Recall: A vector equation

$$x_1\mathbf{v}_1 + \ldots + x_n\mathbf{v}_n = \mathbf{b}$$

has a solution if and only if $\mathbf{b} \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$.

Definition

If A is a matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$:

$$A = \begin{bmatrix} \mathbf{v}_1 & \ldots & \mathbf{v}_n \end{bmatrix}$$

then the set $\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is called the *column space* of A and it is denoted $\mathrm{Col}(A)$.

Upshot. A matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b} \in \mathrm{Col}(A)$.

Recall: A vector equation

$$x_1\mathbf{v}_1 + \ldots + x_n\mathbf{v}_n = \mathbf{b}$$

has only one solution for each $\mathbf{b} \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ if and only if the homogenous equation

$$x_1\mathbf{v}_1 + \ldots + x_n\mathbf{v}_n = \mathbf{0}$$

has only the trivial solution $x_1 = 0, \ldots, x_n = 0$.

Definition

If A is a matrix then the set of solution of the homogenous equation

$$A\mathbf{x} = \mathbf{0}$$

is called the *null space* of A and it is denoted $\mathrm{Nul}(A)$.

Upshot. A matrix equation $A\mathbf{x} = \mathbf{b}$ has only one solution for each $\mathbf{b} \in \mathrm{Col}(A)$ if and only if $\mathrm{Nul}(A) = \{\mathbf{0}\}$.

Example. Find the null space of the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Proposition

$\mathrm{Nul}(A) = \{\mathbf{0}\}$ if and only if the matrix A has a pivot position in every column.

Example. Find the null space of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -2 & -2 & 1 & -5 \\ 1 & 1 & -1 & 3 \end{bmatrix}$$

Note

If A is an $m \times n$ matrix then $\mathrm{Nul}(A)$ can be always described as a span of some vectors in $\mathbb{R}^n$.

Example. Solve the matrix equation $A\mathbf{x} = \mathbf{b}$ where

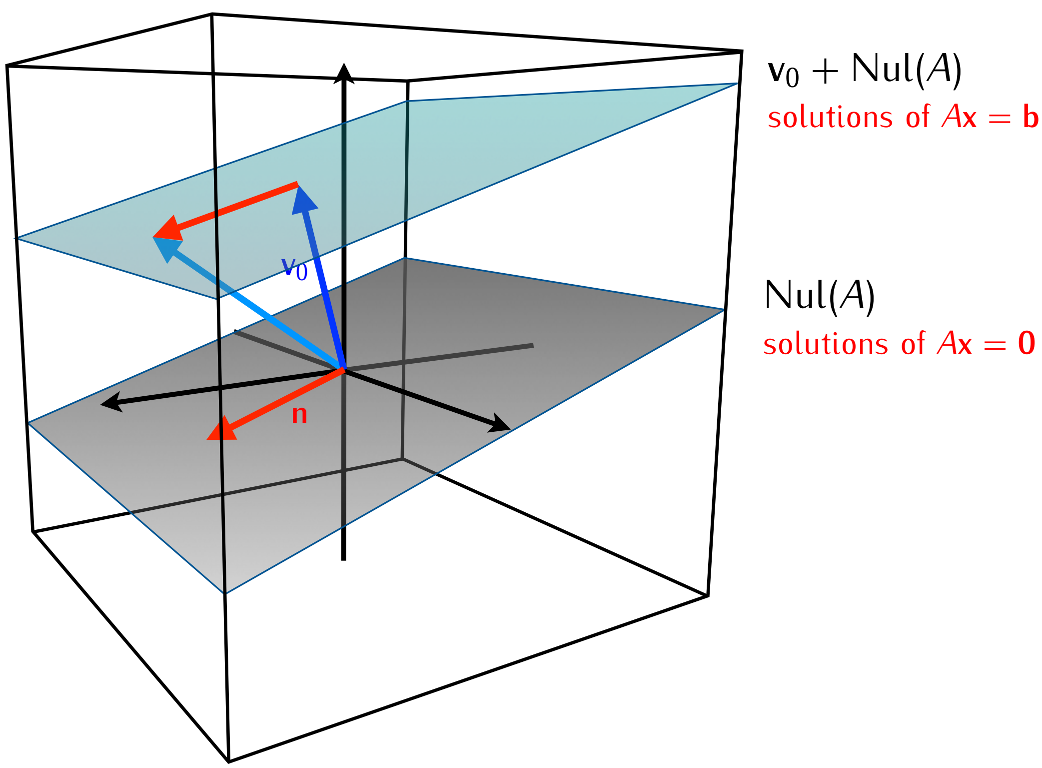
$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -2 & -2 & 1 & -5 \\ 1 & 1 & -1 & 3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Proposition

