Recall:

1) If A is an $m \times n$ matrix then the function

$$T_A\colon \mathbb{R}^n \to \mathbb{R}^m$$

defined by $T_A(\mathbf{v}) = A\mathbf{v}$ is called the matrix transformation associated to A.

2) A function $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if

(ii) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
(ii) $T(c\mathbf{v}) = cT(\mathbf{v})$

3) Every matrix transformation is a linear transformation.

4) Every linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation:

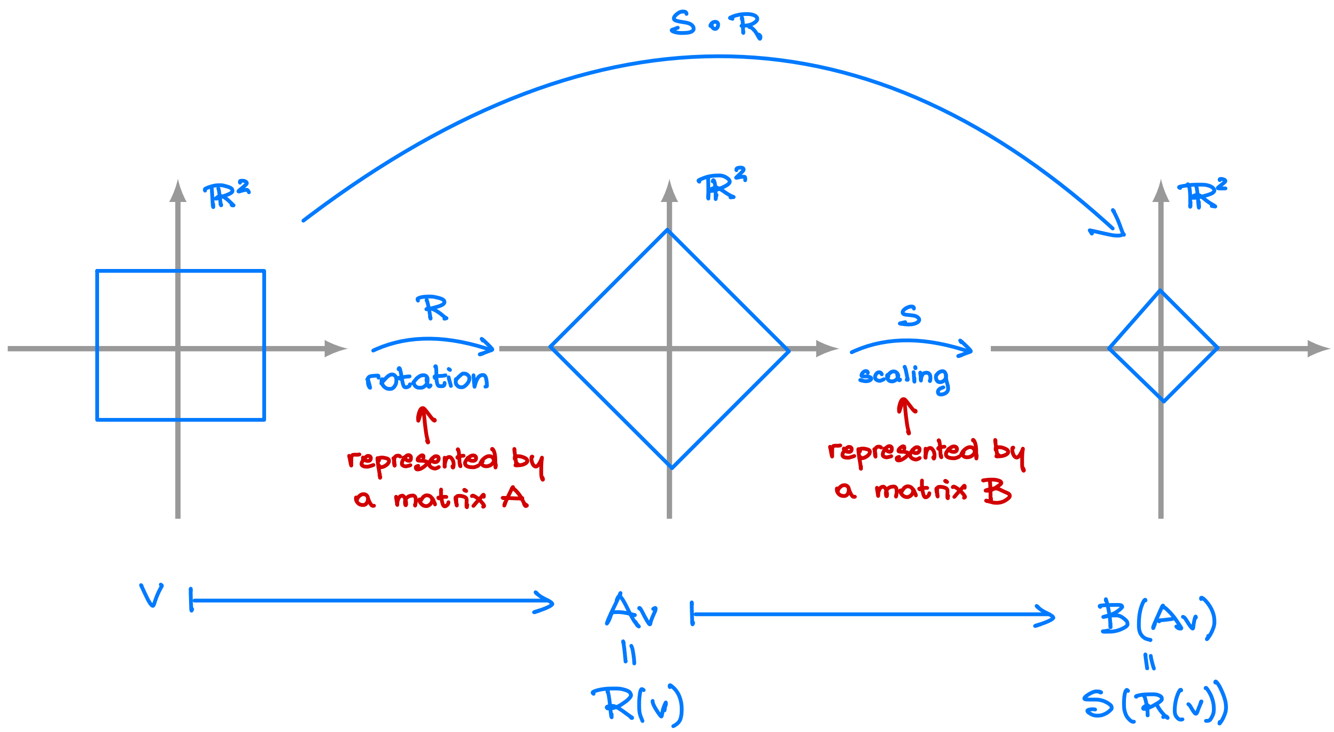
$$T(\mathbf{v}) = A\mathbf{v}$$

where

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \end{bmatrix}$$

The matrix A is called the standard matrix of T.

Composition of linear transformations



Question:

Is there a matrix C such that

$$S \circ R(v) = Cv$$

(or equivalently: $Cv = B(Av)$)?

Theorem

If $S\colon \mathbb{R}^n \to \mathbb{R}^m$ and $T\colon \mathbb{R}^m \to \mathbb{R}^k$ are linear transformation then the composition

$$T \circ S\colon \mathbb{R}^n \to \mathbb{R}^k$$

is also a linear transformation.

Proof: We need to check:

1) $T\circ S(u+v) = T\circ S(u) + T\circ S(v)$

2) $T\circ S(cu) = c\cdot (T\circ S(u))$

1) $T\circ S(u+v) = T(S(u+v)) = T(S(u)+S(v))$

since S is linear

$= T(S(u)) + T(S(v))$

since T is linear

$= T\circ S(u) + T\circ S(v)$

2) Similar.

