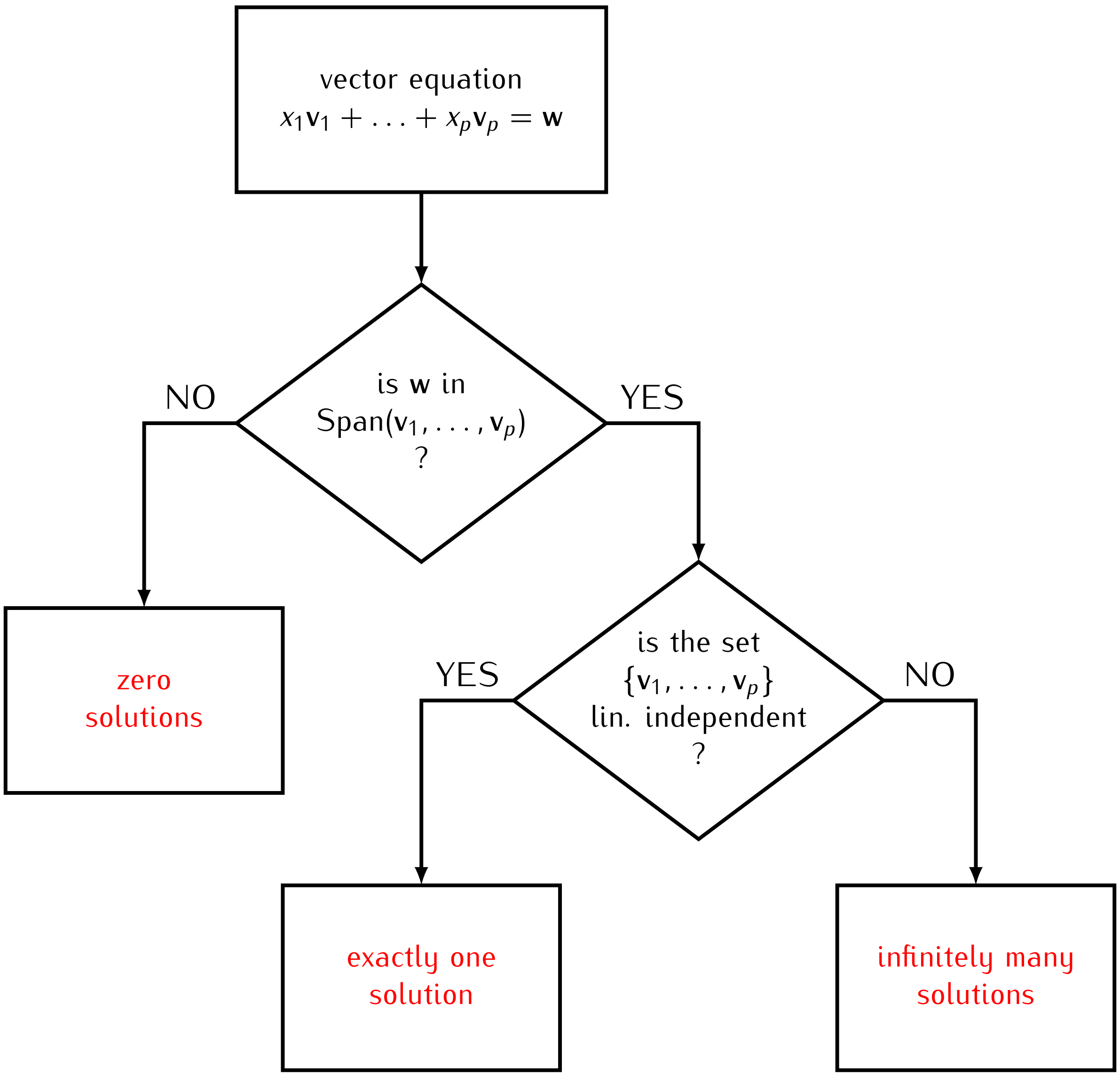
Some properties of linearly (in)dependent sets

1) A set consisting of one vector $\{\mathbf{v}_1\}$ is linearly dependent if and only if $\mathbf{v}_1 = \mathbf{0}$.

2) A set consisting of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if and only if one vector is a scalar multiple of the other.

3) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a set of p vectors in $\mathbb{R}^n$ and $p > n$ then this set is linearly dependent.

Upshot: how to find the number of solutions of a vector equation



Recall:

1) $\mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_p) = \left\{ \begin{array}{c} \text{the set of all} \\ \text{linear combinations} \\ c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p \end{array} \right\}$

2) A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly independent if the equation

$$x_1\mathbf{v}_1 + \dots + x_p\mathbf{v}_p = \mathbf{0}$$

has only one, trivial solution $x_1 = 0, \dots, x_p = 0$.