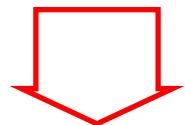
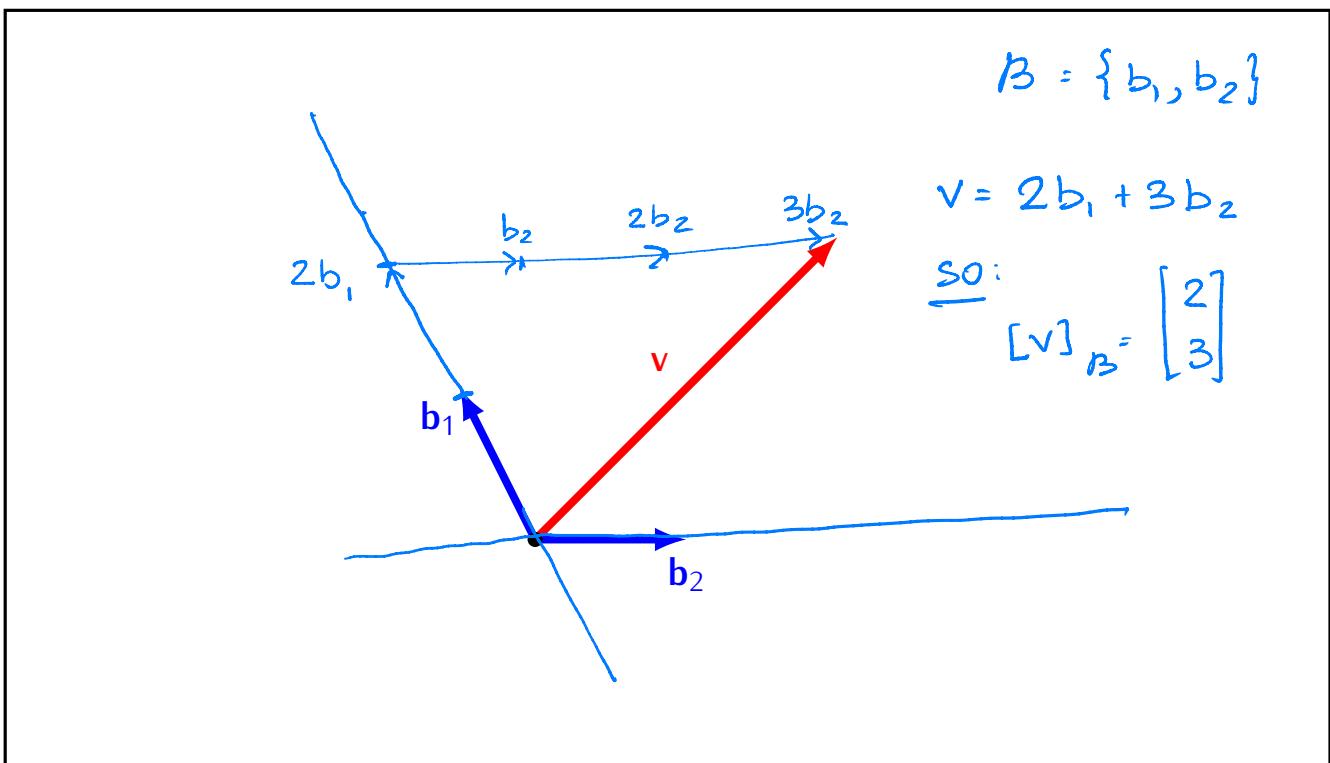


Recall: Any basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ of a vector space V defines a coordinate system:

