$$
\left[\begin{array}{cccc|c}
0 & 4 & -8 & 0 & 4 \\
2 & 6 & -6 & -2 & -4 \\
2 & 7 & -8 & 0 & -1
\end{array}\right]
\xrightarrow{\text{row reduction}}
\left[\begin{array}{cccc|c}
1 & 0 & 3 & 0 & -4 \\
0 & 1 & -2 & 0 & 1 \\
0 & 0 & 0 & 1 & 1
\end{array}\right]
$$

Definition

A *pivot position* in a matrix is a position that after row reduction contains a leading one.

A *pivot column* of a matrix is a column that contains a pivot position.

Theorem

1) A system of linear equations is inconsistent if and only if the last column of its augmented matrix is a pivot column.

2) Free variables of the system correspond to non–pivot columns of the coefficient matrix.

3) The system has only one solution if and only if every column of its augmented matrix is a pivot column, except for the last column.

Proof.

```
                    ┌─────────────────┐
                    │    augmented    │
                    │     matrix      │
                    └─────────────────┘
                             │
                             ▼
                          ╱     ╲
              YES       ╱   Is    ╲      NO
          ┌──────────╱  the last   ╲──────────┐
          │          ╲ column a     ╱         │
          │           ╲pivot column╱          │
          ▼            ╲    ?     ╱            ▼
    ┌──────────┐         ╲     ╱            ╱     ╲
    │   zero   │                   YES    ╱   Is    ╲    NO
    │solutions │              ┌─────────╱   every    ╲────────┐
    └──────────┘              │         ╲ column of   ╱        │
                             │          ╲coeficient  ╱         │
                             ▼           ╲ matrix a  ╱          ▼
                       ┌──────────┐       ╲ pivot   ╱    ┌──────────────┐
                       │exactly one│       ╲column?╱     │infinitely many│
                       │ solution │                      │  solutions   │
                       └──────────┘                      └──────────────┘
```