**Recall:**

If $A = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$ is an $m \times n$ matrix then:

**1)** $\mathrm{Col}(A) = \mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_n)$

**2)** $\mathrm{Nul}(A) = \{\mathbf{v} \in \mathbb{R}^n \mid A\mathbf{v} = \mathbf{0}\}$

## Construction of a basis of Col($A$)

**Lemma**

Let $V$ be a vector space, and let $\mathbf{v}_1, \ldots, \mathbf{v}_p \in V$. If a vector $\mathbf{v}_i$ is a linear combination of the other vectors then

$$\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p) = \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_p)$$

**Upshot.** One can construct a basis of a vector space $V$ as follows:

- Start with a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ such that $\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p) = V$.
- Keep removing vectors without changing the span, until you get a linearly independent set.

**Example.** Find a basis of Col($A$) where $A$ is the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**Example.** Find a basis of Col($A$) where $A$ is the following matrix:

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

## Construction of a basis of Nul($A$)

**Example.** Find a basis of Nul($A$) where $A$ is the following matrix:

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

**Upshot.** If $A$ is matrix then:

$$\dim \mathrm{Col}(A) = \text{the number of pivot columns of } A$$

$$\dim \mathrm{Nul}(A) = \text{the number of non-pivot columns of } A$$

**Definition**

If $A$ is a matrix then:

- the dimension of $\mathrm{Col}(A)$ is called the *rank* of $A$ and it is denoted $\mathrm{rank}(A)$
- the dimension of $\mathrm{Nul}(A)$ is called the *nullity* of $A$.

**The Rank Theorem**

If $A$ is an $m \times n$ matrix then

$$\mathrm{rank}(A) + \dim \mathrm{Nul}(A) = n$$

## Low rank matrices in data analysis

| | exam 1 | exam 2 | exam 3 | exam 4 |
|---|---|---|---|---|
| **student 1** | 2 | 1 | 3 | 10 |
| **student 2** | 4 | 2 | 6 | 20 |
| **student 3** | 10 | 5 | 15 | 50 |
| **student 4** | 2 | 1 | 3 | 10 |
| **student 5** | 0 | 0 | 0 | 0 |
| **student 6** | 6 | 3 | 9 | 40 |