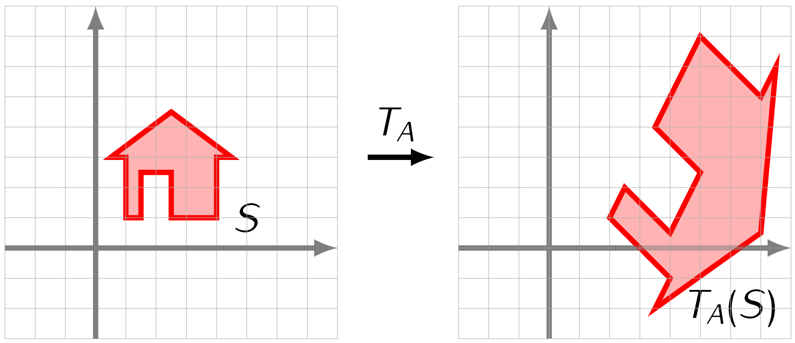
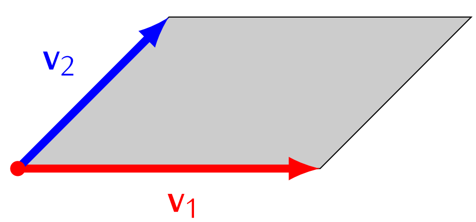
Recall: Determinants indicate how an area (or volume) changes under a matrix transformation:



$$\text{area}(T_A(S)) = |\det A| \cdot \text{area}(S)$$

Note. Any two vectors in $\mathbb{R}^2$ define a parallelogram:



Notation

$$\text{area}(\mathbf{v}_1, \mathbf{v}_2) = \begin{pmatrix} \text{area of the parallelogram} \\ \text{defined by } \mathbf{v}_1 \text{ and } \mathbf{v}_2 \end{pmatrix}$$

Theorem

If $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ then

$$\text{area}(\mathbf{v}_1, \mathbf{v}_2) = \left|\det \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\right|$$

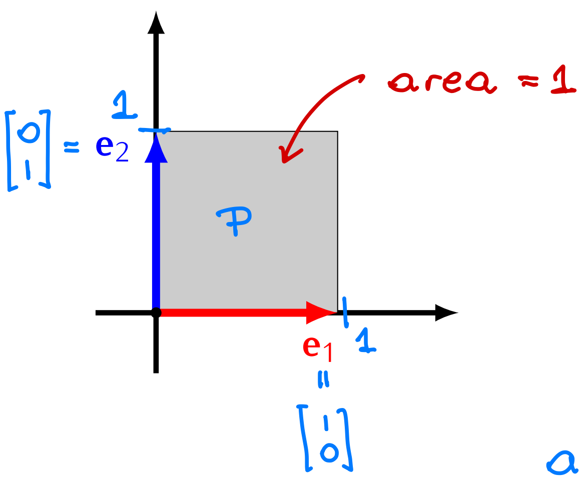
Proof.

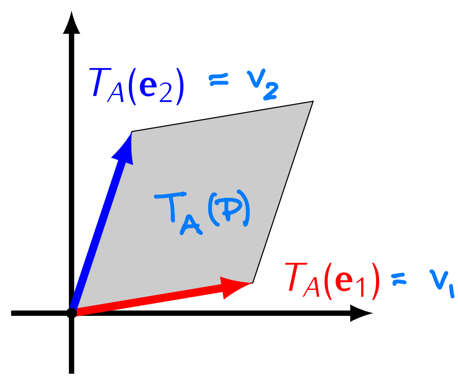
$A = [v_1 \;\; v_2]$

$T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\quad w \longmapsto Aw$

$T_A(e_1) = v_1$

$T_A(e_2) = v_2$

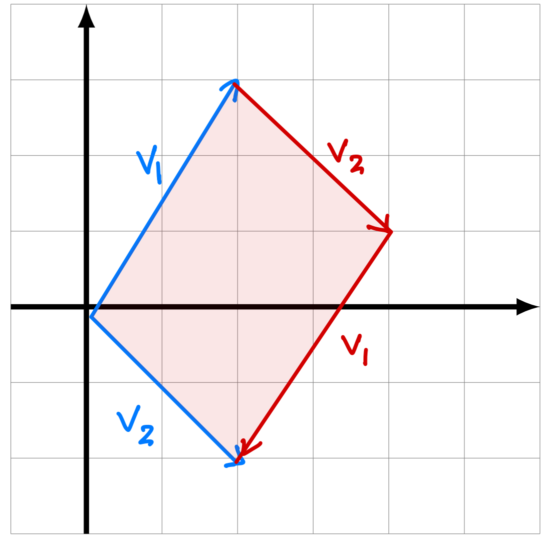




$$\begin{aligned} \text{area}(v_1, v_2) &= \text{area}(T_A(P)) \\ &= |\det A| \cdot \text{area}(P) \\ &= |\det A| \cdot 1 \\ &= |\det A| \end{aligned}$$

Example.