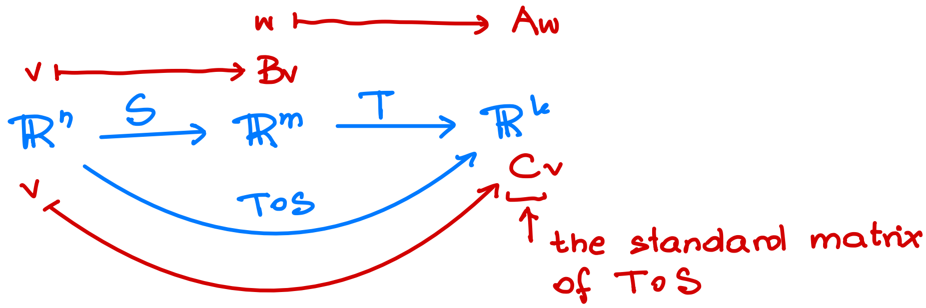
Upshot. The function $T \circ S$ is represented by some matrix C:

$$T \circ S(\mathbf{v}) = C\mathbf{v}$$

Question. Let $S\colon \mathbb{R}^n \to \mathbb{R}^m$ and $T\colon \mathbb{R}^m \to \mathbb{R}^k$ be linear transformations, and let

- B is the standard matrix of S
- A is the standard matrix of T

What if the standard matrix of $T \circ S\colon \mathbb{R}^n \to \mathbb{R}^k$?



We have:

$$C = [\,T\circ S(e_1) \quad T\circ S(e_2) \;\cdots\; T\circ S(e_n)\,]$$
$$= [\,T(S(e_1)) \quad T(S(e_2)) \;\cdots\; T(S(e_n))\,]$$
$$= [\,A(Be_1) \quad A(Be_2) \;\cdots\; A(Be_n)\,]$$

Note: If $B = [v_1 \ v_2 \ \cdots \ v_n]$ then $Be_1 = [v_1 \ v_2 \ \cdots \ v_n] \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = v_1$

In the same way: $Be_2 = v_2, \dots, Be_n = v_n$

We obtain:

$$C = [\,Av_1 \quad Av_2 \;\cdots\; Av_n\,]$$

where $v_1, v_2, \dots, v_n$ – columns of B

Definition

Let

- A be an $k \times m$ matrix
- $B = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}$ be an $m \times n$ matrix

Then $A \cdot B$ is an $k \times n$ matrix given by

$$A \cdot B = \begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & \dots & A\mathbf{v}_n \end{bmatrix}$$

Note:

If $S: \mathbb{R}^n \to \mathbb{R}^m$, $T: \mathbb{R}^m \to \mathbb{R}^k$ - linear transformations

B = the standard matrix of S

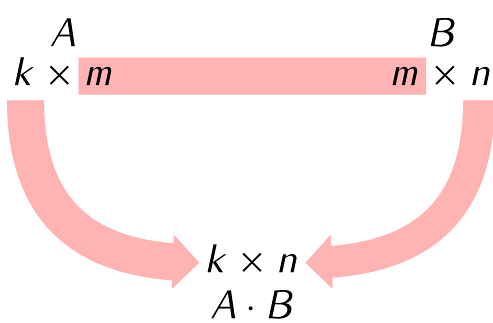
A = the standard matrix of T

then AB is the standard matrix of $T \circ S$:

$$T \circ S(v) = (AB)v$$

Note. The product $A \cdot B$ is defined only if

$$(\text{number of columns of } A) \;=\; (\text{number of rows of } B)$$



Example.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} \qquad B = \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & -1 & 2 & 1 \\ 4 & 5 & 1 & 0 \\ 1 & 2 & 3 & 1 \end{bmatrix} \end{array} = [v_1 \; v_2 \; v_3 \; v_4]$$

2×3 $\qquad$ 3×4

AB is defined
and it is a 2×4 matrix

$$AB = [Av_1 \;\; Av_2 \;\; Av_3 \;\; Av_4]$$

$$Av_1 = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} = 0 \cdot \begin{bmatrix} 0 \\ 3 \end{bmatrix} + 4 \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 1 \cdot \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 21 \end{bmatrix}$$

$$Av_2 = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix} = (-1) \cdot \begin{bmatrix} 0 \\ 3 \end{bmatrix} + 5 \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 2 \cdot \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 9 \\ 27 \end{bmatrix}$$

$$Av_3 = \quad \ldots \quad \ldots \quad = \begin{bmatrix} 7 \\ 25 \end{bmatrix}$$

$$Av_4 = \quad \ldots \quad \ldots \quad = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

<u>We obtain</u>:

$$AB = \begin{bmatrix} 6 & 9 & 7 & 2 \\ 21 & 27 & 25 & 8 \end{bmatrix}$$