

Definition

A function $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a *linear transformation* if it satisfies the following conditions:

- 1) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$
- 2) $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $\mathbf{v} \in \mathbb{R}^n$ and any scalar c .

Proposition

Every matrix transformation is a linear transformation.

Theorem

Every linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a matrix transformation:

$$T = T_A$$

for some matrix A .

Corollary

If $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation then $T = T_A$ where A is the matrix given by

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \dots \quad T(\mathbf{e}_n)]$$

This matrix is called the *standard matrix* of T .

Example. Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the function given by

$$T \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 + x_2 \\ 0 \\ 2x_1 \end{bmatrix}$$

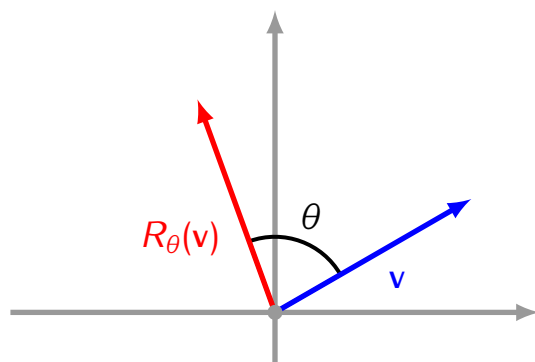
Check if T is a linear transformation. If it is, find its standard matrix.

Example. Let $S: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the function given by

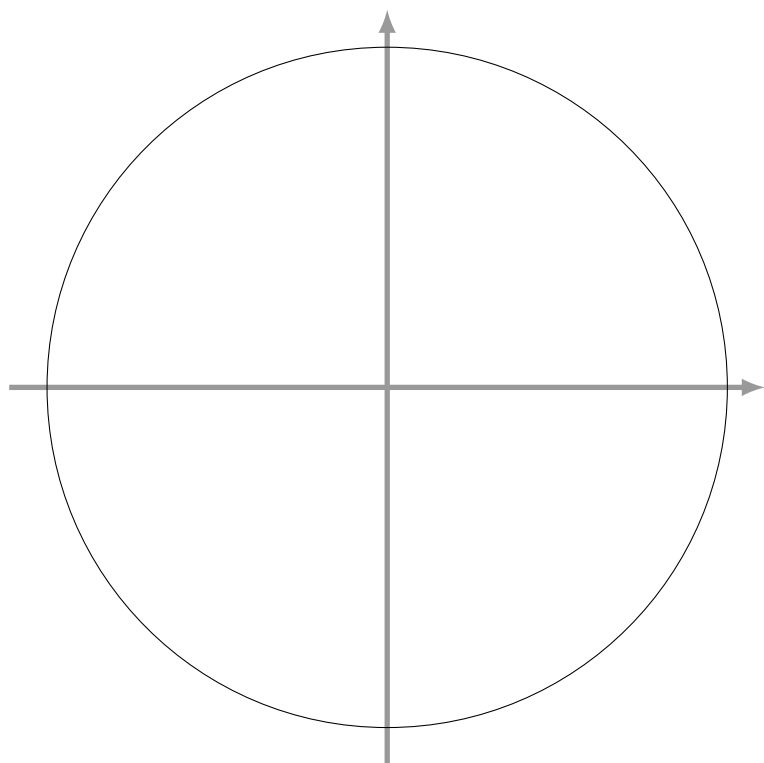
$$S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 + x_2 \\ x_2 \\ 3x_1 \end{bmatrix}$$

Check if S is a linear transformation. If it is, find its standard matrix.

Back to rotations:



$$R_\theta: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$



Proposition

Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ be the standard basis of \mathbb{R}^n . For any vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in \mathbb{R}^m$ there exists one and only one linear transformation

$$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

such that

$$T(\mathbf{e}_1) = \mathbf{v}_1 \quad T(\mathbf{e}_2) = \mathbf{v}_2, \quad \dots, \quad T(\mathbf{e}_n) = \mathbf{v}_n$$

The standard matrix of this linear transformation is given by

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}$$

Recall:

1) If A is an $m \times n$ matrix then the function

$$T_A: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

defined by $T_A(\mathbf{v}) = A\mathbf{v}$ is called the matrix transformation associated to A .

2) A function $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation if

(ii) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

(ii) $T(c\mathbf{v}) = cT(\mathbf{v})$

3) Every matrix transformation is a linear transformation.

4) Every linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a matrix transformation:

$$T(\mathbf{v}) = A\mathbf{v}$$

where

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \dots \quad T(\mathbf{e}_n)]$$

The matrix A is called the standard matrix of T .