Recall:

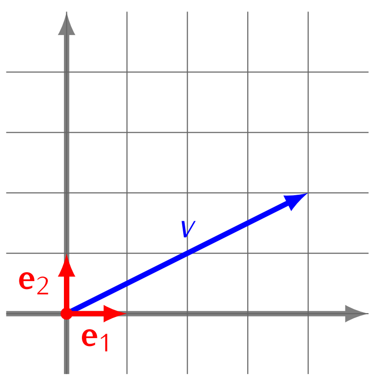
1) A basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ of a vector space V defines a coordinate system:

$$\mathbf{v} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$$

$$\Downarrow$$

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

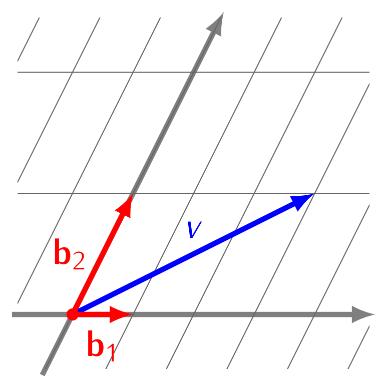
2) Different bases define different coordinate systems.



$\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$

$v = 4e_1 + 2e_2$

So: $[v]_{\mathcal{E}} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$



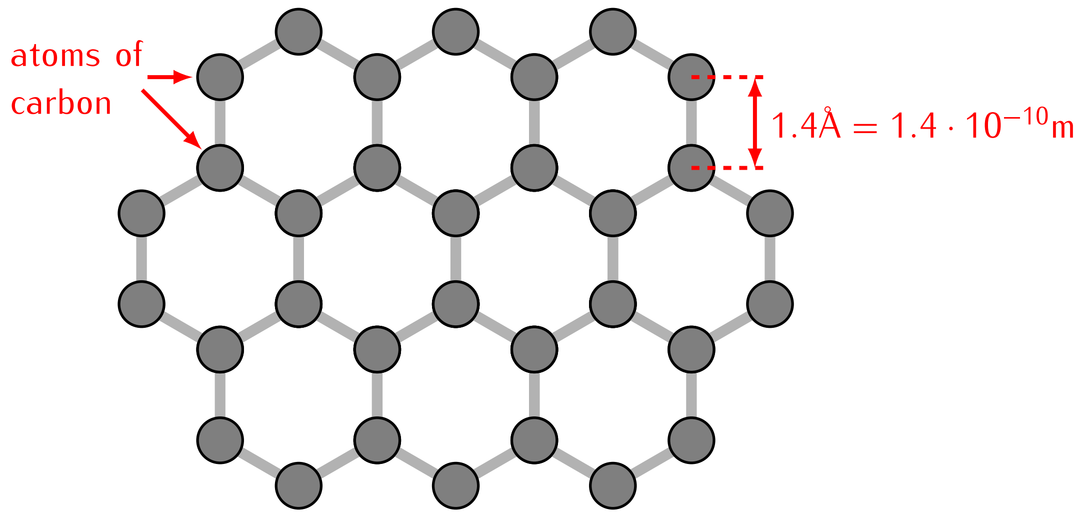
$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$

$v = 3b_1 + 1 \cdot b_2$

So: $[v]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Note. Choosing a convenient basis can simplify computations.

Example. Graphene lattice.



