

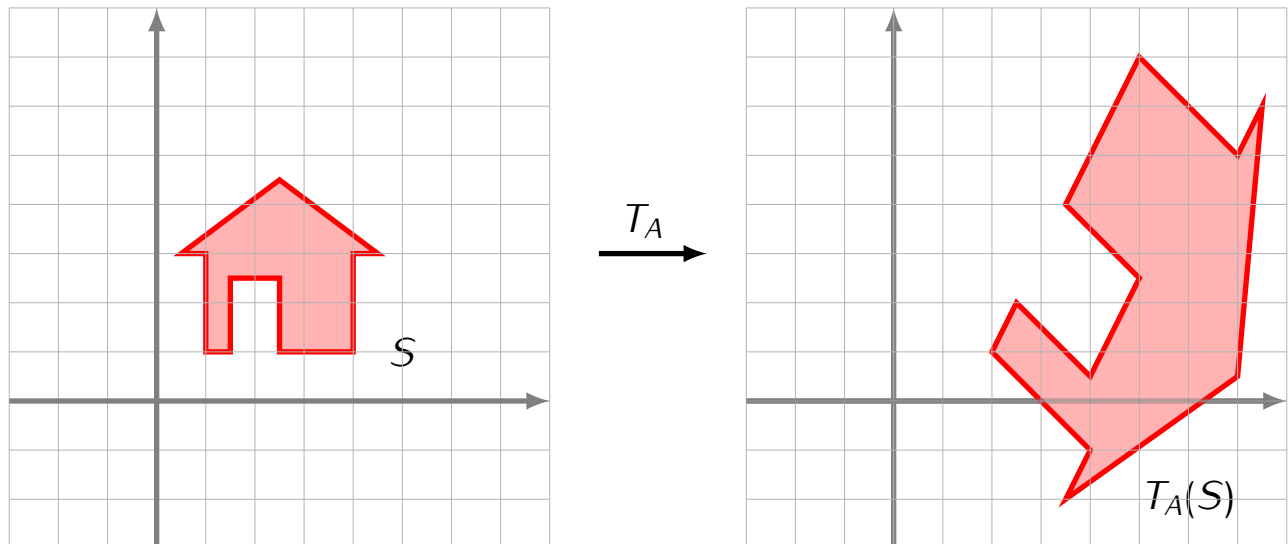
Idea. Given an $n \times n$ matrix A , the determinant $\det A$ is the factor by which the matrix transformation

$$T_A: \mathbb{R}^n \rightarrow \mathbb{R}^n$$

shrinks or expands the volume of each region of \mathbb{R}^n .

Example.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

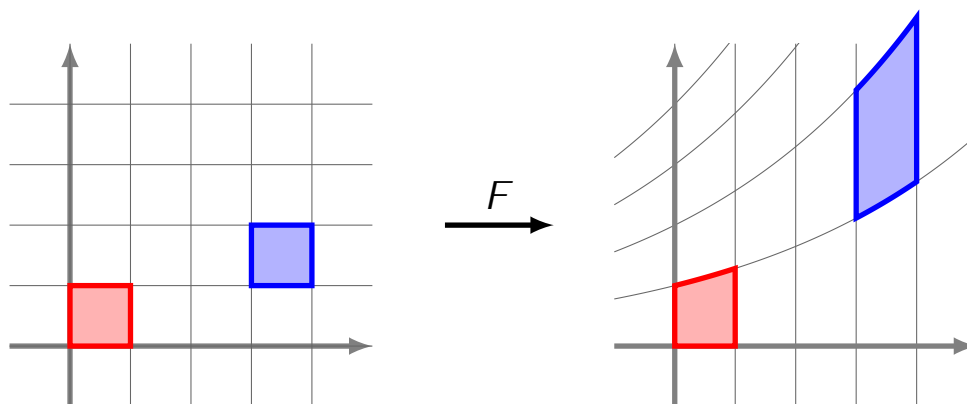


$$\text{area}(T_A(S)) = |\det A| \cdot \text{area}(S)$$

Note. Existence of determinants depends on the fact that a linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ changes the volume of each region of \mathbb{R}^n by the same factor.

