

Recall:

- A basis of a vector space  $V$  is a set of vectors  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  such that
  - 1)  $\text{Span}(\mathbf{b}_1, \dots, \mathbf{b}_n) = V$
  - 2) The set  $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  is linearly independent.

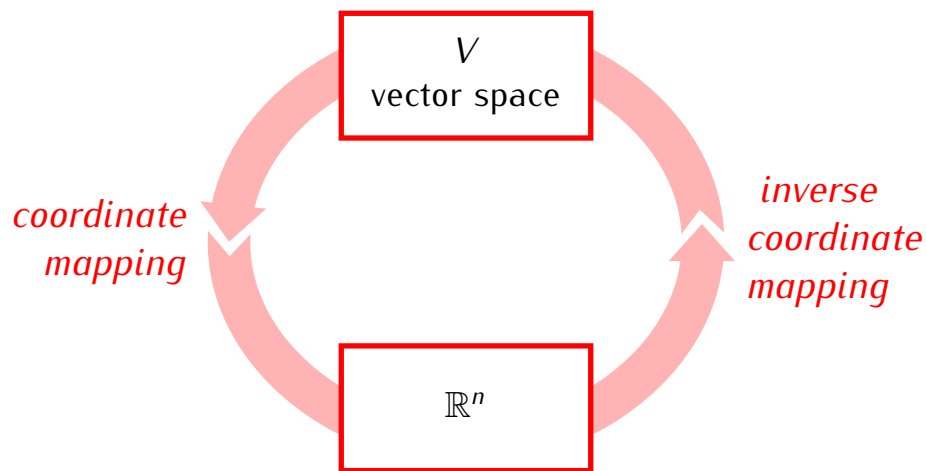
- For  $\mathbf{v} \in V$  let  $c_1, \dots, c_n$  be the unique numbers such that

$$c_1 \mathbf{b}_1 + \dots + c_n \mathbf{b}_n = \mathbf{v}$$

The vector

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$$

is called the *coordinate vector of  $\mathbf{v}$  relative to the basis  $\mathcal{B}$* .



### Theorem

Let  $\mathcal{B}$  be a basis of a vector space  $V$ . If  $\mathbf{v}_1, \dots, \mathbf{v}_p, \mathbf{w} \in V$  then:

- 1) Solutions of the equation  $x_1 \mathbf{v}_1 + \dots + x_p \mathbf{v}_p = \mathbf{w}$  are the same as solutions of the equation  $x_1 [\mathbf{v}_1]_{\mathcal{B}} + \dots + x_p [\mathbf{v}_p]_{\mathcal{B}} = [\mathbf{w}]_{\mathcal{B}}$ .
- 2) The set of vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is linearly independent if and only if the set  $\{[\mathbf{v}_1]_{\mathcal{B}}, \dots, [\mathbf{v}_p]_{\mathcal{B}}\}$  is linearly independent.
- 3)  $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_p) = V$  if and only if  $\text{Span}([\mathbf{v}_1]_{\mathcal{B}}, \dots, [\mathbf{v}_p]_{\mathcal{B}}) = \mathbb{R}^n$ .
- 4)  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is a basis of  $V$  if and only if  $\{[\mathbf{v}_1]_{\mathcal{B}}, \dots, [\mathbf{v}_p]_{\mathcal{B}}\}$  is a basis of  $\mathbb{R}^n$ .

**Example.** Recall that  $\mathbb{P}_2$  is the vector space of polynomials of degree  $\leq 2$ . Consider the following polynomials in  $\mathbb{P}_2$ :

$$p_1(t) = 1 + 2t + t^2$$

$$p_2(t) = 3 + t + 2t^2$$

$$p_3(t) = 1 - 8t - t^2$$

Check if the set  $\{p_1, p_2, p_3\}$  is linearly independent.

### Theorem

Let  $\{v_1, \dots, v_p\}$  be vectors in  $\mathbb{R}^n$ . The set  $\{v_1, \dots, v_p\}$  is a basis of  $\mathbb{R}^n$  if and only if the matrix

$$A = \begin{bmatrix} v_1 & \dots & v_p \end{bmatrix}$$

has a pivot position in every row and in every column (i.e. if  $A$  is an invertible matrix).

### Corollary

Every basis of  $\mathbb{R}^n$  consists of  $n$  vectors.

### Theorem

Let  $V$  be a vector space. If  $V$  has a basis consisting of  $n$  vectors then every basis of  $V$  consists of  $n$  vectors.

### Definition

A vector space has *dimension*  $n$  if  $V$  has a basis consisting of  $n$  vectors. Then we write  $\dim V = n$ .

**Example.**

### Theorem

Let  $V$  be a vector space such that  $\dim V = n$ , and let  $v_1, \dots, v_p \in V$ .

1) If  $\text{Span}(v_1, \dots, v_p) = V$  then  $p \geq n$ .

2) If  $\{v_1, \dots, v_p\}$  is a linearly independent set then  $p \leq n$ .

### Corollary

Let  $V$  be a vector space such that  $\dim V = n$ . If  $W$  be a subspace of  $V$  then  $\dim W \leq n$ . Moreover, if  $\dim W = n$  then  $W = V$ .

**Note.**

- 1) One can show that every vector space has a basis.
- 2) Some vector spaces have bases consisting of infinitely many vectors. If  $V$  is such vector space then we write  $\dim V = \infty$ .

**Example.**