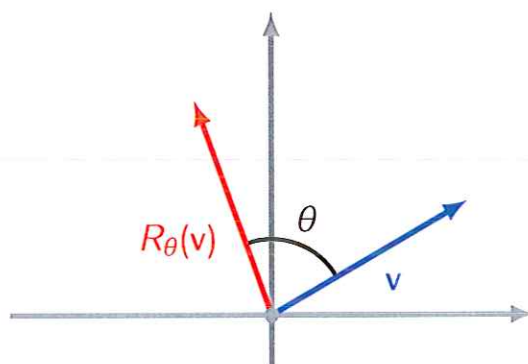


**Problem:** How to recognize if a function  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a matrix transformation?

**Example.** Rotation by an angle  $\theta$ :



$$R_\theta: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

Question:

1) Is  $R_\theta$  a matrix transformation?  
 That is, is there a matrix  $A$  such that  

$$R_\theta(v) = Av$$
  
 for all  $v \in \mathbb{R}^2$ ?

2) If so, what is this matrix  $A$ ?

### Definition

A function  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a *linear transformation* if it satisfies the following conditions:

- 1)  $T(u + v) = T(u) + T(v)$  for all  $u, v \in \mathbb{R}^n$
- 2)  $T(cv) = cT(v)$  for any  $v \in \mathbb{R}^n$  and scalar  $c$ .

### Proposition

Every matrix transformation is a linear transformation.

Proof: Let  $A$  be an  $m \times n$  matrix:

$$\begin{array}{ccc} T_A : \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \\ v & \longmapsto & Av \end{array}$$

We have:

$$1) \quad T_A(u+v) = A(u+v) = Au + Av = T_A(u) + T_A(v)$$

$$2) \quad T_A(cu) = A(cu) = c(Au) = cT_A(u)$$

### Theorem

Every linear transformation  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a matrix transformation:

$$T = T_A$$

for some matrix  $A$ .

Proof:

Let

"standard basis vectors"

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n$$

Take  $A = [T(e_1) \ T(e_2) \ \dots \ T(e_n)]$

We will show that  $T(u) = A \cdot u$  for any  $u \in \mathbb{R}^n$

Indeed:

If  $u = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$  then  $u = c_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \dots + c_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

so:  $u = c_1 e_1 + c_2 e_2 + \dots + c_n e_n$

This gives:

$$T(u) = T(c_1 e_1 + c_2 e_2 + \dots + c_n e_n) = T(c_1 e_1) + T(c_2 e_2) + \dots + T(c_n e_n)$$

$$= c_1 T(e_1) + c_2 T(e_2) + \dots + c_n T(e_n)$$

$$= [T(e_1) \ T(e_2) \ \dots \ T(e_n)] \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = A \cdot u$$

### Corollary

If  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a linear transformation then  $T = T_A$  where  $A$  is the matrix given by

$$A = [ T(e_1) \quad T(e_2) \quad \dots \quad T(e_n) ]$$

This matrix is called the *standard matrix* of  $T$ .

**Example.** Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be the function given by

$$T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 + x_2 \\ 0 \\ 2x_1 \end{bmatrix}$$

Check if  $T$  is a linear transformation. If it is, find its standard matrix.

Solution:

1) Check if  $T(u+v) = T(u) + T(v)$

$$\text{Let } u = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, v = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\left. \begin{aligned} T(u) &= T \left( \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right) = \begin{bmatrix} a_1 + a_2 \\ 0 \\ 2a_1 \end{bmatrix} \\ T(v) &= T \left( \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) = \begin{bmatrix} b_1 + b_2 \\ 0 \\ 2b_1 \end{bmatrix} \end{aligned} \right\} T(u) + T(v) = \begin{bmatrix} a_1 + a_2 + b_1 + b_2 \\ 0 \\ 2a_1 + 2b_1 \end{bmatrix}$$

$$T(u+v) = T \left( \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix} \right) = \begin{bmatrix} (a_1 + b_1) + (a_2 + b_2) \\ 0 \\ 2(a_1 + b_1) \end{bmatrix}$$

This gives:  $T(u) + T(v) = T(u+v)$

2) Similarly we can check that  $T(cu) = cT(u)$ .

This shows that  $T$  is a linear transformation

The standard matrix of T:

$$A = [T(e_1) \ T(e_2)]$$

$$\text{where } e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We get:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 2 & 0 \end{bmatrix}$$

Check:

$$A \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} x_1 + x_2 \\ 0 \\ 2x_1 \end{bmatrix}}_{\substack{= \\ T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)}}$$



Example. Let  $S: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be the function given by

$$S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 + x_2 \\ x_2 \\ 3x_1 \end{bmatrix}$$

Check if  $S$  is a linear transformation. If it is, find its standard matrix.

