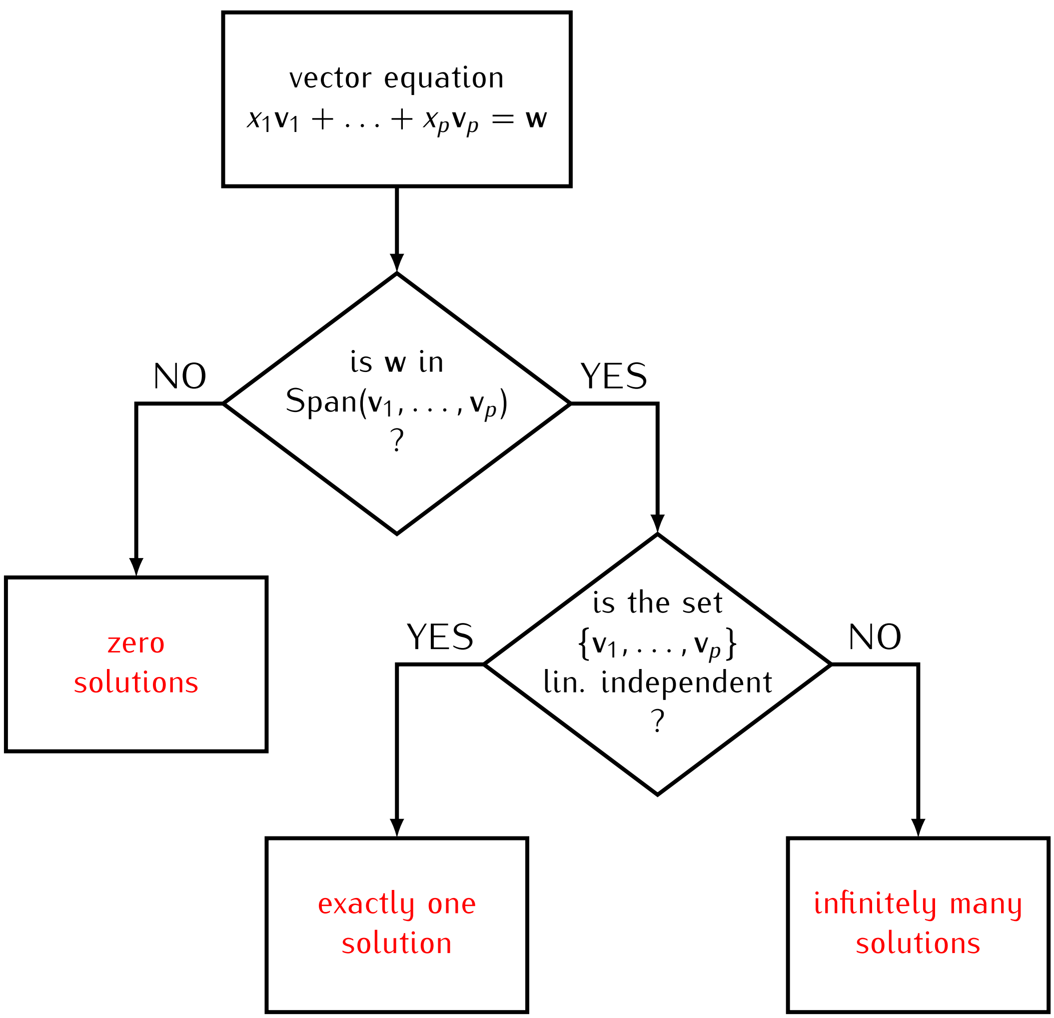
Recall:

1) $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p) = \left\{ \begin{array}{c} \text{the set of all} \\ \text{linear combinations} \\ c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p \end{array} \right\}$

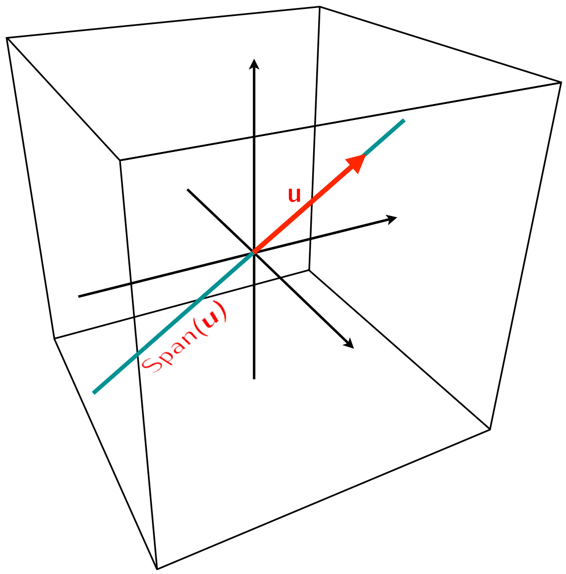
2) A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent if the equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

