Recall:

1) A square matrix A is diagonalizable if there exists an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}$$

2) If A is diagonalizable then it is easy to compute powers of A:

$$A^k = PD^kP^{-1}$$

3) An $n \times n$ matrix A is a diagonalizable if and only if it has n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$. In such case we have:

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \qquad \begin{array}{l} \lambda_1 = \text{eigenvalue corresponding to } \mathbf{v}_1 \\ \lambda_2 = \text{eigenvalue corresponding to } \mathbf{v}_2 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \lambda_n = \text{eigenvalue corresponding to } \mathbf{v}_n \end{array}$$

4) Not every square matrix is diagonalizable.

Definition

An *orthogonal matrix* is square matrix Q such that $Q^TQ = I$ (i.e. $Q^T = Q^{-1}$).

Example.

$$Q = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}$$

$$Q^TQ = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix} \cdot \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Proposition

A square matrix $Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{bmatrix}$ is an orthogonal matrix if and only if $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is an orthonormal set of vectors, i.e.:

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Note. If $\mathbf{v}$, $\mathbf{w}$ are vectors in $\mathbb{R}^n$ then

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T\mathbf{w}$$

Example.

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$v^Tw = \underbrace{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}}_{1\times 3} \cdot \underbrace{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}}_{3\times 1} = 1\cdot 4 + 2\cdot 5 + 3\cdot 6 = v\cdot w$$

Proof of Proposition

$$Q^TQ = \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix} \cdot \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} = \begin{bmatrix} u_1^Tu_1 & u_1^Tu_2 & \cdots & u_1^Tu_n \\ \vdots & & & \vdots \\ u_n^Tu_1 & u_n^Tu_2 & \cdots & u_n^Tu_n \end{bmatrix} = \begin{bmatrix} u_1\cdot u_1 & u_1\cdot u_2 & \cdots & u_1\cdot u_n \\ \vdots & & & \\ u_n\cdot u_1 & u_n\cdot u_2 & \cdots & u_n\cdot u_n \end{bmatrix}$$

It follows that

$$Q^TQ = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \text{ if and only if } u_i \cdot u_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Definition

A square matrix A is *orthogonally diagonalizable* if there exists an orthogonal matrix Q and a diagonal matrix D such that

$$A = QDQ^{-1} = QDQ^T$$

Note. An $n \times n$ matrix A is a orthogonally diagonalizable

$$A = QDQ^T$$

then:

- $Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{bmatrix}$

where $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ are orthonormal eigenvectors:

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

- $D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$

λ_1 = eigenvalue corresponding to $\mathbf{u}_1$
λ_2 = eigenvalue corresponding to $\mathbf{u}_2$
... ...
λ_n = eigenvalue corresponding to $\mathbf{u}_n$

Definition

A square matrix A is *symmetric* if $A^T = A$

Example.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 5 & 6 \\ 3 & 5 & 7 & 8 \\ 4 & 6 & 8 & 9 \end{bmatrix}$$

Proposition

If a matrix A is orthogonally diagonalizable then A is a symmetric matrix.

Proof: If $A = QDQ^T$ then

$$A^T = (QDQ^T)^T = (Q^T)^T D^T Q^T \underset{D^T = D}{=} QDQ^T$$

Spectral Theorem

Every symmetric matrix is orthogonally diagonalizable.

Theorem

If A is a symmetric matrix and λ_1, λ_2 are two different eigenvalues of A, then eigenvectors corresponding to λ_1 are orthogonal to eigenvectors corresponding to λ_2.

