Recall:

• A vector space is a set V equipped with operations of addition and multiplication by scalars. These operations must satisfy some properties.

• Some examples of vector spaces:

1) $\mathbb{R}^n$ = the vector space of column vectors.

2) $\mathcal{F}(\mathbb{R})$ = the vector space of all functions $f\colon \mathbb{R} \to \mathbb{R}$.

3) $C(\mathbb{R})$ = the vector space of all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$.

4) $C^\infty(\mathbb{R})$ = the vector space of all smooth functions $f\colon \mathbb{R} \to \mathbb{R}$.

5) $M_{m,n}(\mathbb{R})$ = the vector space of all $m \times n$ matrices.

6) $\mathbb{P}$ = the vector space of all polynomials.

7) $\mathbb{P}_n$ = the vector space of polynomials of degree $\leq n$.

• If V, W are vector spaces then a linear transformation is a function $T\colon V \to W$ such that

1) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

2) $T(c\mathbf{v}) = cT(\mathbf{v})$

• Many problems involving $\mathbb{R}^n$ can be easily solved using row reduction, matrix multiplication etc.

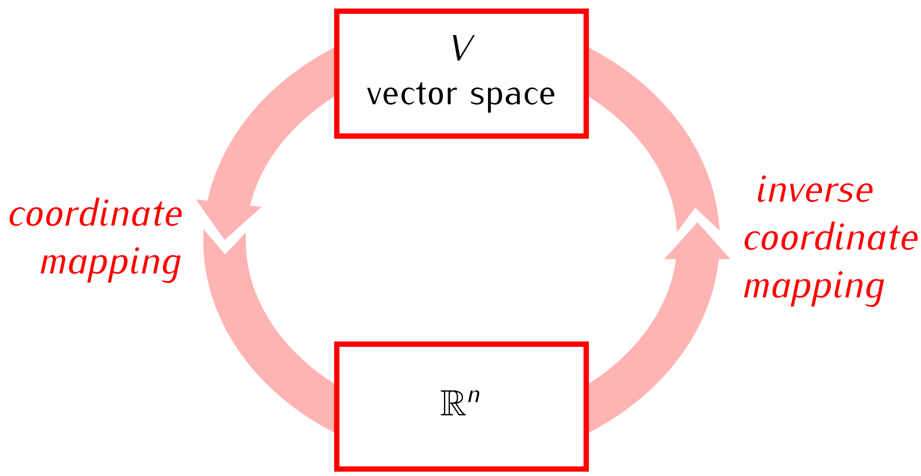
• The same types of problems involving other vector spaces can be difficult to solve.

Next goal:

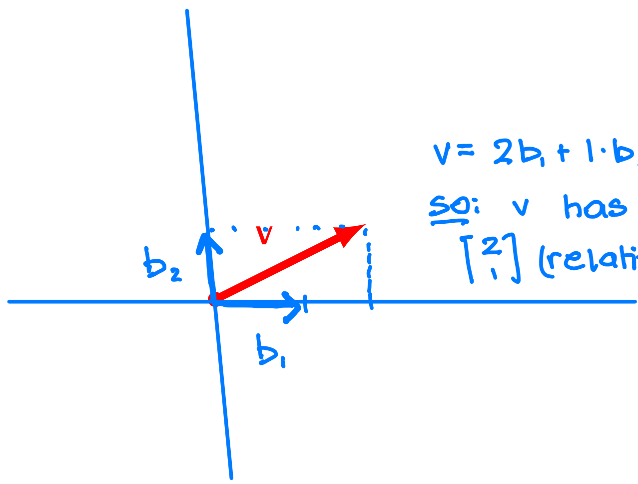
If V is a *finite dimensional* vector space then we can construct a *coordinate mapping*

$$V \to \mathbb{R}^n$$

which lets us turn computations in V into computations in $\mathbb{R}^n$.



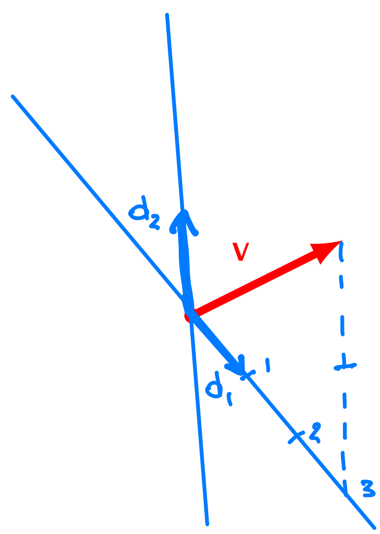
Motivation: How to assign coordinates to vectors



$v = 2b_1 + 1 \cdot b_2$

<u>so:</u> v has coordinates

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (relative to $\{b_1, b_2\}$)



$v = 3d_1 + 2d_2$

<u>so:</u> v has coordinates

$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (relative to $\{d_1, d_2\}$)

<u>Upshot</u>: In order to define a coordinate system in a vector space V, we need to select vectors $b_1, \dots, b_p$ such that any vector v can be written as

$$v = c_1 b_1 + c_2 b_2 + \dots + c_p b_p$$

in a unique way. Then v will have coordinates $\begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$ relative to $\{b_1, b_2, \dots, b_p\}$.

