Definition

Let V, W be vector spaces A *linear transformation* is a function

$$T\colon V \to W$$

which satisfies the following conditions:

1) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$
2) $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $\mathbf{v} \in V$ and any scalar c.

Example:

If A is an $m\times n$ matrix then it defines a linear transformation

$$T_A\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$v \longmapsto Av$$

Example:

Recall: $C^\infty(\mathbb{R}) = \left\{ \begin{array}{l} \text{the vector space of} \\ \text{all smooth functions } f\colon \mathbb{R}\to\mathbb{R} \end{array} \right\}$

Take: $D\colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $D(f) = \underbrace{f'}_{}$

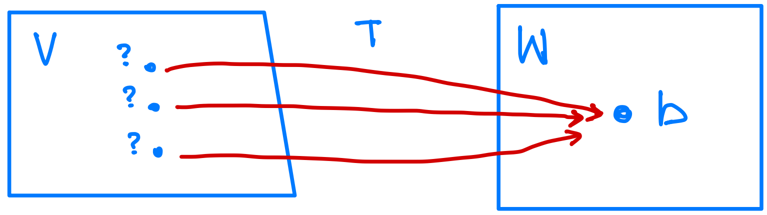
the derivative of f

D is a linear transformation:

1) $D(f+g) = (f+g)' = f' + g' = D(f) + D(g)$
2) $D(cf) = (cf)' = c\cdot f' = c\cdot D(f)$

Note. If $T\colon V \to W$ is a linear transformation then for any vector $\mathbf{b} \in W$ we can consider the equation

$$T(\mathbf{x}) = \mathbf{b}$$



Example

If $T_A\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ – a matrix linear transformation,
$v \longmapsto Av$

then the equation $T_A(x) = b$ is the same as the matrix equation $Ax = b$.

Example:

Take $D\colon C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$
$f(t) \longmapsto f'(t)$

For $g \in C^\infty(\mathbb{R})$ the equation $D(x) = g$ is the same as the differential equation

$$\frac{dx}{dt} = g$$

This equation is solved by integration:

$$x(t) = \int g(t)\,dt$$

Definition

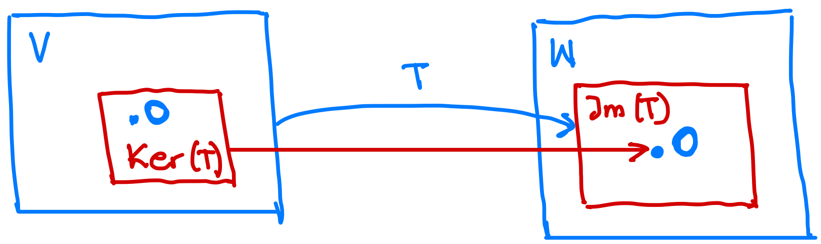
If $T: V \to W$ is a linear transformation then:

1) The *kernel* of T is the set

$$\mathrm{Ker}(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\}$$

2) The *image* of T is the set

$$\mathrm{Im}(T) = \{\mathbf{w} \in W \mid \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V\}$$



Example: A - $m \times n$ matrix, $T_A: \mathbb{R}^n \to \mathbb{R}^m$, $v \mapsto Av$

$$\mathrm{Ker}(T_A) = \{v \in \mathbb{R}^n \mid T_A(v) = \mathbf{0}\}$$
$$= \{v \in \mathbb{R}^n \mid Av = \mathbf{0}\}$$
$$= \mathrm{Nul}(A)$$ ← the null space of A

$$\mathrm{Im}(T_A) = \{w \in \mathbb{R}^m \mid T_A(v) = w \text{ for some } v \in \mathbb{R}^n\}$$
$$= \{w \in \mathbb{R}^m \mid Av = w \text{ for some } v \in \mathbb{R}^n\}$$
$$= \mathrm{Col}(A)$$ ← the column space of A

Example: $D: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $f \mapsto f'$

$$\mathrm{Ker}(D) = \{f \in C^\infty(\mathbb{R}) \mid f' = 0\} = \{\text{the set of all constant functions}\}$$

$$\mathrm{Im}(D) = \{g \in C^\infty(\mathbb{R}) \mid g = f' \text{ for some } f \in C^\infty(\mathbb{R})\} = C^\infty(\mathbb{R})$$

Proposition

If $T: V \to W$ is a linear transformation then:

1) $\mathrm{Ker}(T)$ is a subspace of V

2) $\mathrm{Im}(T)$ is a subspace of W

Theorem

If $T: V \to W$ is a linear transformation and v_0 is a solution of the equation

$$T(\mathbf{x}) = \mathbf{b}$$

then all other solutions of this equation are vectors of the form

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{n}$$

where $\mathbf{n} \in \mathrm{Ker}(T)$.

Proof: If v_0 is a solution of $T(x) = b$ and $n \in \mathrm{Ker}(T)$
then $T(v_0 + n) = T(v_0) + T(n) = b + \mathbf{0} = b$
So $v_0 + n$ is also a solution of $T(x) = b$.

Conversely, if v is some solution of $T(x) = b$
then take $n = v - v_0$. Then we have

$$v = v_0 + (v - v_0) = v_0 + n$$

Also, $T(n) = T(v - v_0) = T(v) - T(v_0) = b - b = \mathbf{0}$
so $n \in \mathrm{Ker}(T)$.

Example.

$$\begin{aligned} D\colon C^\infty(\mathbb{R}) &\longrightarrow C^\infty(\mathbb{R}) \\ f &\longmapsto f' \end{aligned}$$

Recall $\operatorname{Ker}(D) = \{$ the set of all constant functions $\}$

Let $g(t) = t^2$. Solutions of the equation

$$D(x) = g$$

are functions $f(t)$ such that $f'(t) = g(t) = t^2$

This gives: solutions of $D(x) = g$ are functions of the form

$$f(t) = \int t^2\,dt = \underbrace{\tfrac{1}{3}t^3}_{\substack{\text{a particular}\\ \text{solution of}\\ D(x)=g}} + \underbrace{C}_{\substack{\text{a constant function}\\ \text{i.e. a function}\\ \text{from } \operatorname{Ker}(D)}}$$