Recall:

• A vector space is a set V equipped with operations of addition and multiplication by scalars. These operations must satisfy some properties.

• Some examples of vector spaces:

1) $\mathbb{R}^n$ = the vector space of column vectors.

2) $\mathcal{F}(\mathbb{R})$ = the vector space of all functions $f\colon \mathbb{R} \to \mathbb{R}$.

3) $C(\mathbb{R})$ = the vector space of all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$.

4) $C^\infty(\mathbb{R})$ = the vector space of all smooth functions $f\colon \mathbb{R} \to \mathbb{R}$.

5) $M_{m,n}(\mathbb{R})$ = the vector space of all $m \times n$ matrices.

6) $\mathbb{P}$ = the vector space of all polynomials.

7) $\mathbb{P}_n$ = the vector space of polynomials of degree $\leq n$.

• If V, W are vector spaces then a linear transformation is a function $T\colon V \to W$ such that

1) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

2) $T(c\mathbf{v}) = cT(\mathbf{v})$

• Many problems involving $\mathbb{R}^n$ can be easily solved using row reduction, matrix multiplication etc.

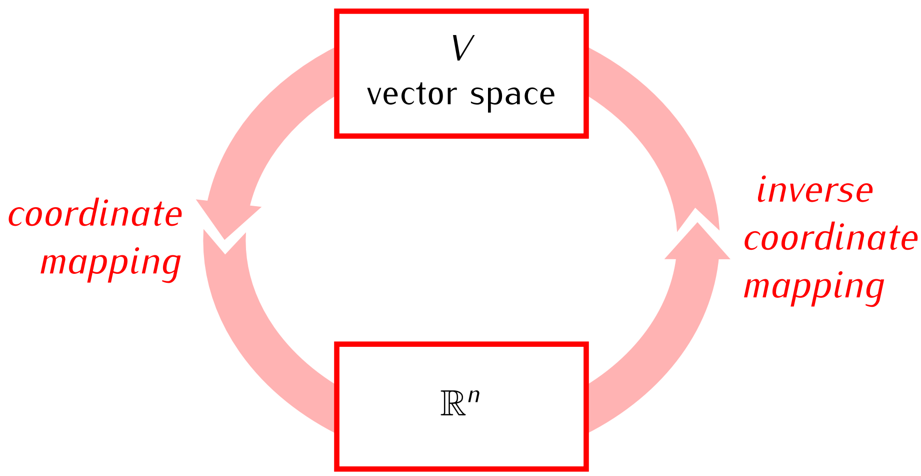
• The same types of problems involving other vector spaces can be difficult to solve.

Next goal:

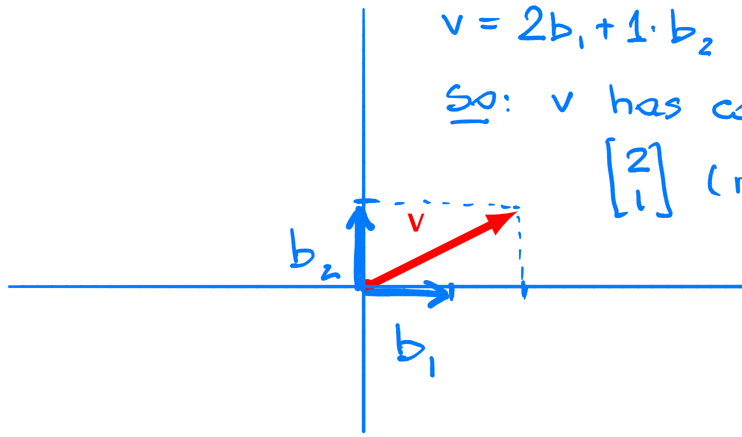
If V is a *finite dimensional* vector space then we can construct a *coordinate mapping*

$$V \to \mathbb{R}^n$$

which lets us turn computations in V into computations in $\mathbb{R}^n$.

