Recall:

1) An orthogonal matrix $Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{bmatrix}$ is a square matrix such that $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is an orthonormal set of vectors, i.e.:

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

2) If Q is an orthogonal matrix then $Q^{-1} = Q^T$

3) A square matrix A is orthogonally diagonalizable if there exist an orthogonal matrix Q and a diagonal matrix D such that

$$A = QDQ^{-1} = QDQ^T$$

4) A matrix A is orthogonally diagonalizable if and only if A is a symmetric matrix (i.e. $A^T = A$).

Yet another view of matrix multiplication

Note. If C is an $n \times 1$ matrix and D is an $1 \times n$ matrix then CD is an $n \times n$ matrix.

e.g.

$$\underset{3\times 1}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} \cdot \underset{1\times 3}{\begin{bmatrix} 4 & 5 & 6 \end{bmatrix}} = \begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}$$

Propostion

Let A be an $n \times n$ matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$, and B be an $n \times n$ matrix with rows $\mathbf{w}_1, \ldots, \mathbf{w}_n$:

$$A = \begin{bmatrix} \mathbf{v}_1 & \ldots & \mathbf{v}_n \end{bmatrix} \qquad B = \begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_n \end{bmatrix}$$

Then

$$AB = \mathbf{v}_1\mathbf{w}_1 + \mathbf{v}_2\mathbf{w}_2 + \ldots + \mathbf{v}_n\mathbf{w}_n$$

Example.

$$A = \begin{array}{c} \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \end{array} \qquad B = \begin{bmatrix} 5 & 1 \\ 7 & 2 \end{bmatrix} \begin{array}{c} w_1 \\ w_2 \end{array}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 & 1 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 1\cdot 5 + 2\cdot 7 & 1\cdot 1 + 2\cdot 2 \\ 3\cdot 5 + 4\cdot 7 & 3\cdot 1 + 4\cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 5 & 1\cdot 1 \\ 3\cdot 5 & 3\cdot 1 \end{bmatrix} + \begin{bmatrix} 2\cdot 7 & 2\cdot 2 \\ 4\cdot 7 & 4\cdot 2 \end{bmatrix}$$

$$= \underset{r_1}{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} \cdot \underset{w_1}{\begin{bmatrix} 5 & 1 \end{bmatrix}} + \underset{r_2}{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} \cdot \underset{w_2}{\begin{bmatrix} 7 & 2 \end{bmatrix}}$$

Theorem

Let A be a symmetric matrix with orthogonal diagonalization

$$A = QDQ^T$$

If

$$Q = \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_n \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & 0 \\ 0 & \dots & \lambda_n \end{bmatrix}$$

then

$$A = \lambda_1(\mathbf{u}_1\mathbf{u}_1^T) + \lambda_2(\mathbf{u}_2\mathbf{u}_2^T) + \dots + \lambda_n(\mathbf{u}_n\mathbf{u}_n^T)$$

Note. The above formula is called the *spectral decomposition* of the matrix A.

Proof

$$A = QDQ^T = [u_1 \ u_2 \ \dots \ u_n] \cdot \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \cdot \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix}$$

$$= [\lambda_1 u_1 \ \ \lambda_2 u_2 \dots \lambda_n u_n] \cdot \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix}$$

$$= \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \dots + \lambda_n u_n u_n^T$$

Example.

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ \underbrace{1/\sqrt{2}}_{u_1} & \underbrace{1/\sqrt{2}}_{u_2} \end{bmatrix} \cdot \begin{bmatrix} \overset{\lambda_1}{4} & 0 \\ 0 & \underset{\lambda_2}{2} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$$

Spectral decomposition of A:

$$A = 4 \cdot u_1 u_1^T + 2 u_2 u_2^T$$

$$4 u_1 u_1^T = 4 \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = 4 \cdot \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$2 u_2 u_2^T = 2 \cdot \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = 2 \cdot \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$4 u_1 u_1^T + 2 u_2 u_2^T = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$