Recall: An $n \times n$ matrix A defines a linear transformation

$$T_A : \mathbb{R}^n \to \mathbb{R}^n$$

given by $T_A(\mathbf{v}) = A\mathbf{v}$.

Next goal: Understand this linear transformation better.

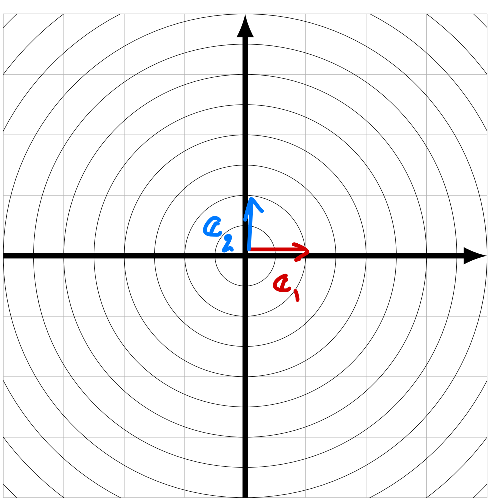
Example.

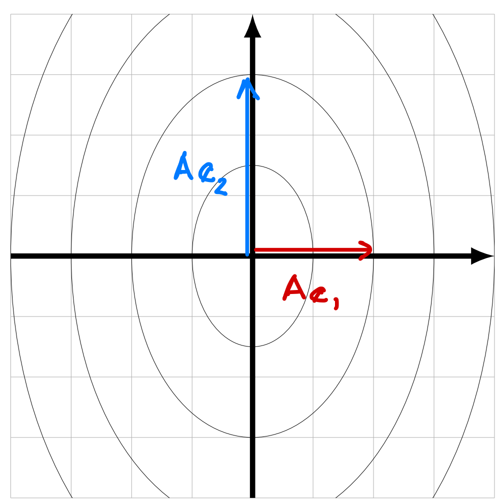
$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$v \longmapsto Av$$

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad Ae_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \cdot e_1$

$e_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ $\quad Ae_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix} = 3 \cdot e_2$





Example.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

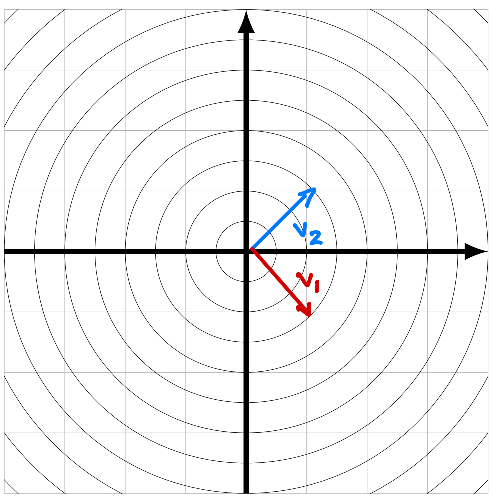
$v \longmapsto Av$

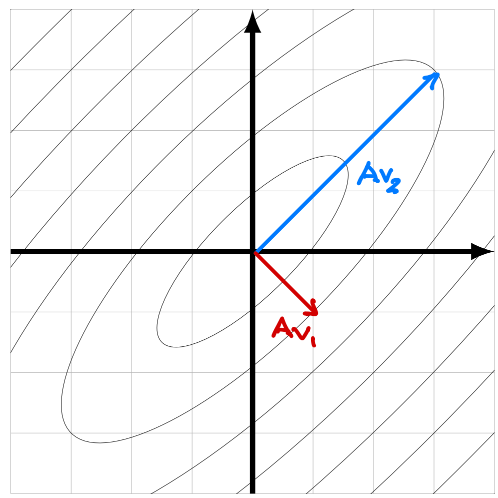
$Ae_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $Ae_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Take $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$Av_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \cdot v_1$

$Av_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3v_2$





Definition

Let A be an $n \times n$ matrix. If $\mathbf{v} \in \mathbb{R}^n$ is a non-zero vector and λ is a scalar such that

$$A\mathbf{v} = \lambda \mathbf{v}$$

then we say that

- λ is an *eigenvalue* of A
- $\mathbf{v}$ is an *eigenvector* of A corresponding to λ.

