- **Linear equations**

1) Three forms of equations:

 – system of linear equations

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \\ \quad \vdots \qquad\qquad\quad \vdots \qquad \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n = b_m \end{cases}$$

 – vector equation

$$x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

 – matrix equation

$$\begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

2) How to solve matrix equations:

 – row reduction (= Gauss-Jordan elimination)

3) Related notions:

 – elementary row operations
 – reduced echelon form of a matrix
 – leading ones
 – pivot positions and pivot columns

- **Vectors**

1) $\mathbb{R}^n$ = the set of all vectors with n entries

2) Operations on vectors in $\mathbb{R}^n$:

- addition
- multiplication by scalars

3) Geometric interpretation of vectors and vector operations.

4) Linear combinations of vectors.

5) Span of a set of vectors.

6) Linear independence of vectors.

- **Matrices**

1) Operations on Matrices:

- addition $A + B$
- multiplication by scalars cA
- matrix multiplication AB
- matrix Transpose A^T
- matrix inverse A^{-1}

2) Properties of the matrix algebra:

❶ $AB \neq BA$
❷ $(AB)^T = B^T A^T$
❸ $(A^T)^T = A$
❹ $(A + B)^T = A^T + B^T$
❺ $(A^T)^{-1} = (A^{-1})^T$
❻ $(AB)^{-1} = B^{-1}A^{-1}$
❼ $(A^{-1})^{-1} = A$
❽ $(A + B)^{-1} \neq A^{-1} + B^{-1}$

3) Column space of a matrix Col(A).

4) Null space of a matrix Nul(A), and its representation as a span of vectors.

- **Matrix transformations and linear transformations**

1) An $m \times n$ matrix defines a matrix transformation

$$T_A \colon \mathbb{R}^n \to \mathbb{R}^m, \qquad T_A(\mathbf{v}) = A\mathbf{v}$$

2) Composition of matrix transformations = matrix multiplication:

$$T_A \circ T_B = T_{AB}$$

3) Linear transformation is a function $T \colon \mathbb{R}^n \to \mathbb{R}^m$ such that

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
(ii) $T(c\mathbf{v}) = cT(\mathbf{v})$

4) Every matrix transformation is a linear transformation.

5) Every linear transformation $T \colon \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation:

$$T = T_A$$

where $A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \end{bmatrix}$ is the standard matrix of A.

6) If $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation then

– $\mathrm{Col}(A)$ = the set of all values of T_A
– $T_A(\mathbf{v}_1) = T_A(\mathbf{v}_2)$ if and only if $\mathbf{v}_1 - \mathbf{v}_2 \in \mathrm{Nul}(A)$

7) A matrix transformation $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$ is onto if any only if:

– $\mathrm{Col}(A) = \mathbb{R}^m$
– A has a pivot position in every row

8) A matrix transformation $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if any only if:

– $\mathrm{Nul}(A) = \{\mathbf{0}\}$
– A has a pivot position in every column

9) If A is an invertible matrix then $T_{A^{-1}}$ is the inverse function of T_A.