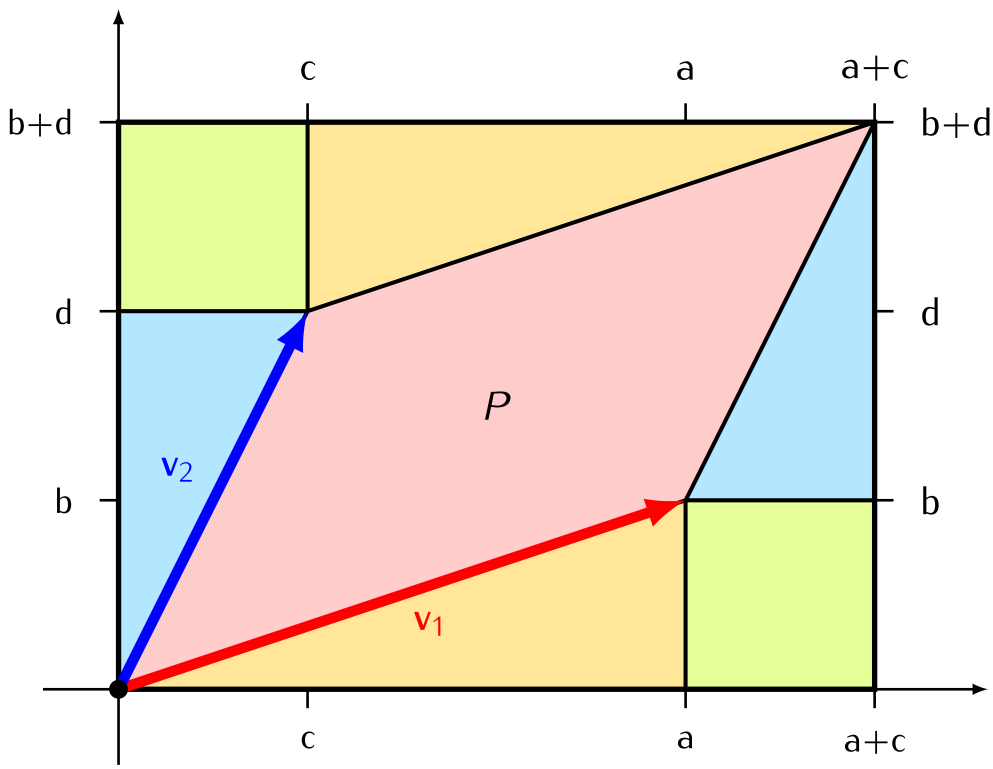
Theorem

If $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ then
$$\mathrm{area}(\mathbf{v}_1, \mathbf{v}_2) = \left|\det \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\right|$$

Idea of the proof.

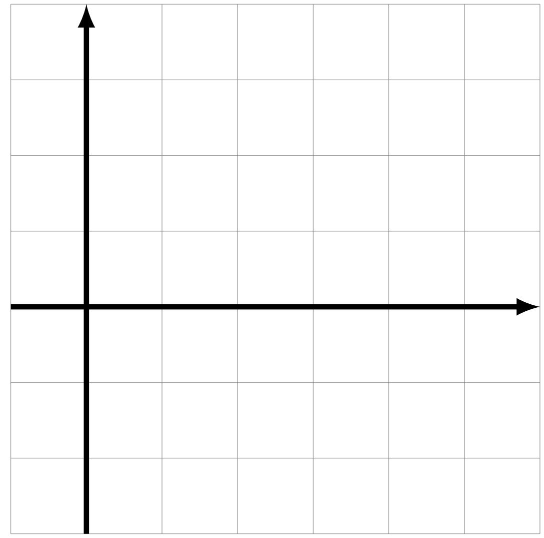
$$\mathbf{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} c \\ d \end{bmatrix}$$



$$\begin{array}{r} \mathrm{area}(P) \\ +\ \frac{1}{2}ab \\ +\ \frac{1}{2}ab \\ +\ \frac{1}{2}cd \\ +\ \frac{1}{2}cd \\ +\ cb \\ +\ cb \\ \hline (a+c)(b+d) \end{array}$$

Example.

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

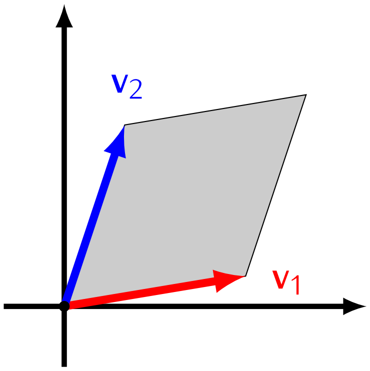


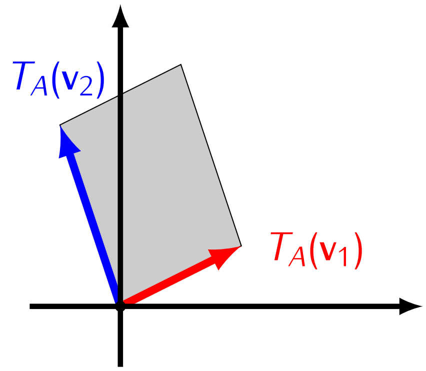
Determinants and linear transformations

Recall: If A is a 2×2 matrix then it defines a linear transformation

$$T_A \colon \mathbb{R}^2 \to \mathbb{R}^2 \qquad T_A(\mathbf{v}) = A\mathbf{v}$$

Note. T_A maps parallelograms to parallelograms:





Theorem

If A is a 2×2 matrix and $\mathbf{v}_1, \mathbf{v}_1 \in \mathbb{R}^2$ then

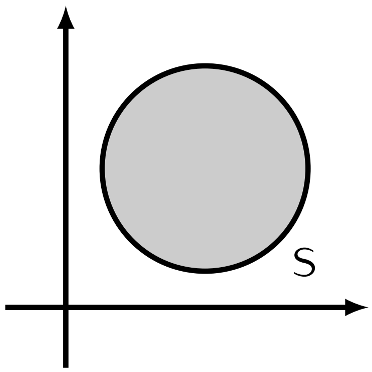
$$\text{area}(T_A(\mathbf{v}_1), T_A(\mathbf{v}_2)) = |\det A| \cdot \text{area}(\mathbf{v}_1, \mathbf{v}_2)$$

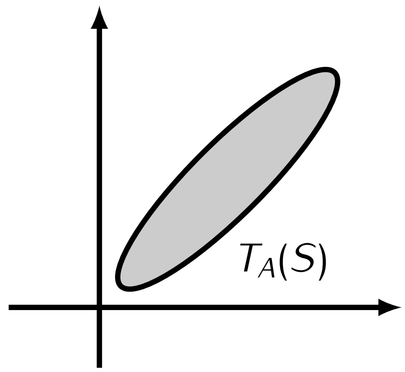
Generalization:

Theorem

If A is a 2×2 matrix then for any region S of $\mathbb{R}^2$ we have:

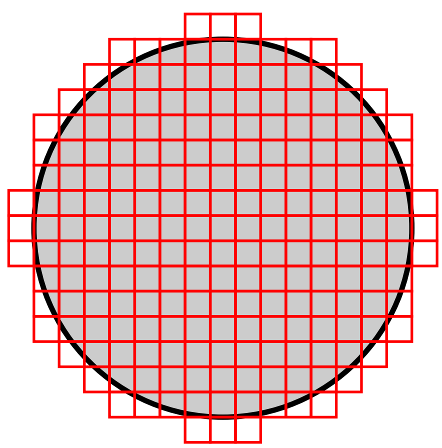
$$\text{area}(T_A(S)) = |\det A| \cdot \text{area}(S)$$





Idea of the proof.

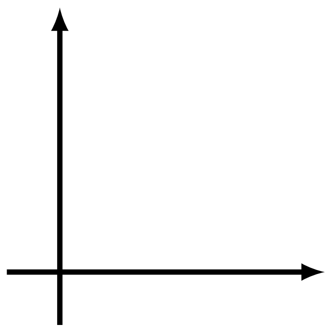
The area of S can be approximated by the sum of small squares covering S.

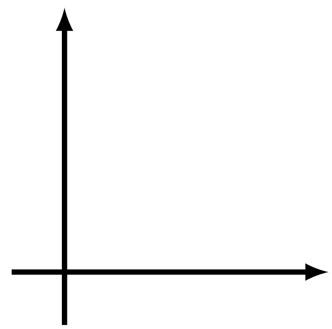


Sign of the determinant

Example.

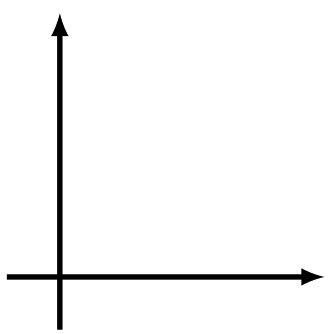
$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

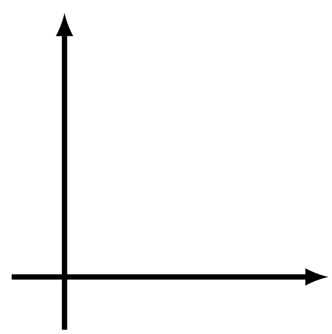




Example.

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$$





Theorem

If A is a 2×2 matrix then the linear transformation $T_A \colon \mathbb{R}^2 \to \mathbb{R}^2$ preserves orientation if $\det A > 0$ and reverses orientation if $\det A < 0$.

Linear Algebra	Calculus
$\mathbb{R}^n = \begin{pmatrix} \text{set of all column vectors} \\ \text{with } n \text{ entries} \end{pmatrix}$	$C^\infty(\mathbb{R}) = \begin{pmatrix} \text{set of all smooth} \\ \text{functions } f\colon \mathbb{R} \to \mathbb{R} \end{pmatrix}$
Column vectors can be added and multiplied by real numbers.	**Functions** can be added and multiplied by real numbers.
Linear transformation is a function $$T\colon \mathbb{R}^n \to \mathbb{R}^m, \quad T(\mathbf{v}) = A\mathbf{v}$$ It satisfies: • $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ • $T(c\mathbf{v}) = cT(\mathbf{v})$	**Differentiation** is a function $$D\colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \quad D(f) = f'$$ It satisfies: • $D(f+g) = D(f) + D(g)$ • $D(cf) = cD(f)$
Typical problem: given a vector $\mathbf{b}$ find all vectors $\mathbf{x}$ such that $$T(\mathbf{x}) = \mathbf{b}$$ (i.e solve the equation $A\mathbf{x} = \mathbf{b}$).	**Typical problem:** given a function g find all functions f such that $$D(f) = g$$ (i.e find antiderivatives of g).
Fact: Such vectors x are of the form $$\mathbf{x} = \mathbf{v}_0 + \mathbf{n}$$ where: • $\mathbf{v}_0$ is some distinguished solution of $A\mathbf{x} = \mathbf{b}$; • $\mathbf{n} \in \mathrm{Nul}(A)$ (i.e. $\mathbf{n}$ is a solution of $A\mathbf{x} = \mathbf{0}$).	**Fact:** Such functions f are of the form $$f = F + C$$ where: • F is some distinguished antiderivative of g; • C is a constant function (i.e. C is a solution of $D(f) = 0$).