

Recall:

1) If

$$\mathbf{u} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

are vectors in  $\mathbb{R}^n$  then:

- $\mathbf{u} \cdot \mathbf{v} = a_1 b_1 + \dots + a_n b_n$
- $\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$
- $\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$

2) Vectors  $\mathbf{u}, \mathbf{v}$  are orthogonal if  $\mathbf{u} \cdot \mathbf{v} = 0$ .3) Pythagorean theorem:  $\mathbf{u}, \mathbf{v}$  are orthogonal if and only if

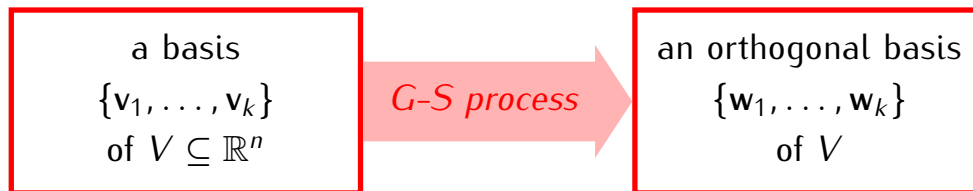
$$\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u} + \mathbf{v}\|^2$$

4) If  $V \subseteq \mathbb{R}^n$  is a subspace then an orthogonal basis of  $V$  is a basis which consists of vectors that are orthogonal to one another.5) If  $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is an orthogonal basis of  $V$  and  $\mathbf{w} \in V$  then

$$[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

where  $c_i = \frac{\mathbf{w} \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i}$ .

6) Gram-Schmidt process:



$$w_1 = v_1$$

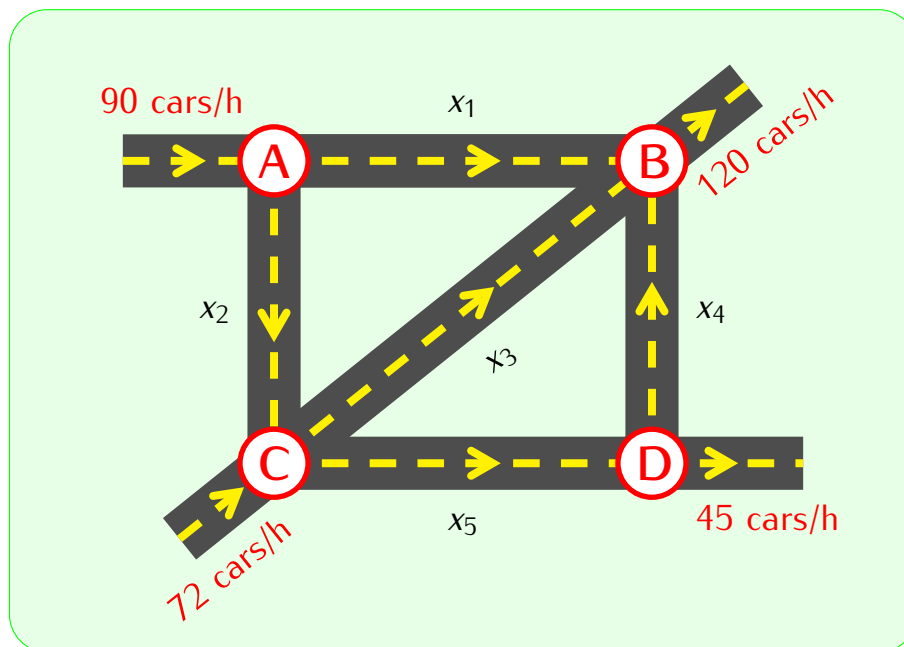
$$w_2 = v_2 - \left( \frac{w_1 \cdot v_2}{w_1 \cdot w_1} \right) w_1$$

$$w_3 = v_3 - \left( \frac{w_1 \cdot v_3}{w_1 \cdot w_1} \right) w_1 - \left( \frac{w_2 \cdot v_3}{w_2 \cdot w_2} \right) w_2$$

... ..

