- **Inner product**

1) Inner product and orthogonality in $\mathbb{R}^n$:

- definitions of $\mathbf{u}\cdot\mathbf{v}$, $||\mathbf{v}||$, $\text{dist}(\mathbf{u},\mathbf{v})$
- orthogonality of vectors
- Pythagorean theorem
- orthogonal sets of vectors
- orthogonal bases and coordinate systems
- Gram-Schmidt process:

$$\boxed{\begin{array}{c}\text{a basis}\\ \{\mathbf{v}_1,\dots,\mathbf{v}_k\}\\ \text{of } V\subseteq\mathbb{R}^n\end{array}} \xrightarrow{\textit{G-S process}} \boxed{\begin{array}{c}\text{an orthogonal basis}\\ \{\mathbf{w}_1,\dots,\mathbf{w}_k\}\\ \text{of } V\end{array}}$$

$$\mathbf{w}_1 = \mathbf{v}_1$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \left(\frac{\mathbf{w}_1\cdot\mathbf{v}_2}{\mathbf{w}_1\cdot\mathbf{w}_1}\right)\mathbf{w}_1$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \left(\frac{\mathbf{w}_1\cdot\mathbf{v}_3}{\mathbf{w}_1\cdot\mathbf{w}_1}\right)\mathbf{w}_1 - \left(\frac{\mathbf{w}_2\cdot\mathbf{v}_3}{\mathbf{w}_2\cdot\mathbf{w}_2}\right)\mathbf{w}_2$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$\mathbf{w}_k = \mathbf{v}_k - \left(\frac{\mathbf{w}_1\cdot\mathbf{v}_k}{\mathbf{w}_1\cdot\mathbf{w}_1}\right)\mathbf{w}_1 - \left(\frac{\mathbf{w}_2\cdot\mathbf{v}_k}{\mathbf{w}_2\cdot\mathbf{w}_2}\right)\mathbf{w}_2 - \dots - \left(\frac{\mathbf{w}_{k-1}\cdot\mathbf{v}_k}{\mathbf{w}_{k-1}\cdot\mathbf{w}_{k-1}}\right)\mathbf{w}_{k-1}$$

2) Orthogonal projections:

- If V is a subspace of $\mathbb{R}^n$ with an orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and $\mathbf{w} \in \mathbb{R}^n$ then

$$\mathrm{proj}_V \mathbf{w} = \left(\frac{\mathbf{w} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1 + \ldots + \left(\frac{\mathbf{w} \cdot \mathbf{v}_k}{\mathbf{v}_k \cdot \mathbf{v}_k}\right) \mathbf{v}_k$$

- main property: $\mathrm{proj}_V \mathbf{w} \in V$, $\mathbf{w} - \mathrm{proj}_V \mathbf{w} \in V^{\perp}$
- Best Approximation Theorem:

$$\mathrm{dist}(\mathbf{w}, \mathrm{proj}_V \mathbf{w}) \leq \mathrm{dist}(\mathbf{w}, \mathbf{v})$$

  for all $\mathbf{v} \in V$.

3) Least square solutions.

- computation:

$$\begin{pmatrix} \text{least square solutions of} \\ A\mathbf{x} = \mathbf{b} \end{pmatrix} = \begin{pmatrix} \text{solutions of} \\ A\mathbf{x} = \mathrm{proj}_{\mathrm{Col}(A)} \mathbf{b} \end{pmatrix}$$

- much faster computation:

$$\begin{pmatrix} \text{least square solutions of} \\ A\mathbf{x} = \mathbf{b} \end{pmatrix} = \begin{pmatrix} \text{solutions of} \\ A^T A\mathbf{x} = A^T \mathbf{b} \end{pmatrix}$$

- application: least square fitting of lines and curves.

4) General inner product spaces.

- **Eigenvalues and eigenvectors**

1) Definition.

2) Computation:

– if A is an $n \times n$ matrix then

$$\text{eigenvalues of } A = \begin{pmatrix} \text{roots of the characteristic polynomial} \\ P(\lambda) = \det(A - \lambda I) \end{pmatrix}$$

– if λ is an eigenvalue of A then

$$(\text{the eigenspace of } A \text{ corresponding to } \lambda) = \mathrm{Nul}(A - \lambda I)$$

3) Diagonalization of matrices:

– A square matrix A is diagonalizable if there exists an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}$$

– An $n \times n$ matrix A is a diagonalizable if and only if it has n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. In such case we have:

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} \qquad \begin{array}{l} \lambda_1 = \text{eigenvalue corresponding to } \mathbf{v}_1 \\ \lambda_2 = \text{eigenvalue corresponding to } \mathbf{v}_2 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \lambda_n = \text{eigenvalue corresponding to } \mathbf{v}_n \end{array}$$

– Not every matrix is diagonalizable.

– If A is diagonalizable, $A = PDP^{-1}$ then

$$A^k = PD^kP^{-1}$$

4) Symmetric matrices and orthogonal diagonalization.

– An orthogonal matrix $Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{bmatrix}$ is a square matrix such that $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is an orthonormal set of vectors, i.e.:

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

– If Q is an orthogonal matrix then $Q^{-1} = Q^T$

– A square matrix A is orthogonally diagonalizable if there exist an orthogonal matrix Q and a diagonal matrix D such that

$$A = QDQ^{-1} = QDQ^T$$

– A matrix A is orthogonally diagonalizable if and only if A is a symmetric matrix (i.e. $A^T = A$).

– Spectral decomposition of a symmetric matrix:

$$A = \lambda_1(\mathbf{u}_1\mathbf{u}_1^T) + \lambda_2(\mathbf{u}_2\mathbf{u}_2^T) + \dots + \lambda_n(\mathbf{u}_n\mathbf{u}_n^T)$$

5) The singular value decomposition of a matrix:

$$A = U\Sigma V^T$$