

**Definition**

A set of vectors  $\{v_1, \dots, v_k\}$  in  $\mathbb{R}^n$  is an *orthogonal set* if each pair each pair of vectors in this set is orthogonal, i.e.

$$v_i \cdot v_j = 0$$

for all  $i \neq j$ .

**Example.**

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$  is an orthogonal set in  $\mathbb{R}^3$ .

**Example.**

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix} \right\}$  is another orthogonal set in  $\mathbb{R}^3$ .

### Proposition

If  $\{v_1, \dots, v_k\}$  is an orthogonal set of non-zero vectors in  $\mathbb{R}^n$  then this set is linearly independent.

**Recall:** Any linearly independent set of  $n$  vectors in  $\mathbb{R}^n$  is a basis of  $\mathbb{R}^n$ .

### Corollary

If  $\{v_1, \dots, v_n\}$  is an orthogonal set of  $n$  non-zero vectors in  $\mathbb{R}^n$  then this set is a basis of  $\mathbb{R}^n$ .

### Definition

If  $V$  is a subspace of  $\mathbb{R}^n$  then we say that a set  $\{v_1, \dots, v_k\}$  is an *orthogonal basis* of  $V$  if

- 1)  $\{v_1, \dots, v_k\}$  is a basis of  $V$  and
- 2)  $\{v_1, \dots, v_k\}$  is an orthogonal set.

**Recall.** If  $\mathcal{B} = \{v_1, \dots, v_k\}$  is a basis of a vector space  $V$  and  $w \in V$  then the coordinate vector of  $w$  relative to  $\mathcal{B}$  is the vector

$$[w]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

where  $c_1, \dots, c_k$  are scalars such that  $c_1 v_1 + \dots + c_k v_k = w$ .

### Proposition

If  $\mathcal{B} = \{v_1, \dots, v_k\}$  is an orthogonal basis of  $V$  and  $w \in V$  then

$$[w]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

where  $c_i = \frac{w \cdot v_i}{v_i \cdot v_i} = \frac{w \cdot v_i}{\|v_i\|^2}$

**Example.** Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix} \right\}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

The set  $\mathcal{B}$  is an orthogonal basis of  $\mathbb{R}^3$ . Compute  $[\mathbf{w}]_{\mathcal{B}}$ .

### Theorem (Gram-Schmidt Process)

Let  $\{v_1, \dots, v_k\}$  be a basis of  $V$ . Define vectors  $\{w_1, \dots, w_k\}$  as follows:

$$w_1 = v_1$$

$$w_2 = v_2 - \left( \frac{w_1 \cdot v_2}{w_1 \cdot w_1} \right) w_1$$

$$w_3 = v_3 - \left( \frac{w_1 \cdot v_3}{w_1 \cdot w_1} \right) w_1 - \left( \frac{w_2 \cdot v_3}{w_2 \cdot w_2} \right) w_2$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$w_k = v_k - \left( \frac{w_1 \cdot v_k}{w_1 \cdot w_1} \right) w_1 - \left( \frac{w_2 \cdot v_k}{w_2 \cdot w_2} \right) w_2 - \dots - \left( \frac{w_{k-1} \cdot v_k}{w_{k-1} \cdot w_{k-1}} \right) w_{k-1}$$

Then the set  $\{w_1, \dots, w_k\}$  is an orthogonal basis of  $V$ .

**Example.** In  $\mathbb{R}^4$  take

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 7 \\ 4 \\ 3 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 7 \\ 7 \\ 8 \end{bmatrix}$$

The set  $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is a basis of some subspace  $V \subseteq \mathbb{R}^4$ . Find an orthogonal basis of  $V$ .

### Definition

An orthogonal basis  $\mathcal{B} = \{\mathbf{w}_1, \dots, \mathbf{w}_k\}$  of  $V$  is called an *orthonormal basis* if  $\|\mathbf{w}_i\| = 1$  for  $i = 1, \dots, k$ .

### Proposition

If  $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is an orthonormal basis of  $V$  and  $\mathbf{w} \in V$  then

$$[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

where  $c_i = \mathbf{w} \cdot \mathbf{v}_i$ .

**Note.** If  $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is an orthogonal basis of  $V$  then

$$\mathcal{C} = \left\{ \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \dots, \frac{\mathbf{v}_k}{\|\mathbf{v}_k\|} \right\}$$

is an orthonormal basis of  $V$ .