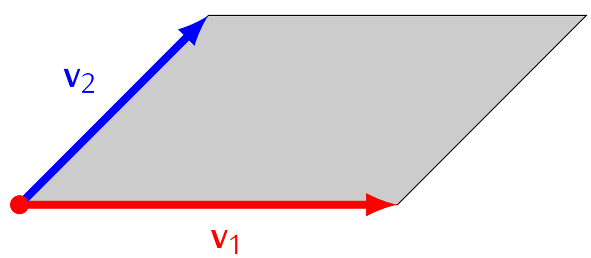
Recall:

A (matrix) $\longrightarrow$ $\det A$ (number)

- $= 0$ if A is not invertible
- $\neq 0$ if A is invertible

Note. Any two vectors in $\mathbb{R}^2$ define a parallelogram:



Notation

$$\text{area}(\mathbf{v}_1, \mathbf{v}_2) = \begin{pmatrix}\text{area of the parallelogram} \\ \text{defined by } \mathbf{v}_1 \text{ and } \mathbf{v}_2\end{pmatrix}$$

Theorem

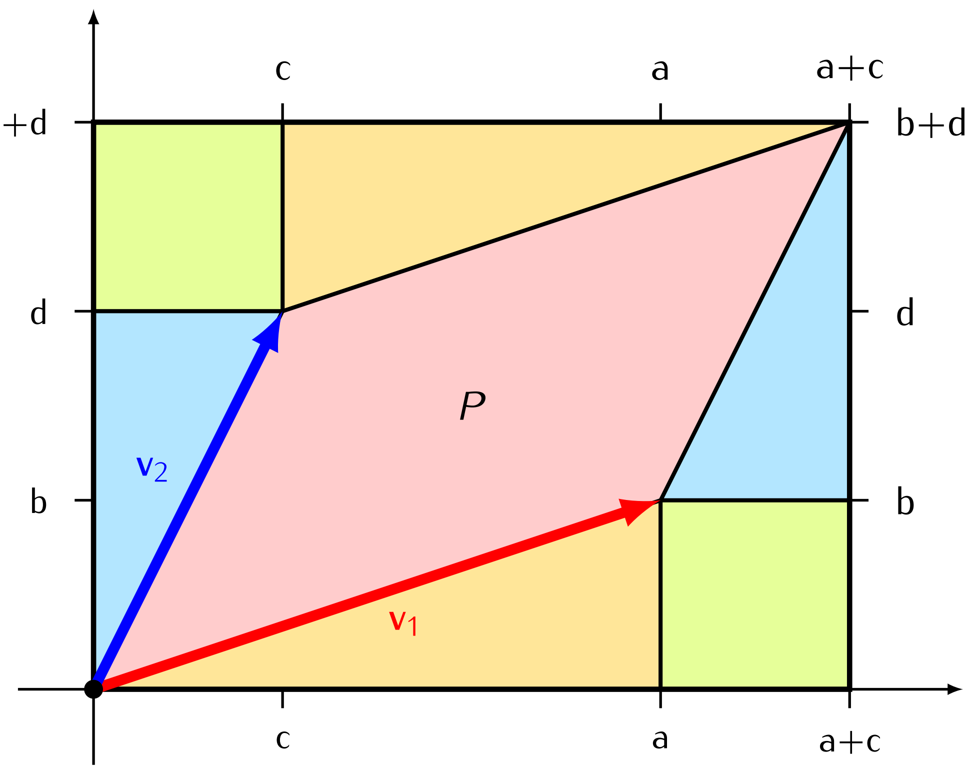
If $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ then

$$\mathrm{area}(\mathbf{v}_1, \mathbf{v}_2) = \left|\det\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\right|$$

Idea of the proof.

$$\mathbf{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} c \\ d \end{bmatrix}$$

$$[v_1 \; v_2] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$



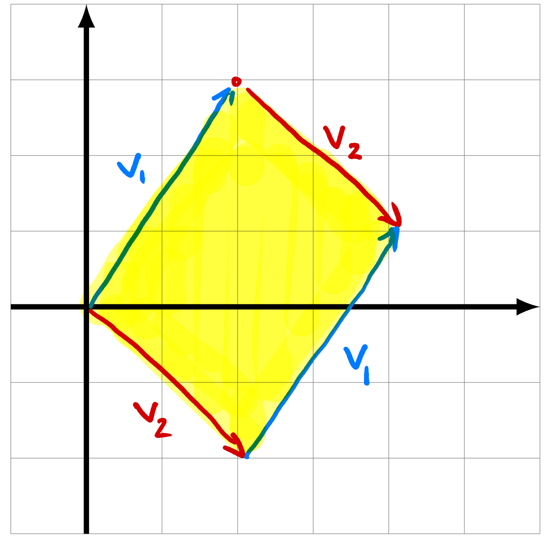
$$\begin{array}{r} \mathrm{area}(P) \\ +\ \frac{1}{2}ab \\ +\ \frac{1}{2}ab \\ +\ \frac{1}{2}cd \\ +\ \frac{1}{2}cd \\ +\ cb \\ +\ cb \\ \hline (a+c)(b+d) \end{array}$$

