

Recall:

1) An orthogonal matrix  $Q = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \dots \quad \mathbf{u}_n]$  is a square matrix such that  $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$  is an orthonormal set of vectors, i.e.:

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

2) If  $Q$  is an orthogonal matrix then  $Q^{-1} = Q^T$

3) A square matrix  $A$  is orthogonally diagonalizable if there exist an orthogonal matrix  $Q$  and a diagonal matrix  $D$  such that

$$A = QDQ^{-1} = QDQ^T$$

4) A matrix  $A$  is orthogonally diagonalizable if and only if  $A$  is a symmetric matrix (i.e.  $A^T = A$ ).

## Yet another view of matrix multiplication

**Note.** If  $C$  is an  $n \times 1$  matrix and  $D$  is an  $1 \times n$  matrix then  $CD$  is an  $n \times n$  matrix.

### Proposition

Let  $A$  be an  $n \times n$  matrix with columns  $\mathbf{v}_1, \dots, \mathbf{v}_n$ , and  $B$  be an  $n \times n$  matrix with rows  $\mathbf{w}_1, \dots, \mathbf{w}_n$ :

$$A = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix} \quad B = \begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_n \end{bmatrix}$$

Then

$$AB = \mathbf{v}_1\mathbf{w}_1 + \mathbf{v}_2\mathbf{w}_2 + \dots + \mathbf{v}_n\mathbf{w}_n$$

**Example.**

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 1 \\ 7 & 2 \end{bmatrix}$$

### Theorem

Let  $A$  be a symmetric matrix with orthogonal diagonalization

$$A = QDQ^T$$

If

$$Q = [\mathbf{u}_1 \quad \dots \quad \mathbf{u}_n] \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & 0 \\ 0 & \dots & \lambda_n \end{bmatrix}$$

then

$$A = \lambda_1(\mathbf{u}_1\mathbf{u}_1^T) + \lambda_2(\mathbf{u}_2\mathbf{u}_2^T) + \dots + \lambda_n(\mathbf{u}_n\mathbf{u}_n^T)$$

**Note.** The above formula is called the *spectral decomposition* of the matrix  $A$ .

**Example.**

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$$

## Spectral decomposition and linear transformations

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$$

