

Recall:

1) If A is an $m \times n$ matrix then the function

$$T_A: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

defined by $T_A(\mathbf{v}) = A\mathbf{v}$ is called the matrix transformation associated to A .

2) A function $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation if

$$(i) \quad T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

$$(ii) \quad T(c\mathbf{v}) = cT(\mathbf{v})$$

3) Every matrix transformation is a linear transformation.

4) Every linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a matrix transformation:

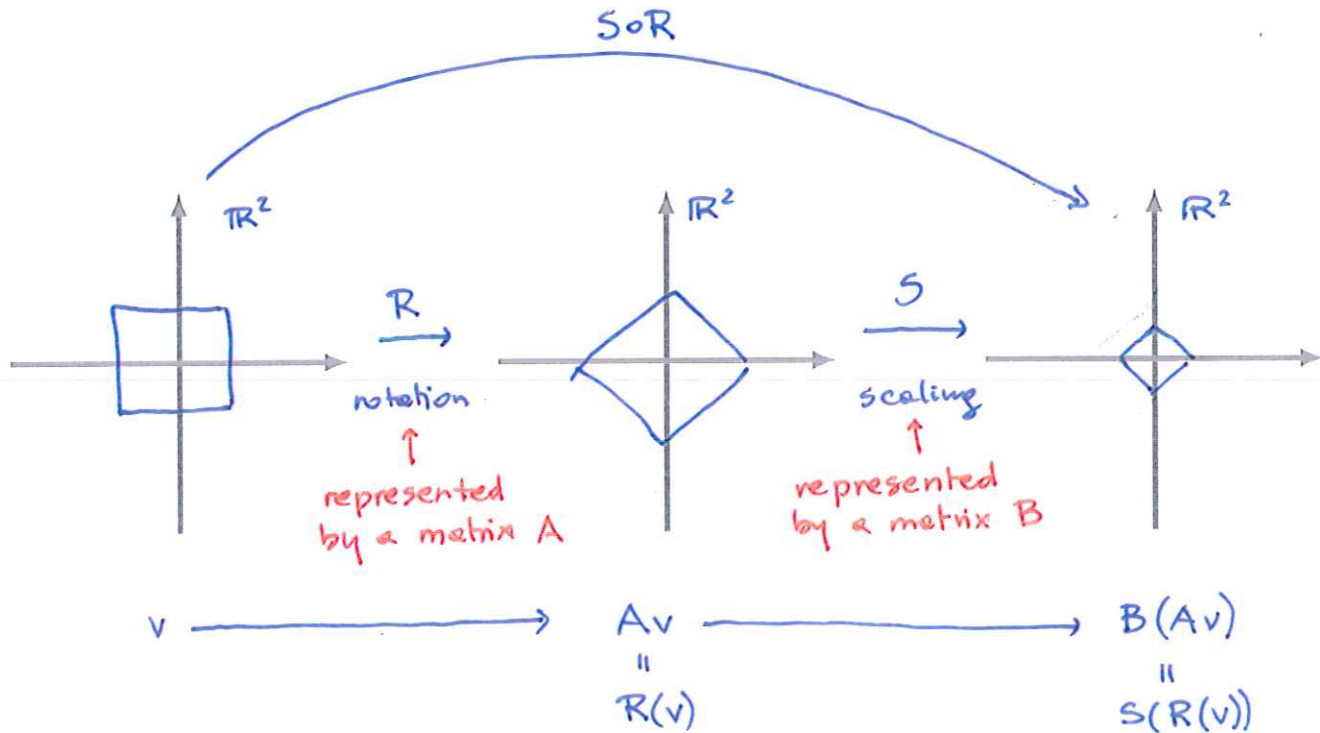
$$T(\mathbf{v}) = A\mathbf{v}$$

where

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \dots \quad T(\mathbf{e}_n)]$$

The matrix A is called the standard matrix of T .

Composition of linear transformations



Question:

Is there a matrix C such that

$$SoR(v) = Cv$$

(or equivalently: $Cv = B(Av)$) ?

Theorem

If $S: \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $T: \mathbb{R}^m \rightarrow \mathbb{R}^k$ are linear transformation then the composition

$$T \circ S: \mathbb{R}^n \rightarrow \mathbb{R}^k$$

is also a linear transformation.

