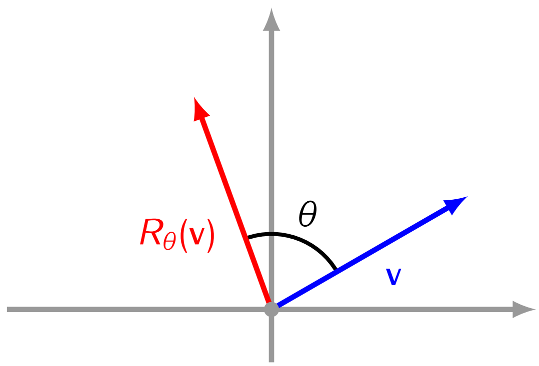
Problem: How to recognize if a function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation?

Example. Rotation by an angle θ:



$$R_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$$

Definition

A function $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a *linear transformation* if it satisfies the following conditions:

1) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$

2) $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $\mathbf{v} \in \mathbb{R}^n$ and any scalar c.

Proposition

Every matrix transformation is a linear transformation.

Theorem

Every linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation:

$$T = T_A$$

for some matrix A.

Corollary

If $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then $T = T_A$ where A is the matrix given by

$$A = \left[\begin{array}{cccc} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \end{array}\right]$$

This matrix is called the *standard matrix* of T.

Example. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the function given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 0 \\ 2x_1 \end{bmatrix}$$

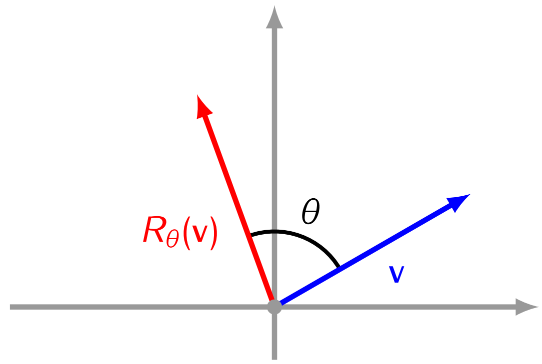
Check if T is a linear transformation. If it is, find its standard matrix.

Example. Let $S\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the function given by

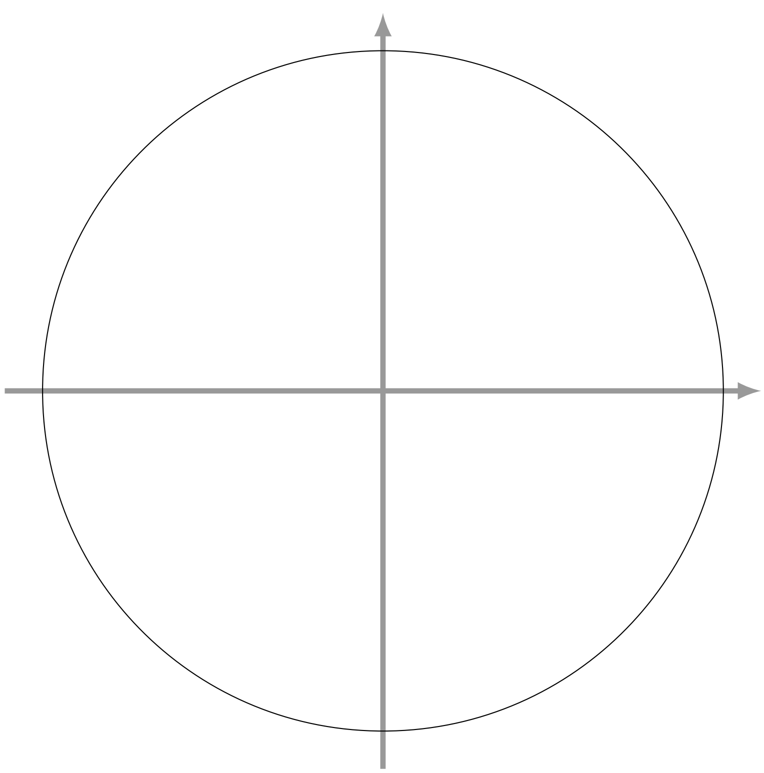
$$S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 + x_2 \\ x_2 \\ 3x_1 \end{bmatrix}$$

Check if S is a linear transformation. If it is, find its standard matrix.

Back to rotations:



$$R_\theta \colon \mathbb{R}^2 \to \mathbb{R}^2$$



Proposition

Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be the standard basis of of $\mathbb{R}^n$. For any vectors $\mathbf{v}_1, \mathbf{v}_n, \ldots, \mathbf{v}_n \in \mathbb{R}^m$ there exists one and only one linear transformation

$$T\colon \mathbb{R}^n \to \mathbb{R}^m$$

such that

$$T(\mathbf{e}_1) = \mathbf{v}_1 \quad T(\mathbf{e}_2) = \mathbf{v}_2, \quad \ldots, \quad T(\mathbf{e}_n) = \mathbf{v}_n$$

The standard matrix of this linear transformation is given by

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}$$