> **Definition**
>
> A vector $\mathbf{w} \in \mathbb{R}^n$ is a *linear combination* of vectors $\mathbf{v}_1, \ldots \mathbf{v}_p \in \mathbb{R}^n$ if there exists scalars $c_1, \ldots, c_p$ such that
>
> $$\mathbf{w} = c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p$$

Equivalently: A vector $\mathbf{w}$ is a linear combination of vectors $\mathbf{v}_1, \ldots \mathbf{v}_p$ is the vector equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{w}$$

has a solution.

Example.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix}$$

Example. Let

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{v_2} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \qquad \mathbf{v_3} = \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 9 \\ 3 \\ 6 \end{bmatrix}$$
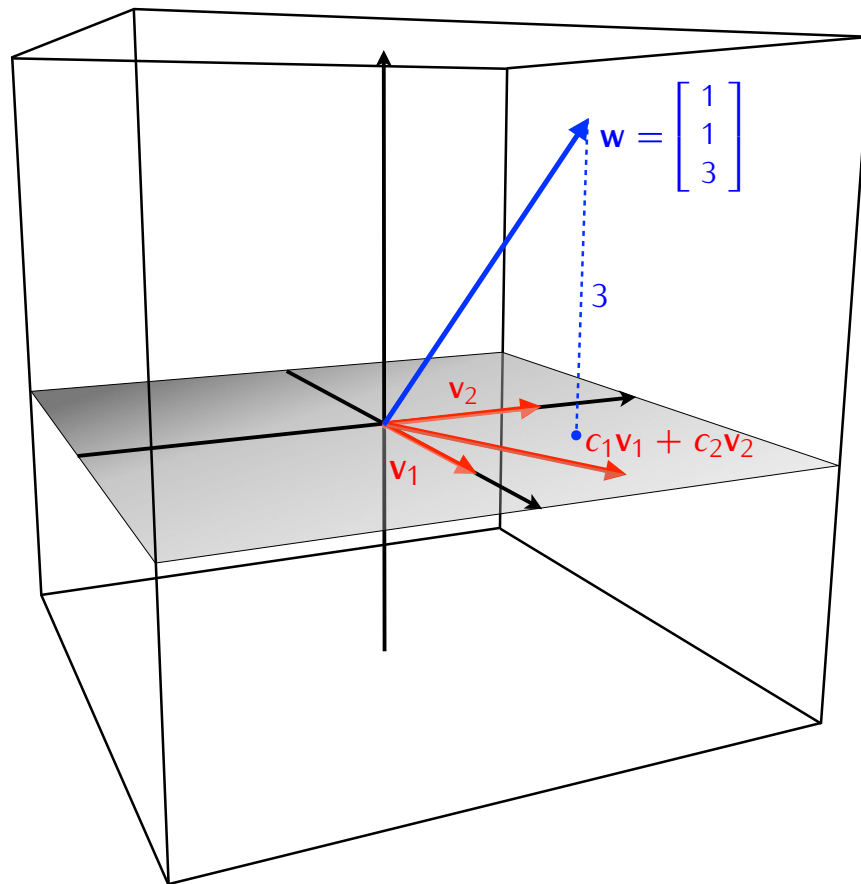
Express $\mathbf{w}$ as a linear combination of $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ or show that this is not possible.

Example. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

Express $\mathbf{w}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$ or show that this is not possible.

Geometric picture of the last example

$\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

3

$\mathbf{v}_2$

$\mathbf{v}_1$

$c_1\mathbf{v}_1 + c_2\mathbf{v}_2$

Definition

If $v_1, \ldots, v_p$ are vectors in $\mathbb{R}^n$ then

$$\text{Span}(v_1, \ldots, v_p) = \left\{ \begin{array}{c} \text{the set of all} \\ \text{linear combinations} \\ c_1 v_1 + \ldots + c_p v_p \end{array} \right\}$$

Example.