Proof: We need to check:

- 1) $T \circ S(u+v) = T \circ S(u) + T \circ S(v)$
- 2) $T \circ S(cu) = c \cdot (T \circ S(u))$

$$\begin{aligned} 1) \quad T \circ S(u+v) &= T(S(u+v)) = T(S(u) + S(v)) \\ &\quad \uparrow \\ &\quad \text{since } S \text{ is linear} \\ &= T(S(u)) + T(S(v)) \\ &\quad \uparrow \\ &\quad \text{since } T \text{ is linear} \\ &= T \circ S(u) + T \circ S(v) \end{aligned}$$

2) similar

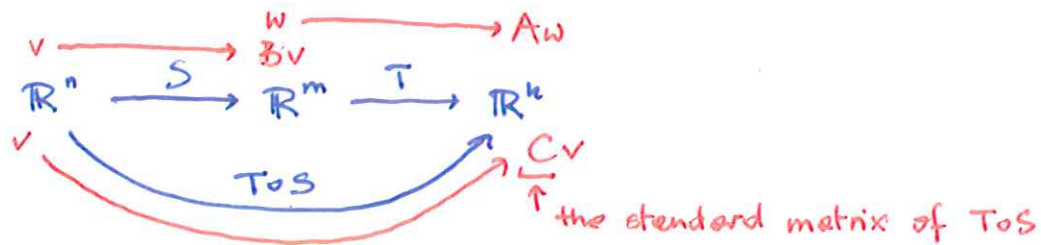
Upshot. The function $T \circ S$ is represented by some matrix C :

$$T \circ S(v) = Cv$$

Question. Let $S: \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $T: \mathbb{R}^m \rightarrow \mathbb{R}^k$ be linear transformations, and let

- B is the standard matrix of S
- A is the standard matrix of T

What if the standard matrix of $T \circ S: \mathbb{R}^n \rightarrow \mathbb{R}^k$?



We have:

$$\begin{aligned} C &= [T \circ S(e_1) \quad T \circ S(e_2) \quad \dots \quad T \circ S(e_n)] \\ &= [T(S(e_1)) \quad T(S(e_2)) \quad \dots \quad T(S(e_n))] \\ &= [A(B e_1) \quad A(B e_2) \quad \dots \quad A(B e_n)] \end{aligned}$$

Note: If $B = [v_1 \ v_2 \ \dots \ v_n]$ then $B e_1 = [v_1 \ v_2 \ \dots \ v_n] \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = v_1$

In the same way: $B e_2 = v_2, \dots, B e_n = v_n$

We get:

$$C = [A v_1 \ A v_2 \ \dots \ A v_n]$$

where v_1, v_2, \dots, v_n - columns of B .

Definition

Let

- A be an $k \times m$ matrix
- $B = [v_1 \ v_2 \ \dots \ v_n]$ be an $m \times n$ matrix

Then $A \cdot B$ is an $k \times n$ matrix given by

$$A \cdot B = [Av_1 \ Av_2 \ \dots \ Av_n]$$

Note: If $S: \mathbb{R}^n \rightarrow \mathbb{R}^m$, $T: \mathbb{R}^m \rightarrow \mathbb{R}^k$ are linear transformations

$B =$ the standard matrix of S

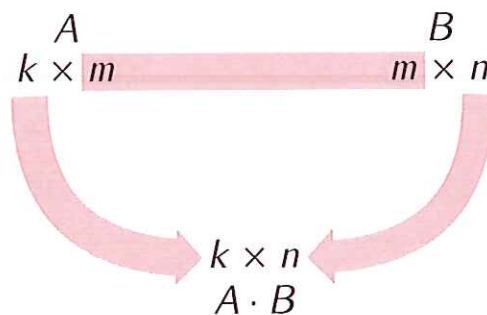
$A =$ the standard matrix of T

then AB is the standard matrix of $T \circ S$:

$$T \circ S(v) = (AB)v$$

Note. The product $A \cdot B$ is defined only if

$$(\text{number of columns of } A) = (\text{number of rows of } B)$$



Example.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -1 & 2 & 1 \\ 4 & 5 & 1 & 0 \\ 1 & 2 & 3 & 1 \end{bmatrix} = [v_1 \ v_2 \ v_3 \ v_4]$$

2×3 3×4
 \uparrow \uparrow
AB is defined
and it is a 2×4 matrix

$$AB = [Av_1 \ Av_2 \ Av_3 \ Av_4]$$

$$Av_1 = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} = 0 \cdot \begin{bmatrix} 0 \\ 3 \end{bmatrix} + 4 \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 1 \cdot \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 21 \end{bmatrix}$$

$$Av_2 = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix} = (-1) \begin{bmatrix} 0 \\ 3 \end{bmatrix} + 5 \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 2 \cdot \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 9 \\ 27 \end{bmatrix}$$

$$Av_3 = \dots = \begin{bmatrix} 7 \\ 25 \end{bmatrix}$$

$$Av_4 = \dots = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

We obtain:

$$AB = \begin{bmatrix} 6 & 9 & 7 & 2 \\ 21 & 27 & 25 & 8 \end{bmatrix}$$