Recall: If A is an $m \times n$ matrix then

$$A \cdot \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}}_{\text{vector in } \mathbb{R}^n} = \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}}_{\text{vector in } \mathbb{R}^m}$$

Example:

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}_{\substack{2\times 3 \\ \text{matrix}}} \cdot \underbrace{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}}_{\text{vector in } \mathbb{R}^3} = 2\cdot\begin{bmatrix} 1 \\ 4 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 1\cdot\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 9 \end{bmatrix}}_{\text{vector in } \mathbb{R}^2}$$

$$\begin{array}{ccc} \mathbb{R}^3 & \xrightarrow{\;A\cdot\;} & \mathbb{R}^2 \\ v & \longmapsto & Av \end{array}$$

Definition

If A is an $m \times n$ matrix then the function

$$T_A \colon \mathbb{R}^n \to \mathbb{R}^m$$

given by $T_A(\mathrm{v}) = A\mathrm{v}$ is called the *matrix transformation associated to* A.

Example.

Let $T_A\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the matrix transformation defined by the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{bmatrix}$$

1) Compute $T_A(\mathbf{v})$ where $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

$$T_A(v) = Av = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

2) Find a vector $\mathbf{v}$ such that $T_A(\mathbf{v}) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

Solution: We need to find a vector $v = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that

$$\underbrace{Av = \begin{bmatrix} 5 \\ 6 \end{bmatrix}}_{\text{matrix equation}}$$

augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 1 & 3 & 3 & 6 \end{array}\right] \xrightarrow{\text{row reduction}} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & \underbrace{0}_{\text{free}} & 1 \end{array}\right] \end{array} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 - 3x_3 \\ 1 \\ x_3 \end{bmatrix}$$

e.g.: $x_3 = 0 \Rightarrow v = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

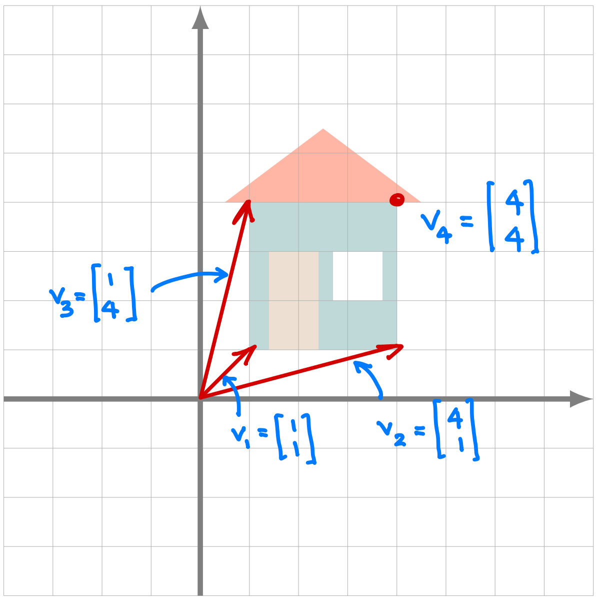
$x_3 = 1 \Rightarrow v = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

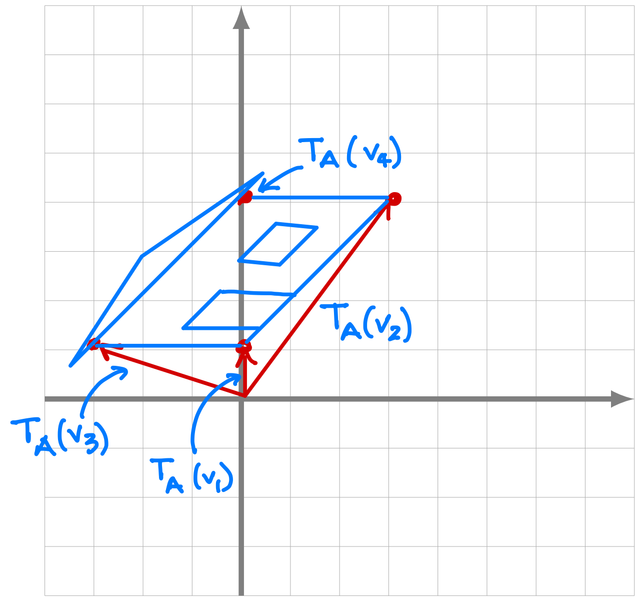
$\vdots$

Geometric interpretation of matrix transformations $\mathbb{R}^2 \to \mathbb{R}^2$

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$v \longmapsto Av$$





$$T_A(v_1) = Av_1 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T_A(v_2) = Av_2 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