Definition

If V is a vector space then vector $\mathbf{w} \in V$ is a *linear combination* of vectors $\mathbf{v}_1, \dots \mathbf{v}_p \in V$ if there exist scalars $c_1, \dots, c_p$ such that

$$\mathbf{w} = c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p$$

Definition

If V is a vector space and $\mathbf{v}_1, \dots, \mathbf{v}_p$ are vectors in V then

$$\mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_p) = \left\{ \begin{array}{c} \text{the set of all} \\ \text{linear combinations} \\ c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p \end{array} \right\}$$

Note: If $v_1, \dots, v_p \in V$ are vectors such that

$$\mathrm{Span}(v_1, \dots, v_p) = V$$

then every vector $w \in V$ can be written as

$$w = c_1 v_1 + c_2 v_2 + \dots + c_p v_p$$

for some $c_1, \dots, c_p \in \mathbb{R}$.

Definition

If V is a vector space and $\mathbf{v}_1, \ldots, \mathbf{v}_p \in V$ then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is *linearly independent* if the homogenous equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

has only one, trivial solution $x_1 = 0, \ldots, x_p = 0$. Otherwise the set is *linearly dependent*.

Theorem

Let V be a vector space, and let $\mathbf{v}_1, \ldots, \mathbf{v}_p \in V$. If the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent then the equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{w}$$

has exactly one solution for any vector $\mathbf{w} \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p)$.

Proof: The same as for vector equations in $\mathbb{R}^n$.

Definition

A *basis* of a vector space V is an ordered set of vectors

$$\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$$

such that

1) $\mathrm{Span}(\mathbf{b}_1, \ldots, \mathbf{b}_n) = V$

2) The set $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is linearly independent.

Example:

In $\mathbb{R}^n$ let $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

The set $\mathcal{E} = \{e_1, e_2, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$.

This basis is called the standard basis of $\mathbb{R}^n$

Example:

In $\mathbb{R}^2$ take $b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The set $\mathcal{B} = \{b_1, b_2\}$ is a basis of $\mathbb{R}^2$.

Check: 1) the set $\{b_1, b_2\}$ is linearly independent

2) $\mathrm{Span}(b_1, b_2) = \mathbb{R}^2$, since if $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2$ then

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (a_2 - a_1) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = a_1 b_1 + (a_2 - a_1) b_2$$

Example:

Let $\mathbb{P}_n$ = the vector space of polynomials of degree $\leq n$

$= \{ a_0 + a_1 t + \ldots + a_n t^n \mid a_i \in \mathbb{R} \}$

The set $\mathcal{E} = \{1, t, t^2, \ldots, t^n\}$ is a basis of $\mathbb{P}_n$.

This basis is called the standard basis of $\mathbb{P}_n$.

Theorem

A set $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis of a vector space V if any only if for each $\mathbf{v} \in V$ the vector equation

$$x_1\mathbf{b}_1 + \ldots + x_n\mathbf{b}_n = \mathbf{v}$$

has a unique solution.

Proof: Since $\mathrm{Span}(b_1, \ldots, b_n) = V$ thus for each $v \in V$ the equation

$$x_1 b_1 + \ldots + x_n b_n = v$$

has a solution.

Since the set $\{b_1, \ldots, b_n\}$ is linearly independent, this solution is unique.

Definition

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis of a vector space V. For $\mathbf{v} \in V$ let $c_1, \ldots, c_n$ be the unique numbers such that

$$c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n = \mathbf{v}$$

Then the vector

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$$

is called the *coordinate vector of* $\mathbf{v}$ *relative to the basis* $\mathcal{B}$ and it is denoted by $[\mathbf{v}]_{\mathcal{B}}$.

Example. Let $\mathcal{E} = \{1, t, t^2\}$ be the standard basis of $\mathbb{P}_2$, and let

$$p(t) = 3 + 2t - 4t^2$$

Find the coordinate vector $[p]_{\mathcal{E}}$.

Solution: We have:

$$p(t) = 3 \cdot \underbrace{1}_{\substack{1^{st}\text{ vector}\\ \text{of } \mathcal{E}}} + 2 \cdot \underbrace{t}_{\substack{2^{nd}\text{ vector}\\ \text{of } \mathcal{E}}} + (-4) \cdot \underbrace{t^2}_{\substack{3^{rd}\text{ vector}\\ \text{of } \mathcal{E}}}$$

so:

$$[p(t)]_{\mathcal{E}} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$$

Example. Let $\mathcal{B} = \{1, 1+t, 1+t+t^2\}$. One can check that $\mathcal{B}$ is a basis of $\mathbb{P}_2$. Let

$$p(t) = 3 + 2t - 4t^2$$

Find the coordinate vector $[p]_{\mathcal{B}}$.

Solution: We have:

$$p(t) = 1 \cdot \underbrace{1}_{\substack{1^{st}\text{ vector}\\ \text{of } \mathcal{B}}} + 6 \cdot \underbrace{(1+t)}_{\substack{2^{nd}\text{ vector}\\ \text{of } \mathcal{B}}} + (-4) \underbrace{(1+t+t^2)}_{\substack{3^{rd}\text{ vector}\\ \text{of } \mathcal{B}}}$$

so:

$$[p(t)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 6 \\ -4 \end{bmatrix}$$

Example. Consider the following vectors in $\mathbb{R}^2$:

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{b}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

One can check that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis of $\mathbb{R}^2$. Find the coordinate vector $[\mathbf{v}]_{\mathcal{B}}$.

Solution:

We have $[v]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ where $c_1 b_1 + c_2 b_2 = v$.

Thus it suffices to solve the equation $x_1 b_1 + x_2 b_2 = v$.

Augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow[\text{reduction}]{\text{row}} \left[\begin{array}{cc|c} 1 & 0 & 1/3 \\ 0 & 1 & 1/3 \end{array}\right]$$

Thus: $\begin{cases} x_1 = 1/3 \\ x_2 = 1/3 \end{cases}$

so:

$$[v]_{\mathcal{B}} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$$