Recall:

Vector equations are equivalent to systems of linear equations:

$$x_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix} \quad \Longleftrightarrow \quad \begin{cases} 2x_1 + 4x_2 = 7 \\ 3x_1 + 2x_2 = 3 \end{cases}$$

vector equation

system of linear equations

Upshot. A vector equation can have either:

- no solutions
- exactly one solution
- infinitely many solutions

Next:

- When does a vector equation have a solution?
- When does it have exactly one solution?

Definition

A vector $\mathbf{w} \in \mathbb{R}^n$ is a *linear combination* of vectors $\mathbf{v}_1, \ldots \mathbf{v}_p \in \mathbb{R}^n$ if there exists scalars $c_1, \ldots, c_p$ such that

$$\mathbf{w} = c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p$$

Equivalently: A vector $\mathbf{w}$ is a linear combination of vectors $\mathbf{v}_1, \ldots \mathbf{v}_p$ is the vector equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{w}$$

has a solution.

Example.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix}$$

Example. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 9 \\ 3 \\ 6 \end{bmatrix}$$

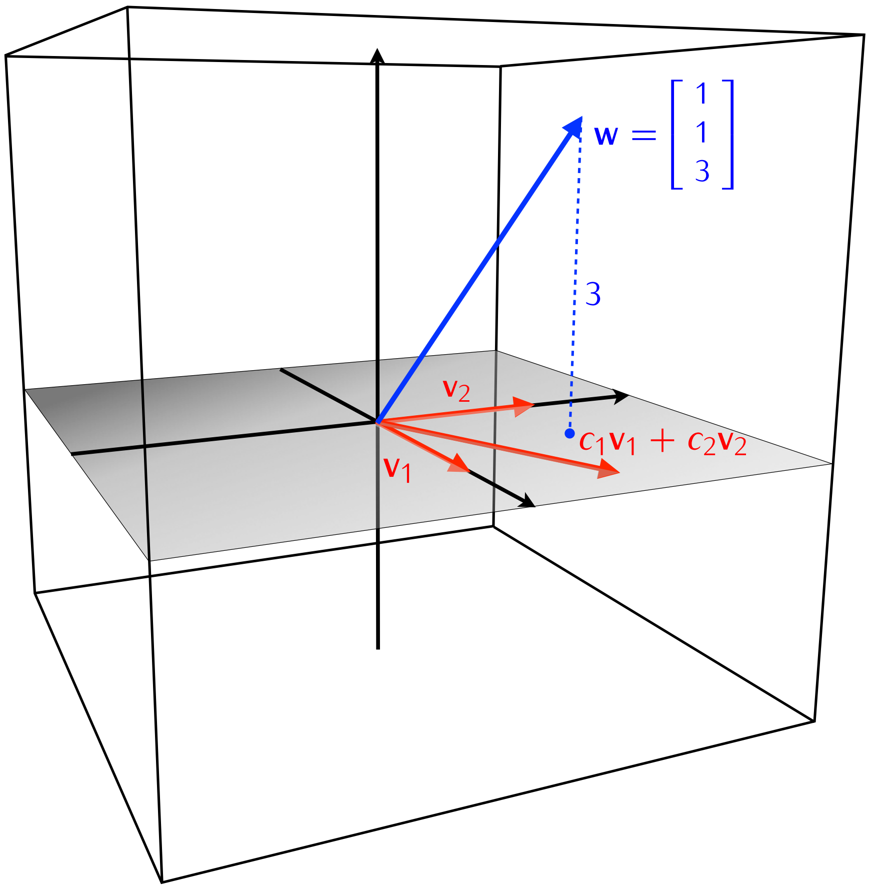
Express $\mathbf{w}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ or show that this is not possible.

Example. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

Express $\mathbf{w}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$ or show that this is not possible.

