Next:

How to solve a system of linear equations

system of equations

$$\begin{cases} -x_1 + 2x_2 + 3x_3 = 4 \\ 2x_1 + 6x_3 = 9 \\ 4x_1 - x_2 - 3x_3 = 0 \end{cases}$$

make a matrix

[augmented matrix]

Gauss–Jordan elimination

[matrix in reduced row echelon form]

read off solutions

solutions

$$\begin{cases} x_1 = \dots \\ x_2 = \dots \\ x_3 = \dots \end{cases}$$

Matrices

matrix = rectangular array of numbers

Example.

$$\overbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}^{\text{3 columns}} \left.\vphantom{\begin{bmatrix} 1 \\ 4 \end{bmatrix}}\right\} \text{2 rows}$$

2×3 matrix

$$\overbrace{\begin{bmatrix} 1 & 2 & 0 \\ 7 & -5 & 1 \\ 8 & 10 & 7 \\ 6 & 4 & 3 \end{bmatrix}}^{\text{3 columns}} \left.\vphantom{\begin{bmatrix} 1 \\ 7 \\ 8 \\ 6 \end{bmatrix}}\right\} \text{4 rows}$$

4×3 matrix

Note

Every system of linear equations can be represented by a matrix.

Example.

$$\begin{cases} -x_1 + 2x_2 + 3x_3 = 4 \\ 2x_1 + 6x_3 = 9 \\ 4x_1 - x_2 - 3x_3 = 0 \end{cases}$$

$$\underbrace{\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \end{array}}_{}$$

$$\underbrace{\left[\begin{array}{ccc|c} -1 & 2 & 3 & 4 \\ 2 & 0 & 6 & 9 \\ 4 & -1 & -3 & 0 \end{array}\right]}_{\text{the augmented matrix of the system of equations}}$$

the coefficients matrix

the column of constants

Elementary row operations:

1) Interchange of two rows.

Example.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 1 \\ 4 & 3 & 0 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 & 3 & 0 & 7 \\ 0 & 1 & 5 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

2) Multiplication of a row by a non-zero number.

Example.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 1 \\ 4 & 3 & 0 & 7 \end{bmatrix}\begin{matrix} \\ \\ \cdot 2 \end{matrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 1 \\ 8 & 6 & 0 & 14 \end{bmatrix}$$

3) Addition of a multiple of one row to another row.

Example.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 1 \\ 4 & 3 & 0 & 7 \end{bmatrix}\begin{matrix} \cdot 3 \\ \\ \\ \end{matrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 7 & 14 & 13 \\ 4 & 3 & 0 & 7 \end{bmatrix}$$

Proposition

Elementary row operations do not change solutions of the system of equations represented by a matrix.

system of linear equations → augmented matrix

elementary row operation

augmented matrix → system of linear equations

different systems same solutions

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ 2x_1 - x_2 + 5x_3 = 0 \\ 3x_1 + 6x_2 - x_3 = 1 \end{cases} \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & -1 & 5 & 0 \\ 3 & 6 & -1 & 1 \end{array}\right]$$

$$\begin{cases} 3x_1 + 6x_2 - x_3 = 1 \\ 2x_1 - x_2 + 5x_3 = 0 \\ x_1 + 2x_2 + 3x_3 = 4 \end{cases} \longleftarrow \left[\begin{array}{ccc|c} 3 & 6 & -1 & 1 \\ 2 & -1 & 5 & 0 \\ 1 & 2 & 3 & 4 \end{array}\right]$$

(interchange of rows) = (interchange of equations)

(multiplication of a row) = (multiplication of an equation)

(addition of a multiple of one row to another) = (addition of a multiple of one equation to another)