Recall:

1) If

$$\mathbf{u} = \left[\begin{array}{c} a_1 \\ \vdots \\ a_n \end{array} \right] \qquad \mathbf{v} = \left[\begin{array}{c} b_1 \\ \vdots \\ b_n \end{array} \right]$$

are vectors in \mathbb{R}^n then:

- $\mathbf{u} \cdot \mathbf{v} = a_1 b_1 + \ldots + a_n b_n$
- $\bullet \ \|u\| = \sqrt{u \cdot u}$
- dist(u, v) = ||u v||
- 2) Vectors \mathbf{u}, \mathbf{v} are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.
- 3) Pythagorean theorem: \mathbf{u}, \mathbf{v} are orthogonal if and only if

$$||\mathbf{u}||^2 + ||\mathbf{v}||^2 = ||\mathbf{u} + \mathbf{v}||^2$$

- **4)** If $V \subseteq \mathbb{R}^n$ is a subspace then an orthogonal basis of V is a basis which consists of vectors that are orthogonal to one another.
- **5)** If $\mathcal{B} = \{\mathbf{v}_1, \dots \mathbf{v}_k\}$ is an orthogonal basis of V and $\mathbf{w} \in V$ then

$$\left[\begin{array}{c}\mathbf{w}\end{array}\right]_{\mathcal{B}} = \left[\begin{array}{c}c_1\\ \vdots\\ c_k\end{array}\right]$$

where
$$c_i = \frac{\mathbf{w} \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i}$$
.

6) Gram-Schmidt process:

a basis
$$\{\mathbf v_1,\dots,\mathbf v_k\} \\ \text{of } V\subseteq\mathbb R^n$$
 an orthogonal basis
$$\{\mathbf w_1,\dots,\mathbf w_k\} \\ \text{of } V$$

$$\mathbf{w}_1 = \mathbf{v}_1$$

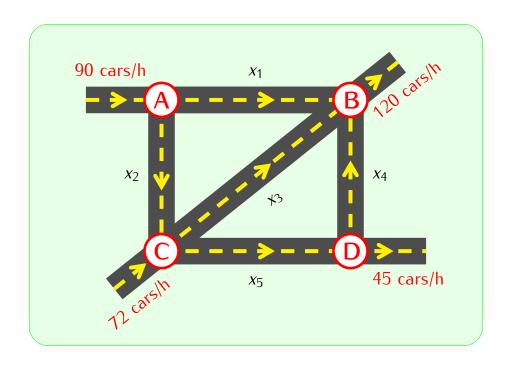
$$\mathbf{w}_2 = \mathbf{v}_2 - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_2}{\mathbf{w}_1 \cdot \mathbf{w}_1}\right) \mathbf{w}_1$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_3}{\mathbf{w}_1 \cdot \mathbf{w}_1}\right) \mathbf{w}_1 - \left(\frac{\mathbf{w}_2 \cdot \mathbf{v}_3}{\mathbf{w}_2 \cdot \mathbf{w}_2}\right) \mathbf{w}_2$$

...

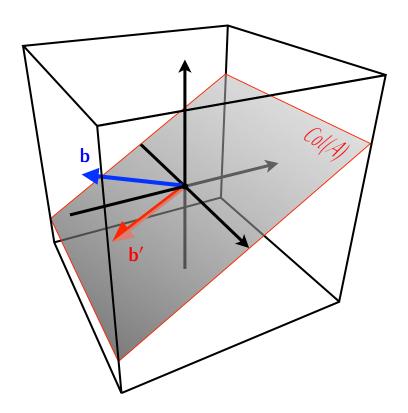
$$\mathbf{w}_k = \mathbf{v}_k - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_k}{\mathbf{w}_1 \cdot \mathbf{w}_1}\right) \mathbf{w}_1 - \left(\frac{\mathbf{w}_2 \cdot \mathbf{v}_k}{\mathbf{w}_2 \cdot \mathbf{w}_2}\right) \mathbf{w}_2 - \ldots - \left(\frac{\mathbf{w}_{k-1} \cdot \mathbf{v}_k}{\mathbf{w}_{k-1} \cdot \mathbf{w}_{k-1}}\right) \mathbf{w}_{k-1}$$

Problem. Find the flow rate of cars on each segment of streets:



Upshot.

- Recall: a matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b} \in \text{Col}(A)$.
- In practical applications we may obtain a matrix equation that has no solutions, i.e. where $\mathbf{b} \notin \text{Col}(A)$.
- In such cases we may look for approximate solutions as follows:
 - replace **b** by a vector **b**' such that $\mathbf{b}' \in \operatorname{Col}(A)$ and $\operatorname{dist}(\mathbf{b}, \mathbf{b}')$ is a as small as possible.
 - then solve $A\mathbf{x} = \mathbf{b}'$



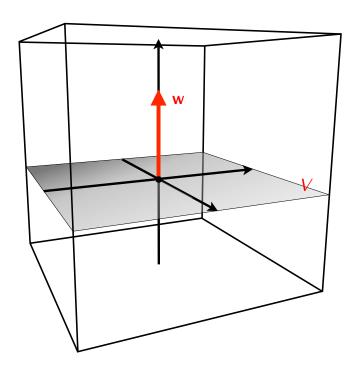
Definition

Given $\mathbf{b}' \in \operatorname{Col}(A)$ as above we will say that a vector \mathbf{v} is a *least square* solution of the equation $A\mathbf{x} = \mathbf{b}$ if \mathbf{v} is a solution of the equation $A\mathbf{x} = \mathbf{b}'$.

