Note. If $T\colon V \to W$ is a linear transformation then for any vector $\mathbf{b} \in W$ we can consider the equation

$$T(\mathbf{x}) = \mathbf{b}$$

Definition

If $T \colon V \to W$ is a linear transformation then:

1) The *kernel* of T is the set

$$\mathrm{Ker}(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}$$

2) The *image* of T is the set

$$\mathrm{Im}(T) = \{\mathbf{w} \in W \mid \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V\}$$

Proposition

If $T : V \to W$ is a linear transformation then:

1) $\mathrm{Ker}(T)$ is a subspace of V

2) $\mathrm{Im}(T)$ is a subspace of W

Theorem

If $T\colon V \to W$ is a linear transformation and v_0 is a solution of the equation

$$T(\mathbf{x}) = \mathbf{b}$$

then all other solutions of this equation are vectors of the form

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{n}$$

where $\mathbf{n} \in \mathrm{Ker}(T)$.

Example.

$$\begin{aligned} D\colon C^\infty(\mathbb{R}) &\longrightarrow C^\infty(\mathbb{R}) \\ f &\longmapsto f' \end{aligned}$$

Proposition

If $T\colon V \to W$ is a linear transformation then

1) T is onto if and only if $\mathrm{Im}(T) = W$

2) T is one-to-one if and only if $\mathrm{Ker}(T) = \{\mathbf{0}\}$.

Recall:

• A vector space is a set V equipped with operations of addition and multiplication by scalars. These operations must satisfy some properties.

• Some examples of vector spaces:

1) $\mathbb{R}^n$ = the vector space of column vectors.

2) $\mathcal{F}(\mathbb{R})$ = the vector space of all functions $f\colon \mathbb{R} \to \mathbb{R}$.

3) $C(\mathbb{R})$ = the vector space of all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$.

4) $C^\infty(\mathbb{R})$ = the vector space of all smooth functions $f\colon \mathbb{R} \to \mathbb{R}$.

5) $M_{m,n}(\mathbb{R})$ = the vector space of all $m \times n$ matrices.

6) $\mathbb{P}$ = the vector space of all polynomials.

7) $\mathbb{P}_n$ = the vector space of polynomials of degree $\leq n$.