MTH 309 Practice Exam 2

1. Let M be a 4×4 matrix

$$M = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$$

Assume that we know that $\det M = 5$. Compute determinants of the following matrices. Justify your answers.

$$A = \begin{bmatrix} e & a & i & m \\ f & b & j & n \\ g & c & k & o \\ g & d & l & p \end{bmatrix}, \qquad B = \begin{bmatrix} a & b & c & d \\ 2e & 2f & 2g & 2h \\ 3i & 3j & 3k & 3l \\ 4m & 4n & 4o & 4p \end{bmatrix}$$

$$C = \begin{bmatrix} a & -b & -c & -d \\ -e & f & g & h \\ -i & j & k & l \\ -m & n & o & p \end{bmatrix}, \qquad E = \begin{bmatrix} 3 & 2 & 1 & 0 & 0 \\ a & b & a & c & d \\ e & f & e & g & h \\ i & j & i & k & l \\ m & n & m & o & p \end{bmatrix}$$

2. Consider the matrix equation $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}$$

a) Compute $\det A$.

b) Use Cramer's rule to compute the value of x_3.

3. Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix}$$

The set $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis of $\mathbb{R}^3$ (you do not need to verify it).

a) Compute $[\mathbf{w}]_{\mathcal{B}}$, the coordinate vector of $\mathbf{w}$ relative to the basis $\mathcal{B}$.

b) Let $\mathbf{u} \in \mathbb{R}^3$ be a vector such that

$$[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Compute the vector $\mathbf{u}$.

4. Decide which of the following sets of vectors are subspaces of $\mathbb{R}^2$. Justify your answers.

a) The set S_2 consisting of all vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ such that $a_1 \geqslant a_2$.

b) The set S_3 consisting of all vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ such that $a_1 \cdot a_2 = 0$.

c) The set S_4 consisting of all vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ such that $a_1 \cdot a_2 \geqslant 0$.

d) The set S_1 consisting of all vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ such that $a_1^2 + a_2^2 = 0$.

5. For each of the statements given below decide if it is true or false. If it is true explain why. If it is false give a counterexample.

a) If A and B are square matrices such that $\det(AB) = 1$ then both matrices A and B must be invertible.

b) If A is a 3×3 matrix and $\mathrm{Col}(A)$ is its column space, then $\dim \mathrm{Col}(A) = 3$.

c) If $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of $\mathbb{R}^3$ and $\mathbf{u}$ is a vector in $\mathbb{R}^3$ such that

$$[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

then $\mathbf{u}$ must be in $\mathrm{Span}(\mathbf{v}_2, \mathbf{v}_3)$.

d) If V is a subspace of $\mathbb{R}^3$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ are vectors such that $\mathbf{u} + \mathbf{v} \in V$ and $\mathbf{u} - \mathbf{v} \in V$ then $\mathbf{u} \in V$.