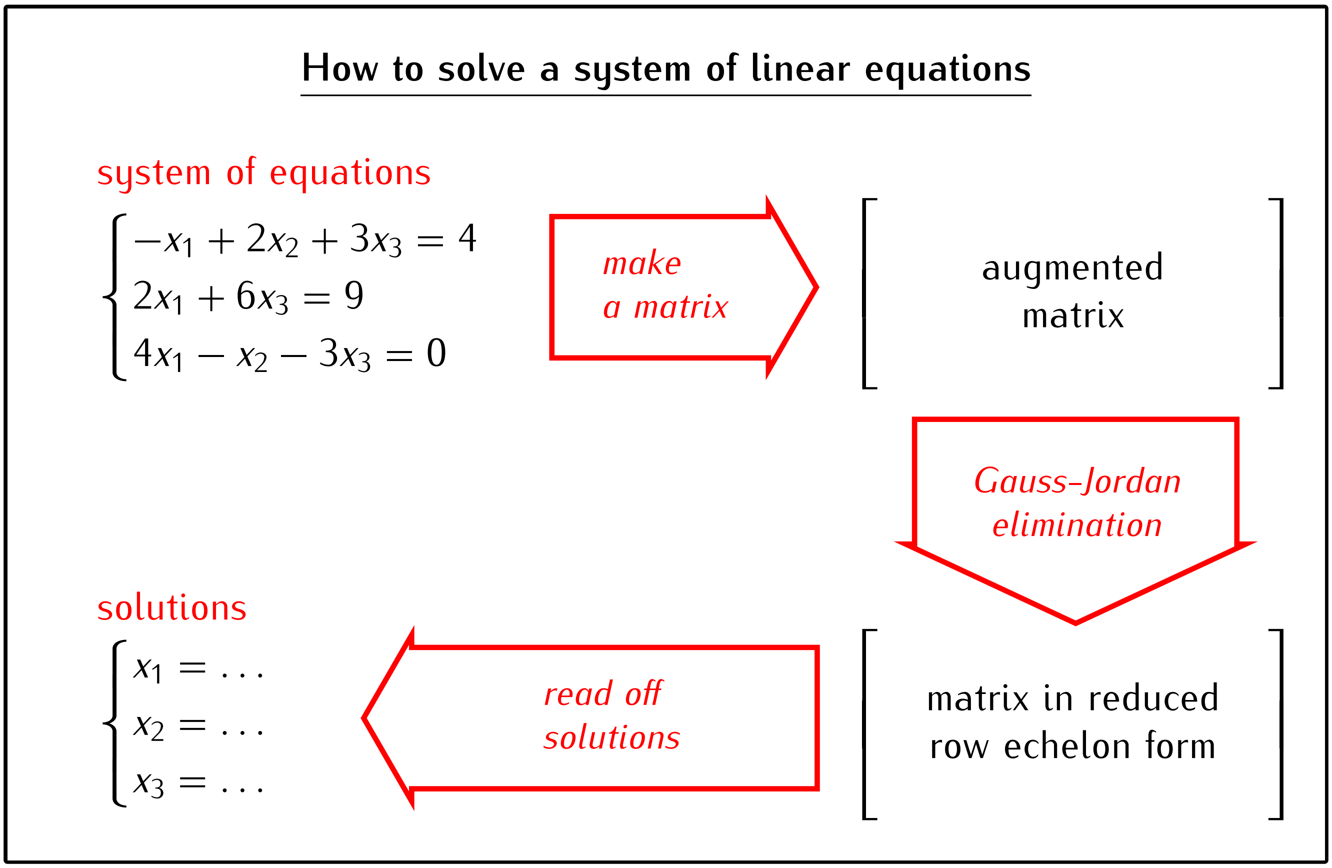
**Recall:**




- Every system of linear equations can be represented by a matrix

- Elementary row operations:
  - interchange of two rows
  - multiplication of a row by a non-zero number
  - addition of a multiple of one row to another row.

- Elementary row operations do not change solutions of systems of linear equations.

## Definition

A matrix is in the *row echelon form* if:

1) the first non-zero entry of each row is a 1 ("a leading one");
2) the leading one in each row is to the right of the leading one in the row above it.

A matrix is in the *reduced row echelon form* if in addition it satisfies:

3) all entries above each leading one are 0.

leading ones

$$\begin{bmatrix} 1 & * & * & * & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

($* =$ any number)

**Example**

$$\begin{bmatrix} 1 & 0 & 4 & 0 & 7 & 0 \\ 0 & 1 & 5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

reduced row echelon form

$$\begin{bmatrix} 1 & 2 & 4 & 6 & 7 & 0 \\ 0 & 1 & 5 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

echelon form

$$\begin{bmatrix} 1 & 0 & 4 & 0 & 7 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 6 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

NOT row echelon form

**Fact**

If a system of linear equations is represented by a matrix in the reduced row echelon form then it is easy to solve the system.

**Example**

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \phantom{0} \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \longrightarrow \begin{cases} x_1 + 3x_3 = 0 \\ x_2 + 7x_3 = 0 \\ x_4 = 0 \\ \boxed{0x_1 + 0x_2 + 0x_3 + 0x_4 = 1} \end{cases}$$

$0 = 1$

impossible, so:

<u>NO SOLUTIONS</u>

**Proposition**

A matrix in the reduced row echelon form represents an inconsistent system if and only if it contains a row of the form

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

i.e. with the leading one in the last column.

**Example**

$$
\begin{array}{c}
\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\
\left[\begin{array}{cccc|c}
1 & 0 & 3 & 0 & 0 \\
0 & 1 & 7 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
$$

no leading one in the column of $x_3$

$$
\begin{cases}
x_1 + 3x_3 = 0 \\
x_2 + 7x_3 = 0 \\
x_4 = 0 \\
0 = 0
\end{cases}
$$

$$
\downarrow
$$

$$
\begin{cases}
x_1 = -3x_3 \\
x_2 = -7x_3 \\
x_3 = \text{any number} \\
x_4 = 0
\end{cases}
$$

$x_3$ is a <u>free variable</u>

$$
\begin{cases}
x_1 = -3x_3 \\
x_2 = -7x_3 \\
x_3 = x_3 \\
x_4 = 0
\end{cases}
$$

<u>INFINITELY MANY SOLUTIONS</u>

**Note**

In an augmented matrix in the reduced row echelon form free variables correspond to columns of the coefficient matrix that do not contain leading ones.

**Example**

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 8 \end{array}\right]$$

$$\begin{cases} x_1 = 5 \\ x_2 = 6 \\ x_3 = 7 \\ x_4 = 8 \end{cases}$$

EXACTLY ONE SOLUTION

**Note**

A matrix in the reduced row echelon form represents a system of equations with exactly one solution if and only if it has a leading one in every column except for the last one.