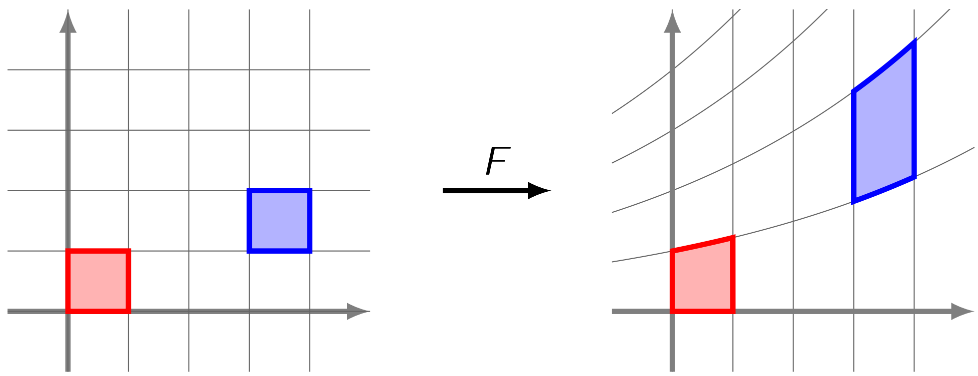
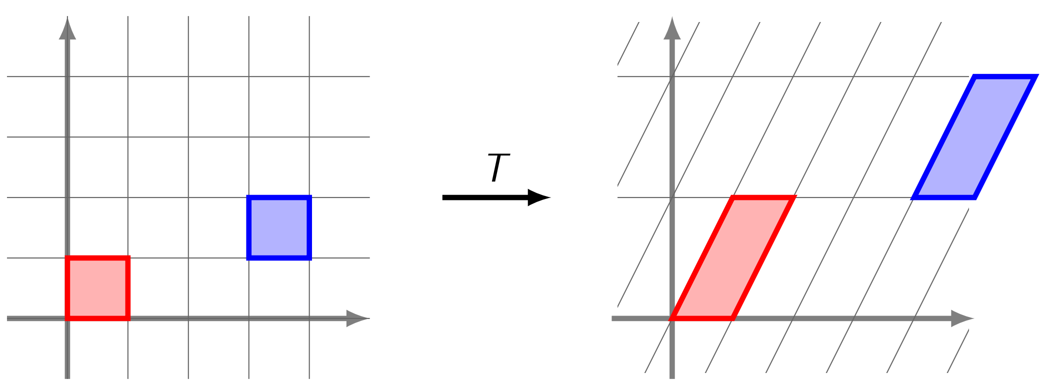
Example. Nonlinear transformation $F\colon \mathbb{R}^2 \to \mathbb{R}^2$

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ ye^x \end{bmatrix}$$



Example. Linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ 2y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$