Example.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

We had:

$$A \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So: $\lambda_1 = 2$, $\lambda_2 = 3$ are eigenvalues of A

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 2$.

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_2 = 3$.

Example.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

We had:

$$A \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad A \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So: $\lambda_1 = 1$, $\lambda_2 = 3$ are eigenvalues of A

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 1$.

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_2 = 3$.

Computation of eigenvalues

Recall: $I_n = n \times n$ identity matrix:

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Note:

$$\lambda I_n = \begin{bmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \lambda \end{bmatrix}$$

For any $v \in \mathbb{R}^n$ we have:

$$(\lambda I_n)v = \lambda(I_n v) = \lambda v$$

Propostiton

If A be an $n \times n$ matrix then $\lambda \in \mathbb{R}$ is an eigenvalue of A if and only if the matrix equation

$$(A - \lambda I_n)\mathbf{x} = \mathbf{0}$$

has a non-trivial solution.

Proof: λ is an eigenvalue of A

$$\Updownarrow$$

there is a vector $v \neq \mathbf{0}$ such that $Av = \lambda v = (\lambda I_n)v$

$$\Updownarrow$$

there is a vector $v \neq \mathbf{0}$ such that $(A - \lambda I_n)v = \mathbf{0}$

Propostiton

If B is an $n \times n$ matrix then equation

$$B\mathrm{x} = \mathbf{0}$$

has a non-trivial solution if and only of the matrix B is not invertible.

Proof: $Bx = 0$ has a non-trivial solution

$\Updownarrow$

not every column of B is a pivot column

$\Updownarrow$

B is not invertible

Propostiton

If A be an $n \times n$ matrix then $\lambda \in \mathbb{R}$ is an eigenvalue of A if and only if

$$\det(A - \lambda I_n) = 0$$

Proof: λ is an eigenvalue of A

$\Updownarrow$

$(A - \lambda I_n)x = 0$ has a non-trivial solution

$\Updownarrow$

$(A - \lambda I_n)$ is not invertible

$\Updownarrow$

$\det(A - \lambda I_n) = 0$

Example. Find all eigenvalues of the following matrix:

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

<u>Solution:</u> We need to find $\lambda \in \mathbb{R}$ such that $\det(A - \lambda I) = 0$

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right)$$

$$= \det \left[\begin{array}{ccc} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{array}\right] \begin{array}{cc} 2-\lambda & 2 \\ 1 & 3-\lambda \\ 1 & 2 \end{array}$$

$$\begin{array}{cccccc} - & - & - & + & + & + \end{array}$$

$$= (2-\lambda)\cdot(3-\lambda)\cdot(2-\lambda) + 2\cdot 1\cdot 2 + 1\cdot 1\cdot 2$$
$$- 1\cdot(3-\lambda)\cdot 1 - (2-\lambda)\cdot 1\cdot 2 - 2\cdot 1\cdot(2-\lambda)$$
$$= -\lambda^3 + 7\lambda^2 - 11\lambda + 5$$

<u>We obtain:</u> λ is an eigenvalue of A if and only if

$$-\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$

<u>Check:</u> The only solutions of this equation are $\lambda_1 = 1$, $\lambda_2 = 5$.

<u>We obtain:</u> The matrix A has two eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 5$.

Definition

If A is an $n \times n$ matrix then

$$P(\lambda) = \det(A - \lambda I_n)$$

is a polynomial of degree n. $P(\lambda)$ is called the *characteristic polynomial* of the matrix A.