<u>We obtain:</u>

$$\begin{aligned} \mathrm{area}(P) &= (a+c)\cdot(b+d) - ab - cd - 2cb \\ &= (\cancel{ab} + ad + \cancel{cb} + \cancel{cd}) - \cancel{ab} - \cancel{cd} - \cancel{2}cb \\ &= ad - cb = \left|\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right| = |\det[v_1 \; v_2]| \end{aligned}$$

Example.

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

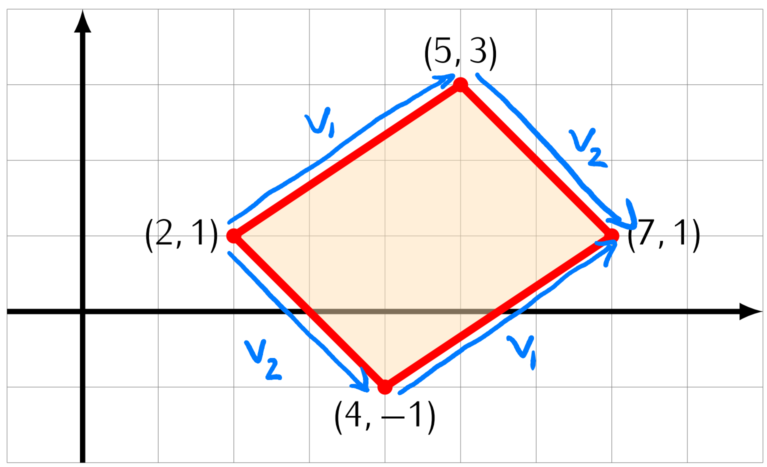


$$\text{area}(v_1, v_2) =$$

$$= \left| \det [v_1 \; v_2] \right|$$

$$= \left| \det \begin{bmatrix} 2 & 2 \\ 3 & -2 \end{bmatrix} \right| = |-4-6| = 10$$

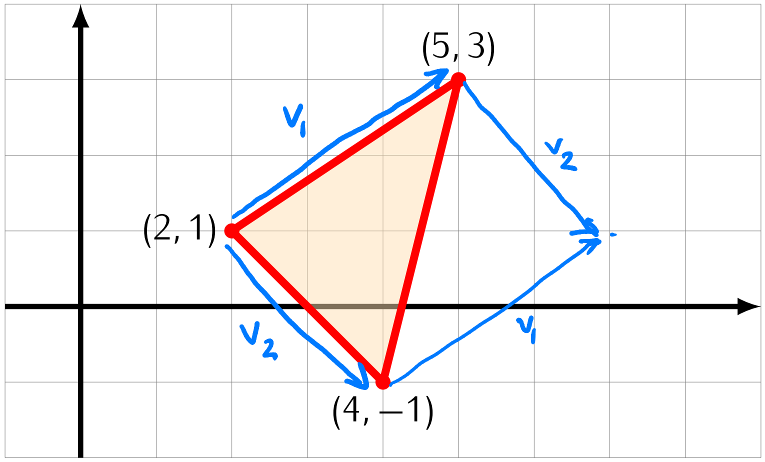
Example. Calculate the area of the parallelogram with vertices at the points $(2,1)$, $(5,3)$, $(7,1)$, $(4,-1)$.



$$v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$\begin{aligned} \text{area}(v_1, v_2) &= \left| \det [v_1 \; v_2] \right| \\ &= \left| \det \begin{bmatrix} 3 & 2 \\ 2 & -2 \end{bmatrix} \right| \\ &= |-6-4| = 10 \end{aligned}$$

Example. Calculate the area of the triangle with vertices at the points $(2,1)$, $(5,3)$, $(4,-1)$.



$$v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$\text{area}(v_1, v_2) = \left| \det [v_1 \ v_2] \right|$$

$$= \left| \det \begin{bmatrix} 3 & 2 \\ 2 & -2 \end{bmatrix} \right|$$

$$= |-6-4| = 10$$

$$\text{area triangle} = \frac{1}{2}\,\text{area}(v_1, v_2) = \frac{1}{2} \cdot 10$$

$$= 5$$

Note. In order to compute areas of other polygons, subdivide them into triangles.