

Recall:

1) If

$$\mathbf{u} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

are vectors in \mathbb{R}^n then:

- $\mathbf{u} \cdot \mathbf{v} = a_1 b_1 + \dots + a_n b_n$
- $\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$
- $\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$

2) Vectors \mathbf{u}, \mathbf{v} are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.3) Pythagorean theorem: \mathbf{u}, \mathbf{v} are orthogonal if and only if

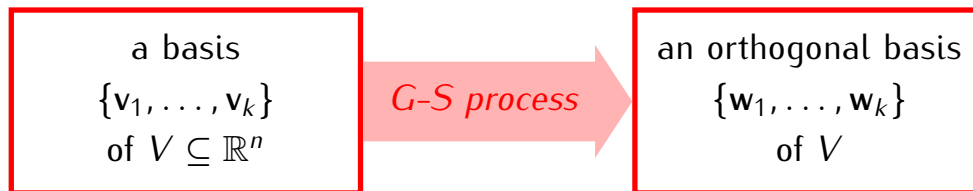
$$\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u} + \mathbf{v}\|^2$$

4) If $V \subseteq \mathbb{R}^n$ is a subspace then an orthogonal basis of V is a basis which consists of vectors that are orthogonal to one another.5) If $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an orthogonal basis of V and $\mathbf{w} \in V$ then

$$[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

where $c_i = \frac{\mathbf{w} \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i}$.

6) Gram-Schmidt process:



$$w_1 = v_1$$

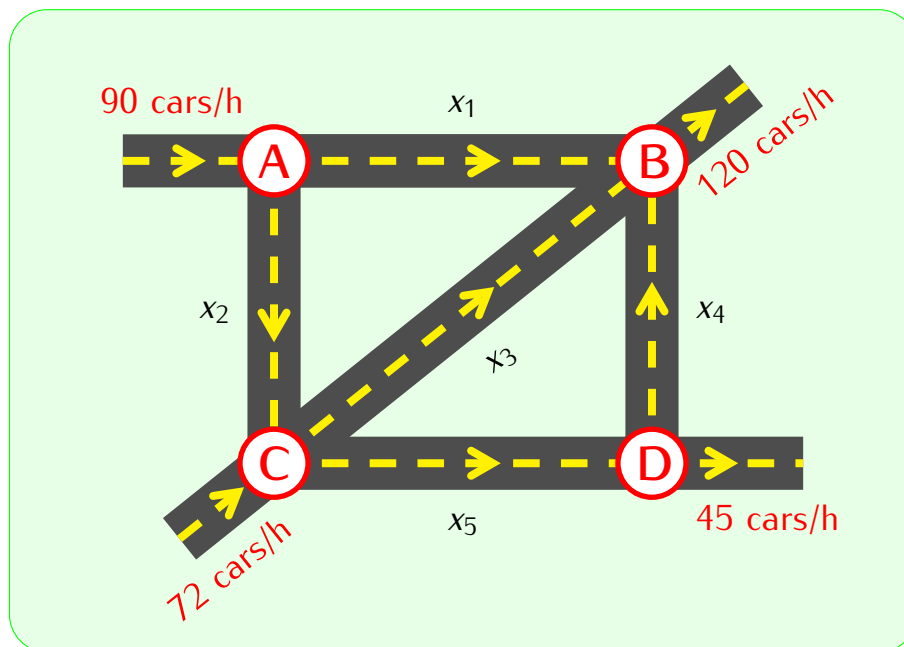
$$w_2 = v_2 - \left(\frac{w_1 \cdot v_2}{w_1 \cdot w_1} \right) w_1$$

$$w_3 = v_3 - \left(\frac{w_1 \cdot v_3}{w_1 \cdot w_1} \right) w_1 - \left(\frac{w_2 \cdot v_3}{w_2 \cdot w_2} \right) w_2$$

... ..

