<u>Next:</u> From systems of linear equations to vector equations.

$$\left\{\begin{array}{rl} x_1 + 2x_2 &= 4 \\ 2x_1 + 7x_2 &= 9 \\ 4x_1 + x_2 &= 0 \end{array}\right. \quad \Longrightarrow \quad x_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 0 \end{bmatrix}$$

Why vectors and vector equations are useful:

- They show up in many applications (velocity vectors, force vectors etc.)
- They give a better geometric picture of systems of linear equations.

Definition

A *column vector* is a matrix with one column.

Example

$$u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad v = \begin{bmatrix} 5 \\ -1 \\ 0 \\ 7 \end{bmatrix}$$

Note. Columns of a matrix are column vectors.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = [v_1 \ v_2 \ v_3]$$

$$v_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Notation

$\mathbb{R}^n$ is the set of all column vectors with n entries.

$$\mathbb{R}^2 = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \,\middle|\, a_1, a_2 \in \mathbb{R} \right\} = \text{the set of all vectors with 2 entries}$$

$$\mathbb{R}^3 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \,\middle|\, a_1, a_2, a_3 \in \mathbb{R} \right\} = \text{the set of all vectors with 3 entries}$$

Operations on vectors in $\mathbb{R}^n$

1) Addition of vectors:

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

e.g.:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 5 \\ -1 \\ 7 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 10 \end{bmatrix}$$

Note: We can add two vectors only if they have the same number of entries!

2) Multiplication by scalars:

scalars = real numbers

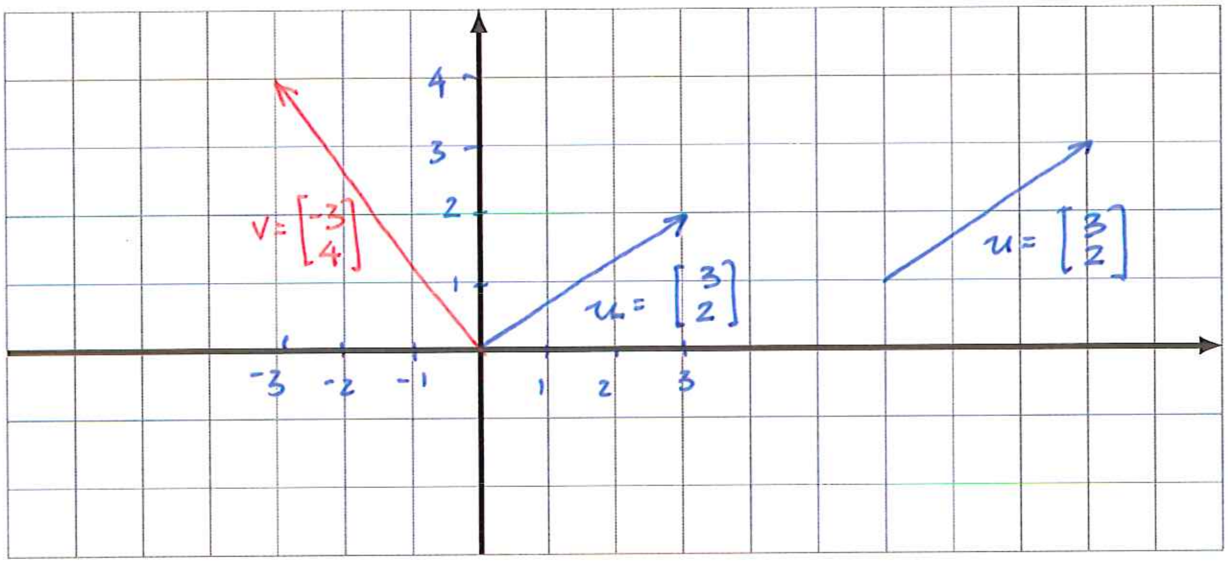
$$c \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}$$

e.g.:

