Definition

A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ in $\mathbb{R}^n$ is an *orthogonal set* if each pair each pair of vectors in this set is orthogonal, i.e.

$$\mathbf{v}_i \cdot \mathbf{v}_j = 0$$

for all $i \neq j$.

Example.

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is an orthogonal set in $\mathbb{R}^3$.

Example.

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix} \right\}$ is another orthogonal set in $\mathbb{R}^3$.

Proposition

If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal set of non-zero vectors in $\mathbb{R}^n$ then this set is linearly independent.

Recall: Any linearly independent set of n vectors in $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$.

Corollary

If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthogonal set of n non-zero vectors in $\mathbb{R}^n$ then this set is a basis of $\mathbb{R}^n$.

Definition

If V is a subspace of $\mathbb{R}^n$ then we say that a set $\{\mathbf{v}_1, \dots \mathbf{v}_k\}$ is an *orthogonal basis* of V if

1) $\{\mathbf{v}_1, \dots \mathbf{v}_k\}$ is a basis of V and
2) $\{\mathbf{v}_1, \dots \mathbf{v}_k\}$ is an orthogonal set.

Recall. If $\mathcal{B} = \{\mathbf{v}_1, \dots \mathbf{v}_k\}$ is a basis of a vector space V and $\mathbf{w} \in V$ then the coordinate vector of $\mathbf{w}$ relative to $\mathcal{B}$ is the vector

$$[\ \mathbf{w}\ ]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

where $c_1, \dots, c_k$ are scalars such that $c_1\mathbf{v}_1 + \dots + c_k\mathbf{v}_k = \mathbf{w}$.

Propostion

If $\mathcal{B} = \{\mathbf{v}_1, \dots \mathbf{v}_k\}$ is an orthogonal basis of V and $\mathbf{w} \in V$ then

$$[\ \mathbf{w}\ ]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

where $c_i = \dfrac{\mathbf{w} \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i} = \dfrac{\mathbf{w} \cdot \mathbf{v}_i}{||\mathbf{v}_i||^2}$

Example. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix} \right\}, \qquad \mathbf{w} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

The set $\mathcal{B}$ is an orthogonal basis of $\mathbb{R}^3$. Compute $\left[\ \mathbf{w}\ \right]_{\mathcal{B}}$.

Theorem (Gram-Schmidt Process)

Let $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a basis of V. Define vectors $\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ as follows:

$$\begin{aligned}
\mathbf{w}_1 &= \mathbf{v}_1 \\
\mathbf{w}_2 &= \mathbf{v}_2 - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_2}{\mathbf{w}_1 \cdot \mathbf{w}_1}\right)\mathbf{w}_1 \\
\mathbf{w}_3 &= \mathbf{v}_3 - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_3}{\mathbf{w}_1 \cdot \mathbf{w}_1}\right)\mathbf{w}_1 - \left(\frac{\mathbf{w}_2 \cdot \mathbf{v}_3}{\mathbf{w}_2 \cdot \mathbf{w}_2}\right)\mathbf{w}_2 \\
\dots &\quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
\mathbf{w}_k &= \mathbf{v}_k - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_k}{\mathbf{w}_1 \cdot \mathbf{w}_1}\right)\mathbf{w}_1 - \left(\frac{\mathbf{w}_2 \cdot \mathbf{v}_k}{\mathbf{w}_2 \cdot \mathbf{w}_2}\right)\mathbf{w}_2 - \dots - \left(\frac{\mathbf{w}_{k-1} \cdot \mathbf{v}_k}{\mathbf{w}_{k-1} \cdot \mathbf{w}_{k-1}}\right)\mathbf{w}_{k-1}
\end{aligned}$$

Then the set $\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ is an orthogonal basis of V.

Example. In $\mathbb{R}^4$ take

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 7 \\ 4 \\ 3 \\ -3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 7 \\ 7 \\ 8 \end{bmatrix}$$

The set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of some subspace $V \subseteq \mathbb{R}^4$. Find an orthogonal basis of V.

Definition

An orthogonal basis $\mathcal{B} = \{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ of V is called an *orthonormal basis* if $||\mathbf{w}_i|| = 1$ for $i = 1, \ldots, k$.

Propostion

If $\mathcal{B} = \{\mathbf{v}_1, \ldots \mathbf{v}_k\}$ is an orthonormal basis of V and $\mathbf{w} \in V$ then

$$[\ \mathbf{w}\ ]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

where $c_i = \mathbf{w} \cdot \mathbf{v}_i$.

Note. If $\mathcal{B} = \{\mathbf{v}_1, \ldots \mathbf{v}_k\}$ is an orthogonal basis of V then

$$\mathcal{C} = \left\{ \frac{\mathbf{v}_1}{||\mathbf{v}_1||}, \ldots, \frac{\mathbf{v}_k}{||\mathbf{v}_k||} \right\}$$

is an orthonormal basis of V.