2025.01.21



**Recall:**

### How to solve a system of linear equations

system of equations

$$\begin{cases} -x_1 + 2x_2 + 3x_3 = 4 \\ 2x_1 + 6x_3 = 9 \\ 4x_1 - x_2 - 3x_3 = 0 \end{cases}$$

*make a matrix* →

$\left[ \text{augmented matrix} \right]$

↓ " row reduction " *Gauss-Jordan elimination*

$\left[ \text{matrix in reduced row echelon form} \right]$

← *read off solutions*

solutions

$$\begin{cases} x_1 = \dots \\ x_2 = \dots \\ x_3 = \dots \end{cases}$$

## Gauss-Jordan elimination process (= row reduction)

$$\begin{bmatrix} \quad \text{matrix} \quad \end{bmatrix} \xrightarrow[\textit{row operations}]{\textit{elementary}} \begin{bmatrix} \text{matrix} \\ \text{in the reduced row} \\ \text{echelon form} \end{bmatrix}$$

1 Interchange rows, if necessary, to bring a non-zero element to the top of the first non-zero column of the matrix.

2 Multiply the first row so that its first non-zero entry becomes 1.

3 Add multiples of the first row to eliminate non-zero entries below the leading one.

4 Ignore the first row; apply steps 1-3 to the rest of the matrix.

5 Eliminate non-zero entries above all leading ones.

**Example.**

$$\begin{bmatrix} 0 & 4 & -8 & 0 & 4 \\ 2 & 6 & -6 & -2 & -4 \\ 2 & 7 & -8 & 0 & -1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 2 & 6 & -6 & -2 & -4 \\ 0 & 4 & -8 & 0 & 4 \\ 2 & 7 & -8 & 0 & -1 \end{bmatrix} \cdot \frac{1}{2}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 3 & -3 & -1 & -2 \\ 0 & 4 & -8 & 0 & 4 \\ 2 & 7 & -8 & 0 & -1 \end{bmatrix} \cdot (-2)$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 3 & -3 & -1 & -2 \\ 0 & 4 & -8 & 0 & 4 \\ 0 & 1 & -2 & 2 & 3 \end{bmatrix} \cdot \frac{1}{4}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 3 & -3 & -1 & -2 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 1 & -2 & 2 & 3 \end{bmatrix} \cdot (-1)$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 3 & -3 & -1 & -2 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix} \cdot \frac{1}{2}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 3 & -3 & -1 & -2 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \cdot (1)$$

matrix in the row echelon form

(not reduced yet)

$$\downarrow$$

$$\begin{bmatrix} 1 & \boxed{3} & -3 & 0 & -1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \cdot (-3)$$

$$\downarrow$$

$$\begin{bmatrix} 1 & \boxed{0} & 3 & 0 & -4 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

matrix in the reduced row echelon form

(THE END)

## How to solve systems of linear equations: example

$$\begin{cases} 4x_2 - 8x_3 = 4 \\ 2x_1 + 6x_2 - 6x_3 - 2x_4 = -4 \\ 2x_1 + 7x_2 - 8x_3 = -1 \end{cases} \xrightarrow{1} \begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \end{array} \left[\begin{array}{cccc|c} 0 & 4 & -8 & 0 & 4 \\ 2 & 6 & -6 & -2 & -4 \\ 2 & 7 & -8 & 0 & -1 \end{array}\right]$$

2 ↓ row reduction (see the previous page for computations)

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & -4 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & \underbrace{0}_{\text{free variable}} & 1 & 1 \end{array}\right]$$

$$\xrightarrow{3} \begin{cases} x_1 + 3x_3 = -4 \\ x_2 - 2x_3 = 1 \\ x_4 = 1 \end{cases}$$

simplify:

$$\begin{cases} x_1 = -4 - 3x_3 \\ x_2 = 1 + 2x_3 \\ x_3 = \text{free} \\ x_4 = 1 \end{cases}$$

(infinitely many solutions)