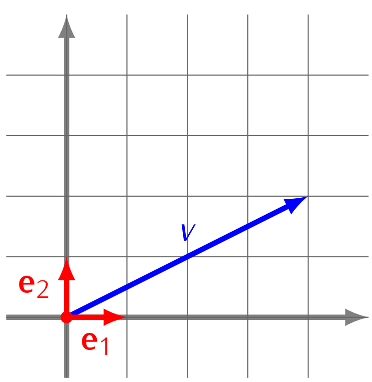
Recall:

1) A basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ of a vector space V defines a coordinate system:

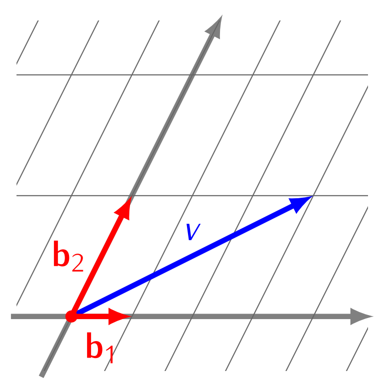
$$\mathbf{v} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$$

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

2) Different bases define different coordinate systems.



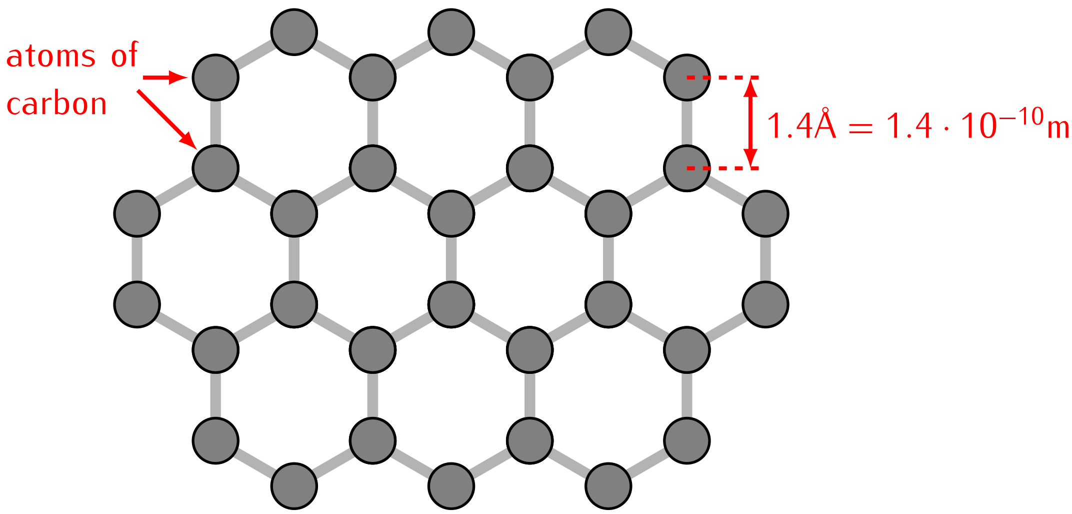
$\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$



$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$

Note. Choosing a convenient basis can simplify computations.

Example. Graphene lattice.



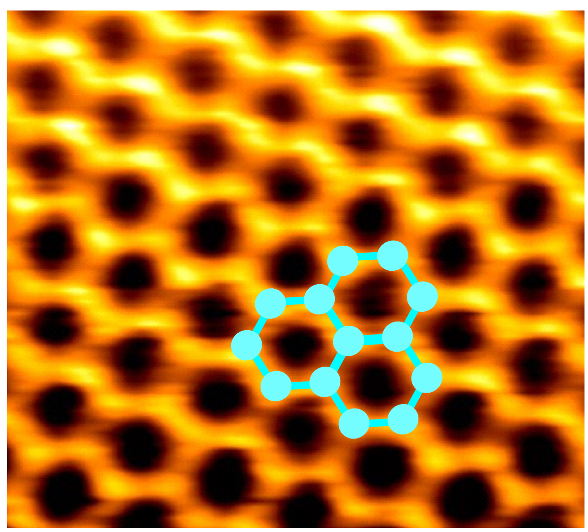


Image of graphene taken with an atomic force microscope.
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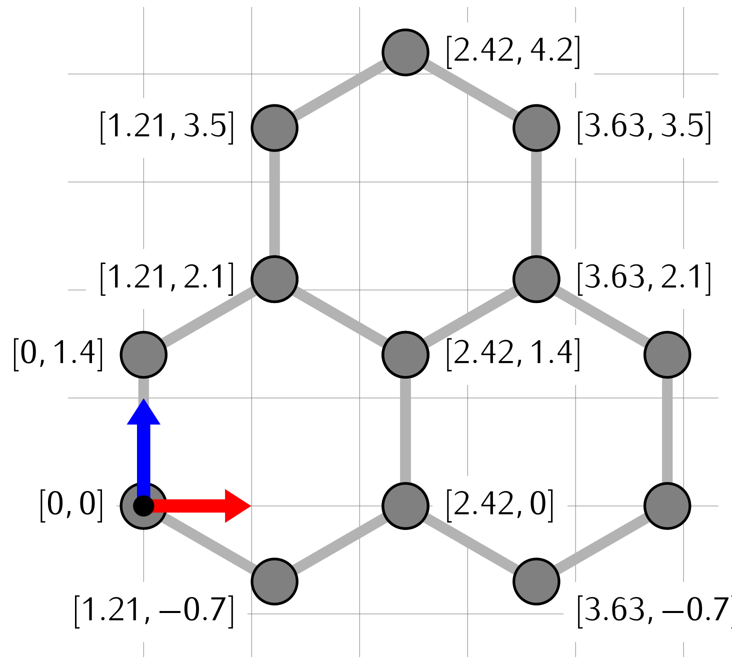
Coordinates of atoms in the graphene lattice

