- **Determinants**

1) Computation:

- by row reduction
- by cofactor expansion

2) Properties:

- a matrix is invertible if and only if $\det A \neq 0$
- determinants and elementary row/column operations
- algebraic properties:

❶ $\det(AB) = \det(A)\det(B)$
❷ $\det(A^{-1}) = (\det A)^{-1}$
❸ $\det(A^T) = \det A$
❹ $\det(A+B) \neq \det A + \det B$

3) Cramer's rule. If A is an $n \times n$ invertible matrix and $\mathbf{b} \in \mathbb{R}^n$ then the solution of the equation

$$A\mathbf{x} = \mathbf{b}$$

is given by

$$\mathbf{x} = \frac{1}{\det A}\begin{bmatrix} \det A_1(\mathbf{b}) \\ \vdots \\ \det A_n(\mathbf{b}) \end{bmatrix}$$

4) If A is an $n \times n$ invertible matrix then

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

where $C_{ij} = (-1)^{i+j} \det A_{ij}$

5) Geometric interpretation of determinants:

- determinants measure how linear transformations change area / volume
- determinants compute areas of polygons
- the sign of a determinant indicates if a linear transformation preserves or reverses orientation

- **General vector spaces**

1) Definition.

2) Examples:

- $\mathbb{R}^n$
- $\mathbb{P}$, $\mathbb{P}_n$ – vector spaces of polynomials
- $M_{m,n}(\mathbb{R})$ – the vector space of $m \times n$ matrices
- $\mathcal{F}(\mathbb{R})$, $C(\mathbb{R})$, $C^\infty(\mathbb{R})$ – vector spaces of functions $f\colon \mathbb{R} \to \mathbb{R}$ (all functions, continuous functions, smooth functions)

3) Subspace of a vector space:

- definition
- subspaces associated to an $m \times n$ matrix A:

  $\mathrm{Nul}(A) \subseteq \mathbb{R}^n$
  $\mathrm{Col}(A) \subseteq \mathbb{R}^m$

4) Linear transformations of vectors spaces:

- definition
- the image $\mathrm{Im}(T)$ and kernel $\mathrm{Ker}(T)$ of a linear transformation T

5) Basis of a vector space

- definition
- computation of bases of $\mathbb{R}^n$, $\mathrm{Col}(A)$ and $\mathrm{Nul}(A)$

– the standard bases of the vector spaces of polynomials ($\mathbb{P}$ and $\mathbb{P}_n$)

6) Coordinates of a vector relative to a basis .

7) Dimension of a vector space:

– definition
– properties

8) Rank of a matrix:

(a) if A is an $m \times n$ matrix then
– rank $A = \dim \mathrm{Col}(A)$
– The rank theorem: rank $A + \dim \mathrm{Nul}(A) = n$.