

Recall: A vector equation

$$x_1 \mathbf{v}_1 + \dots + x_n \mathbf{v}_n = \mathbf{b}$$

has a solution if and only if  $\mathbf{b} \in \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_n)$ .

### Definition

If  $A$  is a matrix with columns  $\mathbf{v}_1, \dots, \mathbf{v}_n$ :

$$A = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$$

then the set  $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_n)$  is called the *column space* of  $A$  and it is denoted  $\text{Col}(A)$ .

**Upshot.** A matrix equation  $A\mathbf{x} = \mathbf{b}$  has a solution if and only if  $\mathbf{b} \in \text{Col}(A)$ .

**Recall:** A vector equation

$$x_1\mathbf{v}_1 + \dots + x_n\mathbf{v}_n = \mathbf{b}$$

has only one solution for each  $\mathbf{b} \in \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_n)$  if and only if the homogenous equation

$$x_1\mathbf{v}_1 + \dots + x_n\mathbf{v}_n = \mathbf{0}$$

has only the trivial solution  $x_1 = 0, \dots, x_n = 0$ .

### Definition

If  $A$  is a matrix then the set of solution of the homogenous equation

$$A\mathbf{x} = \mathbf{0}$$

is called the *null space* of  $A$  and it is denoted  $\text{Nul}(A)$ .

**Upshot.** A matrix equation  $A\mathbf{x} = \mathbf{b}$  has only one solution for each  $\mathbf{b} \in \text{Col}(A)$  if and only if  $\text{Nul}(A) = \{\mathbf{0}\}$ .

**Example.** Find the null space of the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

**Proposition**

$\text{Nul}(A) = \{\mathbf{0}\}$  if and only if every column of  $A$  is a pivot column.

**Example.** Find the null space of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -2 & -2 & 1 & -5 \\ 1 & 1 & -1 & 3 \end{bmatrix}$$

**Note**

If  $A$  is an  $m \times n$  matrix then  $\text{Nul}(A)$  can be always described as a span of some vectors in  $\mathbb{R}^n$ .

**Example.** Solve the matrix equation  $A\mathbf{x} = \mathbf{b}$  where

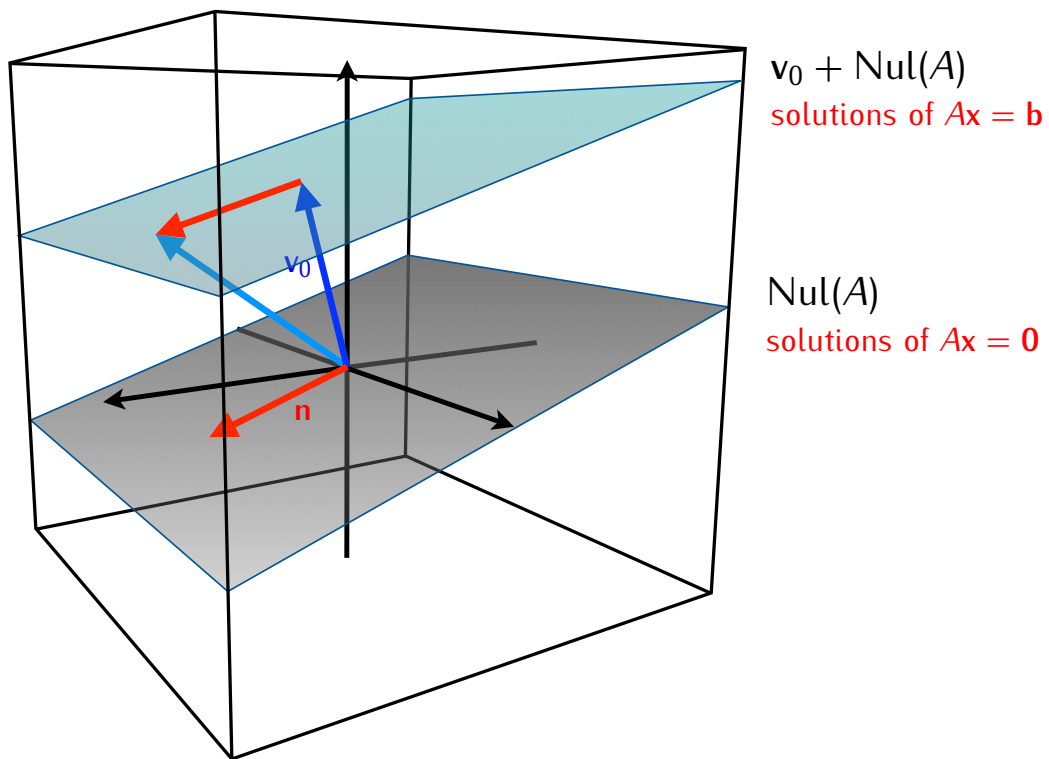
$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -2 & -2 & 1 & -5 \\ 1 & 1 & -1 & 3 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

