### Recall:

1) A square matrix A is diagonalizable if there exists an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}$$

**2)** If *A* is diagonalizable then it is easy to compute powers of *A*:

$$A^k = PD^k P^{-1}$$

3) An  $n \times n$  matrix A is a diagonalizable if and only if it has n linearly independent eigenvectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ . In such case we have:

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \qquad \begin{array}{l} \lambda_1 = \text{eigenvalue corresponding to } \mathbf{v}_1 \\ \lambda_2 = \text{eigenvalue corresponding to } \mathbf{v}_2 \\ \dots & \dots & \dots \\ \lambda_n = \text{eigenvalue corresponding to } \mathbf{v}_n \end{array}$$

4) Not every square matrix is diagonalizable.

# Definition

A square matrix A is symmetric if  $A^T = A$ 

# Theorem

Every symmetric matrix is diagonalizable.

#### **Theorem**

If A is a symmetric matrix and  $\lambda_1, \lambda_2$  are two different eigenvalues of A, then eigenvectors corresponding to  $\lambda_1$  are orthogonal to eigenvectors corresponding to  $\lambda_2$ .

**Note.** If  $\mathbf{v}$ ,  $\mathbf{w}$  are vectors in  $\mathbb{R}^n$  then

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w}$$

Example.

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

### Theorem

If A is an  $n \times n$  symmetric matrix then A has n orthogonal eigenvectors.

## Example.

a) Find three orthogonal eigenvectors of the following symmetric matrix:

$$A = \left[ \begin{array}{ccc} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{array} \right]$$

**b)** Use these eigenvectors to diagonalize this matrix.

<u>Upshot.</u> How to find $n$ orthogonal eigenvectors for a symmetric $n \times n$ matrix $A$ :
1) Find eigenvalues of A.
<ul><li>2) Find a basis of the eigenspace for each eigenvalue.</li><li>3) Use the Gram-Schmidt process to find an orthogonal basis of each eigenspace.</li></ul>
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## Definition

A square matrix  $Q = [\mathbf{u}_1 \ \mathbf{u}_2 \ \dots \ \mathbf{u}_n]$  is an *orthogonal matrix* if  $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$  is an orthonormal set of vectors, i.e.:

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

### **Theorem**

If Q is an orthogonal matrix then Q is invertible and  $Q^{-1} = Q^{T}$ .

Note. If  $P = [v_1 \ v_2 \ \dots \ v_n]$  is a matrix with orthogonal columns, then

$$Q = \left[ \begin{array}{ccc} \mathbf{v}_1 & \mathbf{v}_2 \\ \hline ||\mathbf{v}_1|| & \overline{||\mathbf{v}_2||} & \cdots & \overline{||\mathbf{v}_n||} \end{array} \right]$$

is an orthogonal matrix.

#### **Theorem**

If A is a symmetric matrix then A is orthogonally diagonalizable. That is, there exists an orthogonal matrix Q and a diagonal matrix D such that

$$A = QDQ^{-1} = QDQ^{T}$$

**Example.** Find an orthogonal diagonalization of the following symmetric matrix:

$$A = \left[ \begin{array}{ccc} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{array} \right]$$

**Note.** We have seen that any symmetric matrix is orthogonally diagonalizable. The converse statement is also true:

# Proposition

If a matrix A is orthogonally diagonalizable then A is a symmetric matrix.