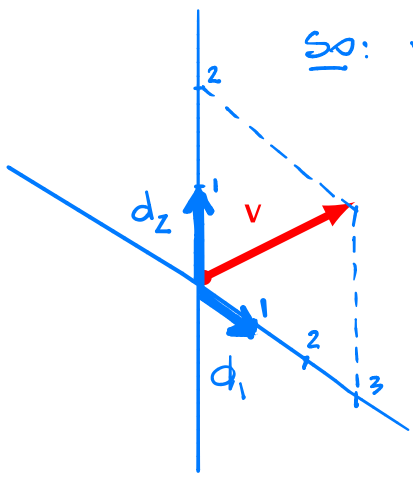


Motivation: How to assign coordinates to vectors



$v = 2b_1 + 1 \cdot b_2$

So: v has coordinates $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (relative to $\{b_1, b_2\}$).



$v = 3d_1 + 2d_2$

So: v has coordinates $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (relative to $\{d_1, d_2\}$).

Upshot: In order to define a coordinate system in a vector space V we need to select vectors $b_1, \dots, b_p$ such that any vector $v \in V$ can be written as

$$v = c_1 b_1 + \dots + c_p b_p$$

in a unique way. Then v will have coordinates $\begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$ relative to $\{b_1, \dots, b_p\}$

Definition

If V is a vector space then vector $\mathbf{w} \in V$ is a *linear combination* of vectors $\mathbf{v}_1, \dots \mathbf{v}_p \in V$ if there exist scalars $c_1, \dots, c_p$ such that

$$\mathbf{w} = c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p$$

Definition

If V is a vector space and $\mathbf{v}_1, \dots, \mathbf{v}_p$ are vectors in V then

$$\mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_p) = \left\{ \begin{array}{c} \text{the set of all} \\ \text{linear combinations} \\ c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p \end{array} \right\}$$

Definition

If V is a vector space and $\mathbf{v}_1, \dots, \mathbf{v}_p$ are vectors in V such that

$$V = \mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_p)$$

the the set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is called the *spanning set* of V.

Note: If $\{v_1, \dots, v_p\}$ is a spanning set of V then every vector $w \in V$ is a linear combination of $v_1, \dots, v_p$:

$$w = c_1 v_1 + \dots + c_p v_p$$

Example.

1) In $\mathbb{R}^3$ the set $\left\{ \underset{e_1}{\begin{bmatrix}1\\0\\0\end{bmatrix}}, \underset{e_2}{\begin{bmatrix}0\\1\\0\end{bmatrix}}, \underset{e_3}{\begin{bmatrix}0\\0\\1\end{bmatrix}} \right\}$

is a spanning set since for any vector $v = \begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}$ we have:

$$v = a_1 e_1 + a_2 e_2 + a_3 e_3$$

2) <u>Recall</u>: $\mathbb{P}_2$ = the vector space of polynomials of degree ≤ 2

$$= \{ a_0 + a_1 t + a_2 t^2 \mid a_i \in \mathbb{R} \}$$

The set $\{ p_0(t) = 1,\ p_1(t) = t,\ p_2(t) = t^2 \}$ is a spanning set of $\mathbb{P}_2$ since if $q(t) = a_0 + a_1 t + a_2 t^2$ then:

$$q(t) = a_0 p_0(t) + a_1 p_1(t) + a_2 p_2(t)$$

3) The set $\{ s_1(t) = t^2 + t + 1,\ s_2(t) = t + 1,\ s_3(t) = 1 \}$ is also a spanning set of $\mathbb{P}_2$ since $q(t) = a_0 + a_1 t + a_2 t^2$ can be written as follows:

$$\begin{aligned} q(t) &= a_2 (t^2 + t + 1) + (a_1 - a_2)(1 + t) + (a_0 - a_1) \cdot 1 \\ &= a_2 s_1(t) + (a_1 - a_2) s_2(t) + (a_0 - a_1) \cdot s_3(t) \end{aligned}$$

Definition

If V is a vector space and $\mathbf{v}_1, \ldots, \mathbf{v}_p \in V$ then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is *linearly independent* if the homogenous equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

has only one, trivial solution $x_1 = 0, \ldots, x_p = 0$. Otherwise the set is *linearly dependent*.