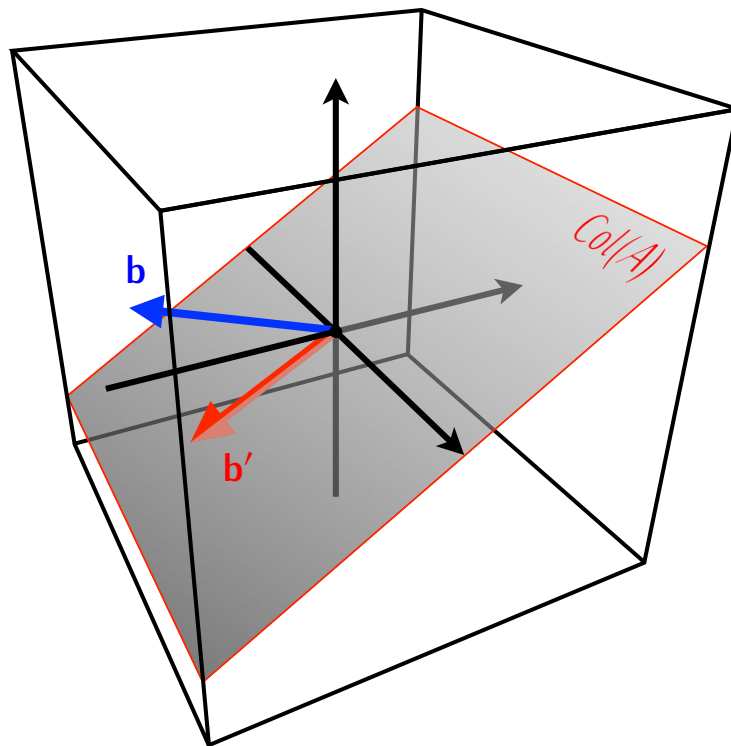
$$w_k = v_k - \left( \frac{w_1 \cdot v_k}{w_1 \cdot w_1} \right) w_1 - \left( \frac{w_2 \cdot v_k}{w_2 \cdot w_2} \right) w_2 - \dots - \left( \frac{w_{k-1} \cdot v_k}{w_{k-1} \cdot w_{k-1}} \right) w_{k-1}$$

**Problem.** Find the flow rate of cars on each segment of streets:



## Upshot.

- Recall: a matrix equation  $Ax = \mathbf{b}$  has a solution if and only if  $\mathbf{b} \in \text{Col}(A)$ .
- In practical applications we may obtain a matrix equation that has no solutions, i.e. where  $\mathbf{b} \notin \text{Col}(A)$ .
- In such cases we may look for approximate solutions as follows:
  - replace  $\mathbf{b}$  by a vector  $\mathbf{b}'$  such that  $\mathbf{b}' \in \text{Col}(A)$  and  $\text{dist}(\mathbf{b}, \mathbf{b}')$  is as small as possible.
  - then solve  $Ax = \mathbf{b}'$



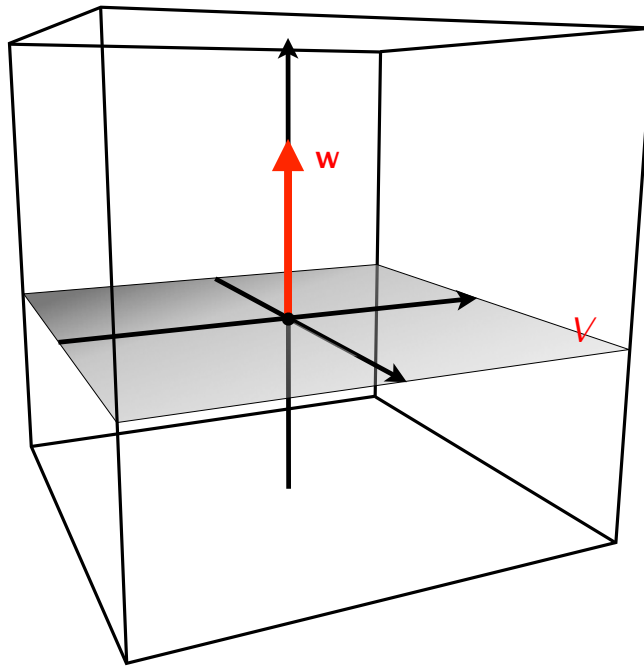
### Definition

Given  $\mathbf{b}' \in \text{Col}(A)$  as above we will say that a vector  $\mathbf{v}$  is a *least square solution* of the equation  $Ax = \mathbf{b}$  if  $\mathbf{v}$  is a solution of the equation  $Ax = \mathbf{b}'$ .

