Recall:

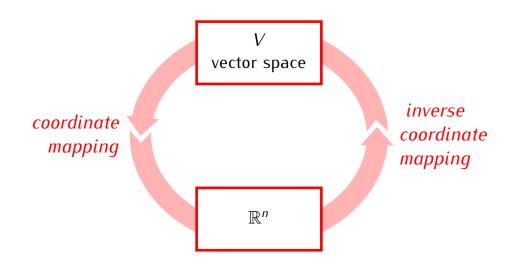
- ullet A vector space is a set V equipped with operations of addition and multiplication by scalars. These operations must satisfy some properties.
- Some examples of vector spaces:
 - 1) \mathbb{R}^n = the vector space of column vectors.
- 2) $\mathcal{F}(\mathbb{R}) = \text{the vector space of all functions } f: \mathbb{R} \to \mathbb{R}.$
- 3) $C(\mathbb{R})$ = the vector space of all continuous functions $f: \mathbb{R} \to \mathbb{R}$.
- 4) $C^{\infty}(\mathbb{R})$ = the vector space of all smooth functions $f: \mathbb{R} \to \mathbb{R}$.
- **5)** $\mathcal{M}_{m,n}(\mathbb{R}) = \text{the vector space of all } m \times n \text{ matrices.}$
- **6)** \mathbb{P} = the vector space of all polynomials.
- 7) \mathbb{P}_n = the vector space of polynomials of degree $\leq n$.
- ullet If V,W are vector spaces then a linear transformation is a function $T\colon V\to W$ such that
 - 1) T(u + v) = T(u) + T(v)
- 2) $T(c\mathbf{v}) = cT(\mathbf{v})$
- ullet Many problems involving \mathbb{R}^n can be easily solved using row reduction, matrix multiplication etc.
- The same types of problems involving other vector spaces can be difficult to solve.

Next goal:

If V is a $\mathit{finite\ dimensional\ vector\ space\ then\ we\ can\ construct\ a\ }\mathit{coordinate\ }\mathit{mapping\ }$

$$V \to \mathbb{R}^n$$

which lets us turn computations in V into computations in \mathbb{R}^n .



Motivation: How to assign coordinates to vectors



Definition

If V is a vector space then vector $\mathbf{w} \in V$ is a *linear combination* of vectors $\mathbf{v}_1, \dots \mathbf{v}_p \in V$ if there exist scalars c_1, \dots, c_p such that

$$\mathbf{w} = c_1 \mathbf{v}_1 + \ldots + c_p \mathbf{v}_p$$

Definition

If V is a vector space and $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are vectors in V then

$$Span(v_1, ..., v_p) = \begin{cases} the set of all \\ linear combinations \\ c_1v_1 + ... + c_pv_p \end{cases}$$

Definition

If V is a vector space and $\mathbf{v}_1,\ldots,\mathbf{v}_p$ are vectors in V such that

$$V = \operatorname{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p)$$

the the set $\{v_1, \ldots, v_p\}$ is called the *spanning set* of V.

Example.

Definition

If V is a vector space and $\mathbf{v}_1, \dots, \mathbf{v}_p \in V$ then the set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is *linearly independent* if the homogenous equation

$$x_1\mathbf{v}_1+\ldots+x_p\mathbf{v}_p=\mathbf{0}$$

has only one, trivial solution $x_1 = 0, ..., x_p = 0$. Otherwise the set is *linearly dependent*.

Theorem

Let V be a vector space, and let $\mathbf{v}_1, \dots, \mathbf{v}_p \in V$. If the set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly independent then the equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{w}$$

has exactly one solution for any vector $\mathbf{w} \in \mathsf{Span}(\mathbf{v}_1, \dots, \mathbf{v}_p)$.

Example.

Recall: $\mathcal{F}(\mathbb{R})$ = the vector space of all functions $f: \mathbb{R} \to \mathbb{R}$. Let $f, g, h \in \mathcal{F}(\mathbb{R})$ be the following functions:

$$f(t) = \sin(t), \quad g(t) = \cos(t), \quad h(t) = \cos^2(t)$$

Check if the set $\{f, g, h\}$ is linearly independent.

Example.

Let f, g, $h \in \mathcal{F}(\mathbb{R})$ be the following functions:

$$f(t) = \sin^2(t), \quad g(t) = \cos^2(t), \quad h(t) = \cos 2t$$

Check if the set $\{f, g, h\}$ is linearly independent.

Definition

A basis of a vector space V is an ordered set of vectors

$$\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$$

such that

- 1) $\operatorname{Span}(\mathbf{b}_1, \ldots, \mathbf{b}_n) = V$
- 2) The set $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is linearly independent.

Theorem

A set $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$ is a basis of a vector space V if any only if for each $\mathbf{v} \in V$ the vector equation

$$x_1\mathbf{b}_1 + \ldots + x_n\mathbf{b}_n = \mathbf{v}$$

has a unique solution.

Definition

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis of a vector space V. For $\mathbf{v} \in V$ let c_1, \dots, c_n be the unique numbers such that

$$c_1\mathbf{b}_1+\ldots+c_n\mathbf{b}_n=\mathbf{v}$$

Then the vector

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$$

is called the *coordinate vector of* v *relative to the basis* $\mathcal B$ and it is denoted by $[v]_{\mathcal B}$.

Example. Let $\mathcal{E} = \{1, t, t^2\}$ be the standard basis of \mathbb{P}_2 , and let

$$p(t) = 3 + 2t - 4t^2$$

Find the coordinate vector $[p]_{\mathcal{E}}$.

Example. Let $\mathcal{B} = \{1, 1+t, 1+t+t^2\}$. One can check that \mathcal{B} is a basis of \mathbb{P}_2 . Let

$$p(t) = 3 + 2t - 4t^2$$

Find the coordinate vector $[p]_{\mathcal{B}}$.