Recall:

Vector equations are equivalent to systems of linear equations:

$$x_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix} \quad \Longleftrightarrow \quad \begin{cases} 2x_1 + 4x_2 = 7 \\ 3x_1 + 2x_2 = 3 \end{cases}$$

vector equation

system of linear equations

Upshot. A vector equation can have either:

- no solutions
- exactly one solution
- infinitely many solutions

Next:

- When does a vector equation have a solution?
- When does it have exactly one solution?

Definition

A vector $\mathbf{w} \in \mathbb{R}^n$ is a *linear combination* of vectors $\mathbf{v}_1, \ldots \mathbf{v}_p \in \mathbb{R}^n$ if there exists scalars $c_1, \ldots, c_p$ such that

$$\mathbf{w} = c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p$$

Equivalently: A vector $\mathbf{w}$ is a linear combination of vectors $\mathbf{v}_1, \ldots \mathbf{v}_p$ is the vector equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{w}$$

has a solution.

Example.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix}$$

Some linear combinations of v_1, v_2, v_3:

$$2v_1 + v_2 - v_3 = 2 \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}$$

$$\underset{\displaystyle v_1 + 0v_2 - v_3}{\underset{\shortparallel}{v_1 - v_3}} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ -2 \end{bmatrix}$$

$$0v_1 + 0v_2 + 0v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 9 \\ 3 \\ 6 \end{bmatrix}$$

Express $\mathbf{w}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ or show that this is not possible.

Solution: We need to solve the equation

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = w$$

augmented matrix

$$[\, v_1 \ v_2 \ v_3 \mid w \,] = \left[\begin{array}{ccc|c} 1 & 3 & 5 & 9 \\ 2 & 1 & 0 & 3 \\ 1 & 2 & 3 & 6 \end{array}\right]$$

$$\Big\downarrow \text{ row reduction}$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \multicolumn{4}{c}{\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]} \\ & & \text{free} & \end{array}$$

$$\begin{cases} x_1 = x_3 \\ x_2 = 3 - 2x_3 \\ x_3 = \text{free} \end{cases}$$

We obtain: w is a linear combination of v_1, v_2, v_3.

e.g. if $x_3 = 2$ then $w = 2v_1 + (-1)v_2 + 2v_3$

$x_3 = 1$ then $w = v_1 + v_2 + v_3$

$\vdots$

Example. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

Express $\mathbf{w}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$ or show that this is not possible.

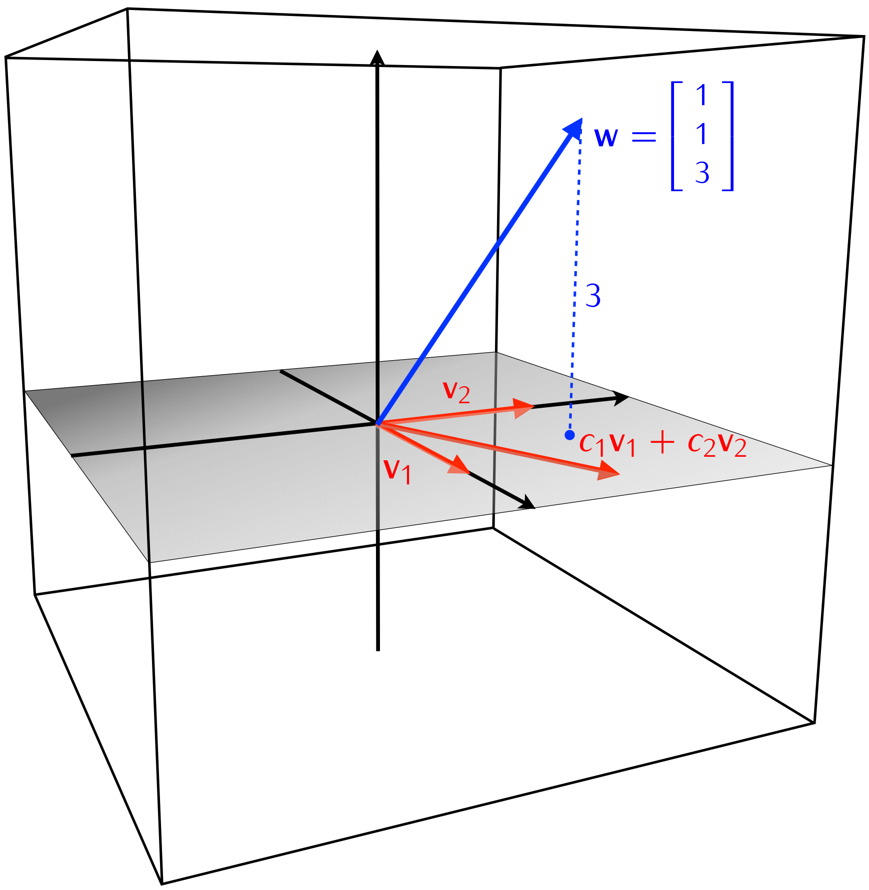
<u>Solution:</u> Linear combinations of v_1, v_2 are vectors of the form:

$$c_1 v_1 + c_2 v_2 = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ 0 \end{bmatrix}$$

the last coordinate is 0.

Since $w = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ is not of this form, it is <u>not</u> a linear combination of v_1, v_2.

