Definition

A *(real) vector space* is a set V together with two operations:

- addition

$$\begin{aligned} V \times V &\longrightarrow V \\ (\mathbf{u}, \ \mathbf{v}) &\longmapsto \mathbf{u} + \mathbf{v} \end{aligned}$$

- multiplication by scalars

$$\begin{aligned} \mathbb{R} \times V &\longrightarrow V \\ (c, \ \mathbf{v}) &\longmapsto c \cdot \mathbf{v} \end{aligned}$$

Moreover the following conditions must be satisfied:

1) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3) there is an element $\mathbf{0} \in V$ such that $\mathbf{0} + \mathbf{u} = \mathbf{u}$ for any $\mathbf{u} \in V$
4) for any $\mathbf{u} \in V$ there is an element $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7) $(cd)\mathbf{u} = c(d\mathbf{u})$
8) $1\mathbf{u} = \mathbf{u}$

Elements of V are called *vectors*.

Theorem

If V is a vectors space then:

1) $c \cdot \mathbf{0} = \mathbf{0}$ where $c \in \mathbb{R}$ and $\mathbf{0} \in V$ is the zero vector;

2) $0 \cdot \mathbf{u} = \mathbf{0}$ where $0 \in \mathbb{R}$, $\mathbf{u} \in V$ and $\mathbf{0}$ is the zero vector;

3) $(-1) \cdot \mathbf{u} = -\mathbf{u}$

Proof of 2)

$$\mathbf{0} \underset{\text{by (4)}}{=} (0\cdot u) + (-(0\cdot u)) = (0+0)\cdot u + (-(0\cdot u))$$

$$\underset{\text{by (6)}}{=} (0\cdot u + 0\cdot u) + (-0\cdot u)$$

$$\underset{\text{by (2)}}{=} 0\cdot u + (0\cdot u + (-(0\cdot u)))$$

$$\underset{\text{by (4)}}{=} 0\cdot u + \mathbf{0}$$

$$\underset{\text{by (1)}}{=} \mathbf{0} + 0\cdot u$$

$$\underset{\text{by (3)}}{=} 0\cdot u$$

So: $\mathbf{0} = 0\cdot u$

Examples of vector spaces.

1) $\mathbb{R}^n = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \;\middle|\; a_1, \dots, a_n \in \mathbb{R} \right\}$

2) $\mathcal{F}(\mathbb{R}) = \{$ the set of all functions $f : \mathbb{R} \to \mathbb{R} \}$

3) $\mathbb{P}(\mathbb{R}) = \{$ the set of all polynomials of variable $t \}$

$= \{ a_0 + a_1 t + \dots + a_n t^n \mid a_i \in \mathbb{R},\ n \geqslant 0 \}$

4) $M_{m,n}(\mathbb{R}) = \{$ the set of all $m \times n$ matrices $\}$

$= \left\{ \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \;\middle|\; a_{ij} \in \mathbb{R} \right\}$