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

> **Proposition**
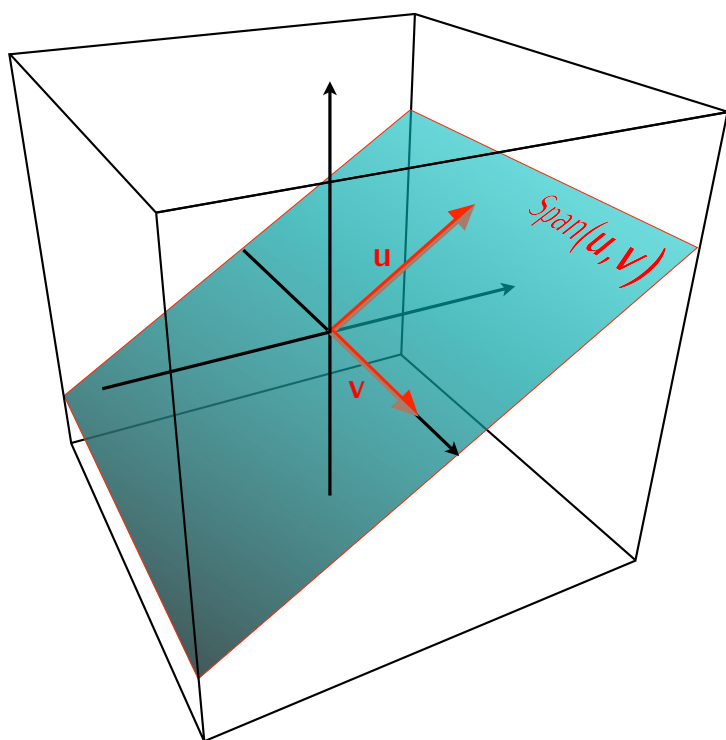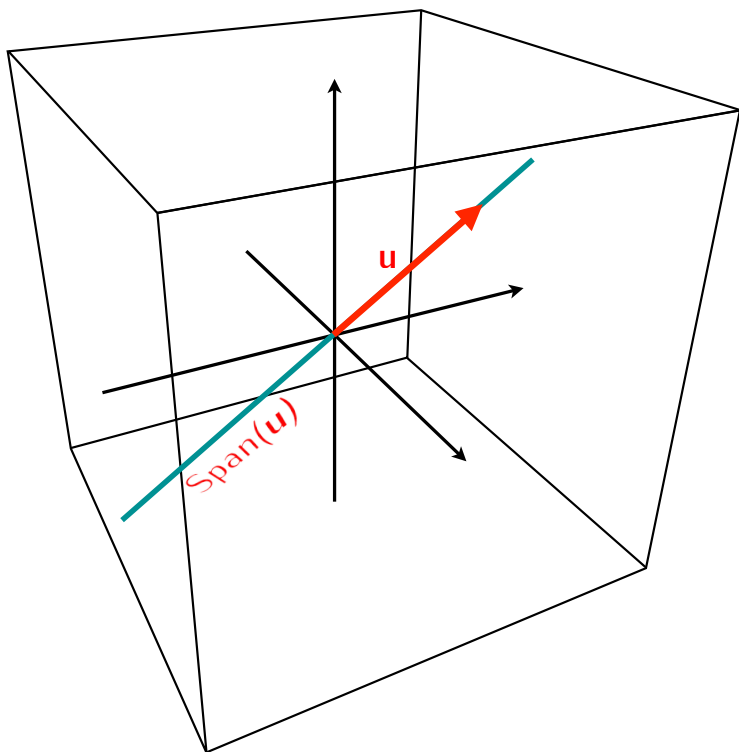>
> A vector $\mathbf{w}$ is in $\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ if and only if the vector equation
>
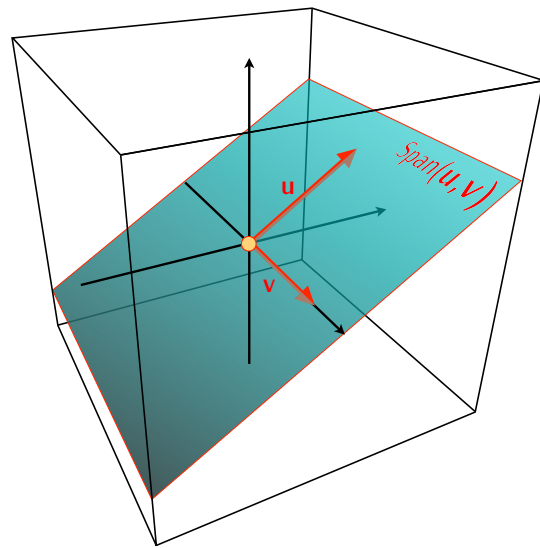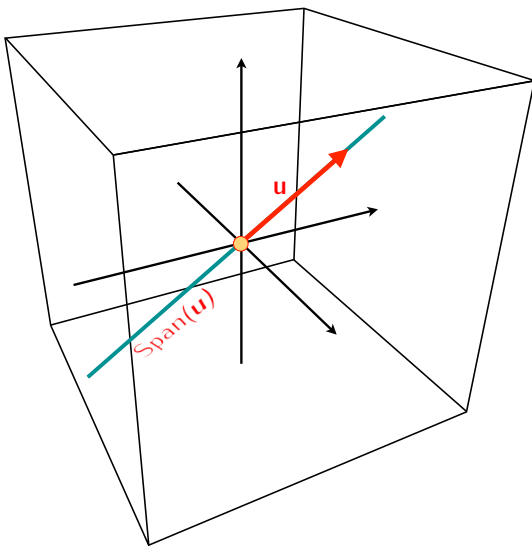> $$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{w}$$
>
> has a solution.

Geometric interpretation of Span

Proposition

For arbitrary vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$ the zero vector $\mathbf{0} \in \mathbb{R}^n$ is in $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p)$.

Definition

A *homogenous vector equation* is a vector equation of the form

$$x_1 \mathbf{v}_1 + \ldots + x_p \mathbf{v}_p = \mathbf{0}$$

(i.e. with the zero vector as the vector of constants).

Definition

Let $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is *linearly independent* if the homogenous equation

$$x_1 \mathbf{v}_1 + \ldots + x_p \mathbf{v}_p = \mathbf{0}$$

has only one, trivial solution $x_1 = 0, \ldots, x_p = 0$. Otherwise the set is *linearly dependent*.