

Recall:

1) If

$$\mathbf{u} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

are vectors in \mathbb{R}^n then:

- $\mathbf{u} \cdot \mathbf{v} = a_1b_1 + \dots + a_nb_n$
- $\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$
- $\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$

2) Vectors \mathbf{u}, \mathbf{v} are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.3) Pythagorean theorem: \mathbf{u}, \mathbf{v} are orthogonal if and only if

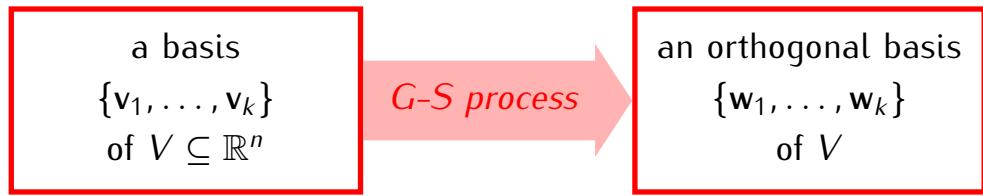
$$\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u} + \mathbf{v}\|^2$$

4) If $V \subseteq \mathbb{R}^n$ is a subspace then an orthogonal basis of V is a basis which consists of vectors that are orthogonal to one another.5) If $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an orthogonal basis of V and $\mathbf{w} \in V$ then

$$[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

where $c_i = \frac{\mathbf{w} \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i}$.

6) Gram-Schmidt process:



$$\mathbf{w}_1 = \mathbf{v}_1$$

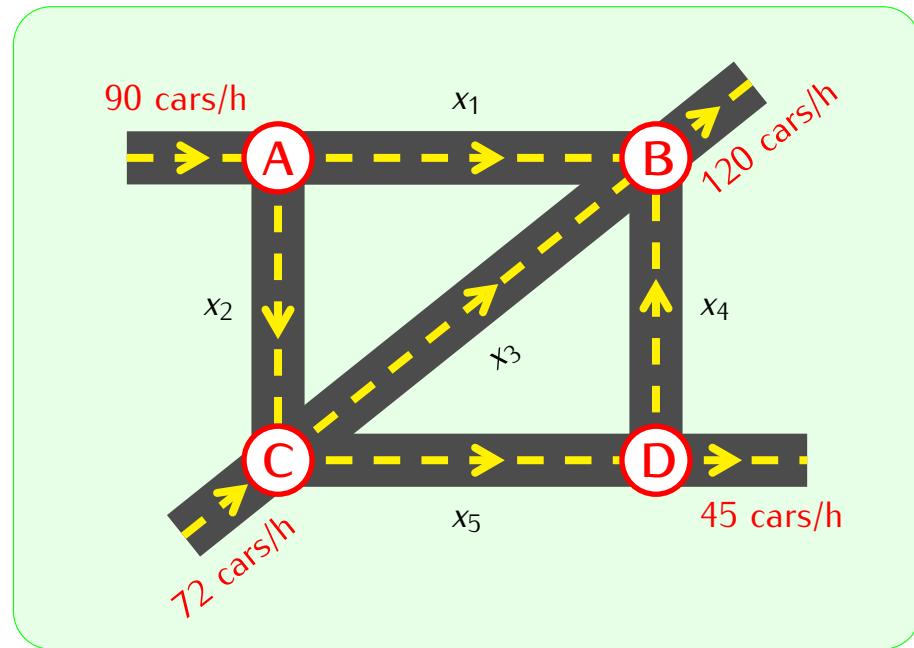
$$\mathbf{w}_2 = \mathbf{v}_2 - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_2}{\mathbf{w}_1 \cdot \mathbf{w}_1} \right) \mathbf{w}_1$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_3}{\mathbf{w}_1 \cdot \mathbf{w}_1} \right) \mathbf{w}_1 - \left(\frac{\mathbf{w}_2 \cdot \mathbf{v}_3}{\mathbf{w}_2 \cdot \mathbf{w}_2} \right) \mathbf{w}_2$$

