Operations on matrices so far:

- addition/subtraction $A \pm B$
- scalar multiplication $c \cdot A$
- matrix multiplication $A \cdot B$
- matrix transpose A^T

Next: How to divide matrices?

Note: If a, b - numbers then:

1) $\frac{a}{b} = a \cdot b^{-1}$

2) b^{-1} is the number such that $bb^{-1} = 1$

Definition

A matrix A is *invertible* if there exists a matrix B such that

$$A \cdot B = B \cdot A = I$$

(where $I =$ the identity matrix). In such case we say that B is the *inverse* of A and we write $B = A^{-1}$.

Example.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ is invertible,} \quad A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Check:

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Note: Not every matrix is invertible.

Example:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Assume that $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a matrix such that $AB = I$

Then:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}$$

The first column gives:

$$\left.\begin{aligned} 1 &= a+c \\ 0 &= a+c \end{aligned}\right\} \leftarrow \text{impossible}$$

Thus A is not invertible.

Matrix inverses and matrix equations

Proposition

If A is an invertible matrix then for any vector $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution.

Proof: If $Ax = b$ then:

$$A^{-1}Ax = A^{-1}b$$
$$Ix = A^{-1}b$$
$$x = \underbrace{A^{-1}b}_{\text{the unique solution of } Ax=b}$$

Example. Solve the following matrix equation:

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{A} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Recall: A is invertible, $A^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$

This gives:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$$

Recall: If A be is $m \times n$ matrix then:

- the matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ if and only if A has a pivot position in every row;
- the matrix equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution for $\mathbf{b} \in \mathrm{Col}(A)$ if and only if A has a pivot position in every column.

$$\left[\begin{array}{c|c} A & b \end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{cc|c} 1 & 0 & \square \\ 0 & 1 & \square \\ 0 & 0 & \square \end{array}\right]$$

no pivot in this row

we can get a leading one here, and then no solutions

$$\left[\begin{array}{c|c} A & b \end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & \square \\ 0 & 0 & 1 & \square \end{array}\right]$$

no pivot in this column
This gives a free variable
and so infinitely many solutions

Theorem

If a matrix A is invertible then it must be a square matrix.

For a square matrix A the following conditions are equivalent:

1) A is an invertible matrix.

2) The matrix A has a pivot position in every row and column.

3) The reduced row echelon form of A is the identity matrix I_n.

If A is invertible then

$$\begin{pmatrix} \text{number of} \\ \text{rows of } A \end{pmatrix} = \begin{pmatrix} \text{number of} \\ \text{pivot positions} \end{pmatrix} = \begin{pmatrix} \text{number of} \\ \text{columns of } A \end{pmatrix}$$

So A is a square matrix.

Proposition

If A is an $n \times n$ invertible matrix then

$$A^{-1} = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \dots & \mathbf{w}_n \end{bmatrix}$$

where $\mathbf{w}_i$ is the solution of $A\mathbf{x} = \mathbf{e}_i$.

Proof: We have:

$$I_n = \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & & e_n \end{array} \\ \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \end{array} = [e_1\ e_2 \dots e_n]$$

This gives:

$$[e_1\ e_2 \dots e_n] = I_n = AA^{-1} = A[w_1\ w_2\ \dots\ w_n] = [Aw_1\ Aw_2\ \dots\ Aw_n]$$

We obtain:

$$Aw_1 = e_1\ ,\quad Aw_2 = e_2\ ,\ \dots\ ,\quad Aw_n = e_n$$

Example.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2\text{, so } A \text{ is invertible}$$

We have:

$$A^{-1} = [w_1\ w_2] \quad \text{where} \quad w_1 = (\text{solution of } Ax = e_1)$$
$$w_2 = (\text{solution of } Ax = e_2)$$

Solve $Ax = e_1$:

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 1 & 0 \end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \end{array}\right] \quad \text{so: } w_1 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

Solve $Ax = e_2$:

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 1 \end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{array}\right] \quad \text{so: } w_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

This gives:

$$A^{-1} = [w_1\ w_2] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Simplification:
How to solve several matrix equations with the same coefficient matrix at the same time

$A\mathbf{x} = \mathbf{b}_1,\ A\mathbf{x} = \mathbf{b}_2,\ \ldots,\ A\mathbf{x} = \mathbf{b}_n$

matrix of equations

$\Downarrow$

$\left[\begin{array}{c|cccc} A & \mathbf{b}_1 & \mathbf{b}_2 & \ldots & \mathbf{b}_n \end{array}\right]$

augmented matrix

row reduction $\Downarrow$

$\left[\begin{array}{c|c} \quad & \qquad\qquad \end{array}\right]$

reduced matrix

read off solutions $\Downarrow$

solutions

Example. Solve the vector equations $A\mathbf{x} = \mathbf{e}_1$ and $A\mathbf{x} = \mathbf{e}_2$ where

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solution:

augmented matrix:

$$\left[\, A \;\middle|\; e_1 \;\; e_2 \,\right] = \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow[\text{reduction}]{\text{row}} \left[\begin{array}{c|c|c} \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{\substack{\text{row reduced} \\ \text{form of } A}} & \underbrace{\begin{matrix} 1/2 \\ -1/2 \end{matrix}}_{\substack{\text{solution} \\ \text{of } Ax=e_1}} & \underbrace{\begin{matrix} 1/2 \\ 1/2 \end{matrix}}_{\substack{\text{solution} \\ \text{of } Ax=e_2}} \end{array}\right]$$

Summary:
How to invert a matrix

Example: $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

1) Augment A by the identity matrix.

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]$$

2) Reduce the augmented matrix.