Geometric picture of the last example



Definition

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are vectors in $\mathbb{R}^n$ then

$$\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p) = \left\{ \begin{array}{c} \text{the set of all} \\ \text{linear combinations} \\ c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p \end{array} \right\}$$

Example.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\mathrm{Span}(v_1, v_2) = \{\text{the set of all vectors } c_1v_1 + c_2v_2\}$

$= \left\{\text{the set of all vectors } \begin{bmatrix} c_1 \\ c_2 \\ 0 \end{bmatrix}\right\}$

e.g.:

$\underset{v_1+2v_2}{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}}, \underset{-3v_1+4v_2}{\begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}}, \underset{0v_1+0v_2}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} \in \mathrm{span}(v_1, v_2)$

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix} \notin \mathrm{Span}(v_1, v_2)$

Proposition

A vector $\mathbf{w}$ is in Span$(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ if and only if the vector equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{w}$$

has a solution.

Proof: $w \in \text{Span}(v_1, \ldots, v_p)$

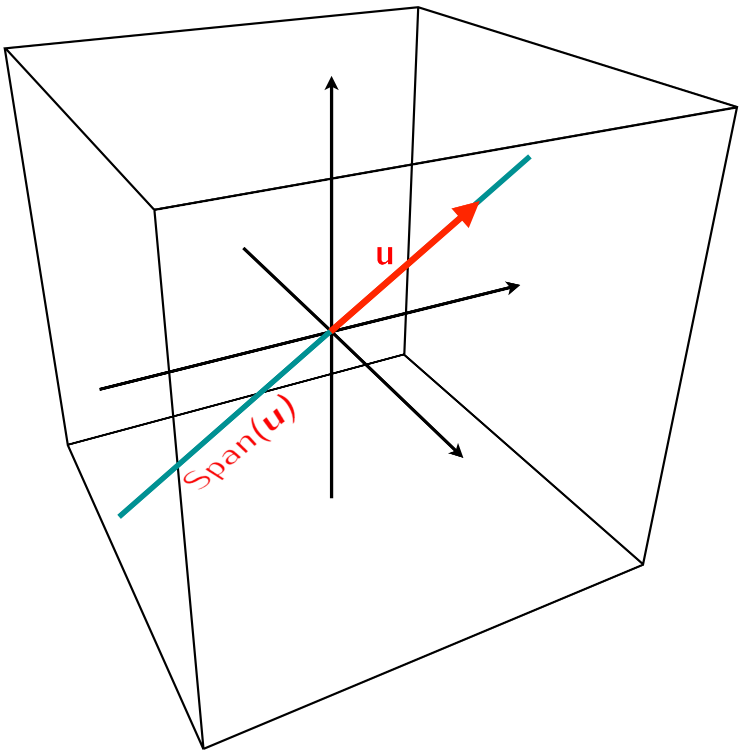
$\Updownarrow$

w is a linear combination of $v_1, \ldots, v_p$

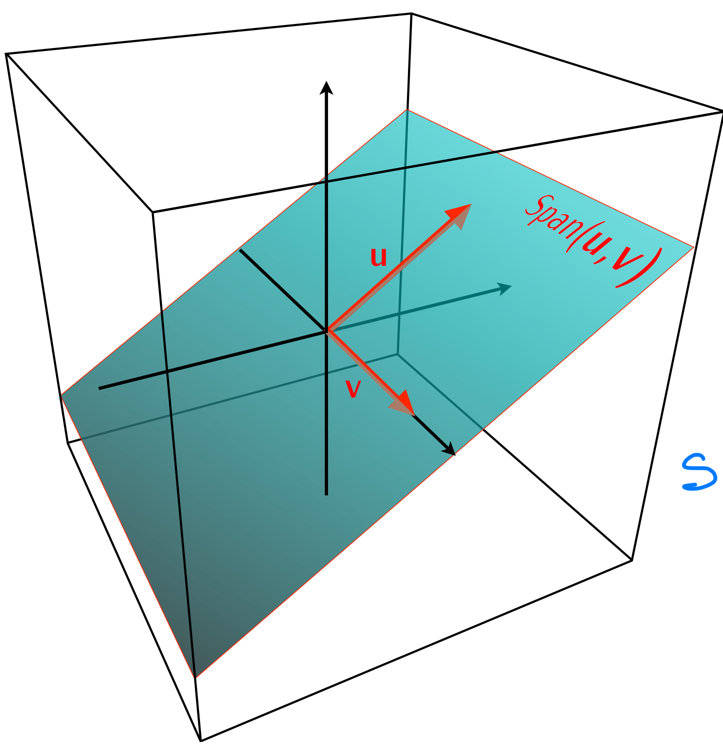
$\Updownarrow$

$x_1 v_1 + \ldots + x_p v_p = w$ has a solution

Geometric interpretation of Span



$$\text{Span}(u) = \{ cu \mid c \in \mathbb{R} \}$$



$$\text{Span}(u, v) = \{ c_1 u + c_2 v \mid c_1, c_2 \in \mathbb{R} \}$$

Proposition

For arbitrary vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$ the zero vector $\mathbf{0} \in \mathbb{R}^n$ is in $\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p)$.

Proof:

$$\underbrace{\mathbf{0}}_{\uparrow \text{ the zero vector}} = 0v_1 + 0v_2 + \ldots + 0v_p$$

so $\mathbf{0}$ is a linear combination of $v_1, \ldots, v_p$

and so $\mathbf{0} \in \mathrm{Span}(v_1, \ldots, v_p)$