Defitnition

Let V be a vector space. A *subspace* of V is a subset $W \subseteq V$ such that

1) $\mathbf{0} \in W$
2) if $\mathbf{u}, \mathbf{v} \in W$ then $\mathbf{u} + \mathbf{v} \in W$
3) if $\mathbf{u} \in W$ and $c \in \mathbb{R}$ then $c\mathbf{u} \in W$.

Example.

Recall: $\mathbb{P} =$ the vector space of all polynomials.

Take $\mathbb{P}_n = \{$the set of all polynomials of degree $\leq n\}$

$\mathbb{P}_n$ is a subspace of $\mathbb{P}$.

Note: Let $S_3 = \{$the set of polynomials of degree equal to $3\}$

S_3 is not a subspace of $\mathbb{P}$

E.g.

$$\left.\begin{array}{l} p(t) = 7 + t - 2t^2 + 3t^3 \\ q(t) = 5 - 4t + 2t^2 - 3t^3 \end{array}\right\} \text{ polynomials in } S_3$$

$$p(t) + q(t) = 12 - 3t \;\} \text{ polynomial of degree 1, not in } S_3$$

Proposition

Let V be a vector space and $W \subseteq V$ is a subspace then W is itself a vector space.

Example.

Recall: $\mathcal{F}(\mathbb{R})$ = the vector space of all functions $f\colon \mathbb{R} \to \mathbb{R}$

Some interesting subspaces of $\mathcal{F}(\mathbb{R})$:

1) $C(\mathbb{R})$ = the subspace of all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$

2) $C^n(\mathbb{R})$ = the subspace of all functions $f\colon \mathbb{R} \to \mathbb{R}$ that are differentiable n or more times.

3) $C^\infty(\mathbb{R})$ = the subspace of all smooth functions $f\colon \mathbb{R} \to \mathbb{R}$ (i.e. functions that have derivatives of all orders: f', f'', f''', ...).

Note: Let $S = \left\{ \begin{array}{l} \text{the set of all functions } f: \mathbb{R} \to \mathbb{R} \\ \text{such that } f(t) \geqslant 0 \text{ for all } t \in \mathbb{R} \end{array} \right\}$

S is not a subspace of $\mathcal{F}(\mathbb{R})$.

E.g. Take $f(t) = t^2$.

Then $f(t) \in S$ but $(-2)\cdot f(t) = -2t^2$ is not in S.

Note. If V is a vector space then:

1) the biggest subspace of V is V itself;

2) the smallest subspace of V is the subspace $\{\mathbf{0}\}$ consisting of the zero vector only;

3) if a subspace of V contains a non-zero vector, then it contains infinitely many vectors.

Indeed, if W is a subspace of V, and $u \in W$ for some $u \neq \mathbf{0}$ then for any $c \in \mathbb{R}$ we have $cu \in W$ and $c_1 u \neq c_2 u$ if $c_1 \neq c_2$.