Next: How to find the vector **b**'?

Definition

Let V be a subspace of \mathbb{R}^n . A vector $\mathbf{w} \in \mathbb{R}^n$ is orthogonal to V if $\mathbf{w} \cdot \mathbf{v} = 0$ for all $\mathbf{v} \in V$.



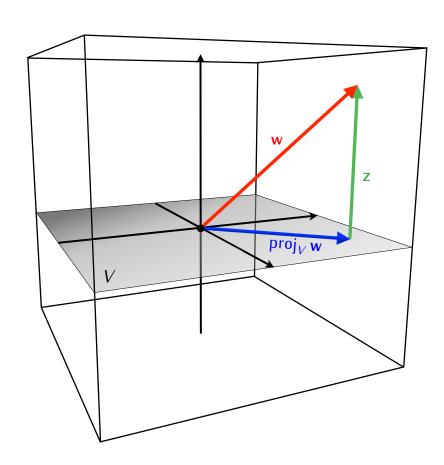
Proposition

If $V = \operatorname{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ then a vector $\mathbf{w} \in \mathbb{R}^n$ is orthogonal to V if and only if $\mathbf{w} \cdot \mathbf{v}_i = 0$ for $i = 1, \dots, k$.

Definition

Let V be a subspace of \mathbb{R}^n and let $\mathbf{w} \in \mathbb{R}^n$ the orthogonal projection of \mathbf{w} onto V is a vector $\operatorname{proj}_V \mathbf{w}$ such that

- 1) $\operatorname{proj}_V \mathbf{w} \in V$
- 2) the vector $\mathbf{z} = \mathbf{w} \operatorname{proj}_{V} \mathbf{w}$ is orthogonal to V.



The Best Approximation Theorem

If V is a subspace of \mathbb{R}^n and $\mathbf{w} \in \mathbb{R}^n$ then $\operatorname{proj}_V \mathbf{w}$ is a vector in V which is closest to \mathbf{w} :

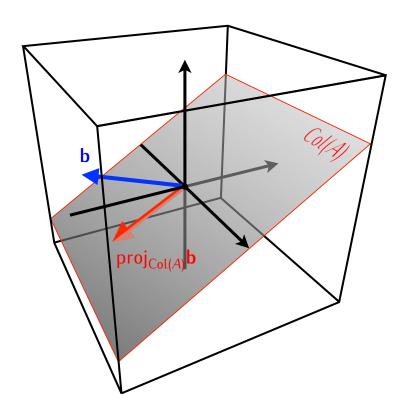
$$dist(\mathbf{w}, proj_V \mathbf{w}) \leq dist(\mathbf{w}, \mathbf{v})$$

for all $\mathbf{v} \in V$.

Corollary

The least square solutions of a matrix equation $A\mathbf{x}=\mathbf{b}$ are solutions of the equation

$$A\mathbf{x} = \operatorname{proj}_{\operatorname{Col}(A)}\mathbf{b}$$



Next: If V is a subspace of \mathbb{R}^n and $\mathbf{w} \in \mathbb{R}^n$ how to compute $\text{proj}_V \mathbf{w}$?

Theorem

If V is a subspace of \mathbb{R}^n with an orthogonal basis $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ and $\mathbf{w}\in\mathbb{R}^n$ then

$$\operatorname{proj}_{V} \mathbf{w} = \left(\frac{\mathbf{w} \cdot \mathbf{v}_{1}}{\mathbf{v}_{1} \cdot \mathbf{v}_{1}}\right) \mathbf{v}_{1} + \ldots + \left(\frac{\mathbf{w} \cdot \mathbf{v}_{k}}{\mathbf{v}_{k} \cdot \mathbf{v}_{k}}\right) \mathbf{v}_{k}$$

Corollary

If V is a subspace of \mathbb{R}^n with an orthonormal basis $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ and $\mathbf{w}\in\mathbb{R}^n$ then

$$\operatorname{proj}_{V} \mathbf{w} = (\mathbf{w} \cdot \mathbf{v}_{1}) \mathbf{v}_{1} + \ldots + (\mathbf{w} \cdot \mathbf{v}_{k}) \mathbf{v}_{k}$$

Example. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1\\2\\0\\3 \end{bmatrix}, \begin{bmatrix} 2\\-4\\5\\2 \end{bmatrix}, \begin{bmatrix} 4\\1\\0\\-2 \end{bmatrix} \right\}, \quad \mathbf{w} = \begin{bmatrix} 1\\2\\2\\1 \end{bmatrix}$$

The set \mathcal{B} is an orthogonal basis of some subspace V of \mathbb{R}^4 . Compute $\operatorname{proj}_V \mathbf{w}$.

Note. In general if V is a subspace of \mathbb{R}^n and $\mathbf{w} \in \mathbb{R}^n$ then in order to find $\operatorname{proj}_V \mathbf{w}$ we need to do the following:

- 1) find a basis of V.
- 2) use the Gram-Schmidt process to get an orthogonal basis of V
- 3) use the orthogonal basis to compute $proj_V \mathbf{w}$.

Example. Consider the following matrix A and vector \mathbf{u} :

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 3 & 4 & 2 \\ 2 & 6 & 3 & -1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

Compute $\text{proj}_{\text{Col}(A)}\mathbf{u}$.

Example. Find least square solutions of the matrix equation $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 90 \\ 120 \\ 72 \\ 45 \end{bmatrix}$$