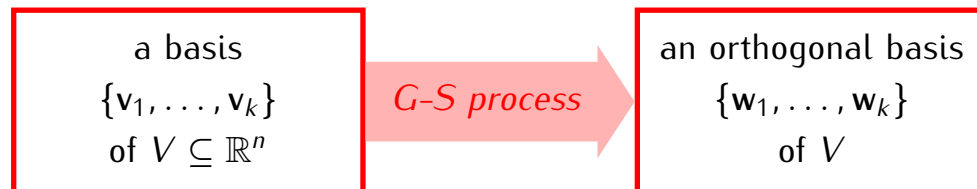- **Inner product**

1) Inner product and orthogonality in $\mathbb{R}^n$:

- definitions of $\mathbf{u} \cdot \mathbf{v}$, $\|\mathbf{v}\|$, dist($\mathbf{u}, \mathbf{v}$)

- orthogonality of vectors

- Pythagorean theorem

- orthogonal sets of vectors

- orthogonal bases and coordinate systems

- Gram–Schmidt process:

a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of $V \subseteq \mathbb{R}^n$	*G–S process*	an orthogonal basis $\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ of V

$$\mathbf{w}_1 = \mathbf{v}_1$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_2}{\mathbf{w}_1 \cdot \mathbf{w}_1} \right) \mathbf{w}_1$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_3}{\mathbf{w}_1 \cdot \mathbf{w}_1} \right) \mathbf{w}_1 - \left(\frac{\mathbf{w}_2 \cdot \mathbf{v}_3}{\mathbf{w}_2 \cdot \mathbf{w}_2} \right) \mathbf{w}_2$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\mathbf{w}_k = \mathbf{v}_k - \left(\frac{\mathbf{w}_1 \cdot \mathbf{v}_k}{\mathbf{w}_1 \cdot \mathbf{w}_1} \right) \mathbf{w}_1 - \left(\frac{\mathbf{w}_2 \cdot \mathbf{v}_k}{\mathbf{w}_2 \cdot \mathbf{w}_2} \right) \mathbf{w}_2 - \dots - \left(\frac{\mathbf{w}_{k-1} \cdot \mathbf{v}_k}{\mathbf{w}_{k-1} \cdot \mathbf{w}_{k-1}} \right) \mathbf{w}_{k-1}$$

2) Orthogonal projections:

    – If V is a subspace of $\mathbb{R}^n$ with an orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and $\mathbf{w} \in \mathbb{R}^n$ then

$$\text{proj}_V \mathbf{w} = \left(\frac{\mathbf{w} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 + \ldots + \left(\frac{\mathbf{w} \cdot \mathbf{v}_k}{\mathbf{v}_k \cdot \mathbf{v}_k} \right) \mathbf{v}_k$$

    – main property: $\text{proj}_V \mathbf{w} \in V$, $\mathbf{w} - \text{proj}_V \mathbf{w} \in V^\perp$

    – Best Approximation Theorem:

$$\text{dist}(\mathbf{w}, \text{proj}_V \mathbf{w}) \leq \text{dist}(\mathbf{w}, \mathbf{v})$$

    for all $\mathbf{v} \in V$.

3) Least square solutions.

    – computation:

$$\begin{pmatrix} \text{least square solutions of} \\ A\mathbf{x} = \mathbf{b} \end{pmatrix} = \begin{pmatrix} \text{solutions of} \\ A\mathbf{x} = \text{proj}_{\text{Col}(A)} \mathbf{b} \end{pmatrix}$$

    – much faster computation:

$$\begin{pmatrix} \text{least square solutions of} \\ A\mathbf{x} = \mathbf{b} \end{pmatrix} = \begin{pmatrix} \text{solutions of} \\ A^T A\mathbf{x} = A^T \mathbf{b} \end{pmatrix}$$

    – application: least square fitting of lines and curves.

• Eigenvalues and eigenvectors

1) Definition.

2) Computation:

  – if A is an $n \times n$ matrix then

$$\text{eigenvalues of } A = \begin{pmatrix} \text{roots of the characteristic polynomial} \\ P(\lambda) = \det(A - \lambda I) \end{pmatrix}$$

  – if λ is an eigenvalue of A then

$$(\text{the eigenspace of } A \text{ corresponding to } \lambda) = \text{Nul}(A - \lambda I)$$

3) Diagonalization of matrices:

  – A square matrix A is diagonalizable if there exists an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}$$

  – An $n \times n$ matrix A is a diagonalizable if and only if it has n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. In such case we have:

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} \quad \begin{array}{l} \lambda_1 = \text{eigenvalue corresponding to } \mathbf{v}_1 \\ \lambda_2 = \text{eigenvalue corresponding to } \mathbf{v}_2 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \lambda_n = \text{eigenvalue corresponding to } \mathbf{v}_n \end{array}$$

  – Not every matrix is diagonalizable.

  – If A is diagonalizable, $A = PDP^{-1}$ then

$$A^k = PD^k P^{-1}$$

4) Symmetric matrices and orthogonal diagonalization.

– An orthogonal matrix $Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_n \end{bmatrix}$ is a square matrix such that $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthonormal set of vectors, i.e.:

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

– If Q is an orthogonal matrix then $Q^{-1} = Q^T$

– A square matrix A is orthogonally diagonalizable if there exist an orthogonal matrix Q and a diagonal matrix D such that

$$A = QDQ^{-1} = QDQ^T$$

– A matrix A is orthogonally diagonalizable if and only if A is a symmetric matrix (i.e. $A^T = A$).

– Spectral decomposition of a symmetric matrix:

$$A = \lambda_1(\mathbf{u}_1\mathbf{u}_1^T) + \lambda_2(\mathbf{u}_2\mathbf{u}_2^T) + \ldots + \lambda_n(\mathbf{u}_n\mathbf{u}_n^T)$$

5) The singular value decomposition of a matrix:

$$A = U\Sigma V^T$$

Singular values decomposition will not be a part of the exam.