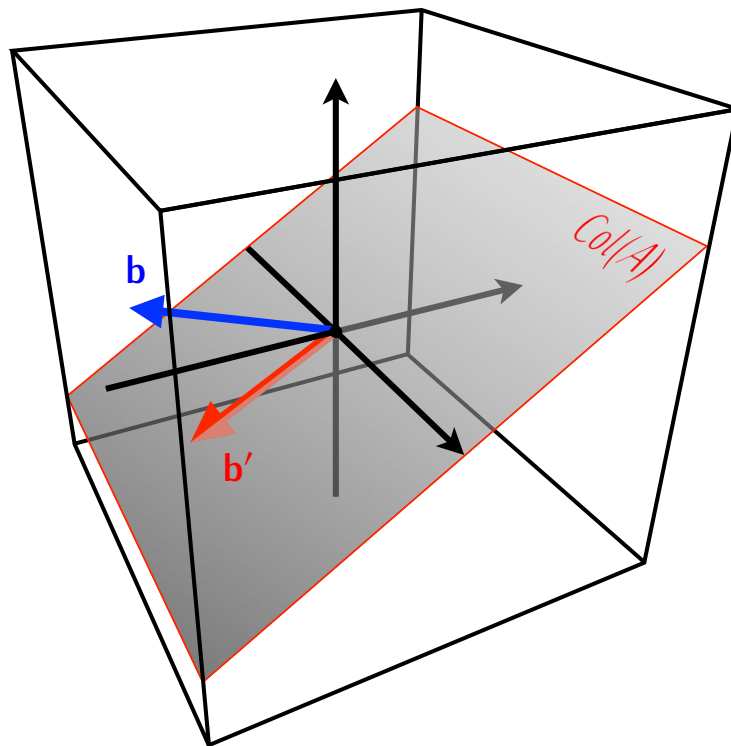
$$w_k = v_k - \left(\frac{w_1 \cdot v_k}{w_1 \cdot w_1} \right) w_1 - \left(\frac{w_2 \cdot v_k}{w_2 \cdot w_2} \right) w_2 - \dots - \left(\frac{w_{k-1} \cdot v_k}{w_{k-1} \cdot w_{k-1}} \right) w_{k-1}$$

Problem. Find the flow rate of cars on each segment of streets:



Upshot.

- Recall: a matrix equation $Ax = \mathbf{b}$ has a solution if and only if $\mathbf{b} \in \text{Col}(A)$.
- In practical applications we may obtain a matrix equation that has no solutions, i.e. where $\mathbf{b} \notin \text{Col}(A)$.
- In such cases we may look for approximate solutions as follows:
 - replace \mathbf{b} by a vector \mathbf{b}' such that $\mathbf{b}' \in \text{Col}(A)$ and $\text{dist}(\mathbf{b}, \mathbf{b}')$ is as small as possible.
 - then solve $Ax = \mathbf{b}'$



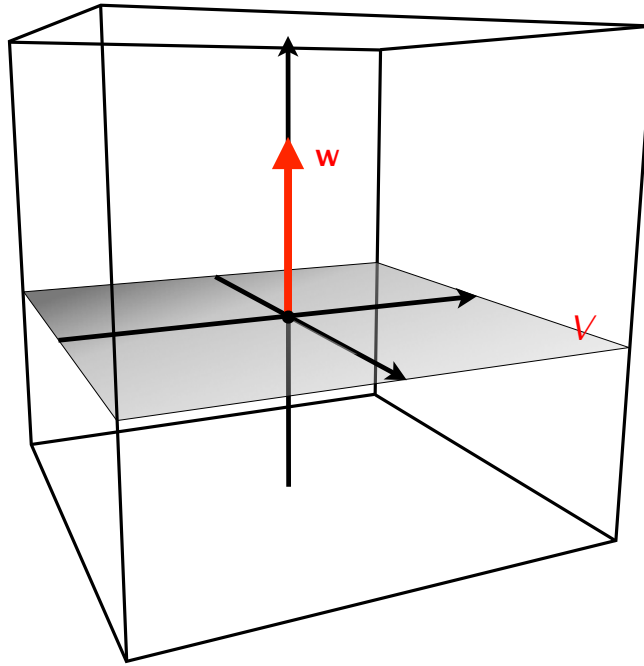
Definition

Given $\mathbf{b}' \in \text{Col}(A)$ as above we will say that a vector \mathbf{v} is a *least square solution* of the equation $Ax = \mathbf{b}$ if \mathbf{v} is a solution of the equation $Ax = \mathbf{b}'$.