Theorem

Let V be a vector space, and let $\mathbf{v}_1, \ldots, \mathbf{v}_p \in V$. If the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent then the equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{w}$$

has exactly one solution for any vector $\mathbf{w} \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p)$.

<u>Proof</u>: The same as for vector equations in $\mathbb{R}^n$.
(see p. 43 of these notes).

Example.

Recall: $\mathcal{F}(\mathbb{R})$ = the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$. Let $f, g, h \in \mathcal{F}(\mathbb{R})$ be the following functions:

$$f(t) = \sin(t), \quad g(t) = \cos(t), \quad h(t) = \cos^2(t)$$

Check if the set $\{f, g, h\}$ is linearly independent.

Solution: We need to find all numbers $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1 f(t) + c_2 g(t) + c_3 h(t) = 0$$

i.e.: $c_1 \cdot \sin(t) + c_2 \cos(t) + c_3 \cos^2(t) = 0$

plug in $t=0$: $c_1 \underbrace{\sin(0)}_{0} + c_2 \underbrace{\cos(0)}_{1} + c_3 \underbrace{\cos^2(0)}_{1} = 0$

so we must have: $\boxed{c_2 + c_3 = 0}$

plug in $t=\pi$: $c_1 \underbrace{\sin(\pi)}_{0} + c_2 \underbrace{\cos(\pi)}_{-1} + c_3 \underbrace{\cos^2(\pi)}_{(-1)^2 = 1} = 0$

so we must have $\boxed{-c_2 + c_3 = 0}$

plug in $t = \frac{\pi}{2}$: $c_1 \underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} + c_2 \underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} + c_3 \underbrace{\cos^2\left(\frac{\pi}{2}\right)}_{0} = 0$

so we must have $\boxed{c_1 = 0}$

We obtain: $\begin{cases} c_2 + c_3 = 0 \\ -c_2 + c_3 = 0 \\ c_1 = 0 \end{cases}$ so: $\begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases}$

This means that the set $\{\sin(t), \cos(t), \cos^2(t)\}$ is linearly independent.

Example.

Let $f, g, h \in \mathcal{F}(\mathbb{R})$ be the following functions:

$$f(t) = \sin^2(t), \quad g(t) = \cos^2(t), \quad h(t) = \cos 2t$$

Check if the set $\{f, g, h\}$ is linearly independent.

Solution: We need to find all numbers $c_1, c_2, c_3 \in \mathbb{R}$ such that $c_1 \sin^2(t) + c_2 \cos^2(t) + c_3 \cos(2t) = 0$

plug in $t=0$: $c_1 \underbrace{\sin^2(0)}_{=0} + c_2 \underbrace{\cos^2(0)}_{=1} + c_3 \underbrace{\cos(0)}_{=1} = 0$

so: $\boxed{c_2 + c_3 = 0}$ or: $\boxed{c_2 = -c_3}$

plug in $t = \frac{\pi}{2}$: $c_1 \underbrace{\sin^2\left(\frac{\pi}{2}\right)}_{=1} + c_2 \underbrace{\cos^2\left(\frac{\pi}{2}\right)}_{=0} + c_3 \underbrace{\cos(\pi)}_{=-1} = 0$

so: $\boxed{c_1 - c_3 = 0}$ or: $\boxed{c_1 = c_3}$

Thus the numbers c_1, c_2, c_3 must satisfy:

$$\begin{cases} c_1 = c_3 \\ c_2 = -c_3 \end{cases}$$

Using trigonomentry we can verify that

$$c_3 \sin^2(t) - c_3 \cos^2(t) + c_3 \cos 2t = 0$$

for any value of c_3.