...

$$\mathbf{w}_k = \mathbf{v}_k - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_k}{\mathbf{w}_1 \cdot \mathbf{w}_1} \right) \mathbf{w}_1 - \left(\frac{\mathbf{w}_2 \cdot \mathbf{v}_k}{\mathbf{w}_2 \cdot \mathbf{w}_2} \right) \mathbf{w}_2 - \dots - \left(\frac{\mathbf{w}_{k-1} \cdot \mathbf{v}_k}{\mathbf{w}_{k-1} \cdot \mathbf{w}_{k-1}} \right) \mathbf{w}_{k-1}$$

Problem. Find the flow rate of cars on each segment of streets:



Solution:

$$\begin{array}{l} \text{FLOW IN} = \text{FLOW OUT} \\ \textcircled{a} A: 90 = x_1 + x_2 \\ \textcircled{b} B: x_1 + x_3 + x_4 = 120 \\ \textcircled{c} C: 72 + x_2 = x_3 + x_5 \\ \textcircled{d} D: x_5 = 45 + x_4 \end{array} \quad \left\{ \begin{array}{l} x_1 + x_2 = 90 \\ x_1 + x_3 + x_4 = 120 \\ -x_2 + x_3 + x_5 = 72 \\ -x_4 + x_5 = 45 \end{array} \right.$$

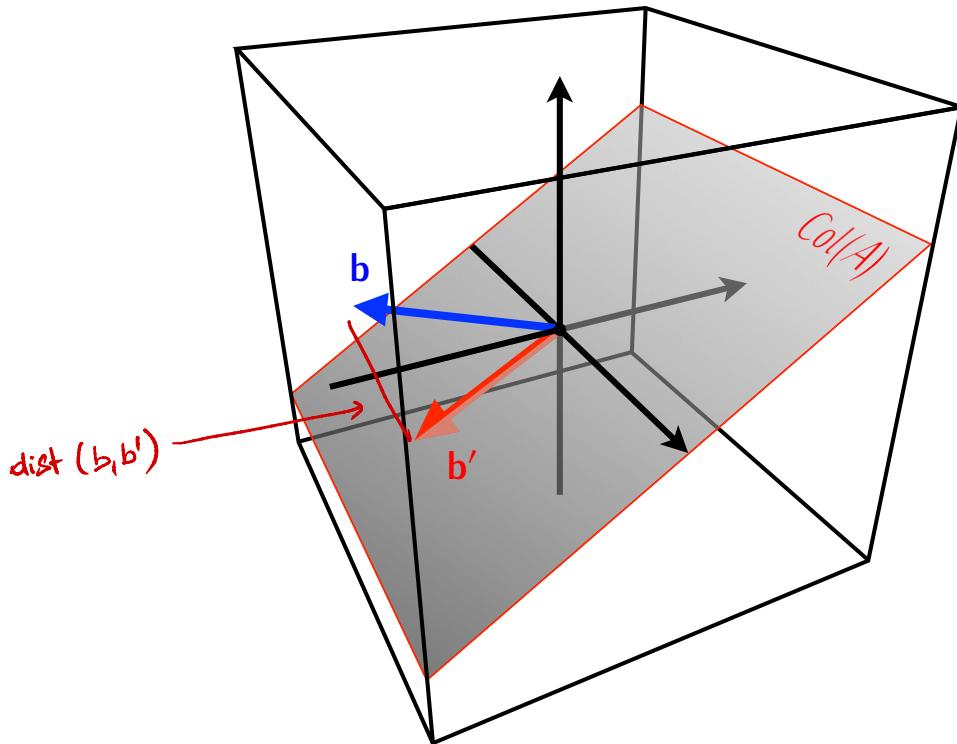
augmented matrix

$$\left[\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \hline 1 & 1 & 0 & 0 & 0 & 90 \\ 1 & 0 & 1 & 1 & 0 & 120 \\ 0 & -1 & 1 & 0 & 1 & 72 \\ 0 & 0 & 0 & -1 & 1 & 45 \end{array} \right] \xrightarrow{\text{row red.}} \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

leading one in the last column, so:
no solutions

Upshot.

