2025.01.21

Next: From systems of linear equations to vector equations.

$$\begin{cases} x_1 + 2x_2 = 4 \\ 2x_1 + 7x_2 = 9 \\ 4x_1 + x_2 = 0 \end{cases} \quad \Longrightarrow \quad x_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 0 \end{bmatrix}$$

Why vectors and vector equations are useful:

- They show up in many applications (velocity vectors, force vectors etc.)
- They give a better geometric picture of systems of linear equations.

Definition

A *column vector* is a matrix with one column.

Example:

$$u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad v = \begin{bmatrix} 5 \\ -1 \\ 0 \\ 7 \end{bmatrix}$$

Note. Columns of a matrix are column vectors.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Notation

$\mathbb{R}^n$ is the set of all column vectors with n entries.

$$\mathbb{R}^2 = \left(\begin{matrix}\text{the set of all} \\ \text{vectors with 2 entries}\end{matrix}\right) = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \;\middle|\; a_1, a_2 \in \mathbb{R} \right\}$$

$$\mathbb{R}^3 = \left(\begin{matrix}\text{the set of all} \\ \text{vectors with 3 entries}\end{matrix}\right) = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \;\middle|\; a_1, a_2, a_3 \in \mathbb{R} \right\}$$

Operations on vectors in $\mathbb{R}^n$

1) Addition of vectors:

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

e.g.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 5 \\ -1 \\ 7 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 10 \end{bmatrix}$$

Note: We can add two vectors only if they have the same number of entries.

2) Multiplication by scalars:

scalars = real numbers

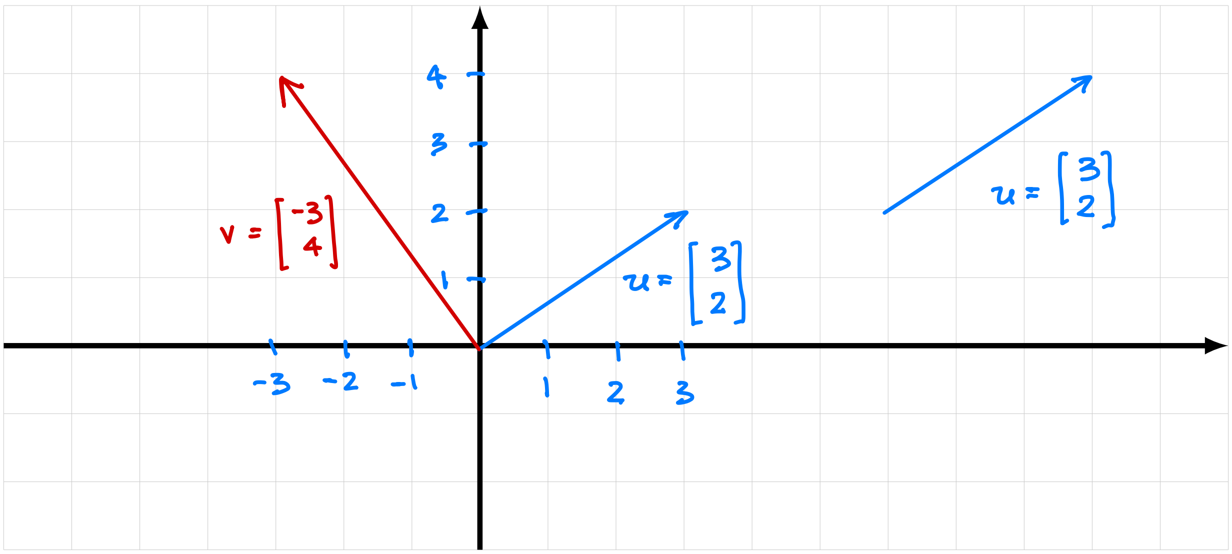
$$c \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}$$

e.g.