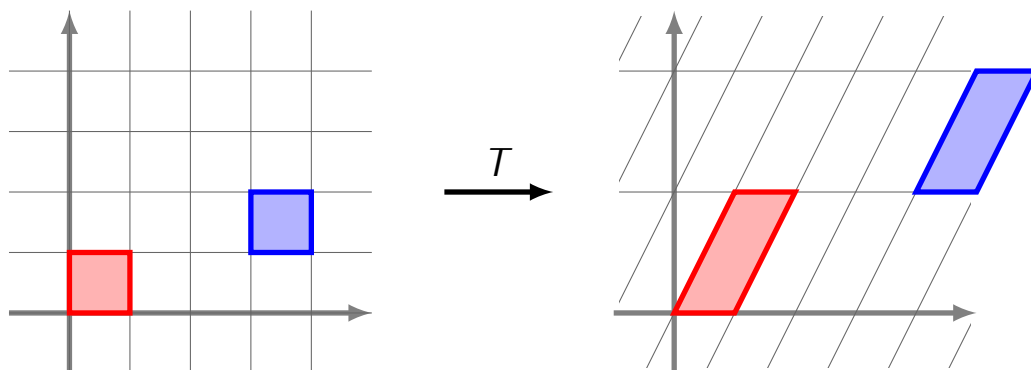
Example. Nonlinear transformation $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ ye^x \end{bmatrix}$$



Example. Linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ 2y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$



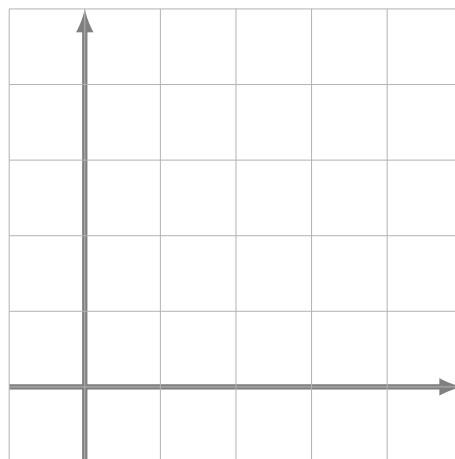
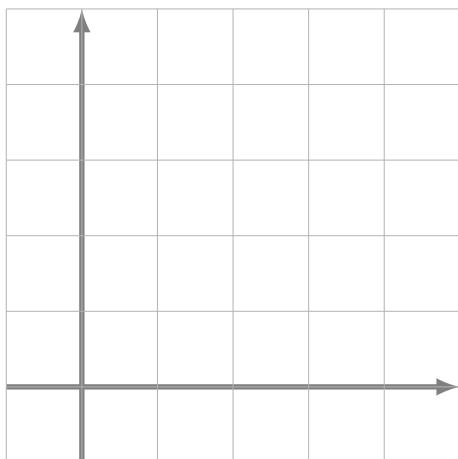
Properties of the determinant

Notation. Given numbers $c_1, c_2, \dots, c_n \in \mathbb{R}$ let $D(c_1, c_2, \dots, c_n)$ denote the $n \times n$ matrix

$$D(c_1, c_2, \dots, c_n) = \begin{bmatrix} c_1 & 0 & \dots & 0 \\ 0 & c_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_n \end{bmatrix}$$

Example.

$$D(2, 3) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$



Property 1. For any numbers $c_1, c_2, \dots, c_n \in \mathbb{R}$ we have

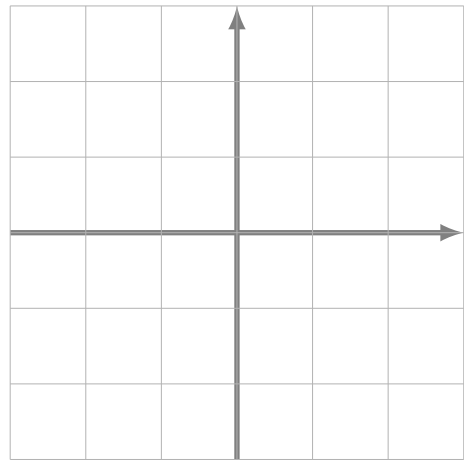
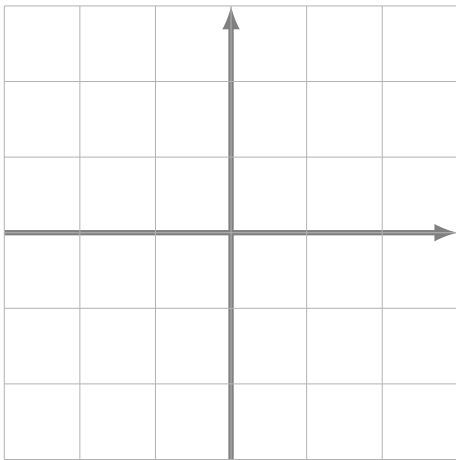
$$\det D(c_1, c_2, \dots, c_n) = c_1 \cdot c_2 \cdot \dots \cdot c_n$$