- Recall: a matrix equation $Ax = \mathbf{b}$ has a solution if and only if $\mathbf{b} \in \text{Col}(A)$.
- In practical applications we may obtain a matrix equation that has no solutions, i.e. where $\mathbf{b} \notin \text{Col}(A)$.
- In such cases we may look for approximate solutions as follows:
 - replace \mathbf{b} by a vector \mathbf{b}' such that $\mathbf{b}' \in \text{Col}(A)$ and $\text{dist}(\mathbf{b}, \mathbf{b}')$ is as small as possible.
 - then solve $Ax = \mathbf{b}'$



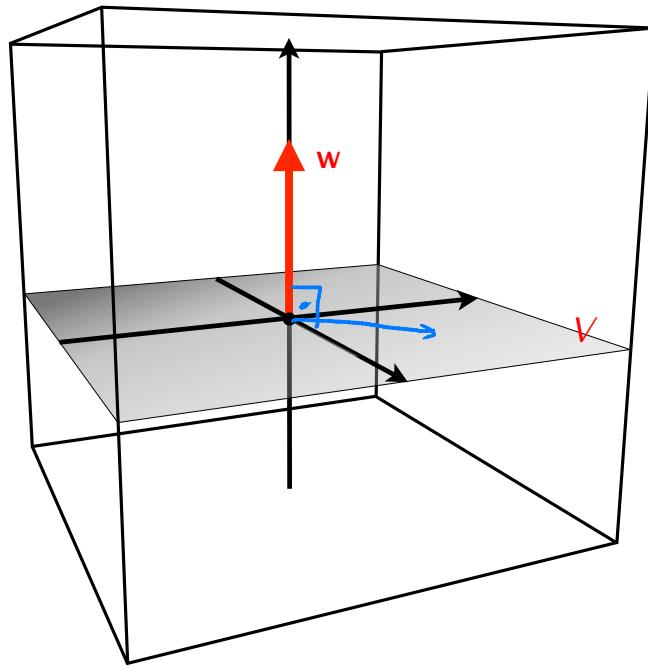
Definition

Given $\mathbf{b}' \in \text{Col}(A)$ as above we will say that a vector \mathbf{v} is a *least square solution* of the equation $Ax = \mathbf{b}$ if \mathbf{v} is a solution of the equation $Ax = \mathbf{b}'$.

Next: How to find the vector \mathbf{b}' ?

Definition

Let V be a subspace of \mathbb{R}^n . A vector $w \in \mathbb{R}^n$ is *orthogonal to V* if $w \cdot v = 0$ for all $v \in V$.



Proposition

If $V = \text{Span}(v_1, \dots, v_k)$ then a vector $w \in \mathbb{R}^n$ is orthogonal to V if and only if $w \cdot v_i = 0$ for $i = 1, \dots, k$.

Proof: Assume that w is orthogonal to v_1, \dots, v_k .

If $v \in \text{Span}(v_1, \dots, v_k)$ then $v = c_1 v_1 + \dots + c_k v_k$.

This gives:

