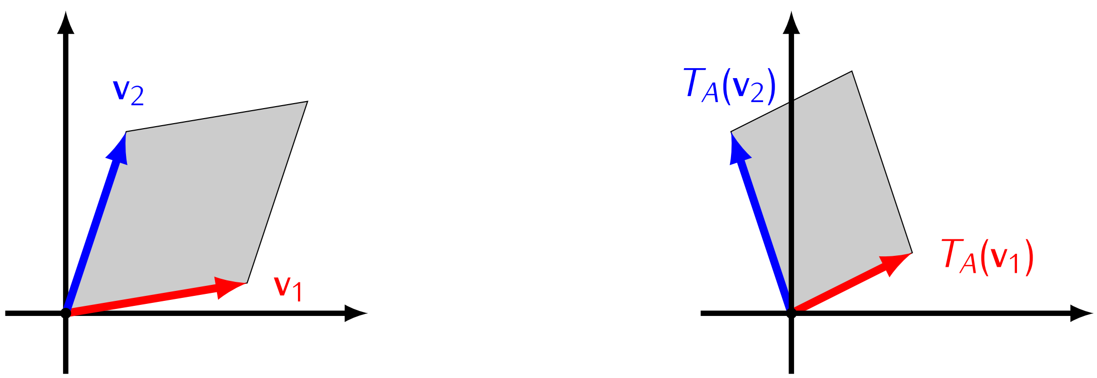
Recall: If A is a 2×2 matrix then it defines a linear transformation

$$T_A\colon \mathbb{R}^2 \to \mathbb{R}^2 \qquad T_A(\mathbf{v}) = A\mathbf{v}$$

Note. T_A maps parallelograms to parallelograms:



Theorem

If A is a 2×2 matrix and $\mathbf{v}_1, \mathbf{v}_1 \in \mathbb{R}^2$ then

$$\text{area}(T_A(\mathbf{v}_1), T_A(\mathbf{v}_2)) = |\det A| \cdot \text{area}(\mathbf{v}_1, \mathbf{v}_2)$$

Proof:

$$\text{area}(T_A(v_1), T_A(v_2)) = \text{area}(Av_1, Av_2)$$
$$= |\det[Av_1 \ Av_2]|$$

Recall: $[Av_1 \ Av_2] = A \cdot [v_1 \ v_2]$

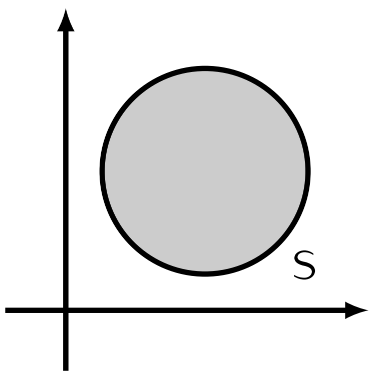
So: $\text{area}(T_A(v_1), T_A(v_2)) = |\det(A \cdot [v_1 \ v_2])|$
$$= |\det(A) \cdot \det[v_1 \ v_2]|$$
$$= |\det(A)| \cdot \text{area}(v_1, v_2)$$

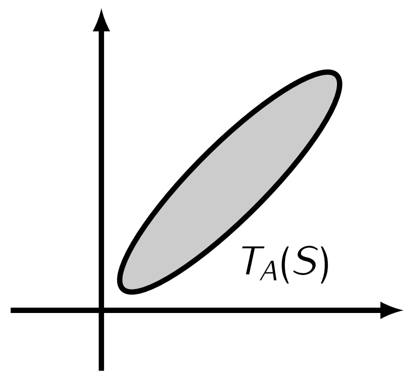
Generalization:

Theorem

If A is a 2×2 matrix then for any region S of $\mathbb{R}^2$ we have:

$$\text{area}(T_A(S)) = |\det A| \cdot \text{area}(S)$$





Idea of the proof.

The area of S can be approximated by the sum of small squares covering S.

$$\text{area}(S) \approx \sum \text{area}(\text{small square})$$

$$\begin{aligned}\text{area}(T_A(S)) &\approx \sum \text{area}(T_A(\text{small square}))\\ &= \sum |\det(A)| \cdot \text{area}(\text{small square})\\ &= |\det(A)| \cdot \sum \text{area}(\text{small square})\\ &\approx |\det(A)| \cdot \text{area}(S)\end{aligned}$$