

Operations on matrices so far:

- addition/subtraction  $A \pm B$
- scalar multiplication  $c \cdot A$
- matrix multiplication  $A \cdot B$
- matrix transpose  $A^T$

Next: How to divide matrices?

**Definition**

A matrix  $A$  is *invertible* if there exists a matrix  $B$  such that

$$A \cdot B = B \cdot A = I$$

(where  $I$  = the identity matrix). In such case we say that  $B$  is the *inverse* of  $A$  and we write  $B = A^{-1}$ .

**Example.**

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ is invertible, } A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

## Matrix inverses and matrix equations

### Proposition

If  $A$  is an invertible matrix then for any vector  $\mathbf{b}$  the equation  $A\mathbf{x} = \mathbf{b}$  has exactly one solution.

**Example.** Solve the following matrix equation:

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

**Recall:** If  $A$  be is  $m \times n$  matrix then:

- the matrix equation  $A\mathbf{x} = \mathbf{b}$  has a solution for every  $\mathbf{b}$  if and only if  $A$  has a pivot position in every row;
- the matrix equation  $A\mathbf{x} = \mathbf{b}$  has exactly one solution for  $\mathbf{b} \in \text{Col}(A)$  if and only if  $A$  has a pivot position in every column.

### Theorem

If a matrix  $A$  is invertible then it must be a square matrix.

For a square matrix  $A$  the following conditions are equivalent:

- 1)  $A$  is an invertible matrix.
- 2) The matrix  $A$  has a pivot position in every row and column.
- 3) The reduced row echelon form of  $A$  is the identity matrix  $I_n$ .

### Proposition

If  $A$  is an  $n \times n$  invertible matrix then

$$A^{-1} = [ \mathbf{w}_1 \quad \mathbf{w}_2 \quad \dots \quad \mathbf{w}_n ]$$

where  $\mathbf{w}_i$  is the solution of  $A\mathbf{x} = \mathbf{e}_i$ .

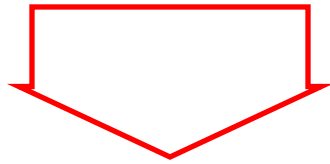
**Example.**

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

**Simplification:**  
How to solve several matrix equations with the same  
coefficient matrix at the same time

$$Ax = \mathbf{b}_1, Ax = \mathbf{b}_2, \dots, Ax = \mathbf{b}_n$$

matrix of equations



$$\left[ A \mid \mathbf{b}_1 \ \mathbf{b}_2 \ \dots \ \mathbf{b}_n \right]$$

augmented matrix



$$\left[ \begin{array}{c|c} & \end{array} \right]$$

reduced matrix



solutions

**Example.** Solve the vector equations  $A\mathbf{x} = \mathbf{e}_1$  and  $A\mathbf{x} = \mathbf{e}_2$  where

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

**Summary:**  
**How to invert a matrix**

**Example:**  $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

1) Augment  $A$  by the identity matrix.

2) Reduce the augmented matrix.

2) If after the row reduction the matrix on the left is the identity matrix, then  $A$  is invertible and

$$A^{-1} = \text{the matrix on the right}$$

Otherwise  $A$  is not invertible.



## Properties of matrix inverses

1) If  $A$  is invertible then  $A^{-1}$  is invertible and

$$(A^{-1})^{-1} = A$$

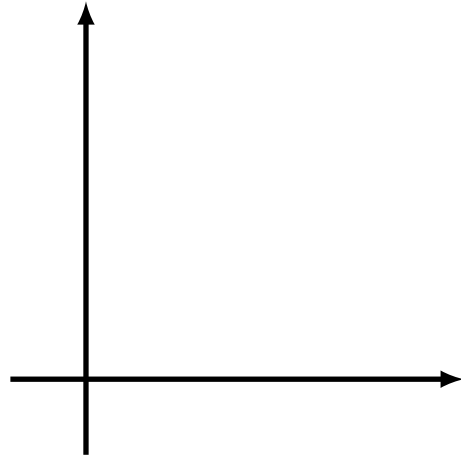
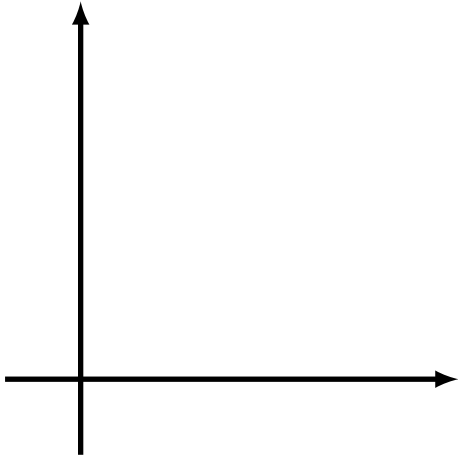
2) If  $A, B$  are invertible then  $AB$  is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

3) If  $A$  is invertible then  $A^T$  is invertible and

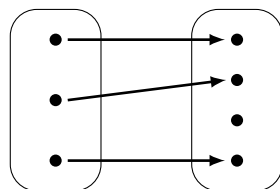
$$(A^T)^{-1} = (A^{-1})^T$$

## Matrix inverses and matrix transformations

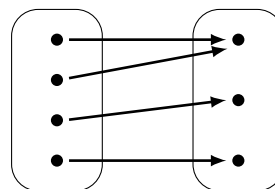


### Note

- If  $A$  is an  $n \times n$  invertible matrix then the matrix transformation  $T_A: \mathbb{R}^n \rightarrow \mathbb{R}^n$  has the inverse function  $T_{A^{-1}}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ .
- As a consequence the function  $T_A$  is both onto and one-to-one.



not onto



not one-to-one

**Example.**

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

