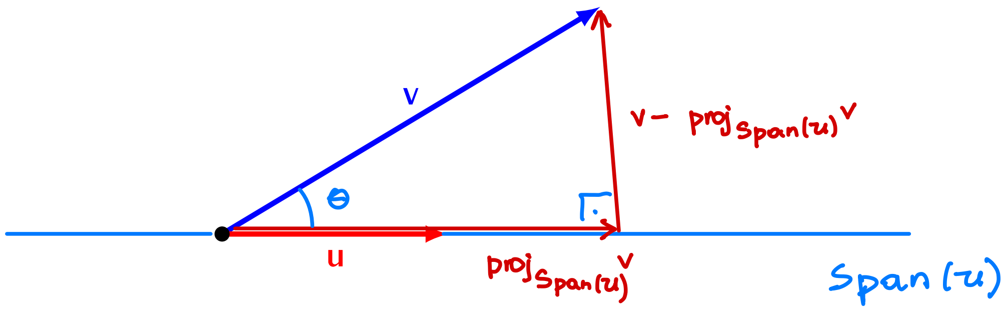
Goal: Given vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ compute the angle between $\mathbf{u}$ and $\mathbf{v}$.



We have:

$$\cos\theta = \frac{\text{proj}_{\text{Span}(u)} v}{\|v\|}$$

Note: $\{u\}$ is an orthogonal basis of Span(u).
This gives:

$$\text{proj}_{\text{Span}(u)} v = \left(\frac{v \cdot u}{u \cdot u}\right) u = \left(\frac{v \cdot u}{\|u\|^2}\right) \cdot u$$

Thus:

$$\|\text{proj}_{\text{Span}(u)} v\| = \frac{v \cdot u}{\|u\|^{\cancel{2}}} \cdot \|\cancel{u}\| = \frac{v \cdot u}{\|u\|}$$

We obtain:

$$\boxed{\cos\theta = \frac{\frac{v \cdot u}{\|u\|}}{\|v\|} = \frac{v \cdot u}{\|v\| \cdot \|u\|}}$$

Proposition

If $\mathbf{u}, \mathbf{v}$ are non-zero vectors in $\mathbb{R}^n$ and θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ then

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\cdot||\mathbf{v}||}$$

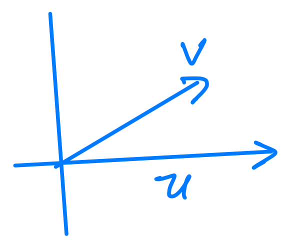
Example. Compute $\cos\theta$, where θ is the angle between the following vectors in $\mathbb{R}^3$:

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution:

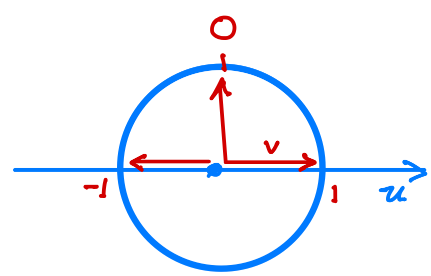
$$\cos\theta = \frac{u\cdot v}{||u||\cdot||v||} = \frac{3}{\sqrt{5}\cdot\sqrt{3}} = \frac{3}{\sqrt{15}} \approx 0.77$$

Note: This gives: $\theta = \arccos\frac{3}{\sqrt{15}} \approx 39.2^\circ$



Note. In data science the number

$$\frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\cdot||\mathbf{v}||}$$



is called the *cosine similarity* between vectors $\mathbf{u}$ and $\mathbf{v}$. The number

$$1 - \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\cdot||\mathbf{v}||}$$

is called the *cosine distance* between $\mathbf{u}$ and $\mathbf{v}$.