Definition

A *homogenous vector equation* is a vector equation of the form

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

(i.e. with the zero vector as the vector of constants).

Definition

Let $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is *linearly independent* if the homogenous equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

has only one, trivial solution $x_1 = 0, \ldots, x_p = 0$. Otherwise the set is *linearly dependent*.

Theorem

Let $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$. Consider the equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{w}$$

where $\mathbf{w} \in \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p)$.

If the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent then this equation has exactly one solution.

If the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent then this equation has infinitely many solutions.

Example. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ -12 \end{bmatrix}$$

Check is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

Note

A set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent if and only if every column of the matrix

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_p \end{bmatrix}$$

is a pivot column.

Some properties of linearly (in)dependent sets

1) A set consisting of one vector $\{\mathbf{v}_1\}$ is linearly dependent if and only if $\mathbf{v}_1 = \mathbf{0}$.

2) A set consisting of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if and only if one vector is a scalar multiple of the other.

3) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a set of p vectors in $\mathbb{R}^n$ and $p > n$ then this set is linearly dependent.

Upshot: how to find the number of solutions of a vector equation