Upshot

If A is a square matrix then

$$\text{eigenvalues of } A = \text{roots of } P(\lambda)$$

Example.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

We had: $P(\lambda) = \det(A - \lambda I) = \underbrace{-\lambda^3 + 7\lambda^2 - 11\cdot\lambda + 5}$

the characteristic polynomial of the matrix A

Corollary

An $n \times n$ matrix can have at most n distinct eigenvalues.

Proof: The characteristic polynomial $P(\lambda)$ of A is a polynomial of degree n, so it can have at most n distinct roots.

Computation of eigenvectors

Recall: If B is a matrix then

$$\text{Nul}(B) = \{v \mid Bv = 0\}$$

Proposition

If λ is an eigenvalue of an $n \times n$ matrix A then

$$\left\{\begin{matrix}\text{eigenvectors of } A \\ \text{corresponding to } \lambda\end{matrix}\right\} = \left\{\begin{matrix}\text{vectors in} \\ \text{Nul}(A - \lambda I_n)\end{matrix}\right\}$$

Proof:

$$v \in \text{Nul}(A - \lambda I)$$
$$\Updownarrow$$
$$(A - \lambda I)v = \mathbf{0}$$
$$\Updownarrow$$
$$Av = (\lambda I)v$$
$$\Updownarrow$$
$$Av = \lambda v$$

Recall: If B is an $m \times n$ matrix then $\mathrm{Nul}(B)$ is a subspace of $\mathbb{R}^n$.

Corollary/Definition

If A is an $n \times n$ matrix and λ is an eigenvalue of A then the set of all eigenvectors corresponding to λ is a subspace of $\mathbb{R}^n$.

This subspace is called the *eigenspace* of A corresponding to λ.

Proposition

If λ is an eigenvalue of an $n \times n$ matrix A then

$$\left\{ \begin{matrix} \text{eigenspace of } A \\ \text{corresponding to } \lambda \end{matrix} \right\} = \mathrm{Nul}(A - \lambda I_n)$$

Example. Consider the following matrix:

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Recall that eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 5$. Compute bases of eigenspaces of A corresponding to these eigenvalues.

Solution.

$\underline{\lambda_1 = 1}$

$$\left(\begin{matrix}\text{eigenspace}\\ \text{of } \lambda_1 = 1\end{matrix}\right) = \text{Nul}(A - 1\cdot I)$$

$$= \text{Nul}\left(\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \text{Nul}\left(\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\right)$$

<u>We need to solve</u>:

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

<u>aug. matrix</u>:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right] \xrightarrow{\text{row reduction}} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \underbrace{0}_{\text{free}} & \underbrace{0}_{\text{free}} & 0 \end{array}\right] \end{array} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

<u>We obtain</u>:

$$\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\} = \left(\begin{matrix}\text{a basis of}\\ \text{Nul}(A - 1\cdot I)\end{matrix}\right) = \left(\begin{matrix}\text{a basis of the eigenspace}\\ \text{of } A \text{ for the eigenvalue } \lambda_1 = 1\end{matrix}\right)$$

$\underline{\lambda_2 = 5}$

$$\begin{pmatrix}\text{eigenspace} \\ \text{of } \lambda_1 = 5\end{pmatrix} = \mathrm{Nul}(A - 5\cdot I)$$

$$= \mathrm{Nul}\left(\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}\right) = \mathrm{Nul}\left(\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\right)$$

<u>We need to solve</u>:

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

<u>aug. matrix</u>:

$$\left[\begin{array}{ccc|c} -3 & 2 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 2 & -3 & 0 \end{array}\right] \xrightarrow{\text{row reduction}} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & \underbrace{0}_{\text{free}} & 0 \end{array}\right] \end{array} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

<u>We obtain</u>:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} = \begin{pmatrix} \text{a basis of} \\ \mathrm{Nul}(A - 5\cdot I) \end{pmatrix} = \begin{pmatrix} \text{a basis of the eigenspace} \\ \text{of } A \text{ for the eigenvalue } \lambda_2 = 5 \end{pmatrix}$$