Definition

Let V, W be vector spaces A *linear transformation* is a function

$$T\colon V \to W$$

which satisfies the following conditions:

1) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$

2) $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $\mathbf{v} \in V$ and any scalar c.

Proposition

If $T\colon V \to W$ is a linear transformation then $T(\mathbf{0}) = \mathbf{0}$.

Note. If $T: V \to W$ is a linear transformation then for any vector $\mathbf{b} \in W$ we can consider the equation

$$T(\mathbf{x}) = \mathbf{b}$$

Definition

If $T\colon V \to W$ is a linear transformation then:

1) The *kernel* of T is the set

$$\mathrm{Ker}(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}$$

2) The *image* of T is the set

$$\mathrm{Im}(T) = \{\mathbf{w} \in W \mid \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V\}$$

Proposition

If $T\colon V \to W$ is a linear transformation then:

1) $\mathrm{Ker}(T)$ is a subspace of V

2) $\mathrm{Im}(T)$ is a subspace of W

Theorem

If $T\colon V \to W$ is a linear transformation and v_0 is a solution of the equation

$$T(\mathbf{x}) = \mathbf{b}$$

then all other solutions of this equation are vectors of the form

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{n}$$

where $\mathbf{n} \in \mathrm{Ker}(T)$.

Example.

$$\begin{aligned} D\colon C^\infty(\mathbb{R}) &\longrightarrow C^\infty(\mathbb{R}) \\ f &\longmapsto f' \end{aligned}$$

Proposition

If $T\colon V \to W$ is a linear transformation then

1) T is onto if and only if $\mathrm{Im}(T) = W$

2) T is one-to-one if and only if $\mathrm{Ker}(T) = \{\mathbf{0}\}$.