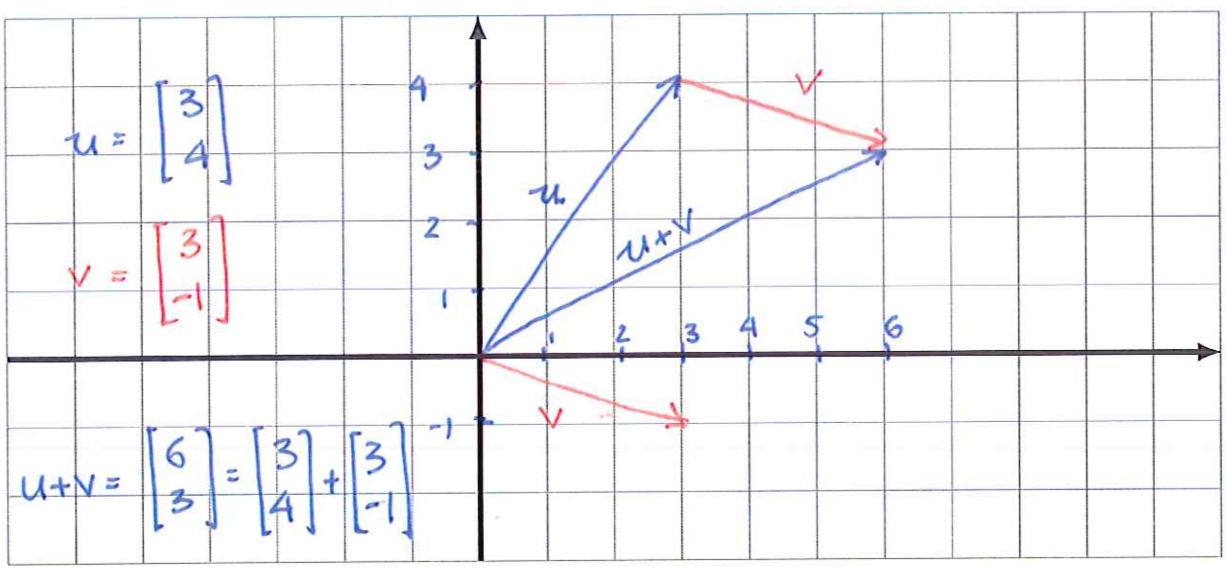
$$3 \cdot \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 15 \end{bmatrix}$$

Geometric interpretation of vectors in $\mathbb{R}^2$

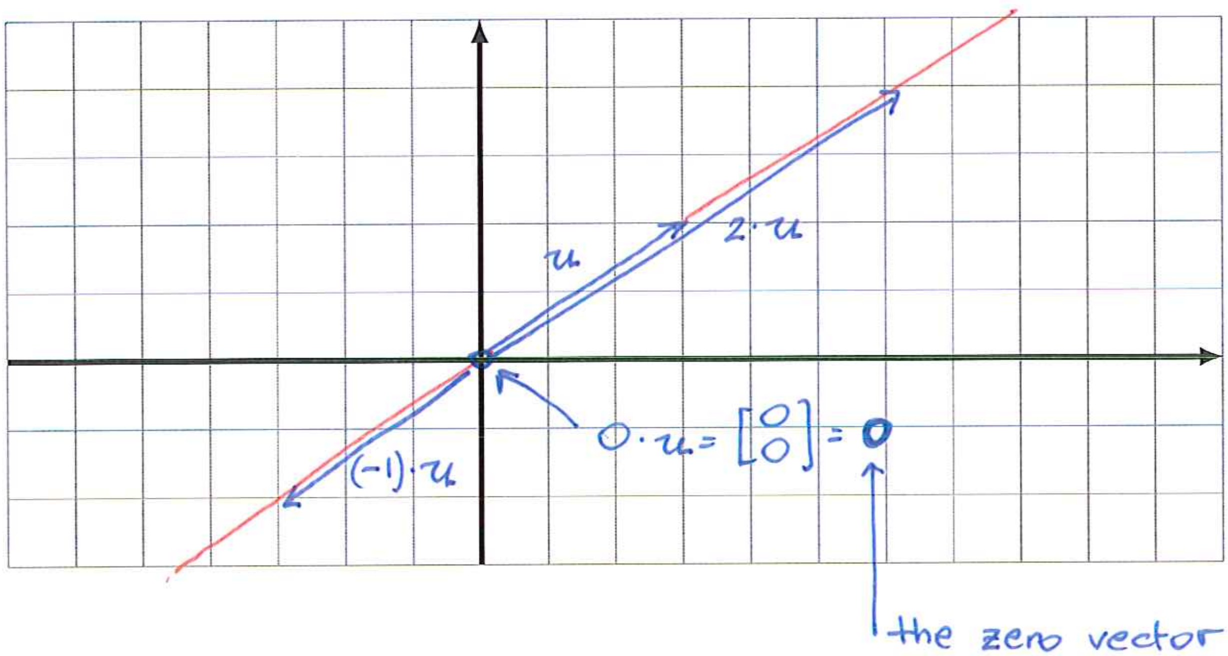
Vector coordinates:



Vector addition:



Scalar multiplication:



Vector equations

$$\underbrace{x_1 \mathbf{v}_1}_{} + x_2 \mathbf{v}_2 + \ldots + x_p \mathbf{v}_p = \mathbf{w}$$

vectors in $\mathbb{R}^n$

Example. Solve the following vector equation:

$$\underbrace{x_1 \overset{v_1}{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + x_2 \overset{v_2}{\begin{bmatrix} 4 \\ -2 \end{bmatrix}}} = \overset{w_2}{\begin{bmatrix} 10 \\ 3 \end{bmatrix}}$$

$$\underbrace{\begin{bmatrix} 2x_1 \\ 3x_1 \end{bmatrix} + \begin{bmatrix} 4x_2 \\ -2x_2 \end{bmatrix}} = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 + 4x_2 \\ 3x_1 - 2x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{cases} 2x_1 + 4x_2 = 10 \\ 3x_1 - 2x_2 = 3 \end{cases}$$

system of linear equations

augmented matrix:

$$\begin{array}{c} \begin{bmatrix} 2 & 4 & | & 10 \\ 3 & -2 & | & 3 \end{bmatrix} \\ v_1 \quad v_2 \quad w \end{array} \xrightarrow{\text{row red.}} \begin{array}{c} x_1 \quad x_2 \\ \begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & 3/2 \end{bmatrix} \end{array} \xrightarrow{\text{solutions}} \begin{cases} x_1 = 2 \\ x_2 = 3/2 \end{cases}$$

