Yet another view of matrix multiplication

Note. If C is an $n \times 1$ matrix and D is an $1 \times n$ matrix then CD is an $n \times n$ matrix.

Propostion

Let A be an $n \times n$ matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$, and B be an $n \times n$ matrix with rows $\mathbf{w}_1, \ldots, \mathbf{w}_n$:

$$A = \begin{bmatrix} \mathbf{v}_1 & \ldots & \mathbf{v}_n \end{bmatrix} \qquad B = \begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_n \end{bmatrix}$$

Then

$$AB = \mathbf{v}_1\mathbf{w}_1 + \mathbf{v}_2\mathbf{w}_2 + \ldots + \mathbf{v}_n\mathbf{w}_n$$

Example.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 1 \\ 7 & 2 \end{bmatrix}$$

Theorem

Let A be a symmetric matrix with orthogonal diagonalization

$$A = QDQ^T$$

If

$$Q = \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_n \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & 0 \\ 0 & \dots & \lambda_n \end{bmatrix}$$

then

$$A = \lambda_1(\mathbf{u}_1\mathbf{u}_1^T) + \lambda_2(\mathbf{u}_2\mathbf{u}_2^T) + \dots + \lambda_n(\mathbf{u}_n\mathbf{u}_n^T)$$

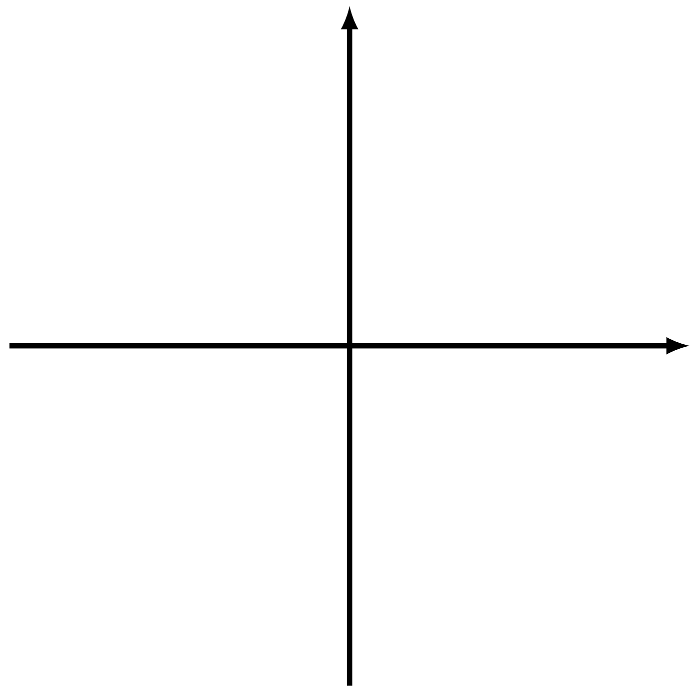
Note. The above formula is called the *spectral decomposition* of the matrix A.

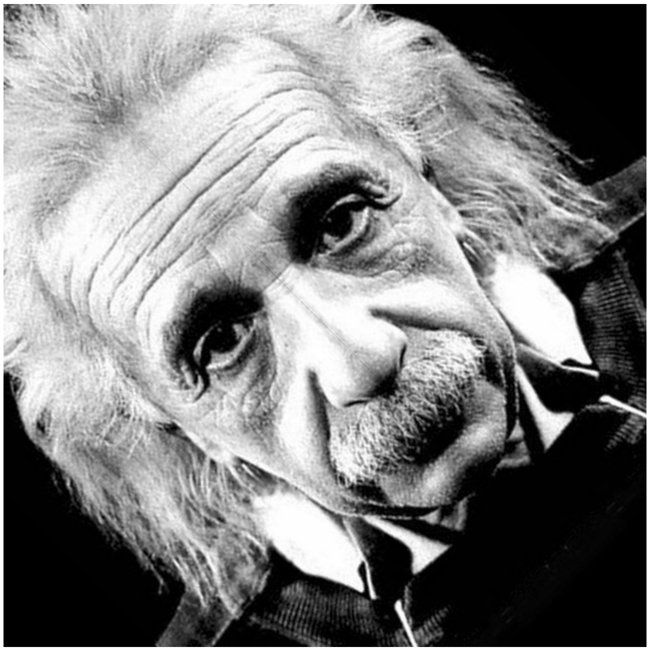
Example.

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$$

Spectral decomposition and linear transformations

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$$





- The size of this image is 1000×1000 pixels.
- The color of each pixel is represented by an integer between 0 (black) and 255 (white).
- The whole image is described by a (symmetric) matrix A consisting of $1000 \times 1000 = 1,000,000$ numbers
- Each number is stored in 1 byte, so the image file size is $1,000,000$ bytes (≈ 1 MB).

How to make the image file smaller:

1) Find the spectral decomposition of the matrix A:

$$A = \lambda_1(\mathbf{u}_1\mathbf{u}_1^T) + \lambda_2(\mathbf{u}_2\mathbf{u}_2^T) + \ldots + \lambda_{1000}(\mathbf{u}_{1000}\mathbf{u}_{1000}^T)$$

where $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_{1000}|$.

2) For $k = 1, \ldots, 1000$ define:

$$B_k = \lambda_1(\mathbf{u}_1\mathbf{u}_1^T) + \lambda_2(\mathbf{u}_2\mathbf{u}_2^T) + \ldots + \lambda_k(\mathbf{u}_k\mathbf{u}_k^T)$$

This matrix approximates the matrix A and can be stored using $k \cdot (1000 + 1)$ numbers (i.e. $k \cdot (1000 + 1)$ bytes).