Next: How to find the vector \mathbf{b}' ?

Definition

Let V be a subspace of \mathbb{R}^n . A vector $\mathbf{w} \in \mathbb{R}^n$ is *orthogonal to V* if $\mathbf{w} \cdot \mathbf{v} = 0$ for all $\mathbf{v} \in V$.



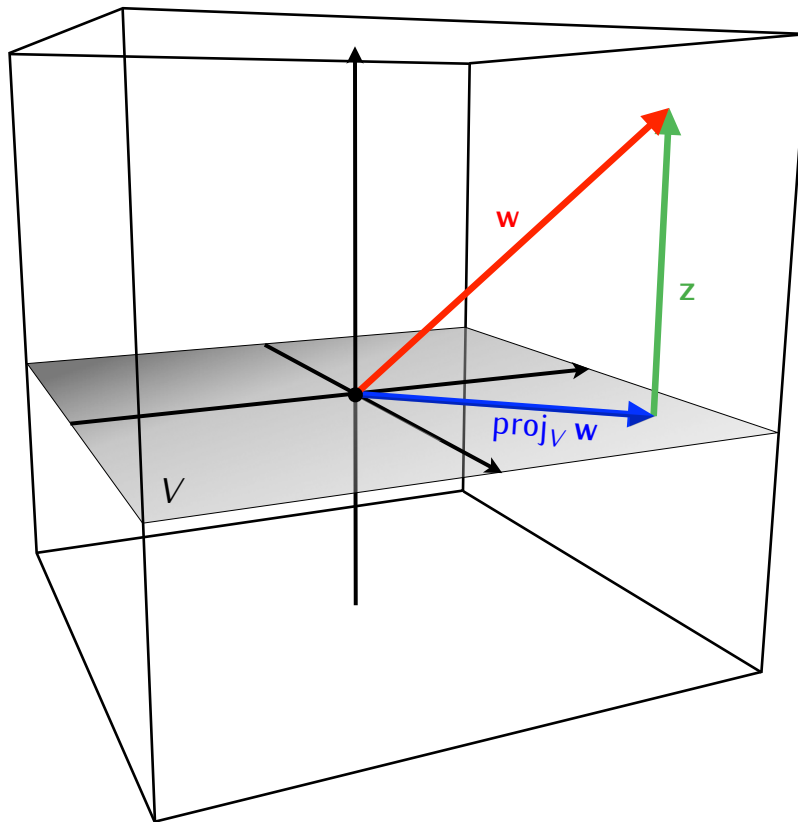
Proposition

If $V = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ then a vector $\mathbf{w} \in \mathbb{R}^n$ is orthogonal to V if and only if $\mathbf{w} \cdot \mathbf{v}_i = 0$ for $i = 1, \dots, k$.

Definition

Let V be a subspace of \mathbb{R}^n and let $\mathbf{w} \in \mathbb{R}^n$ the *orthogonal projection of \mathbf{w} onto V* is a vector $\text{proj}_V \mathbf{w}$ such that

- 1) $\text{proj}_V \mathbf{w} \in V$
- 2) the vector $\mathbf{z} = \mathbf{w} - \text{proj}_V \mathbf{w}$ is orthogonal to V .



