Recall:

1) Let A be an $n \times n$ matrix. If $\mathbf{v} \in \mathbb{R}^n$ is a non-zero vector and λ is a scalar such that

$$A\mathbf{v} = \lambda\mathbf{v}$$

then

- λ is an eigenvalue of A
- $\mathbf{v}$ is an eigenvector of A corresponding to λ.

2) The characteristic polynomial of an $n \times n$ matrix A is the polynomial given by the formula

$$P(\lambda) = \det(A - \lambda I_n)$$

where I_n is the $n \times n$ identity matrix.

3) If A is a square matrix then

$$\text{eigenvalues of } A = \text{roots of } P(\lambda)$$

4) If λ is an eigenvalue of an $n \times n$ matrix A then

$$\left\{\begin{matrix}\text{eigenvectors of } A \\ \text{corresponding to } \lambda\end{matrix}\right\} = \left\{\begin{matrix}\text{vectors in} \\ \text{Nul}(A - \lambda I_n)\end{matrix}\right\}$$

Motivating example: Fibonacci numbers

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233,\ 377,\ \ldots$$

$F_1\ F_2\ F_3\ F_4\ \ldots$

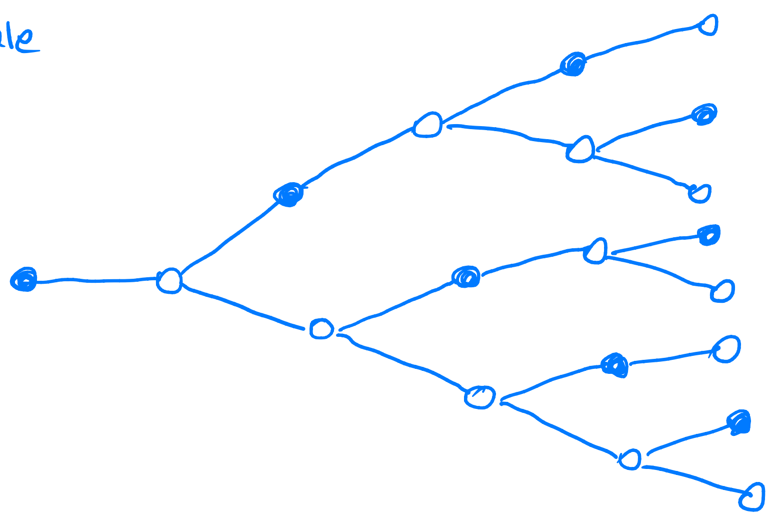
Recursive formula:

$$\begin{cases} F_1 = 1, \quad F_2 = 1 \\ F_{n+1} = F_n + F_{n-1} \quad \text{for } n \geq 2 \end{cases}$$

Fibonacci numbers and the honeybee family tree

● male

○ female



1 1 2 3 5 8 ...

Problem. Find a formula for the n-th Fibonacci number F_n.

Note:

$$\begin{bmatrix}0&1\\1&1\end{bmatrix}\cdot\begin{bmatrix}F_{n-1}\\F_n\end{bmatrix}=\begin{bmatrix}0\cdot F_{n-1}+1\cdot F_n\\1\cdot F_{n-1}+1\cdot F_n\end{bmatrix}=\begin{bmatrix}F_n\\F_{n+1}\end{bmatrix}$$

This gives:

$$\underbrace{\begin{bmatrix}0&1\\1&1\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}\begin{array}{l}F_1\\F_2\end{array}=\begin{bmatrix}1\\2\end{bmatrix}\begin{array}{l}F_2\\F_3\end{array}}_{\begin{bmatrix}0&1\\1&1\end{bmatrix}^1\cdot\begin{bmatrix}1\\1\end{bmatrix}}\quad\underbrace{\begin{bmatrix}0&1\\1&1\end{bmatrix}\cdot\begin{bmatrix}1\\2\end{bmatrix}\begin{array}{l}F_2\\F_3\end{array}=\begin{bmatrix}2\\3\end{bmatrix}\begin{array}{l}F_3\\F_4\end{array}}_{\begin{bmatrix}0&1\\1&1\end{bmatrix}^2\cdot\begin{bmatrix}1\\1\end{bmatrix}}\quad\underbrace{\begin{bmatrix}0&1\\1&1\end{bmatrix}\cdot\begin{bmatrix}2\\3\end{bmatrix}\begin{array}{l}F_3\\F_4\end{array}=\begin{bmatrix}3\\5\end{bmatrix}\begin{array}{l}F_4\\F_5\end{array}}_{\begin{bmatrix}0&1\\1&1\end{bmatrix}^3\cdot\begin{bmatrix}1\\1\end{bmatrix}}$$

In general:

$$\begin{bmatrix}0&1\\1&1\end{bmatrix}^{n-1}\cdot\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}F_n\\F_{n+1}\end{bmatrix}$$

Problem:

$$\begin{bmatrix}0&1\\1&1\end{bmatrix}^{n-1}=\,?$$

General Problem. If A is a square matrix how to compute A^k quickly?

Easy case:

Definition

A square matrix D is *diagonal matrix* if all its entries outside the main diagonal are zeros:

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

e.g.:

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Proposition

If D is a diagonal matrix as above then

$$D^k = \begin{bmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n^k \end{bmatrix}$$

e.g.:

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad D^2 = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \quad D^3 = \begin{bmatrix} 8 & 0 \\ 0 & 27 \end{bmatrix} \quad \dots$$

Definition

A square matrix A is a *diagonalizable* if A is of the form

$$A = PDP^{-1}$$

where D is a diagonal matrix and P is an invertible matrix.

Example.

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ is a diagonalizable matrix:

$$A = \underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix}}_{P} \cdot \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} \cdot \underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix}^{-1}}_{P^{-1}}$$

Proposition

If A is a diagonalizable matrix, $A = PDP^{-1}$, then

$$A^k = PD^kP^{-1}$$

Proof:

$$A^k = \underbrace{A \cdot A \cdot A \cdot \ldots \cdot A}_{k \text{ times}}$$

$$= (P \cdot D \cdot \cancel{P^{-1}}) \cdot (\cancel{P} \cdot D \cdot \cancel{P^{-1}}) \cdot (\cancel{P} \cdot D \cdot \cancel{P^{-1}}) \cdot \ldots \cdot (\cancel{P} \cdot D \cdot P^{-1})$$

$$= P \cdot \underbrace{D \cdot D \cdot D \cdot \ldots \cdot D}_{k \text{ times}} \cdot P^{-1} = P \cdot D^k \cdot P^{-1}$$

Example.

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Compute A^{10}.

Diagonalization Theorem

1) An $n \times n$ matrix A is a diagonalizable if and only if it has n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

2) In such case $A = PDP^{-1}$ where :

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

$\lambda_1 =$ eigenvalue corresponding to $\mathbf{v}_1$
$\lambda_2 =$ eigenvalue corresponding to $\mathbf{v}_2$
$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$
$\lambda_n =$ eigenvalue corresponding to $\mathbf{v}_n$

Example. Diagonalize the following matrix if possible:

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Note. Not every matrix is diagonalizable.

Example. Check if the following matrix is diagonalizable:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

Proposition

If A is an $n \times n$ matrix with n distinct eigenvalues then A is diagonalizable.

Back to Fibonacci numbers:

$$\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{n-1} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$