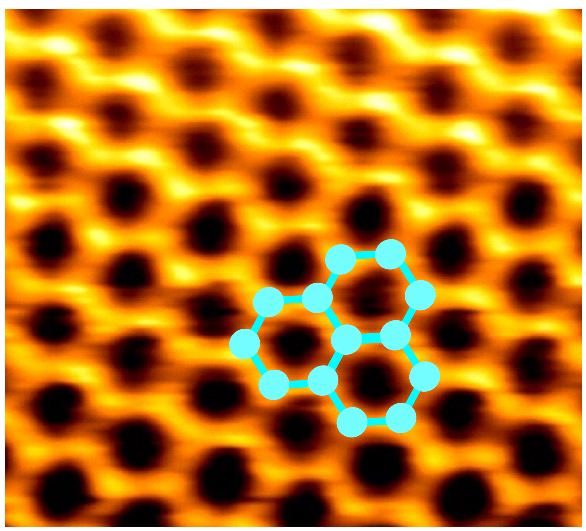


Image of graphene taken with an atomic force microscope.
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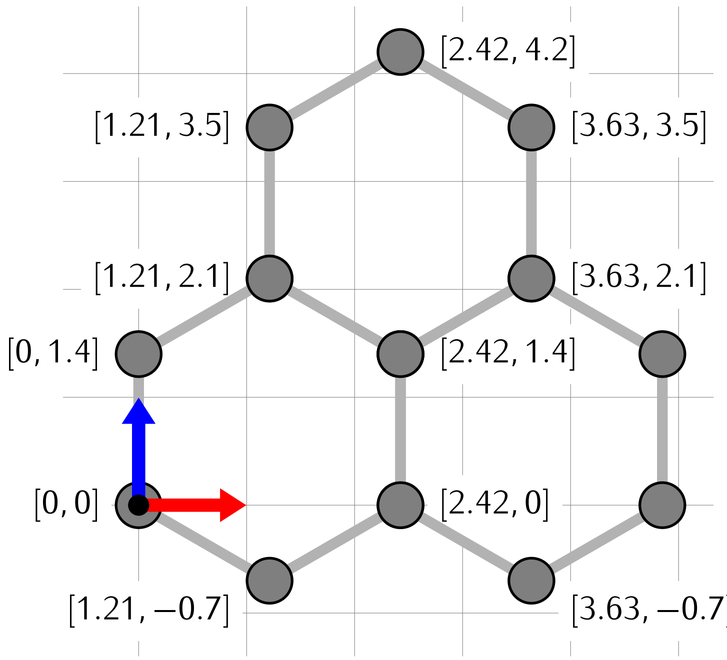
Coordinates of atoms in the graphene lattice

In the standard basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

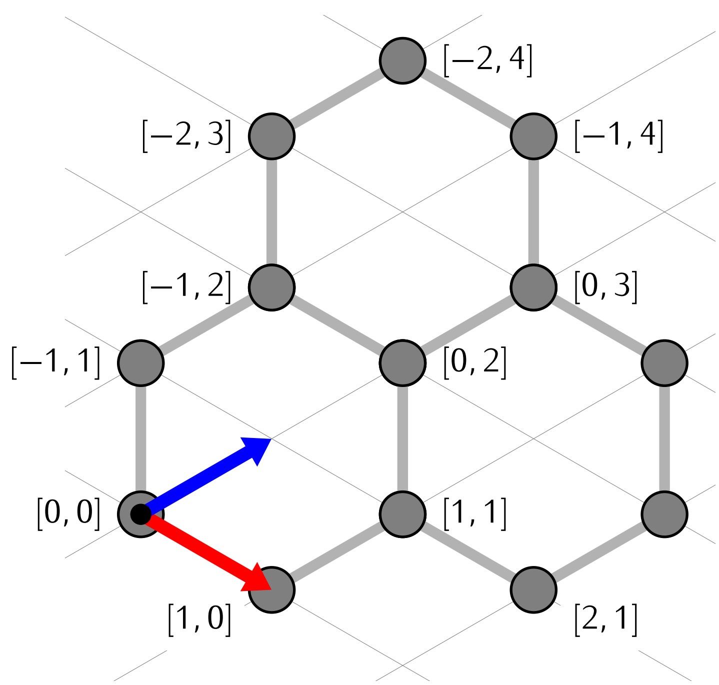
$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



In a more convenient basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$:

$$\mathbf{b}_1 = \begin{bmatrix} 1.21 \\ -0.7 \end{bmatrix}$$

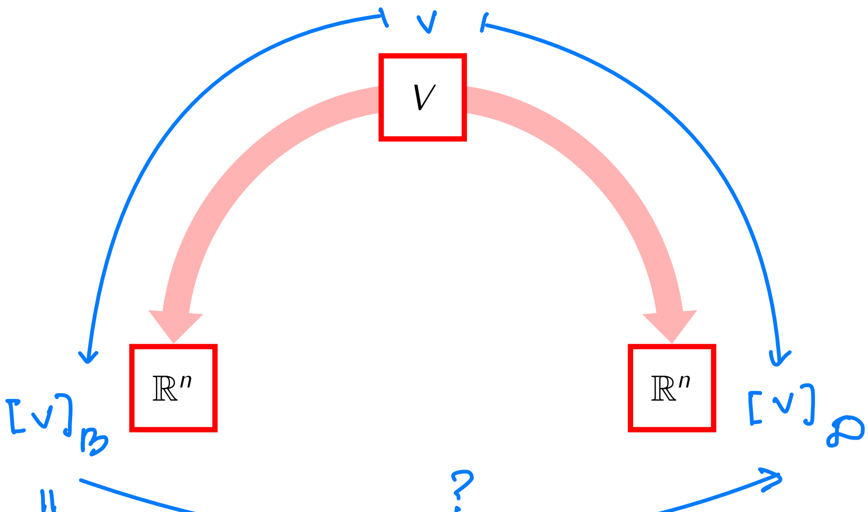
$$\mathbf{b}_2 = \begin{bmatrix} 1.21 \\ 0.7 \end{bmatrix}$$



Problem Let

$$\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}, \qquad \mathcal{D} = \{\mathbf{d}_1, \dots, \mathbf{d}_1\}$$

be two bases of a vector space V, and let $\mathbf{v} \in V$. Assume that we know $[\mathbf{v}]_{\mathcal{B}}$. What is $[\mathbf{v}]_{\mathcal{D}}$?



