Recall:

- A basis of a vector space V is a set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ such that

 1) $\text{Span}(\mathbf{b}_1, \ldots, \mathbf{b}_n) = V$

 2) The set $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is linearly independent.

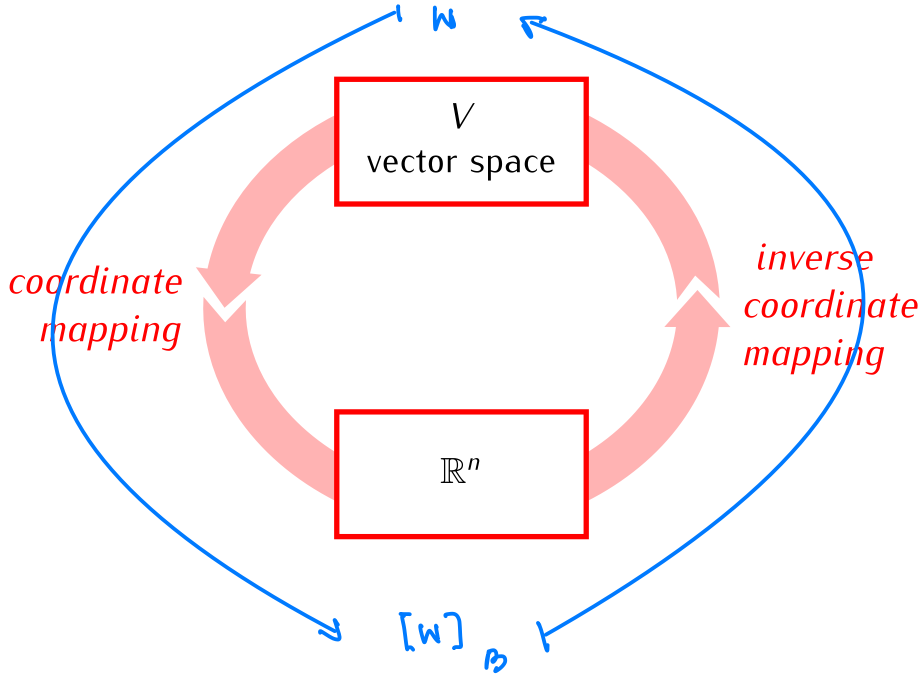
- For $\mathbf{v} \in V$ let $c_1, \ldots, c_n$ be the unique numbers such that

$$c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n = \mathbf{v}$$

The vector

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$$

is called the *coordinate vector of* $\mathbf{v}$ *relative to the basis* $\mathcal{B}$.



Theorem

Let $\mathcal{B}$ be a basis of a vector space V. If $\mathbf{v}_1, \dots \mathbf{v}_p, \mathbf{w} \in V$ then:

1) Solutions of the equation $x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{w}$ are the same as solutions of the equation $x_1 [\mathbf{v}_1]_{\mathcal{B}} + \ldots + x_p [\mathbf{v}_p]_{\mathcal{B}} = [\mathbf{w}]_{\mathcal{B}}$.

2) The set of vectors $\{\mathbf{v}_1, \dots \mathbf{v}_p\}$ is linearly independent if and only if the set $\{[\mathbf{v}_1]_{\mathcal{B}}, \dots, [\mathbf{v}_p]_{\mathcal{B}}\}$ is linearly independent.

3) $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_p) = V$ if any only if $\text{Span}\left([\mathbf{v}_1]_{\mathcal{B}}, \dots, [\mathbf{v}_p]_{\mathcal{B}}\right) = \mathbb{R}^n$.

4) $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a basis of V if and only if $\{[\mathbf{v}_1]_{\mathcal{B}}, \dots, [\mathbf{v}_p]_{\mathcal{B}}\}$ is a basis of $\mathbb{R}^n$.

Example. Recall that $\mathbb{P}_2$ is the vector space of polynomials of degree ≤ 2. Consider the following polynomials in $\mathbb{P}_2$:

$$p_1(t) = 1 + 2t + t^2$$
$$p_2(t) = 3 + t + 2t^2$$
$$p_3(t) = 1 - 8t - t^2$$

Check if the set $\{p_1, p_2, p_3\}$ is linearly independent.

<u>Recall:</u> In $\mathbb{P}_2$ we have the standard basis $\mathcal{E} = \{1, t, t^2\}$

<u>We have:</u>

$$[p_1]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad [p_2]_{\mathcal{E}} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \qquad [p_3]_{\mathcal{E}} = \begin{bmatrix} 1 \\ -8 \\ -1 \end{bmatrix}$$

It suffices to check if the set $\{[p_1]_{\mathcal{E}}, [p_2]_{\mathcal{E}}, [p_3]_{\mathcal{E}}\}$ is linearly independent.

<u>augmented matrix:</u>

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & -8 \\ 1 & 2 & -1 \end{bmatrix} \xrightarrow{\text{row red.}} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

no leading one, so the set $\{[p_1]_{\mathcal{E}}, [p_2]_{\mathcal{E}}, [p_3]_{\mathcal{E}}\}$ is linearly dependent

This shows that the set $\{p_1, p_2, p_3\}$ is linearly dependent.

