Theorem

Let A and B be $n \times n$ matrices. If B is obtained from A by multiplying one row (or one column) of A by a scalar k then

$$\det B = k \cdot \det A$$

Example.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ \boxed{3 \quad 1 \quad 4} \\ 5 & 6 & 7 \end{bmatrix} \cdot 3 \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 9 & 3 & 12 \\ 5 & 6 & 7 \end{bmatrix} = B$$

$$\boxed{\det B = 3 \cdot \det A}$$

Proof: Let

$$D = \begin{array}{cc} & i^{th} \text{ column} \\ & \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & k & \\ 0 & & & & \ddots \\ & & & & 1 \end{bmatrix} \leftarrow i^{th} \text{ row} \end{array}$$

Recall: $\det D = 1 \cdot \ldots \cdot k \cdot \ldots \cdot 1 = k$

Check: $DA = \left(\begin{matrix}\text{the matrix obtained by multiplying} \\ \text{the } i^{th} \text{ row of } A \text{ by } k\end{matrix}\right) = B$

We get: $\det B = \det(DA) = (\det D) \cdot (\det A) = k \cdot \det A$

Also: $AD = \left(\begin{matrix}\text{the matrix obtained by multiplying} \\ \text{the } i^{th} \text{ column of } A \text{ by } k\end{matrix}\right) = B$

This gives: $\det B = \det(AD) = (\det A) \cdot (\det D) = (\det A) \cdot k$

Corollary

If a square matrix A contains a row or column consisting of zeros, then $\det A = 0$.

$$\underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ \boxed{0 \quad 0 \quad 0} \\ 4 & 5 & 6 \end{bmatrix}} \cdot 0 = \underset{B = A}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{bmatrix}}$$

$\det A = \det B = 0 \cdot \det A = 0$

Theorem

Let A and B be $n \times n$ matrices. If B is obtained from A by adding a multiple of one row of A to another row (or adding a multiple of one column to another column) then

$$\det B = \det A$$

Example.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \\ 5 & 6 & 7 \end{bmatrix} \cdot(-3) \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 5 & 6 & 7 \end{bmatrix} = B \qquad \boxed{\det B = \det A}$$

<u>Proof</u>: Recall:

$$E^n_{i,j}(k) = \begin{array}{c} \\ \\ \end{array}\begin{bmatrix} 1 & & & \\ & 1 & & k \\ & & \ddots & \\ & & & 1 \end{bmatrix} \begin{array}{l} \\ \leftarrow i^{th} \text{ row} \\ \\ \\ \end{array}$$

j^{th} column

$$\det E^n_{i,j}(k) = 1$$

<u>Check:</u>

$$E^n_{i,j}(k) \cdot A = \left(\begin{array}{l} \text{the matrix obtained from } A \text{ by} \\ \text{adding } k \cdot (\text{row } j) \text{ to row } i \end{array} \right) = B$$

<u>This gives:</u>

$$\det B = \det\left(E^n_{i,j}(k) \cdot A\right) = \left(\det E^n_{i,j}(k)\right) \cdot \left(\det A\right) = 1 \cdot \det A = \det A$$

<u>Similarly:</u>

$$A \cdot E^n_{i,j}(k) = \left(\begin{array}{l} \text{the matrix obtained from } A \text{ by} \\ \text{adding } k \cdot (\text{column } i) \text{ to column } j \end{array} \right) = B$$

<u>We get:</u>

$$\det B = \det\left(A \cdot E^n_{i,j}(k)\right) = \left(\det A\right) \cdot \left(\det E^n_{i,j}(k)\right) = (\det A) \cdot 1 = \det A$$

Theorem

Let A and B be $n \times n$ matrices. If B is obtained from A by interchanging two rows (or two columns) then

$$\det B = -\det A$$

Example.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \\ 5 & 6 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 5 & 6 & 7 \\ 3 & 1 & 4 \end{bmatrix} = B \qquad \boxed{\det B = -\det A}$$

Proof: Interchange of rows (or columns) can be obtained using the other two elementary operations.

E.g.:

$$A = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \xrightarrow{\cdot(-1)} \begin{bmatrix} r_1 \\ r_2 - r_1 \\ \vdots \\ r_n \end{bmatrix} \xrightarrow{\cdot 1} \begin{bmatrix} r_2 \\ r_2 - r_1 \\ \vdots \\ r_n \end{bmatrix} \xrightarrow{\cdot(-1)} \begin{bmatrix} r_2 \\ -r_1 \\ \vdots \\ r_n \end{bmatrix} \xrightarrow{\cdot(-1)} \begin{bmatrix} r_2 \\ r_1 \\ \vdots \\ r_n \end{bmatrix} = B$$

does not change the determinant

multiplies the determinant by (-1)

This gives: $\det B = (-1) \cdot \det A$

Definition

An square matrix is *upper triangular* is all its entries below the main diagonal are 0.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

Proposition

If A is an upper triangular matrix as above then

$$\det A = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$$

<u>Proof:</u> 1) If $a_{ii} \neq 0$ for all i then

$$\det A = \det \begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix} \leftarrow$$

diagonal matrix,
$\det = a_{11} \cdot \ldots \cdot a_{nn}$

<u>Example:</u>

$$A = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix} \cdot\left(-\tfrac{5}{3}\right) \cdot(-2) \longrightarrow \begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \cdot(-2) \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = B$$

$$\det A = \det B = 1 \cdot 2 \cdot 3$$

2) If $a_{ii} = 0$ for some i, then A has the same determinant as a matrix with a row of zeros.

<u>Example:</u>

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 5 \end{bmatrix} \cdot\left(-\tfrac{4}{5}\right) \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix} = B$$

$$\det A = \det B = 0 = 1 \cdot 0 \cdot 5$$