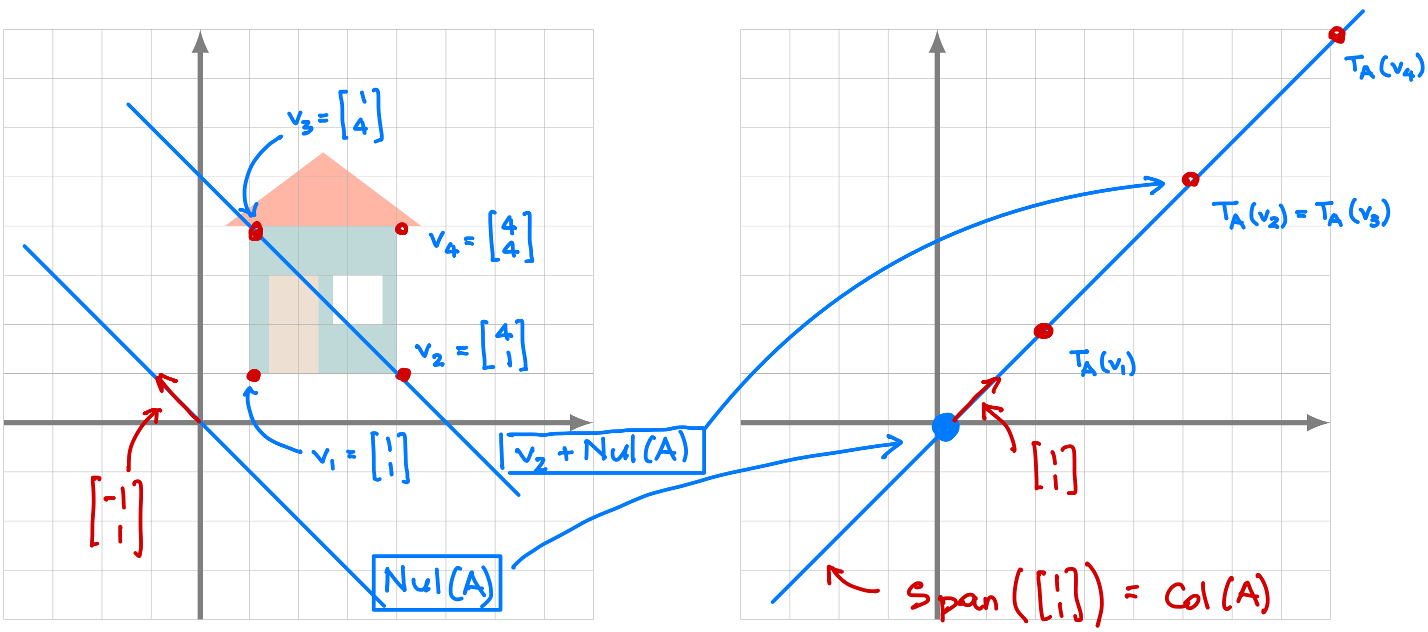
$$T_A(v_3) = Av_3 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$T_A(v_4) = Av_4 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

Example.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$v \longmapsto Av$$



$$T_A(v_1) = Av_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

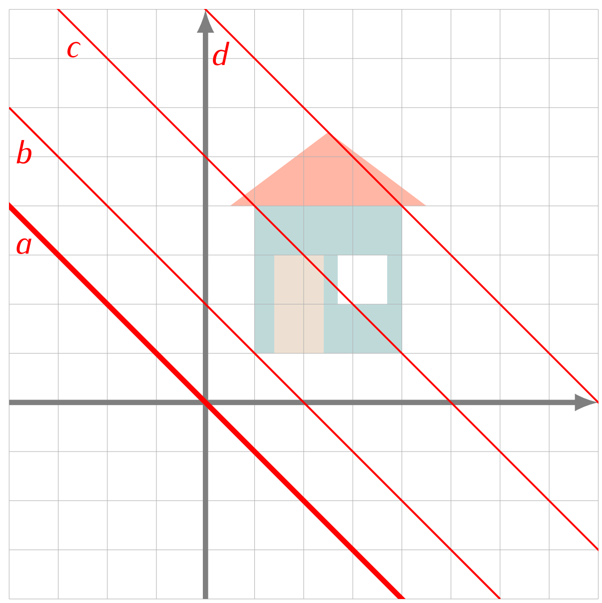
$$T_A(v_2) = Av_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

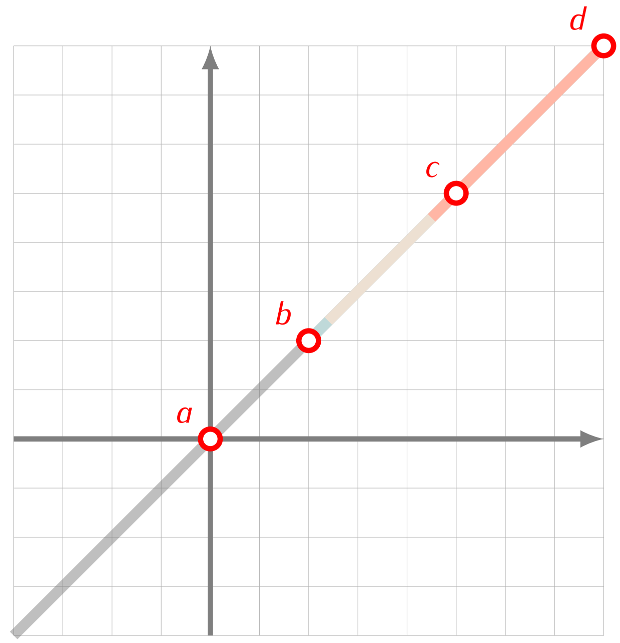
$$T_A(v_3) = Av_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$T_A(v_4) = Av_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$$

$$(\text{Values of } T_A) = (\text{vectors in } \mathrm{Col}(A)) = \mathrm{Span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \mathrm{Span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

$$(\text{Vectors } v \text{ such that } T_A(v) = \mathbf{0}) = \mathrm{Nul}(A) = \mathrm{Span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$$


c
d
b
a


d
c
b
a

Note

If $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation associated to a matrix A then:

- $\mathrm{Col}(A)$ = the set of values of T_A.
- $\mathrm{Nul}(A)$ = the set of vectors $\mathbf{v}$ such that $T_A(\mathbf{v}) = \mathbf{0}$.
- $T_A(\mathbf{v}) = T_A(\mathbf{w})$ if and only if $\mathbf{w} = \mathbf{v} + \mathbf{n}$ for some $\mathbf{n} \in \mathrm{Nul}(A)$.