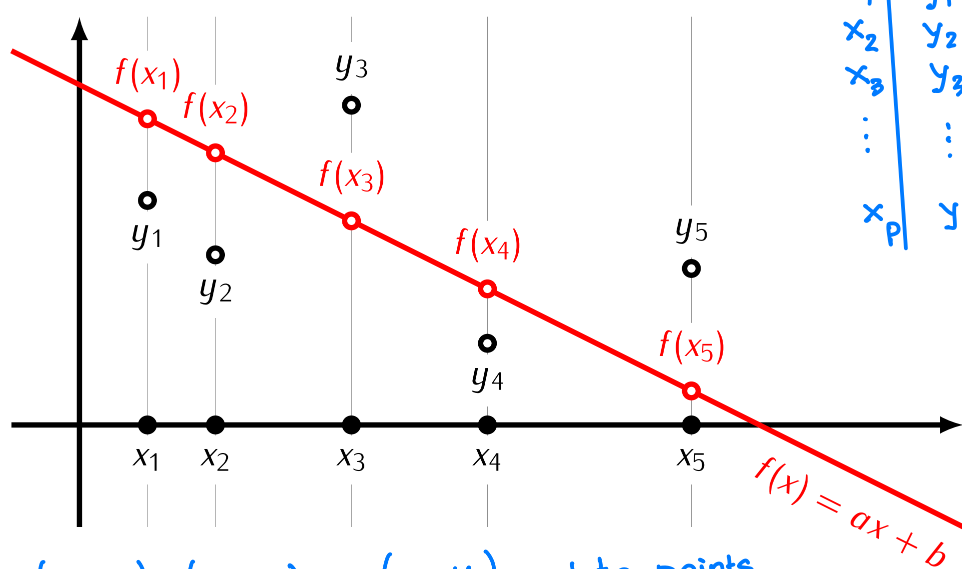
Application: Least square lines

	data values	computed values
x_1	y_1	$f(x_1)$
x_2	y_2	$f(x_2)$
x_3	y_3	$f(x_3)$
$\vdots$	$\vdots$	$\vdots$
x_p	y_p	$f(x_p)$



$(x_1, y_1), (x_2, y_2), \dots, (x_p, y_p)$ – data points

Definition

If $(x_1, y_1), \dots, (x_p, y_p)$ are points on the plane then the *least square line* for these points is the line given by an equation $f(x) = ax + b$ such that the number

$$\operatorname{dist}\left(\begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}, \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_p) \end{bmatrix}\right) = \sqrt{(y_1 - f(x_1))^2 + \ldots + (y_p - f(x_p))^2}$$

is the smallest possible.

Note: If $f(x) = ax+b$ then

$$\begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_p) \end{bmatrix} = \begin{bmatrix} ax_1+b \\ ax_2+b \\ \vdots \\ ax_p+b \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_p & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix}$$

This gives: $f(x) = ax+b$ is the least square line if the distance

$$\text{dist}\left(\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_p & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} \right)$$

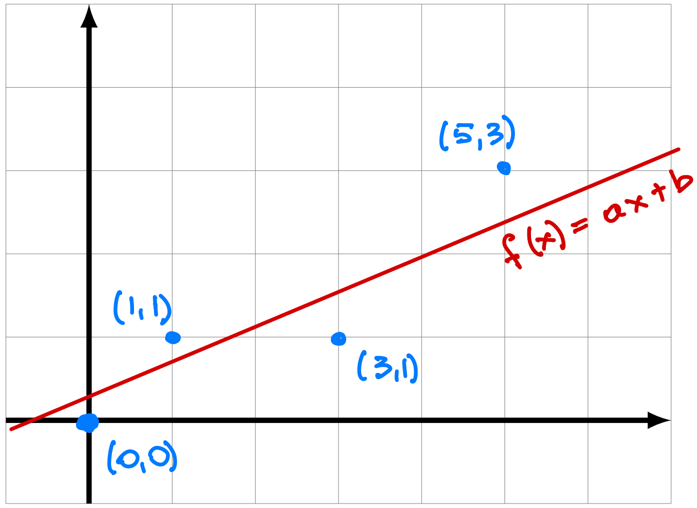
is as small as possible.

