Definition

A *(real) vector space* is a set V together with two operations:

- addition

$$\begin{array}{rcl} V \times V & \longrightarrow & V \\ (\mathbf{u},\ \ \mathbf{v}) & \longmapsto & \mathbf{u}+\mathbf{v} \end{array}$$

- multiplication by scalars

$$\begin{array}{rcl} \mathbb{R} \times V & \longrightarrow & V \\ (c,\ \ \mathbf{v}) & \longmapsto & c\cdot\mathbf{v} \end{array}$$

Moreover the following conditions must be satisfied:

1) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
2) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
3) there is an element $\mathbf{0}\in V$ such that $\mathbf{0}+\mathbf{u}=\mathbf{u}$ for any $\mathbf{u}\in V$
4) for any $\mathbf{u}\in V$ there is an element $-\mathbf{u}\in V$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$
5) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$
6) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$
7) $(cd)\mathbf{u}=c(d\mathbf{u})$
8) $1\mathbf{u}=\mathbf{u}$

Elements of V are called *vectors*.

Theorem

If V is a vectors space then:

1) $c \cdot \mathbf{0} = \mathbf{0}$ where $c \in \mathbb{R}$ and $\mathbf{0} \in V$ is the zero vector;

2) $0 \cdot \mathbf{u} = \mathbf{0}$ where $0 \in \mathbb{R}$, $\mathbf{u} \in V$ and $\mathbf{0}$ is the zero vector;

3) $(-1) \cdot \mathbf{u} = -\mathbf{u}$

Examples of vector spaces.