**Recall:** An  $n \times n$  matrix A defines a linear transformation

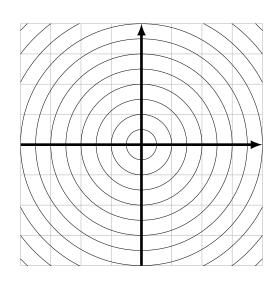
$$T_A \colon \mathbb{R}^n \to \mathbb{R}^n$$

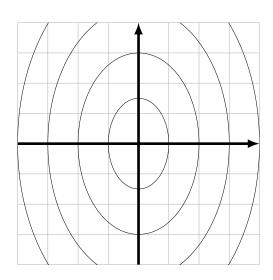
given by  $T_A(\mathbf{v}) = A\mathbf{v}$ .

Next goal: Understand this linear transformation better.

Example.

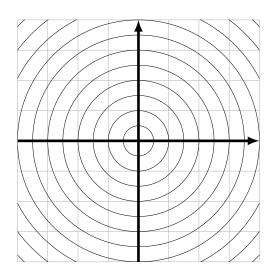
$$A = \left[ \begin{array}{cc} 2 & 0 \\ 0 & 3 \end{array} \right]$$

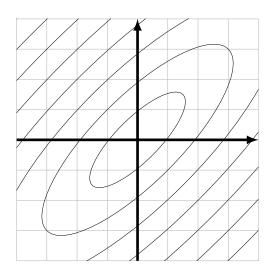




Example.

$$A = \left[ \begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} \right]$$





#### **Definition**

Let A be an  $n \times n$  matrix. If  $\mathbf{v} \in \mathbb{R}^n$  is a non-zero vector and  $\lambda$  is a scalar such that

$$A\mathbf{v} = \lambda \mathbf{v}$$

then we say that

- ullet  $\lambda$  is an eigenvalue of A
- ullet v is an *eigenvector* of A corresponding to  $\lambda$ .

Example.

$$A = \left[ \begin{array}{cc} 2 & 0 \\ 0 & 3 \end{array} \right]$$

Example.

$$A = \left[ \begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} \right]$$

## Computation of eigenvalues

**Recall:**  $I_n = n \times n$  identity matrix:

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

## **Propostiton**

If A be an  $n \times n$  matrix then  $\lambda \in \mathbb{R}$  is an eigenvalue of A if and only if the matrix equation

$$(A - \lambda I_n)\mathbf{x} = \mathbf{0}$$

has a non-trivial solution.

## **Propostiton**

If B is an  $n \times n$  matrix then equation

$$B\mathbf{x} = \mathbf{0}$$

has a non-trivial solution if and only of the matrix  $\boldsymbol{B}$  is not invertible.

# **Propostiton**

If A be an  $n \times n$  matrix then  $\lambda \in \mathbb{R}$  is an eigenvalue of A if and only if

$$\det(A - \lambda I_n) = 0$$

**Example.** Find all eigenvalues of the following matrix:

$$A = \left[ \begin{array}{rrr} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{array} \right]$$

#### **Definition**

If A is an  $n \times n$  matrix then

$$P(\lambda) = \det(A - \lambda I_n)$$

is a polynomial of degree n.  $P(\lambda)$  is called the *characteristic polynomial* of the matrix A.

## **Upshot**

If A is a square matrix then

eigenvalues of 
$$A = \text{roots of } P(\lambda)$$

Example.

$$A = \left[ \begin{array}{rrr} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{array} \right]$$

## Corollary

An  $n \times n$  matrix can have at most n distinct eigenvalues.

# Computation of eigenvectors

# **Proposition**

If  $\lambda$  is an eigenvalue of an  $n \times n$  matrix A then

$$\left\{ \begin{array}{l} \text{eigenvectors of } A \\ \text{corresponding to } \lambda \end{array} \right\} = \left\{ \begin{array}{l} \text{vectors in} \\ \text{Nul}(A - \lambda I_n) \end{array} \right\}$$

## Corollary/Definition

If A is an  $n \times n$  matrix and  $\lambda$  is an eigenvalue of A then the set of all eigenvectors corresponding to  $\lambda$  is a subspace of  $\mathbb{R}^n$ .

This subspace is called the *eigenspace* of A corresponding to  $\lambda$ .

## **Proposition**

If  $\lambda$  is an eigenvalue of an  $n \times n$  matrix A then

$$\begin{cases} \text{eigenspace of } A \\ \text{corresponding to } \lambda \end{cases} = \text{Nul}(A - \lambda I_n)$$

**Example.** Consider the following matrix:

$$A = \left[ \begin{array}{ccc} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{array} \right]$$

Recall that eigenvalues of A are  $\lambda_1=1$  and  $\lambda_2=5$ . Compute bases of eigenspaces of A corresponding to these eigenvalues.

Solution.

$$\underline{\lambda_1 = 1}$$

 $\underline{\lambda_2 = 5}$