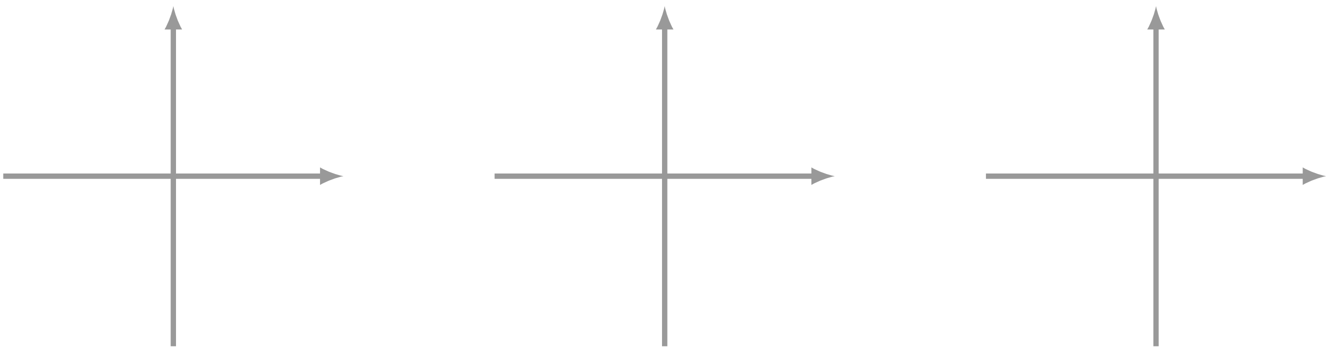
Composition of linear transformations



Theorem

If $S\colon \mathbb{R}^n \to \mathbb{R}^m$ and $T\colon \mathbb{R}^m \to \mathbb{R}^k$ are linear transformation then the composition

$$T \circ S\colon \mathbb{R}^n \to \mathbb{R}^k$$

is also a linear transformation.

Upshot. The function $T \circ S$ is represented by some matrix C:

$$T \circ S(\mathbf{v}) = C\mathbf{v}$$

Question. Let $S\colon \mathbb{R}^n \to \mathbb{R}^m$ and $T\colon \mathbb{R}^m \to \mathbb{R}^k$ be linear transformations, and let

- B is the standard matrix of S
- A is the standard matrix of T

What if the standard matrix of $T \circ S\colon \mathbb{R}^n \to \mathbb{R}^k$?

Definition

Let

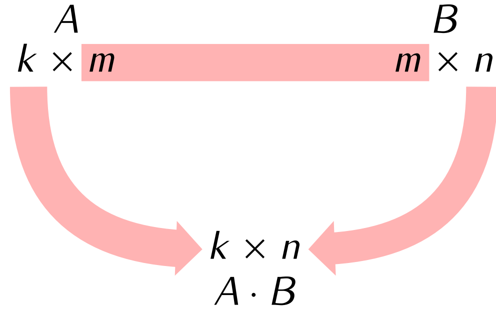
- A be an $k \times m$ matrix
- $B = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}$ be an $m \times n$ matrix

Then $A \cdot B$ is an $k \times n$ matrix given by

$$A \cdot B = \begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & \dots & A\mathbf{v}_n \end{bmatrix}$$

Note. The product $A \cdot B$ is defined only if

$$(\text{number of columns of } A) \;=\; (\text{number of rows of } B)$$



Example.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -1 & 2 & 1 \\ 4 & 5 & 1 & 0 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{km} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \dots & b_{mn} \end{bmatrix}$$

$$AB = \begin{bmatrix} c_{11} & \dots & c_{1m} \\ \vdots & & \vdots \\ c_{k1} & \dots & c_{km} \end{bmatrix}$$

$$c_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{im} \end{bmatrix} \cdot \begin{bmatrix} b_{1j} \\ b_{1j} \\ \vdots \\ b_{1j} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{im}b_{mj}$$