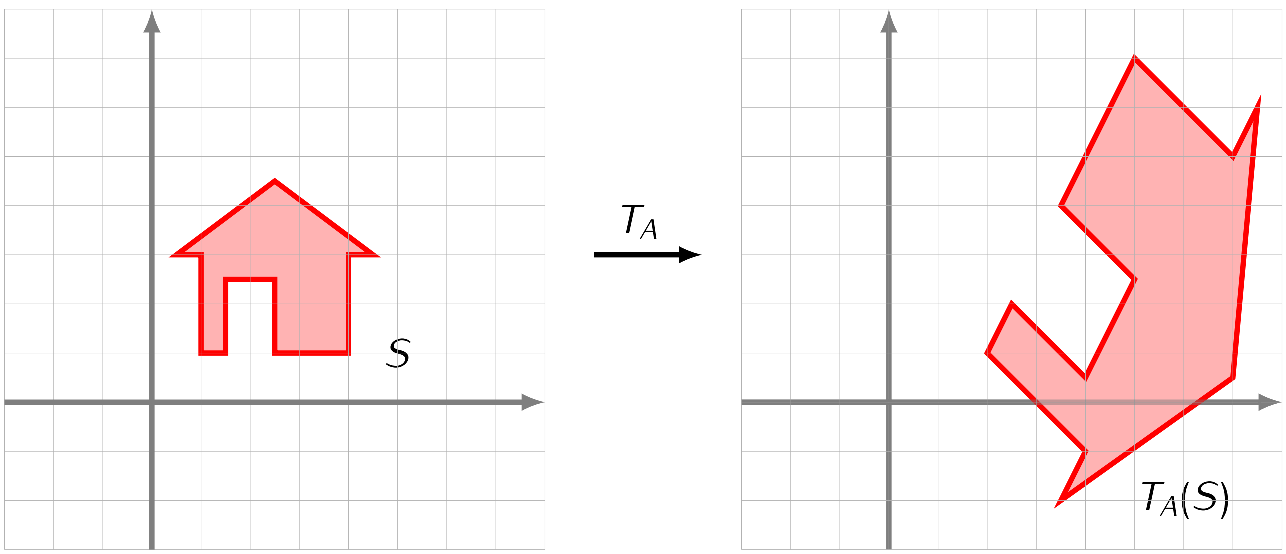
Idea. Given an $n \times n$ matrix A, the determinant $\det A$ is the factor by which the matrix transformation

$$T_A \colon \mathbb{R}^n \to \mathbb{R}^n$$

shrinks or expands the volume of each region of $\mathbb{R}^n$.

Example.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$



$$\text{area}(T_A(S)) = |\det A| \cdot \text{area}(S)$$

Properties of the determinant

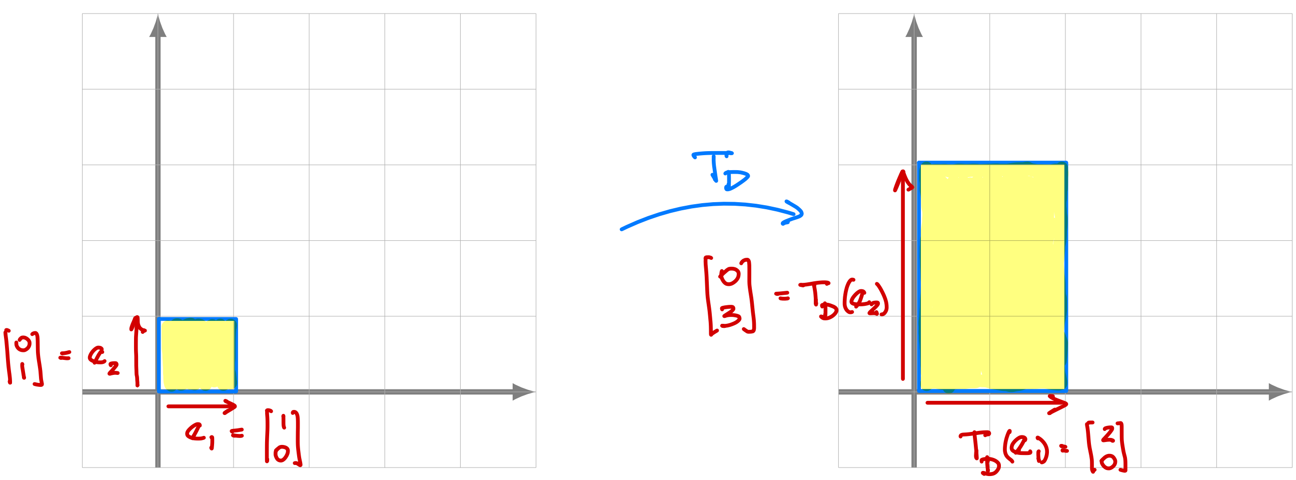
Notation. Given numbers $c_1, c_2, \ldots, c_n \in \mathbb{R}$ let $D(c_1, c_2, \ldots, c_n)$ denote the $n \times n$ matrix

$$D(c_1, c_2, \ldots, c_n) = \begin{bmatrix} c_1 & 0 & \ldots & 0 \\ 0 & c_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & c_n \end{bmatrix}$$