$$w \cdot v = w \cdot (c_1 v_1 + \dots + c_k v_k)$$

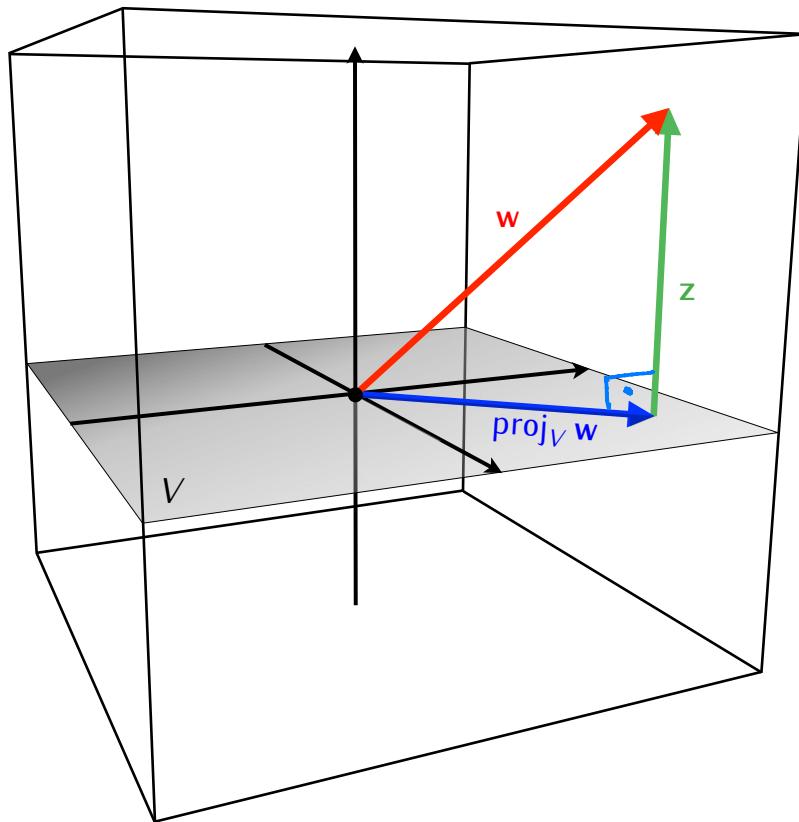
$$= c_1(w \cdot v_1) + \dots + c_k(w \cdot v_k) = 0$$

So: w is orthogonal to every vector $v \in \text{Span}(v_1, \dots, v_k)$.

Definition

Let V be a subspace of \mathbb{R}^n and let $w \in \mathbb{R}^n$ the *orthogonal projection* of w onto V is a vector $\text{proj}_V w$ such that

- 1) $\text{proj}_V w \in V$
- 2) the vector $z = w - \text{proj}_V w$ is orthogonal to V .



The Best Approximation Theorem

If V is a subspace of \mathbb{R}^n and $w \in \mathbb{R}^n$ then $\text{proj}_V w$ is a vector in V which is closest to w :

$$\text{dist}(w, \text{proj}_V w) \leq \text{dist}(w, v)$$

for all $v \in V$.

Proof: Let $v \in V$. We want to show:

$$\underbrace{\text{dist}(w, \text{proj}_V w)}_{\|w - \text{proj}_V w\|} \leq \underbrace{\text{dist}(w, v)}_{\|w - v\|}$$

- Note:
- 1) $w - \text{proj}_V w$ is a vector orthogonal to V (by the definition of $\text{proj}_V w$).
 - 2) $\text{proj}_V w - v \in V$ (since both $\text{proj}_V w$ and v are vectors in V).

This gives:

$$(w - \text{proj}_V w) \cdot (\text{proj}_V w - v) = 0$$

By the Pythagorean Theorem we obtain:

$$\|w - \text{proj}_V w\|^2 + \underbrace{\|\text{proj}_V w - v\|^2}_{0} = \underbrace{\|(w - \text{proj}_V w) + (\text{proj}_V w - v)\|^2}_{\|w - v\|^2}$$

We obtain:

