Recall:

1) If A is an upper triangular matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

then $\det A = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$.

Note. If A is a square matrix then the row echelon form of A is always upper triangular.

2) Let A and B be $n \times n$ matrices.

• If B is obtained from A by interchanging two rows (or two columns) then

$$\det B = -\det A$$

• If B is obtained from A by multiplying one row (or one column) of A by a scalar k then

$$\det B = k \cdot \det A$$

• If B is obtained from A by adding a multiple of one row of A to another row (or adding a multiple of one column to another column) then

$$\det B = \det A$$

Computation of determinants via row reduction

Idea. To compute $\det A$, row reduce A to the row echelon form. Keep track how the determinant changes at each step of the row reduction process.

Example. Compute $\det A$ where

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 4 & 0 & 10 \\ 3 & 4 & 1 & 7 \\ -2 & 5 & 3 & 0 \end{bmatrix}$$

$$\det\begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 4 & 0 & 10 \\ 3 & 4 & 1 & 7 \\ -2 & 5 & 3 & 0 \end{bmatrix} = (-1)\det\begin{bmatrix} 2 & 4 & 0 & 10 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 1 & 7 \\ -2 & 5 & 3 & 0 \end{bmatrix} \cdot \left(\tfrac{1}{2}\right)$$

$$= 2\cdot(-1)\cdot\det\begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 1 & 7 \\ -2 & 5 & 3 & 0 \end{bmatrix} \cdot(-3) \quad \cdot(2)$$

$$= 2\cdot(-1)\cdot\det\begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & -2 & 1 & -8 \\ 0 & 9 & 3 & 10 \end{bmatrix} \cdot(2) \quad \cdot(-9)$$

$$= 2\cdot(-1)\cdot\det\begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -15 & -17 \end{bmatrix} \cdot(3)$$

$$= 2\cdot(-1)\cdot\det\begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & -23 \end{bmatrix} = 2\cdot(-1)\cdot 1\cdot 1\cdot 5\cdot(-23) = 230$$

upper triangular

Theorem

If A is a square matrix then A is invertible if and only if $\det A \neq 0$

Recall: A is invertible if and only if its reduced row echelon form is the identity matrix.

Proof:

We have:

$$\det A = (\text{some non-zero number}) \cdot \det \begin{bmatrix} \text{reduced} \\ \text{echelon} \\ \text{form of } A \end{bmatrix}$$

1) If A is invertible then:

$$\det A = (\text{some non-zero number}) \cdot \overbrace{\det \underbrace{I}_{\text{the identity matrix}}}^{1} \neq 0$$

2) If A is not invertible then:

$$\begin{bmatrix} \text{reduced form} \\ \text{of } A \end{bmatrix} = \begin{pmatrix} \text{an upper triangular matrix} \\ \text{with a zero on the main} \\ \text{diagonal} \end{pmatrix}$$

e.g. $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

This gives:

$$\det \begin{bmatrix} \text{reduced form} \\ \text{of } A \end{bmatrix} = 0$$

so: $\det A = 0$.

A direct way of computing the determinant of a 2×2 matrix

Proposition

If A is a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$-$ $+$

then $\det A = ad - bc$

Proof: (in the case $a \neq 0$)

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \left(-\frac{c}{a}\right) = \det \underbrace{\begin{bmatrix} a & b \\ 0 & d - \frac{c}{a} b \end{bmatrix}}_{\text{upper triangular}}$$

$$= a \cdot \left(d - \frac{c}{a} b\right)$$

$$= \underline{ad - cb}$$

Example.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \det A = 1 \cdot 4 - 2 \cdot 3 = 4 - 6 = -2 \ //$$

A direct way of computing the determinant of a 3×3 matrix

Example.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\left[\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}\right]\begin{array}{cc} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{array}$$

$$- \quad - \quad - \qquad + \quad + \quad +$$

$$\det A = (1\cdot 5\cdot 9) + (2\cdot 6\cdot 7) + (3\cdot 4\cdot 8)$$
$$- (3\cdot 5\cdot 7) - (1\cdot 6\cdot 8) - (2\cdot 4\cdot 9)$$

$$= 45 + 84 + 96$$
$$- 105 - 48 - 72$$

$$= 225 - 225 = 0$$

Some further properties of determinants

1) $\det(A^{-1}) = (\det A)^{-1}$] $\leftarrow$ $\det(A)\cdot\det(A^{-1}) = \det(AA^{-1}) = \det I = 1$

2) $\det(A^T) = \det A$ $\quad$ so : $\det(A^{-1}) = \dfrac{1}{\det A}$

Note. In general $\det(A+B) \neq \det A + \det B$.

<u>Example</u> :

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\qquad$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\det A = 0$ $\qquad$ $\det B = 0$

$A+B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\det A+B = 1$