Recall:

- ullet A basis of a vector space V is a set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ such that
 - 1) Span($\mathbf{b}_1, \ldots, \mathbf{b}_n$) = V
 - 2) The set $\{b_1, \ldots, b_n\}$ is linearly independent.

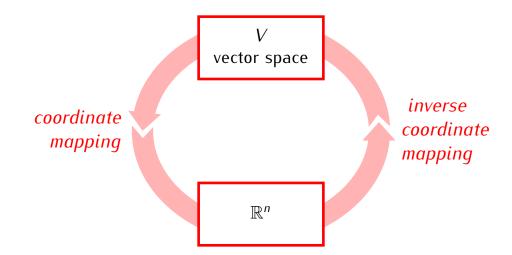
• For $v \in V$ let c_1, \ldots, c_n be the unique numbers such that

$$c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n = \mathbf{v}$$

The vector

$$\begin{bmatrix} \mathbf{v} \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$$

is called the *coordinate vector of* v *relative to the basis* \mathcal{B} .



Let \mathcal{B} be a basis of a vector space V. If $\mathbf{v}_1, \dots \mathbf{v}_p, \mathbf{w} \in V$ then:

- 1) Solutions of the equation $x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{w}$ are the same as solutions of the equation $x_1[\mathbf{v}_1]_{\mathcal{B}} + \ldots + x_p[\mathbf{v}_p]_{\mathcal{B}} = [\mathbf{w}]_{\mathcal{B}}$.
- 2) The set of vectors $\{v_1, \dots v_p\}$ is linearly independent if and only if the set $\{[v_1]_{\mathcal{B}}, \dots, [v_p]_{\mathcal{B}}\}$ is linearly independent.
- 3) Span $(\mathbf{v}_1, \dots, \mathbf{v}_p) = V$ if any only if Span $([\mathbf{v}_1]_{\mathcal{B}}, \dots, [\mathbf{v}_p]_{\mathcal{B}}) = \mathbb{R}^n$.
- 4) $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a basis of V if and only if $\{[\mathbf{v}_1]_{\mathcal{B}}, \dots, [\mathbf{v}_p]_{\mathcal{B}}\}$ is a basis of \mathbb{R}^n .

Example. Recall that \mathbb{P}_2 is the vector space of polynomials of degree ≤ 2 . Consider the following polynomials in \mathbb{P}_2 :

$$p_1(t) = 1 + 2t + t^2$$

$$p_2(t) = 3 + t + 2t^2$$

$$p_3(t) = 1 - 8t - t^2$$

Check if the set $\{p_1, p_2, p_3\}$ is linearly independent.

Let $\{\mathbf v_1,\ldots,\mathbf v_p\}$ be vectors in $\mathbb R^n$. The set $\{\mathbf v_1,\ldots,\mathbf v_p\}$ is a basis of $\mathbb R^n$ if and only if the matrix

$$A = [\mathbf{v}_1 \dots \mathbf{v}_p]$$

has a pivot position in every row and in every column (i.e. if A is an invertible matrix).

Corollary

Every basis of \mathbb{R}^n consists of n vectors.

Let V be a vector space. If V has a basis consisting of n vectors then every basis of V consists of n vectors.

Definition

A vector space has dimension n if V has a basis consisting of n vectors. Then we write dim V=n.

Example.

Let V be a vector space such that dim V = n, and let $\mathbf{v}_1, \dots \mathbf{v}_p \in V$.

- 1) If $\{v_1, \ldots, v_p\}$ is a spanning set of V then $p \ge n$.
- 2) If $\{v_1, \ldots, v_p\}$ is a linearly independent set then $p \leq n$.

Corollary

Let V be a vector space such that $\dim V = n$. If W be a subspace of V then $\dim W \leq n$. Moreover, if $\dim W = n$ then W = V.

Note.

- 1) One can show that every vector space has a basis.
- 2) Some vector spaces have bases consisting of infinitely many vectors. If V is such vector space then we write dim $V=\infty$.

Example.