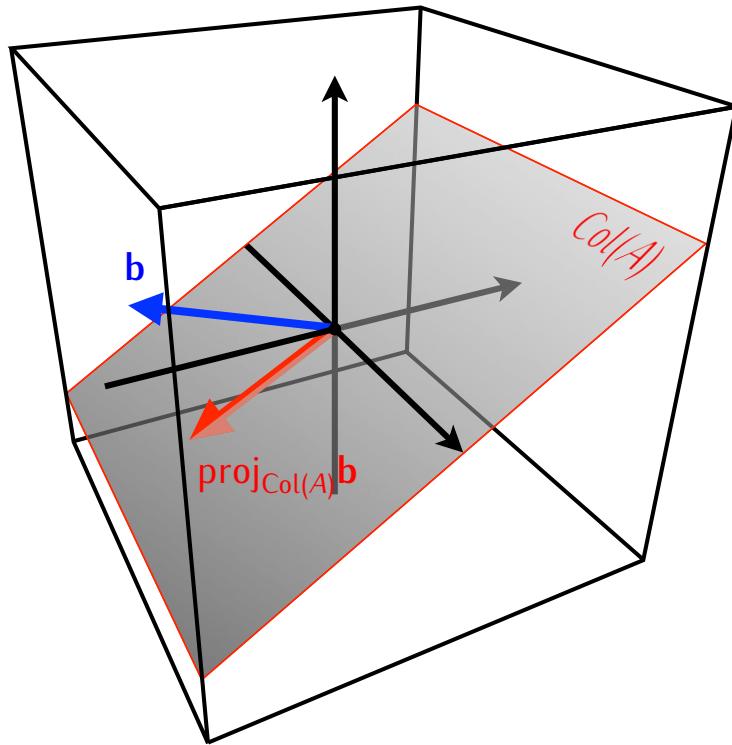
$$\|w - \text{proj}_V w\|^2 \leq \|w - v\|^2$$

$$\text{so: } \|w - \text{proj}_V w\| \leq \|w - v\|$$

Corollary

The least square solutions of a matrix equation $Ax = \mathbf{b}$ are solutions of the equation

$$A\mathbf{x} = \text{proj}_{\text{Col}(A)} \mathbf{b}$$



Next: If V is a subspace of \mathbb{R}^n and $\mathbf{w} \in \mathbb{R}^n$ how to compute $\text{proj}_V \mathbf{w}$?

Theorem

If V is a subspace of \mathbb{R}^n with an orthogonal basis $\{v_1, \dots, v_k\}$ and $w \in \mathbb{R}^n$ then

$$\text{proj}_V w = \left(\frac{w \cdot v_1}{v_1 \cdot v_1} \right) v_1 + \dots + \left(\frac{w \cdot v_k}{v_k \cdot v_k} \right) v_k$$

Proof: We need to check:

1) $\text{proj}_V w \in V$

2) $w - \text{proj}_V w$ is orthogonal to V

1) By the formula in the theorem we have

$$\text{proj}_V w \in \text{Span}(v_1, \dots, v_k) = V$$

↑
since $\{v_1, \dots, v_k\}$
is a basis of V

2) Since $V = \text{Span}(v_1, \dots, v_k)$, it suffices to check that $w - \text{proj}_V w$ is orthogonal to v_1, \dots, v_k .

E.g. for v_1 : $(w - \text{proj}_V w) \cdot v_1 = w \cdot v_1 - (\text{proj}_V w) \cdot v_1 =$
 $= w \cdot v_1 - \left[\underbrace{\left(\frac{w \cdot v_1}{v_1 \cdot v_1} \right) v_1 \cdot v_1}_{=0} + \underbrace{\left(\frac{w \cdot v_2}{v_2 \cdot v_2} \right) v_2 \cdot v_1}_{=0} + \dots + \underbrace{\left(\frac{w \cdot v_k}{v_k \cdot v_k} \right) v_k \cdot v_1}_{=0} \right] = 0$

Corollary

If V is a subspace of \mathbb{R}^n with an orthonormal basis $\{v_1, \dots, v_k\}$ and $w \in \mathbb{R}^n$ then

$$\text{proj}_V w = (w \cdot v_1) v_1 + \dots + (w \cdot v_k) v_k$$

Example. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 0 \\ -2 \end{bmatrix} \right\}, \quad w = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

The set \mathcal{B} is an orthogonal basis of some subspace V of \mathbb{R}^4 . Compute $\text{proj}_V w$.

