So far:

$$\begin{cases} 2x_1 + 4x_2 + 6x_3 + 3x_4 = 7 \\ 3x_1 + 2x_2 + 2x_3 + 9x_4 = 3 \\ 5x_1 + 8x_2 + 3x_3 + 3x_4 = 9 \end{cases}$$

system of
linear equations

$$\Updownarrow$$

$$x_1 \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 2 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 9 \end{bmatrix}$$

vector equation

Next:

$A \cdot x = b$

$$\begin{bmatrix} 2 & 4 & 6 & 3 \\ 3 & 2 & 2 & 9 \\ 5 & 8 & 3 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 9 \end{bmatrix}$$

matrix equation

Definition

Let A be an $m \times n$ matrix with columns $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ and let $\mathbf{w}$ be a vector in $\mathbb{R}^n$:

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

The product $A\mathbf{w}$ is a vector in $\mathbb{R}^m$ given by

$$A\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

Example.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$$A\mathbf{w} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix} + (-2)\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 1 \cdot \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 12 \end{bmatrix} + \begin{bmatrix} -4 \\ -10 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

Properties of matrix-vector multiplication

1) The product $A\mathbf{w}$ is defined only if

$$(\text{number of columns of } A) = (\text{number of entries of } \mathbf{w})$$

$$\begin{array}{ccc} A & & \mathbf{w} \\ m \times n & & n \times 1 \\ & m \times 1 & \\ & A \cdot \mathbf{w} & \end{array}$$

e.g.:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 7 \\ 0 \\ 5 \end{bmatrix}$$

$$\underbrace{2 \times 3 \qquad 4 \times 1}$$

no match, so this multiplication is not defined

2) $A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \left(\begin{bmatrix} 7 \\ -1 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

3) If c is a scalar then $A(c\mathbf{w}) = c(A\mathbf{w})$.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \left(5 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) = 5 \cdot \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$$

Example. Solve the matrix equation

$$\underbrace{\overset{A}{\begin{bmatrix} 1 & 1 & -4 \\ 1 & -2 & 3 \\ 3 & -3 & 0 \end{bmatrix}} \cdot \overset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}}_{\shortparallel} = \overset{b}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$$

$$x_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

vector equation

augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & -4 & 1 \\ 1 & -2 & 3 & 2 \\ 3 & -3 & 0 & 3 \end{array}\right] \xrightarrow{\text{row reduction}} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \end{array}$$

$$\underbrace{}_{A} \quad \underbrace{}_{b}$$

solutions:

$$\begin{cases} x_1 = 3 \\ x_2 = 2 \\ x_1 = 1 \end{cases}$$

in vector form:

$$x = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Check:

$$\begin{bmatrix} 1 & 1 & -4 \\ 1 & -2 & 3 \\ 3 & -3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

How to solve a matrix equation

$$A\mathbf{x} = \mathbf{b}$$

matrix equation

$$\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$$

augmented matrix

row reduction

$$\left[\text{ reduced matrix }\right]$$

read off solutions

$$\mathbf{x} = \ldots$$

solutions