The Best Approximation Theorem

If V is a subspace of \mathbb{R}^n and $\mathbf{w} \in \mathbb{R}^n$ then $\text{proj}_V \mathbf{w}$ is a vector in V which is closest to \mathbf{w} :

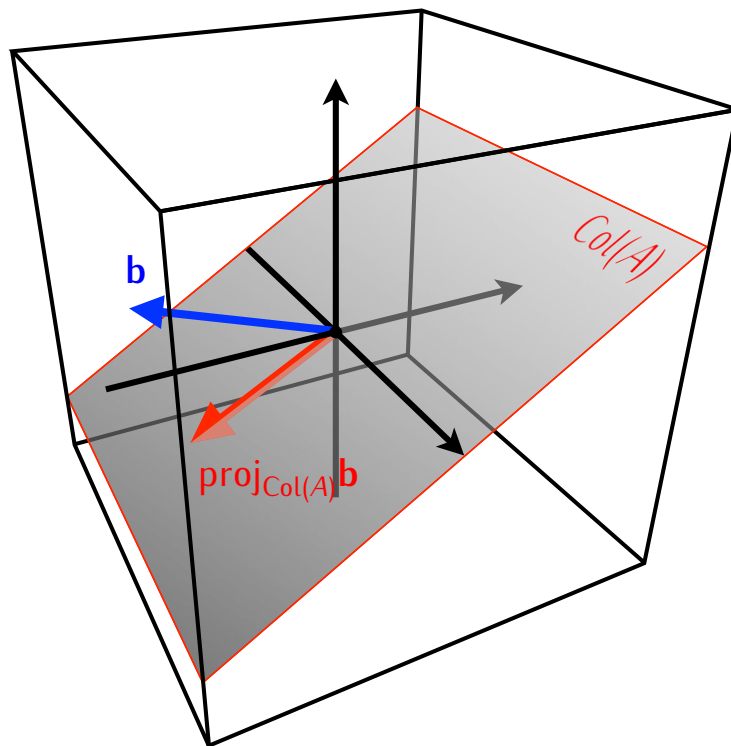
$$\text{dist}(\mathbf{w}, \text{proj}_V \mathbf{w}) \leq \text{dist}(\mathbf{w}, \mathbf{v})$$

for all $\mathbf{v} \in V$.

Corollary

The least square solutions of a matrix equation $Ax = \mathbf{b}$ are solutions of the equation

$$Ax = \text{proj}_{\text{Col}(A)} \mathbf{b}$$



Next: If V is a subspace of \mathbb{R}^n and $\mathbf{w} \in \mathbb{R}^n$ how to compute $\text{proj}_V \mathbf{w}$?

Theorem

If V is a subspace of \mathbb{R}^n with an orthogonal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ and $\mathbf{w} \in \mathbb{R}^n$ then

$$\text{proj}_V \mathbf{w} = \left(\frac{\mathbf{w} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 + \dots + \left(\frac{\mathbf{w} \cdot \mathbf{v}_k}{\mathbf{v}_k \cdot \mathbf{v}_k} \right) \mathbf{v}_k$$

Corollary

If V is a subspace of \mathbb{R}^n with an orthonormal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ and $\mathbf{w} \in \mathbb{R}^n$ then

$$\text{proj}_V \mathbf{w} = (\mathbf{w} \cdot \mathbf{v}_1) \mathbf{v}_1 + \dots + (\mathbf{w} \cdot \mathbf{v}_k) \mathbf{v}_k$$

Example. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 0 \\ -2 \end{bmatrix} \right\}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

The set \mathcal{B} is an orthogonal basis of some subspace V of \mathbb{R}^4 . Compute $\text{proj}_V \mathbf{w}$.

Note. In general if V is a subspace of \mathbb{R}^n and $\mathbf{w} \in \mathbb{R}^n$ then in order to find $\text{proj}_V \mathbf{w}$ we need to do the following:

- 1) find a basis of V .
- 2) use the Gram-Schmidt process to get an orthogonal basis of V
- 3) use the orthogonal basis to compute $\text{proj}_V \mathbf{w}$.

Example. Consider the following matrix A and vector \mathbf{u} :

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 3 & 4 & 2 \\ 2 & 6 & 3 & -1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

Compute $\text{proj}_{\text{Col}(A)} \mathbf{u}$.

Example. Find least square solutions of the matrix equation $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 90 \\ 120 \\ 72 \\ 45 \end{bmatrix}$$