$$\mathbf{v} = c_1 \mathbf{b}_1 + \dots + c_n \mathbf{b}_n = \mathbf{v}$$

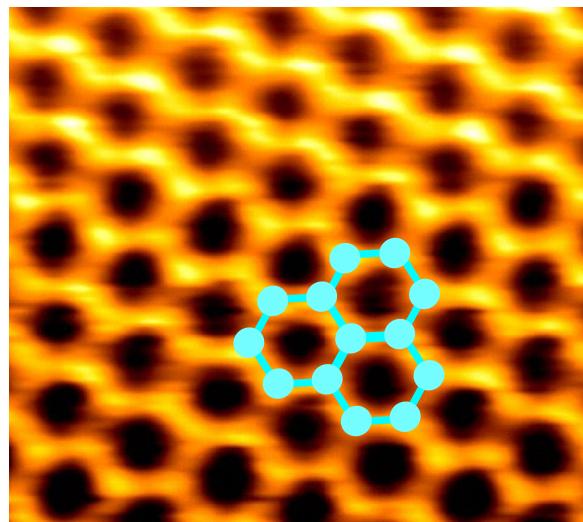
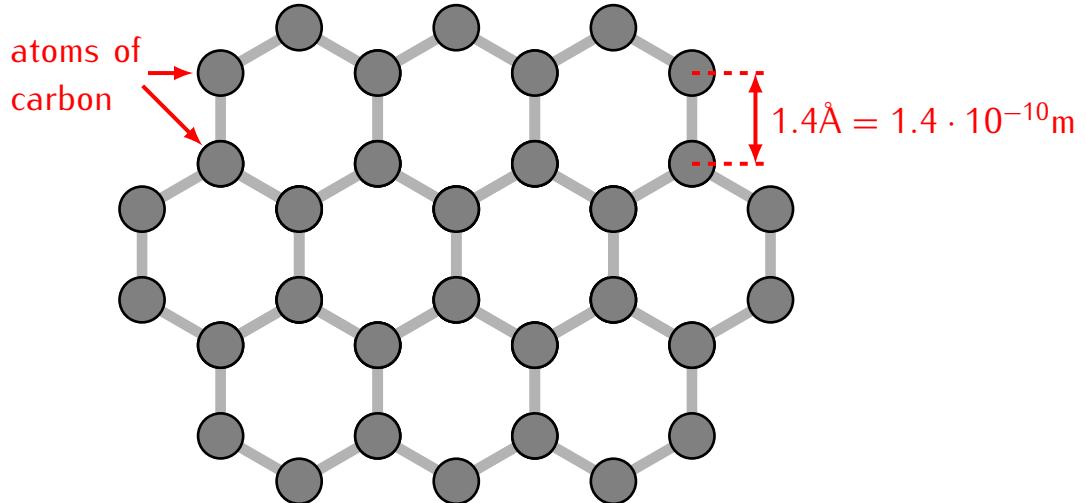


$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$



Note. Choosing a convenient basis can simplify computations.

Example. Graphene lattice.



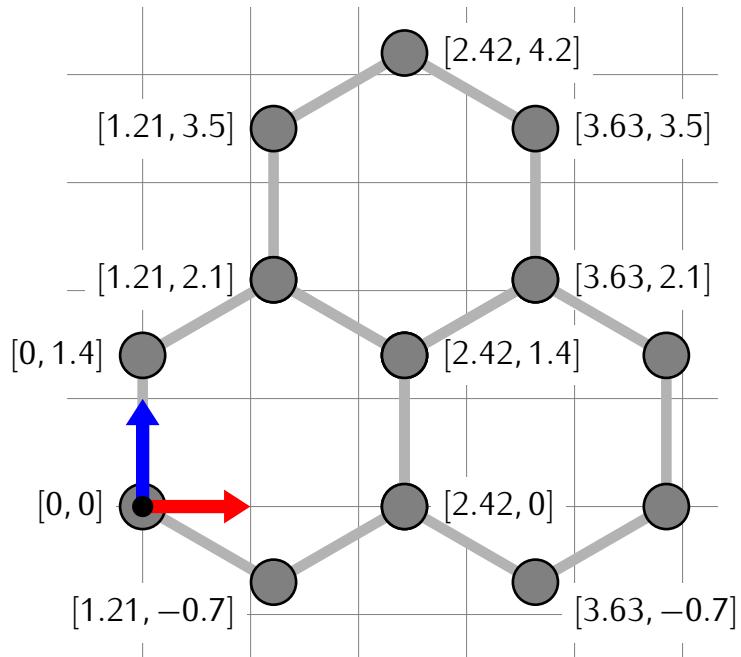
*Image of graphene taken with an atomic force microscope.
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Coordinates of atoms in the graphene lattice