This gives that the set $\{\sin^2(t), \cos^2(t), \cos(2t)\}$ is not linearly independent.

Definition

A *basis* of a vector space V is an ordered set of vectors

$$\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$$

such that

1) $\mathrm{Span}(\mathbf{b}_1, \ldots, \mathbf{b}_n) = V$
2) The set $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is linearly independent.

Example:

In $\mathbb{R}^n$ let $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, ..., $e_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$.

The set $\mathcal{E} = \{e_1, e_2, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$.
This basis is called the standard basis of $\mathbb{R}^n$.

Example:

In $\mathbb{R}^2$ take $b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The set $\mathcal{B} = \{b_1, b_2\}$ is a basis of $\mathbb{R}^2$. (Check: 1) $\mathrm{Span}(b_1, b_2) = \mathbb{R}^2$
2) $\{b_1, b_2\}$ is linearly independent)

Example:

Let $\mathbb{P}_n$ = the vector space of polynomials of degree $\leq n$

$$= \{a_0 + a_1 t + \ldots + a_n t^n \mid a_i \in \mathbb{R}\}$$

The set $\mathcal{E} = \{1, t, t^2, \ldots, t^n\}$ is a basis of $\mathbb{P}_n$.
This basis is called the standard basis of $\mathbb{P}_n$.

Theorem

A set $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis of a vector space V if any only if for each $\mathbf{v} \in V$ the vector equation

$$x_1\mathbf{b}_1 + \ldots + x_n\mathbf{b}_n = \mathbf{v}$$

has a unique solution.

Proof: Since $\mathrm{Span}(b_1, \ldots, b_n) = V$, thus for each $v \in V$ the equation

$$x_1 b_1 + \ldots + x_n b_n = v$$

has a solution.

Since the set $\{b_1, \ldots, b_n\}$ is linearly independent this solution is unique.

Definition

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis of a vector space V. For $\mathbf{v} \in V$ let $c_1, \ldots, c_n$ be the unique numbers such that

$$c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n = \mathbf{v}$$

Then the vector

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$$

is called the *coordinate vector of* $\mathbf{v}$ *relative to the basis* $\mathcal{B}$ and it is denoted by $[\mathbf{v}]_{\mathcal{B}}$.

Example. Let $\mathcal{E} = \{1, t, t^2\}$ be the standard basis of $\mathbb{P}_2$, and let

$$p(t) = 3 + 2t - 4t^2$$

Find the coordinate vector $[p]_{\mathcal{E}}$.

Solution: We have:

$$p(t) = 3 \cdot \underbrace{1}_{\substack{1^{st} \text{ vector} \\ \text{of } \mathcal{E}}} + 2 \cdot \underbrace{t}_{\substack{2^{nd} \text{ vector} \\ \text{of } \mathcal{E}}} + (-4) \cdot \underbrace{t^2}_{\substack{3^{rd} \text{ vector} \\ \text{of } \mathcal{E}}}$$

so:

$$[p]_{\mathcal{E}} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$$

Example. Let $\mathcal{B} = \{1, 1 + t, 1 + t + t^2\}$. One can check that $\mathcal{B}$ is a basis of $\mathbb{P}_2$. Let

$$p(t) = 3 + 2t - 4t^2$$

Find the coordinate vector $[p]_{\mathcal{B}}$.

Solution: We have:

$$p(t) = 1 \cdot \underbrace{1}_{\substack{1^{st} \text{ vector} \\ \text{of } \mathcal{B}}} + 6 \cdot \underbrace{(1+t)}_{\substack{2^{nd} \text{ vector} \\ \text{of } \mathcal{B}}} + (-4) \underbrace{(1+t+t^2)}_{\substack{3^{rd} \text{ vector} \\ \text{of } \mathcal{B}}}$$

so:

$$[p]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 6 \\ -4 \end{bmatrix}$$