Proposition

The line $f(x) = ax+b$ is the least square line for points $(x_1, y_1), \ldots, (x_p, y_p)$ if the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ is the least square solution of the equation

$$\begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_p & 1 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}$$

Example. Find the equation of the least square line for the points $(0,0)$, $(1,1)$, $(3,1)$, $(5,3)$.



Solution: The least square solution is given by

$$f(x) = ax + b$$

where $\begin{bmatrix} a \\ b \end{bmatrix}$ is a least square solution of

$$\begin{array}{c} \begin{array}{cc} x & 1 \end{array} \\ \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 5 & 1 \end{bmatrix}}_{A} \end{array} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{array}{c} y \\ \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix}}_{b} \end{array}$$

$$A^T A = \begin{bmatrix} 35 & 9 \\ 9 & 4 \end{bmatrix} \qquad A^T b = \begin{bmatrix} 19 \\ 5 \end{bmatrix}$$

Normal equation:

$$\begin{bmatrix} 35 & 9 \\ 9 & 4 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 19 \\ 5 \end{bmatrix}$$

aug. matrix

$$\left[\begin{array}{cc|c} 35 & 9 & 19 \\ 9 & 4 & 5 \end{array}\right] \xrightarrow{\text{row red.}} \left[\begin{array}{cc|c} 1 & 0 & 31/59 \\ 0 & 1 & 4/59 \end{array}\right] \qquad \begin{cases} a = 31/59 \\ b = 4/59 \end{cases}$$

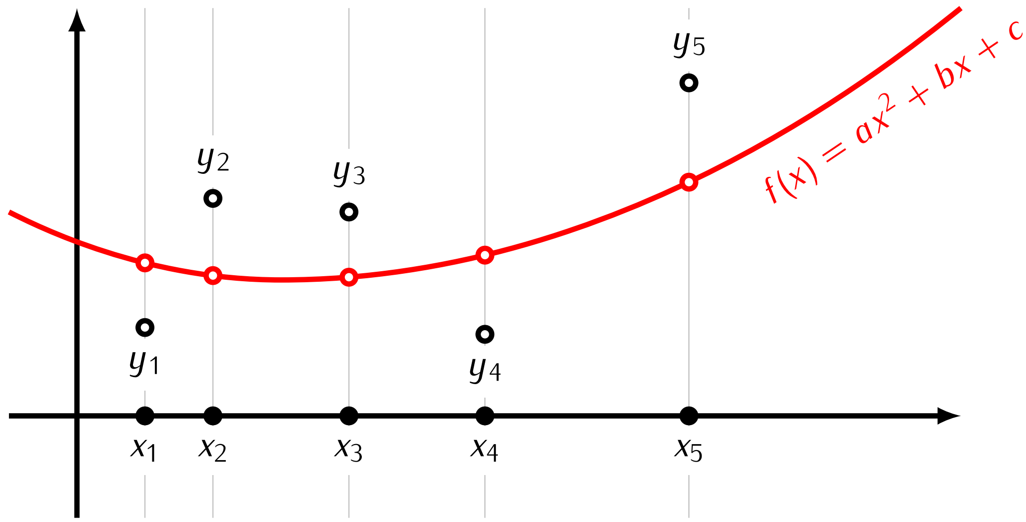
The least square line:

$$f(x) = \frac{31}{59}x + \frac{4}{59}$$

Application: Least square curves

The above procedure can be used to determine curves other than lines that fit a set of points in the least square sense.

