Recall: A vector equation

$$x_1\mathbf{v}_1 + \ldots + x_n\mathbf{v}_n = \mathbf{b}$$

has a solution if and only if $\mathbf{b} \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$.

Equivalently: If $A = [v_1 \ \ldots \ v_n]$ then the matrix equation

$$Ax = b$$

has a solution if and only if $b \in \mathrm{Span}(v_1, \ldots, v_n)$.

Definition

If A is a matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$:

$$A = \begin{bmatrix} \mathbf{v}_1 & \ldots & \mathbf{v}_n \end{bmatrix}$$

then the set $\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is called the *column space* of A and it is denoted $\mathrm{Col}(A)$.

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \mathrm{Col}(A) = \mathrm{Span}\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}\right)$$

Upshot. A matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b} \in \mathrm{Col}(A)$.

Recall: A vector equation

$$x_1\mathbf{v}_1 + \ldots + x_n\mathbf{v}_n = \mathbf{b}$$

has only one solution for each $\mathbf{b} \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ if and only if the homogenous equation

$$x_1\mathbf{v}_1 + \ldots + x_n\mathbf{v}_n = \mathbf{0}$$

has only the trivial solution $x_1 = 0, \ldots, x_n = 0$.

Reformulation for matrix equations:

A matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has only one solution for each $\mathbf{b} \in \mathrm{Col}(A)$ if and only if the homogenous equation

$$A\mathbf{x} = \mathbf{0}$$

has only the trivial solution $\mathbf{x} = \underbrace{\mathbf{0}}_{\text{the zero vector}}$.

Definition

If A is a matrix then the set of solution of the homogenous equation

$$A\mathbf{x} = \mathbf{0}$$

is called the *null space* of A and it is denoted $\mathrm{Nul}(A)$.

Upshot. A matrix equation $A\mathbf{x} = \mathbf{b}$ has only one solution for each $\mathbf{b} \in \mathrm{Col}(A)$ if and only if $\mathrm{Nul}(A) = \{\mathbf{0}\}$.

Example. Find the null space of the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Solution: We need to solve

$$Ax = \mathbf{0}$$

augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{array}\right] \xrightarrow{\text{row reduction}} \begin{array}{ccc} x_1 & x_2 & \\ \hline \end{array}\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

solutions:

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$$

in vector form:

$$x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So: $\mathrm{Nul}(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} = \mathrm{Span}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} \right)$

Proposition

$\mathrm{Nul}(A) = \{\mathbf{0}\}$ if and only if the matrix A has a pivot position in every column.

Example. Find the null space of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -2 & -2 & 1 & -5 \\ 1 & 1 & -1 & 3 \end{bmatrix}$$

Solution: We need to solve $A\mathbf{x} = \mathbf{0}$.

augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ -2 & -2 & 1 & -5 & 0 \\ 1 & 1 & -1 & 3 & 0 \end{array}\right] \xrightarrow{\text{row reduction}} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & \underbrace{0}_{\text{free}} & 0 & \underbrace{0}_{\text{free}} & 0 \end{array}\right] \end{array}$$

Solutions:

$$\begin{cases} x_1 = -x_2 - 2x_4 \\ x_2 = x_2 \\ x_3 = x_4 \\ x_4 = x_4 \end{cases}$$

vector form:

$$\mathbf{x} = \begin{bmatrix} -x_2 - 2x_4 \\ x_2 \\ x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2x_4 \\ 0 \\ x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

This gives:

$$\mathrm{Nul}(A) = \mathrm{Span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right)$$

Note

If A is an $m \times n$ matrix then $\mathrm{Nul}(A)$ can be always described as a span of some vectors in $\mathbb{R}^n$.

