

Recall:

Vector equations are equivalent to systems of linear equations:

$$x_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} 2x_1 + 4x_2 = 7 \\ 3x_1 + 2x_2 = 3 \end{cases}$$

vector equation system of linear equations

**Upshot.** A vector equation can have either:

- no solutions
- exactly one solution
- infinitely many solutions

Next:

- When does a vector equation have a solution?
- When does it have exactly one solution?

### Definition

A vector  $\mathbf{w} \in \mathbb{R}^n$  is a *linear combination* of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_p \in \mathbb{R}^n$  if there exists scalars  $c_1, \dots, c_p$  such that

$$\mathbf{w} = c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p$$

**Equivalently:** A vector  $\mathbf{w}$  is a linear combination of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_p$  is the vector equation

$$x_1\mathbf{v}_1 + \dots + x_p\mathbf{v}_p = \mathbf{w}$$

has a solution.

**Example.**

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix}$$

**Example.** Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 9 \\ 3 \\ 6 \end{bmatrix}$$

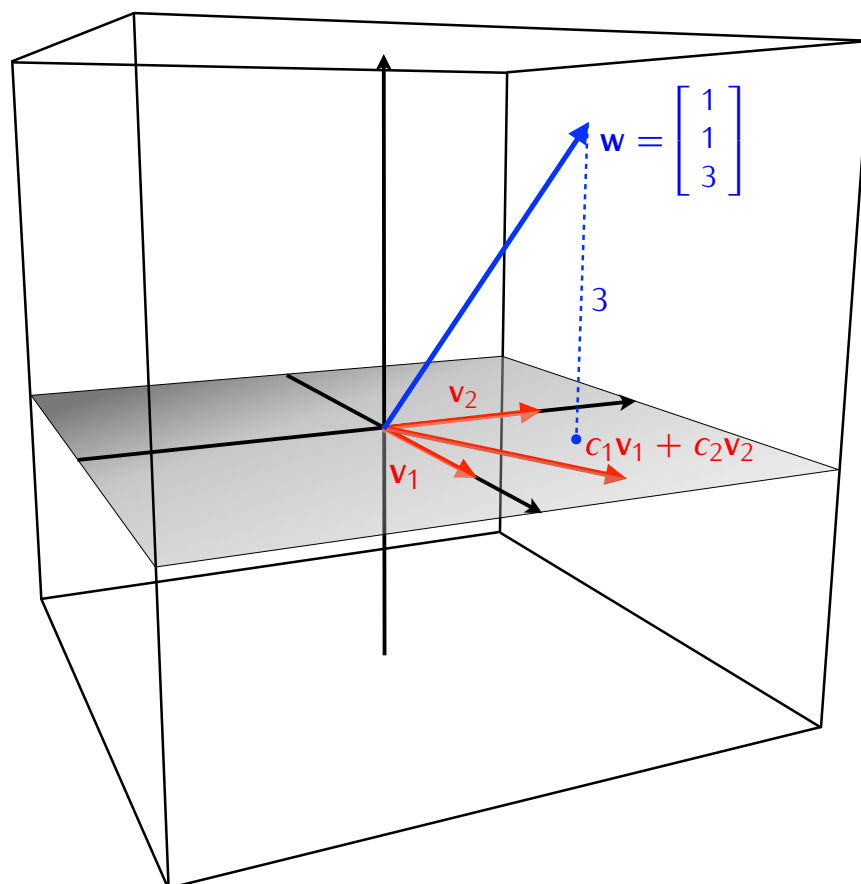
Express  $\mathbf{w}$  as a linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  or show that this is not possible.

**Example.** Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

Express  $\mathbf{w}$  as a linear combination of  $\mathbf{v}_1, \mathbf{v}_2$  or show that this is not possible.