Example.

$$\underset{\substack{\parallel \\ D}}{D(2,3)} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$T_D \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$v \longmapsto Dv$$



$$\det(D(2,3)) = 6 = 2 \cdot 3$$

Property 1. For any numbers $c_1, c_2, \ldots, c_n \in \mathbb{R}$ we have

$$\det D(c_1, c_2, \ldots, c_n) = c_1 \cdot c_2 \cdot \ldots \cdot c_n$$

Notation. Given integers $1 \le i, j \le n$ such that $i \neq j$ and a number $c \in \mathbb{R}$ let $E^n_{i,j}(c)$ denote the $n \times n$ matrix which has:

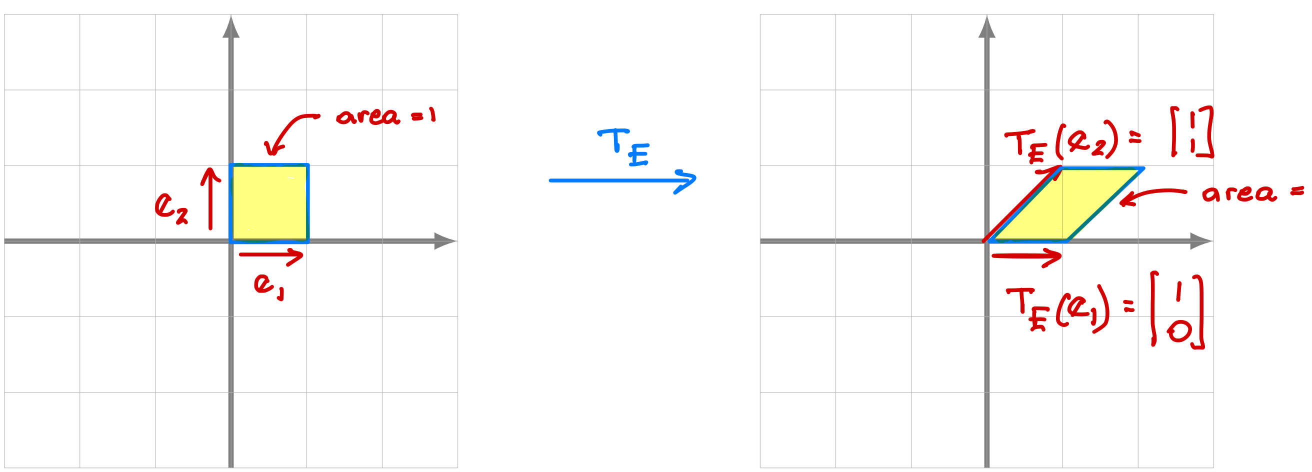
- all entries on the main diagonal equal to 1
- the entry in the i-th row and the j-th column equal to c
- all other entries equal to 0.

Example:

$$E^3_{2,3}(5) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \qquad E^4_{3,1}(7) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 7 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example.

$$\underset{\substack{\shortparallel \\ E}}{E^2_{1,2}(1)} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad T_E \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad v \longmapsto Ev$$



