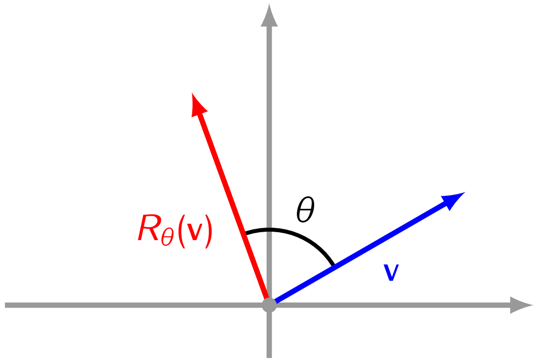
**Problem:** How to recognize if a function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation?

**Example.** Rotation by an angle $\theta$:




$$R_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$$

Questions:

1) Is $R_\theta$ a matrix transformation?
   That is, is there a matrix $A$ such that $R_\theta(v) = Av$ for all $v \in \mathbb{R}^2$?

2) If so, what is this matrix $A$?

**Definition**

A function $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a *linear transformation* if it satisfies the following conditions:

1) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$
2) $T(c\mathbf{v}) = cT(\mathbf{v})$ for any $\mathbf{v} \in \mathbb{R}^n$ and any scalar $c$.

**Proposition**

Every matrix transformation is a linear transformation.

Proof: Let $A$ be an $m \times n$ matrix

$$T_A\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$v \longmapsto Av$$

We have:

1) $T_A(u+v) = A(u+v) = Au + Av = T_A(u) + T_A(v)$

2) $T_A(cu) = A(cu) = c(Au) = c \cdot T_A(u)$

**Theorem**

Every linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation:

$$T = T_A$$

for some matrix $A$.

Proof:

Denote:

$$\underbrace{e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ \dots,\ e_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}_{\text{"standard basis vectors"}} \in \mathbb{R}^n$$

Take $A = [\,T(e_1)\ \ T(e_2)\ \ \dots\ \ T(e_n)\,]$

We will show that $T(u) = Au$ for all $u \in \mathbb{R}^n$

Indeed:

if $u = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ then $u = c_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \dots + c_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

$= c_1 e_1 + c_2 e_2 + \dots + c_n e_n$

This gives:

$$\begin{aligned} T(u) &= T(c_1 e_1 + c_2 e_2 + \dots + c_n e_n) \\ &= T(c_1 e_1) + T(c_2 e_2) + \dots + T(c_n e_n) \\ &= c_1 T(e_1) + c_2 T(e_2) + \dots + c_n T(e_n) \\ &= [\,T(e_1)\ \ T(e_2)\ \ \dots\ \ T(e_n)\,] \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \\ &= Au \end{aligned}$$

**Corollary**

If $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then $T = T_A$ where $A$ is the matrix given by

$$A = \left[\; T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \ldots \quad T(\mathbf{e}_n) \;\right]$$

This matrix is called the *standard matrix* of $T$.

**Example.** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the function given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 0 \\ 2x_1 \end{bmatrix}$$

Check if $T$ is a linear transformation.

Solution:

1) Check if $T(u+v) = T(u)+T(v)$

Let $u = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $v = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. We have:

$$\left.\begin{aligned} T(u) &= T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} a_1 + a_2 \\ 0 \\ 2a_1 \end{bmatrix} \\ T(v) &= T\left(\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\right) = \begin{bmatrix} b_1 + b_2 \\ 0 \\ 2b_1 \end{bmatrix} \end{aligned}\right\} \quad T(u)+T(v) = \begin{bmatrix} a_1 + a_2 + b_1 + b_2 \\ 0 \\ 2a_1 + 2b_1 \end{bmatrix}$$

$$T(u+v) = T\left(\begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}\right) = \begin{bmatrix} a_1 + b_1 + a_2 + b_2 \\ 0 \\ 2(a_1 + b_1) \end{bmatrix}$$

This gives: $T(u) + T(v) = T(u+v)$

2) Similarly we can check that $T(cu) = cT(u)$

1)+2) show that $T$ is a linear transformation

**Example.** Find the standard matrix for the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ from the previous example.

Solution: Recall: $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 0 \\ 2x_1 \end{bmatrix}$

Standard matrix: $A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$

where $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We get:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 2 & 0 \end{bmatrix}$$

Check:

$$A \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + x_2 \\ 0 \\ 2x_1 \end{bmatrix} = T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

**Example.** Let $S\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the function given by

$$S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 + x_2 \\ x_2 \\ 3x_1 \end{bmatrix}$$

Check if $S$ is a linear transformation. If it is, find its standard matrix.