Next: How to find the vector  $\mathbf{b}'$ ?

### Definition

Let  $V$  be a subspace of  $\mathbb{R}^n$ . A vector  $\mathbf{w} \in \mathbb{R}^n$  is *orthogonal to  $V$*  if  $\mathbf{w} \cdot \mathbf{v} = 0$  for all  $\mathbf{v} \in V$ .



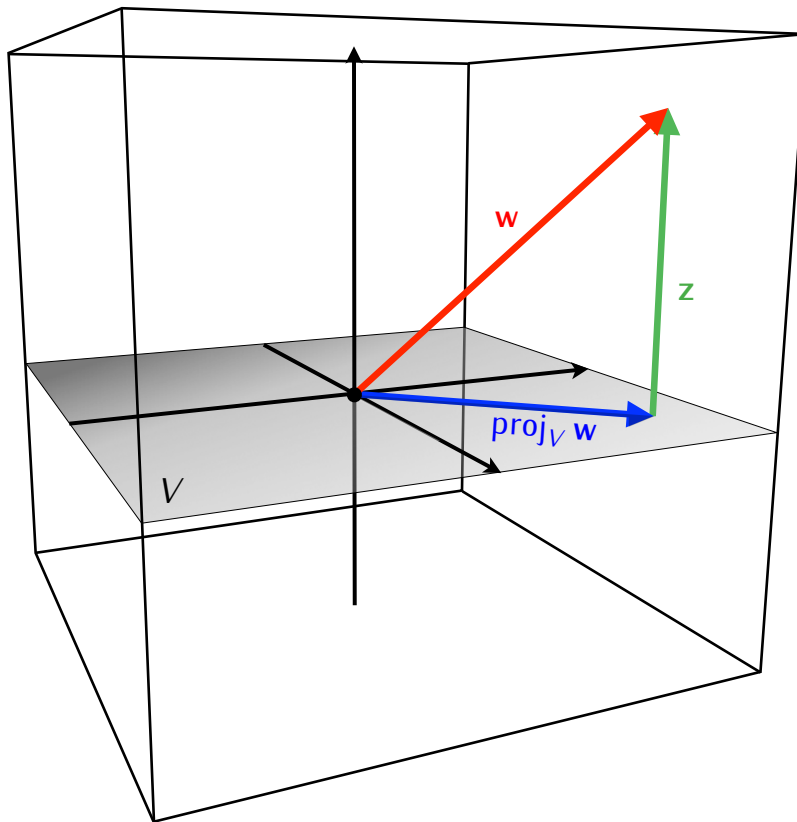
### Proposition

If  $V = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$  then a vector  $\mathbf{w} \in \mathbb{R}^n$  is orthogonal to  $V$  if and only if  $\mathbf{w} \cdot \mathbf{v}_i = 0$  for  $i = 1, \dots, k$ .

## Definition

Let  $V$  be a subspace of  $\mathbb{R}^n$  and let  $\mathbf{w} \in \mathbb{R}^n$  the *orthogonal projection of  $\mathbf{w}$  onto  $V$*  is a vector  $\text{proj}_V \mathbf{w}$  such that

- 1)  $\text{proj}_V \mathbf{w} \in V$
- 2) the vector  $\mathbf{z} = \mathbf{w} - \text{proj}_V \mathbf{w}$  is orthogonal to  $V$ .