Notation. Given integers $1 \leq i, j \leq n$ such that $i \neq j$ and a number $c \in \mathbb{R}$ let $E_{i,j}^n(c)$ denote the $n \times n$ matrix which has:

- all entries on the main diagonal equal to 1
- the entry in the i -th row and the j -th column equal to c
- all other entries equal to 0.

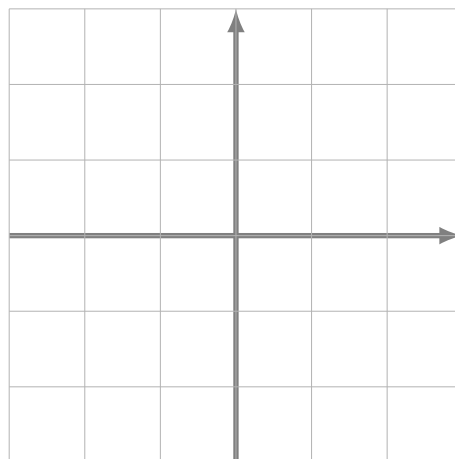
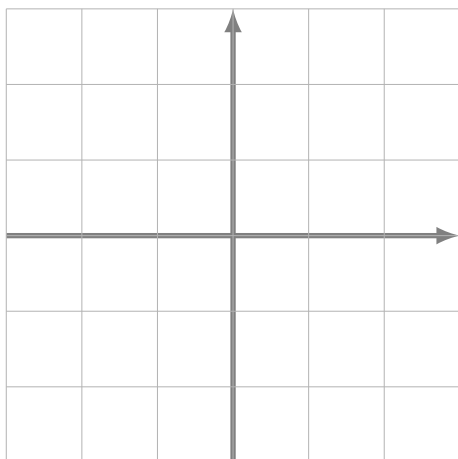
Example.

$$E_{1,2}^2(1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$



Example.

$$E_{2,1}^2(-2) = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

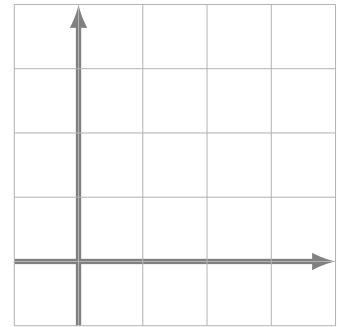
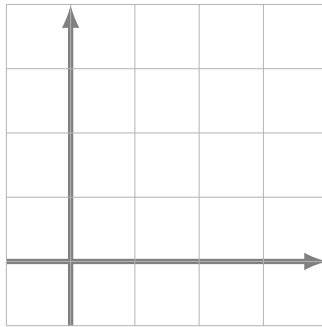
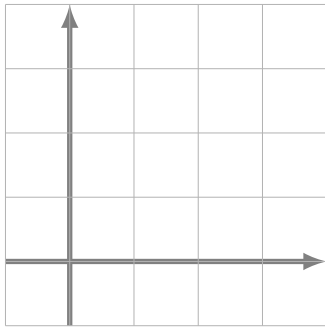


Property 2. For any integers $1 \leq i, j \leq n$, $i \neq j$ and a number $c \in \mathbb{R}$ we have

$$\det E_{i,j}^n(c) = 1$$

Example.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$



Property 3. $\det(AB) = \det(A) \cdot \det(B)$

Theorem

There is exactly one assignment which associates to each $n \times n$ matrix A a number $\det A$ and which satisfies properties 1, 2, and 3.