Solution:

1) Check if $S(u+v) = S(u) + S(v)$

Let $u = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $v = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. We have:

$$S(u) = S\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} 1 + a_2 \\ a_2 \\ 3a_1 \end{bmatrix}$$

$$S(v) = S\left(\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\right) = \begin{bmatrix} 1 + b_2 \\ b_2 \\ 3b_1 \end{bmatrix}$$

$$S(u) + S(v) = \begin{bmatrix} 2 + a_2 + b_2 \\ a_2 + b_2 \\ 3a_1 + 3b_1 \end{bmatrix}$$

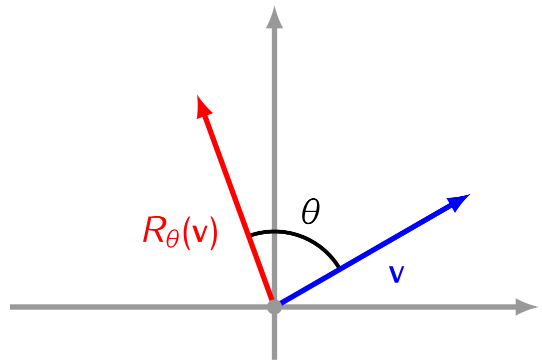
$$S(u+v) = S\left(\begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}\right) = \begin{bmatrix} 1 + a_2 + b_2 \\ a_2 + b_2 \\ 3(a_1 + b_1) \end{bmatrix}$$

not equal

We get: $S(u) + S(v) \neq S(u+v)$

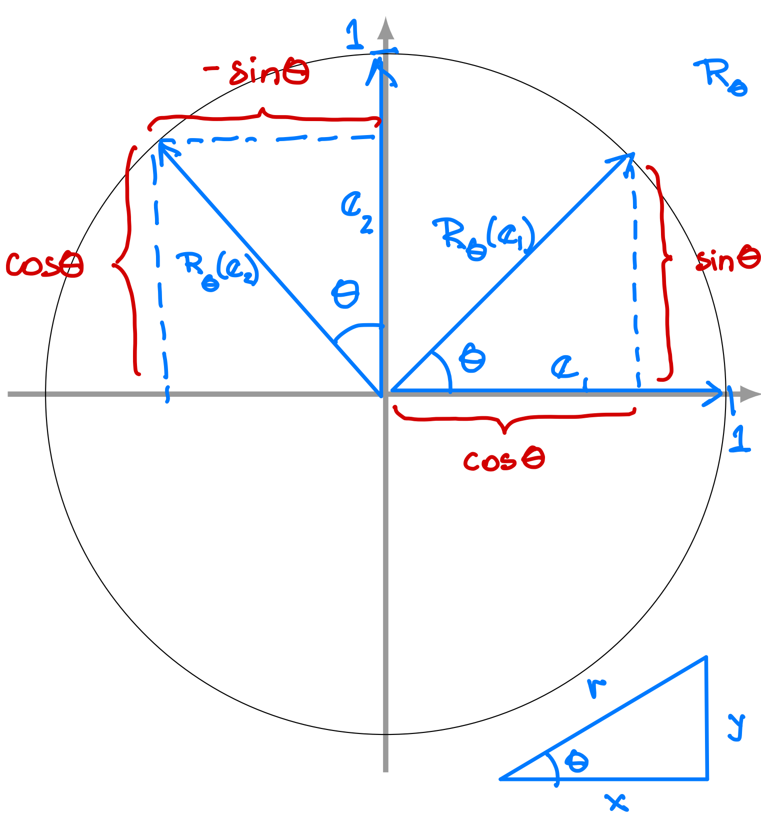
This shows that $S$ is not a linear transformation and thus it can't be represented by a matrix.

**Back to rotations:**




$$R_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$$

- One can check that $R_\theta$ is a linear transformation.
- The standard matrix of $R_\theta$: $A = [R_\theta(e_1) \;\; R_\theta(e_2)]$




We have:

$$R_\theta(e_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$R_\theta(e_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

This gives:

$$\boxed{A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$$

the standard matrix of the rotation by an angle $\theta$

$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

**Proposition**

Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be the standard basis of of $\mathbb{R}^n$. For any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^m$ there exists one and only one linear transformation

$$T \colon \mathbb{R}^n \to \mathbb{R}^m$$

such that

$$T(\mathbf{e}_1) = \mathbf{v}_1 \quad T(\mathbf{e}_2) = \mathbf{v}_2, \quad \ldots, \quad T(\mathbf{e}_n) = \mathbf{v}_n$$

The standard matrix of this linear transformation is given by

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}$$