Defitnition

Let V be a vector space. A *subspace* of V is a subset $W \subseteq V$ such that

1) $\mathbf{0} \in W$
2) if $\mathbf{u}, \mathbf{v} \in W$ then $\mathbf{u} + \mathbf{v} \in W$
3) if $\mathbf{u} \in W$ and $c \in \mathbb{R}$ then $c\mathbf{u} \in W$.

Example.

Recall: $\mathbb{P}$ = the vector space of all polynomials.

Proposition

Let V be a vector space and $W \subseteq V$ is a subspace then W is itself a vector space.

Example.

Recall: $\mathcal{F}(\mathbb{R})$ = the vector space of all functions $f\colon \mathbb{R} \to \mathbb{R}$

Some interesting subspaces of $\mathcal{F}(\mathbb{R})$:

1) $C(\mathbb{R})$ = the subspace of all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$

2) $C^n(\mathbb{R})$ = the subspace of all functions $f\colon \mathbb{R} \to \mathbb{R}$ that are differentiable n or more times.

3) $C^\infty(\mathbb{R})$ = the subspace of all smooth functions $f\colon \mathbb{R} \to \mathbb{R}$ (i.e. functions that have derivatives of all orders: f', f'', f''', $\dots$).

Note. If V is a vector space then:

1) the biggest subspace of V is V itself;

2) the smallest subspace of V is the subspace $\{\mathbf{0}\}$ consisting of the zero vector only;

3) if a subspace of V contains a non-zero vector, then it contains infinitely many vectors.