Theorem

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be vectors in $\mathbb{R}^n$. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a basis of $\mathbb{R}^n$ if and only if the matrix

$$A = \begin{bmatrix} \mathbf{v}_1 & \ldots & \mathbf{v}_p \end{bmatrix}$$

has a pivot position in every row and in every column (i.e. if A is an invertible matrix).

Proof: By definition, $\{v_1, \ldots, v_p\}$ is a basis of $\mathbb{R}^n$ if and only if

1) the set $\{v_1, \ldots, v_p\}$ is linearly independent (i.e. $[v_1 \ldots v_p]$ has a pivot position in every column)

2) $\mathrm{Span}(v_1, \ldots, v_p) = \mathbb{R}^n$ (i.e. $[v_1 \ldots v_p]$ has a leading one in every row).

Example: $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $v_3 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

$$[v_1 \ v_2 \ v_3] = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \xrightarrow{\text{row red.}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$\mathrm{Span}(v_1, v_2, v_3) = \mathbb{R}^2$ but $\{v_1, v_2, v_3\}$ is not linearly independent. Thus $\{v_1, v_2, v_3\}$ is <u>not</u> a basis of $\mathbb{R}^2$.

Corollary

Every basis of $\mathbb{R}^n$ consists of n vectors.

Theorem

Let V be a vector space. If V has a basis consisting of n vectors then every basis of V consists of n vectors.

Proof: Let $\mathcal{B} = \{b_1, b_2, \dots, b_n\}$, $\mathcal{D} = \{d_1, d_2, \dots, d_m\}$ be two bases of V.

We have:

1) For each $v \in V$, the coordinate vector $[v]_{\mathcal{B}}$ is a vector in $\mathbb{R}^n$.
2) Since $\{d_1, d_2, \dots, d_m\}$ is a basis of V, the set $\{[d_1]_{\mathcal{B}}, [d_2]_{\mathcal{B}}, \dots, [d_m]_{\mathcal{B}}\}$ is a basis of $\mathbb{R}^n$.

Since every basis of $\mathbb{R}^n$ consists of n vectors, we obtain $m = n$.

Definition

A vector space has *dimension* n if V has a basis consisting of n vectors. Then we write $\dim V = n$.

Example.

1) In $\mathbb{R}^n$ take the standard basis:

$$\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

Since this basis consists of n vectors, we obtain $\dim \mathbb{R}^n = n$.

2) <u>Recall</u>: $\mathbb{P}^n$ = the vector space of polynomials of degree $\leq n$

The standard basis of $\mathbb{P}_n$:

$$\mathcal{E} = \{1, t, t^2, \dots, t^n\}$$

Since $\mathcal{E}$ consists of $n+1$ vectors, we obtain: $\dim \mathbb{P}_n = n+1$.

Theorem

Let V be a vector space such that $\dim V = n$, and let $\mathbf{v}_1, \dots \mathbf{v}_p \in V$.

1) If $\mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_p) = V$ then $p \geq n$.

2) If $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a linearly independent set then $p \leq n$.

Proof: It is enough to check this for $V = \mathbb{R}^n$.

1) If $v_1, \dots, v_p$ are vectors in $\mathbb{R}^n$ and $p < n$ then the matrix $[v_1 \dots v_p]$ can't have a pivot position in every row, so $\mathrm{Span}(v_1, \dots, v_p) \neq \mathbb{R}^n$.

2) If $v_1, \dots, v_p$ are vectors in $\mathbb{R}^n$ and $p > n$ then the matrix $[v_1 \dots v_p]$ can't have a pivot position in every column, so the set $\{v_1, \dots, v_p\}$ is not linearly independent.

Corollary

Let V be a vector space such that $\dim V = n$. If W be a subspace of V then $\dim W \leq n$. Moreover, if $\dim W = n$ then $W = V$.

Proof: If $\dim W = m$ then W has a basis consisting of m vectors: Since the set $\{w_1, \dots, w_m\}$ is a linearly independent set in V, by the Theorem above we obtain:

$$\dim W = m \leq n = \dim V$$

Next, assume that $\dim W = n = \dim V$ and that $\{w_1, \dots, w_n\}$ is a basis of W. If $W \neq V$, we can find a vector $v \in V$, such that $v \notin W$. Then $\{w_1, \dots, w_n, v\}$ is a linearly independent set consisting of $n+1$ vectors of V. By the above theorem this is impossible.

Note.

1) One can show that every vector space has a basis.

2) Some vector spaces have bases consisting of infinitely many vectors. If V is such vector space then we write $\dim V = \infty$.

Example.

1) Recall: $\mathbb{P} = \{$the vector space of all polynomials$\}$
$= \{a_0 + a_1 t + \ldots + a_n t^n \mid a_i \in \mathbb{R},\ n \geqslant 0\}$

The set $\mathcal{E} = \{1, t, t^2, \ldots\}$ is a basis of $\mathbb{P}$. Since $\mathcal{E}$ consists of infinitely many vectors, we get that $\dim \mathbb{P} = \infty$

2) Recall: $C^\infty(\mathbb{R}) = \{$the vector space of all functions $f: \mathbb{R} \to \mathbb{R}\}$

Since $\mathbb{P}$ is a subspace of $C^\infty(\mathbb{R})$ and $\dim \mathbb{P} = \infty$, we get that $\dim C^\infty(\mathbb{R}) = \infty$

It is not possible to write explicitly a basis of $C^\infty(\mathbb{R})$.