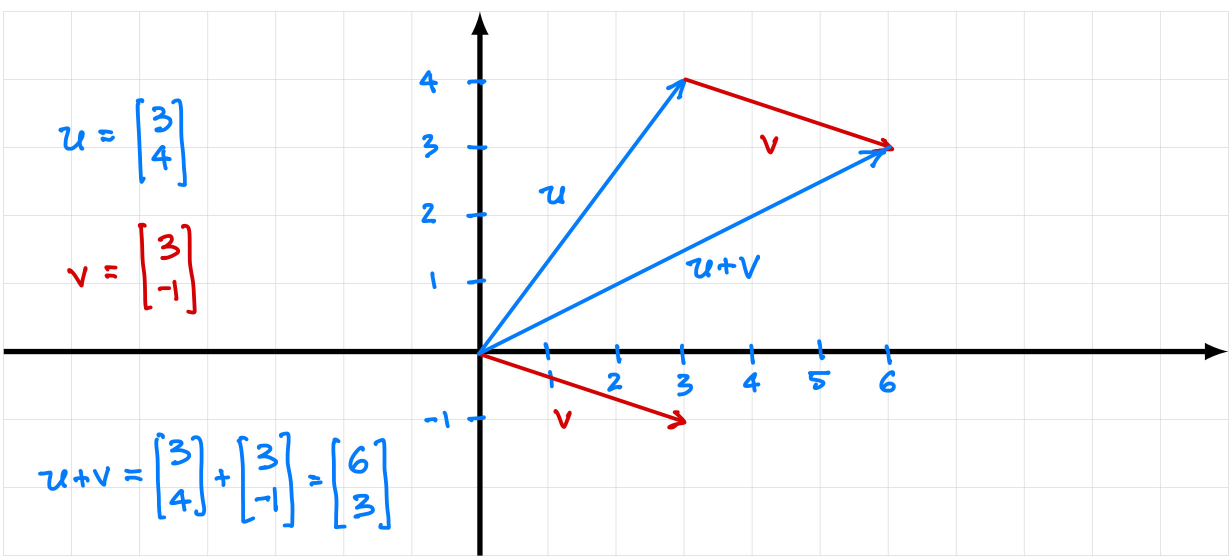
$$3 \cdot \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 15 \end{bmatrix}$$

<u>Geometric interpretation of vectors in $\mathbb{R}^2$</u>

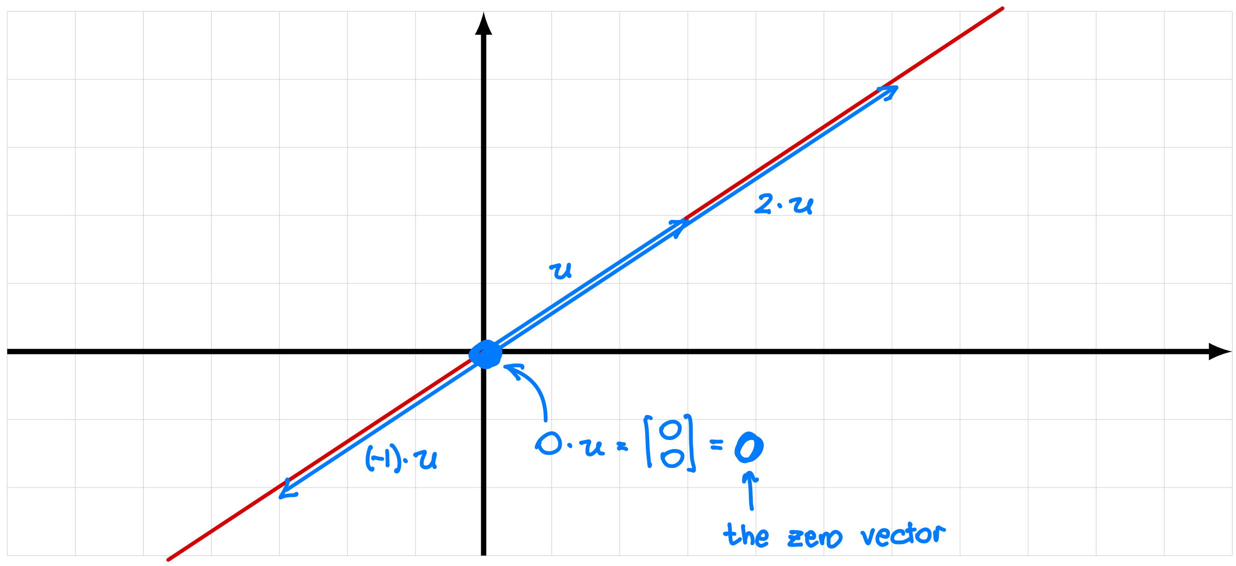
Vector coordinates:



Vector addition:



Scalar multiplication:



Vector equations

$$\underbrace{x_1\mathbf{v}_1}_{} + \underbrace{x_2\mathbf{v}_2}_{} + \ldots + x_p\underbrace{\mathbf{v}_p}_{} = \underbrace{\mathbf{w}}_{}$$

vectors in $\mathbb{R}^n$

Example. Solve the following vector equation:

$$\underbrace{x_1 \overset{v_1}{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} + x_2 \overset{v_2}{\begin{bmatrix} 4 \\ -2 \end{bmatrix}}}_{\parallel} = \overset{w}{\begin{bmatrix} 10 \\ 3 \end{bmatrix}}$$