In the standard basis
 $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

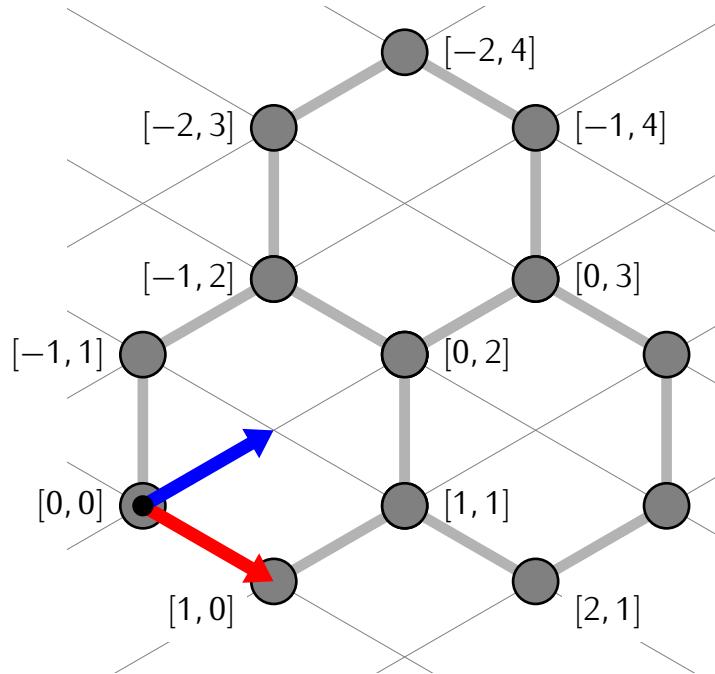
$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



In a more convenient basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$:

$$\mathbf{b}_1 = \begin{bmatrix} 1.21 \\ -0.7 \end{bmatrix}$$

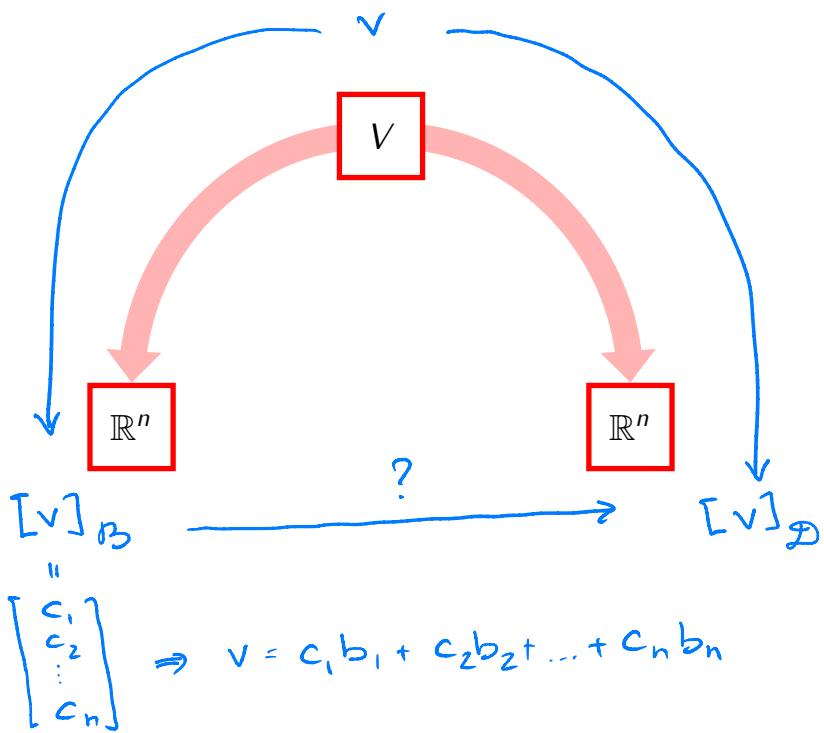
$$\mathbf{b}_2 = \begin{bmatrix} 1.21 \\ 0.7 \end{bmatrix}$$



Problem Let

$$\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}, \quad \mathcal{D} = \{\mathbf{d}_1, \dots, \mathbf{d}_n\}$$

be two bases of a vector space V , and let $\mathbf{v} \in V$. Assume that we know $[\mathbf{v}]_{\mathcal{B}}$. What is $[\mathbf{v}]_{\mathcal{D}}$?