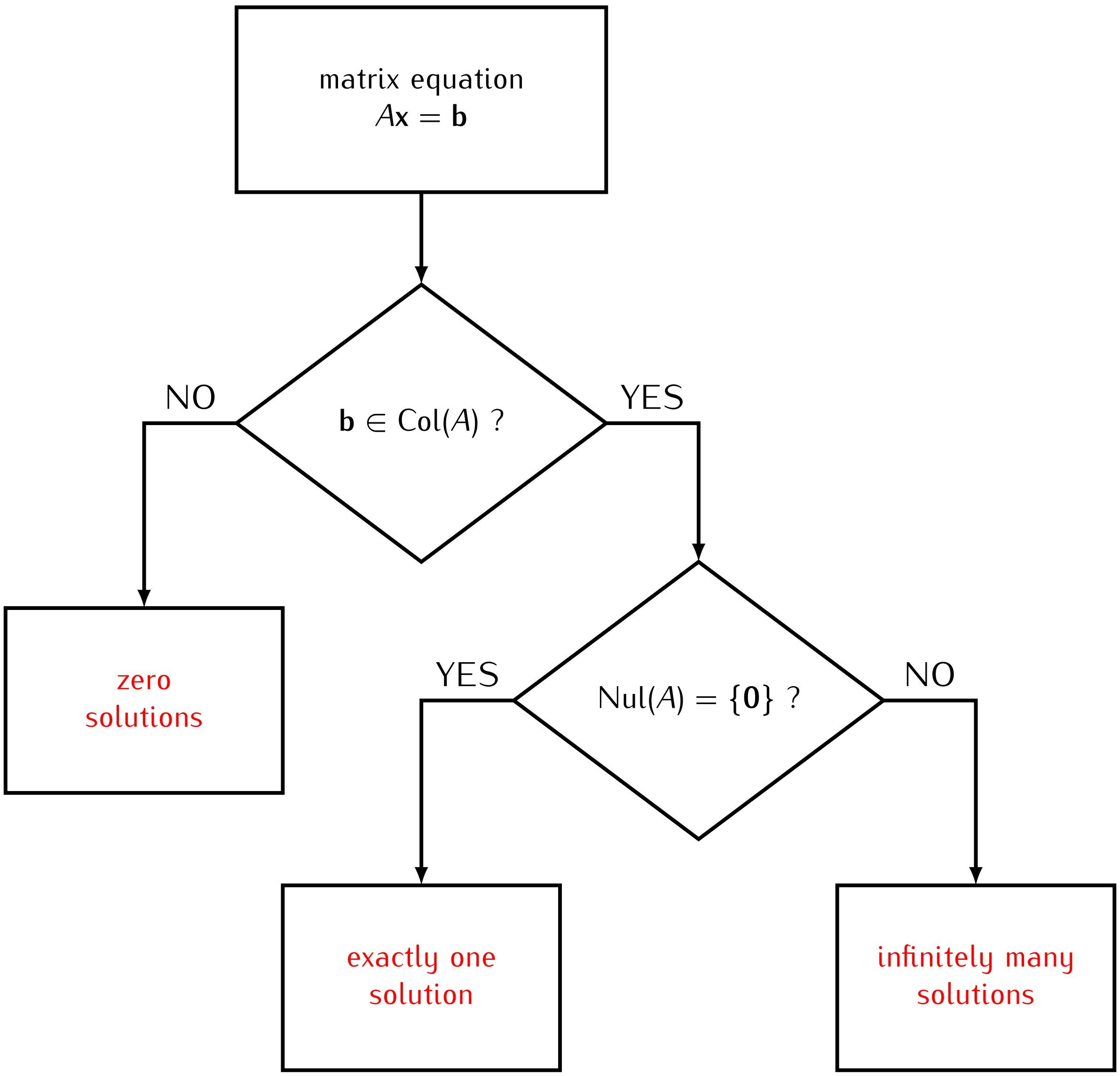
Let $\mathbf{v}_0$ be some chosen solution of a matrix equation $A\mathbf{x} = \mathbf{b}$. Then any other solution $\mathbf{v}$ of this equation is of the form

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{n}$$

where $\mathbf{n} \in \mathrm{Nul}(A)$.



Upshot: how to find the number of solutions of a matrix equation



Question: What conditions on the matrix A guarantee that the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every vector $\mathbf{b}$?

Example.

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \xrightarrow{\textit{row reduction}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Example.

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 7 \end{bmatrix} \xrightarrow{\textit{row reduction}} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Proposition

A matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution for every vector $\mathbf{b}$ if and only if A has a pivot position in every row.

In such case $\mathrm{Col}(A) = \mathbb{R}^m$, where m is the number of rows of A.