Example. Solve the matrix equation $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -2 & -2 & 1 & -5 \\ 1 & 1 & -1 & 3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Solution:

augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 1 \\ -2 & -2 & 1 & -5 & 0 \\ 1 & 1 & -1 & 3 & -1 \end{array}\right] \xrightarrow{\text{row reduction}} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \hline \multicolumn{5}{c}{\left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & \underbrace{0}_{\text{free}} & 0 & \underbrace{0}_{\text{free}} & 0 \end{array}\right]} \end{array}$$

Solutions:

$$\begin{cases} x_1 = 1 - x_2 - 2x_4 \\ x_2 = x_2 \\ x_3 = 2 + x_4 \\ x_4 = x_4 \end{cases}$$

vector form:

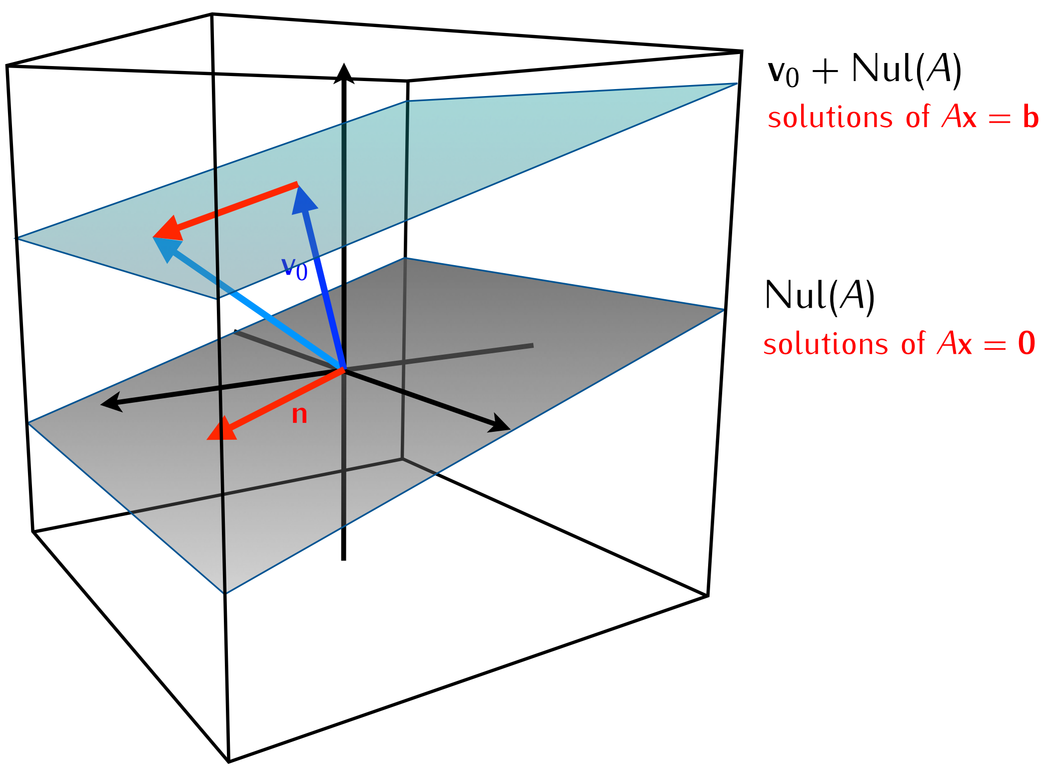
$$\mathbf{x} = \begin{bmatrix} 1 - x_2 - 2x_4 \\ x_2 \\ 2 + x_4 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}}_{\text{a particular solution of } A\mathbf{x}=\mathbf{b}} + \underbrace{x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}}_{\text{a vector from Nul}(A)}$$