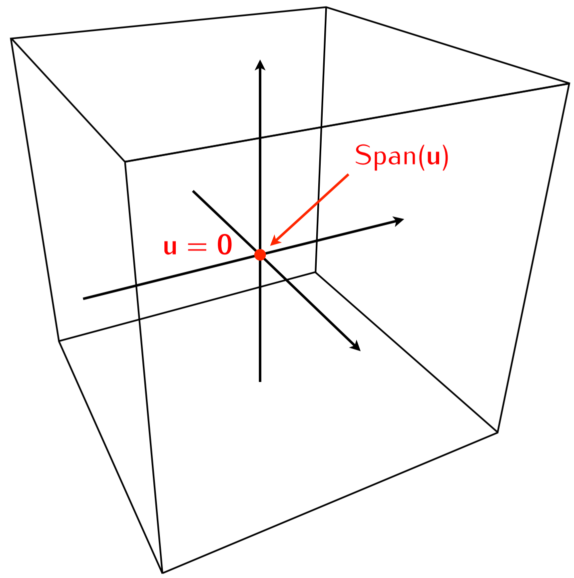
has only one, trivial solution $x_1 = 0, \ldots, x_p = 0$.



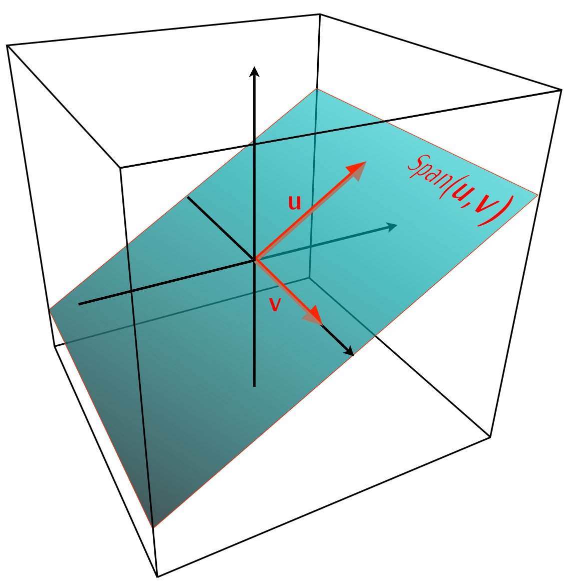
Linear independence vs. Span



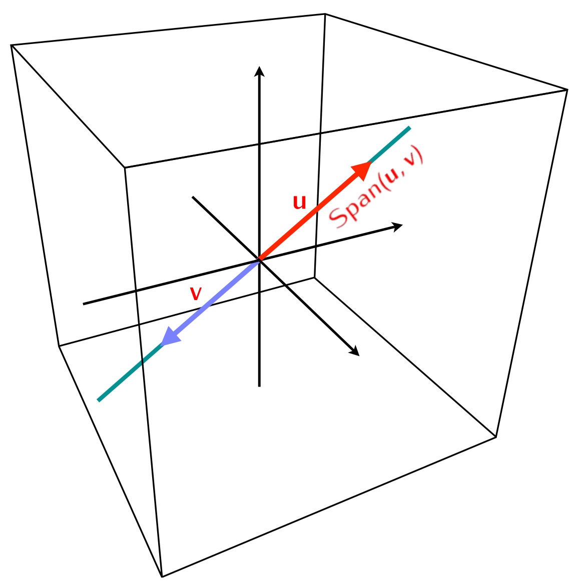
$\{\mathbf{u}\}$ linearly independent



$\{\mathbf{u}\}$ linearly dependent



$\{\mathbf{u}, \mathbf{v}\}$ linearly independent



$\{\mathbf{u}, \mathbf{v}\}$ linearly dependent

Theorem

If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a linearly dependent set of vectors, then for some $\mathbf{v}_i$ we have
$$\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p) = \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_p)$$

Example.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The set $\{v_1, v_2, v_3\}$ is linearly dependent e.g.

$$2v_1 - v_2 + 0v_3 = \mathbf{0}$$

This gives:

$$(*) \qquad v_2 = 2v_1 + 0v_3$$

Now assume that $w \in \text{Span}(v_1, v_2, v_3)$

Then $w = c_1v_1 + c_2v_2 + c_3v_3$.

Using $(*)$ we obtain:

$$w = c_1v_1 + c_2(2v_1 + 0v_3) + c_3v_3$$
$$= (c_1 + 2c_2)v_1 + c_3v_3$$

So: $w \in \text{Span}(v_1, v_3)$