## Proposition

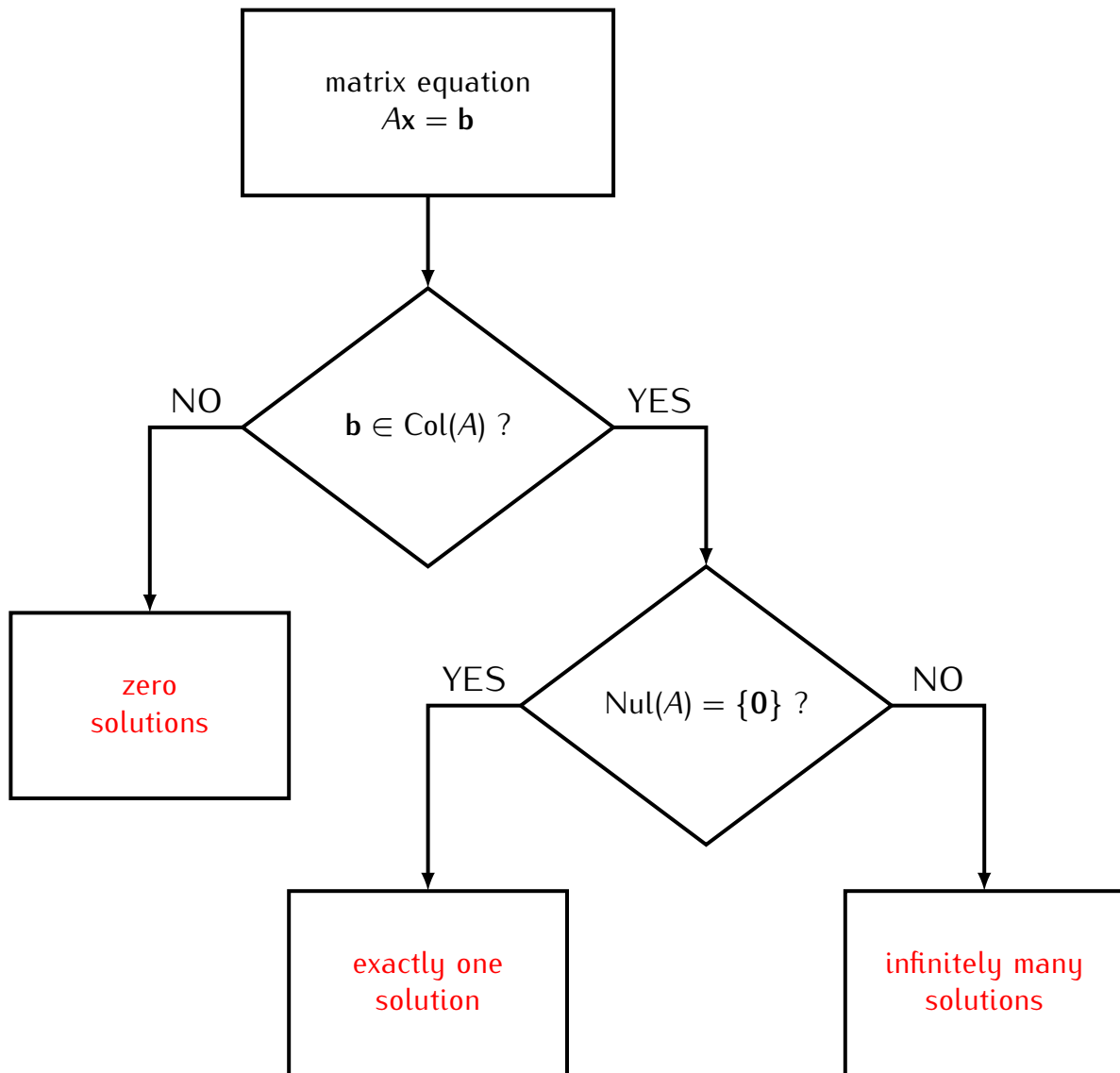
Let  $\mathbf{v}_0$  be some chosen solution of a matrix equation  $A\mathbf{x} = \mathbf{b}$ . Then any other solution  $\mathbf{v}$  of this equation is of the form

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{n}$$

where  $\mathbf{n} \in \text{Nul}(A)$ .



Upshot: how to find the number of solutions of a matrix equation



**Question:** What conditions on the matrix  $A$  guarantee that the equation  $A\mathbf{x} = \mathbf{b}$  has a solution for every vector  $\mathbf{b}$ ?

**Example.**

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

**Example.**

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 7 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



### Proposition

A matrix equation  $A\mathbf{x} = \mathbf{b}$  has a solution for every vector  $\mathbf{b}$  if and only if  $A$  has a pivot position in every row.

In such case  $\text{Col}(A) = \mathbb{R}^m$ , where  $m$  is the number of rows of  $A$ .