Solution

1) Check if  $S(u) + S(v) = S(u+v)$

$$u = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad v = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$S(u) = S\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} 1 + a_2 \\ a_2 \\ 3a_1 \end{bmatrix}$$

$$S(v) = S\left(\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\right) = \begin{bmatrix} 1 + b_2 \\ b_2 \\ 3b_1 \end{bmatrix}$$

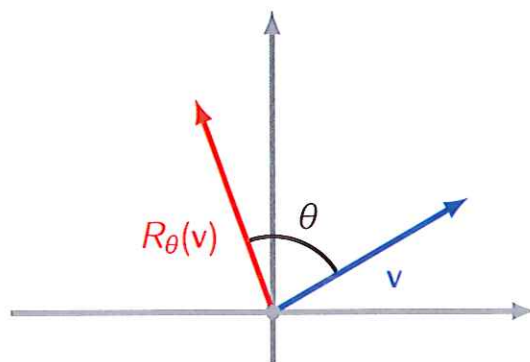
$$S(u) + S(v) = \begin{bmatrix} \textcircled{2} + a_2 + b_2 \\ a_2 + b_2 \\ 3a_1 + 3b_1 \end{bmatrix}$$

$$S(u+v) = S\left(\begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}\right) = \begin{bmatrix} \textcircled{1} + (a_2 + b_2) \\ a_2 + b_2 \\ 3(a_1 + b_1) \end{bmatrix} \quad \leftarrow \text{not equal}$$

We get:  $S(u) + S(v) \neq S(u+v)$

This shows that  $S$  is not a linear transformation and thus it can't be represented by a matrix.

## Back to rotations:



$$R_\theta: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

- One can check that  $R_\theta$  is a linear transformation
- The standard matrix of  $R_\theta$ :

$$A = [R_\theta(e_1) \quad R_\theta(e_2)]$$

We get:

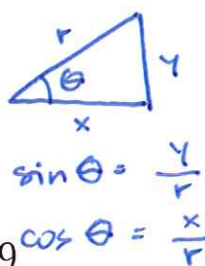
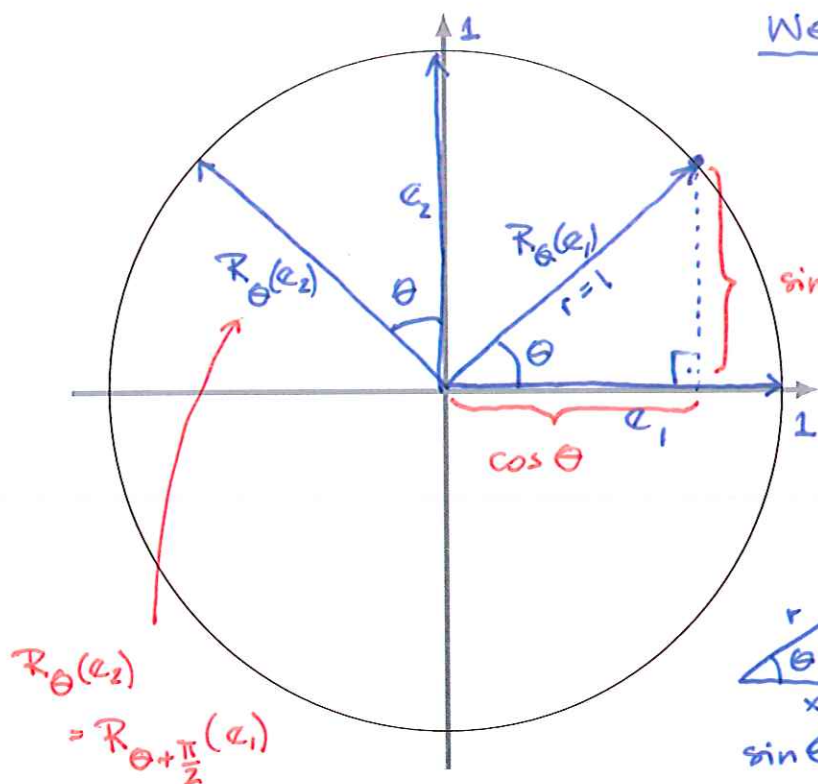
$$R_\theta(e_1) = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$$

$$R_\theta(e_2) = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$$

We get:

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

↑  
the standard matrix  
of the rotation by  
an angle  $\theta$



### Proposition

Let  $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$  be the standard basis of  $\mathbb{R}^n$ . For any vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in \mathbb{R}^m$  there exists one and only one linear transformation

$$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

such that

$$T(\mathbf{e}_1) = \mathbf{v}_1 \quad T(\mathbf{e}_2) = \mathbf{v}_2, \quad \dots, \quad T(\mathbf{e}_n) = \mathbf{v}_n$$

The standard matrix of this linear transformation is given by

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}$$