How to solve a vector equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{w}$$

vector of equation

make a matrix

$$\left[\begin{array}{ccc|c} \mathbf{v}_1 & \ldots & \mathbf{v}_p & \mathbf{w} \end{array}\right]$$

augmented matrix

row reduction

$$\left[\text{ reduced matrix }\right]$$

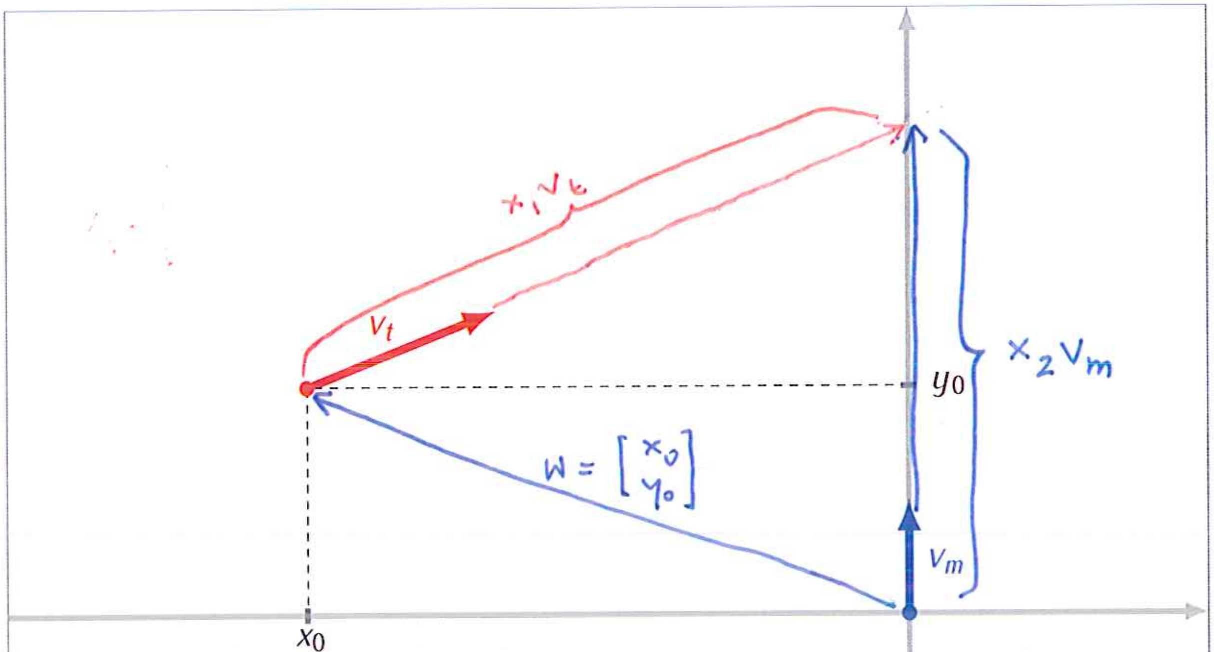
read off solutions

$$\begin{cases} x_1 & = \ldots \\ \ldots & \ldots \\ x_p & = \ldots \end{cases}$$

solutions

Example: Target shooting.

At time $t = 0$ a target is observed at the position (x_0, y_0) moving in the direction of the vector v_t. The target is moving at such speed, that it travels the length of v_t in one second. Find t_0 such that a missile positioned at the point $(0, 0)$ will intercept the target if it is fired at the time t_0. The missile travels the length of the vector v_m in one second.



x_1 = duration of flight of the target
x_2 = duration of flight of the missile

$$\boxed{t_0 = x_1 - x_2}$$

so, it is enough to find x_1, x_2

$$W + x_1 V_t = x_2 V_m$$

$$\boxed{x_1 V_t - x_2 V_m = -W} \leftarrow \text{solve for } x_1, x_2$$

e.g.:

$V_t = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$, $V_m = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$, $W = \begin{bmatrix} -165 \\ 25 \end{bmatrix}$

$$x_1 \begin{bmatrix} 11 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 165 \\ -25 \end{bmatrix} \xrightarrow{\text{aug. matrix}} \left[\begin{array}{cc|c} 11 & 0 & 165 \\ 1 & -4 & -25 \end{array}\right] \xrightarrow{\text{row red}} \begin{array}{ccc} x_1 & x_2 & \\ \end{array} \left[\begin{array}{cc|c} 1 & 0 & 15 \\ 0 & 1 & 10 \end{array}\right]$$

$$\begin{cases} x_1 = 15 \\ x_2 = 10 \end{cases} \Rightarrow t_0 = 15 - 10 = 5$$