$$[v]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \Rightarrow v = c_1 b_1 + c_2 b_2 + \dots + c_n b_n$$

<u>Note:</u>

$$\begin{aligned} [v]_{\mathcal{D}} &= [c_1 b_1 + c_2 b_2 + \dots + c_n b_n]_{\mathcal{D}} \\ &= c_1 [b_1]_{\mathcal{D}} + c_2 [b_2]_{\mathcal{D}} + \dots + c_n [b_n]_{\mathcal{D}} \\ &= \big[[b_1]_{\mathcal{D}} \ [b_2]_{\mathcal{D}} \ \dots \ [b_n]_{\mathcal{D}} \big] \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \end{aligned}$$

<u>We obtain:</u>

$$\boxed{[v]_{\mathcal{D}} = \big[[b_1]_{\mathcal{D}} \ [b_2]_{\mathcal{D}} \ \dots \ [b_n]_{\mathcal{D}} \big] \cdot [v]_{\mathcal{B}}}$$

Definition

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{D} = \{\mathbf{d}_1, \dots, \mathbf{d}_1\}$ be two bases of a vector space V. The matrix

$$P_{\mathcal{D}\leftarrow\mathcal{B}} = \left[\ \left[\mathbf{b}_1\right]_{\mathcal{D}} \ \ \left[\mathbf{b}_2\right]_{\mathcal{D}} \quad \dots \quad \left[\mathbf{b}_n\right]_{\mathcal{D}}\ \right]$$

is called the *change of coordinates matrix* from the basis $\mathcal{B}$ to the basis $\mathcal{D}$.

Propostion

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{D} = \{\mathbf{d}_1, \dots, \mathbf{d}_1\}$ be two bases of a vector space V. For any vector $\mathbf{v} \in V$ we have

$$\left[\mathbf{v}\right]_{\mathcal{D}} = P_{\mathcal{D}\leftarrow\mathcal{B}} \cdot \left[\mathbf{v}\right]_{\mathcal{B}}$$

Example. Let $\mathbb{P}_2$ be the vector space of polynomials of degree ≤ 2. Consider two bases of $\mathbb{P}_2$:

$$\mathcal{B} = \{1, 1+t, 1+t+t^2\}$$
$$\mathcal{D} = \{1+t, 1-5t, 2+t^2\}$$

1) Compute the change of coordinates matrix $P_{\mathcal{D}\leftarrow\mathcal{B}}$.

2) Let $p \in \mathbb{P}_2$ be a polynomial such that

$$[p]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

Compute $[p]_{\mathcal{D}}$.

Solution:

1) $P_{\mathcal{D}\leftarrow\mathcal{B}} = \left[\, [1]_{\mathcal{D}} \;\; [1+t]_{\mathcal{D}} \;\; [1+t+t^2]_{\mathcal{D}} \,\right]$

We have:

$1 = \frac{5}{6}(1+t) + \frac{1}{6}(1-5t) + 0\cdot(2+t^2)$ so: $[1]_{\mathcal{D}} = \begin{bmatrix} 5/6 \\ 1/6 \\ 0 \end{bmatrix}$

$1+t = 1\cdot(1+t) + 0\cdot(1-5t) + 0\cdot(2+t^2)$ so: $[1+t]_{\mathcal{D}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$1+t+t^2 = -\frac{2}{3}(1+t) - \frac{1}{3}(1-5t) + 1\cdot(2+t^2)$ so: $[1+t+t^2] = \begin{bmatrix} -2/3 \\ -1/3 \\ 1 \end{bmatrix}$

This gives:

$$P_{\mathcal{D}\leftarrow\mathcal{B}} = \begin{bmatrix} 5/6 & 1 & -2/3 \\ 1/6 & 0 & -1/3 \\ 0 & 0 & 1 \end{bmatrix}$$

2) $$[p_{\mathcal{D}}] = P_{\mathcal{D}\leftarrow\mathcal{B}} \cdot [p]_{\mathcal{B}} = \begin{bmatrix} 5/6 & 1 & -2/3 \\ 1/6 & 0 & -1/3 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 19/6 \\ -7/6 \\ 5 \end{bmatrix}$$

Proposition

If $\mathcal{B}, \mathcal{D}, \mathcal{E}$ are three bases of a vector space V then:

1) $P_{\mathcal{B}\leftarrow\mathcal{D}} = (P_{\mathcal{D}\leftarrow\mathcal{B}})^{-1}$
2) $P_{\mathcal{E}\leftarrow\mathcal{B}} = P_{\mathcal{E}\leftarrow\mathcal{D}} \cdot P_{\mathcal{D}\leftarrow\mathcal{B}}$