Recall:
Let A be a matrix with a singular value decomposition

$$A = U\Sigma V^T$$

If

$$U = \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_m \end{bmatrix} \qquad V = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$$

and $\sigma_1, \dots, \sigma_r$ are singular values of A then then

$$A = \sigma_1(\mathbf{u}_1\mathbf{v}_1^T) + \sigma_2(\mathbf{u}_2\mathbf{v}_2^T) + \dots + \sigma_r(\mathbf{u}_r\mathbf{v}_r^T)$$

Example: Movie ratings:

$$\begin{array}{c} \\ \textbf{user\_1} \\ \textbf{user\_2} \\ \textbf{user\_3} \\ \textbf{user\_4} \\ \textbf{user\_5} \\ \textbf{user\_6} \\ \textbf{user\_7} \end{array} \begin{array}{c} \begin{array}{ccccc} \textbf{Matrix} & \textbf{Amelie} & \textbf{Alien} & \textbf{Casablanca} & \textbf{Interstellar} \end{array} \\ \begin{bmatrix} 5 & 0 & 5 & 0 & 4 \\ 5 & 0 & 3 & 0 & 5 \\ 0 & 5 & 0 & 5 & 1 \\ 1 & 5 & 0 & 4 & 0 \\ 4 & 0 & 4 & 0 & 3 \\ 0 & 5 & 0 & 4 & 0 \\ 3 & 0 & 3 & 0 & 2 \end{bmatrix} \end{array} = A$$

Singular value decomposition of the matrix of movie ratings:

$$U = \begin{array}{c} \begin{matrix} u_1 & u_2 & & & & & \end{matrix} \\ \begin{bmatrix} -0.6 & 0.1 & -0.3 & -0.2 & 0.2 & -0.7 & -0.2 \\ -0.5 & 0.1 & 0.8 & 0.2 & 0.1 & 0.1 & 0.1 \\ -0.1 & -0.6 & 0.2 & -0.7 & -0.4 & 0.0 & 0.0 \\ -0.1 & -0.5 & -0.1 & 0.7 & -0.4 & -0.1 & -0.2 \\ -0.5 & 0.1 & -0.3 & -0.1 & -0.1 & 0.7 & -0.4 \\ -0.1 & -0.6 & -0.1 & 0.0 & 0.8 & 0.1 & 0.2 \\ -0.3 & 0.1 & -0.3 & 0.0 & -0.3 & 0.1 & 0.8 \end{bmatrix} \end{array} \quad \Sigma = \begin{array}{c} \begin{matrix} \sigma_1 & \sigma_2 & & & \end{matrix} \\ \begin{bmatrix} 13.6 & 0 & 0 & 0 & 0 \\ 0 & 11.4 & 0 & 0 & 0 \\ 0 & 0 & 1.9 & 0 & 0 \\ 0 & 0 & 0 & 1.1 & 0 \\ 0 & 0 & 0 & 0 & 0.3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$V = \begin{array}{c} \begin{matrix} v_1 & v_2 & & & \end{matrix} \\ \begin{bmatrix} -0.6 & 0.1 & 0.0 & 0.7 & -0.4 \\ -0.1 & -0.7 & -0.1 & 0.3 & 0.6 \\ -0.5 & 0.1 & -0.7 & -0.4 & 0.2 \\ -0.1 & -0.6 & 0.0 & -0.4 & -0.7 \\ -0.5 & 0.1 & 0.7 & -0.4 & 0.3 \end{bmatrix} \end{array}$$

$$A \approx \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \cdot \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \cdot \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}$$

tell us how each user feels about sci-fi and romance movies

" 1st principal " component

"2nd principal " component

picks up sci-fi movies

$$\begin{bmatrix} 5 & 0 & 5 & 0 & 4 \\ 5 & 0 & 3 & 0 & 5 \\ 0 & 5 & 0 & 5 & 1 \\ 1 & 5 & 0 & 4 & 0 \\ 4 & 0 & 4 & 0 & 3 \\ 0 & 5 & 0 & 4 & 0 \\ 3 & 0 & 3 & 0 & 2 \end{bmatrix} \approx \begin{array}{r} \text{user1} \\ \text{user2} \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{bmatrix} -0.6 & 0.1 \\ -0.5 & 0.1 \\ -0.1 & -0.6 \\ -0.1 & -0.5 \\ -0.5 & 0.1 \\ -0.1 & -0.6 \\ -0.3 & 0.1 \end{bmatrix} \cdot \begin{bmatrix} 13.6 & 0 \\ 0 & 11.4 \end{bmatrix} \cdot \begin{array}{c} \begin{matrix} \text{Matrix} & \text{Amelie} & \text{Alien} & \text{Casablanca} & \text{Interstellar} \end{matrix} \\ \begin{bmatrix} -0.6 & -0.1 & -0.5 & -0.1 & -0.5 \\ 0.1 & -0.7 & 0.1 & -0.6 & 0.1 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \text{" 1st principal axis "} \\ \leftarrow \text{"2nd principal axis "} \end{array}$$

picks up romance movies