Example: Least square parabolas



Definition

If $(x_1, y_1), \dots, (x_p, y_p)$ are points on the plane then the *least square parabola* for these points is the parabola given by an equation $f(x) = ax^2 + bx + c$ such that the number

$$\operatorname{dist}\left(\begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}, \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_p) \end{bmatrix}\right) = \sqrt{(y_1 - f(x_1))^2 + \ldots + (y_p - f(x_p))^2}$$

is the smallest possible.

Note: If $f(x) = ax^2 + bx + c$ then

$$\begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_p) \end{bmatrix} = \begin{bmatrix} ax_1^2 + bx_1 + c \\ ax_2^2 + bx_2 + c \\ \vdots \\ ax_p^2 + bx_p + c \end{bmatrix} = \begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ \vdots & \vdots & \vdots \\ x_p^2 & x_p & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

This gives: $f(x) = ax^2 + bx + c$ is the least square parabola if the distance

$$\operatorname{dist}\left(\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ \vdots & \vdots & \vdots \\ x_p^2 & x_p & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} \right)$$

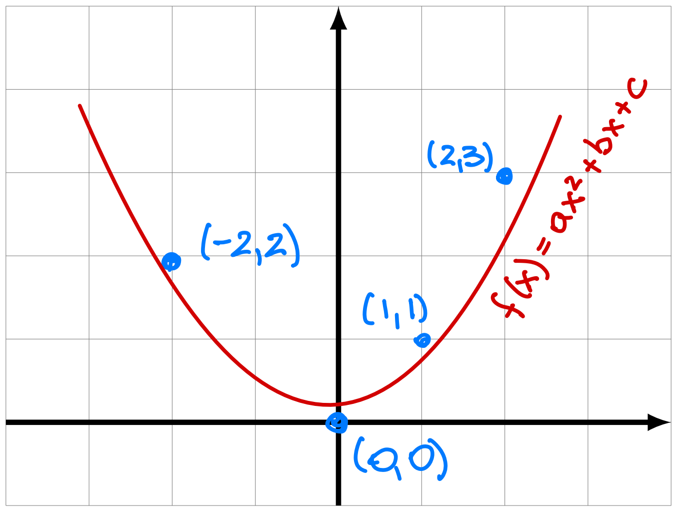
is as small as possible.

Proposition

The parabola $f(x) = ax^2 + bx + c$ is the least square parabola for points $(x_1, y_1), \ldots, (x_p, y_p)$ if the vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is the least square solution of the equation

$$\begin{bmatrix} x_1^2 & x_1 & 1 \\ \vdots & \vdots & \\ x_p^2 & x_p & 1 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}$$

Example. Find the equation of the least square parabola for the points $(-2, 2)$, $(0, 0)$, $(1, 1)$, $(2, 3)$.



Solution: We need to find a least square solution of the equation

$$\begin{array}{c} \begin{array}{ccc} x^2 & x & 1 \end{array} \\ \begin{bmatrix} (-2)^2 & -2 & 1 \\ 0^2 & 0 & 1 \\ 1^2 & 1 & 1 \\ 2^2 & 2 & 1 \end{bmatrix} \end{array} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{array}{c} y \\ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix} \end{array}$$

Normal equation:

$$\underbrace{\begin{bmatrix} 33 & 1 & 9 \\ 1 & 9 & 1 \\ 9 & 1 & 4 \end{bmatrix}}_{A^TA} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} 21 \\ 3 \\ 6 \end{bmatrix}}_{A^Tb}$$

aug. matrix

$$\left[\begin{array}{ccc|c} 33 & 1 & 9 & 21 \\ 1 & 9 & 1 & 3 \\ 9 & 1 & 4 & 6 \end{array}\right] \xrightarrow{\text{row red.}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 27/37 \\ 0 & 1 & 0 & 51/185 \\ 0 & 0 & 1 & -39/185 \end{array}\right] \qquad \begin{cases} a = 27/37 \\ b = 51/185 \\ c = -39/185 \end{cases}$$

The least square parabola: $f(x) = \frac{27}{37}x^2 + \frac{51}{185}x - \frac{39}{185}$