### The Best Approximation Theorem

If  $V$  is a subspace of  $\mathbb{R}^n$  and  $\mathbf{w} \in \mathbb{R}^n$  then  $\text{proj}_V \mathbf{w}$  is a vector in  $V$  which is closest to  $\mathbf{w}$ :

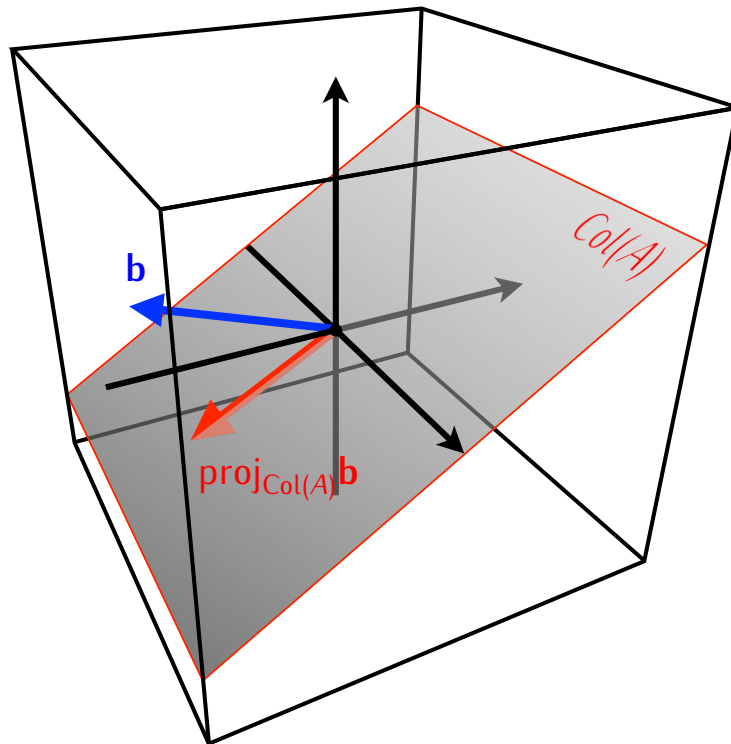
$$\text{dist}(\mathbf{w}, \text{proj}_V \mathbf{w}) \leq \text{dist}(\mathbf{w}, \mathbf{v})$$

for all  $\mathbf{v} \in V$ .

### Corollary

The least square solutions of a matrix equation  $Ax = \mathbf{b}$  are solutions of the equation

$$Ax = \text{proj}_{\text{Col}(A)} \mathbf{b}$$



Next: If  $V$  is a subspace of  $\mathbb{R}^n$  and  $\mathbf{w} \in \mathbb{R}^n$  how to compute  $\text{proj}_V \mathbf{w}$ ?



### Theorem

If  $V$  is a subspace of  $\mathbb{R}^n$  with an orthogonal basis  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  and  $\mathbf{w} \in \mathbb{R}^n$  then

$$\text{proj}_V \mathbf{w} = \left( \frac{\mathbf{w} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 + \dots + \left( \frac{\mathbf{w} \cdot \mathbf{v}_k}{\mathbf{v}_k \cdot \mathbf{v}_k} \right) \mathbf{v}_k$$

### Corollary

If  $V$  is a subspace of  $\mathbb{R}^n$  with an orthonormal basis  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  and  $\mathbf{w} \in \mathbb{R}^n$  then

$$\text{proj}_V \mathbf{w} = (\mathbf{w} \cdot \mathbf{v}_1) \mathbf{v}_1 + \dots + (\mathbf{w} \cdot \mathbf{v}_k) \mathbf{v}_k$$

**Example.** Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 0 \\ -2 \end{bmatrix} \right\}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

The set  $\mathcal{B}$  is an orthogonal basis of some subspace  $V$  of  $\mathbb{R}^4$ . Compute  $\text{proj}_V \mathbf{w}$ .

**Note.** In general if  $V$  is a subspace of  $\mathbb{R}^n$  and  $\mathbf{w} \in \mathbb{R}^n$  then in order to find  $\text{proj}_V \mathbf{w}$  we need to do the following:

- 1) find a basis of  $V$ .
- 2) use the Gram-Schmidt process to get an orthogonal basis of  $V$
- 3) use the orthogonal basis to compute  $\text{proj}_V \mathbf{w}$ .

**Example.** Consider the following matrix  $A$  and vector  $\mathbf{u}$ :

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 3 & 4 & 2 \\ 2 & 6 & 3 & -1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

Compute  $\text{proj}_{\text{Col}(A)} \mathbf{u}$ .

**Example.** Find least square solutions of the matrix equation  $A\mathbf{x} = \mathbf{b}$  where

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 90 \\ 120 \\ 72 \\ 45 \end{bmatrix}$$