Theorem

Any A an $m \times n$ matrix can be written as a product

$$A = U\Sigma V^T$$

where:

- $U = \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_m \end{bmatrix}$ is an $m \times m$ orthogonal matrix.
- $V = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$ is an $n \times n$ orthogonal matrix.
- Σ is an $m \times n$ matrix of the following form:

$$\begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \sigma_n \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \sigma_m & 0 & \cdots & 0 \end{bmatrix}$$

(if $n \leq m$) (if $n \geq m$)

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$.

Note.

- The numbers $\sigma_1, \sigma_2, \ldots$ are called *singular values* of A.
- The vectors $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are called *left singular vectors* of A.
- Then the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are called *right singular vectors* of A.
- The formula $A = U\Sigma V^T$ is called a *singular value decomposition (SVD)* of A.
- The matrix Σ is uniquely determined, but U and V depend on some choices.

Theorem

Let A be a matrix with a singular value decomposition

$$A = U\Sigma V^T$$

If

$$U = \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_m \end{bmatrix} \qquad V = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$$

and $\sigma_1, \dots, \sigma_r$ are singular values of A then then

$$A = \sigma_1(\mathbf{u}_1\mathbf{v}_1^T) + \sigma_2(\mathbf{u}_2\mathbf{v}_2^T) + \dots + \sigma_r(\mathbf{u}_r\mathbf{v}_r^T)$$

Application: Image compression

- The size of this image is 800×700 pixels.
- The color of each pixel is represented by an integer between 0 (black) and 255 (white).
- The whole image is described by a matrix A consisting of $800 \times 700 = 560,000$ numbers.
- Each number is stored in 1 byte, so the image file size is $560,000$ bytes (≈ 0.53 MB).

How to make the image file smaller:

1) Compute SVD of the matrix A:

$$A = U\Sigma V^T$$

where

$$U = \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_m \end{bmatrix} \qquad V = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$$

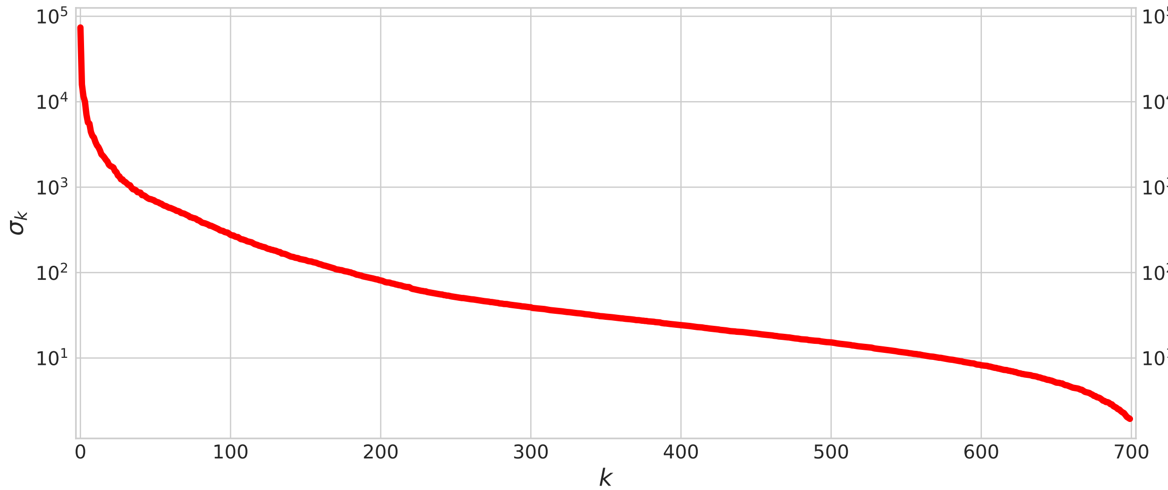
and $\sigma_1, \dots, \sigma_r$ are singular values of A.

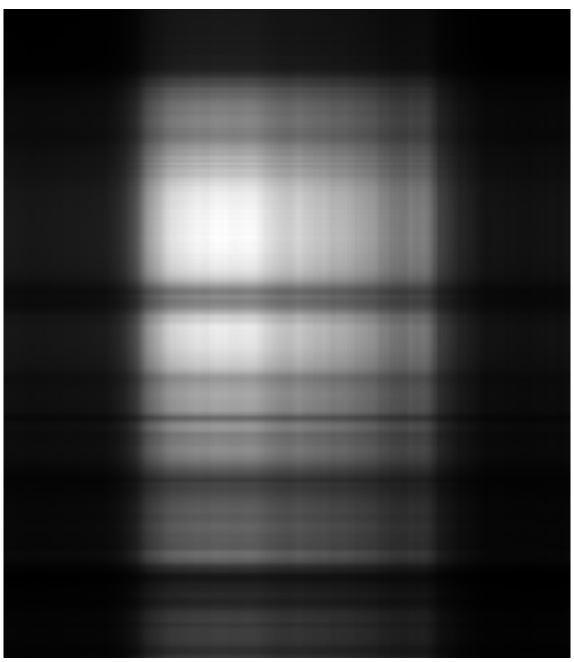
2) Replace A by the matrix

$$B_k = \sigma_1(\mathbf{u}_1\mathbf{v}_1^T) + \ldots + \sigma_k(\mathbf{u}_k\mathbf{v}_k^T)$$

for some $1 \leq k \leq 700$. This matrix can be stored using $k \cdot (800 + 700 + 1)$ numbers.

Singular values of the matrix A





matrix B_1
1501 bytes
compression 374:1

matrix B_5
7905 bytes
compression 75:1

matrix B_{10}
15,010 bytes
compression 37:1

matrix B_{20}
30,020 bytes
compression 18:1

matrix B_{50}
75,050 bytes
compression 7:1

matrix B_{100}
150,100 bytes
compression 4:1

How to compute SVD of a matrix A

How to compute SVD of a matrix A

1) Compute an orthogonal diagonalization of the symmetric $n \times n$ matrix $A^T A$:

$$A^T A = QDQ^T$$

such that eigenvalues on the diagonal of the matrix D are arranged from the largest to the smallest. We set $V = Q$.

2) If

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

then $\sigma_i = \sqrt{\lambda_i}$. This gives the matrix Σ.

Note: if $n > m$ then we use only $\lambda_1, \dots, \lambda_m$. The remaining eigenvalues $\lambda_{m+1}, \dots, \lambda_n$ of D will be equal to 0 in this case.

3) Let $V = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$, and let $\sigma_1, \dots, \sigma_r$ be non-zero singular values of A. The first r columns of the matrix $U = \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_m \end{bmatrix}$ are given by

$$\mathbf{u}_i = \frac{1}{\sigma_i} A\mathbf{v}_i$$

The remaining columns $\mathbf{u}_{r+1}, \dots, \mathbf{u}_m$ can be added arbitrarily so that U is an orthogonal matrix (i.e.$\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$) is an orthonormal basis of $\mathbb{R}^m$.

Example. Find SVD of the following matrix:

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$