Definition

If A is an $n \times n$ matrix then for $1 \leq i, j \leq n$ the (i,j)*-minor* of A is the matrix A_{ij} obtained by deleting the i^{th} row and j^{th} column of A.

Example.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$$

$$A_{23} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$$

Definition

If A is an $n \times n$ matrix and $1 \leq i, j \leq n$ then the ij*-cofactor of* A is the number

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

Example.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$C_{11} = (-1)^{1+1} \cdot \det A_{11} = (-1)^2 \cdot \det \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} = (45-48) = -3$$

$$C_{23} = (-1)^{2+3} \cdot \det A_{23} = (-1)^5 \cdot \det \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix} = (-1)\cdot(8-14) = (-1)\cdot(-6) = 6$$

Theorem

Let A be an $n \times n$ matrix.

1) For any $1 \leq i \leq n$ we have

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

(cofactor expansion across the i^{th} row).

2) For any $1 \leq j \leq n$ we have

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$

(cofactor expansion down the j^{th} column).

Example.

$$A = \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 0 \\ 2 \\ 0 \end{matrix}} & \begin{matrix} 3 & 0 & 4 \\ 4 & 6 & 1 \\ 1 & 0 & 3 \\ 5 & 0 & 0 \end{matrix} \end{bmatrix}$$

Cofactor expansion down the 1^{st} column:

$$\det A = 1 \cdot C_{11} + 0 \cdot C_{21} + 2 \cdot C_{31} + 0 \cdot C_{41} = C_{11} + 2 \cdot C_{31}$$

$$C_{11} = (-1)^{1+1} \cdot \det \begin{bmatrix} 4 & 6 & 1 \\ 1 & 0 & 3 \\ 5 & 0 & 0 \end{bmatrix} \begin{matrix} 4 & 6 \\ 1 & 0 \\ 5 & 0 \end{matrix} = 1 \cdot (4 \cdot 0 \cdot 0 + 6 \cdot 3 \cdot 5 + 1 \cdot 1 \cdot 0 - 1 \cdot 0 \cdot 5 - 4 \cdot 3 \cdot 0 - 6 \cdot 1 \cdot 0) = 90$$

$$C_{31} = (-1)^{3+1} \det \begin{bmatrix} 3 & 0 & 4 \\ 4 & 6 & 1 \\ 5 & 0 & 0 \end{bmatrix} \begin{matrix} 3 & 0 \\ 4 & 6 \\ 5 & 0 \end{matrix} = 1 \cdot (3 \cdot 6 \cdot 0 + 0 \cdot 1 \cdot 5 + 4 \cdot 4 \cdot 0 - 4 \cdot 6 \cdot 5 - 3 \cdot 1 \cdot 0 - 0 \cdot 4 \cdot 0) = -120$$

We obtain:

$$\det A = 90 + 2 \cdot (-120) = 90 - 240 = -150$$

Example. Compute the determinant of the following matrix:

$$\begin{bmatrix}
1 & 0 & 0 & 3 & 0 & 0 & 2 & 0 & 3 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & \pi & 0 & 0 & 0 & 6 & 0 & 0 & 5 & 6 & 0 & 2 & 0 & 7 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 11 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 2 & 0 & 4 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 2 & 1 & 2 & 3 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 2 & 7 & 0 & -4 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 1 & 0 & 4 & 3 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 8 & 7 & 7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 2 & 8 & 9 & 0 & 3 & 3 & 2 & 5 & 6 & 3 & 8 & 9 & 2 & 6 & 2 & 2 & 1
\end{bmatrix}$$

$$\det A = 1 \cdot (-1)^{1+1} \cdot 2 \cdot (-1)^{1+1} \cdot 1 \cdot (-1)^{1+1} \cdot \frac{1}{2} \cdot (-1)^{1+1} \cdot 2 \cdot (-1)^{17+1} \cdot \ldots$$