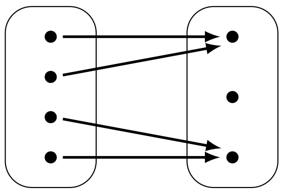
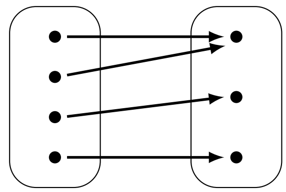
MTH 309 — 14. Onto and one-to-one transformations

Recall: A function $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is:

- *onto* if for each $\mathbf{b} \in \mathbb{R}^m$ there is $\mathbf{v} \in \mathbb{R}^n$ such that $F(\mathbf{v}) = \mathbf{b}$;

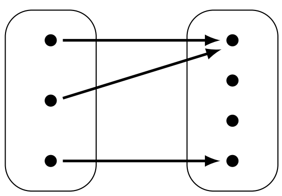


not onto

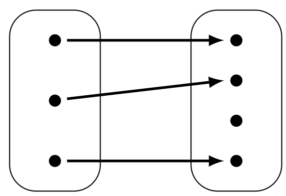


onto

- *one-to-one* if for any $\mathbf{v}_1, \mathbf{v}_2$ such that $\mathbf{v}_1 \neq \mathbf{v}_2$ we have $F(\mathbf{v}_2) \neq F(\mathbf{v}_2)$.



not one-to-one



one-to-one

Proposition

Let A be an $m \times n$ matrix. The following conditions are equivalent:

1) The matrix transformation $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ is onto.

2) $\mathrm{Col}(A) = \mathbb{R}^m$.

3) The matrix A has a pivot position in every row.

Proposition

Let A be an $m \times n$ matrix. The following conditions are equivalent:

1) The matrix transformation $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one.

2) $\mathrm{Nul}(A) = \{0\}$.

3) The matrix A has a pivot position in every column.

Example. For the following 2×2 matrix A check if the matrix transformation $T_A\colon \mathbb{R}^2 \to \mathbb{R}^2$ is onto and if it is one-to-one.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\Big\downarrow \text{ row reduction}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

pivot position in every row $\Rightarrow$ $\mathrm{Col}(A) = \mathbb{R}^2$

$\Rightarrow$ T_A is onto

pivot position in every column $\Rightarrow$ $\mathrm{Nul}(A) = \{\mathbf{0}\}$

$\Rightarrow$ T_A is one-to-one

Example. For the following 3×4 matrix A check if the matrix transformation $T_A\colon \mathbb{R}^4 \to \mathbb{R}^3$ is onto and if it is one-to-one.

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -2 & -2 & 1 & -5 \\ 1 & 1 & -1 & 4 \end{bmatrix}$$

$$\Big\downarrow \text{ row reduction}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

pivot position in every row $\Rightarrow$ $\mathrm{Col}(A) = \mathbb{R}^2$

$\Rightarrow$ T_A is onto

no pivot position in the second column $\Rightarrow$ $\mathrm{Nul}(A) \neq \{\mathbf{0}\}$

$\Rightarrow$ T_A is <u>not</u> one-to-one

Proposition

Let A be an $m \times n$ matrix. If the matrix transformation $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ is both onto and one-to-one then we must have $m = n$ (i.e. A must be a square matrix).