Note: $[\mathbf{v}]_{\mathcal{D}} = [\mathbf{c}_1 \mathbf{b}_1 + \mathbf{c}_2 \mathbf{b}_2 + \dots + \mathbf{c}_n \mathbf{b}_n]_{\mathcal{D}}$

$$= \mathbf{c}_1 [\mathbf{b}_1]_{\mathcal{D}} + \mathbf{c}_2 [\mathbf{b}_2]_{\mathcal{D}} + \dots + \mathbf{c}_n [\mathbf{b}_n]_{\mathcal{D}}$$

$$= [[\mathbf{b}_1]_{\mathcal{D}} \ [\mathbf{b}_2]_{\mathcal{D}} \ \dots \ [\mathbf{b}_n]_{\mathcal{D}}] \cdot \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \vdots \\ \mathbf{c}_n \end{bmatrix}$$

We obtain:

$$[\mathbf{v}]_{\mathcal{D}} = [[\mathbf{b}_1]_{\mathcal{D}} \ [\mathbf{b}_2]_{\mathcal{D}} \ \dots \ [\mathbf{b}_n]_{\mathcal{D}}] \cdot [\mathbf{v}]_{\mathcal{B}}$$

Definition

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{D} = \{\mathbf{d}_1, \dots, \mathbf{d}_1\}$ be two bases of a vector space V . The matrix

$$P_{\mathcal{D} \leftarrow \mathcal{B}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{D}} & [\mathbf{b}_2]_{\mathcal{D}} & \cdots & [\mathbf{b}_n]_{\mathcal{D}} \end{bmatrix}$$

is called the *change of coordinates matrix* from the basis \mathcal{B} to the basis \mathcal{D} .

Propostion

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{D} = \{\mathbf{d}_1, \dots, \mathbf{d}_1\}$ be two bases of a vector space V . For any vector $\mathbf{v} \in V$ we have

$$[\mathbf{v}]_{\mathcal{D}} = P_{\mathcal{D} \leftarrow \mathcal{B}} \cdot [\mathbf{v}]_{\mathcal{B}}$$

Example. Let \mathbb{P}_2 be the vector space of polynomials of degree ≤ 2 . Consider two bases of \mathbb{P}_2 :

$$\mathcal{B} = \{1, 1+t, 1+t+t^2\}$$

$$\mathcal{D} = \{1+t, 1-5t, 2+t^2\}$$

1) Compute the change of coordinates matrix $P_{\mathcal{D} \leftarrow \mathcal{B}}$.

2) Let $p \in \mathbb{P}_2$ be a polynomial such that

$$[p]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

Compute $[p]_{\mathcal{D}}$.

Solution:

$$1) P_{\mathcal{D} \leftarrow \mathcal{B}} = [[1]_{\mathcal{D}} \ [1+t]_{\mathcal{D}} \ [1+t+t^2]_{\mathcal{D}}]$$

We have:

$$1 = \frac{5}{6}(1+t) + \frac{1}{6}(1-5t) + 0 \cdot (2+t^2)$$

$$\text{so: } [1]_{\mathcal{D}} = \begin{bmatrix} \frac{5}{6} \\ \frac{1}{6} \\ 0 \end{bmatrix}$$

$$1+t = 1 \cdot (1+t) + 0 \cdot (1-5t) + 0 \cdot (2+t^2)$$

$$\text{so: } [1+t]_{\mathcal{D}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$1+t+t^2 = -\frac{2}{3}(1+t) - \frac{1}{3}(1-5t) + 1 \cdot (2+t^2)$$

$$\text{so: } [1+t+t^2]_{\mathcal{D}} = \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

This gives:

$$P_{\mathcal{D} \leftarrow \mathcal{B}} = \begin{bmatrix} \frac{5}{6} & 1 & -\frac{2}{3} \\ \frac{1}{6} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$2) [p]_{\mathcal{D}} = P_{\mathcal{D} \leftarrow \mathcal{B}} \cdot [p]_{\mathcal{B}} = \begin{bmatrix} \frac{5}{6} & 1 & -\frac{2}{3} \\ \frac{1}{6} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{19}{6} \\ -\frac{7}{6} \\ 5 \end{bmatrix}$$

Proposition

If $\mathcal{B}, \mathcal{D}, \mathcal{E}$ are three bases of a vector space V then:

- 1) $P_{\mathcal{B} \leftarrow \mathcal{D}} = (P_{\mathcal{D} \leftarrow \mathcal{B}})^{-1}$
- 2) $P_{\mathcal{E} \leftarrow \mathcal{B}} = P_{\mathcal{E} \leftarrow \mathcal{D}} \cdot P_{\mathcal{D} \leftarrow \mathcal{B}}$