This gives: $\det(E^2_{1,2}(1)) = 1$

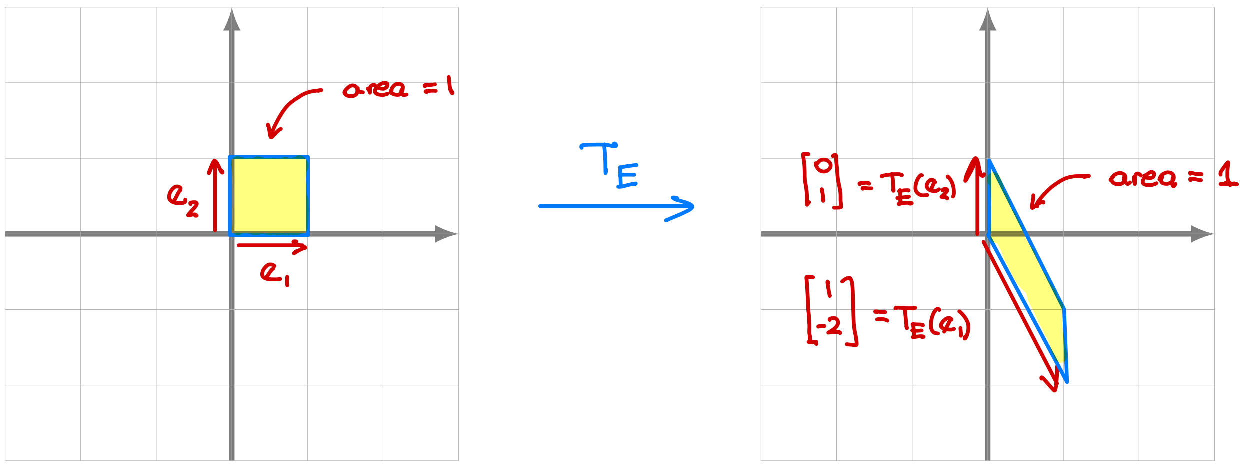
Example.

$$E^2_{2,1}(-2) = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$\parallel$

E

$$T_E : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$v \longmapsto Ev$$



We obtain: $\det E^2_{2,1}(-2) = 1$

Property 2. For any integers $1 \leq i, j \leq n$, $i \neq j$ and a number $c \in \mathbb{R}$ we have

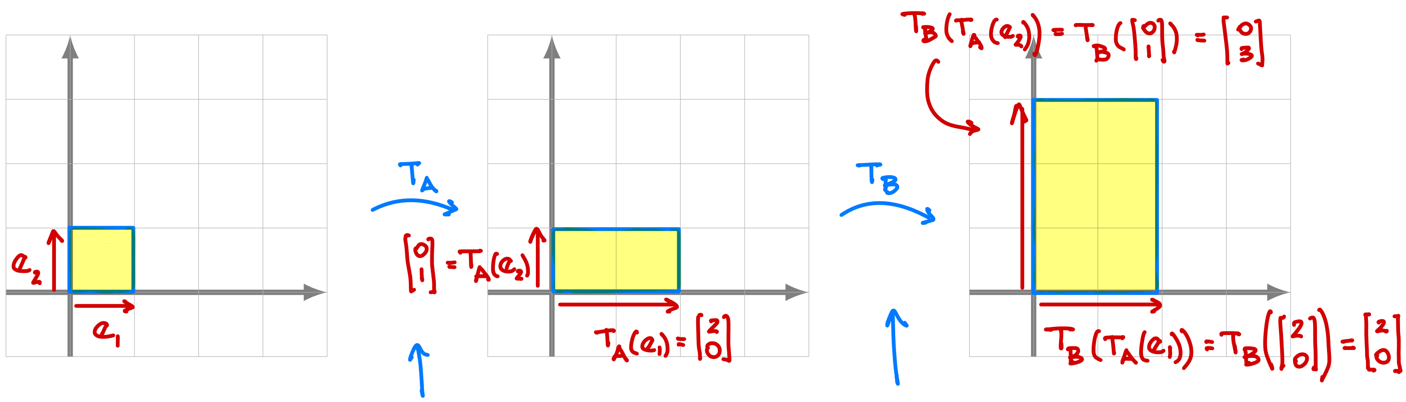
$$\det E^n_{i,j}(c) = 1$$

Example.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad v \longmapsto Av$$

$$T_B : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad v \longmapsto Bv$$



$\det A = 2$ $\qquad$ $\det B = 3$

$$T_B \circ T_A = T_{BA}$$

$$\det BA = 6 = (\det B) \cdot (\det A)$$

Property 3. $\det(AB) = \det(A) \cdot \det(B)$

Theorem

There is exactly one assignment which associates to each $n \times n$ matrix A a number $\det A$ and which satisfies properties 1, 2, and 3.