## Geometric picture of the last example



### Definition

If  $\mathbf{v}_1, \dots, \mathbf{v}_p$  are vectors in  $\mathbb{R}^n$  then

$$\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_p) = \left\{ \begin{array}{l} \text{the set of all} \\ \text{linear combinations} \\ c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p \end{array} \right\}$$

**Example.**

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

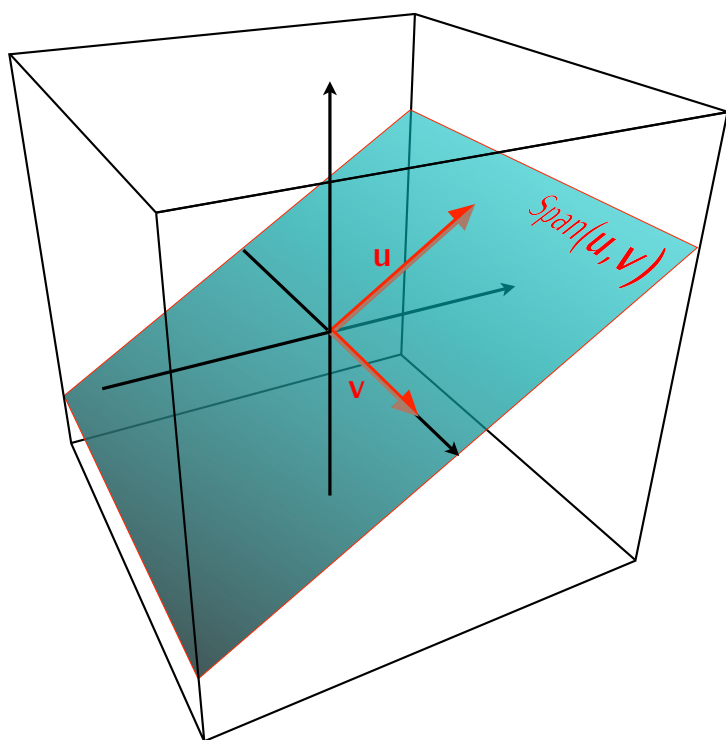
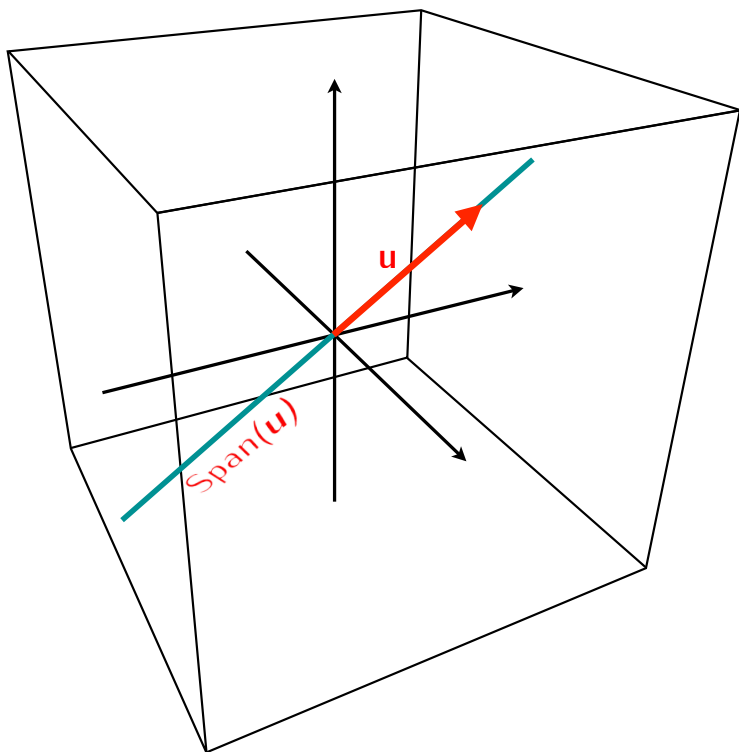
### Proposition

A vector  $\mathbf{w}$  is in  $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_p)$  if and only if the vector equation

$$x_1\mathbf{v}_1 + \dots + x_p\mathbf{v}_p = \mathbf{w}$$

has a solution.

## Geometric interpretation of Span





## Proposition

For arbitrary vectors  $v_1, \dots, v_p \in \mathbb{R}^n$  the zero vector  $\mathbf{0} \in \mathbb{R}^n$  is in  $\text{Span}(v_1, \dots, v_p)$ .