$$\underbrace{\begin{bmatrix} 2x_1 \\ 3x_1 \end{bmatrix} + \begin{bmatrix} 4x_2 \\ -2x_2 \end{bmatrix}}_{\parallel} = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 + 4x_2 \\ 3x_1 - 2x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{cases} 2x_1 + 4x_2 = 10 \\ 3x_1 - 2x_2 = 3 \end{cases}$$

system of linear equations

augmented matrix:

$$\begin{array}{ccc} \left[\begin{array}{cc|c} 2 & 4 & 10 \\ 3 & -2 & 3 \end{array}\right] \\ v_1 \quad v_2 \quad w \end{array} \xrightarrow{\text{row reduction}} \begin{array}{ccc} x_1 & x_2 & \\ \end{array} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3/2 \end{array}\right] \longrightarrow \begin{cases} x_1 = 2 \\ x_2 = 3/2 \end{cases}$$

How to solve a vector equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{w}$$

vector of equation

make a matrix

$$\left[\begin{array}{ccc|c} \mathbf{v}_1 & \ldots & \mathbf{v}_p & \mathbf{w} \end{array}\right]$$

augmented matrix

row reduction

$$\left[\ \text{reduced matrix}\ \right]$$

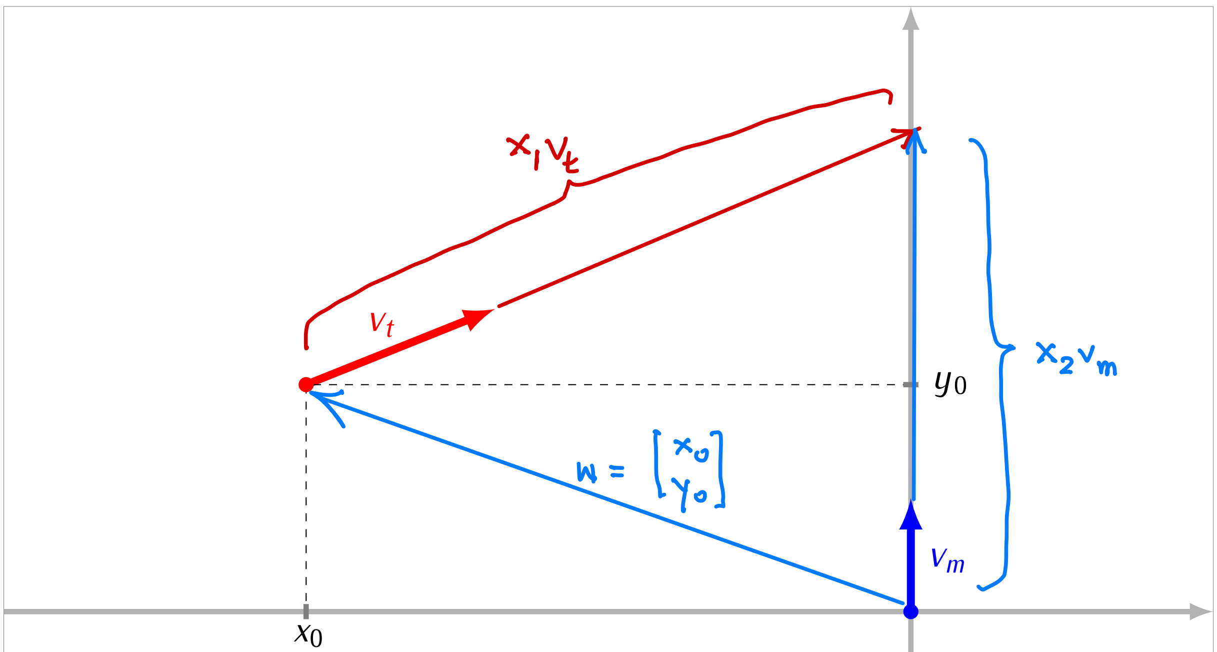
read off solutions

$$\begin{cases} x_1 & = \ldots \\ \ldots & \quad \ldots \\ x_p & = \ldots \end{cases}$$

solutions

Example: Target shooting.

At time $t = 0$ a target is observed at the position (x_0, y_0) moving in the direction of the vector v_t. The target is moving at such speed, that it travels the length of v_t in one second. A missile is positioned at the point $(0, 0)$. When fired, it will move vertically at such speed, that it will travel the length of the vector v_m in one second. After how many seconds should the missile be fired in order to intercept the target?



x_1 = duration of flight of the target
x_2 = duration of flight of the missile

$\boxed{t_0 = x_1 - x_2}$

So it is enough to find x_1 and x_2.

$w + x_1 v_t = x_2 v_m$

$\boxed{x_1 v_t - x_2 v_m = -w} \longleftarrow$ solve for x_1, x_2