Solution :

$$\text{proj}_V w = \left(\frac{w \cdot v_1}{v_1 \cdot v_1} \right) v_1 + \left(\frac{w \cdot v_2}{v_2 \cdot v_2} \right) v_2 + \left(\frac{w \cdot v_3}{v_3 \cdot v_3} \right) v_3$$

$$w \cdot v_1 = 1 \cdot 1 + 2 \cdot 2 + 2 \cdot 0 + 1 \cdot 3 = 8$$

$$v_1 \cdot v_1 = 1^2 + 2^2 + 0^2 + 3^2 = 14$$

$$w \cdot v_2 = 1 \cdot 2 + 2 \cdot (-4) + 2 \cdot 5 + 1 \cdot 2 = 6$$

$$v_2 \cdot v_2 = 2^2 + (-4)^2 + 5^2 + 2^2 = 49$$

$$w \cdot v_3 = 1 \cdot 4 + 2 \cdot 1 + 2 \cdot 0 + 1 \cdot (-2) = 4$$

$$v_3 \cdot v_3 = 4^2 + 1^2 + 0^2 + (-2)^2 = 21$$

This gives :

$$\text{proj}_V w = \frac{8}{14} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} + \frac{6}{49} \cdot \begin{bmatrix} 2 \\ -4 \\ 5 \\ 2 \end{bmatrix} + \frac{4}{21} \cdot \begin{bmatrix} 4 \\ 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 232/147 \\ 124/147 \\ 90/147 \\ 232/147 \end{bmatrix}$$

Note. In general if V is a subspace of \mathbb{R}^n and $w \in \mathbb{R}^n$ then in order to find $\text{proj}_V w$ we need to do the following:

- 1) find a basis of V .
- 2) use the Gram-Schmidt process to get an orthogonal basis of V
- 3) use the orthogonal basis to compute $\text{proj}_V w$.

Example. Consider the following matrix A and vector u :

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 3 & 4 & 2 \\ 2 & 6 & 3 & -1 \end{bmatrix}, \quad u = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

Compute $\text{proj}_{\text{Col}(A)} u$.

Solution:

- ① Find a basis of $\text{Col}(A)$:

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 3 & 4 & 2 \\ 2 & 6 & 3 & -1 \end{bmatrix} \xrightarrow{\text{row red}} \begin{bmatrix} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ basis of } \text{Col}(A) = \left\{ \begin{bmatrix} v_1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} v_2 \\ 4 \\ 3 \end{bmatrix} \right\}$$

- ② Use the G-S process to get an orthogonal basis of $\text{Col}(A)$:

$$w_1 = v_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad w_2 = v_2 - \frac{(w_1 \cdot v_2)}{(w_1 \cdot w_1)} w_1 = v_2 - \frac{10}{5} w_1 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} - 2 \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\{w_1, w_2\} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$$

- ③ Use the orthogonal basis to compute $\text{proj}_{\text{Col}(A)} u$:

$$\begin{aligned} \text{proj}_{\text{Col}(A)} u &= \left(\frac{u \cdot w_1}{w_1 \cdot w_1} \right) w_1 + \left(\frac{u \cdot w_2}{w_2 \cdot w_2} \right) w_2 = \frac{3}{5} w_1 + \frac{6}{6} w_2 = \frac{3}{5} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 13/5 \\ 12/5 \\ -1 \end{bmatrix} \end{aligned}$$

Example. Find least square solutions of the matrix equation $Ax = b$ where

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 90 \\ 120 \\ 72 \\ 45 \end{bmatrix}$$

Solution

Exercise:

- ① Find a basis of $\text{Col}(A)$
- ② Use the Gram-Schmidt process to get an orthogonal basis of $\text{Col}(A)$.
- ③ Use the orthogonal basis to compute $\text{proj}_{\text{Col}(A)} b$.
- ④ Solve the equation
 $Ax = \text{proj}_{\text{Col}(A)} b$.

Solutions of this equation are the least square solutions of $Ax = b$.

Next: How to simplify this procedure?