Definition

A *homogenous vector equation* is a vector equation of the form

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

(i.e. with the zero vector as the vector of constants).

Note: A homogenous equation always has at least one, trivial solution: $x_1 = 0, x_2 = 0, \ldots, x_p = 0$.

This leaves two possibilities for homogoneous equations:

① only one solution

e.g.

$$x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

② infinitely many solutions

e.g.

$$x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}-1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Definition

Let $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is *linearly independent* if the homogenous equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

has only one, trivial solution $x_1 = 0, \ldots, x_p = 0$. Otherwise the set is *linearly dependent*.

e.g. the set

$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ is linearly independent

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Theorem

Let $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$. Consider the equation

$$x_1\mathbf{v}_1 + \ldots + x_p\mathbf{v}_p = \mathbf{w}$$

where $\mathbf{w} \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p)$.

If the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent then this equation has exactly one solution.

If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent then this equation has infinitely many solutions.

Proof: Assume that $\{v_1, \ldots, v_p\}$ - lin. independent and that $x_1v_1 + \ldots + x_pv_p = w$ has two solutions:

$$c_1v_1 + \ldots + c_pv_p = w$$
$$d_1v_1 + \ldots + d_pv_p = w$$

then: $(c_1-d_1)v_1 + \ldots + (c_p-d_p)v_p = w - w = \mathbf{0}$

By linear independence we get: $(c_1-d_1)=0$
$\vdots$
$(c_p-d_p)=0$

So: $c_1 = d_1, \ldots, c_p = d_p$.

Conversely, assume that $\{v_1, \ldots, v_p\}$ - lin. dependent.

Then we have $c_1v_1 + \ldots + c_pv_p = \mathbf{0}$ where $c_i \neq 0$ for some i.

If $d_1v_1 + \ldots + d_pv_p = w$ then

$(c_1+d_1)v_1 + \ldots + (c_p+d_p)v_p =$
$(c_1v_1 + \ldots + c_pv_p) + (d_1v_1 + \ldots + d_pv_p) = \mathbf{0} + w = w$

It follows that the equation $x_1v_1 + \ldots + x_pv_p = w$ has two different solutions:

$$\begin{cases} x_1 = d_1 \\ \vdots \\ x_p = d_p \end{cases} \quad \text{and} \quad \begin{cases} x_1 = d_1 + c_1 \\ \vdots \\ x_p = d_p + c_p \end{cases}$$

Example. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ -12 \end{bmatrix}$$

Check is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

Solution

We need to solve

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = \mathbf{0}$$

augmented matrix:

$$[\, v_1 \;\; v_2 \;\; v_3 \mid \mathbf{0} \,] = \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ -2 & 4 & -12 & 0 \end{array}\right] \xrightarrow[\text{reduction}]{\text{row}} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & \underbrace{0} & 0 \end{array}\right]$$

free variable so infinitely many solutions

Thus the set $\{v_1, v_2, v_3\}$ is not linearly independent.

Note

A set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent if and only if every column of the matrix

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_p \end{bmatrix}$$

is a pivot column.

Some properties of linearly (in)dependent sets

1) A set consisting of one vector $\{\mathbf{v}_1\}$ is linearly dependent if and only if $\mathbf{v}_1 = \mathbf{0}$.

- if $v_1 \neq \mathbf{0}$ then $x_1 v_1 = \mathbf{0}$ has only one solution $x_1 = 0$,
  so $\{v_1\}$ is lin. independent.

- if $v_1 = \mathbf{0}$ then $x_1 v_1 = \mathbf{0}$ holds for any value of x_1,
  so $\{v_1\}$ is lin. dependent.

2) A set consisting of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if and only if one vector is a scalar multiple of the other.

If $\{v_1, v_2\}$ is lin. dependent then

$$c_1 v_1 + c_2 v_2 = \mathbf{0}$$

for some c_1, c_2 such that $c_1 \neq 0$ or $c_2 \neq 0$.

Say $c_1 \neq 0$. Then:

$$c_1 v_1 = -c_2 v_2$$

$$v_1 = \left(-\frac{c_2}{c_1}\right) v_2$$

So v_1 is a multiple of v_2.

3) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a set of p vectors in $\mathbb{R}^n$ and $p > n$ then this set is linearly dependent.

We need to show that if $p > n$ then not every column of the matrix

$$[\, v_1 \ \cdots \ v_p \,]$$

is a pivot column.

E.g. $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $v_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ $v_3 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

$$[\, v_1 \ v_2 \ v_3 \,] = \underbrace{\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}}_{\substack{p = 3 \\ \text{columns}}} \Big\} \ \underbrace{n = 2 \ \text{rows}}$$

This means that there are at most 2 leading ones, so at most 2 pivot columns. Thus we will have a non-pivot column.

Upshot: how to find the number of solutions of a vector equation