In the standard basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

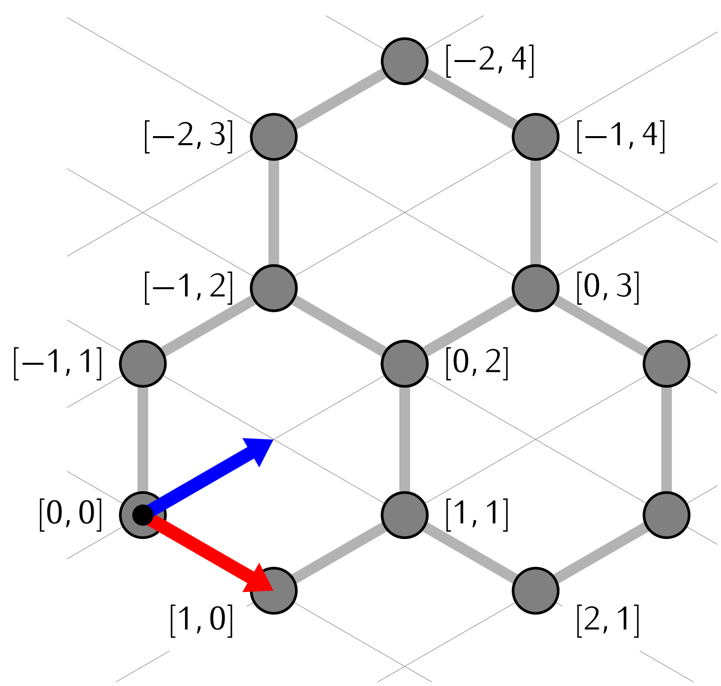
$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



In a more convenient basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$:

$$\mathbf{b}_1 = \begin{bmatrix} 1.21 \\ -0.7 \end{bmatrix}$$

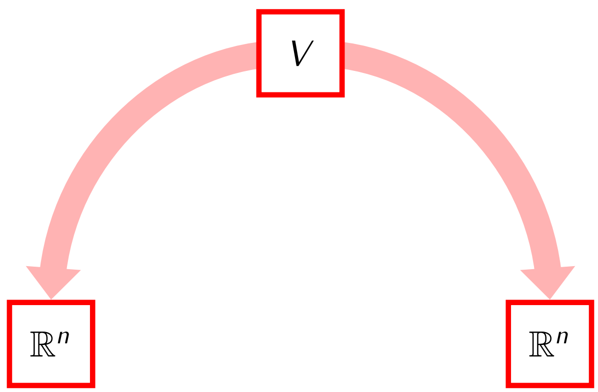
$$\mathbf{b}_2 = \begin{bmatrix} 1.21 \\ 0.7 \end{bmatrix}$$



Problem Let

$$\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}, \qquad \mathcal{D} = \{\mathbf{d}_1, \ldots, \mathbf{d}_n\}$$

be two bases of a vector space V, and let $\mathbf{v} \in V$. Assume that we know $\big[\mathbf{v}\big]_{\mathcal{B}}$. What is $\big[\mathbf{v}\big]_{\mathcal{D}}$?



Definition

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{D} = \{\mathbf{d}_1, \ldots, \mathbf{d}_n\}$ be two bases of a vector space V. The matrix

$$P_{\mathcal{D}\leftarrow\mathcal{B}} = \left[\ [\mathbf{b}_1]_{\mathcal{D}} \quad [\mathbf{b}_2]_{\mathcal{D}} \quad \ldots \quad [\mathbf{b}_n]_{\mathcal{D}}\ \right]$$

is called the *change of coordinates matrix* from the basis $\mathcal{B}$ to the basis $\mathcal{D}$.

Propostion

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{D} = \{\mathbf{d}_1, \ldots, \mathbf{d}_1\}$ be two bases of a vector space V. For any vector $\mathbf{v} \in V$ we have

$$[\mathbf{v}]_{\mathcal{D}} = P_{\mathcal{D}\leftarrow\mathcal{B}} \cdot [\mathbf{v}]_{\mathcal{B}}$$

Example. Let $\mathbb{P}_2$ be the vector space of polynomials of degree ≤ 2. Consider two bases of $\mathbb{P}_2$:

$$\mathcal{B} = \{1, 1+t, 1+t+t^2\}$$
$$\mathcal{D} = \{1+t, 1-5t, 2+t^2\}$$

1) Compute the change of coordinates matrix $P_{\mathcal{D}\leftarrow\mathcal{B}}$.

2) Let $p \in \mathbb{P}_2$ be a polynomial such that

$$\big[p\big]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

Compute $\big[p\big]_{\mathcal{D}}$.

Proposition

If $\mathcal{B}, \mathcal{D}, \mathcal{E}$ are three bases of a vector space V then:

1) $P_{\mathcal{B}\leftarrow\mathcal{D}} = (P_{\mathcal{D}\leftarrow\mathcal{B}})^{-1}$
2) $P_{\mathcal{E}\leftarrow\mathcal{B}} = P_{\mathcal{E}\leftarrow\mathcal{D}} \cdot P_{\mathcal{D}\leftarrow\mathcal{B}}$