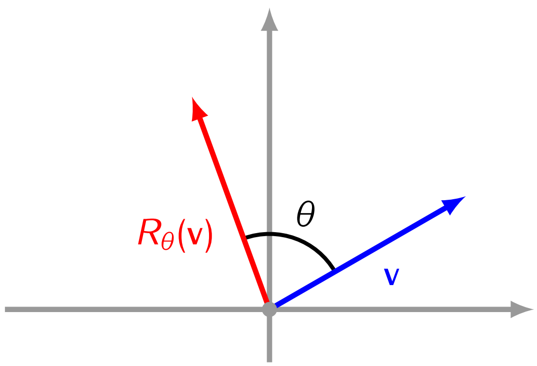
Proposition

Let A be an $m \times n$ matrix. If the matrix transformation $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$ is both onto and one-to-one then we must have $m = n$ (i.e. A must be a square matrix).

Problem: How to recognize if a function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation?

Example. Rotation by an angle θ:



$$R_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$$

Definition

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a *linear transformation* if it satisfies the following conditions:

1) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$
2) $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $\mathbf{v} \in \mathbb{R}^n$ and any scalar c.

Proposition

Every matrix transformation is a linear transformation.

Theorem

Every linear transformation $T \colon \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation:

$$T = T_A$$

for some matrix A.

Corollary

If $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then $T = T_A$ where A is the matrix given by

$$A = \left[\begin{array}{cccc} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \end{array}\right]$$

This matrix is called the *standard matrix* of T.

Example. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the function given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 0 \\ 2x_1 \end{bmatrix}$$

Check if T is a linear transformation. If it is, find its standard matrix.

Example. Let $S\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the function given by

$$S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 + x_2 \\ x_2 \\ 3x_1 \end{bmatrix}$$

Check if S is a linear transformation. If it is, find its standard matrix.