Recall: If $\mathbf{v}$, $\mathbf{w}$ are vectors in $\mathbb{R}^n$ then

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w}$$

Proof of Theorem:

Let v = an eigenvector of A corresponding to λ_1

w = ______ " ______ " ______ λ_2

We have:

$$\lambda_1(v \cdot w) = (\lambda_1 v) \cdot w = (Av) \cdot w = (Av)^T w \underset{A^T = A}{=} v^T A w$$

$$= v^T(\lambda_2 w) = \lambda_2(v^T w) = \lambda_2(v \cdot w)$$

This gives:

$$\lambda_1(v \cdot w) = \lambda_2(v \cdot w)$$

$$(\lambda_1 - \lambda_2)(v \cdot w) = 0$$

Since $\lambda_1 \neq \lambda_2$, we have $\lambda_1 - \lambda_2 \neq 0$, so $v \cdot w = 0$.

Example.
Find three orthogonal eigenvectors of the following symmetric matrix:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Solution:

1) Find eigenvalues of A:

$$P(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{bmatrix}$$

$$= -\lambda^3 + 6\lambda^2 - 9\lambda + 4$$

$$(\text{eigenvalues of } A) = (\text{roots of } P(\lambda)) = (\lambda_1 = 4,\ \lambda_2 = 1)$$

2) Find a basis of the eigenspace for each eigenvalue

$$\begin{pmatrix} \text{eigenspace} \\ \text{for } \lambda_1 = 4 \end{pmatrix} = \text{Nul}(A - 4I) \qquad \underline{\text{basis}}: \left\{ \overset{v_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \right\}$$

$$\begin{pmatrix} \text{eigenspace} \\ \text{for } \lambda_2 = 1 \end{pmatrix} = \text{Nul}(A - 1 \cdot I) \qquad \underline{\text{basis}}: \left\{ \overset{v_2}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}, \overset{v_3}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}} \right\}$$

Note:
$v_1 \cdot v_2 = 0$
$v_1 \cdot v_3 = 0$
$v_2 \cdot v_3 = 1$

<u>Upshot</u>: We have 3 eigenvectors:

$$\begin{array}{ccccc} v_1 & & v_2 & & v_3 \\ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\lambda_1 = 4} & , & \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \; , \; \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{\lambda_2 = 1} \end{array}$$

- v_1 is orthogonal to v_2 and v_3 (since it corresponds to a different eigenvalue).
- v_2, v_3 are not orthogonal to each other.

  To fix this, we use G-S process to find an orthogonal basis of the eigenspace for $\lambda_1 = 1$:

$$w_2 = v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$w_3 = v_3 - \left(\frac{w_2 \cdot v_3}{w_2 \cdot w_2}\right) w_2 = \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix}$$

<u>We obtain 3 orthogonal eigenvectors of A</u>:

$$\begin{array}{ccccc} v_1 & & w_2 & & w_3 \\ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\lambda_1 = 4} & , & \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \; , \; \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix}}_{\lambda_2 = 1} \end{array}$$

<u>Upshot.</u> How to find n orthogonal eigenvectors for a symmetric $n \times n$ matrix A:

1) Find eigenvalues of A.

2) Find a basis of the eigenspace for each eigenvalue.

3) Use the Gram-Schmidt process to find an orthogonal basis of each eigenspace.

Example. Find an orthogonal diagonalization of the following symmetric matrix:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Solution:

The previous example gives a diagonalization of A:

$$A = PDP^{-1} \qquad P = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & -1 & -1/2 \\ 1 & 0 & 1 \\ 1 & 1 & -1/2 \end{bmatrix} \end{array} \qquad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This not an orthogonal diagonalization, since P is not an orthogonal matrix:

$$P^TP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3/2 \end{bmatrix}$$

To get an orthogonal matrix, replace P by $Q = \left[\frac{v_1}{\|v_1\|} \ \frac{v_2}{\|v_2\|} \ \frac{v_3}{\|v_3\|} \right]$:

$$\|v_1\| = \sqrt{3}$$
$$\|v_2\| = \sqrt{2}$$
$$\|v_3\| = \sqrt{\tfrac{3}{2}} = \frac{\sqrt{6}}{2}$$

so:

$$Q = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$$

We obtain: $A = QDQ^{-1} = QDQ^T$ where D is the same as above:

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$