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$



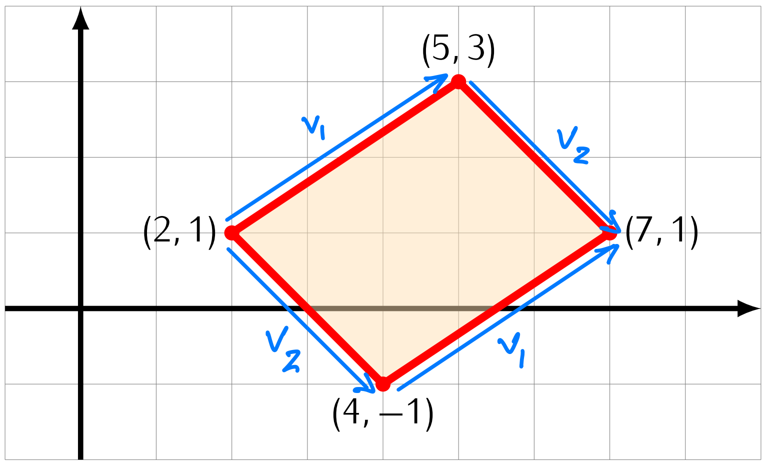
$\text{area}(v_1, v_2) =$

$= \left| \det [v_1 \;\; v_2] \right|$

$= \left| \det \begin{bmatrix} 2 & 2 \\ 3 & -2 \end{bmatrix} \right| = |-4-6|$

$= |-10| = 10$

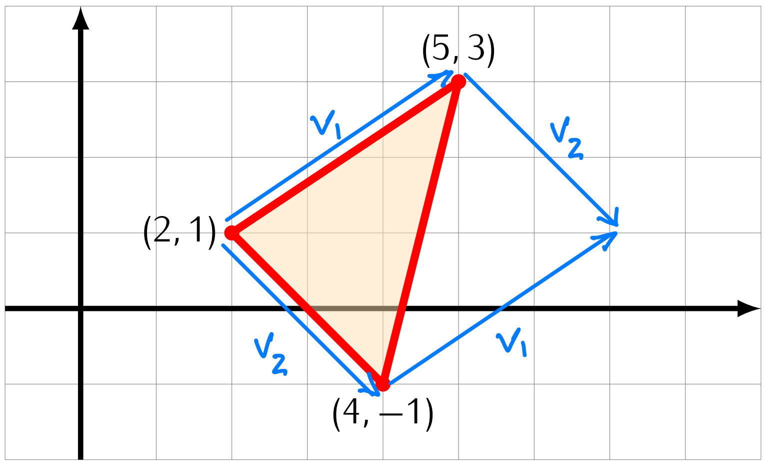
Example. Calculate the area of the parallelogram with vertices at the points $(2,1)$, $(5,3)$, $(7,1)$, $(4,-1)$.



$$v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$\begin{aligned} \text{area}(v_1, v_2) &= |\det [v_1 \ v_2]| \\ &= \left|\det \begin{bmatrix} 3 & 2 \\ 2 & -2 \end{bmatrix}\right| \\ &= |-6-4| = 10 \end{aligned}$$

Example. Calculate the area of the triangle with vertices at the points $(2, 1)$, $(5, 3)$, $(4, -1)$.



(area of the triangle) = $\frac{1}{2}$ (area of the parallelogram)

$v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

area $(v_1, v_2) = \left| \det \begin{bmatrix} 3 & 2 \\ 2 & -2 \end{bmatrix} \right| = |-10| = 10$

area triangle $= \frac{1}{2}$ area $(v_1, v_2) = \frac{1}{2} \cdot 10 = 5$ //

Note. In order to compute areas of other polygons, subdivide them into triangles.