Proposition

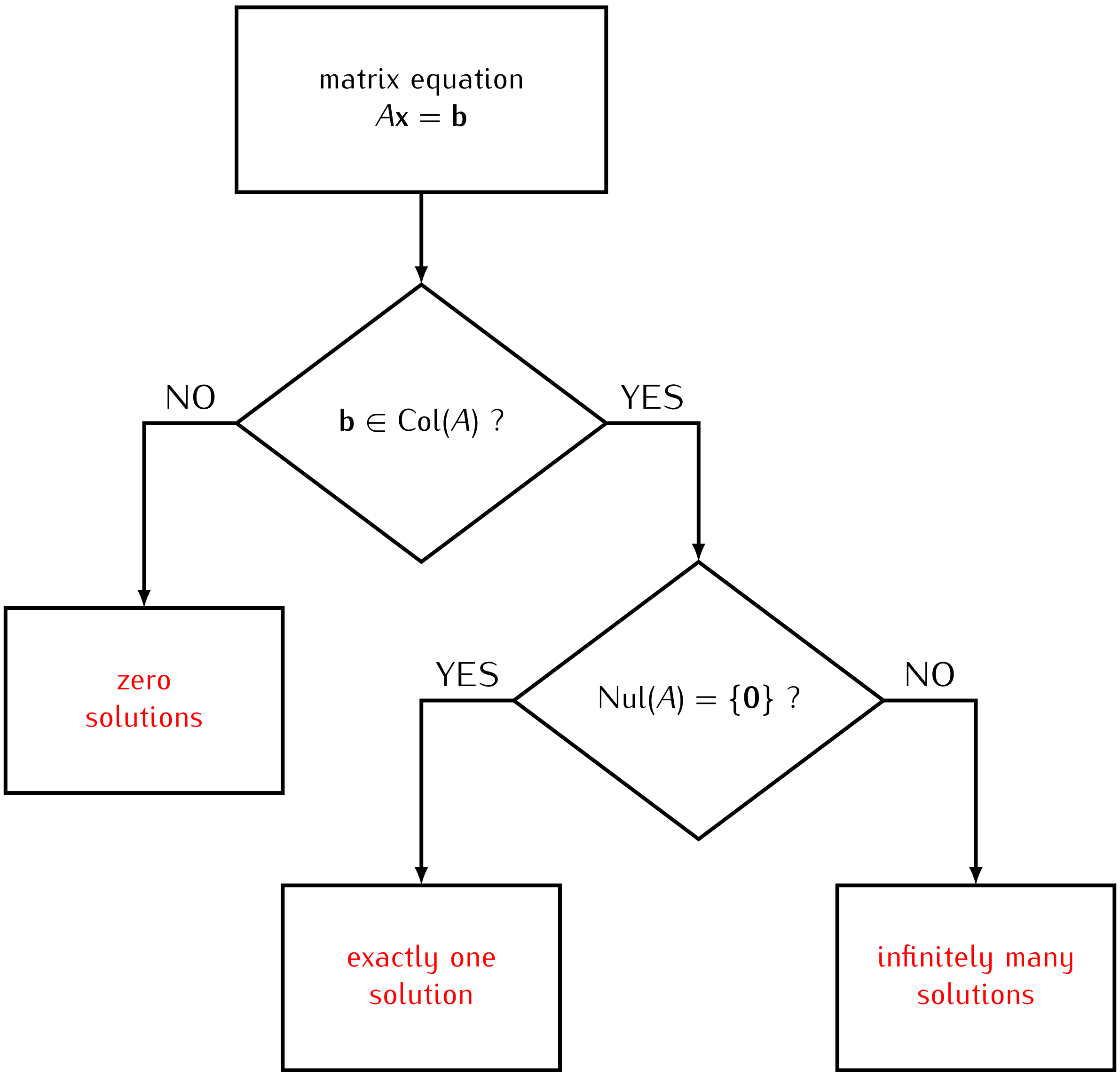
Let $\mathbf{v}_0$ be some chosen solution of a matrix equation $A\mathbf{x} = \mathbf{b}$. Then any other solution $\mathbf{v}$ of this equation is of the form

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{n}$$

where $\mathbf{n} \in \mathrm{Nul}(A)$.



Upshot: how to find the number of solutions of a matrix equation



Question: What conditions on the matrix A guarantee that the equation $A\mathbf{x} = \mathbf{b}$ has a solution for an arbitrary vector $\mathbf{b}$?

Example.

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \xrightarrow{\textit{row reduction}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

augmented matrix of $Ax = b$:

$$[\,A \mid b\,] \xrightarrow{\text{row reduction}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \square \\ 0 & 1 & 0 & -1 & \square \\ 0 & 0 & 1 & 2 & \square \end{array}\right]$$

no place for a leading one in this column, so $Ax = b$ will always have a solution

Example.

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 7 \end{bmatrix} \xrightarrow{\textit{row reduction}} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

augmented matrix of $Ax = b$:

$$[\,A \mid b\,] \xrightarrow{\text{row reduction}} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & \square \\ 0 & 1 & 1 & 1 & \square \\ 0 & 0 & 0 & 0 & \square \end{array}\right]$$

By choosing an appropriate vector b we will get a leading one in this column. Thus $Ax = b$ will have no solution for some b.

Proposition

A matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b}$ if and only if A has a pivot position in every row.

In such case $\mathrm{Col}(A) = \mathbb{R}^m$, where m is the number of rows of A.