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

2) If after the row reduction the matrix on the left is the identity matrix, then A is invertible and

$$A^{-1} = \text{the matrix on the right}$$

Otherwise A is not invertible.

In our example A is invertible and $A^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

Properties of matrix inverses

1) If A is invertible then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

2) If A, B are invertible then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

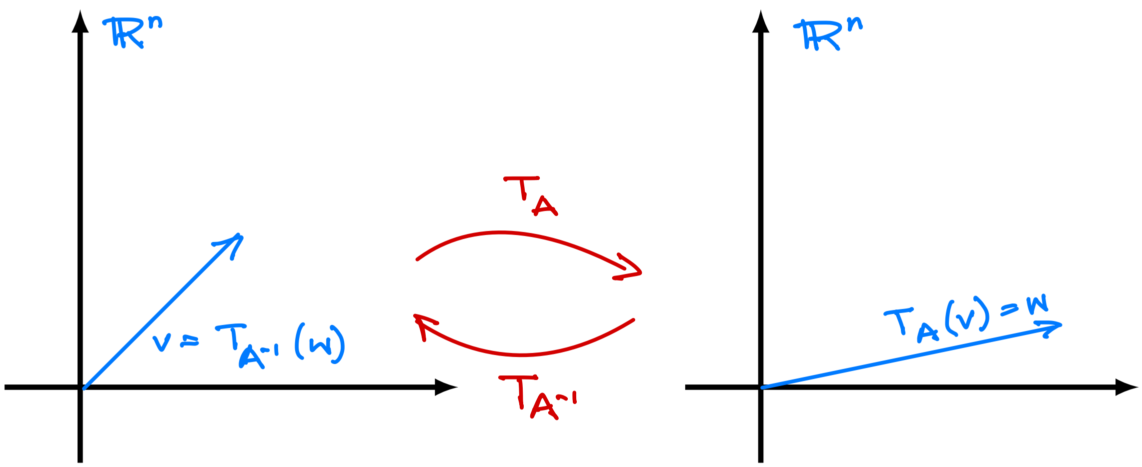
3) If A is invertible then A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

Matrix inverses and matrix transformations

A - invertible $n \times n$ matrix

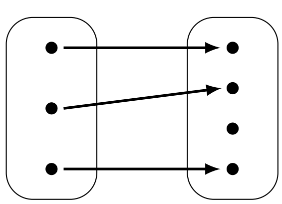
$$T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n \qquad T_{A^{-1}} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$v \longmapsto Av \qquad\qquad v \longmapsto A^{-1}v$$



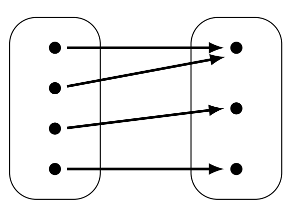
$$T_{A^{-1}}(T_A(v)) = T_{A^{-1}}(Av) = A^{-1}(Av) = (A^{-1}A)v = Iv = v$$
$$T_A(T_{A^{-1}}(w)) = \quad \dots \qquad \dots \qquad = w$$

Note

- If A is an $n \times n$ invertible matrix then the matrix transformation $T_A \colon \mathbb{R}^n \to \mathbb{R}^n$ has the inverse function $T_{A^{-1}} \colon \mathbb{R}^n \to \mathbb{R}^n$.
- As a consequence the function T_A is both onto and one-to-one.



not onto



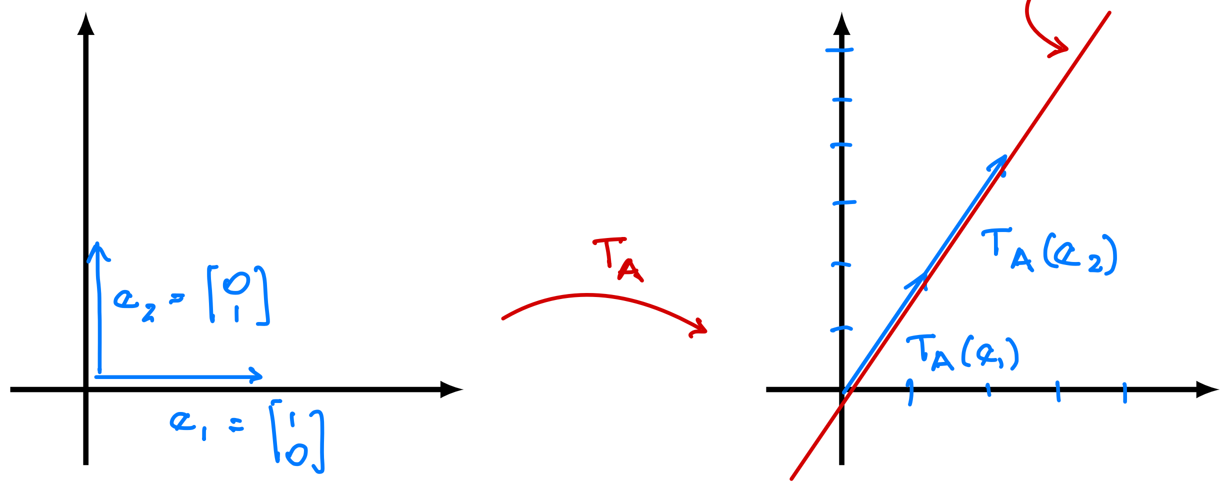
not one-to-one

Example.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$T_A : \mathbb{R}^2 \to \mathbb{R}^2$$
$$v \longmapsto Av$$

$\mathrm{Col}(A)$ = the space of all possible values of T_A



$$T_A(e_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$T_A(e_2) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

- T_A is not onto, so A is not invertible
- T_A is also not one-to-one since e.g.

$$T_A\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = T_A\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$