Geometric picture of the last example



Definition

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are vectors in $\mathbb{R}^n$ then

$$\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p) = \left\{ \begin{array}{c} \text{the set of all} \\ \text{linear combinations} \\ c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p \end{array} \right\}$$

Example.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

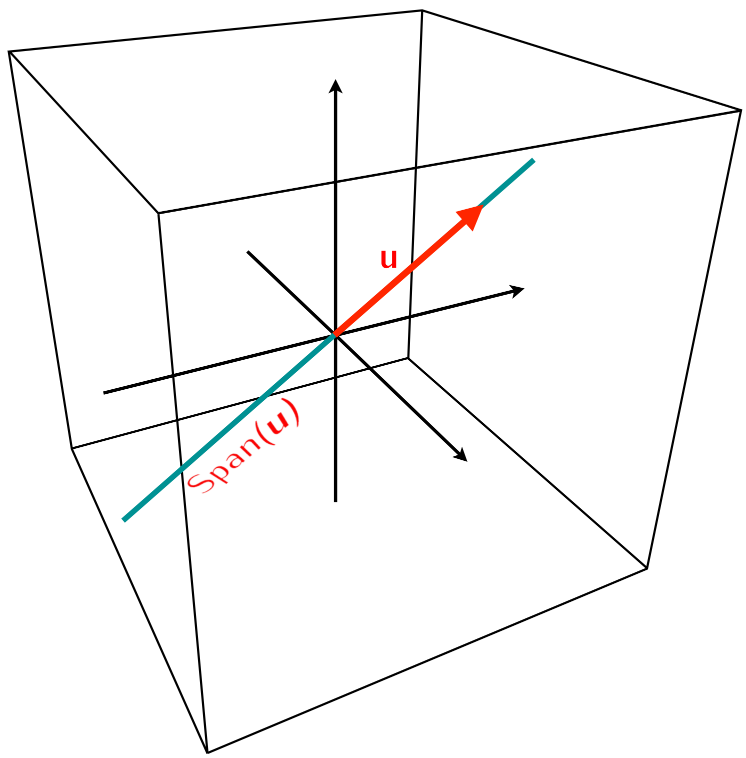
Proposition

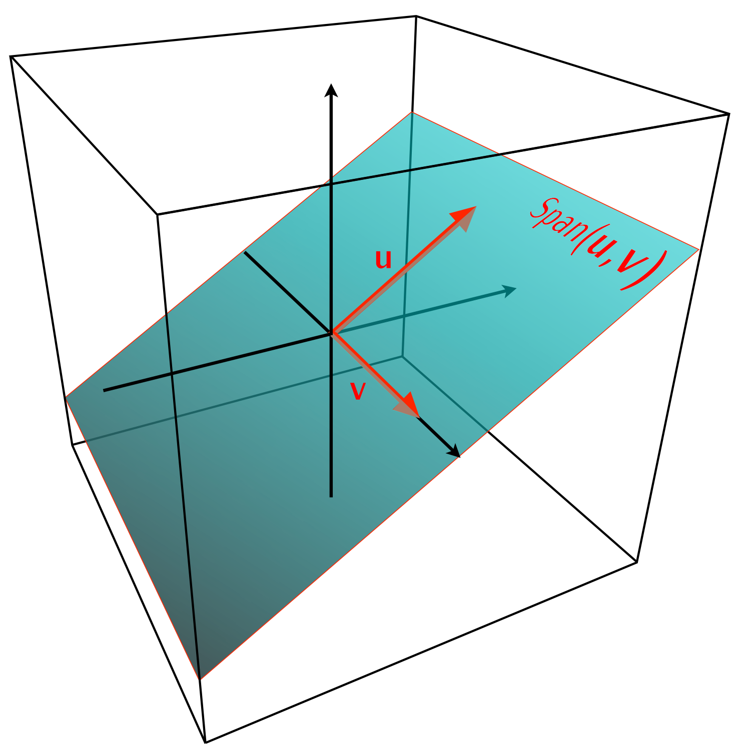
A vector $\mathbf{w}$ is in $\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ if and only if the vector equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{w}$$

has a solution.

Geometric interpretation of Span





Proposition

For arbitrary vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$ the zero vector $\mathbf{0} \in \mathbb{R}^n$ is in $\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p)$.