

**Definition**

Let  $V, W$  be vector spaces A *linear transformation* is a function

$$T: V \rightarrow W$$

which satisfies the following conditions:

- 1)  $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$  for all  $\mathbf{u}, \mathbf{v} \in V$
- 2)  $T(c\mathbf{v}) = cT(\mathbf{v})$  for any  $\mathbf{v} \in V$  and any scalar  $c$ .

**Note.** If  $T: V \rightarrow W$  is a linear transformation then for any vector  $\mathbf{b} \in W$  we can consider the equation

$$T(\mathbf{x}) = \mathbf{b}$$

### Definition

If  $T: V \rightarrow W$  is a linear transformation then:

1) The *kernel* of  $T$  is the set

$$\text{Ker}(T) = \{v \in V \mid T(v) = \mathbf{0}\}$$

2) The *image* of  $T$  is the set

$$\text{Im}(T) = \{w \in W \mid w = T(v) \text{ for some } v \in V\}$$

### Proposition

If  $T: V \rightarrow W$  is a linear transformation then:

- 1)  $\text{Ker}(T)$  is a subspace of  $V$
- 2)  $\text{Im}(T)$  is a subspace of  $W$

### Theorem

If  $T: V \rightarrow W$  is a linear transformation and  $v_0$  is a solution of the equation

$$T(x) = \mathbf{b}$$

then all other solutions of this equation are vectors of the form

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{n}$$

where  $\mathbf{n} \in \text{Ker}(T)$ .

**Example.**

$$\begin{aligned} D: C^\infty(\mathbb{R}) &\longrightarrow C^\infty(\mathbb{R}) \\ f &\longmapsto f' \end{aligned}$$